MAXIMALITY OF GALOIS ACTIONS FOR ABELIAN AND HYPER-KÄHLER VARIETIES

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Abstract

Let $\{\rho_\ell\}_\ell$ be the system of ℓ-adic representations arising from the ith ℓ-adic cohomology of a proper smooth variety X defined over a number field K. Let Γ_ℓ and $\mathbf{G}_\ell$ be, respectively, the image and the algebraic monodromy group of ρ_ℓ. We prove that the reductive quotient of $\mathbf{G}_\ell^\circ$ is unramified over every degree 12 totally ramified extension of $\mathbb{Q}_\ell$ for all sufficiently large ℓ. We give a necessary and sufficient condition $(*)$ on $\{\rho_\ell\}_\ell$ such that, for all sufficiently large ℓ, the subgroup Γ_ℓ is in some sense maximal compact in $\mathbf{G}_\ell(\mathbb{Q}_\ell)$. This is used to deduce Galois maximality results for ℓ-adic representations arising from abelian varieties (for all i) and hyper-Kähler varieties ($i = 2$) defined over finitely generated fields over $\mathbb{Q}$.

1. Introduction

1.1. Galois maximality conjecture

The starting point of this article is the well-known theorem of Serre [33, Théorème 2] asserting that, for every elliptic curve E defined over a number field K with $\mathrm{End}_{\overline{K}}(E) = \mathbb{Z}$, the ℓ-adic Galois representation $\mathrm{Gal}_K \to \mathrm{GL}_2(\mathbb{Z}_\ell)$, given by the Galois action on the ℓ-adic Tate module $T_\ell(E)$, is surjective for all ℓ sufficiently large. In his 1984–1985 Collège de France course [39], Serre extended this result to abelian varieties X of dimension g with $\mathrm{End}_{\overline{K}}(X) = \mathbb{Z}$, when g is odd (or $g \in \{2, 6\}$). For $\ell \gg 0$, the image of Gal_K in $\mathrm{Aut}(T_\ell(X))$ is then $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$. The hypothesis on g ensures that the only semisimple group admitting an irreducible, minuscule, symplectic representation of degree $2g$ is Sp_{2g}, and the only representation of this form is the standard one. (The relevance of this to abelian varieties is due, to the best of our knowledge, to Ribet in [32].) When $g = 4m$, for example, $\mathrm{Sp}_{2m} \times \mathrm{SL}_2 \times \mathrm{SL}_2$ has an irreducible, minuscule, symplectic representation of degree $2g$, namely, the

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external tensor product of the natural representations of the three factors, and a new idea is needed.

Wintenberger in [44, Théorème 2] succeeded in proving a general result for abelian varieties, which can be formulated as follows. Let X be an abelian variety of dimension g over a number field K. Let Γ_ℓ be the image of Gal_K in $\mathrm{Aut}(T_\ell(X))$, and let $\mathbf{G}_\ell$ be the Zariski closure of Γ_ℓ in GL_{2g} over $\mathbb{Q}_\ell$. Let $\mathbf{G}_\ell^{\mathrm{sc}}$ denote the universal covering group of the derived group of the identity component $\mathbf{G}_\ell^\circ$. Then for all sufficiently large ℓ, the group Γ_ℓ contains the image of a hyperspecial maximal compact subgroup of $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$. A key ingredient in Wintenberger's argument was Falting's proof of Tate's conjecture for abelian varieties (see [14, Satz 4]), which guarantees the existence of certain algebraic cycles predicted by Tate's general conjecture.

Our goal in the following is to show that it is feasible to prove theorems of this type with less powerful information from arithmetic geometry: instead of assuming that certain algebraic cycles exist, it is enough to assume equality between certain dimensions of ℓ-adic and (mod ℓ) Tate cycles or to deduce new cases from known cases by establishing suitable Tate cycles. This not only gives a new proof of Wintenberger's theorem for abelian varieties (where we know the needed algebraic cycles exist, as graphs of endomorphisms), but it gives new Galois maximality results (e.g., for hyper-Kähler varieties), where we do not know it. Beyond these special cases, it offers a general approach to proving the maximality of Galois images without first proving a version of the Tate conjecture or the Mumford–Tate conjecture. The price for doing this is harder work on the group theory side.

Let X be a proper smooth variety X defined over a finitely generated subfield $K \subset \mathbb{C}$. Let $\overline{K}$ denote the algebraic closure of K in $\mathbb{C}$, and let $X_{\overline{K}} := X \times_K \overline{K}$. For a fixed nonnegative integer i and a varying rational prime ℓ, each ℓ-adic étale cohomology group $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is a $\mathbb{Q}_\ell$-vector space with a continuous $\mathrm{Gal}_K := \mathrm{Gal}(\overline{K}/K)$-action. Let n be the common dimension of $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ for all ℓ. We obtain a system of ℓ-adic representations

$$\{\rho_\ell : \mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{Q}_\ell)\}_\ell, \tag{1}$$

which in the case that K is a number field is (by the main theorem of Deligne from [12]) a *strictly compatible system* in the sense of Serre from [36, Chapter 1]. The image $\Gamma_\ell := \rho_\ell(\mathrm{Gal}_K)$ is called the *monodromy group* of ρ_ℓ; it is a compact ℓ-adic Lie subgroup of $\mathrm{GL}_n(\mathbb{Q}_\ell)$.

The *algebraic monodromy group* of ρ_ℓ, denoted by $\mathbf{G}_\ell$, is defined to be the Zariski closure of Γ_ℓ in $\mathrm{GL}_{n,\mathbb{Q}_\ell}$. There exists a finite extension L/K such that $\rho_\ell(\mathrm{Gal}_L) \subset \mathbf{G}_\ell^\circ(\mathbb{Q}_\ell)$ for all ℓ (see [37, pp. 6, 17], [39, Section 2.2.3]). When X/K is projective, the conjectural theory of motives, together with the celebrated conjectures of Hodge, of Tate, and of Mumford–Tate, predicts the existence of a common connected reductive

$\mathbb{Q}$-form $\mathbf{G}_{\mathbb{Q}}$ of $\mathbf{G}_\ell^\circ$ for all ℓ (see [35, Section 3]). Then Serre's conjectures on maximal motives in [35, Conjectures 11.4, 11.8] imply that, if $\mathscr{G}$ denotes any extension of $\mathbf{G}_{\mathbb{Q}}$ to a group scheme over $\mathbb{Z}[1/N]$ for some nonzero integer N, then the compact subgroup $\rho_\ell(\mathrm{Gal}_L)$ is in a suitable sense maximal in $\mathscr{G}(\mathbb{Z}_\ell)$ if ℓ is sufficiently large.

Denote $\Gamma_\ell \cap \mathbf{G}_\ell^\circ(\mathbb{Q}_\ell)$ by Γ_ℓ°, denote the derived group of $\mathbf{G}_\ell^\circ$ by $\mathbf{G}_\ell^{\mathrm{der}}$, denote the intersection $\Gamma_\ell \cap \mathbf{G}_\ell^{\mathrm{der}}(\mathbb{Q}_\ell)$ by $\Gamma_\ell^{\mathrm{der}}$, denote the quotient of $\mathbf{G}_\ell^\circ$ by its radical by $\mathbf{G}_\ell^{\mathrm{ss}}$, and denote the image of Γ_ℓ° under the quotient map $\mathbf{G}_\ell^\circ(\mathbb{Q}_\ell) \to \mathbf{G}_\ell^{\mathrm{ss}}(\mathbb{Q}_\ell)$ by $\Gamma_\ell^{\mathrm{ss}}$. Since $\mathbf{G}_\ell^{\mathrm{ss}}$ is connected semisimple, it has a universal covering group, which we denote $\mathbf{G}_\ell^{\mathrm{sc}}$; we write $\Gamma_\ell^{\mathrm{sc}}$ for the preimage of $\Gamma_\ell^{\mathrm{ss}}$ under the map $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell) \to \mathbf{G}_\ell^{\mathrm{ss}}(\mathbb{Q}_\ell)$. The following statement, due to the second author, is a weak version of Serre's maximality conjecture with the feature that it can be formulated without assuming the Mumford–Tate conjecture. The connections between these conjectures are explored further in [23].

CONJECTURE 1.1 (see [26])
Let $\{\rho_\ell\}_\ell$ be the system of ℓ-adic representations arising from the ith ℓ-adic cohomology of a proper smooth variety X/K. Then the ℓ-adic Lie group $\Gamma_\ell^{\mathrm{sc}}$ is a hyperspecial maximal compact subgroup of $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ for all sufficiently large ℓ.

It is proved in [26, Theorem 3.17] that the assertion on $\Gamma_\ell^{\mathrm{sc}}$ holds for a density 1 subset of primes ℓ. These primes lie in an infinite union of sets defined by Chebotarev-type conditions, and there seems no hope of showing by this method that this thin set of possible exceptional primes is in fact finite. In [22, Theorem 1], we proved Conjecture 1.1 for "type A" Galois representations. What is special about type A is that semisimple groups of this type contain no proper semisimple subgroups of equal rank. For instance, a new idea would be needed to rule out possibilities like

$$\Gamma_\ell^{\mathrm{sc}} = \{\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z}_\ell) \mid \bar{\gamma} \in \mathrm{Sp}_{2m}(\mathbb{F}_\ell) \times \mathrm{Sp}_{2n-2m}(\mathbb{F}_\ell)\},$$

where $\bar{\gamma}$ denotes (mod ℓ) reduction. To rule out this kind of behavior, we introduce a new hypothesis $(*)$ below.

1.2. Main results of the paper
It is convenient to replace K with a finite extension so that we may assume that $\mathbf{G}_\ell$ is connected for all ℓ. Denote by ρ_ℓ^{ss} the semisimplification of ρ_ℓ, and denote by $\mathbf{G}_\ell^{\mathrm{red}}$ the quotient of $\mathbf{G}_\ell^\circ$ by its unipotent radical. The group $\mathbf{G}_\ell^{\mathrm{red}}$ is also the image of $\mathbf{G}_\ell^\circ$ under the semisimplification of $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$. The image of $H^i(X_{\overline{K}}, \mathbb{Z}_\ell)$ is a lattice in $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$. Let

$$\bar{\rho}_\ell^{\mathrm{ss}} : \mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{F}_\ell)$$

be the *semisimple reduction* of ρ_ℓ, that is, the semisimplification of the representation obtained by reducing ρ_ℓ modulo ℓ, and denote the image of $\bar{\rho}_\ell^{\mathrm{ss}}$ by G_ℓ. By the Brauer–Nesbitt theorem, this does not depend on the choice of lattice. For every sufficiently large ℓ, there exists a connected reductive subgroup $\underline{G_\ell}$ (called the *algebraic envelope*[1] of G_ℓ) of $\mathrm{GL}_{n,\mathbb{F}_\ell}$ such that G_ℓ is a subgroup of $\underline{G_\ell}(\mathbb{F}_\ell)$ of index bounded above by a constant independent of ℓ (see Section 4.1).

The central result of the present article is Theorem 1.2, which gives an arithmetic condition equivalent to Conjecture 1.1.

THEOREM 1.2
Let $\{\rho_\ell\}_\ell$ be the system of ℓ-adic representations arising from the ith ℓ-adic cohomology of a proper smooth variety X defined over a number field K such that $\mathbf{G}_\ell$ is connected for all ℓ. Then for sufficiently large ℓ, the subgroup $\Gamma_\ell^{\mathrm{sc}}$ is a hyperspecial maximal compact subgroup in $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ (and $\mathbf{G}_\ell^{\mathrm{red}}$ is unramified) for all sufficiently large ℓ if and only if the commutants of $\Gamma = \rho_\ell^{\mathrm{ss}}(\mathrm{Gal}_K)$ and $G = \bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}_K)$ on the ambient spaces have the same dimension:

$$\dim_{\mathbb{Q}_\ell}\big(\mathrm{End}_\Gamma(\mathbb{Q}_\ell^n)\big) = \dim_{\mathbb{F}_\ell}\big(\mathrm{End}_G(\mathbb{F}_\ell^n)\big). \qquad (*)$$

An even-dimensional, projective smooth, simply connected variety Y defined over K is said to be *hyper-Kähler* if the space of holomorphic 2-forms $H^0(Y(\mathbb{C}), \Omega^2_{Y(\mathbb{C})})$ is of dimension 1 and is generated by a form that is nondegenerate everywhere on $Y(\mathbb{C})$. Examples include Hilbert schemes of points on K3 surfaces (including K3 surfaces themselves) and generalized Kummer varieties. By using Theorem 1.2, we prove the following.

THEOREM 1.3
Let $\{\rho_\ell\}_\ell$ be the system of ℓ-adic representations arising from the ith ℓ-adic cohomology of a proper smooth variety X defined over a subfield K of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$. For all sufficiently large ℓ, the group $\Gamma_\ell^{\mathrm{sc}}$ is a hyperspecial maximal compact subgroup of $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ and $\mathbf{G}_\ell^\circ$ is reductive and unramified under either of the following hypotheses:
(a) *X is an abelian variety,*
(b) *X is a hyper-Kähler variety and $i = 2$.*

Remark 1.4
For the special case that Serre originally considered in [33], additional work is needed to deduce his result from Theorem 1.3(a). Namely, one must translate between the

[1]The algebraic envelope is denoted by $\bar{\mathbf{G}}_\ell$ in [20]. Here we follow the notation of [38].

Galois action on the Tate module and the dual action on H^1, note that $\bar{\rho}_\ell = \bar{\rho}_\ell^{\mathrm{ss}}$ for all ℓ sufficiently large, and check that $\det \rho_\ell$ is surjective. Serre also presented his ℓ-adic result as a consequence of an "adelic openness" result: the image of Gal_K in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ is open. Theorem 1.3 also has an "adelic" version, which requires some care to state since we do not know that the groups $\mathbf{G}_\ell$ come from a common algebraic group over $\mathbb{Q}$. (Details are given in [23].)

We have already mentioned that Wintenberger proved Theorem 1.3(a) in the key case where K is a number field and $i = 1$. In the case of K3 surfaces, Theorem 1.3(b) is due to Cadoret and Moonen from [8, Theorem B], conditional on the Mumford–Tate conjecture.

1.3. Ingredients and structure

The key ingredient in proving Theorem 1.2 is the following purely group-theoretic result.

