IRRATIONALITY OF MOTIVIC ZETA FUNCTIONS

MICHAEL J. LARSEN and VALERY A. LUNTS

Abstract

Let $K_0(\mathrm{Var}_{\mathbb{Q}})[1/\mathbb{L}]$ *denote the Grothendieck ring of* $\mathbb{Q}$*-varieties with the Lefschetz class inverted. We show that there exists a K3 surface* X *over* $\mathbb{Q}$ *such that the motivic zeta function* $\zeta_X(t) := \sum_n [\mathrm{Sym}^n X] t^n$ *regarded as an element in* $K_0(\mathrm{Var}_{\mathbb{Q}})[1/\mathbb{L}][[t]]$ *is not a rational function in* t*, thus disproving a conjecture of Denef and Loeser.*

1. Introduction

Let k be a field. We denote by $K_0(\mathrm{Var}_k)$ the Grothendieck group of k-varieties, that is, the free abelian group generated by isomorphism classes of k-varieties modulo the cutting-and-pasting relations $[X] = [Y] + [X \setminus Y]$ for all pairs (X, Y) consisting of a variety X and a closed subvariety Y. It is endowed with a commutative ring structure characterized by $[X][Y] = [X \times Y]$. Note that we use *variety* to mean a reduced separated scheme of finite type over k, but the Grothendieck ring would not be changed if we allowed nonreduced schemes or nonseparated schemes, or limited ourselves to affine schemes (and closed subschemes with affine complement).

Following Kapranov [14], we define the *motivic zeta function*

$$\zeta_X(t) := \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n \in K_0(\mathrm{Var}_k)[[t]],$$

where $\mathrm{Sym}^n X$ is the symmetric nth power X^n / Σ_n.

By a *motivic measure*, we mean a homomorphism $\mu\colon K_0(\mathrm{Var}_k) \to A$, where A is a commutative ring. We write $\mu(\zeta_X(t))$ for the image of $\zeta_X(t)$ in $A[[t]]$. If k is a finite field, then $\mu\colon [X] \mapsto |X(k)|$ defines a motivic measure with values in $\mathbb{Z}$. The image $\mu(\zeta_X(t)) \in \mathbb{Z}[[t]]$ is the usual zeta function of X and therefore rational as a function of t by Dwork's theorem (see [8]). In [14, Remarks 1.3.5], Kapranov asked whether this rationality holds for the motivic zeta function itself. He proved that this is so when X is a curve with at least one k-point, even if k is not a finite field, and Litt [20] generalized the result to all curves. (Since $K_0(\mathrm{Var}_k)$ is not an integral

DUKE MATHEMATICAL JOURNAL
Vol. 169, No. 1, © 2020 DOI 10.1215/00127094-2019-0035
Received 7 September 2018. Revision received 10 May 2019.
First published online 21 November 2019.
2010 *Mathematics Subject Classification*. Primary 14G10; Secondary 11F80, 14F42, 14K15.

domain by [23], there is a question of exactly what this means, which we settle for the purposes of this paper by saying that $\zeta_X(t)$ rational means that there exists a polynomial $B(t) = 1 + b_1 t + \cdots + b_n t^n$ such that $B(t)\zeta_X(t) \in K_0(\mathrm{Var}_k)[t]$.)

In [16], we proved that, in general, $\zeta_X(t)$ is *not* rational when X is a surface. This does not quite finish the question, since for many purposes (especially motivic integration), the natural object to consider is not $K_0(\mathrm{Var}_k)$ but $K_0(\mathrm{Var}_k)[1/\mathbb{L}]$, where $\mathbb{L} := [\mathbb{A}^1]$. It is known that $\mathbb{L}$ is a zero-divisor (see [4]; see also [26] for an analysis of the annihilator of $\mathbb{L}$). One might still hope, therefore, that $\zeta_X(t)$ may be rational as a power series over $K_0(\mathrm{Var}_k)[1/\mathbb{L}]$. No variant of the method of [16] can possibly test this, since the motivic measures constructed in that paper are birationally invariant and therefore vanish on $\mathbb{L}$. This made possible the conjecture of Denef and Loeser (see [7, Conjecture 7.5.1]) predicting that $\zeta_X(t)$ should satisfy this weaker rationality condition. In this paper, we show that in general it does not.

To explain our strategy, we begin by discussing certain motivic measures which *cannot* detect the irrationality of zeta functions. A reference for the following discussion is [17]. We endow $K_0(\mathrm{Var}_k)$ with the λ-structure in which the $[X] \to [\mathrm{Sym}^n X]$ operations play the role of symmetric powers; in other words, $\lambda^n([X])$ is defined to be the t^n coefficient of $\zeta_X(t)^{-1}$. If A is a finite λ-ring (in the sense that every element $a \in A$ can be written $a = b - c$, where $\lambda^n b = \lambda^n c = 0$ for n sufficiently large), then every λ-homomorphism $\mu\colon K_0(\mathrm{Var}_k) \to A$ is a motivic measure for which $\mu(\zeta_X(t))$ is rational for all X/k.

Here is an example. Let $K(G_k, \mathbb{Q}_\ell)$ denote the Grothendieck ring of (virtual) finite-dimensional continuous representations of G_k, where, as usual, $0 \to V_1 \to V_2 \to V_3 \to 0$ implies $[V_2] = [V_1] + [V_3]$. Then $K(G_k, \mathbb{Q}_\ell)$ is a λ-ring (even a special λ-ring), and

$$[X] \mapsto \sum_{i=0}^{2\dim X} (-1)^i \big[H^i(\bar{X}, \mathbb{Q}_\ell)\big],$$

where $H^i(\bar{X}, \mathbb{Q}_\ell)$ denotes the ith ℓ-adic étale cohomology group of $\bar{X}$ as a G_k-representation, defines a ring homomorphism μ. It is a consequence of the Künneth formula and the isomorphism

$$H^i(\mathrm{Sym}^n \bar{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(\bar{X}^n, \mathbb{Q}_\ell)^{\Sigma_n}$$

that μ is a λ-homomorphism. Thus $\mu(\zeta_X(t))$ is rational in t for all X, where the degree of numerator and denominator depend only on the dimension of the cohomology of $\bar{X}$.

In particular, if X is a K3 surface, then $\mu(\zeta_X(t))^{-1}$ is a polynomial of degree 24, the product of a degree 22 polynomial corresponding to the H^2-term and the factors

$(1-t)(1-\mu(\mathbb{L})^2 t)$, corresponding to the H^0 and H^4 terms. We consider K3 surfaces of Picard number 18, in which the H^2 factor further decomposes $(1-\mu(\mathbb{L})t)^{18}\Lambda(t)$.

We modify this construction in three ways. First, we consider coefficients in $\bar{\mathbb{F}}_\ell$ instead of $\mathbb{Q}_\ell$. Second, we use a modified Grothendieck ring $K_\ell^{\mathrm{sp}}(G_k)$ of Galois representations, in which we identify $[V_2]$ with $[V_1]+[V_3]$ only when $V_2 \cong V_1 \oplus V_3$ as G_k-modules. This is essential, since the essence of our construction is to distinguish $\bar{\mathbb{F}}_\ell$-valued Galois representations which have the same semisimplification. Third, we replace k by $k(\zeta_\ell)$ in order to trivialize the cyclotomic character $G_k \to \mathbb{F}_\ell^\times$ (so that $\mathbb{L}$ maps to 1). Up to the t^ℓ coefficient, everything works as before, but the expression for $\mu(\zeta_X(t))$ as a rational function breaks down at the t^ℓ coefficient. No one ℓ value necessarily excludes the possibility of rationality, but by taking values of ℓ tending to infinity, we can prove that $\zeta_X(t)$ cannot be rational.

Assuming the characteristic of k is 0, we can define ν_ℓ so that for every nonsingular projective k-variety X, we have $\nu_\ell([X]) = [H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)]$ in the Grothendieck ring $K_\ell^{\mathrm{sp}}(G_{k(\zeta_\ell)})$. It is easy to calculate the semisimplification of $H^\bullet(\overline{\mathrm{Sym}^n X}, \bar{\mathbb{F}}_\ell)$ as a $G_{k(\zeta_\ell)}$-representation, but as $\mathrm{Sym}^n X$ is in general singular, we do not know when

$$\nu_\ell\big([\mathrm{Sym}^n X]\big) = \big[\mathrm{Sym}^n H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)\big].$$

However, we show that this holds when all the cohomology of X is in even degree and ℓ is sufficiently large compared to n. If ℓ is large compared to the degrees of the numerator and denominator of $\zeta_X(t)$, then the linear recurrence satisfied by the $\nu_\ell([\mathrm{Sym}^i X])$ ultimately implies that $\nu_\ell([\mathrm{Sym}^\ell X])$ is noneffective. This is a result of the breakdown of the correspondence between the (mod ℓ) representation theory of $\mathrm{SL}_2(\mathbb{F}_\ell)$ and the complex representation theory of $\mathrm{SL}_2(\mathbb{C})$ which occurs in dimension ℓ.

Unfortunately, we do not know how to compute the value $\nu_\ell([\mathrm{Sym}^\ell X])$ directly, but, using a generalization to arbitrary fields of Göttsche's relation [12] in $K_0(\mathrm{Var}_k)$ between the classes $[X^{[i]}]$ of the Hilbert schemes of X and the classes of the symmetric powers of X, we can show that $\nu_\ell([X^{[\ell]}])$ is also noneffective. This is absurd, since $X^{[\ell]}$ is projective and nonsingular.

We remark that as this proof depends on showing that images of certain Galois representations are "as large as possible," it breaks down for some fields k, especially algebraically closed fields. It would be particularly interesting to have an argument which works over $\mathbb{C}$.

In Section 2, we discuss Grothendieck groups of representations of finite groups, especially $\mathrm{SL}_2(\mathbb{F}_\ell)$ and $\mathrm{SL}_2(\mathbb{F}_\ell)^2$. In Section 3, we use the method of Bittner [2] to construct ν_ℓ. In Section 4, we discuss some variants of the category of Chow motives which enable us to show that if ℓ is large compared to n, then $\mathrm{Sym}^n X$ behaves like a nonsingular variety as far as ν_ℓ is concerned. In Section 5, we show that there exists a

K3 surface over $\mathbb{Q}$ with the desired Galois-theoretic properties. The proof of the main theorem is in Section 6. The generalization of Göttsche's theorem to an arbitrary base field is given in the Appendix.

2. Grothendieck rings of representations

We fix an odd prime ℓ and an algebraic closure $\bar{\mathbb{F}}_\ell$ of the prime field $\mathbb{F}_\ell$, which we regard as a space with the discrete topology. For any topological group G, we denote by $K_\ell^{\mathrm{sp}}(G)$ the Grothendieck ring of the exact category given by *split* short exact sequences of continuous $\bar{\mathbb{F}}_\ell[G]$-modules which are finite-dimensional over $\bar{\mathbb{F}}_\ell$.

We claim that, as an additive group, $K_\ell^{\mathrm{sp}}(G)$ is the free $\mathbb{Z}$-module on indecomposable continuous $\bar{\mathbb{F}}_\ell[G]$-modules. To see this, recall from [15] that an additive category is *Krull–Schmidt* if every object is a finite direct sum of indecomposable objects whose endomorphism rings are local. As every finite-dimensional G-module has finite length, the category of such modules is Krull–Schmidt (see [15, Section 5]).

By the Krull–Remak–Schmidt theorem, this implies that the factors appearing in any decomposition into indecomposables, together with their multiplicities, are uniquely determined. We say that an element of $K_\ell^{\mathrm{sp}}(G)$ is *effective* if it is a nonnegative linear combination of indecomposable classes.

