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Edge orbits and cyclic and r-pyramidal decompositions of complete uniform hypergraphs

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ABSTRACT

Let $\mathbb{Z}_n$ denote the group of integers modulo n and let $\mathcal{E}_n^{(k)}$ be the set of all k-element subsets of $\mathbb{Z}_n$ where $1 \le k < n$. If $E \in \mathcal{E}_n^{(k)}$, let $[E] = \{E + r : r \in \mathbb{Z}_n\}$. Then $[E]$ is the orbit of E where $\mathbb{Z}_n$ acts on $\mathcal{E}_n^{(k)}$ via $(r, E) \mapsto E + r$. Furthermore, $\{[E] : E \in \mathcal{E}_n^{(k)}\}$ is a partition of $\mathcal{E}_n^{(k)}$ into $\mathbb{Z}_n$-orbits. In this article, we count the total number of $\mathbb{Z}_n$-orbits of $\mathcal{E}_n^{(k)}$, count the number of orbits of each size, determine the corresponding results when fixed points are introduced, and give an application to cyclic and r-pyramidal decompositions of complete uniform hypergraphs into isomorphic subgraphs.

1. Introduction

If $(G, \oplus)$ is a group with identity 0, X is a set, and f is a function from $G \times X$ into X such that $f(0, x) = x$ for all $x \in X$ and $f(g_1 \oplus g_2, x) = f(g_1, f(g_2, x))$ for all $g_1, g_2 \in G$ and $x \in X$, then it is said that G *acts on* X. Furthermore, f partitions X into G-*orbits*, where two elements $x, y \in X$ are in the same orbit if and only if $x = f(g, y)$ for some $g \in G$. For $g \in G$, let $\text{Fix}(g) = \{x \in X : f(g, x) = x\}$. Thus $\text{Fix}(g)$ is the set of elements of X that are fixed by g. Burnside's lemma (see [10]) gives the number of G-orbits of X.

Lemma 1 (*Burnside's Lemma*)**.** *Let a finite group G act on a set X. The number of orbits that G induces is given by:*

$$\frac{1}{|G|}\sum_{g\in G}|\text{Fix}(g)|.$$

Let $\mathbb{Z}_n$ denote the group of integers modulo n and let $\mathcal{E}_n^{(k)}$ be the set of all k-element subsets of $\mathbb{Z}_n$ where $1 < k < n$. If $E \in \mathcal{E}_n^{(k)}$ and $r \in \mathbb{Z}_n$, let $E + r$ be formed by replacing each element $x \in E$ with $x + r$; so $(r, E) \mapsto E + r$ maps $\mathbb{Z}_n \times \mathcal{E}_n^{(k)}$ into $\mathcal{E}_n^{(k)}$. It can be seen that the group $\mathbb{Z}_n$ acts on the set $\mathcal{E}_n^{(k)}$ partitioning it into $\mathbb{Z}_n$-orbits, where $E_1, E_2 \in \mathcal{E}_n^{(k)}$ are in the same orbit if and only if $E_1 + r = E_2$ for some $r \in \mathbb{Z}_n$. We define $[E]$ to be $\{E + r : r \in \mathbb{Z}_n\}$, which we refer to as the $\mathbb{Z}_n$-*orbit of* E. If $S \subseteq \mathcal{E}_n^{(k)}$ and $r \in \mathbb{Z}_n$, let $S + r = \{E + r : E \in S\}$. By *clicking* S, we shall mean replacing S with $S + 1$.

1.1. Applications in hypergraphs

The set $\mathcal{E}_n^{(k)}$ can be thought of as being the edge set of the complete k-uniform hypergraph $K_n^{(k)}$ with vertex set $\mathbb{Z}_n$. The $\mathbb{Z}_n$-orbit of an edge E can be viewed as the set resulting from successively clicking E.

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Let K and G be k-uniform hypergraphs with G a subgraph of K. A *G-decomposition* of K is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i. In this case, we may refer to the elements of Γ as *G-blocks*. A G-decomposition of K is also known as a *(K, G)-design*. A $(K_m^{(k)}, K_n^{(k)})$-design is known as a *Steiner system* $S(k, m, n)$. A summary of results on Steiner systems $S(k, m, n)$ can be found in [6].

Let G be a subgraph of $K_n^{(k)}$, where $V(K_n^{(k)}) = \mathbb{Z}_n$, and let Γ be a G-decomposition of $K_n^{(k)}$. Then Γ is said to be *cyclic* if Γ is closed under clicking. Thus if $G_i \in \Gamma$, then $G_i + 1 \in \Gamma$. If we partition $\mathcal{E}_n^{(k)}$ into m distinct $\mathbb{Z}_n$-orbits each of size n and if G a subgraph of $K_n^{(k)}$ consisting of one edge from each of the m distinct $\mathbb{Z}_n$-orbits, then $\Gamma = \{G + i : i \in \mathbb{Z}_n\}$ is a cyclic G-decomposition of $K_n^{(k)}$. For example, if G is the subgraph of $K_8^{(3)}$ with edge-set $S = \big\{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 1, 5\}, \{0, 1, 6\}, \{0, 2, 4\}, \{0, 2, 5\}\big\}$, then $\Gamma = \{G + i : i \in \mathbb{Z}_8\}$ constitutes a cyclic G-decomposition of $K_8^{(3)}$.