THEOREM 1.5

Let $\mathbf{G} \subset \mathrm{GL}_{n,\mathbb{Q}_\ell}$ be a connected reductive subgroup, let $\underline{G} \subset \mathrm{GL}_{n,\mathbb{F}_\ell}$ be a connected reductive subgroup with $\underline{G}^{\mathrm{der}}$ as derived group and $\underline{Z}$ as connected center, let Γ be a closed subgroup of $\mathbf{G}(\mathbb{Q}_\ell) \cap \mathrm{GL}_n(\mathbb{Z}_\ell)$, and let $\phi\colon \Gamma \to \mathrm{GL}_n(\mathbb{F}_\ell)$ be a semisimple continuous representation with $G := \phi(\Gamma) \subset \underline{G}(\mathbb{F}_\ell)$. Assume that this data satisfies the following conditions.

(a) *The subgroup Γ is Zariski-dense in $\mathbf{G}$.*

(b) *There is an equality of semisimple ranks:* $\mathrm{rank}\, \mathbf{G}^{\mathrm{der}} = \mathrm{rank}\, \underline{G}^{\mathrm{der}}$.

(c) *The derived group $\underline{G}^{\mathrm{der}}$ is exponentially generated (see Section 2.1).*

(d) *For all $\gamma \in \Gamma$, the (mod ℓ) reduction of the characteristic polynomial of γ is the characteristic polynomial of $\phi(\gamma)$.*

(e) *The index $[\underline{G}(\mathbb{F}_\ell) : G]$ is bounded by $k \in \mathbb{N}$.*

(f) *The formal character of $(\underline{Z}, \mathbb{F}_\ell^n)$ is bounded by $N \in \mathbb{N}$, where $\underline{Z}$ is the connected center of $\underline{G}$ (see Section 2.6).*

(g) *Condition $(*)$ holds for Γ and G, that is,*

$$\dim_{\mathbb{Q}_\ell}\big(\mathrm{End}_\Gamma(\mathbb{Q}_\ell^n)\big) = \dim_{\mathbb{F}_\ell}\big(\mathrm{End}_G(\mathbb{F}_\ell^n)\big).$$

If ℓ is sufficiently large in terms of the data in (a)–(g), then the reduction representation $\Gamma \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_\ell) \to \mathrm{GL}_n(\mathbb{F}_\ell)$ and ϕ are conjugate, Γ^{sc} is a hyperspecial maximal compact subgroup of $\mathbf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$, and $\mathbf{G}^{\mathrm{der}}$ is unramified. Hypotheses (a)–(f) of Theorem 1.5 suffice to imply that $\mathbf{G}^{\mathrm{der}}$ splits over some finite unramified extension of $\mathbb{Q}_\ell$ and is unramified over every degree 12 *totally ramified extension of $\mathbb{Q}_\ell$.*

Once Theorem 1.5 is established, Theorem 1.2 follows by checking that (after restricting ρ_ℓ to some open subgroup Gal_L of Gal_K and semisimplifying) there exist $n,k,N \in \mathbb{N}$ such that the conditions (a)–(f) in Theorem 1.5 are verified for the monodromy group $\Gamma := \rho_\ell(\mathrm{Gal}_L)$, $G := \rho_\ell^{\mathrm{ss}}(\mathrm{Gal}_L)$, $\mathbf{G} := \mathbf{G}_\ell^\circ$, and $\underline{G}$ is the algebraic envelope of $\rho_\ell^{\mathrm{ss}}(\mathrm{Gal}_L)$ if ℓ is sufficiently large. This verification uses the main results in [20]. Unfortunately, condition (g) is in a different category, and further progress on Conjecture 1.1 seems to require it.

Theorem 1.3(a) is a consequence of Theorem 1.2, a (mod ℓ) version of the Tate conjecture for abelian varieties for $\ell \gg 0$ proved by Faltings in [15, Theorem 4.2] for the condition $(*)$ in Theorem 1.2, and an ℓ-independence result of algebraic monodromy groups under specialization (see [18, Corollary 2.7]). Theorem 1.3(b) is mainly a consequence of Theorem 1.3(a), the Kuga–Satake construction, and André's results on motivated cycles from [1] and [2]. In fact, we will see that the condition that $(*)$ holds for all sufficiently large ℓ is stable under duals, tensor products, and passage to subrepresentations (see Lemma 4.7 and the proof of Theorem 1.2).

The last claim of Theorem 1.5 is obtained mainly by Bruhat–Tits theory, which determines the possibilities for a connected semisimple group $\mathbf{G}/\mathbb{Q}_\ell$ whose group of $\mathbb{Q}_\ell$-points contains a maximal compact subgroup whose total ℓ-rank (see Section 2.3) equals the rank of $\mathbf{G}$.

To indicate the idea behind the rest of Theorem 1.5, we consider a particularly favorable case. Suppose that $n = 2g$ is even and that $\Gamma \subset \mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$ is Zariski-dense in $\mathbf{G} = \mathrm{GSp}_{2g,\mathbb{Q}_\ell}$. We note that not all maximal compact subgroups of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ are of this form $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$; in general, we can achieve an embedding of Γ in a maximal compact subgroup of this kind only after passing to a (totally ramified) finite extension of $\mathbb{Q}_\ell$, but we assume this for the purpose of illustration. We further assume that the (mod ℓ) reduction $\Gamma \to \mathrm{GL}_{2g}(\mathbb{F}_\ell)$ is semisimple, so the reduction map can be identified, after conjugation, with ϕ. The goal is to show that the inclusion $\Gamma' := [\Gamma,\Gamma] \subset \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ is an equality if ℓ is large enough with respect to n. By reducing (mod ℓ), we obtain $G' \subset \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$.

By applying Nori's theory to G' (see Section 2.1), we obtain a connected algebraic group

$$\underline{S} \subset \mathrm{Sp}_{2g,\mathbb{F}_\ell}$$

such that G' is of bounded index, independent of ℓ, in $\underline{S}(\mathbb{F}_\ell)$. However, Theorem 1.5(e) implies that G' is also of bounded index in $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$, and this, given that both $\underline{S}$ and $\underline{G}^{\mathrm{der}}$ are generated by additive algebraic groups ($\underline{S}$ by construction, $\underline{G}^{\mathrm{der}}$ by Theorem 1.5(c)) and therefore connected and also, in some sense, of bounded complexity, implies that $\underline{S} = \underline{G}^{\mathrm{der}}$ if ℓ is sufficiently large.

By Theorem 1.5(b), the ranks of $\underline{G}^{\mathrm{der}}$ and $\mathbf{G}^{\mathrm{der}} = \mathrm{Sp}_{2g,\mathbb{Q}_\ell}$ coincide, so both must be $g-1$. Since Γ is Zariski-dense in $\mathrm{GSp}_{2g,\mathbb{Q}_\ell}$, which acts irreducibly on $\mathbb{Q}_\ell^{2g}$, the group G acts irreducibly on $\mathbb{F}_\ell^{2g}$ by $(*)$. It follows that $\underline{G}$ and therefore $\underline{G}^{\mathrm{der}} = \underline{S}$ also act irreducibly.

In the inclusion of connected semisimple groups $\underline{S} \subset \mathrm{Sp}_{2g,\mathbb{F}_\ell}$, both have the same rank by Theorem 1.5(b) and the same commutant in $\mathrm{End}_{2g,\mathbb{F}_\ell}$, namely, the scalars. In characteristic 0, the equality $\underline{S} = \mathrm{Sp}_{2g,\mathbb{F}_\ell}$ would be an immediate consequence of the Borel–de Siebenthal theorem from [3]; this is known to hold also in positive characteristic except for characteristic 2 and 3 (see [16]).

Now $\underline{S} = \mathrm{Sp}_{2g,\mathbb{F}_\ell}$ is simply connected, so it has no proper subgroups of bounded index as ℓ grows without bound. Thus, for sufficiently large ℓ, we obtain

$$G' = \underline{S}(\mathbb{F}_\ell) = \mathrm{Sp}_{2g}(\mathbb{F}_\ell). \tag{2}$$

Finally, the result of Serre [38, Lemme 1], subsequently generalized by Vasiu in [43], asserts that any closed subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ which maps onto $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ is all of $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$. Applying this to Γ', we get the theorem in this case.

Various group-theoretic technicalities arise in implementing this idea in the general case. The condition $(*)$ gives a loose comparison between the *reductive* groups $\underline{G}$ and $\mathbf{G}$, whereas what is needed for the Borel–de Siebenthal theorem is a comparison between some *semisimple* groups $\underline{S}$ and $\underline{I}$, where the first comes from G' via Nori's construction and the second comes from $\mathbf{G}^{\mathrm{der}}$ via Bruhat–Tits theory. Bruhat–Tits theory works best for simply connected semisimple groups, but it is also useful for the groups we work with to be subgroups of GL_n. Much of the technical work here justifies moving back and forth between a reductive group, its derived group, and the universal cover of the derived group.

Section 2 assembles results from group theory that are needed in Section 3, including Nori's theory, our theory of ℓ-dimensions and ℓ-ranks, Bruhat–Tits theory, and some results about centralizers, formal characters, and regular elements. Section 3 proves the purely group-theoretic Theorem 1.5. Section 4 presents the main results on the algebraic envelopes $\underline{G}_\ell$ and proves Theorems 1.2 and 1.3.

Conventions for schemes and groups

The symbol ℓ always denotes a rational prime. Suppose that $R \to S$ is a homomorphism of commutative rings with unity, and suppose that X is a scheme over $\mathrm{Spec}(R)$ (or simply R). Denote the fiber product $X \times_R S := X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ also by X_S.

A semisimple algebraic group will always be assumed to be connected. A simple algebraic group over F is a semisimple group over F which has no proper, connected, closed, normal subgroup defined over F.

In order to keep track of the various kinds of groups that arise in this paper, we use the following system. An algebraic group over a field F is always assumed to be a smooth affine group variety over F. The bold letters, for example, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{S}$, denote algebraic groups over fields of characteristic 0 unless otherwise stated. The underlined letters, for example, $\underline{G}$, $\underline{H}$, $\underline{S}$, always denote algebraic groups over finite fields. Given a homomorphism $f : \underline{G} \to \underline{H}$, we denote by $f(\underline{G})$ the image of f in $\underline{H}$ endowed with the unique structure of reduced closed subscheme; the induced morphism $\underline{G} \to f(\underline{G})$ is assumed to be smooth in this paper. Group schemes over rings of dimension 1 (e.g., $\mathbb{Z}$, $\mathbb{Z}_\ell$) are denoted by $\mathcal{G}$ and $\mathcal{H}$. Capital Greek letters denote infinite groups, which are generally ℓ-adic Lie groups, whereas capital Roman letters denote finite groups.

Simple complex Lie algebras are denoted by $\mathfrak{g}$ and $\mathfrak{h}$. We identify such algebras with their Dynkin diagrams, so instead of saying that $\mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{SU}_n(\mathbb{F}_q)$ are both of type A_{n-1}, we may say they are of type $\mathfrak{g} = \mathfrak{sl}_n$. Let $\mathbf{G}$ denote an algebraic group over a field F of any characteristic, and let $\Gamma \subset \mathbf{G}(F)$ denote a subgroup. The rank of $\mathbf{G}$, denoted by $\operatorname{rank} \mathbf{G}$, always means the dimension of a maximal torus of $\mathbf{G} \times_F \overline{F}$. We denote by

$\mathbf{G}^{\mathrm{der}}$	the derived group of $\mathbf{G}$,
$\mathbf{G}^{\circ}$	the identity component of $\mathbf{G}$,
$\mathbf{G}^{\mathrm{ss}}$	the quotient of $\mathbf{G}^{\circ}$ by its radical,
$\mathbf{G}^{\mathrm{sc}}$	the universal cover of $\mathbf{G}^{\mathrm{ss}}$,
$\mathbf{G}^{\mathrm{red}}$	the quotient of $\mathbf{G}^{\circ}$ by its unipotent radical,
$\dim \mathbf{G}$	the dimension of $\mathbf{G}$ as an F-variety,
$\operatorname{rk} \mathfrak{g}$	the rank of the simple Lie algebra $\mathfrak{g}$,
$\dim \mathfrak{g}$	the dimension of $\mathfrak{g}$ as a $\mathbb{C}$-vector space,
Γ^{ss}	the image of $\Gamma^{\circ} := \Gamma \cap \mathbf{G}^{\circ}(F)$ under the quotient map $\mathbf{G}^{\circ} \to \mathbf{G}^{\mathrm{ss}}$,
Γ^{sc}	the preimage of Γ^{ss} under the map $\mathbf{G}^{\mathrm{sc}}(F) \to \mathbf{G}^{\mathrm{ss}}(F)$,
$M_n(R)$	the ring of $n \times n$ matrices with entries in a ring R,
$\mathrm{GL}_n(R)$	the group of units of $M_n(R)$.

2. Group-theoretic preliminaries

2.1. Nori's theory

Let n be a positive integer, and suppose that $\ell \geq n$. Let G be a subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$. Nori's theory from [31] produces a connected $\mathbb{F}_\ell$-algebraic subgroup $\underline{S}$ of $\mathrm{GL}_{n,\mathbb{F}_\ell}$ that approximates G if ℓ is larger than a constant depending only on n.

Let $G[\ell] := \{x \in G \mid x^\ell = 1\}$. The subgroup of G generated by $G[\ell]$ is denoted by G^+ and is normal in G. Define exp and log by

$$\exp(x) := \sum_{i=0}^{\ell-1} \frac{x^i}{i!} \qquad \text{and} \qquad \log(x) := -\sum_{i=1}^{\ell-1} \frac{(1-x)^i}{i}.$$

Denote by $\underline{S}$ the (connected) algebraic subgroup of $\mathrm{GL}_{n,\mathbb{F}_\ell}$, defined over $\mathbb{F}_\ell$, generated by the one-parameter subgroups

$$t \mapsto x^t := \exp\big(t \cdot \log(x)\big) \tag{3}$$

for all $x \in G[\ell]$. The $\mathbb{F}_\ell$-subgroup $\underline{S}$ is called the *Nori group* of $G \subset \mathrm{GL}_n(\mathbb{F}_\ell)$. An algebraic subgroup of $\mathrm{GL}_{n,\mathbb{F}_\ell}$ is said to be *exponentially generated* if it is generated by the one-parameter subgroups x^t in (3) for some set of unipotent elements $x \in \mathrm{GL}_n(\mathbb{F}_\ell)$. Since $\underline{S}$ is exponentially generated, $\underline{S}$ is an extension of a semisimple group by a unipotent group (see [31, Section 3]). If $y \in M_n(\mathbb{F}_\ell)$ commutes with x, then it also commutes with $\log x$ and therefore with the algebraic group x^t. Thus,

$$Z_{G^+}\big(M_n(\mathbb{F}_\ell)\big) = Z_{\underline{S}(\mathbb{F}_\ell)}\big(M_n(\mathbb{F}_\ell)\big) = Z_{\underline{S}(\overline{\mathbb{F}}_\ell)}\big(M_n(\overline{\mathbb{F}}_\ell)\big) \cap M_n(\mathbb{F}_\ell). \tag{4}$$

The following theorem approximates G^+ by $\underline{S}(\mathbb{F}_\ell)$.

