WARING'S PROBLEM FOR RATIONAL FUNCTIONS IN ONE VARIABLE

BO-HAE IM AND MICHAEL LARSEN

Abstract. Let $f \in \mathbb{Q}(x)$ be a non-constant rational function. We consider "Waring's Problem for $f(x)$," i.e., whether every element of $\mathbb{Q}$ can be written as a bounded sum of elements of $\{f(a) \mid a \in \mathbb{Q}\}$. For rational functions of degree 2, we give necessary and sufficient conditions. For higher degrees, we prove that every polynomial of odd degree and every odd Laurent polynomial satisfies Waring's Problem. We also consider the "Easier Waring's Problem": whether every element of $\mathbb{Q}$ can be represented as a bounded sum of elements of $\{\pm f(a) \mid a \in \mathbb{Q}\}$.

1. Introduction

The classical Waring's Problem (WP) asks if, for every positive integer d, there exists N such that every natural number can be written as the sum of N dth powers of natural numbers. This was settled in the affirmative by Hilbert [3]. Shortly afterward, Erich Kamke [4] proved that for every polynomial $f(x) \in \mathbb{Z}[x]$ with positive leading coefficient there exists N such that every sufficiently large integer satisfying an obvious congruence condition (depending on $f(x)$) can be written as a sum of N values of the form $f(x_i)$, where the x_i are natural numbers.

In this paper, we propose to consider the analogous problem for rational functions. Since in this setting, we can in general only expect $f(x_i)$ to belong to $\mathbb{Q}$, we consider the question of whether every rational number, or every sufficiently positive rational number, can be written as a sum of values $f(x_i)$, $x_i \in \mathbb{Q}$.

In 1934, Edward Wright [7] introduced the Easier Waring's Problem (EWP): to represent an integer as a sum or difference of a fixed number of dth powers, i.e., as $\pm x_1^d \pm x_2^d \cdots \pm x_N^d$.

Bjorn Poonen [6] considered both the original WP and the EWP for rational functions over $\mathbb{Q}$. He showed that if f has at most three poles, all simple, and all in $\mathbb{P}^1(\mathbb{Q})$, then f satisfies WP. Moreover if all the poles of

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f lie in $\mathbb{P}^1(\mathbb{Q})$, then it at least satisfies EWP. The latter result holds more generally for number fields.

We revisit this question, proving that every polynomial f of odd degree satisfies WP, while for polynomials of even degree, either every sufficiently positive rational or every sufficiently negative rational can be expressed as a bounded sum of values of f. Laurent polynomials satisfy EWP over any field of characteristic zero (for number fields, this is covered by [6]), so in particular, odd Laurent polynomials satisfy WP. We use two different methods, one completely elementary, and one based on a theorem of Browning and Heath-Brown [2] estimating the number of integral solutions of a system of polynomial equations in a box.

This paper was written simultaneously with the paper [5] of the second named author and Dong Quan Ngoc Nguyen on Waring's Problem for unipotent algebraic groups over number fields. Since a basic idea behind that paper is that the proper setting for Waring-type problems is polynomial-valued maps, the fact that one can prove such results for Laurent polynomials came as something of a surprise to us.

We would like to thank Arno Fehm for calling our attention to [6]. The second named author would like to acknowledge useful conversations with Nguyen.

2. Generalities

Throughout this section, X denotes a subset of $\mathbb{Q}$.