Any continuous homomorphism $G \to H$ induces a restriction homomorphism $K_\ell^{\mathrm{sp}}(H) \to K_\ell^{\mathrm{sp}}(G)$, which maps effective classes to effective classes. If $G \to H$ is surjective, then Res_G^H is injective because distinct indecomposable representations restrict to distinct indecomposable representations of G. A class in $K_\ell^{\mathrm{sp}}(H)$ is effective if and only if its restriction to $K_\ell^{\mathrm{sp}}(G)$ is so.

If G is profinite, then $K_\ell^{\mathrm{sp}}(G)$ is the direct limit of $K_\ell^{\mathrm{sp}}(G/H)$ as H ranges over open normal subgroups of G. In this section, we consider only finite groups G endowed with the discrete topology, so the continuity condition will play no role.

PROPOSITION 2.1
If H_1 and H_2 are finite groups, then the external tensor product defines an injective homomorphism $K_\ell^{\mathrm{sp}}(H_1) \otimes K_\ell^{\mathrm{sp}}(H_2) \to K_\ell^{\mathrm{sp}}(H_1 \times H_2)$.

Proof
We need to show that if $\rho_1\colon H_1 \to \mathrm{GL}(V_1)$ and $\rho_2\colon H_2 \to \mathrm{GL}(V_2)$ are indecomposable representations, then $\rho_{12}\colon H_1 \times H_2 \to \mathrm{GL}(V_1 \boxtimes V_2)$ is an indecomposable representation of $H_1 \times H_2$ and that, moreover, the isomorphism class of the representation $V_1 \boxtimes V_2$ determines the isomorphism classes of V_1 and V_2. The second claim follows immediately by applying Krull–Remak–Schmidt to the restriction of $V_1 \boxtimes V_2$ to $G_1 \times \{1\}$ and $\{1\} \times G_2$.

To prove that ρ_{12} is indecomposable, it suffices to prove that the centralizer Z_{12} of the $\bar{\mathbb{F}}_\ell$-span of $\rho_{12}(H_1 \times H_2)$ in $\mathrm{End}(V_1 \otimes V_2)$ is a local $\bar{\mathbb{F}}_\ell$-algebra. To commute with $\rho_{12}(H_1 \times H_2)$ is the same as to commute with $\rho_{12}(H_1 \times \{1\})$ and $\rho_{12}(\{1\} \times H_2)$. If Z_i denotes the centralizer of $\rho_i(H_i)$ in $\mathrm{End}(V_i)$, and Z_i' is any $\bar{\mathbb{F}}_\ell$-linear complement of Z_i in $\mathrm{End}(V_i)$, then the centralizer of $\rho_{12}(H_1 \times \{1\})$ is

$$Z_1 \otimes \mathrm{End}(V_2) = Z_1 \otimes (Z_2 \oplus Z_2') = Z_1 \otimes Z_2 \oplus Z_1 \otimes Z_2',$$

the centralizer of $\rho_{12}(\{1\} \times H_2)$ is

$$\mathrm{End}(V_1) \times Z_2 = (Z_1 \oplus Z_1') \otimes Z_2 = Z_1 \otimes Z_2 \oplus Z_1' \otimes Z_2,$$

and the intersection of these two centralizers is $Z_1 \otimes Z_2$.

Each finite-dimensional representation is indecomposable if and only if its endomorphism ring is local (see [15, Proposition 5.4]). The tensor product of finite-dimensional local algebras over an algebraically closed field is again local (see [19, Theorem 4]), and this proves the proposition. ☐

If V is a G-representation, then we define $\zeta_V(t) \in K_\ell^{\mathrm{sp}}(G)[[t]]$ as

$$\zeta_V(t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n V] t^n,$$

where $\mathrm{Sym}^n V$ denotes the space of Σ_n-coinvariants of the tensor product $V^{\otimes n}$. Note that if ℓ is a prime, Σ is any group of order prime to ℓ, and V is a finite-dimensional Σ-representation over $\bar{\mathbb{F}}_\ell$, then the map

$$\bar{v} \mapsto \sum_{\sigma \in \Sigma} \sigma v$$

induces a natural isomorphism $V_\Sigma \to V^\Sigma$, from coinvariants to invariants. We may therefore identify the symmetric nth power with the symmetric tensors of rank n when $n < \ell$.

PROPOSITION 2.2

For every group G and every representation V,

$$\Big(\sum_{i=0}^{\dim V} (-1)^i [\wedge^i V] t^i\Big) \zeta_V(t) \equiv 1 \pmod{t^\ell}.$$

Proof

Equivalently, we claim that, for $1 \le k < l$, we have

$$\sum_{i+j=k} (-1)^i [\wedge^i V \otimes \mathrm{Sym}^j V] = 0. \tag{2.1}$$

For every object W of a λ-ring, we have the identity

$$\sum_{i+j=k} \lambda^i(W)\lambda^j(-W) = 0.$$

If W is a finite-dimensional complex vector space regarded as an object of the representation ring of $\mathrm{GL}(W)$, then it is easy to see by the splitting principle that $(-1)^j\lambda^j(-W)$ corresponds to $\mathrm{Sym}^j W$. If $\epsilon_{i,j}$ in the group ring $\mathbb{Z}[1/k!][S_k]$ denotes the projector which maps $W^{\otimes k} = W^{\otimes(i+j)}$ onto $\wedge^i W \otimes \mathrm{Sym}^j W$, this implies

$$\sum_{i+j=k} (-1)^i \epsilon_{i,j} = (-1)^{i+j} \sum_{i+j=k} (-1)^j \epsilon_{i,j} = 0.$$

As $k!$ is invertible (mod ℓ), this reduces to the same identity over $\mathbb{F}_\ell$, which implies the identity (2.1) for group representations in characteristic ℓ. □

We will eventually be interested in the case $G = \mathrm{SL}_2(\mathbb{F}_\ell)^2$, but we start with $H = \mathrm{SL}_2(\mathbb{F}_\ell)$. We denote by V_i the ith symmetric power of the natural 2-dimensional $\bar{\mathbb{F}}_\ell$-representation of H (where the 0th symmetric power is understood to be the trivial representation) and by $\mathbf{W}$ the representation $V_1 \otimes V_{\ell-1}$.

PROPOSITION 2.3
We define

$$F_n K_\ell^{\mathrm{sp}}(H) = \begin{cases} \mathrm{Span}_{\mathbb{Z}}([V_0],\ldots,[V_n]) & \text{if } n \le \ell-1, \\ \mathrm{Span}_{\mathbb{Z}}([V_0],\ldots,[V_{\ell-1}],[\mathbf{W}]) & \text{if } n = \ell, \\ K_\ell^{\mathrm{sp}}(H) & \text{if } n > \ell. \end{cases}$$

We have the following facts:

(1) *The representation* $\mathbf{W}$ *is indecomposable.*

(2) *The product on* $K_\ell^{\mathrm{sp}}(H)$ *is compatible with the filtration* F_i *in the sense that*

$$\big(F_i K_\ell^{\mathrm{sp}}(H)\big)\big(F_j K_\ell^{\mathrm{sp}}(H)\big) \subseteq F_{i+j} K_\ell^{\mathrm{sp}}(H).$$

Proof
The representation V_1 is the restriction of the tautological 2-dimensional representation $\tilde{V}_1$ of $\mathrm{SL}_2(\bar{\mathbb{F}}_\ell)$. Applying [1, Lemma 3.1, Proposition 3.3(iii)] with $\lambda = \ell - 2$, we know that $\tilde{V}_1 \otimes \mathrm{Sym}^{\ell-1}\tilde{V}_1$ is indecomposable, and by [1, Lemma 4.1(a)], the restriction $\mathbf{W}$ of this representation to $\mathrm{SL}_2(\mathbb{F}_\ell)$ is the injective hull of an irreducible representation of $\mathrm{SL}_2(\mathbb{F}_\ell)$ and therefore indecomposable. (This fact can also be read off from Table 1 of the same paper.) This gives claim (1).

By [1, Lemma 2.5] (or Proposition 2.2), for $1 \le i \le \ell - 2$, we have

$$V_i \otimes V_1 \cong V_{i-1} \oplus V_{i+1}. \tag{2.2}$$

By induction on j, this implies that, for $i, j \ge 0$ and $i + j \le \ell - 1$, we have the Clebsch–Gordan formula

$$V_i \otimes V_j \cong V_{i+j} \oplus V_{i+j-2} \oplus \cdots \oplus V_{|i-j|}.$$

For $i + j = \ell$ and $0 < i < j$, we claim that

$$V_i \otimes V_j = \mathbf{W} \oplus \bigoplus_{k=2}^{i} V_{\ell-2k}. \tag{2.3}$$

The statement is trivial for $i = 1$, and for $i \ge 2$,

$$\begin{aligned} V_{i-2} \otimes V_j \oplus V_i \otimes V_j &\cong (V_{i-2} \oplus V_i) \otimes V_j \\ &\cong V_1 \otimes (V_{i-1} \otimes V_j) \\ &\cong V_1 \otimes (V_{\ell-1} \oplus V_{\ell-3} \oplus \cdots \oplus V_{\ell+1-2i}) \\ &\cong \mathbf{W} \oplus (V_{\ell-2} \oplus V_{\ell-4}) \oplus \cdots \oplus (V_{\ell+2-2i} \oplus V_{\ell-2i}). \end{aligned}$$

As

$$V_{i-2} \otimes V_j = V_{\ell-2} \oplus V_{\ell-4} \oplus \cdots \oplus V_{\ell+2-2i},$$

Krull–Schmidt implies our claim, which in turn implies (2). □

Let

$$\Lambda_{V_1}(t) := 1 - [V_1]t + t^2 \in K_\ell^{\mathrm{sp}}(H)[t].$$

The analogy between the (mod ℓ) representation theory of H and the (complex) representation theory of $\mathrm{SL}_2(\mathbb{C})$ might suggest the possibility that $\zeta_{V_1}(t) = \Lambda_{V_1}(t)^{-1}$, that is, that the congruence in Proposition 2.2 is actually an equality, but this turns out not to be true. Instead, (2.2) and (2.3) imply

$$\Lambda_{V_1}(t)\zeta_{V_1}(t) \equiv 1 + \big([V_{\ell-2}] + [V_\ell] - [\mathbf{W}]\big)t^\ell \pmod{t^{\ell+1}}. \tag{2.4}$$

Note that since $\mathbf{W}$ is indecomposable, the t^ℓ coefficient of $\Lambda(t)\zeta_{V_1}(t)$ is nonzero. This phenomenon, as it arises in the case of the representation $V_1 \boxtimes V_1$ of $\mathrm{SL}_2(\mathbb{F}_\ell) \times \mathrm{SL}_2(\mathbb{F}_\ell)$, is the key to our proof of irrationality.

Henceforth, $G = \mathrm{SL}_2(\mathbb{F}_\ell) \times \mathrm{SL}_2(\mathbb{F}_\ell)$. For nonnegative integers n, we define

$$F_n K_\ell^{\mathrm{sp}}(G) = \mathrm{Span}_{\mathbb{Z}}\big\{[V_i \boxtimes V_j] \,\big|\, 0 \leq i, j \leq n\big\}.$$

By Proposition 2.3(2), we have

$$\big(F_i K_\ell^{\mathrm{sp}}(G)\big)\big(F_j K_\ell^{\mathrm{sp}}(G)\big) \subseteq F_{i+j} K_\ell^{\mathrm{sp}}(G) \tag{2.5}$$

for all nonnegative integers i and j.

We define $\mathcal{A}_G$ to be the set of power series $\sum_{i \geq 0} c_i t^t$ with $c_i \in F_i K_\ell^{\mathrm{sp}}(G)$.