THEOREM 2.1
There is a constant $C_1(n)$ depending only on n such that if $\ell > C_1(n)$ and G is a subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$, then the following assertions hold.

(i) *If $\underline{S}$ is the Nori group of G, then $G^+ = \underline{S}(\mathbb{F}_\ell)^+$.*

(ii) *If $\underline{S}$ is a Nori group, the quotient $\underline{S}(\mathbb{F}_\ell)/\underline{S}(\mathbb{F}_\ell)^+$ is a commutative group of order at most 2^{n-1}.*

(iii) *If $\underline{G} \subset \mathrm{GL}_{n,\mathbb{F}_\ell}$ is exponentially generated, then the Nori group of $G = \underline{G}(\mathbb{F}_\ell)$ is $\underline{G}$.*

Proof
This is due to Nori: parts (i) and (iii) come from [31, Theorem B] and part (ii) comes from [31, Remark 3.6]. □

A theorem of Jordan [25, p. 91] says that every finite subgroup G of $\mathrm{GL}_n(\mathbb{C})$ has an abelian subgroup Z such that the index $[G : Z]$ is bounded by a constant depending only on n. The following theorem is a variant of Jordan's theorem in positive characteristic.

THEOREM 2.2 ([31, Theorem C])
Let G be a subgroup of $\mathrm{GL}_n(\mathbb{F})$, where $\mathbb{F}$ is a finite field of characteristic $\ell \geq n$. Then G has a commutative subgroup Z of prime to ℓ order such that $Z \cdot G^+$ is normal in G and

$$[G : Z \cdot G^+] \le C_2(n),$$

where $C_2(n)$ is a constant depending only on n (and not on $\mathbb{F}$, ℓ, G).

Note that the statement of [31, Theorem C] does not explicitly assert that the order of Z is prime to ℓ, but that fact is stated in the Introduction to Nori's article and it is immediate from the construction of Z (see [31, p. 270]). Nori's theory does a good job of algebraically approximating the "semisimple" and "unipotent" parts of a finite subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$, but not the toric part; in general, there is no reason to expect, for large ℓ, that Z will be well approximated by the $\mathbb{F}_\ell$-points of *any* torus in $\mathrm{GL}_{n,\mathbb{F}_\ell}$. For Theorem 1.5, we hypothesize that it can be well approximated, moreover, by a torus whose complexity is bounded in a sense to be made precise in Section 2.6. For images of (mod ℓ) Galois representations arising from cohomology of a given projective nonsingular variety, we will see that this additional hypothesis holds.

We have the following result due to Serre for $\underline{S}$ if G acts semisimply on the ambient space.

PROPOSITION 2.3 ([20, Proposition 2.1.2])
Suppose that G acts semisimply on $\mathbb{F}_\ell^n$. There is a constant $C_3(n)$ depending only on n such that if $\ell > C_3(n)$, then the following assertions hold.

(i) *The Nori group $\underline{S}$ is a semisimple $\mathbb{F}_\ell$-subgroup of $\mathrm{GL}_{n,\mathbb{F}_\ell}$.*

(ii) *The representation $\underline{S} \to \mathrm{GL}_{n,\mathbb{F}_\ell}$ is semisimple.*

2.2. *Galois cohomology*

We begin with an estimate in Galois cohomology.

PROPOSITION 2.4
For $k \in \mathbb{N}$ there exists a constant $C_4(k)$ depending only on k such that, if F is a finite extension of $\mathbb{Q}_\ell$ with $\ell > k$ and $\mathbf{C}$ is a finite commutative group scheme over F with $|\mathbf{C}(\overline{F})| \le k$, then

$$\left|H^1\big(F, \mathbf{C}(\overline{F})\big)\right| \le C_4(k).$$

Proof
Consider the inflation-restriction sequence

$$1 \to H^1\big(\mathrm{Gal}(L/F), \mathbf{C}(\overline{F})\big) \to H^1\big(F, \mathbf{C}(\overline{F})\big) \to H^1\big(L, \mathbf{C}(\overline{F})\big)$$

where Gal_L acts trivially on $\mathbf{C}(\overline{F})$ and $[L : F] \le k!$. Since the size of $H^1(\mathrm{Gal}(L/F), \mathbf{C}(\overline{F}))$ is bounded above by some constant depending only on k, it suffices to bound

$H^1(L, \mathbf{C}(\overline{F}))$. Let S be the set of abelian extensions of L of degree bounded above by k. For every element ϕ of

$$H^1\big(L, \mathbf{C}(\overline{F})\big) \cong \mathrm{Hom}\big(\mathrm{Gal}_L, \mathbf{C}(\overline{F})\big) \cong \mathrm{Hom}\big(\mathrm{Gal}_L^{\mathrm{ab}}, \mathbf{C}(\overline{F})\big),$$

$\ker\phi$ corresponds to an element of S. Since $|\mathbf{C}(\overline{F})| \leq k$, we have

$$\left|\mathrm{Hom}\big(\mathrm{Gal}_L^{\mathrm{ab}}, \mathbf{C}(\overline{F})\big)\right| \leq |S| \cdot k!.$$

Let $\mathbb{F}_q$ be the residue field of L. By local class field theory, S corresponds to the set of open subgroups U of

$$L^* = O_L^* \times \mathbb{Z} = \text{pro-}\ell \times \mathbb{F}_q^* \times \mathbb{Z}$$

such that $[L^* : U] \leq k$. Hence, the possibilities of U are bounded above by some constant depending only on k if $\ell > k$. □

COROLLARY 2.5
Let F be a finite extension of $\mathbb{Q}_\ell$, and let $\alpha : \mathbf{G} \to \mathbf{H}$ be a central isogeny of degree at most k of connected reductive groups over F. If Γ is a subgroup of $\mathbf{H}(F)$, then the quotient

$$\Gamma/\alpha\big(\alpha^{-1}(\Gamma) \cap \mathbf{G}(F)\big)$$

is an abelian group with size bounded above by $C_4(k)$ if $\ell > k$.

Proof
Consider the long exact sequence in Galois cohomology

$$1 \to \mathbf{C}(F) \to \mathbf{G}(F) \to \mathbf{H}(F) \to H^1\big(F, \mathbf{C}(\overline{F})\big) \to \cdots,$$

where $\mathbf{C} := \ker\alpha$. The claim is an immediate consequence of Proposition 2.4. □

PROPOSITION 2.6
Let $\mathbf{G}$ be a simply connected semisimple group that is an inner twist of a split group over a finite extension F of $\mathbb{Q}_\ell$, and let d denote the order of the center of $\mathbf{G}(\overline{F})$. For every finite extension F' of F of degree divisible by d, the group $\mathbf{G}$ splits over F'.

Proof
Let $\mathbf{G}_0$ be the split form of $\mathbf{G}$, and let $\mathbf{C}_0$ denote the center of $\mathbf{G}_0$. Now $\mathbf{G}$ is the twist of $\mathbf{G}_0$ by a class in $H^1(F, \mathbf{G}_0(\overline{F})/\mathbf{C}_0(\overline{F}))$. The nonabelian cohomology sequence of the central extension

$$1 \to \mathbf{C}_0(\overline{F}) \to \mathbf{G}_0(\overline{F}) \to \mathbf{G}_0(\overline{F})/\mathbf{C}_0(\overline{F}) \to 1$$

gives an exact sequence

$$H^1\big(F, \mathbf{G}_0(\overline{F})\big) \to H^1\big(F, \mathbf{G}_0(\overline{F})/\mathbf{C}_0(\overline{F})\big) \to H^2\big(F, \mathbf{C}_0(\overline{F})\big).$$

Thus, since $H^1(F, \mathbf{G}_0(\overline{F})) = 0$ by [4, Théorème 4.7], it suffices to prove that d dividing $[F' : F]$ implies that

$$H^2\big(F, \mathbf{C}_0(\overline{F})\big) \to H^2\big(F', \mathbf{C}_0(\overline{F})\big)$$

is the zero map. As $\mathbf{G}_0$ is split, $\mathbf{C}_0$ is a product of groups of the form μ_n, where n divides d. Thus, it suffices to prove that every class in $\mathrm{Br}(F)_n$ lies in $\ker(\mathrm{Br}(F) \to \mathrm{Br}(F'))$ for every extension F'/F such that d divides $[F' : F]$. This follows from the fact (see [34, Section XIII, Proposition 7]) that, at the level of invariants, the map $\mathrm{Br}(F) \to \mathrm{Br}(F')$ is just multiplication by $[F' : F]$. □

2.3. ℓ-Dimension and ℓ-ranks

In this section, we review the definitions of the ℓ-dimension and the ℓ-ranks (i.e., the total ℓ-ranks and the $\mathfrak{h}$-type ℓ-rank for varying simple Lie type $\mathfrak{h}$) of finite groups and profinite groups with open prosolvable subgroups (see [20], [22]) and state the results relating the dimension and the ranks of an algebraic group $\underline{G}/\mathbb{F}_q$ to, respectively, the ℓ-dimension and the ℓ-ranks of $\underline{G}(\mathbb{F}_q)$ (see [22]).

2.3.1

Let G be a finite simple group of Lie type in characteristic $\ell \geq 5$. The condition on ℓ rules out the possibility of Suzuki or Ree groups, so there exists a (unique) adjoint simple group $\underline{G}/\mathbb{F}_{\ell^f}$ so that

$$G = \big[\underline{G}(\mathbb{F}_{\ell^f}), \underline{G}(\mathbb{F}_{\ell^f})\big] = \mathrm{im}\big(\underline{G}^{\mathrm{sc}}(\mathbb{F}_{\ell^f}) \to \underline{G}(\mathbb{F}_{\ell^f})\big).$$

We define the ℓ-dimension of G to be

$$\dim_\ell G := f \cdot \dim \underline{G}.$$

Let $\mathfrak{g}$ denote the unique simple complex Lie algebra whose root system is a factor of the root system of $\underline{G}_{\overline{\mathbb{F}}_\ell}$. If $\mathfrak{h}$ is a simple complex Lie algebra, the $\mathfrak{h}$-type ℓ-rank of G is

$$\mathrm{rk}_\ell^{\mathfrak{h}} G := \begin{cases} f \cdot \mathrm{rank}\, \underline{G} & \text{if } \mathfrak{h} = \mathfrak{g}, \\ 0 & \text{otherwise.} \end{cases}$$

For example, $G = \mathrm{PSL}_n(\mathbb{F}_{\ell^f})$ (resp., $\mathrm{PSU}_n(\mathbb{F}_{\ell^f})$) has $f(n^2-1)$ as the ℓ-dimension and $f(n-1)$ as A_{n-1}-type ℓ-rank.

For simple groups which are not of Lie type in characteristic ℓ (including simple groups of order less than ℓ and abelian simple groups like $\mathbb{Z}/\ell\mathbb{Z}$), we define the ℓ-dimension and $\mathfrak{h}$-type ℓ-rank to be 0. We extend the definitions to arbitrary finite groups G by defining the ℓ-dimension (resp., $\mathfrak{h}$-type ℓ-rank) to be the sum of the ℓ-dimensions (resp., $\mathfrak{h}$-type ℓ-ranks) of its composition factors. We define the total ℓ-rank of G to be

$$\mathrm{rk}_\ell\, G := \sum_{\mathfrak{h}} \mathrm{rk}_\ell^{\mathfrak{h}},$$

where the sum is taken over all simple complex Lie algebras.

This makes it clear that $\dim_\ell$, $\mathrm{rk}_\ell^{\mathfrak{h}}$, and rk_ℓ are additive on short exact sequences of groups. In particular, the ℓ-dimension and the total ℓ-rank of every solvable finite group are 0, and neither passing to a central extension nor to the derived group affects the ℓ-dimension or any ℓ-rank. For instance the ℓ-dimension and ℓ-rank of $\mathrm{GL}_n(\mathbb{F}_{\ell^f})$, $\mathrm{PGL}_n(\mathbb{F}_{\ell^f})$, and $\mathrm{PSL}_n(\mathbb{F}_{\ell^f})$ are all the same.

Our basic results on $\dim_\ell$, $\mathrm{rk}_\ell^{\mathfrak{h}}$, and rk_ℓ of finite groups are the following.

LEMMA 2.7
For $k \in \mathbb{N}$ there exists a constant $C_5(k)$ such that, if $\ell > C_5(k)$ and H is a subgroup of G of index at most k, then the ℓ-dimension and ℓ-ranks of G and H are the same.

Proof
At the cost of replacing k by $k!$, we may assume that H is normal in G. If ℓ is large enough, then the ℓ-dimension and ℓ-ranks of G/H are 0, and the lemma follows by additivity. □

PROPOSITION 2.8
Let G be a subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$, and let $\underline{S}$ be the Nori group of G. There exists a constant $C_6(n)$ depending only on n such that if $\ell > C_6(n)$, then the ℓ-dimension and the ℓ-rank of G and $\underline{S}(\mathbb{F}_\ell)$ are identical.

Proof
The assertion follows directly from Theorems 2.1 and 2.2 and Lemma 2.7. □

PROPOSITION 2.9 ([22, Proposition 4][2])
Let $\underline{G}$ be a connected algebraic group over $\mathbb{F}_{\ell^f}$ with $\ell \geq 5$. The composition factors of $\underline{G}(\mathbb{F}_{\ell^f})$ are cyclic groups and finite simple groups of Lie type in characteristic

[2]The rank of an algebraic group $\mathbf{G}/F$ in [22] is defined to be the usual rank of $\mathbf{G}^{\mathrm{ss}} \times_F \overline{F}$ (see [22, Section 2]).

ℓ. Moreover, let $m_{\mathfrak{g}}$ be the number of factors of $\underline{G}^{\mathrm{sc}}_{\mathbb{F}_\ell}$ of type $\mathfrak{g}$. Then the following equations hold:

(i) $\mathrm{rk}^{\mathfrak{g}}_\ell(\underline{G}(\mathbb{F}_{\ell^f})) = m_{\mathfrak{g}} f \cdot \mathrm{rk}\,\mathfrak{g}$,

(ii) $\mathrm{rk}_\ell(\underline{G}(\mathbb{F}_{\ell^f})) = f \cdot \mathrm{rank}\,\underline{G}^{\mathrm{ss}}$,

(iii) $\dim_\ell(\underline{G}(\mathbb{F}_{\ell^f})) = f \sum_{\mathfrak{g}}(m_{\mathfrak{g}} \cdot \dim \mathfrak{g}) = f \cdot \dim \underline{G}^{\mathrm{ss}}$.