The requirement that the hypergraph G in the previous paragraph must contain exactly one edge from each of the different $\mathbb{Z}_n$-orbits of size n can be viewed as an extension of the notion of a ρ-labeling of G in the case $k = 2$ as introduced by Rosa in [11]. Although stated differently in [11], a subgraph G of $K_{2m+1}^{(2)}$ with m edges admits a ρ-labeling if each of the m edges of G belongs to a different orbit under the action of $\mathbb{Z}_{2m+1}$. In [9], Meszka and Rosa extend this definition to 3-uniform hypergraphs.

Most of the work on G-decompositions of $K_n^{(k)}$ focuses on the case $k = 2$ (see [1] for a summary of known results). In general, little is known when $k \geq 3$. Some of the focus has been on G-decompositions of $K_n^{(3)}$, where G is a graph with a relatively small number of edges (see for example, [4] and [5]). Perhaps the best known general result on decompositions of complete k-uniform hypergraphs is Baranyai's result [3] on the existence of 1-factorizations of $K_{mk}^{(k)}$ for all positive integers m. There are, however, several articles on decompositions of complete k-uniform hypergraphs (see [2] and [8]) and of k-uniform k-partite hypergraphs (see [7] and [12]) into variations on the concept of a Hamilton cycle.

In this article, we count the total number of $\mathbb{Z}_n$-orbits of $\mathcal{E}_n^{(k)}$, count the number of orbits of each size, determine the corresponding results when fixed points are introduced, and give an application to cyclic and r-pyramidal decompositions of complete uniform hypergraphs into isomorphic subgraphs.

2. Counting orbits

In this paper lowercase letters represent integers. If a and b are integers, we define $[a, b]$ to be $\{r \in \mathbb{Z} : a \leq r \leq b\}$. In sums of the form $\sum_{e|m} f(e)$ we assume e is restricted to positive divisors. Let $d = \gcd(n, k)$, with $n = dn_0$ and $k = dk_0$, so $\gcd(n_0, k_0) = 1$. By ϕ and μ we mean the Euler ϕ-function and the Möbius function.

Lemma 2. *If $r > 0$ and $\gcd(r, n) = e$, then $\text{Fix}(r) = \text{Fix}(e)$.*

Proof. There exists a positive integer s such that $r = se$. First suppose $E \in \text{Fix}(e)$, so $E + e = E$. Then $E + r = E + se = E + e + e + \cdots + e = E$, and so $E \in \text{Fix}(r)$. Conversely, suppose $E \in \text{Fix}(r)$. Since $\gcd(r, n) = e$, we can find integers $x > 0$ and y such that $xr + yn = e$. Then, performing addition in $\mathbb{Z}_n$, we have $E + e = E + xr + yn = E + xr + 0 = E + r + r + \cdots + r = E$, and so $E \in \text{Fix}(e)$. □

Lemma 3. *Let n and k be positive integers and set $d = \gcd(n, k)$, $n = dn_0$, and $k = dk_0$. Suppose r is a positive integer such that $r \mid n$ and $n \mid kr$ (so $n_0 \mid r$). For any positive integer h, we have $h \mid r$ and $n \mid hk$ if and only if $h = en_0$, where $e \mid \frac{r}{n_0}$. Moreover, if $h \mid n$ then the number of integers r, $0 \leq r < n$, with $\gcd(n, r) = h$ is $\phi(n/h)$.*

Proof. First suppose $h \mid r$ and $n \mid hk$. Now $n_0 \mid hk_0$, so $n_0 \mid h$. Let $h = en_0$, and let $r = hh'$. Then $r/n_0 = hh'/n_0 = eh'$, so $e \mid r/n_0$.

Conversely, suppose $h = en_0$, where $e \mid r/n_0$. Then $h = en_0 \mid (r/n_0)n_0 = r$. Also $kh = ken_0 = (k/d)e(dn_0) = k_0en$, so $n \mid kh$.