LEMMA 2.4
We have the following:

(1) *$\mathcal{A}_G$ is a subring of $K_\ell^{\mathrm{sp}}(G)[[t]]$.*
(2) *$(1 + tK_\ell^{\mathrm{sp}}(G)[[t]]) \cap \mathcal{A}_G$ and $1 + t\mathcal{A}_G$ are multiplicative groups.*
(3) *If $1 + \sum_{i=1}^{\infty} a_i t^i$ and $1 + \sum_{i=1}^{\infty} b_i t^i$ are elements of $\mathcal{A}_G$ which represent the same $(1 + t\mathcal{A}_G)^\times$-coset, then $a_i \equiv b_i \pmod{F_{i-1} K_\ell^{\mathrm{sp}}(G)}$ for all $i \geq 1$.*

Proof
Part (1) follows immediately from (2.5). For (2), the sets $(1 + tK_\ell^{\mathrm{sp}}(G)[[t]]) \cap \mathcal{A}_G$ and $1 + t\mathcal{A}_G$ are obviously both multiplicative monoids. To show that both sets admit multiplicative inverses, we note that the power series expansion for $(1 + a)^{-1}$ converges t-adically whenever $a \in tK_\ell^{\mathrm{sp}}(G)[[t]]$, and both $(1 + tK_\ell^{\mathrm{sp}}(G)[[t]]) \cap \mathcal{A}_G$ and $1 + t\mathcal{A}_G$ are closed in the t-adic topology on $K_\ell^{\mathrm{sp}}(G)[[t]]$. For (3), if $1 + t\alpha, 1 + t\beta \in \mathcal{A}_G$ belong to the same coset, then

$$(1 + t\beta) = (1 + t\alpha)(1 + t\gamma)$$

for some $\gamma \in \mathcal{A}_G$, so $\beta - \alpha = (1 + t\alpha)\gamma$, where $\alpha\gamma \in \mathcal{A}_G$. This is equivalent to the congruence condition $a_i \equiv b_i \pmod{F_{i-1} K_\ell^{\mathrm{sp}}(G)}$ for all i. □

PROPOSITION 2.5
For $0 \leq n \leq \ell - 1$, we have

$$\mathrm{Sym}^n(V_1 \boxtimes V_1) \cong \sum_{i=0}^{\lfloor n/2 \rfloor} V_{n-2i} \boxtimes V_{n-2i}.$$

Proof
First of all, the symmetric power is a quotient of

$$(V_1 \boxtimes V_1)^{\otimes n} = V_1^{\otimes n} \boxtimes V_1^{\otimes n},$$

which by (2.2) and induction on n is a direct sum of expressions of the form $V_i \boxtimes V_j$ with $i, j \leq n$. Thus $\mathrm{Sym}^n(V_1 \boxtimes V_1)$ is itself a direct sum of such expressions. Writing

$$\mathrm{Sym}^n(V_1 \boxtimes V_1) = \bigoplus_{0\leq i,j\leq n} (V_i \boxtimes V_j)^{a_{i,j}},$$

it remains to prove that $a_{i,j}$ is 0 except when $i = j \in \{n, n-2, n-4, \ldots\}$, in which case it is 1.

Restricting to $H \times \{1\}$, we obtain the isomorphism of H-modules

$$\bigoplus_{0\leq i,j\leq n} V_i^{a_{i,j}(j+1)} \cong \mathrm{Sym}^n(V_1 \oplus V_1) \cong \bigoplus_{a+b=n} V_a \otimes V_b \cong \bigoplus_{k=0}^{\lfloor n/2\rfloor} V_{n-2k}^{n-2k+1}, \quad (2.6)$$

the last isomorphism following from (2.3). Thus, $a_{i,j}(j+1) \leq i+1$ for all $i, j \leq n$. By symmetry, also $a_{i,j}(i+1) \leq j+1$. Thus, $a_{i,j} \leq 1$ with equality only if $i = j$. Comparing with (2.6), we see that $a_{i,i} = 1$ exactly for $i \in \{n, n-2, n-4, \ldots\}$. □

PROPOSITION 2.6

Define

$$\Lambda_{V_1\boxtimes V_1}(t) := 1 - [V_1 \boxtimes V_1]t + \big([V_2 \boxtimes V_0] + [V_0 \boxtimes V_2]\big)t^2 - [V_1 \boxtimes V_1]t^3 + t^4. \quad (2.7)$$

Then

$$\Lambda_{V_1\boxtimes V_1}(t)\zeta_{V_1\boxtimes V_1}(t) \equiv 1 \pmod{t^\ell}.$$

Proof

This follows easily from Proposition 2.2. We may assume that $\ell > 2$, so $\mathrm{SL}_2(\mathbb{F}_\ell)$ is perfect, which implies that the top exterior power of $V_1 \boxtimes V_1$ is trivial. As $V_1 \boxtimes V_1$ is self-dual, it follows that it is equal to its own exterior cube. Finally, restricting to $H \times 1$ and $1 \times H$, we see that

$$\wedge^2(V_1 \boxtimes V_1) \cong V_2 \boxtimes V_0 \oplus V_0 \boxtimes V_2. \qquad \square$$

We now come to the key proposition.

PROPOSITION 2.7

Let R be a ring containing $K_\ell^{\mathrm{sp}}(G)$. Let $A(t), B(t) \in R[t]$ denote polynomials with $A(0) = B(0) = 1$, let k be a nonnegative integer, and let $\bar{\mathbb{F}}_\ell^k$ denote the trivial representation of G of dimension k. Fix $M \geq \max(\deg A + k + 4, \deg B)$, and assume that $\ell \geq M + 1$.

If $a_i \in R$ for all $i \geq 0$,

$$A(t) = B(t)(a_0 + a_1 t + a_2 t^2 + \cdots), \tag{2.8}$$

and

$$a_i = \left[\mathrm{Sym}^i\left((V_1 \boxtimes V_1) \oplus \bar{\mathbb{F}}_\ell^k\right)\right] \in K_\ell^{\mathrm{sp}}(G) \tag{2.9}$$

for $i \leq M$, then

$$(1-t)^k \Lambda_{V_1 \boxtimes V_1}(t) \sum_{i=0}^{\infty} a_i t^i = 1 \tag{2.10}$$

and

$$a_\ell \equiv [\mathbf{W} \boxtimes \mathbf{W}] - [\mathbf{W} \boxtimes V_{\ell-2}] - [V_{\ell-2} \boxtimes \mathbf{W}] \pmod{F_{\ell-1} K_\ell^{\mathrm{sp}}(G)}. \tag{2.11}$$

Proof
Combining (2.8) and (2.9), we obtain

$$A(t) \equiv B(t) \sum_{i=0}^{\infty} \left[\mathrm{Sym}^i\left((V_1 \boxtimes V_1) \oplus \bar{\mathbb{F}}_\ell^k\right)\right] \pmod{t^{M+1}}. \tag{2.12}$$

For any G-representation V and any nonnegative integer $n < \ell$,

$$\mathrm{Sym}^n(V \oplus \bar{\mathbb{F}}_\ell) \cong \bigoplus_{i=0}^{n} \mathrm{Sym}^i V.$$

Thus,

$$(1-t) \sum_{i=0}^{\infty} \left[\mathrm{Sym}^i(V \oplus \bar{\mathbb{F}}_\ell)\right] t^i \equiv \sum_{i=0}^{\infty} [\mathrm{Sym}^i V] t^i \pmod{t^\ell}. \tag{2.13}$$

Applying this for $V = V_1 \boxtimes V_1$ and iterating,

$$(1-t)^k \sum_{i=0}^{\infty} \left[\mathrm{Sym}^i\left((V_1 \boxtimes V_1) \oplus \bar{\mathbb{F}}_\ell^k\right)\right] t^i \equiv \sum_{i=0}^{\infty} \left[\mathrm{Sym}^i(V_1 \boxtimes V_1)\right] t^i \pmod{t^\ell}.$$

Since $\ell \geq M + 1$, (2.12) implies

$$\begin{aligned} A(t)(1-t)^k &\equiv B(t)(1-t)^k \sum_{i=0}^{\infty} \left[\mathrm{Sym}^i\left((V_1 \boxtimes V_1) \oplus \bar{\mathbb{F}}_\ell^k\right)\right] t^i \\ &\equiv B(t) \sum_{i=0}^{\infty} \left[\mathrm{Sym}^i(V_1 \boxtimes V_1)\right] t^i \pmod{t^{M+1}}. \end{aligned} \tag{2.14}$$

Multiplying (2.14) by $\Lambda_{V_1 \boxtimes V_1}(t)$, as defined in (2.7), we deduce

$$\begin{aligned} A(t)(1-t)^k \Lambda_{V_1 \boxtimes V_1}(t) &\equiv B(t)\Lambda_{V_1 \boxtimes V_1}(t) \sum_{i=0}^{\infty} \big[\mathrm{Sym}^i(V_1 \boxtimes V_1)\big] t^i \\ &\equiv B(t) \pmod{t^{M+1}}. \end{aligned} \tag{2.15}$$

As

$$\deg\big(A(t)(1-t)^k \Lambda_{V_1 \boxtimes V_1}(t) - B(t)\big) \le \max(\deg A + k + 4, \deg B) \le M,$$

the congruence (2.15) implies

$$A(t)(1-t)^k \Lambda_{V_1 \boxtimes V_1}(t) = B(t),$$

so

$$A(t)\Big(1 - (1-t)^k \Lambda_{V_1 \boxtimes V_1}(t) \sum_{i=0}^{\infty} a_i t^i\Big) = A(t) - B(t) \sum_{i=0}^{\infty} a_i t^i = 0.$$

As $A(t)$ is invertible, this implies (2.10).

By Lemma 2.4(2), $\Lambda_{V_1 \boxtimes V_1}(t)^{-1}$ belongs to $\mathcal{A}_G$. Let c_i denote its t^i coefficient. Thus, $c_i \in F_i K_\ell^{\mathrm{sp}}(G)$ for all $i \ge 0$, and the equation

$$\Lambda_{V_1 \boxtimes V_1}(t) \sum_{i=0}^{\infty} c_i t^i = 1 \tag{2.16}$$

shows that the terms c_i satisfy a linear recurrence of degree 4. Matching t^ℓ coefficients in (2.16), we get the recurrence relation

$$c_\ell = c_1 c_{\ell-1} - \big([V_2 \boxtimes V_0] + [V_0 \boxtimes V_2]\big) c_{\ell-2} + c_1 c_{\ell-3} - c_{\ell-4}.$$

Modulo classes in $F_{\ell-1} K_\ell^{\mathrm{sp}}(G)$, the right-hand side reads

$$[V_1 \boxtimes V_1][V_{\ell-1} \boxtimes V_{\ell-1}] - \big([V_2 \boxtimes V_0] + [V_0 \boxtimes V_2]\big)[V_{\ell-2} \boxtimes V_{\ell-2}],$$

which, by (2.3), further reduces modulo $F_{\ell-1} K_\ell^{\mathrm{sp}}(G)$ to

$$[\mathbf{W} \boxtimes \mathbf{W}] - [\mathbf{W} \boxtimes V_{\ell-2}] - [V_{\ell-2} \boxtimes \mathbf{W}].$$

As $1 - t \in (1 + t\mathcal{A}_G)^\times$, by (2.10), (2.16), and Lemma 2.4(3), we have

$$a_\ell \equiv c_\ell \equiv [\mathbf{W} \boxtimes \mathbf{W}] - [\mathbf{W} \boxtimes V_{\ell-2}] - [V_{\ell-2} \boxtimes \mathbf{W}] \pmod{F_{\ell-1} K_\ell^{\mathrm{sp}}(G)}. \qquad \square$$

PROPOSITION 2.8

Let R be a ring containing $K_\ell^{\mathrm{sp}}(G)$, let k be a nonnegative integer, and let $b_0, b_1, b_2, \ldots$ be elements of R. If $\ell \geq k+5$,

$$\sum_{i=0}^{\infty} a_i t^i = (1-t)^{-k} \Lambda_{V_1 \boxtimes V_1}(t)^{-1},$$

and

$$\sum_{i=0}^{\infty} b_i t^i = \prod_{r=1}^{\infty} \sum_{j=0}^{\infty} a_j t^{jr}, \tag{2.17}$$

then $b_i \in K_\ell^{\mathrm{sp}}(G)$ for all $i \geq 0$, and

$$b_\ell \equiv [\mathbf{W} \boxtimes \mathbf{W}] - [\mathbf{W} \boxtimes V_{\ell-2}] - [V_{\ell-2} \boxtimes \mathbf{W}] \pmod{F_{\ell-1} K_\ell^{\mathrm{sp}}(G)}.$$

Proof

By Proposition 2.2,

$$\Lambda_{V_1 \boxtimes V_1}(t)^{-1} \equiv \sum_{i=0}^{\infty} \bigl[\mathrm{Sym}^i(V_1 \boxtimes V_1)\bigr] t^i \pmod{t^\ell},$$

and so applying (2.13) k times, (2.9) holds for $0 \leq i < \ell$. Setting $A(t) := 1$, $B(t) := (1-t)^k \Lambda_{V_1 \boxtimes V_1}(t)$, and $M := k+4$, the hypotheses of Proposition 2.7 are satisfied, and so a_ℓ satisfies the congruence (2.11).