2.3.2

Let F be a finite extension of $\mathbb{Q}_\ell$ with the ring of integers O_F and the residue field $\mathbb{F}_q$. The definitions above are extended to certain infinite profinite groups, including compact subgroups of $\mathrm{GL}_n(F)$, as follows. If Γ is a finitely generated profinite group which contains an open prosolvable subgroup, then we define

$$\dim_\ell \Gamma := \dim_\ell(\Gamma/\Delta), \qquad \mathrm{rk}^{\natural}_\ell \Gamma := \mathrm{rk}^{\natural}_\ell(\Gamma/\Delta), \qquad \text{and} \qquad \mathrm{rk}_\ell \Gamma := \mathrm{rk}_\ell(\Gamma/\Delta)$$

for any normal, pro-ℓ, open subgroup Δ of Γ. The ℓ-dimension and ℓ-rank of every pro-ℓ group is 0. (So, in particular, the ℓ-dimension of an ℓ-adic Lie group can be strictly smaller than its dimension in the sense of ℓ-adic manifolds.) By additivity,

$$\dim_\ell \Gamma = \dim_\ell G, \qquad \mathrm{rk}^{\natural}_\ell \Gamma = \mathrm{rk}^{\natural}_\ell G, \qquad \text{and} \qquad \mathrm{rk}_\ell \Gamma = \mathrm{rk}_\ell G,$$

where G denotes the image in $\mathrm{GL}_n(\mathbb{F}_q)$ under the reduction of Γ with respect to an O_F-lattice in F^n stabilized by Γ. If Γ is a compact subgroup of $\mathrm{GL}_n(F)$ and Δ is a closed normal subgroup, then

$$\dim_\ell \Gamma = \dim_\ell \Delta + \dim_\ell(\Gamma/\Delta),$$
$$\mathrm{rk}^{\natural}_\ell \Gamma = \mathrm{rk}^{\natural}_\ell \Delta + \mathrm{rk}^{\natural}_\ell(\Gamma/\Delta),$$

and

$$\mathrm{rk}_\ell \Gamma = \mathrm{rk}_\ell \Delta + \mathrm{rk}_\ell(\Gamma/\Delta).$$

LEMMA 2.10

Let $\Gamma \subseteq \Pi$ be compact subgroups of $\mathrm{GL}_n(\mathbb{Q}_\ell)$ (resp., $\mathrm{GL}_n(\mathbb{F}_\ell)$). There exists a constant $C_7(n)$ depending only on n such that if $\ell > C_7(n)$, then

$$\mathrm{rk}_\ell \Gamma \le \mathrm{rk}_\ell \Pi,$$
$$\dim_\ell \Gamma \le \dim_\ell \Pi.$$

Proof

Fix a Π-stable lattice Λ in $\mathbb{Q}_\ell^n$. By the fact that the ℓ-ranks of prosolvable groups are 0, it suffices to prove the same inequality for the finite groups $G \subseteq P \subseteq \mathrm{GL}_n(\mathbb{F}_\ell)$

obtained by reducing modulo Λ. The Nori group of G is generated by a subset of the collection of unipotent groups generating the Nori group of P and is therefore a closed subgroup of that algebraic group. Both the dimension and semisimple rank of a subgroup of any algebraic group are less than or equal to those of the ambient group, so the lemma follows from Propositions 2.8 and 2.9. □

2.4. Bruhat–Tits theory

We briefly recall some basic facts from Bruhat–Tits theory, mainly from [42]. The main goal of this section is Theorem 2.11.

2.4.1

Let F be a finite extension of $\mathbb{Q}_\ell$ with residue field $\mathbb{F}_q$, and let $\mathbf{G}$ be a connected, semisimple algebraic group defined over F. The Bruhat–Tits building $\mathcal{B}(\mathbf{G}, F)$ is a polysimplicial complex (see [42, Section 2.2.1]), endowed with a $\mathbf{G}(F)$-action that is linear on each facet. If F' is a finite extension of F, then there is a corresponding continuous injection of buildings

$$\iota_{F',F} : \mathcal{B}(\mathbf{G}, F) \to \mathcal{B}(\mathbf{G}, F'),$$

which is equivariant with respect to $\mathbf{G}(F) \subset \mathbf{G}(F')$ and maps vertices of $\mathcal{B}(\mathbf{G}, F)$ to vertices of $\mathcal{B}(\mathbf{G}, F')$. If $F \subset F' \subset F''$ are finite extensions of fields, then

$$\iota_{F'',F'} \circ \iota_{F',F} = \iota_{F'',F}.$$

For every point $x \in \mathcal{B}(\mathbf{G}, F)$, the stabilizer $\mathbf{G}(F)^x$ is a compact subgroup of $\mathbf{G}(F)$. There exist a smooth affine group scheme $\mathcal{G}_x$ over the ring of integers O_F of F and an isomorphism i from the generic fiber of $\mathcal{G}_x$ to $\mathbf{G}$ such that $i(\mathcal{G}_x(O_F)) = \mathbf{G}(F)^x$ and if F' is a finite unramified extension of F, then (see [42, Section 3.4.1])

$$i\big(\mathcal{G}_x(O_{F'})\big) = \mathbf{G}(F')^{\iota_{F',F}(x)}.$$

If the special fiber $\mathcal{G}_{x,\mathbb{F}_q}$ of $\mathcal{G}_x$ is reductive, we say that x is *hyperspecial* and $\mathbf{G}(F)^x$ is a *hyperspecial maximal compact subgroup* (or simply *hyperspecial*) of $\mathbf{G}(F)$ (see [42, Section 3.8.1]).

Every maximal compact subgroup of $\mathbf{G}(F)$ is the stabilizer $\mathbf{G}(F)^x$ of a point $x \in \mathcal{B}(\mathbf{G}, F)$ by [42, Section 3.2]. We may always take x to be the centroid of some facet. Moreover, if $\mathbf{G}$ is in addition simply connected, then x is a vertex (see [42, Section 3.2]) and the special fiber $\mathcal{G}_{x,\mathbb{F}_q}$ is connected (see [42, Section 3.5.2]).

2.4.2

Let F^{nr} be the maximal unramified extension of F in $\overline{F}$. The group $\mathbf{G}$ determines a map of $\mathrm{Gal}(F^{\mathrm{nr}}/F)$-diagrams: the *relative local Dynkin diagram* Δ_F (i.e., the local Dynkin diagram of $\mathbf{G}/F$) with trivial $\mathrm{Gal}(F^{\mathrm{nr}}/F)$-action, the *absolute local Dynkin diagram* $\Delta_{F^{\mathrm{nr}}}$ (i.e., the local Dynkin diagram of $\mathbf{G}_{F^{\mathrm{nr}}}/F^{\mathrm{nr}}$) with an action of $\mathrm{Gal}(F^{\mathrm{nr}}/F)$, and a $\mathrm{Gal}(F^{\mathrm{nr}}/F)$-map $\Delta_{F^{\mathrm{nr}}} \to \Delta_F$ (see [42, Section 1.11]). The Dynkin diagram of $\mathscr{G}^{\mathrm{red}}_{x,\overline{\mathbb{F}}_\ell}$ (the reductive quotient of $\mathscr{G}_{x,\overline{\mathbb{F}}_\ell}$, see conventions for groups) can be constructed by deleting from $\Delta_{F^{\mathrm{nr}}}$ all the vertices (together with all the edges connected to them) mapping to the vertices in Δ_F associated to x. Moreover, if the minimal facet containing x is a chamber (e.g., when $\mathbf{G}/F$ is anisotropic, in which case Δ_F is empty), then $\mathscr{G}^{\mathrm{red}}_{x,\mathbb{F}_q}$ is a torus (see [42, Section 3.5.2]).

A semisimple group $\mathbf{G}$ over a local field F is *unramified* if $\mathbf{G}$ has a Borel subgroup over F and $\mathbf{G}$ splits over an unramified extension of F. The group $\mathbf{G}$ is unramified if and only if $\mathscr{B}(\mathbf{G}, F)$ has a hyperspecial point (see [42, Section 1.10.2] for the "only if" part and [10, Corollary 5.2.14] for the "if" part). And the latter condition is equivalent to the local Dynkin diagram Δ_F having a hyperspecial vertex (see [42, Sections 1.9, 1.10]).

2.4.3
The main theorem of this section is as follows.

THEOREM 2.11
Let F be a finite extension of $\mathbb{Q}_\ell$ with residue degree $f := [\mathbb{F}_q : \mathbb{F}_\ell]$, and let $\ell \geq 5$. Let $\mathbf{G}$ be a semisimple group of rank r over F, and let Π be a maximal compact subgroup of $\mathbf{G}(F)$. Then the following assertions hold.

(i) *The total ℓ-rank of Π is at most fr.*
(ii) *If $\mathrm{rk}_\ell\, \Pi = fr$, then $\mathbf{G}$ splits over a finite unramified extension of F.*
(iii) *If $\mathrm{rk}_\ell\, \Pi = fr$, then $\mathbf{G}$ is unramified over every degree 12 totally ramified extension F^{t}/F.*
(iv) *If $\mathrm{rk}_\ell\, \Pi = fr$, then there exist a totally ramified extension F'/F and a hyperspecial maximal compact subgroup $\Omega \subset \mathbf{G}(F')$ such that $\Pi \subset \Omega$.*

Proof
Since Π^{sc} is maximal compact in $\mathbf{G}^{\mathrm{sc}}(F)$ and the total ℓ-ranks of Π and Π^{sc} are equal, we may assume that $\mathbf{G}$ is simply connected. It therefore factors as a product of groups $\mathbf{G}_i$ which are simply connected and simple. Let x denote a vertex of the building $\mathscr{B}(\mathbf{G}, F)$ stabilized by Π, and let $\mathscr{G}_x$ denote the smooth affine group scheme over O_F in Section 2.4.1. The building of $\mathbf{G}$ is the product of the buildings of the $\mathbf{G}_i$'s (see [42, Section 2.1]), so the vertex $x = (x_1, \ldots, x_k)$, and

$$\mathscr{G}_x = \prod_i (\mathscr{G}_i)_{x_i}.$$

As rank is additive in products, it suffices to prove the theorem in the simple case.

Thus, there exist a finite extension F'/F and an absolutely simple group $\mathbf{G}'/F'$ such that $\mathbf{G} = \mathrm{Res}_{F'/F}\,\mathbf{G}'$. Then Π is a maximal compact subgroup of $\mathbf{G}'(F') = \mathbf{G}(F)$. Denote the rank of $\mathbf{G}'$ by r', the order of the residue field of F' by $\ell^{f'}$, and the ramification degree of F'/F by e. Since we have

$$r = [F' : F]r' = e(f'/f)r',$$

the inequality $\mathrm{rk}_\ell\,\Pi \leq f'r'$ implies that $\mathrm{rk}_\ell\,\Pi \leq fr$, with strict inequality if $e > 1$. Let F^{t}/F be the totally ramified extension described in (iii). If $e = 1$, then it follows that

- F'/F is unramified,
- the composition $F'F^{\mathrm{t}}$ is totally ramified over F' of degree 12, and
- $\mathbf{G}_{F^{\mathrm{t}}} = (\mathrm{Res}_{F'/F}\,\mathbf{G}') \times_F F^{\mathrm{t}} = \mathrm{Res}_{F'F^{\mathrm{t}}/F^{\mathrm{t}}}(\mathbf{G}' \times_{F'} F'F^{\mathrm{t}})$.

Hence, if $\mathbf{G}'$ splits over an unramified extension F'' of F', then $\mathbf{G}$ (resp., $\mathbf{G}_{F^{\mathrm{t}}}$) also splits over F'' (resp., $F''F^{\mathrm{t}}$), which is unramified over F (resp., F^{t}). If $\mathbf{G}'$ is quasisplit over $F'F^{\mathrm{t}}$, then it has a Borel subgroup $\mathbf{B}'$ defined over $F'F^{\mathrm{t}}$, and the restriction of scalars $\mathrm{Res}_{F'F^{\mathrm{t}}/F^{\mathrm{t}}}\,\mathbf{B}'$ is a Borel subgroup of $\mathbf{G}_{F^{\mathrm{t}}}$. Thus, if (i)–(iii) hold for $(\mathbf{G}', F')$, they hold for $(\mathbf{G}, F)$, and without loss of generality, we may assume that $\mathbf{G}$ is absolutely simple.

As the kernel of $\mathscr{G}_x(O_F) \to \mathscr{G}_x(\mathbb{F}_q)$ is pro-ℓ, the total ℓ-ranks of Π and $\mathscr{G}_x(\mathbb{F}_q)$ are equal. Since $\mathbf{G}$ is simply connected, $\mathscr{G}_x(\mathbb{F}_q)$ is the group of $\mathbb{F}_q$-points of an algebraic group which is the extension of the reductive group $\mathscr{G}^{\mathrm{red}}_{x,\mathbb{F}_q}$ by a unipotent group. Thus, $\mathrm{rk}_\ell(\mathscr{G}_x(\mathbb{F}_q))$ is f times $\mathrm{rank}\,\mathscr{G}^{\mathrm{ss}}_{x,\mathbb{F}_q}$, the semisimple rank of $\mathscr{G}^{\mathrm{red}}_{x,\mathbb{F}_q}$. We claim that this is less than or equal to fr, or equivalently,

$$\mathrm{rank}\,\mathscr{G}^{\mathrm{ss}}_{x,\mathbb{F}_q} \leq r, \tag{5}$$

with equality only if $\mathbf{G}$ splits over an unramified extension and has a Borel subgroup over F^{t} in (iii).