To see the second statement note that $\gcd(r, n) = h$ if and only if $\gcd(r/h, n/h) = 1$, $0 \leq r/h < n/h$. □

Theorem 4. *Let k and n be positive integers, $d = \gcd(n, k)$, and let $\mathbb{Z}_n$ act on $\mathcal{E}_n^{(k)}$. Then the number t of $\mathbb{Z}_n$-orbits of $\mathcal{E}_n^{(k)}$ is given by*

$$t = \frac{1}{n}\sum_{e|d} \phi(e)\binom{n/e}{k/e}.$$

Proof. Suppose $\text{Fix}(q) \neq \varnothing$. Then by Lemma 2, $\text{Fix}(q) = \text{Fix}(h)$, where $h = \gcd(q, n)$. Suppose $E \in \text{Fix}(h)$. This implies that if $A \subseteq [0, h-1]$ and $A \subseteq E$, then $A + jh \subseteq E$ for $j \in [0, n/h - 1]$. Hence E must have the form

$$A \cup (h + A) \cup (2h + A) \cup \cdots \cup \big((n/h - 1)h + A\big).$$

Then $(n/h)|A| = |E| = k$, so $|A| = hk/n$ and $n \mid hk$. The number of ways of choosing $|A|$ elements from $\{0, 1, \ldots, h-1\}$ is $\binom{h}{|A|} = \binom{h}{hk/n}$, and this is the same for all the $\phi(n/h)$ values of r such that $\gcd(r, n) = h$. Thus by Burnside's Lemma and the first part of Lemma 3 with $r = n$,

$$t = \frac{1}{n}\sum_{\substack{h|n\\ n|hk}} \phi(n/h)\binom{h}{hk/n} = \frac{1}{n}\sum_{e|d}\phi\left(\tfrac{n}{en_0}\right)\binom{en_0}{en_0k/n}$$
$$= \frac{1}{n}\sum_{e|d}\phi(d/e)\binom{n/(d/e)}{k/(d/e)}.$$

Now if we notice that as e runs through the positive divisors of d, so does d/e, we get the formula of the theorem. □

Example 1. When $n = 30$ and $k = 24$, we have $d = 6$, so

$$t = \frac{1}{30}\sum_{e|6}\phi(e)\binom{30/e}{24/e}$$
$$= \frac{1}{30}\left(\phi(1)\binom{30}{24} + \phi(2)\binom{15}{12} + \phi(3)\binom{10}{8} + \phi(6)\binom{5}{4}\right)$$
$$= \frac{1}{30}(1\cdot 593{,}775 + 1\cdot 455 + 2\cdot 45 + 2\cdot 5) = 19{,}811.$$

3. Difference vectors

If $E \in \mathcal{E}_n^{(k)}$, we can write E uniquely as $\{a_1, a_2, \ldots, a_k\}$ where $0 \le a_1 < a_2 < \cdots < a_k < n$. By the *difference vector* of E, we mean the k-tuple $\Delta E = (a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, n + a_1 - a_k)$. Note that the components of ΔE are positive and sum to n. Also, distinct elements of a $\mathbb{Z}_n$-orbit may yield distinct difference vectors. For example, if $E = \{0, 3, 4\} \in \mathcal{E}_5^{(3)}$, then $\Delta E = (3, 1, 1)$, while $\Delta(E + 2) = \Delta\{0, 1, 2\} = (1, 1, 3)$. However, it is easy to see that edges are in the same $\mathbb{Z}_n$-orbit if and only if they have difference vectors that are cyclic permutations of each other. By the *reduced* difference vector $\Delta' E$ of an edge E, we mean the difference vector among the cyclic permutations of ΔE that is lexicographically first. In our example, the cyclic permutations of ΔE are $(3, 1, 1), (1, 3, 1)$, and $(1, 1, 3)$, and thus $\Delta' E = (1, 1, 3) = \Delta'(E + 2)$.

Now let $\mathcal{D}_n^{(k)}$ be all ordered k-tuples of positive integers with sum n. Using standard counting techniques, it can be proved that the number of sequences of u positive integers with sum v is $\binom{v-1}{u-1}$. In particular, $\left|\mathcal{D}_n^{(k)}\right| = \binom{n-1}{k-1}$. If $D = (b_1, b_2, \ldots, b_k)$ is a difference vector in $\mathcal{D}_n^{(k)}$ and $r \in \mathbb{Z}_k$, we define $D \oplus r$ to be $(b_{1+r}, b_{2+r}, \ldots, b_{k+r})$, where the subscripts are taken modulo k. Then the group $\mathbb{Z}_k$ acts on $\mathcal{D}_n^{(k)}$, and we will denote the $\mathbb{Z}_k$-orbit containing $D \in \mathcal{D}_n^{(k)}$ by $[D]$. There is a one-to-one correspondence between the $\mathbb{Z}_k$-orbits of $\mathcal{D}_n^{(k)}$ with respect to $\oplus$ and the $\mathbb{Z}_n$-orbits of $\mathcal{E}_n^{(k)}$ with respect to $+$. In particular, if $E \in \mathcal{E}_n^{(k)}$, then the $\mathbb{Z}_n$-orbit $[E]$ contained in $\mathcal{E}_n^{(k)}$ corresponds to the $\mathbb{Z}_k$-orbit $[\Delta E]$ contained in $\mathcal{D}_n^{(k)}$.