By Lemma 2.4(2), $\sum_{j=0}^{\infty} a_j t^j \in \mathcal{A}_G$. Therefore, for $r \geq 2$, $\sum_{j=0}^{\infty} a_j t^{jr} \in (1 + t\mathcal{A}_G)^\times$. As the product

$$\prod_{r=2}^{\infty} \prod_{j=0}^{\infty} a_j t^{jr}$$

is t-adically convergent and $(1 + t\mathcal{A}_G)^\times$ is t-adically closed, it follows that this product lies in $(1 + t\mathcal{A}_G)^\times$. By parts (1) and (3) of Lemma 2.4,

$$b_\ell \equiv a_\ell \pmod{F_{\ell-1} K_\ell^{\mathrm{sp}}(G)}.$$

The proposition now follows from (2.11). □

We note for future reference that the relationship (2.17) between the a_i's and the b_i's is significant because it expresses the relationship between the motives of the symmetric powers of a surface X and the Hilbert schemes of X.

3. A family of motivic measures

In this section, we construct the motivic measures needed for the proof of our main theorem.

Let k be a field of characteristic 0, let $\bar{k}$ be an algebraic closure of k, and let $G_k := \mathrm{Gal}(\bar{k}/k)$. We define $\bar{X} := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$ for any variety X/k. We regard the étale cohomology groups $H^i(\bar{X}, \bar{\mathbb{F}}_\ell)$ and $H^i_c(\bar{X}, \bar{\mathbb{F}}_\ell)$ as G_k-representations. They are obtained by extension of scalars from the G_k-representations $H^i(\bar{X}, \mathbb{F}_\ell)$ and $H^i_c(\bar{X}, \mathbb{F}_\ell)$, respectively.

Our construction depends on the Bittner construction (see [2]). In order to carry it out, we make use of the following two results, which come from [13, Chapitre VII, Théorème 8.1] and [13, Chapitre VII, Corollaire 2.2.4]. We are grateful to the referee for providing these references.

THEOREM 3.1
Let X be a nonsingular projective variety over k, let $Y \subset X$ be a nonsingular closed subvariety of codimension r, let X' be the blowup of X along Y, and let Y' be the inverse image of Y in X'. Then for any q there is a natural isomorphism

$$H^q(\bar{X}', \bar{\mathbb{F}}_\ell) = H^q(\bar{X}, \bar{\mathbb{F}}_\ell) \oplus \bigoplus_{j=1}^{r-1} H^{q-2j}\big(\bar{Y}, \bar{\mathbb{F}}_\ell(-j)\big).$$

PROPOSITION 3.2
Let $\bar{Y}$ be a smooth projective variety over $\bar{k}$, and let $\mathcal{E}$ be a vector bundle of rank r over $\bar{Y}$. Let $\mathbb{P}(\mathcal{E}) \to \bar{Y}$ be the corresponding projective bundle. Then for each q there is a natural isomorphism

$$H^q\big(\mathbb{P}(\mathcal{E}), \bar{\mathbb{F}}_\ell\big) = \bigoplus_{j \geq 0} H^{q-2j}\big(\bar{Y}, \bar{\mathbb{F}}_\ell(-j)\big),$$

where the summand for $j = 0$ is the image of the map $p^*\colon H^q(\bar{Y}, \bar{\mathbb{F}}_\ell) \to H^q(\mathbb{P}(\mathcal{E}), \bar{\mathbb{F}}_\ell)$. This isomorphism is G_k-equivariant if both $\bar{Y}$ and $\mathcal{E}$ are defined over k.

THEOREM 3.3
For each prime ℓ and every field k of characteristic 0, there exists a unique motivic measure $\mu_\ell\colon K_0(\mathrm{Var}_k) \to K_\ell^{\mathrm{sp}}(G_k)$ satisfying

$$\mu_\ell\big([X]\big) = \sum_{i=0}^{2\dim X} \big[H^i(\bar{X}, \bar{\mathbb{F}}_\ell)\big],$$

for all projective nonsingular varieties X.

Proof

By Bittner's theorem (see [2, Theorem 3.1]), it suffices to prove that $\mu_\ell([X \times Y]) = \mu_\ell([X])\mu_\ell([Y])$ whenever X and Y are nonsingular projective varieties, and that whenever X is a nonsingular projective variety, Y is a nonsingular closed subvariety, X' is the blowup of X along Y, and Y' is the inverse image of Y in X', we have

$$\mu_\ell\big([X']\big) - \mu_\ell\big([X]\big) = \mu_\ell\big([Y']\big) - \mu_\ell\big([Y]\big).$$

The first property follows immediately from the Künneth formula (see [22, Chapter VI, Corollary 8.13]). The second follows from Theorem 3.1 and Proposition 3.2. □

Definition 3.4

We define the motivic measure $\nu_\ell \colon K_0(\mathrm{Var}_k) \to K_\ell^{\mathrm{sp}}(G_{k(\zeta_\ell)})$ to be the composition of μ_ℓ with the restriction map $K_\ell^{\mathrm{sp}}(G_k) \to K_\ell^{\mathrm{sp}}(G_{k(\zeta_\ell)})$.

In the application to the main theorem, we will always take $k = \mathbb{Q}$.

4. Chow motives and finite Galois modules

Fix a field k, and denote by $V(k)$ the category of smooth, projective, irreducible k-varieties and arbitrary morphisms of such varieties. Given $X \in V(k)$ of dimension d, we consider the graded Chow ring $A^*(X) = \bigoplus_{r=0}^{d} A^{d-r}(X)$ of cycles on X modulo rational equivalence, where the group $A^{d-r}(X) = A_r(X)$ consists of classes of cycles of dimension r (see [10]). Let us recall a version of the category of Chow motives that is appropriate for our needs. First consider the additive category $\mathrm{Cor}(k)$ whose objects are the objects of $V(k)$ and whose morphisms are the degree 0 Chow correspondences. That is, given $X, Y \in \mathrm{Cor}(k)$, X being of pure dimension d, we set

$$\mathrm{Hom}_{\mathrm{Cor}(k)}(X, Y) := A^d(X \times Y).$$

The composition of morphisms is the composition of correspondences (see [21]). The category $\mathrm{Cor}(k)$ is the "additivization" of the category $V(k)$. Next, one defines the category $\mathrm{Chow}(k)$ of Chow motives as the idempotent completion of $\mathrm{Cor}(k)$. Explicitly, the objects of $\mathrm{Chow}(k)$ are pairs (X, p), where $X \in V(k)$ and $p \in \mathrm{End}_{\mathrm{Cor}(k)}(X)$ is a projector: $p^2 = p$. Morphisms between (X, p) and (Y, q) form the group $q \cdot \mathrm{Hom}_{\mathrm{Cor}(k)}(X, Y) \cdot p$. There is a canonical contravariant functor $V(k) \to \mathrm{Chow}(k)$ which sends $X \in V(k)$ to $(X, \mathbf{1})$ and a morphism $f : X \to Y$ to its graph $\Gamma_f \subset Y \times X$. Let $e \in \mathrm{Chow}(k)$ be the image of $\mathrm{Spec}\, k$. The category $\mathrm{Chow}(k)$ is a tensor category with the product

$$(X, p) \otimes (Y, q) = (X \times Y, p \otimes q).$$

There exists an object $\mathbf{L} \in \mathrm{Chow}(k)$, called the *Tate motive*, such that $\mathbb{P}^1 = e \oplus \mathbf{L}$ (see [21]). For $(X, p) \in \mathrm{Chow}(k)$, we denote as usual the product $(X, p) \otimes \mathbf{L}$ by $(X, p)(-1)$.

Given a nonzero integer n, we denote by $\mathrm{Chow}(k)[1/n]$ the localization at n of the additive category $\mathrm{Chow}(k)$; that is, for $A, B \in \mathrm{Chow}(k)$, we have

$$\mathrm{Hom}_{\mathrm{Chow}(k)[1/n]}(A, B) = \mathrm{Hom}_{\mathrm{Chow}(k)}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n].$$

So $\mathrm{Chow}(k)[1/n]$ is a $\mathbb{Z}[1/n]$-linear tensor category. We also consider the category $\mathrm{Chow}(k)_{\mathbb{Q}}$ of rational Chow motives constructed in a similar way.

Example 4.1
Let $X \in V(k)$ be a variety of pure dimension d with an action of a finite group G of order n. Then $p := \frac{1}{n} \sum_{g \in G} \Gamma_g \in A^d(X \times X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$ is a projector. Hence $(X, p) \in \mathrm{Chow}(k)[1/n]$.

Given a field extension $k \subset k'$, we obtain the obvious functors $V(k) \to V(k')$, $\mathrm{Cor}(k) \to \mathrm{Cor}(k')$, $\mathrm{Chow}(k) \to \mathrm{Chow}(k')$, and so on induced by the extension of scalars $X \mapsto X_{k'} = X \times_k k'$ of varieties (see [10, Example 6.2.9]). If $k' = \bar{k}$, as usual, we denote the variety $X \times_k \overline{k}$ by $\bar{X}$. For a prime $\ell \neq \mathrm{char}(k)$, let ζ_ℓ be a primitive ℓth root of 1 in $\bar{k}$.

PROPOSITION 4.2
Let n be a nonzero integer, and let ℓ be a prime number not dividing n and different from the characteristic of the base field k. Then the assignment

$$X \mapsto H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell), \quad X \in V(k),$$

extends to a tensor (contravariant) functor from the category $\mathrm{Chow}(k)[1/n]$ *to the abelian tensor category of finite-dimensional* $\bar{\mathbb{F}}_\ell$*-modules with a continuous* Gal_k*-action:*

$$\Phi_\ell : \mathrm{Chow}(k)[1/n] \to \bar{\mathbb{F}}_\ell\text{-}\mathrm{Gal}_k\text{-mod}.$$

If k contains a primitive ℓth root of unity ζ_ℓ, then the module $\Phi_\ell(\mathbf{L})$ is a 1*-dimensional trivial* $\bar{\mathbb{F}}_\ell$-Gal_k*-module.*

We do not claim originality for this proposition, but, for lack of a reference, we provide a proof.