Since $\mathrm{rank}\,\mathscr{G}^{\mathrm{red}}_{x,\mathbb{F}_q}$ and the relative rank of $\mathbf{G}_{F^{\mathrm{nr}}}$ (the rank of a maximal F^{nr}-split torus of $\mathbf{G}_{F^{\mathrm{nr}}}$) are equal (see [42, Section 3.5]), the inequality (5) holds in general, which is assertion (i), and the equality holds only if $\mathbf{G}$ splits over F^{nr}, which is assertion (ii). Let $\mathbf{G}^{\mathrm{sp}}/F$ be a split form of $\mathbf{G}$. By definition, the number of vertices in the absolute local Dynkin diagram $\Delta_{F^{\mathrm{nr}}}$ of $\mathbf{G}$ is 1 greater than the relative rank of $\mathbf{G}_{F^{\mathrm{nr}}}$. If the equality in (5) holds, then it follows by (ii) and Section 2.4.2 that

(A) $\Delta_{F^{\mathrm{nr}}}$ coincides with the (relative) local Dynkin diagram of $\mathbf{G}^{\mathrm{sp}}/F$,
(B) $\Delta_{F^{\mathrm{nr}}}$ contains at least one $\mathrm{Gal}(F^{\mathrm{nr}}/F)$-stable vertex, and
(C) $\mathbf{G}$ is not anisotropic; that is, $\Delta_F \neq \emptyset$.

To list the cases when the conditions (A), (B), and (C) hold, we consult the tables from [42, Sections 4.2, 4.3]; the possible types are all split types in [42, Section 4.2] together with the following possibilities in [42, Section 4.3]:

$$^2A_n', {}^2B_s, {}^2C_{2m}, {}^2D_t, {}^2D_t', {}^2D_{2s}'', {}^3D_4, {}^4D_{2n}, {}^2E_6, {}^3E_6, {}^2E_7,$$

where $m \geq 1$, $n \geq 2$, $s \geq 3$, and $t \geq 4$ are integers. From the tables, every split type has a hyperspecial vertex in Δ_F and is thus unramified. Similarly, the groups ${}^2A_n'$, ${}^2D_t$, ${}^3D_4$, and ${}^2E_6$ also have hyperspecial vertices in Δ_F and are therefore unramified.

The cases ${}^2C_{2m}$ and ${}^2E_7$ are inner forms of split groups of types C_{2m} and E_7, respectively, by (A) and the fact that C_{2m} and E_7 have no nontrivial outer automorphisms. They therefore split over every even-degree extension by Proposition 2.6, because their $\overline{F}$-centers are of order 2.

Similarly, ${}^3E_6$ is an inner form of a split group of type E_6 by (A) and the fact that its index ${}^1E_{6,2}^{16}$ has presuperscript 1 (see [42, Section 4.3]; meaning that the image of $\mathrm{Gal}(F^{\mathrm{nr}}/F) \to \mathrm{Aut}(\Delta_{F^{\mathrm{nr}}})$ is of order 1 by [41, Table II: Indices]). Thus, it splits over every extension whose degree is divisible by 3 by Proposition 2.6 since its $\overline{F}$-center is of order 3.

To see that, in the remaining cases ${}^2B_s$, ${}^2D_t'$, ${}^2D_{2s}''$, ${}^4D_{2n}$, the group $\mathbf{G}$ becomes unramified over every F^{t} in (iii), we examine the explicit descriptions from [42, Section 4.4] which classify every central isogeny class of absolutely simple groups over F. The quaternionic orthogonal groups ${}^2D_{2s}''$ and ${}^4D_{2n}$ become ordinary orthogonal groups after passage to any ramified quadratic extension of F since each such extension splits the quaternion algebra over F. This leaves the cases of orthogonal groups of quadratic forms including ${}^2B_s$ and ${}^2D_t'$.

By passing to any ramified quadratic extension F'/F, we may assume that the form Q defining $\mathbf{G}$ is $u_1x_1^2 + \cdots + u_nx_n^2$, where the u_i's are units in $O_{F'}$. We claim that there exists a plane hyperbolic with respect to Q contained in the 3-dimensional locus $x_4 = x_5 = \cdots = x_n = 0$. Indeed, the quadratic form $u_1x_1^2 + u_2x_2^2 + u_3x_3^2$ defines a form of SO(3); the space $(F')^3$ of triples (x_1, x_2, x_3) contains a hyperbolic plane if and only if this form is split, that is, if and only if there is a nonzero isotropic vector. A nontrivial solution of the equation $\bar{u}_1x_1^2 + \bar{u}_2x_2^2 + \bar{u}_3x_3^2 = 0$ over the residue field $\mathbb{F}_{q'}$ of F' exists by Chevalley–Warning, which lifts by Hensel's lemma to a nonzero isotropic vector in $(F')^3$. It follows by induction that Q defines a quadratic form of Witt index $n' \geq n/2 - 1$. Hence, $\mathbf{G}_{F'}$ can only be $D_{n'}$, $B_{n'}$, ${}^2D_{n'+1}$ by [42, Section 4.4] and (A), and it is thus unramified, as its relative local Dynkin diagram has a hyperspecial vertex (see [42, Section 4.2, 4.3]).

Since $\ell \geq 5$, every degree 12 totally ramified extension F^{t} of F is tame and thus contains subextensions of all possible degrees dividing 12. We conclude that $\mathbf{G}$ is unramified over F^{t}, and (iii) is obtained.

For assertion (iv), let F^{t} be a field in (iii), and let x^{t} be the centroid of a facet of $\mathcal{B}(\mathbf{G}_{F^{\mathrm{t}}}, F^{\mathrm{t}})$ whose stabilizer is a maximal compact subgroup of $\mathbf{G}(F^{\mathrm{t}})$ containing Π. Since $\mathbf{G}_{F^{\mathrm{t}}}$ is semisimple and unramified by (iii), there exists a totally ramified extension F' of F^{t} such that $x' := \iota_{F',F^{\mathrm{t}}}(x^{\mathrm{t}})$ is a hyperspecial point of $\mathcal{B}(\mathbf{G}_{F'}, F')$ (see [26, Lemma 2.4]). The stabilizer $\mathbf{G}(F')^{x'}$ is the desired group Ω. □

2.5. Commutants and semisimplicity

Let $F/\mathbb{Q}_\ell$ be a finite field extension, let V be an n-dimensional F-vector space, let Λ be an O_F-lattice in V, and let Γ be a closed subgroup of $\mathrm{GL}(\Lambda) \cong \mathrm{GL}_n(O_F) \subset \mathrm{GL}_n(F) \cong \mathrm{GL}(V)$. If F' is a finite extension of F, we can regard Γ also as a subgroup of $\mathrm{GL}(\Lambda') \cong \mathrm{GL}_n(O_{F'})$, where $\Lambda' = \Lambda \otimes_{O_F} O_{F'}$. Let π (resp., π') be a uniformizer of O_F (resp., $O_{F'}$), and define $V' := V \otimes_F F'$. We have the following results in this setting.

LEMMA 2.12

The group Γ acts semisimply on V' if and only if it acts semisimply on V, in which case we have

$$\dim_{F'}(\mathrm{End}_\Gamma V') = \dim_F(\mathrm{End}_\Gamma V).$$

Likewise, Γ acts semisimply on the reduction $\mathrm{L} := \Lambda/\pi\Lambda$ if and only if it acts semisimply on $\mathrm{L}' := \Lambda'/\pi'\Lambda'$, in which case we have

$$\dim_{O_F/(\pi)}(\mathrm{End}_\Gamma \mathrm{L}) = \dim_{O_{F'}/(\pi')}(\mathrm{End}_\Gamma \mathrm{L}').$$

Proof

The proof is clear. □

LEMMA 2.13

Let M be a free O_F-module of finite rank, and let Γ be a subgroup of $\mathrm{Aut}_{O_F} M$. Then for all $k \geq 1$, the inclusion

$$M^\Gamma/\pi^k M^\Gamma \subset (M/\pi^k M)^\Gamma$$

is either proper for all $k \geq 1$ or is an equality for all $k \geq 1$.

Proof

We use the following diagram of cohomology sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^\Gamma & \xrightarrow{\pi} & M^\Gamma & \longrightarrow & (M/\pi M)^\Gamma & \longrightarrow & H^1(\Gamma, M) \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi^{k-1}} & & \downarrow{\scriptstyle \pi^{k-1}} & & \downarrow{\scriptstyle 1} \\
0 & \longrightarrow & M^\Gamma & \xrightarrow{\pi^k} & M^\Gamma & \longrightarrow & (M/\pi^k M)^\Gamma & \longrightarrow & H^1(\Gamma, M).
\end{array}$$

The inclusion follows from the second row. As the rightmost vertical arrow is an isomorphism, $M^\Gamma/\pi M^\Gamma \subsetneq (M/\pi M)^\Gamma$ implies that $M^\Gamma/\pi^k M^\Gamma \subsetneq (M/\pi^k M)^\Gamma$ for all $k \geq 1$. Conversely, if $M^\Gamma/\pi M^\Gamma = (M/\pi M)^\Gamma$, then the cohomology sequence

$$0 \to (M/\pi M)^\Gamma \to (M/\pi^k M)^\Gamma \to (M/\pi^{k-1} M)^\Gamma \to \cdots$$

implies by induction on k that

$$\left|(M/\pi^k M)^\Gamma\right| \leq \left|(M/\pi M)^\Gamma\right|^k = |M^\Gamma/\pi M^\Gamma|^k = |M^\Gamma/\pi^k M^\Gamma|$$

for all $k \geq 1$, which implies that $(M/\pi^k M)^\Gamma = M^\Gamma/\pi^k M^\Gamma$. □

LEMMA 2.14

Let V be a finite-dimensional vector space over a field F, and let $H \subset G \subset \mathrm{GL}(V)$ be subgroups. Let V^{ss} be the semisimplification of G on V. The following assertions hold.

(i) $\dim_F(\mathrm{End}_G V) \leq \dim_F(\mathrm{End}_G V^{\mathrm{ss}})$.

(ii) *If* $\dim_F(\mathrm{End}_G V) = \dim_F(\mathrm{End}_G V^{\mathrm{ss}})$, *then G acts semisimply on V.*

(iii) *If H acts semisimply on V and* $\dim_F(\mathrm{End}_G V) = \dim_F(\mathrm{End}_H V)$, *then G acts semisimply on V.*

(iv) *If H acts semisimply on V and* $\dim_F(\mathrm{End}_G V) = \dim_F(\mathrm{End}_H V)$, *then H is absolutely irreducible on every absolutely irreducible subrepresentation $W \leq V$ of G.*

Proof

Assertions (i) and (ii) are just [7, Lemma 3.6.1.1]. Let H^{red} and G^{red} be the images of H and G, respectively, in $\mathrm{GL}(V^{\mathrm{ss}})$. Since H acts semisimply on V, the representations $H \to \mathrm{GL}(V)$ and $H^{\mathrm{red}} \to \mathrm{GL}(V^{\mathrm{ss}})$ are isomorphic. This implies that

$$\begin{aligned}\dim_F(\mathrm{End}_G V) = \dim_F(\mathrm{End}_H V) &= \dim_F(\mathrm{End}_{H^{\mathrm{red}}} V^{\mathrm{ss}}) \\ &\geq \dim_F(\mathrm{End}_{G^{\mathrm{red}}} V^{\mathrm{ss}}).\end{aligned}$$

Then (iii) follows from (i) and (ii).

For assertion (iv), G is semisimple on V by (iii). The absolute irreducibility of W and the condition $\dim_F(\mathrm{End}_G V) = \dim_F(\mathrm{End}_H V)$ force $1 = \dim_F(\mathrm{End}_G W) = \dim_F(\mathrm{End}_H W)$. We are done since H is semisimple on W. □

PROPOSITION 2.15

Let F be a characteristic 0 local field with valuation ring O_F and residue field $\mathbb{F}_q$. Let V be a finite-dimensional vector space over F, and let Γ be a compact subgroup of $\mathrm{GL}(V)$ which acts semisimply on V. The following assertions are equivalent.

(i) *For some* Γ*-stable lattice* Λ *of* V*, we have*

$$\dim_F(\mathrm{End}_\Gamma V) = \dim_{\mathbb{F}_q}\big(\mathrm{End}_\Gamma(\Lambda \otimes_{O_F} \mathbb{F}_q)^{\mathrm{ss}}\big).$$

(ii) *For every* Γ*-subrepresentation* W *of* V *and every* Γ*-stable lattice* Λ_W *of* W*,* $\Lambda_W \otimes_{O_F} \mathbb{F}_q$ *is semisimple, and*

$$\dim_F(\mathrm{End}_\Gamma W) = \dim_{\mathbb{F}_q}\big(\mathrm{End}_\Gamma(\Lambda_W \otimes_{O_F} \mathbb{F}_q)\big).$$

(iii) *The following two assertions hold.*

(a) *If* W *is an irreducible* Γ*-subrepresentation of* V*, and* Λ_W *is a* Γ*-stable lattice of* W*, then* $\Lambda_W \otimes_{O_F} \mathbb{F}_q$ *is semisimple and*

$$\dim_F(\mathrm{End}_\Gamma W) = \dim_{\mathbb{F}_q}\big(\mathrm{End}_\Gamma(\Lambda_W \otimes_{O_F} \mathbb{F}_q)\big).$$

(b) *If* W_1 *and* W_2 *are nonisomorphic irreducible* Γ*-subrepresentations of* V *and* Λ_1 *and* Λ_2 *are* Γ*-stable lattices of* W_1 *and* W_2*, respectively, then* $\Lambda_1 \otimes_{O_F} \mathbb{F}_q$ *and* $\Lambda_2 \otimes_{O_F} \mathbb{F}_q$ *have no common irreducible* Γ*-subrepresentation.*

Proof

Assume assertion (i), let W and W' be any subrepresentations of V, and let Λ and Λ' be stable lattices in W and W', respectively. Applying and Lemma 2.13 to $M = \mathrm{Hom}_{O_F}(\Lambda, \Lambda') \subset \mathrm{Hom}_\Gamma(W, W')$, we obtain

$$\begin{aligned}\dim_F\big(\mathrm{Hom}_\Gamma(W, W')\big) &= \mathrm{rk}_{O_F}\big(\mathrm{Hom}_\Gamma(\Lambda, \Lambda')\big)\\ &\le \dim_{\mathbb{F}_q}\big(\mathrm{Hom}_\Gamma(\Lambda \otimes_{O_F} \mathbb{F}_q, \Lambda' \otimes_{O_F} \mathbb{F}_q)\big).\end{aligned} \tag{6}$$

Let W_1 and W_2 be, respectively, complementary Γ-subrepresentations of V with Γ-stable lattices Λ_1 and Λ_2. The Brauer–Nesbitt theorem implies that $(\Lambda \otimes_{O_F} \mathbb{F}_q)^{\mathrm{ss}}$ is the semisimplification of $(\Lambda_1 \oplus \Lambda_2) \otimes_{O_F} \mathbb{F}_q$. It follows by (i) that

$$\begin{aligned}\dim_{\mathbb{F}_q}\big(\mathrm{End}_\Gamma(\Lambda \otimes_{O_F} \mathbb{F}_q)^{\mathrm{ss}}\big) &\ge \dim_{\mathbb{F}_q}\big(\mathrm{End}_\Gamma\big((\Lambda_1 \oplus \Lambda_2) \otimes_{O_F} \mathbb{F}_q\big)\big)\\ &= \sum_{i=1}^{2}\sum_{j=1}^{2} \dim_{\mathbb{F}_q}\big(\mathrm{Hom}_\Gamma(\Lambda_i \otimes_{O_F} \mathbb{F}_q, \Lambda_j \otimes_{O_F} \mathbb{F}_q)\big)\\ &\ge \sum_{i=1}^{2}\sum_{j=1}^{2} \dim_F\big(\mathrm{Hom}_\Gamma(\Lambda_i \otimes_{O_F} F, \Lambda_j \otimes_{O_F} F)\big)\\ &= \dim_F(\mathrm{End}_\Gamma V),\end{aligned}$$

where equality holds only if $(\Lambda_1 \oplus \Lambda_2) \otimes_{O_F} \mathbb{F}_q = (\Lambda_1 \otimes_{O_F} \mathbb{F}_q) \oplus (\Lambda_2 \otimes_{O_F} \mathbb{F}_q)$ is semisimple (by Lemma 2.14(ii)) and equality holds in (6) for $W = W' = W_1$ and $W = W' = W_2$. This implies (ii).