Definition 2.1. *We say X is a* base *(resp.* positive base*,* negative base*, or* open base*) if for some positive integer N,*

$$\underbrace{X + X + \cdots + X}_{N}$$

is all of $\mathbb{Q}$ (resp. contains $(a, \infty) \cap \mathbb{Q}$ for some a, contains $(-\infty, b) \cap \mathbb{Q}$ for some b, or contains $(a, b) \cap \mathbb{Q}$ for some $a < b$). We say X is a virtual base *if*

$$\underbrace{\pm X \pm X \pm \cdots \pm X}_{N} = \mathbb{Q}.$$

Clearly, all of these properties are invariant under translation of X or multiplication of X by any positive rational scale factor. The following properties are also immediate:

Lemma 2.2. *For $X \subseteq \mathbb{Q}$, we have*

(a) *If X is a base, it is both a positive base and a negative base.*
(b) *If X is a positive base or a negative base, then it is both an open base and a virtual base.*
(c) *If X is a positive base and unbounded below or a negative base and unbounded above, then it is a base.*

(d) *If X is a base (resp. positive base, negative base, open base, or virtual base), then $-X$ is a base (resp. negative base, positive base, open base, or virtual base).*

The following lemma gives obvious obstructions to a set X being a base (or positive base, etc.)

Lemma 2.3. *For $X \subseteq \mathbb{Q}$, we have*

(a) *If X is a positive base, it cannot be bounded above.*
(b) *If X is a negative base, it cannot be bounded below.*
(c) *If X is a virtual base, it cannot be bounded.*
(d) *If X is any kind of base, it cannot be p-adically bounded for any prime p.*

If $f(x) \in \mathbb{Q}(x)$ is a rational function and F is any field of characteristic 0, we denote by $f(F)$ the set of values $f(a)$ as a ranges over all elements of F which are not poles of f.

Definition 2.4. *We say f* satisfies WP *if $f(\mathbb{Q})$ is a base. We say it* satisfies the positive *(resp.* negative*)* WP *if $f(\mathbb{Q})$ is a positive (resp. negative) base. We say it* satisfies the EWP *if $f(\mathbb{Q})$ is a virtual base.*

Proposition 2.5. *If $f(x) \in \mathbb{Q}(x)$ is a rational function then*

(a) *For f to satisfy WP, it is necessary for it to have at least two distinct poles in $\mathbb{RP}^1$ or one pole of odd order in $\mathbb{RP}^1$.*
(b) *If $f(\mathbb{Q})$ is an open base and f has at least one pole of odd order in $\mathbb{RP}^1$, then f satisfies WP.*
(c) *For f to satisfy the EWP, it is necessary for f to have a pole in $\mathbb{RP}^1$.*
(d) *For f to satisfy the EWP, it is necessary that for each prime p, f has a pole in $\mathbb{Q}_p\mathbb{P}^1$.*

Proof. Part (a) follows from parts (a) and (b) of Lemma 2.3. Parts (c) and (d) follow from parts (c) and (d) of Lemma 2.3 respectively. For part (b), we note that if f has at least one pole of odd order in $\mathbb{RP}^1$, then the closure of $f(\mathbb{Q})$ contains a neighborhood of ∞ in $\mathbb{RP}^1$, i.e., contains all real numbers of absolute value $> B$ for some B. If

$$(a,b) \cap \mathbb{Q} \subset \underbrace{f(\mathbb{Q}) + \cdots + f(\mathbb{Q})}_{N}$$

and $M(b-a) > 2B$, then setting $Y := f(\mathbb{Q}) \cap (-\infty, -B)$ and $Z := f(\mathbb{Q}) \cap (B, \infty)$, we have

$$\mathbb{Q} \subseteq ((Y \cup Z) + (Ma, Mb) \cap \mathbb{Q}) \subseteq \underbrace{f(\mathbb{Q}) + \cdots + f(\mathbb{Q})}_{1+MN}.$$

□

The following proposition shows that the property of being a base is not affected by any finite subset of elements.