Proof
Since the category $\bar{\mathbb{F}}_\ell$-Gal_k-mod is closed under idempotent completion and its localization $(\bar{\mathbb{F}}_\ell\text{-}\mathrm{Gal}_k\text{-mod})[1/n]$ is equivalent to $\bar{\mathbb{F}}_\ell$-Gal_k-mod, it suffices to construct a

functor from the additive category $\mathrm{Cor}(k)$ to $\bar{\mathbb{F}}_\ell$-Gal_k-mod. We construct this functor as the composition of the extension of scalars functor $\mathrm{Cor}(k) \to \mathrm{Cor}(\bar{k})$ with a functor

$$\Psi_\ell : \mathrm{Cor}(\bar{k}) \to \bar{\mathbb{F}}_\ell\text{-vect},$$

where $\bar{\mathbb{F}}_\ell$-vect is the category of $\bar{\mathbb{F}}_\ell$-vector spaces. The functor Ψ_ℓ is defined as follows. Let X and Y be smooth projective varieties (over $\bar{k}$), X being of pure dimension d, and let $C \in A^d(X \times Y)$ be a correspondence of degree 0. Consider the projections $X \overset{p_X}{\leftarrow} X \times Y \overset{p_Y}{\to} Y$. Then, given an element $a \in H^i(Y, \bar{\mathbb{F}}_\ell)$, we put

$$\Psi_\ell(C)(a) = p_{X*}\big(\mathrm{cl}_{X\times Y}(C) \cup p_Y^*(a)\big) \in H^i(X, \bar{\mathbb{F}}_\ell),$$

where $\mathrm{cl}_{X\times Y} : A^s(X \times Y) \to H^{2s}(X \times Y, \bar{\mathbb{F}}_\ell)$ is the *cycle map* (see [6], [22, Chapter VI, Section 9]) and p_Y^* and p_{X*} are the pullback and the pushforward maps on cohomology (see [22, Chapter VI, Remark 11.6]). In order for Ψ_ℓ to be a functor, the cycle map has to satisfy the following properties for morphisms of smooth and projective varieties:

- cl is a morphism of contravariant functors from $V(\bar{k})$ to the category of rings;
- cl commutes with exterior products;
- cl is a morphism of covariant functors from $V(\bar{k})$ to the category of abelian groups.

The first two properties are proved in [6, Cycle, Remarque 2.3.9 and (2.3.8.3)], and the last one is in [18, Theorem 6.1].

Once the functor Ψ_ℓ is constructed, it is clear that its composition with the extension of scalars $\mathrm{Cor}(k) \to \mathrm{Cor}(\bar{k})$ will give the desired functor Φ_ℓ, since for $X \in \mathrm{Cor}(k)$ the vector space $H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)$ is a Gal_k-module and morphisms in $\mathrm{Cor}(k)$ act as morphisms of Gal_k-modules. Also, the last assertion of the proposition is obvious. This proves Proposition 4.2. □

Example 4.3

Let $(X, p) \in \mathrm{Chow}(k)[1/n]$ be as in Example 4.1, let ℓ be prime to n, and let $l \neq \mathrm{char}(k)$. Then $\Phi_\ell((X, p)) = H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)^G$ as $\bar{\mathbb{F}}_\ell$-Gal_k-modules.

COROLLARY 4.4

Assume that in $\mathrm{Chow}(k)_{\mathbb{Q}}$ *we have an isomorphism of objects $A \simeq B$. Then, for a divisible enough integer n, the objects A and B belong to the essential image of the category* $\mathrm{Chow}(k)[1/n]$ *and are isomorphic in* $\mathrm{Chow}(k)[1/n]$. *Fix one such n, let ℓ be a prime not dividing n, and let $l \neq$* $\mathrm{char}(k)$. *Then the* $\bar{\mathbb{F}}_\ell$-Gal_k*-modules $\Phi_\ell(A)$ and $\Phi_\ell(B)$ are defined and are isomorphic.*

Proof
Indeed, an isomorphism in $\mathrm{Chow}(k)_{\mathbb{Q}}$ between A and B is witnessed by a finite diagram of objects and correspondences with denominators. Hence this diagram exists in $\mathrm{Chow}(k)[1/n]$ for a divisible enough n. So $A \simeq B$ in such a category $\mathrm{Chow}(k)[1/n]$. The last assertion now follows from Proposition 4.2. □

We remark that the measures μ_ℓ and ν_ℓ defined in Section 4 factor through $K_0(\mathrm{Chow}(k)[1/n])$ if $\ell \nmid n$. Indeed, it follows from Bittner's presentation of the group $K_0(\mathrm{Var}_k)$ in [2] that the correspondence $X \mapsto (X, \mathbf{1})$ for a smooth and projective X extends to a group homomorphism $\theta : K_0(\mathrm{Var}_k) \to K_0(\mathrm{Chow}(k))$, where $K_0(\mathrm{Chow}(k))$ is the Grothendieck group of the additive category $\mathrm{Chow}(k)$. Denote by $\theta[1/n]$ the composition of θ with the obvious homomorphism $K_0(\mathrm{Chow}(k)) \to K_0(\mathrm{Chow}(k))[1/n]$. Similarly for $\theta_{\mathbb{Q}}$.

The additive functor Φ_ℓ of Proposition 4.2 induces the group homomorphism

$$K_0(\Phi_\ell) : K_0\big(\mathrm{Chow}(k)[1/n]\big) \to K_\ell^{\mathrm{sp}}(\bar{\mathbb{F}}_\ell\text{-}\mathrm{Gal}_k)$$

such that we have the equality

$$\mu_\ell = K_0(\Phi_\ell) \circ \theta[1/n] : K_0(\mathrm{Var}_k) \to K_\ell^{\mathrm{sp}}(\bar{\mathbb{F}}_\ell\text{-}\mathrm{Gal}_k) \tag{4.1}$$

and hence also

$$\nu_\ell = \mathrm{Res}^{\mathrm{Gal}_k}_{\mathrm{Gal}_{k(\zeta_\ell)}} \circ K_0(\Phi_\ell) \circ \theta[1/n] : K_0(\mathrm{Var}_k) \to K_\ell^{\mathrm{sp}}(\bar{\mathbb{F}}_\ell\text{-}\mathrm{Gal}_{k(\zeta_\ell)}).$$

Remark 4.5
For G, k, and (X, p) as in Example 4.1, we have

$$\theta_{\mathbb{Q}}\big([X/G]\big) = \big[(X, p)\big] \in K_0\big(\mathrm{Chow}(k)_{\mathbb{Q}}\big). \tag{4.2}$$

This follows from Corollary 2.4 in [5].

We obtain the following important corollary, which is used in the proof of our main theorem (Theorem 6.1) below.

COROLLARY 4.6
Let X be a smooth projective variety over k with an action of a finite group G. Then for all sufficiently large primes ℓ, we have an equality

$$\mu_\ell\big([X/G]\big) = \big[H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)^G\big] \in K_\ell^{\mathrm{sp}}(\bar{\mathbb{F}}_\ell\text{-}\mathrm{Gal}_k)$$

and therefore also

$$\nu_\ell\big([X/G]\big) = \big[H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)^G\big] \in K_\ell^{\mathrm{sp}}(\bar{\mathbb{F}}_\ell\text{-}\mathrm{Gal}_{k(\zeta_\ell)}).$$

Proof
By Remark 4.5 and Corollary 4.4 for a divisible enough m, we have

$$\theta[1/m]\big([X/G]\big) = \big[(X,p)\big] \in K_0\big(\mathrm{Chow}(k)[1/m]\big).$$

Choose one such m and a prime ℓ not dividing m. Then

$$\begin{aligned}\mu_\ell\big([X/G]\big) &= K_0(\Phi_\ell)\circ\theta[1/m]\big([X/G]\big) = K_0(\Phi_\ell)\big(\big[(X,p)\big]\big)\\ &= \big[\Phi_\ell\big((X,p)\big)\big] = \big[H^\bullet(\bar{X},\bar{\mathbb{F}}_\ell)^G\big],\end{aligned}$$

where the last equality is by Example 4.3. This proves the corollary. □

Remark 4.7
We believe that the assertions of Corollary 4.6 are true for any ℓ which does not divide the order of the group G. To check this, one needs to consider a refinement of the category $\mathrm{Chow}'(k)_{\mathbb{Q}}$ with bounded denominators similar to the category $\mathrm{Chow}(k)[1/n]$. Since Corollary 4.6 suffices for our needs, we decided not to do it.

5. Galois representations

PROPOSITION 5.1
There exist elliptic curves E_1 and E_2 over $\mathbb{Q}$ such that for all sufficiently large primes ℓ, there exist linearly disjoint Galois extensions K_1 and K_2 of $\mathbb{Q}(\zeta_\ell)$ such that the (mod ℓ) Galois representations of $G_{\mathbb{Q}(\zeta_\ell)}$ acting on $H^1(\bar{E}_i,\mathbb{F}_\ell)$ have kernels G_{K_i} and images isomorphic to $\mathrm{SL}_2(\mathbb{F}_\ell)$.

Proof
Fix primes $q,r \geq 5$. Let E_1 and E_2 be any elliptic curves over $\mathbb{Q}$ with multiplicative reduction at q and such that E_1 and E_2 have, respectively, good ordinary reduction and good supersingular reduction at r. (For instance, if $q = 11$ and $r = 5$, the curves given in Cremona notation by $E_1 :=$ `33a1` and $E_2 :=$ `11a1` satisfy these conditions.) Let ρ_i^ℓ denote the homomorphism from the absolute Galois group $G_{\mathbb{Q}}$ to $\mathrm{GL}(H^1(\bar{E}_i,\mathbb{F}_\ell)) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$.

Neither E_1 nor E_2 can have complex multiplication, since every CM curve has integral j-invariant (see [25, Chapter II, Theorem 6.1]), while an elliptic curve with multiplicative reduction at q cannot have q-adically integral j-invariant (see [25, Table 4.1]). By Serre's theorem (see [24, Théorème 2]), for ℓ sufficiently large, the representations ρ_i^ℓ are surjective. As the determinant of ρ_ℓ^i is the (mod ℓ) cyclotomic character, the image of $G_{\mathbb{Q}(\zeta_\ell)}$ in $\mathrm{GL}(H^1(\bar{E}_i,\mathbb{F}_\ell))$ is $\mathrm{SL}_2(\mathbb{F}_\ell)$. We assume this holds and that $\ell \geq 5$. Let $\bar{\rho}_i^\ell\colon G_{\mathbb{Q}} \to \mathrm{PGL}_2(\mathbb{F}_\ell)$ denote the composition of ρ_i^ℓ with the quotient map $\mathrm{GL}_2(\mathbb{F}_\ell) \to \mathrm{PGL}_2(\mathbb{F}_\ell)$.

Suppose that $\rho_1^\ell|_{G_{\mathbb{Q}(\zeta_\ell)}} = \rho_2^\ell|_{G_{\mathbb{Q}(\zeta_\ell)}}$. As the common image of the two representations has trivial centralizer in $\mathrm{PGL}_2(\mathbb{F}_\ell)$, it follows that $\bar\rho_1^\ell = \bar\rho_2^\ell$. Thus, $\rho_1^\ell = \rho_2^\ell \otimes \chi$ for some character χ of $\mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$. Taking the determinant of both sides, we have $\chi^2 = 1$.

The representations ρ_i^ℓ are both unramified at r, so $\mathrm{Tr}(\rho_i^\ell(\mathrm{Frob}_r))$ is well defined, and the two traces are related by a factor of $\chi(\mathrm{Frob}_r) = \pm 1$. This is impossible since the trace of Frob_r is zero for E_2 but not for E_1.

Now ρ_1^ℓ and ρ_2^ℓ together give an injective homomorphism ρ_{12}^ℓ,

$$\mathrm{Gal}\big(K_1K_2/\mathbb{Q}(\zeta_\ell)\big) \to \mathrm{SL}_2(\mathbb{F}_\ell) \times \mathrm{SL}_2(\mathbb{F}_\ell),$$

whose image projects onto $\mathrm{SL}_2(\mathbb{F}_\ell)$ on both factors. As the only normal subgroups of $\mathrm{SL}_2(\mathbb{F}_\ell)$ are the group itself, $\{\pm 1\}$, and $\{1\}$, applying Goursat's lemma to the image of ρ_{12}^l, either this image is all of $\mathrm{SL}_2(\mathbb{F}_\ell) \times \mathrm{SL}_2(\mathbb{F}_\ell)$, in which case ρ_{12}^ℓ is an isomorphism, or $\bar\rho_1$ and $\bar\rho_2$ coincide on $\mathrm{Gal}(K_1K_2/\mathbb{Q}(\zeta_\ell))$. We have seen that the latter is impossible, so the proposition follows. □

We remark that, assuming the Frey–Mazur conjecture is true, Proposition 5.1 is true (in fact, for all $\ell \geq 17$) for any two non-CM elliptic curves which are not isogenous over $\bar{\mathbb{Q}}$.