Assertion (ii) implies (iii.a) trivially and (iii.b) by setting $W = W_1 + W_2$.

Given assertion (iii), if $W_1^{a_1} \oplus \cdots \oplus W_k^{a_k}$ is a decomposition of V into pairwise nonisomorphic Γ-representations, then choosing for each summand $W_i^{a_i}$ a Γ-stable lattice of the form $\Lambda_i^{a_i}$ and setting $\Lambda = \sum_i \Lambda_i^{a_i}$, we see that $\Lambda \otimes_{O_F} \mathbb{F}_q$ is a direct sum of isotypic semisimple representations $(\Lambda_i \otimes_{O_F} \mathbb{F}_q)^{a_i}$, where the representations $\Lambda_i \otimes_{O_F} \mathbb{F}_q$ are pairwise without common irreducible factor. Thus, $\Lambda \otimes_{O_F} \mathbb{F}_q$ is semisimple, and

$$\begin{aligned}\dim_F(\mathrm{End}_\Gamma V) &= \sum_{i=1}^k a_i^2 \dim_F(\mathrm{End}_\Gamma W_i) = \sum_{i=1}^k a_i^2 \dim_{\mathbb{F}_q}\big(\mathrm{End}_\Gamma(\Lambda_i \otimes_{O_F} \mathbb{F}_q)\big) \\ &= \dim_{\mathbb{F}_q}\big(\mathrm{End}_\Gamma(\Lambda \otimes_{O_F} \mathbb{F}_q)\big).\end{aligned}$$

□

COROLLARY 2.16

Let V be a finite-dimensional vector space over $\mathbb{Q}_\ell$, and let Γ be a compact subgroup of GL(V) *which acts semisimply on V. The following assertions are equivalent.*

(i) *For some Γ-stable lattice Λ of V, we have*

$$\dim_{\mathbb{Q}_\ell}(\mathrm{End}_\Gamma V) = \dim_{\mathbb{F}_\ell}\big(\mathrm{End}_\Gamma(\Lambda \otimes \mathbb{F}_\ell)^{\mathrm{ss}}\big).$$

(ii) *If F is a finite extension of $\mathbb{Q}_\ell$ with residue field $\mathbb{F}_q$ such that every irreducible Γ-subrepresentation of $V \otimes F$ is absolutely irreducible, then the following two assertions hold.*
   (a) *If W is an irreducible Γ-subrepresentation of $V \otimes F$ and Λ_W is a Γ-stable O_F-lattice of W, then $\Lambda_W \otimes_{O_F} \mathbb{F}_q$ is absolutely irreducible.*
   (b) *If W_1 and W_2 are nonisomorphic irreducible Γ-subrepresentations of $V \otimes F$ and Λ_1 and Λ_2 are Γ-stable O_F-lattices of W_1 and W_2, respectively, then $\Lambda_1 \otimes_{O_F} \mathbb{F}_q$ and $\Lambda_2 \otimes \mathbb{F}_q$ are not isomorphic.*

Proof

Let F be the finite extension of $\mathbb{Q}_\ell$ in assertion (ii). Tensoring by O_F over $\mathbb{Z}_\ell$, we see by Lemma 2.12 that assertion (i) is equivalent to assertion (i) of Proposition 2.15. Regarding Γ as a subgroup of $\mathrm{Aut}_F(V \otimes F)$, by absolute irreducibility, assertions (iii.a) and (iii.b) of Proposition 2.15 correspond to assertions (ii.a) and (ii.b), respectively. □

LEMMA 2.17

Let F be a finite totally ramified extension of $\mathbb{Q}_\ell$ with ring of integers O_F and residue *field $\mathbb{F}_\ell$, let Λ be a finitely generated free O_F-module, and let $\Gamma \subset \mathrm{Aut}_{O_F} \Lambda$ be a closed subgroup such that the action of Γ on $\Lambda \otimes \mathbb{F}_\ell$ is semisimple. Then*

$$\ker\bigl(\Gamma \to \mathrm{Aut}_{\mathbb{F}_\ell}(\Lambda \otimes \mathbb{F}_\ell)\bigr)$$

is the maximal normal pro-ℓ subgroup of Γ.

Proof

The kernel is a closed subgroup of the pro-ℓ group

$$\ker\bigl(\mathrm{Aut}_{O_F} \Lambda \to \mathrm{Aut}_{\mathbb{F}_\ell}(\Lambda \otimes \mathbb{F}_\ell)\bigr)$$

and, therefore, is again pro-ℓ. So it suffices to prove it is maximal among normal pro-ℓ subgroups of Γ. If not, the image of any normal pro-ℓ subgroup not contained in the kernel is a nontrivial normal ℓ-subgroup of the image of $\Gamma \to \mathrm{Aut}_{\mathbb{F}_\ell}(\Lambda \otimes \mathbb{F}_\ell)$. However, a subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$ which acts semisimply cannot have a nontrivial normal ℓ-subgroup, since a semisimple representation of an ℓ-group over $\mathbb{F}_\ell$ is necessarily trivial. ☐

2.6. Formal characters and regular elements

2.6.1

We work over a field F of any characteristic. Suppose at first that F is algebraically closed. Let $\mathbf{T} \subset \mathrm{GL}_n$ be a torus of rank r. By *weights of* $\mathbf{T}$, we mean the weights of the ambient representation $\mathbf{T} \to \mathrm{GL}_n$, that is, the characters $\chi \in X^*(\mathbf{T})$ appearing in the decomposition of the ambient representation into irreducible factors. We define m_χ to be the multiplicity of the weight χ and $\sum_\chi m_\chi[\chi] \in \mathbb{Z}[X^*(\mathbf{T})]$ to be the *formal character of* $\mathbf{T}$ (as a subgroup of GL_n).

For $N \in \mathbb{N}$, let I_N denote the set of integers in the interval $[-N, N]$. Given an isomorphism $i : \mathbb{Z}^r \to X^*(\mathbf{T})$, the formal character is *bounded by N with respect to i* if $m_\chi > 0$ only for $\chi \in i(I_N^r)$. We say it is *bounded by N* if this is true for some choice of i (see [20, Definition 4]). In this case we say that $\mathbf{T}$ is an *N-bounded* torus.

For any connected algebraic subgroup $\mathbf{G}$ of $\mathrm{GL}_{n,F}$, we define the *formal character of* $\mathbf{G}$ as the formal character of any maximal torus $\mathbf{T} \subset \mathbf{G}$, and we say that $\mathbf{G}$ is N-bounded if $\mathbf{T}$ is N-bounded; since the maximal tori of $\mathbf{G}$ are conjugate to one another, this does not depend on the choice of $\mathbf{T}$. We say the formal characters of connected algebraic subgroups $\mathbf{G}_1 \subset \mathrm{GL}_{n,F_1}$ and $\mathbf{G}_2 \subset \mathrm{GL}_{n,F_2}$ (where F_1 and F_2 may even have different characteristics) are the *same* if there exist maximal tori $\mathbf{T}_1$, $\mathbf{T}_2$ (of $\mathbf{G}_1$, $\mathbf{G}_2$, respectively) and an isomorphism $X^*(\mathbf{T}_1) \to X^*(\mathbf{T}_2)$ mapping the formal character of $\mathbf{T}_1 \subset \mathrm{GL}_{n,F_1}$ to that of $\mathbf{T}_2 \subset \mathrm{GL}_{n,F_2}$. This is equivalent to the existence of $g_1 \in \mathrm{GL}_n(F_1)$ and $g_2 \in \mathrm{GL}_n(F_2)$ such that $g_1^{-1}\mathbf{T}_1 g_1 \subset \mathbb{G}_{m,F_1}^n$ and

$g_2^{-1}\mathbf{T}_2 g_2 \subset \mathbb{G}_{m,F_2}^n$ are diagonal tori cut out by the same set of characters in $X^*(\mathbb{G}_m^n)$ (see [20, Proposition 2.0.1]). If F is not algebraically closed, the formal character of $\mathbf{G} \subset \mathrm{GL}_{n,F}$ is defined to be the formal character of $\mathbf{G}_{\overline{F}} \subset \mathrm{GL}_{n,\overline{F}}$, and we say the former is N-bounded if the latter is N-bounded.

2.6.2
Let $\mathbf{T} \subset \mathbb{G}_m^n$ be a rank r diagonal torus over a field F. The corresponding map of character groups $f : \mathbb{Z}^n = X^*(\mathbb{G}_m^n) \to X^*(\mathbf{T})$ determines an ordered n-tuple $(f(e_1), \ldots, f(e_n)) \in X^*(\mathbf{T})^n$, where the e_i's are the standard generators of $\mathbb{Z}^n$. The number of occurrences of a character $\chi \in X^*(\mathbf{T})$ in this n-tuple equals m_χ, so $\mathbf{T}$ determines the element of $X^*(\mathbf{T})^n$ up to permutation. As $\ker f$ is finitely generated, there exists $M \in \mathbb{N}$ such that $I_M^n \cap \ker f$ generates $\ker f$. This is equivalent to the fact that $\mathbf{T}$ is the intersection in $\mathbb{G}_m^n$ of $\ker \chi$ over some collection of $\chi \in I_M^n \subset \mathbb{Z}^n = X^*(\mathbb{G}_m^n)$. There are finitely many homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^r$ sending each e_i to an element of I_N^r, so there exists a constant $C_8(n, N)$ depending only on $n, N \in \mathbb{N}$ (independent of the field F) such that this property holds for all N-bounded subtori of $\mathbb{G}_m^n$ whenever $M \geq C_8(n, N)$.

Let $\mathbf{T} \subset \mathrm{GL}_{n,F}$ be a torus, and let $t \in \mathbf{T}(F)$. There exists $g \in \mathrm{GL}_n(\overline{F})$ such that $g^{-1}\mathbf{T}g \subset \mathbb{G}_m^n$ is a diagonal torus. If $h \in \mathrm{GL}_n(\overline{F})$ is another element such that $h^{-1}\mathbf{T}h \subset \mathbb{G}_m^n$, then a permutation of coordinates of $\mathbb{G}_m^n$ maps $g^{-1}\mathbf{T}g$ to $h^{-1}\mathbf{T}h$. We say that $t \in \mathbf{T}(F)$ is *m-regular* if whenever $\chi \in I_m^n \subset X^*(\mathbb{G}_m^n)$ is a character such that $g^{-1}tg \in \ker \chi$, then $g^{-1}\mathbf{T}g \subset \ker \chi$. As I_m^n is stable under permutation of coordinates, this does not depend on the choice of g conjugating $\mathbf{T}$ into a diagonal torus.

If $\mathbf{T}$ is a maximal torus of a connected reductive subgroup $\mathbf{G} \subset \mathrm{GL}_{n,F}$ and $t \in \mathbf{T}(F)$ is 1-regular, then t is a regular semisimple element of $\mathbf{G}$; this follows from the fact that the adjoint representation of $\mathbf{G}$ is a subrepresentation of the restriction to $\mathbf{G}$ of the adjoint representation of $\mathrm{GL}_{n,F}$. If $t \in \mathbf{G}(F)$ is regular semisimple, we say that t is m-regular if t is m-regular with respect to the unique maximal torus $\mathbf{T} \subset \mathbf{G}$ (defined over F) containing t. The following lemmas are fundamental.

LEMMA 2.18
Let F_1, F_2 be fields, and let $M \geq C_8(n, N)$ be an integer. If $\mathbf{T}_1 \subset \mathbb{G}_{m,F_1}^n$ and $\mathbf{T}_2 \subset \mathbb{G}_{m,F_2}^n$ are N-bounded diagonal tori of the same rank and $t \in \mathbf{T}_1(F_1)$ is an M-regular element such that, for all $\chi \in I_M^n \cap X^*(\mathbb{G}_m^n)$, the inclusion $\mathbf{T}_2 \subset \ker \chi$ implies that $\chi(t) = 1$, then $\mathbf{T}_1$ and $\mathbf{T}_2$ are cut out by the same set of characters in $I_M^n \cap X^*(\mathbb{G}_m^n)$.

Proof
The proof is immediate. □

LEMMA 2.19

Let F be a field, and let $M \geq C_8(n, N)$ be an integer. If $\mathbf{T}_1$ and $\mathbf{T}_2$ are N-bounded tori of $\mathrm{GL}_{n,F}$ and $t \in \mathbf{T}_1(F) \cap \mathbf{T}_2(F)$ is M-regular for $\mathbf{T}_1$, then $\mathbf{T}_1 \subset \mathbf{T}_2$.

Proof

Without loss of generality, we may assume that F is algebraically closed, $\mathbf{T}_2 \subset \mathbb{G}_m^n$, and $g^{-1}\mathbf{T}_1 g \subset \mathbb{G}_m^n$ for some $g \in \mathrm{GL}_n(F)$. As $g^{-1}tg$ and t are both diagonal, g can be written as zh, where z commutes with t and h normalizes $\mathbb{G}_m^n$ in GL_n. Thus, we may take $g = z$. Since $t \in z^{-1}\mathbf{T}_1 z \subset \mathbb{G}_m^n$ is at least 1-regular, it follows that $\mathbf{T}_1$ is also diagonal. Since t is M-regular, every character $\chi \in I_M^n \cap X^*(\mathbb{G}_m^n)$ which annihilates $\mathbf{T}_2$ sends t to 1 and is therefore trivial on $\mathbf{T}_1$. By the definition of M, we obtain $\mathbf{T}_1 \subset \mathbf{T}_2$. □

2.6.3

Let A be an abelian group, and let B be a subgroup of A. The *saturation of* B is the subgroup of elements $a \in A$ such that $ma \in B$ for some nonzero $m \in \mathbb{Z}$. If B is equal to its saturation, then B is said to be *saturated*. We focus on algebraic groups over finite fields $\mathbb{F}_\ell$.