We can count the $\mathbb{Z}_k$-orbits of $\mathcal{D}_n^{(k)}$ using Burnside's Lemma. This yields the formula

$$t = \frac{1}{k}\sum_{e|d}\phi(e)\binom{n/e - 1}{k/e - 1},$$

where as before $d = \gcd(n, k)$. This formula gives the same result as that of Theorem 4.

Example 2. If $n = 30$ and $k = 24$ so $d = 6$, we have

$$t = \frac{1}{24}\sum_{e|6}\phi(e)\binom{30/e - 1}{24/e - 1}$$
$$= \frac{1}{24}\left(\phi(1)\binom{29}{23} + \phi(2)\binom{14}{11} + \phi(3)\binom{9}{7} + \phi(6)\binom{4}{3}\right)$$
$$= \frac{1}{24}(1\cdot 475{,}020 + 1\cdot 364 + 2\cdot 36 + 2\cdot 4) = 19{,}811.$$

4. Orbits of a given size

If $\gcd(n, k) = 1$ the sum in Theorem 4 has a single term, and the number of $\mathbb{Z}_n$-orbits of $\mathcal{E}_n^{(k)}$ is $\frac{1}{n}\binom{n}{k}$. Since $\left|\mathcal{E}_n^{(k)}\right| = \binom{n}{k}$, every orbit in this case has size n. If $\gcd(n, k) > 1$ however, there will be orbits of various sizes. In this section we count the number of orbits of each size.

We consider the $\mathbb{Z}_n$-orbits of $\mathcal{E}_n^{(k)}$. Let $N(h)$ be the number $\mathbb{Z}_n$-orbits in $\mathcal{E}_n^{(k)}$ with exactly h elements. By the *order* of $E \in \mathcal{E}_n^{(k)}$ we mean the least positive integer h such that $E + h = E$. Since $E + h = E + \gcd(h, n)$, this means $h \mid n$. Clearly the order of E is $\left|[E]\right|$. Recall that $\mathrm{Fix}(h) = \{E \in \mathcal{E}_n^{(k)} : E + h = E\}$.

Lemma 5. *Let s be a positive divisor of n. Then $E \in \mathrm{Fix}(s)$ if and only if E has order h, where $h \mid s$ and $n \mid hk$.*

Proof. Let $E \in \mathrm{Fix}(s)$ have order h, and let $s = q'h + r'$, where $0 \le r' < h$. Then $E = E + s = (E + h + h + \cdots + h) + r' = E + r'$, and so by the definition of order we must have $r' = 0$. Thus $h \mid s$. Since $\mathrm{Fix}(h) \neq \varnothing$ we have $n \mid hk$ as in the proof of Theorem 4.

Now assume the order of E is h, where $h \mid s$ and $n \mid hk$. Say $s = qh$. Then $E + s = E + h + h + \cdots + h = E$, so $E \in \mathrm{Fix}(s)$. □

Since $\mathrm{Fix}(n) = \mathcal{E}_n^{(k)}$, by taking $s = n$ in the previous lemma we see that $m(h) > 0$ if and only if $h \mid n$ and $n \mid hk$, and by Lemma 3 with $r = n$ this happens if and only if $h = en_0$, where $e \mid n/n_0 = d$. This is the first conclusion of the following theorem.

Theorem 6. *Let n and k be positive integers, and consider $\mathcal{E}_n^{(k)}$. Set $d = \gcd(n, k)$, $n = dn_0$, $k = dk_0$. The values of s such that $N(s) > 0$ are exactly the integers $s = en_0$, where e runs through the positive divisors of d. In this case*

$$N(s) = \frac{1}{s} \sum_{f \mid s_0} \mu(s_0/f) \binom{n_0 f}{k_0 f},$$

where $s_0 = s/n_0$.

Proof. Let s be a positive divisor of n such that $\mathrm{Fix}(s) \neq \varnothing$. Then as in the proof of Theorem 4 we have $n \mid sk$. Since if E has order h, then $\left|[E]\right| = h$, the total number of edges of order h is $hN(h)$. By Lemma 5 we have

$$|\mathrm{Fix}(s)| = \sum_{\substack{h \mid s \\ n \mid hk}} hN(h) = \sum_{e \mid s_0} en_0 N(en_0),$$

where in the second sum we have applied the first part of Lemma 3 and set $s_0 = s/n_0$. In the proof of Theorem 4 we found that $|\mathrm{Fix}(s)| = \binom{s}{sk/n}$, so we have

$$\sum_{e \mid s_0} en_0 N(en_0) = \binom{s}{sk/n} = \binom{n_0 s_0}{k_0 s_0}.$$

The last equation has the form $G(s_0) = \sum_{f \mid s_0} g(f)$, where $G(s_0) = \binom{n_0 s_0}{k_0 s_0}$, and $g(f) = fn_0 N(fn_0)$. By the Möbius inversion formula we have

$$g(s_0) = \sum_{f \mid s_0} \mu(s_0/f) G(f) \quad \text{or} \quad s_0 n_0 N(s_0 n_0) = sN(s) = \sum_{f \mid s_0} \mu(s_0/f) \binom{n_0 f}{k_0 f},$$

so

$$N(s) = \frac{1}{s} \sum_{f \mid s_0} \mu(s_0/f) \binom{n_0 f}{k_0 f}. \quad \square$$