Note that if $H^\bullet(\bar X, \bar{\mathbb{F}}_\ell)$ is zero in odd degrees, then the action of Σ_n is the usual permutation action on tensor factors, and the symmetric nth power can therefore be taken in the usual sense of G_k-representations. There is no distinction between the alternating sum of cohomology and the total cohomology, so we can work with Galois representations rather than virtual representations.

THEOREM 5.2
Let E_1, E_2, K_1, K_2 be as in Proposition 5.1. Let X denote the K3 surface obtained by blowing up the nodes of the Kummer surface

$$X' := (E_1 \times E_2)/\langle\iota\rangle,$$

where ι is multiplication by -1. For ℓ sufficiently large, the inclusion map

$$\mathrm{Gal}\big(K_1K_2/\mathbb{Q}(\zeta_\ell)\big) \to \mathrm{SL}_2(\mathbb{F}_\ell) \times \mathrm{SL}_2(\mathbb{F}_\ell) \tag{5.1}$$

is an isomorphism, and

$$\nu_\ell\big([X]\big) = \mathrm{Res}^{\mathrm{SL}_2(\mathbb{F}_\ell)\times\mathrm{SL}_2(\mathbb{F}_\ell)}_{G_{\mathbb{Q}(\zeta_\ell)}}[\bar{\mathbb{F}}_\ell^{20} \oplus V_1 \boxtimes V_1].$$

Proof
The action of ι on the ℓ-torsion of $E_1 \times E_2$ and therefore on

$$H^1(\overline{E_1 \times E_2}, \bar{\mathbb{F}}_\ell) \xrightarrow{\sim} \operatorname{Hom}\bigl(\bar{E}_1[\ell] \times \bar{E}_2[\ell], \bar{\mathbb{F}}_\ell\bigr)$$

is by multiplication by -1; as the cohomology of every abelian variety is generated by H^1, ι acts on $H^q(\overline{E_1 \times E_2}, \bar{\mathbb{F}}_\ell)$ by multiplication by $(-1)^q$. Assuming that ℓ is sufficiently large, it follows from Corollary 4.6 that $H^q(\bar{X}', \bar{\mathbb{F}}_\ell)$ is zero for q odd and is

$$H^2(\overline{E_1 \times E_2}, \bar{\mathbb{F}}_\ell) \cong \bar{\mathbb{F}}_\ell(1) \oplus H^1(\bar{E}_1, \bar{\mathbb{F}}_\ell) \otimes H^1(\bar{E}_2, \bar{\mathbb{F}}_\ell) \oplus \bar{\mathbb{F}}_\ell(1)$$

for $q = 2$. For $q = 0$ and $q = 4$, we get $\bar{\mathbb{F}}_\ell$ and $\bar{\mathbb{F}}_\ell(2)$, respectively.

Let Y' denote the set of 16 double points on X', and let Y be the inverse image of Y' in X, consisting of 16 copies of $\mathbb{P}^1$. Let $U := X \setminus Y \cong X' \setminus Y'$. The excision sequence for $U \subset X'$ gives $H^i_c(\bar{U}, \bar{\mathbb{F}}_\ell) \xrightarrow{\sim} H^i(\bar{X}', \bar{\mathbb{F}}_\ell)$ for $i \geq 2$, and if ℓ is sufficiently large, then the excision sequence for $U \subset X$ gives a short exact sequence of $G_{\mathbb{Q}}$-modules (and therefore of $G_{\mathbb{Q}(\zeta_\ell)}$-modules)

$$0 \to H^2(\bar{X}', \bar{\mathbb{F}}_\ell) \to H^2(\bar{X}, \bar{\mathbb{F}}_\ell) \to \bar{\mathbb{F}}_\ell(1)^{16} \to 0$$

and therefore

$$0 \to H^1(\bar{E}_1, \bar{\mathbb{F}}_\ell) \boxtimes H^1(\bar{E}_2, \bar{\mathbb{F}}_\ell) \to H^2(\bar{X}, \bar{\mathbb{F}}_\ell) \to \bar{\mathbb{F}}_\ell(1)^{18} \to 0.$$

Regarding $H^2(\bar{X}, \bar{\mathbb{F}}_\ell)$ as a representation of $G_{\mathbb{Q}(\zeta_\ell)}$, it factors through the Galois group $\mathrm{Gal}(K_1K_2/\mathbb{Q}(\zeta_\ell))$, which is isomorphic to $\mathrm{SL}_2(\mathbb{F}_\ell)^2$. As an $\mathrm{SL}_2(\mathbb{F}_\ell)^2$-representation, it is an extension of an 18-dimensional trivial representation by $V_1 \boxtimes V_1$. If ℓ is sufficiently large, then this extension is trivial, since all indecomposable $\bar{\mathbb{F}}_\ell$-representations of $\mathrm{SL}_2(\mathbb{F}_\ell)$ which are not irreducible have dimension at least $\ell - 2$ (see [1, Corollary 4.3]). As $H^0(\bar{X}, \bar{\mathbb{F}}_\ell)$ and $H^4(\bar{X}, \bar{\mathbb{F}}_\ell)$ are trivial 1-dimensional representations of $G_{\mathbb{Q}(\zeta_\ell)}$ and $H^1(\bar{X}, \bar{\mathbb{F}}_\ell) = H^3(\bar{X}, \bar{\mathbb{F}}_\ell) = 0$, the theorem follows. □

6. The main theorem

In this section, we prove the main result of this paper.

THEOREM 6.1
For any K3 surface $X/\mathbb{Q}$ of the type in Theorem 5.2,

$$\zeta_X(t) \in K_0[\mathrm{Var}_{\mathbb{Q}}][1/\mathbb{L}][[t]]$$

is irrational in the sense that if $B_{\mathrm{mot}}(t)$ is a polynomial with coefficients in $K_0[\mathrm{Var}_{\mathbb{Q}}][1/\mathbb{L}][t]$ and $B_{\mathrm{mot}}(0) = 1$, then $B_{\mathrm{mot}}(t)\zeta_X(t)$ is not a polynomial.

Proof
Choose X to be the variety defined in Proposition 5.2. We assume that $\zeta_X(t)$ is a rational function and choose $B_{\mathrm{mot}}(t)$ with $B_{\mathrm{mot}}(0) = 1$ such that $A_{\mathrm{mot}}(t) := B_{\mathrm{mot}}(t)\zeta_X(t)$ is a polynomial. Let

$$M := \max(\deg A_{\mathrm{mot}} + 24, \deg B_{\mathrm{mot}}).$$

We fix a prime $\ell > M$ sufficiently large such that:

(1) The homomorphism (5.1) is an isomorphism; that is, $\mathrm{Gal}(K_1K_2/\mathbb{Q}(\zeta_\ell))$ is isomorphic to $G := \mathrm{SL}_2(\mathbb{F}_\ell)^2$.

(2) For all $i \le M$, we have $\nu_\ell([\mathrm{Sym}^i X]) = [\mathrm{Sym}^i H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)]$.

For large enough ℓ, (1) holds by Theorem 5.2, and

$$\begin{aligned}\nu_\ell\big([\mathrm{Sym}^i X]\big) &= \nu_\ell\big([X^i/\Sigma_i]\big) = \big[H^\bullet(\bar{X}^i, \bar{\mathbb{F}}_\ell)^{\Sigma_i}\big] = \big[H^\bullet(\bar{X}^i, \bar{\mathbb{F}}_\ell)_{\Sigma_i}\big] \\ &= \big[\big(H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)^{\otimes i}\big)_{\Sigma_i}\big] = \big[\mathrm{Sym}^i H^\bullet(\bar{X}, \bar{\mathbb{F}}_\ell)\big]\end{aligned}$$

by the definition of Sym^i of a variety, Corollary 4.6, the semisimplicity of $\bar{\mathbb{F}}_\ell[\Sigma_i]$, the Künneth formula, and the definition of Sym^i of a vector space.

We define $R := K_\ell^{\mathrm{sp}}(G_{\mathbb{Q}(\zeta_\ell)})$. By condition (1) on ℓ, we can identify $K_\ell^{\mathrm{sp}}(G)$ with a subring of R via $\mathrm{Res}^G_{\mathbb{Q}(\zeta_\ell)}$.

For all nonnegative integers i, we define $a_i := \nu_\ell([\mathrm{Sym}^i X])$, which belongs to this subring and satisfies

$$a_i = \mathrm{Res}^G_{G_{\mathbb{Q}(\zeta_\ell)}}\big[\mathrm{Sym}^i(\bar{\mathbb{F}}_\ell^{20} \oplus V_1 \boxtimes V_1)\big]$$

for $0 \le i \le M$ by condition (2) on ℓ and Theorem 5.2. For all $i \ge 0$, we define $b_i = \nu_\ell(X^{[i]})$, where $X^{[i]}$ denotes the Hilbert scheme of points of length i on X. In particular, $X^{[0]}$ is defined to be $\mathrm{Spec}\,\mathbb{Q}$. Note that b_i is effective for all i by Theorem 3.3, since it is the class of a nonsingular projective variety.

Let $A := \nu_\ell(A_{\mathrm{mot}})$, and let $B := \nu_\ell(B_{\mathrm{mot}})$. Thus, $A(0) = B(0) = 1$, $M \ge \max(\deg(A) + 24, \deg(B))$, and

$$B(t)\sum_{i=0}^{\infty} a_i t^i = A(t).$$

Applying ν_ℓ to both sides of the identity (A.1) proved in the Appendix, the elements $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ satisfy the identities (2.17). By Proposition 2.7, $\sum_{i\ge 0} a_i t^i = (1-t)^{-20}\Lambda_{V_1\boxtimes V_1}(t)^{-1}$. Proposition 2.8, then implies that $b_\ell = \nu_\ell(X^{[\ell]})$ lies in $K_\ell^{\mathrm{sp}}(G)$ and is not effective, which is a contradiction. □

Appendix. Hilbert schemes of surfaces

This section is devoted to a proof of an identity relating the classes of the Hilbert schemes of a nonsingular surface to those of symmetric powers of the surface.

THEOREM A.1

If X is a nonsingular surface over a field k, then we have an identity of power series in $K_0(\mathrm{Var}_k)$ as follows:

$$\sum_{n=0}^{\infty}[X^{[n]}]t^n = \prod_{i=1}^{\infty}\zeta_X(\mathbb{L}^{i-1}t^i). \tag{A.1}$$

This theorem is due to Göttsche [12, Theorem 1.1] in the case that k is algebraically closed and of characteristic 0. Almost all of the proof goes through for arbitrary fields. We briefly recall his argument, ignoring the combinatorial details, which do not depend on field.

Göttsche considers the Hilbert–Chow morphism $X^{[n]} \to \mathrm{Sym}^n X$ and pulls back the stratification of $\mathrm{Sym}^n X$ by partitions of n. For each such partition $\pi = (1^{a_1}\cdots n^{a_n})$, he realizes the π-stratum of $X^{[n]}$ as the open part of a stratification of $\prod_{i=1}^n \mathrm{Sym}^{a_i} X_{(i)}^{[i]}$, where $X_{(i)}^{[i]}$ is the closed stratum of $X^{[i]}$, that is, consists of length i subschemes of X supported on a single point. Over any field, the natural morphism $X_{(i)}^{[i]} \to X$ mapping a local subscheme to its point of support is Zariski-locally trivial with fiber R_i, where R_i denotes the ith punctual Hilbert scheme, that is, the (reduced) Hilbert scheme of codimension i ideals of $k[[x,y]]$ (see [11, Lemma 2.1.4]). Thus, $[X_{(i)}^{[i]}] = [R_i][X]$.