PROPOSITION 2.20

If ℓ and N are sufficiently large in terms of n, then every exponentially generated subgroup of $\mathrm{GL}_{n,\mathbb{F}_\ell}$ has N-bounded formal character.

Proof

Let $\mathbf{T} \subset \mathbb{G}_m^n$ be a torus of rank r over an algebraically closed field F with $f : X^*(\mathbb{G}_m^n) \to X^*(\mathbf{T})$. For $\Sigma \subset \{1, \ldots, n\}$ of cardinality r, let $\mathbf{T}_\Sigma \subset \mathbb{G}_m^n$ be the rank $n - r$ torus with 1 in the σ-coordinate for all $\sigma \in \Sigma$ with $f_\Sigma : X^*(\mathbb{G}_m^n) \to X^*(\mathbf{T}_\Sigma)$. The fiber product over $\mathbb{G}_m^n$ of two subtori is cut out by the sum of the subgroups of $X^*(\mathbb{G}_m^n)$ cutting out each of the tori. The closed subscheme $\mathbf{C}$ of $\mathbb{G}_m^n$ cut out by a subgroup of $X^*(\mathbb{G}_m^n)$ of index D is reduced and satisfies $|\mathbf{C}(F)| = D$ if and only if D is not divisible by the characteristic of F. If $\mathbf{T} \times_{\mathbb{G}_m^n} \mathbf{T}_\Sigma$ is reduced, then $|\mathbf{T}(F) \cap \mathbf{T}_\Sigma(F)|$ and the index

$$\left[\Lambda^n X^*(\mathbb{G}_m^n) : \Lambda^r(\ker f) \wedge \Lambda^{n-r}(\ker f_\Sigma)\right]$$

have the same cardinality. Thus, if B is fixed, $\mathbf{T} \times_{\mathbb{G}_m^n} \mathbf{T}_\Sigma$ is reduced for all Σ, and $|\mathbf{T}(F) \cap \mathbf{T}_\Sigma(F)| \leq B$ for all Σ for which the intersection is finite, then there are only a finite number of possibilities for the top exterior power $\Lambda^r(\ker f) \subset \Lambda^r X^*(\mathbb{G}_m^n)$ and therefore a finite number of possibilities for $(\ker f) \otimes \mathbb{Q}$ as a subspace of $X^*(\mathbb{G}_m^n) \otimes \mathbb{Q} = \mathbb{Q}^n$. As $\ker f$ is saturated, $(\ker f) \otimes \mathbb{Q}$ determines $\ker f$ as a subgroup of $X^*(\mathbb{G}_m^n)$ and therefore determines the formal character of $\mathbf{T}$.

By [28, Proposition 3] and the fact that Hilbert schemes are of finite type, the exponentially generated subgroups of $\mathrm{GL}_{n,\mathbb{F}_\ell}$ for all sufficiently large ℓ form a constructible family $\mathcal{G} \subset \mathrm{GL}_{n,\mathcal{S}}$ in the sense of [30]; that is, $\mathcal{G}$ is a closed subscheme of the general linear scheme over a scheme $\mathcal{S}$ of finite type over $\operatorname{Spec} \mathbb{Z}$ and, for every algebraically closed field F of characteristic 0 or sufficiently large positive characteristic, every exponentially generated subgroup of $\mathrm{GL}_{n,F}$ is of the form $\mathcal{G}_x$ for some $x \in \mathcal{S}(F)$.

Let $\mathcal{T}$ denote the closed subscheme of $\mathrm{GL}_{n,\mathcal{S}}$ consisting of diagonal matrices, and for any subset $\Sigma \subset \{1, 2, \ldots, n\}$, let $\mathcal{T}_\Sigma$ denote the closed subscheme of $\mathcal{T}$ for which the σ-coordinate is 1 for all $\sigma \in \Sigma$. The fiber product $\mathcal{G} \times_{\mathrm{GL}_{n,\mathcal{S}}} \mathcal{T}_\Sigma$ is a group scheme over $\mathcal{S}$ and, therefore, reduced over every point in characteristic 0 and, therefore, reduced over every point in sufficiently large finite characteristic (see [17, Théorème 9.7.7(iii)]). Moreover, by [17, Corollaire 9.7.9], there is an upper bound for the cardinality of any finite fiber, and this implies that there are only finitely many possibilities for the formal character of any fiber of $\mathcal{G}$. □

PROPOSITION 2.21

There exists a constant $C_9(r,k,N)$ depending only on $r,k,N \in \mathbb{N}$ such that, if $\underline{T}$ is a rank r torus over $\mathbb{F}_\ell$ with $\ell > C_9(r,k,N)$ and the $\mathrm{Gal}_{\mathbb{F}_\ell}$-orbit of $\chi \in X^*(\underline{T}_{\overline{\mathbb{F}}_\ell})$ is N-bounded with respect to some isomorphism $i : \mathbb{Z}^r \to X^*(\underline{T}_{\overline{\mathbb{F}}_\ell})$, then

$$\left|\left\{t \in \underline{T}(\mathbb{F}_\ell) \;\middle|\; \chi(t) = 1\right\}\right| < k^{-1}\left|\underline{T}(\mathbb{F}_\ell)\right|. \tag{7}$$

Proof

Let X be the subgroup of $X^*(\underline{T}_{\overline{\mathbb{F}}_\ell})$ generated by the Galois orbit O_χ of χ. Then the number of possibilities for $i^{-1}(X)$ is bounded by a constant depending only on N and r. Therefore, there exists a positive integer s, depending only on N and r, such that, for all ξ in the saturation of X, we have $s\xi \in X$. It follows that if $t \in \bigcap_{\xi \in X} \ker \xi$, then t^s belongs to the subtorus $\mathbf{T}_{O_\chi}$ of $\mathbf{T}_{\overline{\mathbb{F}}_\ell}$ cut out by the saturation of X. Since any element of $\underline{T}(\mathbb{F}_\ell)$ in $\ker \chi$ is in $\ker \chi^\sigma$ for all $\chi^\sigma \in O_\chi$, it follows that $\{t \in \underline{T}(\mathbb{F}_\ell) \mid \chi(t) = 1\}$ lies in the union of at most s^r translates of a proper $\mathbb{F}_\ell$-subtorus of $\underline{T}$. A proper subtorus has at most $(\ell+1)^{r-1}$ $\mathbb{F}_\ell$-points, while $|\underline{T}(\mathbb{F}_\ell)| \geq (\ell-1)^r$, and the proposition follows. □

COROLLARY 2.22

There exists a constant $C_{10}(\epsilon,m,n,N)$ depending only on $\epsilon > 0$ and $m,n,N \in \mathbb{N}$ such that if $\underline{T} \subset \mathrm{GL}_{n,\mathbb{F}_\ell}$ is an N-bounded torus with $\ell > C_{10}(\epsilon,m,n,N)$, then the number of elements of $\underline{T}(\mathbb{F}_\ell)$ which fail to be m-regular is less than $\epsilon|\underline{T}(\mathbb{F}_\ell)|$.

Proof
By taking

$$C_{10}(\epsilon, m, n, N) = \max_{0 \le r \le n} C_9\big(r, \lceil |I_m^n|/\epsilon \rceil, mN\big),$$

this follows immediately. □

PROPOSITION 2.23
There exists a constant $C_{11}(\epsilon, n, N)$ depending only on $\epsilon > 0$ and $n, N \in \mathbb{N}$ such that if $\underline{G}_1$ and $\underline{G}_2$ are N-bounded connected subgroups of $\mathrm{GL}_{n,\mathbb{F}_\ell}$ with $\underline{G}_1$ reductive, $\ell > C_{11}(\epsilon, n, N)$, and

$$\big|\underline{G}_1(\mathbb{F}_\ell) \cap \underline{G}_2(\mathbb{F}_\ell)\big| > \epsilon \big|\underline{G}_1(\mathbb{F}_\ell)\big|,$$

then $\underline{G}_1 \subset \underline{G}_2$.

Proof
Let r be the rank of $\underline{G}_1$, and let $M \ge C_8(n, N)$ be an integer. By Corollary 2.22, every maximal torus $\underline{T}$ of $\underline{G}_1$ defined over $\mathbb{F}_\ell$ contains $o((\ell+1)^r)$ elements which fail to be M-regular. Each regular semisimple element belongs to a unique $\underline{T}$, so the number of maximal tori containing a regular semisimple element defined over $\mathbb{F}_\ell$ is $O(\ell^{-r}|\underline{G}_1(\mathbb{F}_\ell)|)$. Thus, the number of regular semisimple elements of $\underline{G}_1(\mathbb{F}_\ell)$ which are not M-regular is $o(|\underline{G}_1(\mathbb{F}_\ell)|)$. By Lang–Weil and the fact that the root datum of the connected reductive $\underline{G}_{1,\overline{\mathbb{F}}_\ell}$ has finitely many possibilities (depending on n), the number of elements of $\underline{G}_1(\mathbb{F}_\ell)$ which are not regular semisimple is also $o(|\underline{G}_1(\mathbb{F}_\ell)|)$. We conclude that if ℓ is sufficiently large, then more than $(1-\epsilon/2)|\underline{G}_1(\mathbb{F}_\ell)|$ elements x of $\underline{G}_1(\mathbb{F}_\ell)$ are regular semisimple and are M-regular and, therefore, do not lie in $\underline{G}_2(\mathbb{F}_\ell)$ unless the unique maximal torus of $\underline{G}_1$ containing x is contained in some maximal torus of $\underline{G}_2$ by Lemma 2.19.

It follows that $\underline{G}_1$ and $\underline{G}_2$ have at least $(\epsilon/3)|\underline{G}_1(\mathbb{F}_\ell)|$ elements in common which are regular semisimple and M-regular for $\underline{G}_1$ if ℓ is sufficiently large. The group generated by the unique maximal tori of $\underline{G}_1$ containing these elements is a closed connected subgroup of $\underline{G}_1 \cap \underline{G}_2$ containing at least $(\epsilon/3)|\underline{G}_1(\mathbb{F}_\ell)|$ regular semisimple elements. However, if it is a proper subgroup of $\underline{G}_1$, its dimension is at most $\dim \underline{G}_1 - 1$, so if ℓ is sufficiently large, it contains less than $(2/\ell)|\underline{G}_1(\mathbb{F}_\ell)|$ elements. Thus, $\underline{G}_1 \subset \underline{G}_2$. □

3. Maximality of compact subgroups

3.1. Theorem 1.5
The main goal of this section is to establish the following theorem.

THEOREM 1.5

Let $\mathbf{G} \subset \mathrm{GL}_{n,\mathbb{Q}_\ell}$ *be a connected reductive subgroup, let* $\underline{G} \subset \mathrm{GL}_{n,\mathbb{F}_\ell}$ *be a connected reductive subgroup with* $\underline{G}^{\mathrm{der}}$ *as derived group and* $\underline{Z}$ *as connected center, let* Γ *be a closed subgroup of* $\mathbf{G}(\mathbb{Q}_\ell) \cap \mathrm{GL}_n(\mathbb{Z}_\ell)$*, and let* $\phi\colon \Gamma \to \mathrm{GL}_n(\mathbb{F}_\ell)$ *be a semisimple continuous representation with* $G := \phi(\Gamma) \subset \underline{G}(\mathbb{F}_\ell)$*. Assume that this data satisfies the following conditions.*

(a) *The subgroup* Γ *is Zariski-dense in* $\mathbf{G}$.

(b) *There is an equality of semisimple ranks*: $\operatorname{rank} \mathbf{G}^{\mathrm{der}} = \operatorname{rank} \underline{G}^{\mathrm{der}}$.

(c) *The derived group* $\underline{G}^{\mathrm{der}}$ *is exponentially generated.*

(d) *For all* $\gamma \in \Gamma$*, the (mod* ℓ*) reduction of the characteristic polynomial of* γ *is the characteristic polynomial of* $\phi(\gamma)$.

(e) *The index* $[\underline{G}(\mathbb{F}_\ell) : G]$ *is bounded by* $k \in \mathbb{N}$.

(f) *The formal character of* $(\underline{Z}, \mathbb{F}_\ell^n)$ *is bounded by* $N \in \mathbb{N}$*, where* $\underline{Z}$ *is the connected center of* $\underline{G}$.

(g) *Condition* $(*)$ *holds for* Γ *and* G*; that is,*

$$\dim_{\mathbb{Q}_\ell}\bigl(\mathrm{End}_\Gamma(\mathbb{Q}_\ell^n)\bigr) = \dim_{\mathbb{F}_\ell}\bigl(\mathrm{End}_G(\mathbb{F}_\ell^n)\bigr).$$

If ℓ *is sufficiently large in terms of the data in (a)–(g), then the reduction representation* $\Gamma \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_\ell) \to \mathrm{GL}_n(\mathbb{F}_\ell)$ *and* ϕ *are conjugate,* Γ^{sc} *is a hyperspecial maximal compact subgroup of* $\mathbf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$*, and* $\mathbf{G}^{\mathrm{der}}$ *is unramified. Hypotheses (a)–(f) of Theorem 1.5 suffice to imply that* $\mathbf{G}^{\mathrm{der}}$ *splits over some finite unramified extension of* $\mathbb{Q}_\ell$ *and is unramified over every degree* 12 *totally ramified extension of* $\mathbb{Q}_\ell$.

3.2. The condition $(*)$

Suppose the conditions (a)–(f) of Theorem 1.5 hold. The goal of this section is reduce the condition $(*)$ of Theorem 1.5(g) to the *semisimple part*, that is, the condition $(*')$ in Proposition 3.2.