Proposition 2.6. *If $X, Y \subseteq \mathbb{Q}$ and $X \setminus Y$ and $Y \setminus X$ are finite, then X is a base if and only if Y is a base.*

Proof. Without loss of generality, we may assume $Y = X \cup \{y\}$. Translating, we may assume $y = 0$. The non-trivial direction is that if Y is a base, the same is true for X. Let

$$X_n := \underbrace{X + X + \cdots + X}_{n}, Y_n := \underbrace{Y + Y + \cdots + Y}_{n}.$$

As Y is a base, $Y_N = \mathbb{Q}$ for some $N > 0$. Let $x \in X$ be any non-zero element. For any positive integer m, $-x/m \in Y_N$, so $-x/m \in X_i$ for some positive integer $i \leq N$, and this implies $0 = x + m(-x/m) \in X_{1+im}$. Letting $M := 1 + im$, and applying the same reasoning to $-x/M \in Y_N$, we see that $0 \in X_{1+jM}$ for some positive integer j. The set of positive integers k such that $0 \in X_k$ is a semigroup, and it contains the relatively prime elements M and $1 + jM$, so it contains all integers $\geq K$ for some integer K. Thus,

$$X_{K+N} \supset \{0\} \cup X_1 \cup \cdots \cup X_N = Y_N = \mathbb{Q}.$$

□

Corollary 2.7. *If $g(x) \in \mathbb{Q}(x)$ is a fractional linear transformation, then $f(g(\mathbb{Q}))$ is a base if and only if $f(\mathbb{Q})$ is a base.*

Proof. As $g(\mathbb{Q}) \setminus \mathbb{Q}$ and $\mathbb{Q} \setminus g(\mathbb{Q})$ have at most one element each, the corollary follows immediately. □

3. The EWP for Laurent polynomials in characteristic 0

Throughout this section, K will denote a field of characteristic 0. Our main result is the following theorem.

Theorem 3.1. *If $f(x) \in K\left[x, \frac{1}{x}\right]$ is a non-constant Laurent polynomial, then there exists a positive integer N such that*

$$\underbrace{\pm f(K) \pm f(K) \pm \cdots \pm f(K)}_{N} = K.$$

In particular, $f(x)$ satisfies the EWP.

In fact, we prove the following stronger result.

Theorem 3.2. *Let S be a finite set of non-zero integers. Let K^S denote the K-algebra of functions $S \to K$ and $\mathbf{g}^S : K^* \to K^S$ denote the function defined by*

$$\mathbf{g}^S(x) := s \mapsto x^s.$$

Let

$$X_k^S := \underbrace{\mathbf{g}^S(K^*) + \cdots + \mathbf{g}^S(K^*)}_{k} \underbrace{-\mathbf{g}^S(K^*) - \cdots - \mathbf{g}^S(K^*)}_{k}.$$

Then, there exists a positive integer N such that

$$X_N^S = K^S.$$

In particular, we can take

$$N \leq 3^{-2+2^{|S|\max_{s\in S}|s|}}.$$

We defer the proof of Theorem 3.2, starting instead with the following proposition:

Proposition 3.3. *Theorem 3.2 implies Theorem 3.1.*

Proof. For a given $f(x) \in K\left[x, \frac{1}{x}\right]$, let S denote the set of non-zero exponents of monomials occurring in $f(x)$. As $f(x)$ is not constant, S is non-empty. We write

$$f(x) = a_0 + \sum_{s\in S} a_s x^s,$$

where a_0 may be zero but $a_s \neq 0$ for all $s \in S$. For $x_i, y_j \in K^*$ and $c \in K$,

$$\sum_{i=1}^{N} f(x_i) - \sum_{j=1}^{N} f(y_j) = c,$$

if and only if

$$\sum_{s\in S} a_s \left(\sum_{i=1}^{N} x_i^s - \sum_{j=1}^{N} y_j^s \right) = c. \tag{1}$$

Choose $c_s \in K$ for each $s \in S$ such that

$$\sum_{s\in S} a_s c_s = c. \tag{2}$$

If we assume that Theorem 3.2 is true for some positive integer N, then there exist $x_i, y_j \in K^*$ such that for all $s \in S$,

$$c_s = \sum_{i=1}^{N} x_i^s - \sum_{j=1}^{N} y_j^s. \tag{3}$$

Thus, (1) follows from (2) and (3). This proves Theorem 3.1. □

To prove Theorem 3.2, we need the following lemma.