For $k=\mathbb{C}$, Ellingsrud and Strømme [9, Theorem 1.1(iv)] give a decomposition of R_i into strata which are affine spaces. The proof uses the Białynicki-Birula theorem, which assumes that k is algebraically closed, so this needs to be checked for general k. Göttsche, following an idea of Totaro, shows that $[\mathrm{Sym}^a(\mathbb{A}^b\times X)] = \mathbb{L}^{ab}[\mathrm{Sym}^a X]$; this depends only on étale descent of vector bundles and the fact that $\mathrm{Sym}^i\mathbb{A}^1 \cong \mathbb{A}^i$, both of which hold over arbitrary fields.

So what remains to be verified is the following.

PROPOSITION A.2

Let k be any field, and let n be any positive integer. Then R_n has a stratification into locally closed strata indexed by the set $P(n)$ of partitions β of n such that the stratum associated to β is isomorphic to $\mathbb{A}^{n-|\beta|}$, where $|\beta|$ denotes the number of parts of β.

We prove the proposition by giving an explicit "cell decomposition" of R_n and explicit parameterizations of the cells. Toward this end, we introduce the following notation. Let β and λ be mutually dual partitions (i.e., partitions whose Ferrers diagrams are transpose to one another) with

$$r = \beta_1 \geq \beta_2 \geq \cdots \geq \beta_s > \beta_{s+1} = 0,$$
$$s = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = 0.$$

Thus,

$$\beta_{\lambda_i+1} < i \leq \beta_{\lambda_i}$$

for $1 \leq i \leq r$, and

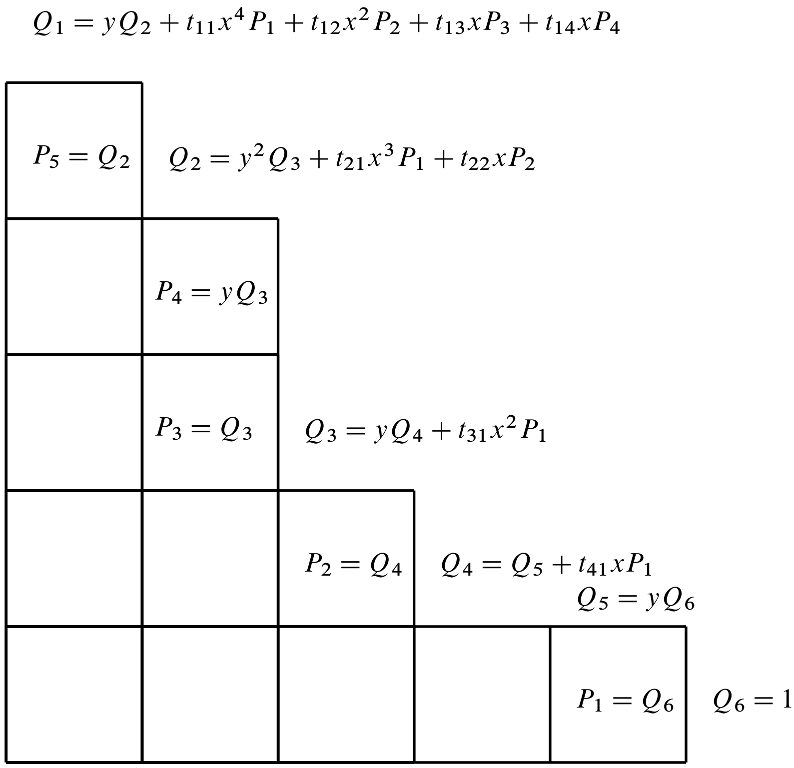
$$\lambda_{\beta_j+1} < j \leq \lambda_{\beta_j}$$

for $1 \leq j \leq s$. For β (and therefore λ) fixed, we define the polynomial ring $\mathcal{A}_\beta := \mathbb{Z}[t_{ij}]$, where $1 \leq i < r$ and $1 \leq j \leq \lambda_{i+1}$, and we recursively define (working from bottom right to top left as in the example, with $\lambda = (5, 4, 2, 1, 1)$, $\beta = (5, 3, 2, 2, 1)$, and $r = s = 5$, depicted below) the finite sequences of polynomials $Q_1, Q_2, \ldots, Q_{r+1} = 1$ and $P_1 = 1, P_2, \ldots P_s$ in $\mathcal{A}_\beta[x, y]$ as follows: for $1 \leq i \leq r$,

$$Q_i := y^{\lambda_i - \lambda_{i+1}} Q_{i+1} + \sum_{j=1}^{\lambda_{i+1}} t_{ij} x^{\beta_j - i} P_j,$$

and for $1 \leq j \leq s$,

$$P_j := y^{j - \lambda_{\beta_j+1} - 1} Q_{\beta_j+1}.$$



As $\beta_j \geq i + 1$ when $j \leq \lambda_{i+1}$, by descending induction, for $1 \leq i \leq r + 1$,

$$Q_i \in y^{\lambda_i} + (x),$$

and by (standard) induction it follows that

$$P_j \in y^{j-1} + (x)$$

for $1 \leq j \leq s$. For $1 \leq i \leq r + 1$, we define

$$\mathcal{I}_i = (Q_i, xQ_{i+1}, x^2Q_{i+2}, \dots, x^{r+1-i}Q_{r+1}).$$

LEMMA A.3

For any field F and ring homomorphism $\phi\colon \mathcal{A}_\beta \to F$, $I_1 := \mathcal{I}_1 \otimes_{\mathcal{A}_\beta,\phi} F$ is an (x, y)-primary ideal of $F[x, y]$ of codimension n. A linear complement for I_1 in $F[x, y]$ is given by

$$\mathrm{Span}\{x^{i-1}y^j \mid 1 \leq i \leq r, 0 \leq j < \lambda_i\}.$$

Moreover, every (x, y)-primary ideal of $F[x, y]$ of codimension n satisfying

$$\dim(I_1 : x^k)/(I_1 : x^{k-1}) = \lambda_k, k = 1, \ldots, r$$

arises from one and only one ϕ.

Proof
Setting $I_k := \mathcal{I}_k \otimes_{\mathcal{A},\phi} F$, we have

$$I_k = (\bar{Q}_k, x\bar{Q}_{k+1}, \ldots, x^{r+1-k}\bar{Q}_{r+1}),$$

where $\bar{Q}_k := Q_k \otimes 1$ belongs to $y^{\lambda_k} + (x) \subset F[x, y]$. As

$$\bar{Q}_k = y^{\lambda_k - \lambda_{k+1}}\bar{Q}_{k+1} + \sum_{j=1}^{\lambda_{k+1}} a_{kj} x^{\beta_j - k} y^{j - \lambda_{\beta_j+1} - 1}\bar{Q}_{\beta_j+1},$$

where $a_{kj} := t_{kj} \otimes 1 = \phi(t_{kj})$, we have

$$x\bar{Q}_k \in (x\bar{Q}_{k+1}, \ldots, x^{r+1-k}\bar{Q}_{r+1}) = xI_{k+1},$$

so $R\bar{Q}_k \in xI_{k+1}$ if and only if $R \in (x)$. This means that an element of I_k belongs to (x) if and only if it belongs to $xI_{k+1} \subset I_k$; that is,

$$(I_k : x) = I_{k+1}$$

for $1 \le k \le r$. By induction, $(I_k : x^j) = I_{k+j}$ for $1 \le k < k + j \le r + 1$. As I_{r+1} is the unit ideal, $x^r \in I_1$, so the image of x in $F[x, y]/I_1$ is nilpotent. As y^{λ_1} is divisible by x (mod I_1), it follows that y is nilpotent in $F[x, y]/I_1$. Thus, I_1 is (x, y)-primary.

The composition of maps $I_k \hookrightarrow F[x, y] \twoheadrightarrow F[y]$ sends $x^i\bar{Q}_{k+i}$ to 0 for $i > 0$ and sends $\bar{Q}_k$ to y^{λ_k}. Thus, we have an isomorphism

$$F[x, y]/\big(I_k + (x)\big) \xrightarrow{\sim} F[y]/(y^{\lambda_k}). \tag{A.2}$$

We prove by descending induction that the span of

$$\{x^{i-k}y^j \mid k \le i \le r, 0 \le j < \lambda_i\} \tag{A.3}$$

is complementary to I_k in $F[x, y]$. This is trivial for $k = r + 1$. Multiplication by x gives an isomorphism

$$F[x, y]/I_{k+1} = F[x, y]/(I_k : x) \to (x)/\big(I_k \cap (x)\big).$$

By (A.2), the short exact sequence

$$0 \to (x)/\big(I_k \cap (x)\big) \to F[x, y]/I_k \to F[x, y]/\big(I_k + (x)\big) \to 0$$

can be rewritten as

$$0 \to F[x,y]/I_{k+1} \to F[x,y]/I_k \to F[y]/(y^{\lambda_k}) \to 0.$$

By induction,

$$\dim F[x,y]/I_k = \lambda_k + \lambda_{k+1} + \cdots + \lambda_r.$$

To prove that the image of (A.3) spans $F[x,y]/I_k$, we assume the corresponding statement for $k+1$. Then

$$\{x^{i-k} y^j \mid k+1 \le i \le r, 0 \le j < \lambda_i\}$$

spans $(x)/(I_{k+1} \cap (x))$ and the image of

$$\{y^j \mid 0 \le j < \lambda_k\}$$

spans $F[y]/(y^{\lambda_k})$, so the image of (A.3) spans $F[x,y]/I_k$.

Next we claim that I_1 determines ϕ. Equivalently, I_1 determines $a_{ij} = \phi(t_{ij})$. We prove by descending induction that I_k determines a_{ij} for all $i \ge k$. This is trivial for $k \ge r+1$. Assume it holds for $k+1$. As $I_{k+1} = (I_k : x)$ determines a_{ij} for $i \ge k+1$ (and therefore determines $\bar{Q}_{k+1}, \dots, \bar{Q}_{r+1}$), we need only consider the case $i = k$. It suffices to prove that

$$I_k \cap \operatorname{Span}\{x^{\beta_j - k} \bar{P}_j \mid 1 \le j \le \lambda_{k+1}\} = \{0\}.$$

Indeed, if

$$\sum_{j=1}^{\lambda_{k+1}} c_j x^{\beta_j - k} \bar{P}_j \in I_k$$

and $m := \min\{\beta_j \mid c_j \ne 0\}$, then this linear combination lies in $I_k \cap (x^{m-k}) = x^{m-k} I_m$, and we have

$$\sum_{j=1}^{\lambda_m} c_j x^{\beta_j - m} \bar{P}_j \in I_m.$$

Reducing (mod x), we have a nontrivial linear combination of y^{j-1} for $j \le \lambda_{\beta_j} \le \lambda_m$ belonging to (y^{λ_m}), which is impossible.

Finally, we claim that every (x,y)-primary codimension-n ideal of in $F[x,y]$ can be expressed as I_1 for some partition λ of n and some ϕ. Defining

$$\lambda_i = \dim(I : x^i)/(I : x^{i-1}),$$

we have $\lambda_1 \geq \lambda_2 \geq \cdots$ since multiplication by x defines an injection

$$(I : x^{i+1})/(I : x^i) \hookrightarrow (I : x^i)/(I : x^{i-1}), i \geq 1,$$

and $\sum_{i=1}^{\infty} \lambda_i = n$ since $(I : x^m) = F[x, y]$ for m sufficiently large. This determines λ, and now we must show that the parameters a_{ij} can be chosen so that $I_1 = I$. We use induction on the number of parts in the partition.