Example 3. If $n = 30$ and $k = 24$ so that $d = 6$, we have $n_0 = 5$ and $k_0 = 4$. The divisors of d are 1, 2, 3, and 6, the corresponding values of s are 5, 10, 15, and 30, and of s_0 are 1, 2, 3, and 6. Then

$$N(5) = \frac{1}{5} \sum_{f \mid 1} \mu(1/f) \binom{5f}{4f} = \frac{1}{5}(1)\binom{5}{4} = 1,$$

$$N(10) = \frac{1}{10} \sum_{f \mid 2} \mu(2/f) \binom{5f}{4f} = \frac{1}{10}\left((-1)\binom{5}{4} + (1)\binom{10}{8}\right) = 4,$$

$$N(15) = \frac{1}{15} \sum_{f \mid 3} \mu(3/f) \binom{5f}{4f} = \frac{1}{15}\left((-1)\binom{5}{4} + (1)\binom{15}{12}\right) = 30,$$

$$\begin{aligned} N(30) &= \frac{1}{30} \sum_{f \mid 6} \mu(6/f) \binom{5f}{4f} \\ &= \frac{1}{30}\left((1)\binom{5}{4} + (-1)\binom{10}{8} + (-1)\binom{15}{12} + (1)\binom{30}{24}\right) \\ &= 19{,}776. \end{aligned}$$

Note that $1 + 4 + 30 + 19{,}776 = 19{,}811$, which agrees with the results in Examples 1 and 2.

If $E \in \mathcal{E}_n^{(k)}$ and $D \in \mathcal{D}_n^{(k)}$ are in corresponding orbits, then $\big|[D]\big| = (k/n)\big|[E]\big|$. Thus if we define $M(h)$ to be the number of orbits of $\mathcal{D}_n^{(k)}$ of size h, then $M(h) = N(nh/k)$. For example $\mathcal{D}_{30}^{(24)}$ has 1 orbit of size 4, 4 orbits of size 8, 30 orbits of size 12, and 19,776 orbits of size 24.

5. Orbits under *r*-Pyramidal Actions

Let n, k, and r be integers with $0 \le r < k < n$ and let $I_r = \{\infty_1, \infty_2, \dots, \infty_r\}$. Also, let $\mathcal{E}_{n,r}^{(k)}$ be the set of all k-element subsets of $\mathbb{Z}_{n-r} \cup I_r$. Then the set $\mathcal{E}_{n,r}^{(k)}$ can be thought of as being the edge set of the complete k-uniform hypergraph $K_n^{(k)}$ with vertex set $\mathbb{Z}_{n-r} \cup I_r$. If $\infty_i \in I_r$ and $s \in \mathbb{Z}_{n-r}$, we define $\infty_i + s$ to be ∞_i. Furthermore if $E \in \mathcal{E}_{n,r}^{(k)}$, we let $E + s = \{x + s : x \in E\}$.

It is easy to see that the group $\mathbb{Z}_{n-r}$ acts on $\mathcal{E}_{n,r}^{(k)}$ via $(s, E) \to E + s$, and so $\mathcal{E}_{n,r}^{(k)}$ is partitioned into $\mathbb{Z}_{n-r}$-orbits under this *r-pyramidal action*. As before if $s \in \mathbb{Z}_{n-r}$, we let $\mathrm{Fix}(s) = \{E \in \mathcal{E}_{n,r}^{(k)} : E + s = E\}$. Clearly if E and E' are in the same orbit, then $E \cap I_r = E' \cap I_r$. Suppose $E \in \mathcal{E}_{n,r}^{(k)}$ and $E \cap I_r = J$, where $|J| = j$. Let $E^* = E \setminus I_r$, so $|E^*| = k - j$. Then $E + s = E$ exactly when $E^* + s = E^*$.

Let $\mathcal{E}_J = \{E \in \mathcal{E}_{n,r}^{(k)} : E \cap I_r = J\}$. Then $\mathbb{Z}_{n-r}$ acts on $\mathcal{E}_J$, and by Theorem 4 the number of $\mathbb{Z}_{n-r}$-orbits in $\mathcal{E}_J$ is

$$\frac{1}{n-r} \sum_{e \mid \gcd(n-r,k-j)} \phi(e) \binom{(n-r)/e}{(k-j)/e}.$$

Let $\mathcal{E}_j = \{E \in \mathcal{E}_{n,r}^{(k)} : |E \cap I_r| = j\}$. Since there are $\binom{r}{j}$ subsets of I_r with j elements, the number of $\mathbb{Z}_{n-r}$-orbits in $\mathcal{E}_j$ is

$$\binom{r}{j} \frac{1}{n-r} \sum_{e \mid \gcd(n-r,k-j)} \phi(e) \binom{(n-r)/e}{(k-j)/e}.$$

Noting that j can vary from 0 to r gives the following.