Lemma 3.4. *Following the notations in Theorem 3.2,*

(*a*) $X_k^S + X_\ell^S = X_{k+\ell}^S$.
(*b*) $X_k^S - X_\ell^S = X_{k+\ell}^S$.
(*c*) $X_k^S X_\ell^S \subseteq X_{2k\ell}^S$.

Proof. Parts (a) and (b) are trivial. For (c),

$$\begin{aligned}\left(\sum_{i=1}^{k}\mathbf{g}^S(x_i)-\sum_{j=1}^{k}\mathbf{g}^S(y_j)\right)\left(\sum_{r=1}^{\ell}\mathbf{g}^S(w_r)-\sum_{s=1}^{\ell}\mathbf{g}^S(z_s)\right)\\ =\left(\sum_{i=1}^{k}\sum_{r=1}^{\ell}\mathbf{g}^S(x_iw_r)+\sum_{j=1}^{k}\sum_{s=1}^{\ell}\mathbf{g}^S(y_jz_s)\right)\\ -\left(\sum_{j=1}^{k}\sum_{r=1}^{\ell}\mathbf{g}^S(y_jw_r)+\sum_{i=1}^{k}\sum_{s=1}^{\ell}\mathbf{g}^S(x_iz_s)\right),\end{aligned}$$

which is in $X^S_{2k\ell}$. □

Proof of Theorem 3.2. We use induction on $|S|$.

If $|S|=1$, i.e. $S=\{s\}$, $s\neq 0$, and $\{a^s \mid a\in K^*\}=\{a^{-s}\mid a\in K^*\}$, so without loss of generality we may assume s is a positive integer. We claim $X^S_N=K$ for $N=3^{2s-2}$. If $s=1$, then clearly

$$X^S_1=K.$$

So we assume that $s\geq 2$. Let $\delta\colon K[x,1/x]\to K[x,1/x]$ denote the difference operator:

$$(\delta f)(x):=f(x+1)-f(x).$$

By induction on s, we see that for $f(x):=x^s$,

$$(\delta^{s-1}f)(x)=s!\,x+\frac{(s-1)s!}{2}.$$

Thus,

$$s!\,x+\frac{(s-1)s!}{2}=\sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}(x+s-1-i)^s\in X^S_{2^{s-2}},$$

as long as $x\notin\{0,-1,-2,\dots,1-s\}$. In particular, $K\setminus\mathbb{Z}\subset X^S_{2^{s-2}}$. Since X^S_1 is not contained in a single $\mathbb{Z}$-coset of K (for instance, it contains 0 and $1-(1/2)^s$), it follows that

$$K\subseteq X^S_1+X^S_{2^{s-2}}=X^S_{1+2^{s-2}}\subset X^S_{3^{2s-2}}.$$

Suppose the claim of the theorem is true for some S, and let us prove it for $S\cup\{t\}$. We know that for

$$N:=3^{-2+2^{|S|}\max_{s\in S\cup\{t\}}|s|}\geq\max(3^{-2+2^{|S|}\max_{s\in S}|s|},3^{-2+2|t|}),\tag{4}$$

we have

$$X^S_N=K^S\text{ and }X^{\{t\}}_N=K.$$

Let $p^{S,t}_k\colon X^{S\cup\{t\}}_k\to X^S_k$ and $q^{S,t}_k\colon X^{S\cup\{t\}}_k\to X^{\{t\}}_k$ denote projection maps. In particular, $p^{S,t}_k$ and $q^{S,t}_k$ are surjective for $k\geq N$.

Then we have two cases:
Case 1. There exist $v, w \in X_{N+1}^{S\cup\{t\}}$ such that $v \neq w$ but

$$p_{N+1}^{S,t}(v) = p_{N+1}^{S,t}(w).$$

Case 2. No such $v, w \in X_{N+1}^{S\cup\{t\}}$ exist, or (equivalently), $p_{N+1}^{S,t}$ is injective (and therefore bijective).