Given I with associated partition $\lambda_1 \geq \lambda_2 \geq \cdots$, let $J := (I : x)$, which is associated to $\lambda_2 \geq \lambda_3 \geq \cdots$. By the induction hypothesis, there exist $a_{ij} \in F$ for $2 \leq i < r$, $1 \leq j \leq \lambda_{i+1}$ such that

$$I_2 = (\bar{Q}_2, x\bar{Q}_3, \ldots, x^{r-1}\bar{Q}_{r+1})$$

coincides with J. The image of I by the (mod (x)) reduction map $F[x, y] \to F[y]$ is (y^{λ_1}), so $I = (\bar{Q}_1) + xI_2$ for some $\bar{Q}_1$ of the form $y^{\lambda_1-\lambda_2}\bar{Q}_2 + x\alpha$, where

$$x\alpha \in (x) \cap J = (x) \cap I_2 = xI_3;$$

that is, $\alpha \in I_3$. On the other hand, if $\alpha - \beta \in I_2$, then

$$(y^{\lambda_1-\lambda_2}\bar{Q}_2 + x\alpha) + xI_2 = (y^{\lambda_1-\lambda_2}\bar{Q}_2 + x\beta) + xI_2.$$

It suffices to prove that every class in I_3/I_2 is represented by some α of the form $\sum_{j=1}^{\lambda_2} a_{1j} x^{\beta_j-2} y^{j-\lambda_{\beta_j+1}-1} \bar{Q}_{\beta_j+1}$. Composing the map $F^{\lambda_2} \to I_3$ given by

$$(a_{11}, \ldots, a_{1\lambda_2}) \mapsto \sum_{j=1}^{\lambda_2} a_{1j} x^{\beta_j-2} y^{j-\lambda_{\beta_j+1}-1} \bar{Q}_{\beta_j+1} \in I_3$$

with the quotient map $I_3 \twoheadrightarrow I_3/I_2$, we get an injective map between vector spaces of dimension λ_2, which must therefore be surjective. □

Now we can prove Proposition A.2.

Proof
It suffices to prove the equivalent form

$$[R_n] = \sum_{\lambda \in P(n)} [\mathbb{A}^{n-\lambda_1}].$$

As $\mathcal{I}_1$ contains $(x, y)^n$, if M denotes the free $\mathcal{A}_\beta$-module of polynomials in $\mathcal{A}_\beta$ of degree $< n$, then we have an isomorphism of $\mathcal{A}_\beta$-modules $M/M \cap \mathcal{I}_1 \xrightarrow{\sim} \mathcal{A}_\beta[x, y]/\mathcal{I}_1$. The $\mathcal{A}_\beta$-linear map

$$\mathrm{Span}_{\mathcal{A}_\beta}\{x^i y^j \mid 0 \le i < r, 0 \le j < \lambda_{i+1}\} \to \mathcal{A}_\beta[x,y]/\mathcal{I}_1$$

becomes an isomorphism after tensoring by any residue field of $\mathcal{A}_\beta$, so by Nakayama's lemma, it must be an isomorphism. Thus $\mathcal{A}_\beta[x,y]/\mathcal{I}_1$ is a free $\mathcal{A}_\beta$-module, and this remains true after tensoring over $\mathbb{Z}$ with k. If $S = \mathrm{Spec}\, \mathcal{A}_\beta \otimes_{\mathbb{Z}} k$ and $Z = \mathrm{Spec}\, \mathcal{A}_\beta[x,y]/\mathcal{I}_1 \otimes_{\mathbb{Z}} k$, then $Z \to S$ is flat and therefore defines an S-point of the Hilbert scheme $(\mathbb{A}^2)^{[n]}$, and since every geometric point of S corresponds to a (x,y)-primary ideal, it follows that S maps to R_n. At the level of F-points, this map gives a bijection between (x,y)-primary ideals associated to λ and F-points of S. The proposition now follows from the following lemma. □

LEMMA A.4
Let k be a field, and let $\phi\colon Y \to X$ be a morphism of k-varieties. If for all extension fields F of k, ϕ defines a bijection from $Y(F)$ to $X(F)$, then $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$.

Proof
Suppose that K is a field and that Z is a K-variety such that, for every extension field L of K, there is a unique morphism $\mathrm{Spec}\, L \to Z$ lifting $\mathrm{Spec}\, L \to \mathrm{Spec}\, K$. If y_1, y_2 are points on Z with residue fields K_1 and K_2 over K, then we can choose a field Ω in which K_1 and K_2 both embed as subfields, so Z has at least two distinct Ω-points, contrary to assumption. Thus Z has a single point, so it is affine: $Z = \mathrm{Spec}\, A$ for some reduced K-algebra A. The nilradical corresponds to the unique maximal ideal, and it is zero since Z is a variety, so A is a field extension L/K. On the other hand, the identity map $\mathrm{Spec}\, K \to \mathrm{Spec}\, K$ lifts to $\mathrm{Spec}\, K \to \mathrm{Spec}\, L$, so the extension $K \to L$ has an inverse, which means it is trivial.

We apply this in the case that K is the residue field of the generic point η of a component of X and $Z := Y_\eta$ is the fiber of Y over η. The conclusion is that there exists a point η' in Y over η for which ϕ gives an isomorphism of residue fields. Thus, there exist open neighborhoods U of η in X and U' of η' in Y such that $\phi^{-1}(U) = U'$ and ϕ induces an isomorphism $U' \to U$. Replacing Y and X by $Y \setminus U'$ and $X \setminus U$, respectively, the restriction of ϕ induces a map on F-points for all extensions F of k, and the lemma follows by Noetherian induction. □

Acknowledgments. We would like to thank both referees for their careful reading of this manuscript and many helpful suggestions. Among other interesting things, we learned from them the result of Mikhail Bondarko [3] that the conservativity conjecture implies the rationality of the motivic zeta-function of any variety in characteristic 0 with values in the K-group of numerical motives. We would also like to gratefully acknowledge helpful conversations with Pierre Deligne, Vladimir Drinfeld, Lothar Göttsche, Luc Illusie, Mircea Mustaţă, and Geordie Williamson. In addition, we would like to thank Sailun Zhan for correcting several inaccuracies in the Appendix.

Larsen's work was partially supported by National Science Foundation grant DMS-1702152. Lunts's work was partially supported by the Laboratory of Mirror Symmetry, National Research University, Higher School of Economics, Russian Federation Government grant ag. 14.641.31.0001.

References

[1] H. H. ANDERSEN, J. JØRGENSEN, and P. LANDROCK, *The projective indecomposable modules of* $\mathrm{SL}(2, p^n)$, Proc. Lond. Math. Soc. (3) **46** (1983), no. 1, 38–52. MR 0684821. DOI 10.1112/plms/s3-46.1.38. *(6, 7, 20)*

[2] F. BITTNER, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. **140** (2004), no. 4, 1011–1032. MR 2059227. DOI 10.1112/S0010437X03000617. *(3, 13, 14, 17)*

[3] M. V. BONDARKO, *Conservativity of realizations implies that numerical motives are Kimura-finite and motivic zeta functions are rational*, preprint, arXiv:1807.10791v2 [math.AG]. *(28)*

[4] L. A. BORISOV, *The class of the affine line is a zero divisor in the Grothendieck ring*, J. Algebraic Geom. **27** (2018), no. 2, 203–209. MR 3764275. DOI 10.1090/jag/701. *(2)*

[5] S. DEL BAÑO ROLLIN and V. NAVARRO AZNAR, *On the motive of a quotient variety*, Collect. Math. **49** (1998), no. 2–3, 203–226. MR 1677089. *(17)*

[6] P. DELIGNE, *Cohomologie étale*, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4.5, Lecture Notes in Math. **569**, Springer, Berlin, 1977. MR 0463174. DOI 10.1007/BFb0091526. *(16)*

[7] J. DENEF and F. LOESER, "On some rational generating series occurring in arithmetic geometry in *Geometric Aspects of Dwork Theory, Vol. I, II*, de Gruyter, Berlin, 2004, 509–526. MR 2099079. *(2)*

[8] B. DWORK, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631–648. MR 0140494. DOI 10.2307/2372974. *(1)*

[9] G. ELLINGSRUD and S. A. STRØMME, *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. **87** (1987), no. 2, 343–352. MR 0870732. DOI 10.1007/BF01389419. *(22)*

[10] W. FULTON, *Intersection Theory*, Ergeb. Math. Grenzgeb. (3) **2**, Springer, Berlin, 1984. MR 0732620. DOI 10.1007/978-3-662-02421-8. *(14, 15)*

[11] L. GÖTTSCHE, *Hilbert Schemes of Zero-Dimensional Subschemes of Smooth Varieties*, Lecture Notes in Math. **1572**, Springer, Berlin, 1994. MR 1312161. DOI 10.1007/BFb0073491. *(22)*

[12] ———, *On the motive of the Hilbert scheme of points on a surface*, Math. Res. Lett. **8** (2001), no. 5–6, 613–627. MR 1879805. DOI 10.4310/MRL.2001.v8.n5.a3. *(3, 22)*

[13] L. ILLUSIE, ed., *Cohomologie l-adique et fonctions L*, Séminaire de Géometrie Algébrique du Bois-Marie 1965–66 (SGA 5), Lecture Notes in Math. **589**, Springer, Berlin, 1977. MR 0491704. *(13)*

[14] M. KAPRANOV, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, preprint, arXiv:math/0001005v2 [math.AG]. *(1)*

[15] H. KRAUSE, *Krull-Schmidt categories and projective covers*, Expo. Math. **33** (2015), no. 4, 535–549. MR 3431480. DOI 10.1016/j.exmath.2015.10.001. *(4, 5)*

[16] M. LARSEN and V. A. LUNTS, *Motivic measures and stable birational geometry*, Mosc. Math. J. **3** (2003), no. 1, 85–95. MR 1996804. DOI 10.17323/1609-4514-2003-3-1-85-95. *(2)*

[17] ———, *Rationality criteria for motivic zeta functions*, Compos. Math. **140** (2004), no. 6, 1537–1560. MR 2098401. DOI 10.1112/S0010437X04000764. *(2)*

[18] G. LAUMON, “Homologie étale” in *Séminaire de géométrie analytique (Paris, 1974–75)*, Astérisque **36–37**, Soc. Math. France, Paris, 1976, 163–188. MR 0444667. *(16)*

[19] J. LAWRENCE, *When is the tensor product of algebras local?, II*, Proc. Amer. Math. Soc. **58** (1976), 22–24. MR 0409551. DOI 10.2307/2041353. *(5)*

[20] D. LITT, *Zeta functions of curves with no rational points*, Michigan Math. J. **64** (2015), no. 2, 383–395. MR 3359031. DOI 10.1307/mmj/1434731929. *(1)*

[21] J. I. MANIN, *Correspondences, motifs and monoidal transformations* (in Russian), Mat. Sb. (N.S.) **77 (119)** (1968), 475–507; English translation in Sb. Math. **6** (1968), 439–470. MR 0258836. *(14, 15)*

[22] J. S. MILNE, *Étale Cohomology*, Princeton Math. Ser. **33**, Princeton Univ. Press, Princeton, 1980. MR 0559531. *(14, 16)*

[23] B. POONEN, *The Grothendieck ring of varieties is not a domain*, Math. Res. Lett. **9** (2002), no. 4, 493–497. MR 1928868. DOI 10.4310/MRL.2002.v9.n4.a8. *(2)*

[24] J.-P. SERRE, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), no. 4, 259–331. MR 0387283. DOI 10.1007/BF01405086. *(18)*

[25] J. H. SILVERMAN, *Advanced Topics in the Arithmetic of Elliptic Curves*, Grad. Texts in Math. **151**, Springer, New York, 1994. MR 1312368. DOI 10.1007/978-1-4612-0851-8. *(18)*

[26] I. ZAKHAREVICH, *The annihilator of the Lefschetz motive*, Duke Math. J. **166** (2017), no. 11, 1989–2022. MR 3694563. DOI 10.1215/00127094-0000016X. *(2)*

Larsen

Department of Mathematics, Indiana University, Bloomington, Indiana, USA; mjlarsen@indiana.edu

Lunts

Department of Mathematics, Indiana University, Bloomington, Indiana, USA; vlunts@gmail.com