Theorem 7. *Let n, k, and r be integers with $0 \le r < k < n$ and consider the complete k-uniform hypergraph $K_n^{(k)}$ with vertex set $\mathbb{Z}_{n-r} \cup I_r$. Let $\mathbb{Z}_{n-r}$ act on the set of edges $\mathcal{E}_{n,r}^{(k)}$ as described above. Then the number of $\mathbb{Z}_{n-r}$-orbits is*

$$\frac{1}{n-r} \sum_{j=0}^{r} \left(\binom{r}{j} \sum_{e \mid \gcd(n-r,k-j)} \phi(e) \binom{(n-r)/e}{(k-j)/e} \right).$$

Example 4. If $n = 30$, $k = 24$, and $r = 3$, the number of $\mathbb{Z}_{27}$-orbits of $\mathcal{E}_{30,3}^{(24)}$ is

$$\begin{aligned}
&\frac{1}{27} \sum_{j=0}^{3} \left(\binom{3}{j} \sum_{e \mid \gcd(27,24-j)} \phi(e) \binom{27/e}{(24-j)/e} \right) \\
&= \frac{1}{27} \left(\binom{3}{0} \sum_{e \mid 3} \phi(e) \binom{27/e}{24/e} + \binom{3}{1} \sum_{e \mid 1} \phi(e) \binom{27/e}{23/e} \right. \\
&\qquad \left. + \binom{3}{2} \sum_{e \mid 1} \phi(e) \binom{27/e}{22/e} + \binom{3}{3} \sum_{e \mid 3} \phi(e) \binom{27/e}{21/e} \right) \\
&= \frac{1}{27} \left(\left(\binom{27}{24} + 2\binom{9}{8} \right) + 3\binom{27}{23} + 3\binom{27}{22} + \left(\binom{27}{21} + 2\binom{9}{7} \right) \right) \\
&= \frac{1}{27} (2943 + 52{,}650 + 242{,}190 + 296{,}082) = 21{,}995.
\end{aligned}$$

Now we consider the number of orbits of a given size. With general n, k, and r as before the example consider a fixed set $J \subseteq I_r$ with $|J| = j$. The action of $\mathbb{Z}_{n-r}$ on $\mathcal{E}_J = \{E \in \mathcal{E}_{n,r}^{(k)} : E \cap I_r = J\}$ amounts to the action of $\mathbb{Z}_{n-r}$ on $\mathcal{E}_{n-r}^{(k-j)}$. Set $n' = n - r$, $k' = k - j$, $d' = \gcd(n', k')$, $n' = d'n_0'$, $k' = d'k_0'$. By Theorem 6 the orbit sizes are exactly the integers $s = en_0'$, where e runs through the positive divisors of d'. Then for such an integer e the number of $\mathbb{Z}_{n-r}$-orbits of $\mathcal{E}_J$ of size en_0' is

$$\frac{1}{s} \sum_{f \mid s_0} \mu(s_0/f) \binom{n_0' f}{k_0' f},$$

where $s_0 = s/n_0'$. Letting J run through the $\binom{r}{j}$ subsets of I_r of size j gives the following.

Theorem 8. *Let $n, k, r,$ and j be integers with $n > k$, $0 \le r < k$, and $0 \le j \le r$, and consider the complete k-uniform hypergraph $K_n^{(k)}$ with vertex set $\mathbb{Z}_{n-r} \cup I_r$. Set $n' = n - r$, $k' = k - j$, $d' = \gcd(n', k')$, $n' = dn'_0$, and $k' = d'k'_0$. The achievable sizes of orbits of $\mathcal{E}_{n,r}^{(k)}$ with exactly j elements in I_r are the integers $s = en'_0$, where e is a positive divisor of d', and the number $N_j(s)$ of such orbits is*

$$N_j(s) = \binom{r}{j}\frac{1}{s}\sum_{f\mid s_0}\mu(s_0/f)\binom{n'_0 f}{k'_0 f},$$

where $s_0 = s/n'_0$.