In Case 1, $0 \neq v - w \in X_{2N+2}^{S\cup\{t\}}$ and $p_{2N+2}^{S,t}(v-w) = 0$. Regarding $v - w$ as a function $f\colon S \cup \{t\} \to K$, we have $f(s) = 0$ for all $s \in S$ but $f(t) \neq 0$. As $q_N^{S,t}$ maps onto K, multiplying $X_N^{S\cup\{t\}}$ by $v - w \in X_{2N+2}^{S\cup\{t\}}$, the set $X_{2N(2N+2)}^{S\cup\{t\}}$ contains all functions $S \cup \{t\} \to K$ which vanish identically on S. Since $X_N^S = K^S$, it follows that

$$X_{2N(2N+2)+N}^{S\cup\{t\}} = K^{S\cup\{t\}}.$$

By (4),

$$3^{-2+2^{|S|\cup\{t\}}\max_{s\in S\cup\{t\}}|s|} = 9N^2 \geq 4N^2 + 5N = 2N(2N+2) + N,$$

we obtain Theorem 3.2 for $S \cup \{t\}$.

It therefore suffices to consider Case 2. Now $X_N^{S\cup\{t\}} \subseteq X_{N+1}^{S\cup\{t\}}$. If equality does not hold, since $p_N^{S,t}$ is surjective, there exist $v \in X_N^{S\cup\{t\}}$ and $w \in X_{N+1}^{S\cup\{t\}} \setminus X_N^{S\cup\{t\}}$ with $p_{N+1}^{S,t}(v) = p_{N+1}^{S,t}(w)$. This is impossible in Case 2. Thus, $X_N^{S\cup\{t\}} = X_{N+1}^{S\cup\{t\}}$. Then, by Lemma 3.4, we have that

$$X_N^{S\cup\{t\}} + X_1^{S\cup\{t\}} = X_{N+1}^{S\cup\{t\}} = X_N^{S\cup\{t\}},$$

so by induction we have

$$X_k^{S\cup\{t\}} = X_N^{S\cup\{t\}}, \text{ for all } k \geq N.$$

We set $X^{S\cup\{t\}} := X_N^{S\cup\{t\}}$ and write $p^{S,t}$ (resp. $q^{S,t}$) for the projection map $p_N^{S,t}$ (resp. $q_N^{S,t}$). By Lemma 3.4, $X^{S\cup\{t\}}$ is closed under addition, subtraction and multiplication, i.e., it is a (possibly non-unital) ring. As we are in Case 2, $p^{S,t}$ is an isomorphism of non-unital rings, and it follows that $X^{S\cup\{t\}}$ is a unital ring and $p^{S,t}$ is an isomorphism of unital rings.

Now, $q^{S,t}\colon X^{S\cup\{t\}} \to X^{\{t\}} = K$ is a surjective non-unital homomorphism of rings and therefore a ring homomorphism. Composing it with the inverse of $p^{S,t}$, we obtain a ring homomorphism $\phi : K^S \to K$ which expresses the value of $f \in X^{S\cup\{t\}}$ at t in terms of the restriction of f to S. Letting e_s denote the idempotent of K^S which is 0 on $S \setminus \{s\}$ and 1 on s, $\phi(e_s) \in \{0, 1\}$ for all s, and as ϕ maps the multiplicative identity $\sum_{s\in S} e_s$ to 1, it follows that $\phi(e_s) = 1$ for some $s \in S$. This implies that $\phi(f) = \phi(fe_s)$ for all $f \in K^S$, i.e., that the homomorphism ϕ factors through the projection $K^S \to K$ given by evaluation at s. Thus, there exists an endomorphism ψ of K such that $\phi(f) = \psi(f(s))$, and $X^{S\cup\{t\}}$ must consist of all functions

$f\colon S \cup \{t\} \to K$ such that $f(t) = \psi(f(s))$. This is absurd because, for instance, $f := \mathbf{g}^{S\cup\{t\}}(2) - \mathbf{g}^{S\cup\{t\}}(1) \in X^{S\cup\{t\}}$ satisfies

$$2^t - 1 = f(t) = \psi(f(s)) = \psi(2^s - 1) = 2^s - 1$$

although $t > s$.