Example 5. Consider $n = 30$, $k = 24$, $r = 3$. If $j = 0$, we have $n' = 27$, $k' = 24$, $d' = 3$, $n'_0 = 9$, $k'_0 = 8$, and $s = 9e$, where $e \mid 3$. Thus s is 9 or 27, and

$$N_0(9) = \binom{3}{0}\frac{1}{9}\sum_{f\mid 1}\mu(1/f)\binom{9f}{8f} = 1,$$

$$N_0(27) = \binom{3}{0}\frac{1}{27}\sum_{f\mid 3}\mu(3/f)\binom{9f}{8f} = \frac{1}{27}\left((-1)\binom{9}{8} + \binom{27}{24}\right) = 108.$$

Note that n' does not change with j. If $j = 1$, we have $k' = 23$, $d' = 1$, $n'_0 = 27$, $k'_0 = 23$, and $s = 27e$, where $e \mid 1$. Thus $s = 27$, and

$$N_1(27) = \binom{3}{1}\frac{1}{27}\sum_{f\mid 1}\mu(1/f)\binom{27f}{23f} = 1950.$$

Likewise if $j = 2$, then $n' = 27$, $k' = 22$, $d' = 1$, $n'_0 = 27$, $k'_0 = 22$, and $s = 27e$, where $e \mid 1$. Thus $s = 27$, and

$$N_2(27) = \binom{3}{2}\frac{1}{27}\sum_{f\mid 1}\mu(1/f)\binom{27f}{22f} = \frac{1}{9}\binom{27}{22} = 8970.$$

Finally if $j = 3$, then $n' = 27$, $k' = 21$, $d' = 3$, $n'_0 = 9$, $k'_0 = 7$, and $s = 9e$, where $e \mid 3$. Thus s is 9 or 27, and

$$N_3(9) = \binom{3}{3}\frac{1}{9}\sum_{f\mid 1}\mu(1/f)\binom{9f}{7f} = \frac{1}{9}\binom{9}{7} = 4,$$

$$N_3(27) = \binom{3}{3}\frac{1}{27}\sum_{f\mid 3}\mu(3/f)\binom{9f}{7f} = \frac{1}{27}\left((-1)\binom{9}{7} + \binom{27}{21}\right) = 10,962.$$

Notice that there are $1 + 4 = 5$ orbits of size 9 and $108 + 1950 + 8970 + 10{,}962 = 21{,}990$ of size 27, for a total of 21,995 orbits, just as we computed in Example 4 using Theorem 7. These include $5 \cdot 9 + 21{,}990 \cdot 27 = 593{,}775$ edges, which is $\binom{30}{24}$.

6. Forcing same-sized orbits

In order to decompose a complete uniform hypergraph $K_n^{(k)}$ into isomorphic subgraphs, it is convenient to use a group action that yields same-sized orbits. As mentioned in the beginning of Section 4, if $\gcd(n, k) = 1$ the action described in Section 1 achieves same-sized orbits. If $\gcd(n, k) > 1$, it may be possible to find an r-pyramidal action that yields same-sized orbits. We call $K_n^{(k)}$ *balancing* if there exists an integer $r \in [0, k-1]$, such that $\mathcal{E}_{n,r}^{(k)}$ has all orbits of size $n - r$.

Theorem 9. *The hypergraph $K_n^{(k)}$ is balancing if and only if $\gcd(n-r, k-j) = 1$ whenever $0 \le j \le r$.*

Proof. Suppose $\gcd(n-r, k-j) = 1$ whenever $0 \le j \le r$. For a fixed such j by Theorem 8, the achievable sizes of orbits of edges containing exactly j elements of I_r are the integers $s = en'_0$, where e is a positive divisor of $d' = \gcd(n-r, k-j) = 1$. Thus the only possible orbit size is $n'_0 = n'/d' = n - r$. Since this number does not depend on j, all orbits must be the same size.

Conversely, assume that for some j, $0 \le j \le r$, we have $\gcd(n-r, k-j) = d' > 1$. Achievable orbit sizes for orbits containing exactly j elements of I_r include n'_0 and $d'n'_0$ by Theorem 8. Thus there are orbits of at least two different sizes. □

If $\gcd(n, k) = 1$, then there exists a graph G with $\binom{n}{k}/n$ edges such that $K_n^{(k)}$ admits a cyclic G-decomposition. Let G be a subgraph of $K_n^{(k)}$, where $V(K_n^{(k)}) = \mathbb{Z}_{n-r} \cup I_r$ and let Γ be a G-decomposition of $K_n^{(k)}$. Then Γ is said to be r-*pyramidal* if Γ is closed under clicking. Thus among decompositions, 0-pyramidal is equivalent to cyclic. If we partition $\mathcal{E}_{n,r}^{(k)}$ into m distinct $\mathbb{Z}_{n-r}$-orbits each of size $n - r$ and if G with $E(G) \subseteq \mathcal{E}_{n,r}^{(k)}$ is a subgraph of $K_n^{(k)}$ with edge-set containing exactly one edge from each of the m distinct $\mathbb{Z}_{n-r}$-orbits, then $\Gamma = \{G + i : i \in \mathbb{Z}_{n-r}\}$ is an r-pyramidal G-decomposition of $K_n^{(k)}$. For example, if G is the subgraph of $K_9^{(3)}$, with $E(K_9^{(3)}) = \mathcal{E}_{9,2}^{(3)}$, such that $E(G) =$

$\{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 1, 5\}, \{0, 2, 4\}, \{0, 1, \infty_1\}, \{0, 2, \infty_1\}, \{0, 3, \infty_1\}, \{0, 1, \infty_2\}, \{0, 2, \infty_2\}, \{0, 3, \infty_2\}, \{0, \infty_1, \infty_2\}\}$, then $\Gamma = \{G + i : i \in \mathbb{Z}_7\}$ constitutes a 2-pyramidal G-decomposition of $K_9^{(3)}$.