□

Corollary 3.5. *Over every characteristic* 0 *field, every odd Laurent polynomial satisfies WP.*

4. Waring's problem for polynomials over $\mathbb{Q}$

The main result in this section is the following:

Theorem 4.1. *Let $f(x) \in \mathbb{Q}[x]$ be a non-constant polynomial. If f is of odd degree, it satisfies WP. If f is of even degree, $f(\mathbb{Q})$ is a positive base or a negative base, according to whether the leading coefficient of f is positive or negative.*

Let $\mathbf{g}\colon \mathbb{Q} \to \mathbb{Q}^d$ denote the map $\mathbf{g}(x) = (x, x^2, \dots, x^d)$. We begin with the following lemma:

Proposition 4.2. *Let m, d, and r denote positive integers, $m \geq d + r$. Let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m$, and let $\mathbf{F} = (F_1, \dots, F_r)$ denote an r-tuple of linear forms in $\mathbf{x} = (x_1, \dots, x_m)$. For fixed $\mathbf{a}$ satisfying*

$$\big|\{a_1, \dots, a_m\}\big| \geq d, \tag{5}$$

the set of $\mathbf{F}$ such that the morphism $\mathbb{A}^m \to \mathbb{A}^{d+r} = \mathbb{A}^d \times \mathbb{A}^r$ given by

$$\mathrm{x} \mapsto \big(\sum_i \mathbf{g}(x_i), \mathbf{F}(\mathrm{x})\big)$$

is smooth at $\mathbf{a}$ forms a dense open subset of the variety $\mathbb{A}^{rm}$ of r-tuples of linear forms in m variables.

Proof. By (5), without loss of generality we may assume that $a_1, \dots, a_d$ are pairwise distinct. Thus the Vandermonde determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_d \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{d-1} & a_2^{d-1} & \cdots & a_d^{d-1} \end{pmatrix}$$

is non-zero, and

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{d-1} & a_2^{d-1} & \cdots & a_m^{d-1} \end{pmatrix}$$

has rank d. It follows that for a generic choice of $r \times m$ matrices b_{ij}, the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{d-1} & a_2^{d-1} & \cdots & a_m^{d-1} \\ b_{11} & b_{12} & \cdots & b_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rm} \end{pmatrix}$$

has rank $d+r$. The proposition now follows from the Jacobian condition for smoothness. □

Proof of Theorem 4.1. Let $d := \deg f$, and let m be an integer greater than d. Fix pairwise distinct rational numbers $a_1, \ldots, a_m$, and let $b_j = \sum_i a_i^j$. Let $\mathbf{b}$ denote the vector $(b_1, \ldots, b_d)$. Consider the closed subscheme V of $\mathbb{P}^m$ over Spec $\mathbb{Q}[t_1, \ldots, t_d]$ defined by the system of d homogeneous equations

$$\begin{aligned} X_1 + \cdots + X_m &= t_1 X_0 \\ X_1^2 + \cdots + X_m^2 &= t_2 X_0^2 \\ &\vdots \\ X_1^d + \cdots + X_m^d &= t_d X_0^d. \end{aligned} \tag{6}$$

Let $V^{\mathbf{c}}$ denote the fiber of V over $\mathbf{c} = (c_1, \ldots, c_d)$.

Using the Jacobian criterion for smoothness and the Vandermonde determinant as before, $(X_0 : \cdots : X_m)$ is a non-singular point of $V^{\mathbf{c}}$ as long as there are at least $d+1$ distinct values among $X_0, \ldots, X_m$. Thus, for all $\mathbf{c}$, the singular locus of $V^{\mathbf{c}}$ has dimension at most $d-1$, and $(1 : a_1 : \cdots : a_m)$ is a non-singular point of $V^{\mathbf{b}}$.

By Bertini's theorem as formulated by Zariski [8], the intersection of $V^{\mathbf{c}}$ with a generic hyperplane $G_1 = 0$ in $\mathbb{P}^m$ can be singular only at a subvariety of $V^{\mathbf{c}}$ of dimension less than that of $\mathrm{Sing}(V^{\mathbf{c}})$. We may choose G_1 to have coefficients in $\mathbb{Q}$ since the rational hyperplanes are dense in the projective space of all real hyperplanes in the real topology and therefore in the Zariski topology. Iteratively choosing $G_2, \ldots, G_d$ generically, the intersection $W_{\mathbf{G}}^{\mathbf{c}}$ of $V^{\mathbf{c}}$ with the locus $G_1 = \cdots = G_d = 0$ is non-singular.

Writing $G_i = b_{i0}X_0 + b_{i1}X_1 + \cdots + b_{im}X_m$, we define $F_i := b_{i1}x_1 + \cdots + b_{im}x_m$, where the $x_i := X_i/X_0$ are affine coordinates on the affine open subset $\mathbb{A}^m$ of $\mathbb{P}^m$ given by $X_0 \neq 0$. Fixing b_{ij} for $j \geq 1$ and letting b_{i0} vary, for a generic d-tuple $(b_{10}, \ldots, b_{d0})$ and generic $\mathbf{c}$, $W_{\mathbf{G}}^{\mathbf{c}}$ is non-singular; the

complement of $X_0 = 0$ is then given by the system of equations

$$\begin{aligned} x_1 + \cdots + x_m &= c_1 \\ x_1^2 + \cdots + x_m^2 &= c_2 \\ &\vdots \\ x_1^d + \cdots + x_m^d &= c_d \\ F_1(x_1, \ldots, x_m) &= -b_{10} \\ &\vdots \\ F_d(x_1, \ldots, x_m) &= -b_{d0}. \end{aligned}$$

Applying Proposition 4.2 and the implicit function theorem, we conclude that there is a d-tuple G and a non-empty open set $U \subset \mathbb{R}^d$ such that for all $\mathbf{c} \in U$, $W_{\mathbf{G}}^{\mathbf{c}}$ has a real point.

If $\dim W_{\mathbf{G}}^{\mathbf{c}} = m - 2d$ is large enough, by a theorem of Brauer [1], $W_{\mathbf{G}}^{\mathbf{c}}(\mathbb{Z}_p) = W_{\mathbf{G}}^{\mathbf{c}}(\mathbb{Q}_p)$ is non-empty. By a theorem of Timothy Browning and Roger Heath-Brown [2, Theorem 1.7], this implies that $W_{\mathbf{G}}^{\mathbf{c}}(\mathbb{Q}) \subset V^{\mathbf{c}}(\mathbb{Q})$ is non-empty.

If $f(x) = a_0 + a_1 x + \ldots + a_d x^d$,

$$\{a_0 m + a_1 c_1 + \cdots + a_d c_d \mid \mathbf{c} \in U \cap \mathbb{Q}^d\} \subset \underbrace{f(\mathbb{Q}) + \cdots + f(\mathbb{Q})}_{m}.$$

Thus $f(\mathbb{Q})$ is an open base. If d is odd, then $f(\mathbb{Q})$ is dense in $(-\infty, -B) \cup (B, \infty)$ for some B, and it follows as in the proof of Proposition 2.5 that $f(\mathbb{Q})$ is a base. If d is even and $a_d > 0$ (resp. $a_d < 0$), then $f(\mathbb{Q})$ is dense in (B, ∞) (resp. $(-\infty, -B)$) for some B, and it follows that $f(\mathbb{Q})$ is a positive (resp. negative) base.