The requirement that the graph G in the previous paragraph contains exactly one edge from each of the m different $\mathbb{Z}_{n-r}$-orbits of size $n - r$ can again be viewed as an extension of the notion of a ρ-labeling of G. Suppose $K_n^{(k)}$ is balancing for some $r \in [0, k-1]$. Let $K_n^{(k)}$ have edge set $\mathcal{E}_{n,r}^{(k)}$. Then a subgraph G of $K_n^{(k)}$ with $m = \binom{n}{k}/(n-r)$ edges is said to admit an *r-pyramidal ρ-labeling* if each of the m edges of G belongs to a different orbit under the action of $\mathbb{Z}_{n-r}$.

We call the integer $k > 1$ *completely balancing* if $K_n^{(k)}$ is balancing for all $n > k$.

Theorem 10. *Let ≥ 2. If k is completely balancing, then for every integer $n > k$ there exists r with $0 \leq r \leq k-1$ and a graph G with $\binom{n}{k}/(n-r)$ edges such that $K_n^{(k)}$ admits an r-pyramidal G-decomposition.*

The following lemma allows us to decide whether k is completely balancing by checking a finite number of cases.

Lemma 11. *Let π_k denote the product of the primes that are at most k. If $K_n^{(k)}$ is balancing for all $n \in [k+1, k+\pi_k]$, then k is completely balancing.*

Proof. Suppose $K_n^{(k)}$ is balancing for all $n \in [k+1, k+\pi_k]$. Now let $n > k$. Then there exists $n' \in [k+1, k+\pi_k]$ such that $n' \equiv n \pmod{\pi_k}$. That is, $n = s\pi_k + n'$ for some $s \geq 0$. By the assumption and Theorem 9, there exists $r \in [0, k-1]$, such that $\gcd(n'-r, k-j) = 1$ for $0 \leq j \leq r$. That is, any prime that divides $k - j$ does not divide $n' - r$. Since any prime dividing $k - j$ also divides π_k, we have $\gcd(n - r, k - j) = \gcd(s\pi_k + n' - r, k - j) = \gcd(n' - r, k - j) = 1$ for $0 \leq j \leq r$. □

One can easily verify that every k with $2 \leq k \leq 6$ is completely balancing. For example, $k = 2$ is completely balancing with $r = 0$ when n is odd and with $r = 1$ when n is even. Similarly, $k = 3$ is completely balancing with $r = 0$ when $n \equiv 1$ or $2 \pmod{3}$, and with $r = 1$ when $n \equiv 0 \pmod{6}$ and $r = 2$ when $n \equiv 3 \pmod{6}$.

Using Theorem 9 and Lemma 11 and a computer, one can easily verify that every $k \leq 14$ is completely balancing. Thus we have the following.

Theorem 12. *For each k with $2 \leq k \leq 14$ and each $n > k$, there exists an r with $0 \leq r \leq k-1$ and a graph G with $\binom{n}{k}/(n-r)$ edges such that $K_n^{(k)}$ admits an r-pyramidal G-decomposition.*

It is simple to verify that $K_n^{(15)}$ is balancing for all $n < 2199$. To show that $K_{2199}^{(15)}$ is not balancing, it suffices to note that for $r \in \{0, 3, 4, 6, 9, 12, 14\}$, we have $\gcd(2199 - r, 15) > 1$; for $r \in \{1, 5, 7, 8, 11, 13\}$, we have $\gcd(2199 - r, 14) > 1$; for $r = 2$, we have $\gcd(2199 - r, 13) > 1$; and for $r = 10$, we have $\gcd(2199 - r, 11) > 1$. In fact, we have verified that $K_n^{(15)}$ is balancing if and only if $n \not\equiv b \pmod{30030}$ where $b \in \{2199, 2200, 5765, 5766, 9125, 9126, 9455, 9456, 9459, 9460, 13{,}355, 13{,}356, 20{,}585, 20{,}586, 20{,}589, 20{,}590, 20{,}919, 20{,}920, 27{,}845, 27{,}846\}$. We have also verified that no $k \in [15, 50]$ is completely balancing and conjecture that no $k > 14$ is completely balancing.

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