□

Theorem 4.3. *If $f(x) \in \mathbb{Q}(x)$ is a rational function of degree 2, then we have the following:*

(a) If f has two distinct poles in $\mathbb{QP}^1$, it satisfies WP.
(b) If f has one pole in $\mathbb{QP}^1$, it satisfies the EWP but not WP.
(c) If f has no pole in $\mathbb{QP}^1$, it does not satisfy even the EWP.

Note that (a) is a special case of [6, Theorem 1.2].

Proof. Let g be a fractional linear transformation over $\mathbb{Q}$ mapping 0 and ∞ to the two poles of f. Then $f(g(x))$ is a rational function of degree two with poles 0 and ∞ and must therefore be of the form $\frac{ax^2+bx+c}{x}$. Let $h(x) = f(g(x)) - b = ax + c/x$. By Corollary 3.5, there exists N such that every rational number is a sum of N terms in $h(\mathbb{Q})$. It follows that every rational number is a sum of N terms of $f(g(x))$. Thus $h(\mathbb{Q})$ is a base, and by Corollary 2.7, the same is true of $f(\mathbb{Q})$.

For part (b), let g be a fractional linear transformation over $\mathbb{Q}$ mapping ∞ to the pole of f (which must be double). Then $h(x) = f(g(x)) = ax^2 + bx + c$

for some $a, b, c \in \mathbb{Q}$. Rescaling and translating, we may assume $h(x)$ is of the form $x^2 + d$. Now, $h(\mathbb{Q})$ is bounded below, so $h(\mathbb{Q})$ is not a base, and therefore that $f(\mathbb{Q})$ is not a base. On the other hand, every value of $h(\mathbb{Q})$ except d is achieved twice, so either $f(\mathbb{Q}) = h(\mathbb{Q})$ or $f(\mathbb{Q}) = h(\mathbb{Q}) \setminus \{d\}$. Since every positive rational number is the sum of four squares of positive rationals, it follows that $h(\mathbb{Q}) + h(\mathbb{Q}) + h(\mathbb{Q}) + h(\mathbb{Q})$ contains all rational numbers in $(4d, \infty)$.

For part (c), let K be the quadratic extension of $\mathbb{Q}$ generated by the poles of f. By Chebotarev density, there exists a prime p such that the prime p is inert in K and therefore f has no pole in $\mathbb{Q}_p$. Thus $f(\mathbb{Q}_p)$ is bounded, and by Lemma 2.3, $f(\mathbb{Q})$ is not a virtual base. □

References

[1] Brauer, Richard: A note on systems of homogeneous algebraic equations. *Bull. Amer. Math. Soc.* **51**, (1945), 749–755.

[2] Browning, Timothy; Heath-Brown, Roger: Forms in many variables and differing degrees, *J. Eur. Math. Soc.* **9** (2017), 357–394.

[3] Hilbert, David: Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n^{ter} Potenzen (Waringsches Problem). *Math. Ann.* **67** (1909), no. 3, 281–300.

[4] Kamke, E.: Verallgemeinerungen des Waring-Hilbertschen Satzes. *Math. Ann.* **83** (1921), no. 1–2, 85–112.

[5] Larsen, Michael; Dong Quan, Nguyen Ngoc: Waring's problem for unipotent groups over number fields, to appear in *Ann. Inst. Fourier*.

[6] Poonen, Bjorn: Sums of values of a rational function. *Acta Arith.* **112** (2004), no. 4, 333–343.

[7] Wright, E. Maitland: An Easier Waring's Problem. *J. London Math. Soc.* S1–9 (1934), no. 4, 267.

[8] Zariski, Oscar: The theorem of Bertini on the variable singular points of a linear system of varieties. *Trans. Amer. Math. Soc.* **56** (1944), 130–140.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 34141, South Korea

E-mail address: bhim@kaist.ac.kr

Department of Mathematics, Indiana University, Bloomington, Indiana 47405, USA

E-mail address: mjlarsen@indiana.edu