PROPOSITION 3.1

There exists a constant $C_{12}(k,n,N)$ *depending only on* $k,n,N \in \mathbb{N}$ *such that, if* $\underline{G} \subset \mathrm{GL}_{n,\mathbb{F}_\ell}$ *is an* N*-bounded connected reductive subgroup with* $\ell > C_{12}(k,n,N)$ *and derived group* $\underline{G}^{\mathrm{der}}$ *also* N*-bounded,* G *is a subgroup of index bounded by* k *in* $\underline{G}(\mathbb{F}_\ell)$*, and* $\underline{S}$ *is the Nori group of* G*, then the following assertions hold.*

(i) *The ambient representation* $G \to \mathrm{GL}_n(\mathbb{F}_\ell)$ *is semisimple.*

(ii) *The ambient representation* $\underline{G} \to \mathrm{GL}_{n,\mathbb{F}_\ell}$ *is semisimple.*

(iii) *The derived group of* $\underline{G}$ *is* $\underline{S}$.

(iv) *The commutant of* G *in* $M_n(\mathbb{F}_\ell)$ *consists of the* $\mathbb{F}_\ell$*-points of the commutant of* $\underline{G}$ *in* $M_{n,\mathbb{F}_\ell}$.

Parts (i) and (ii) hold without the N*-bounded assumption.*

Proof
If $\ell > k$, then $G \cap \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$ contains $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)[\ell]$ and therefore $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)^+$. If ℓ is sufficiently large in terms of n, then every characteristic ℓ representation of $\underline{G}(\mathbb{F}_\ell)$ is semisimple (see [24]). The restriction of a semisimple representation to a normal subgroup is always semisimple, so $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)^+$ acts semisimply on $\mathbb{F}_\ell^n$. As $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)^+$ is normal in $G \cap \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$ and of prime-to-ℓ index, by [11, Section 10, Exercise 8], the latter also acts semisimply on $\mathbb{F}_\ell^n$. On the other hand, $G \cap \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$ is the kernel of a homomorphism from G to the group of $\mathbb{F}_\ell$-points of the torus $\underline{G}/\underline{G}^{\mathrm{der}}$. It is therefore a normal subgroup of prime-to-ℓ index in G, so by applying [11, Section 10, Exercise 8] again, G acts semisimply on $\mathbb{F}_\ell^n$, proving (i).

Part (ii) is true for any connected reductive algebraic group if ℓ is large compared to n (see [27, Theorem 3.5] when $\underline{S}$ is semisimple and [24] in general).

For part (iii), we note that the formal character of $\underline{G}^{\mathrm{der}}$ is bounded by hypothesis, while the formal character of $\underline{S}$ is bounded by Proposition 2.20. As $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)^+ = \underline{S}(\mathbb{F}_\ell)^+$, this group is of bounded index in both $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$ and $\underline{S}(\mathbb{F}_\ell)$. By Lemma 2.7 and Proposition 2.9(iii), this implies that

$$\dim(\underline{G}^{\mathrm{der}}) = \dim_\ell\big(\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)\big) = \dim_\ell\big(\underline{S}(\mathbb{F}_\ell)^+\big) = \dim_\ell\big(\underline{S}(\mathbb{F}_\ell)\big) = \dim \underline{S}$$

for sufficiently large ℓ. Thus, Proposition 2.23 gives $\underline{G}^{\mathrm{der}} = \underline{S}$.

Let $x \in M_n(\mathbb{F}_\ell)$ commute with G, and let its centralizer in $\mathrm{GL}_{n,\mathbb{F}_\ell}$ be $\underline{Z}_x$. Then $G \subset \underline{G}(\mathbb{F}_\ell) \cap \underline{Z}_x(\mathbb{F}_\ell)$. Now, $\underline{Z}_x$ is the complement in a linear subvariety of $n \times n$ matrices of the zero locus of the determinant, so it is irreducible. Moreover, centralizers form a constructible family, so their formal characters are N-bounded (e.g., by the proof of Proposition 2.20). Part (iv) follows by applying Proposition 2.23 as $\underline{G}_1 = \underline{G}$ and $\underline{G}_2$ ranges over all groups $\underline{Z}_x$. □

PROPOSITION 3.2
Under the hypotheses of Theorem 1.5, there exists a constant $C_{13}(k,n,N)$ depending only on $k,n,N \in \mathbb{N}$ such that, if $\ell > C_{13}(k,n,N)$, then the following statements hold.

(i) *For any finite extension F of $\mathbb{Q}_\ell$ with uniformizer π and residue field $\mathbb{F}_{\ell^f}$ and any $\mathcal{O}_F$-lattice Λ of $F^n := \mathbb{Q}_\ell^n \otimes_{\mathbb{Q}_\ell} F$ fixed by Γ, the reduction representation*

$$\Gamma \hookrightarrow \mathrm{GL}(\Lambda) \to \mathrm{GL}(\Lambda/\pi\Lambda)$$

is isomorphic to $\phi \otimes \mathbb{F}_{\ell^f}$ and thus semisimple.

(ii) *The formal characters of $\underline{G}^{\mathrm{der}}$ and $\mathbf{G}^{\mathrm{der}}$ coincide.*

(iii) *The commutator subgroup G' acts semisimply on $\mathbb{F}_\ell^n$ and*

$$\dim_{\mathbb{Q}_\ell}\big(\mathrm{End}_{\Gamma'}(\mathbb{Q}_\ell^n)\big) = \dim_{\mathbb{F}_\ell}\big(\mathrm{End}_{G'}(\mathbb{F}_\ell^n)\big), \qquad (*')$$

where Γ' is the commutator subgroup of Γ, that is, the closure of the group generated by commutators.

Proof

For assertion (i), the Brauer–Nesbitt theorem and Theorem 1.5(d) imply that the semisimplification of $(\Gamma, \Lambda/\pi\Lambda)$ is isomorphic to $\phi \otimes \mathbb{F}_{\ell^f}$. Lemma 2.13 (for $k = 1$) and Lemma 2.14(i) produce the inequalities

$$\dim\bigl(\mathrm{End}_\Gamma(F^n)\bigr) \le \dim\bigl(\mathrm{End}_\Gamma(\Lambda/\pi\Lambda)\bigr) \le \dim\bigl(\mathrm{End}_G(\mathbb{F}^n_{\ell^f})\bigr),$$

which, by Lemma 2.12 and $(*)$, are actually equalities. Then the $\mathbb{F}_{\ell^f}$-representation $\Lambda/\pi\Lambda$ of Γ is semisimple by Lemma 2.14(ii).

For parts (ii) and (iii), we first note that, for some N' depending only on n and N, the three groups $\underline{G}^{\mathrm{der}}$, $\underline{G}$, and $\mathbf{G}^{\mathrm{der}}$ are N'-bounded. Indeed, the N'-boundedness of the first is due to Theorem 1.5(c) and Proposition 2.20; the N'-boundedness of the second is due to the N'-boundedness of the first and Theorem 1.5(f); and the N'-boundedness of the third follows since, in characteristic 0, by the Weyl dimension formula, there are only finitely many possibilities for formal characters of semisimple groups which admit a faithful n-dimensional representation.

Now for (ii), choose an integer $M \ge C_8(n, N')$ (defined in Section 2.6.2). Let $\underline{T}$ be a maximal torus of $\underline{G}^{\mathrm{der}}$. Then the index $[\underline{T}(\mathbb{F}_\ell) : \underline{T}(\mathbb{F}_\ell) \cap G']$ is bounded by a constant depending only on k and n. By Corollary 2.22, if ℓ is sufficiently large in terms of n, N', M, and k, then there exists $g \in \underline{T}(\mathbb{F}_\ell) \cap G'$ which is M-regular in $\underline{T}$. Let $\gamma \in \Gamma' \subset \mathbf{G}^{\mathrm{der}}(\mathbb{Q}_\ell)$ be any lift of g, and let $\gamma_{\mathrm{ss}} \in \mathbf{G}^{\mathrm{der}}(\mathbb{Q}_\ell)$ be its semisimple part. Let $\mathbf{T}$ denote a maximal torus of $\mathbf{G}^{\mathrm{der}}$ which contains γ_{ss}. Let $\mathbf{T}_{\mathrm{GL}}$ be a maximal torus of $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ containing $\mathbf{T}$, and let $h \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be an element such that $h^{-1}\mathbf{T}_{\mathrm{GL}}h$ is diagonal. Thus,

$$h^{-1}\gamma_{\mathrm{ss}}h = \mathrm{diag}(\lambda_1, \dots, \lambda_n), \tag{8}$$

where the λ_i's are the eigenvalues of γ_{ss}. They are integral over $\mathbb{Z}_\ell$, so they reduce to $\bar{\lambda}_1, \dots, \bar{\lambda}_n \in \overline{\mathbb{F}}_\ell$, the eigenvalues of g. Define

$$\mathbf{T}_2 := h^{-1}\mathbf{T}h \tag{9}$$

as the diagonal torus, and define $\mathbf{T}_1$ to be some diagonalization of $\underline{T}_{\overline{\mathbb{F}}_\ell}$ so that $g \in \underline{T}(\mathbb{F}_\ell)$ goes to $\mathrm{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_n)$. Since $\mathbf{T}_1$ and $\mathbf{T}_2$ have the same rank by Theorem 1.5(b), it follows by Lemma 2.18 that they are cut out by the same set of characters in $I^n_M \cap X^*(\mathbb{G}^n_m)$, which implies (ii).

To prove (iii), we use Corollary 2.16 to replace $(*)$ and $(*')$ by assertions 2.16(ii.a) and 2.16(ii.b). We fix a finite extension F of $\mathbb{Q}_\ell$ over which F^n decomposes as a direct sum of absolutely irreducible representations for Γ and Γ'. By Zariski density, any decomposition of F^n as a direct sum of irreducible $\mathbf{G}$-representations gives a decomposition into irreducible Γ-representations, and likewise, a decomposition into $\mathbf{G}^{\mathrm{der}}$-irreducibles gives a decomposition into Γ'-irreducibles. As every

$\mathbf{G}$-irreducible restricts to a $\mathbf{G}^{\mathrm{der}}$-irreducible, the same is true for Γ-irreducibles and Γ'-irreducibles.

By hypothesis, G is of bounded index in $\mathbf{G}(\mathbb{F}_\ell)$. Thus, $G \cap \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$ is of bounded index in $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$, and its inverse image in $\underline{G}^{\mathrm{sc}}(\mathbb{F}_\ell)$ is of bounded index and therefore equal to $\underline{G}^{\mathrm{sc}}(\mathbb{F}_\ell)$ if ℓ is sufficiently large. Thus, G' contains the image of $\underline{G}^{\mathrm{sc}}(\mathbb{F}_\ell) \to \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$, which is of bounded index in $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$. Applying Proposition 3.1 and Lemma 2.14(iv) to $G \subset \underline{G}(\mathbb{F}_\ell)$ and $G' \subset \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$, we conclude that an $\overline{\mathbb{F}}_\ell$-subspace of $\overline{\mathbb{F}}_\ell^n$ is invariant and irreducible for G if and only if it is so for $\underline{G}(\overline{\mathbb{F}}_\ell)$ if and only if it is so for $\underline{G}^{\mathrm{der}}(\overline{\mathbb{F}}_\ell)$ if and only if it is so for G'. Hence, we obtain Corollary 2.16(ii.a) for Γ'.

For Corollary 2.16(ii.b), it suffices to show that if $W_1 \ncong W_2$ are irreducible subrepresentations of Γ' (equivalently $\mathbf{G}^{\mathrm{der}}$) in $\overline{\mathbb{Q}}_\ell^n$, then their reductions as irreducible representations of G' are nonisomorphic for ℓ larger than some constant depending only on k, n, N. Since $\mathrm{diag}(\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$ (the reduction of (8)) and the diagonal torus in (9) are annihilated by the same set of characters in I_M^n, the actions of γ_{ss} on the reductions of W_1 and W_2 are isomorphic if and only if the actions of $\mathbf{T}$ (a maximal torus of $\mathbf{G}^{\mathrm{der}}$) on W_1 and W_2 are isomorphic. We are done. □

PROPOSITION 3.3

Let $\Gamma \subset \mathrm{GL}_n(\mathbb{Q}_\ell)$ be a compact subgroup, let $\Lambda \subset \mathbb{Q}_\ell^n$ be a Γ-stable lattice, let Δ be a closed normal subgroup of Γ, and let $\gamma \in \Gamma$. Assume the following conditions hold:

(a) *γ is a semisimple element of $\mathrm{GL}_n(\mathbb{Q}_\ell)$;*

(b) *every element in $M_n(\mathbb{Q}_\ell)$ which commutes with Δ and with γ commutes with Γ;*

(c) *if $\lambda_1, \lambda_2 \in \overline{\mathbb{Q}}_\ell$ are distinct eigenvalues of γ, then $\lambda_1 - \lambda_2$ is an ℓ-adic unit.*

Then

$$\dim_{\mathbb{Q}_\ell}\big(\mathrm{End}_\Delta(\mathbb{Q}_\ell^n)\big) = \dim_{\mathbb{F}_\ell}\big(\mathrm{End}_\Delta(\Lambda \otimes \mathbb{F}_\ell)^{\mathrm{ss}}\big) \tag{10}$$

implies that

$$\dim_{\mathbb{Q}_\ell}\big(\mathrm{End}_\Gamma(\mathbb{Q}_\ell^n)\big) = \dim_{\mathbb{F}_\ell}\big(\mathrm{End}_\Gamma(\Lambda \otimes \mathbb{F}_\ell)^{\mathrm{ss}}\big). \tag{11}$$

Proof

The left-hand side of (11) is the dimension of the centralizer of Γ in $M_n(\mathbb{Q}_\ell)$, which by (b) is the dimension of the centralizer of γ in $\mathrm{End}_\Delta(\mathbb{Q}_\ell^n)$. By (a), this is the dimension of the 1-eigenspace of γ acting on $\mathrm{End}_\Delta(\mathbb{Q}_\ell^n) \subset M_n(\mathbb{Q}_\ell)$ by conjugation.

Defining $M := \mathrm{End}_\Delta \Lambda$, we have $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \mathrm{End}_\Delta(\mathbb{Q}_\ell^n)$. Equation (10) and Lemma 2.14(ii) imply that Δ is semisimple on $\Lambda \otimes \mathbb{F}_\ell$. Conditions (a) and (c) imply that γ is semisimple on $M \otimes \mathbb{F}_\ell$. Hence, the right-hand side of (11) is bounded above by the $\mathbb{F}_\ell$-dimension of the 1-eigenspace of γ acting on $M \otimes \mathbb{F}_\ell$. By (c) and the first paragraph, this is equal to the left-hand side of (11), which implies that the two are equal. □

3.3. *Proof of Theorem 1.5*

Proof of Theorem 1.5
We assume that $\ell > k$, which means that every element of $\underline{G}(\mathbb{F}_\ell)$ of order ℓ lies in G. As $\underline{G}(\mathbb{F}_\ell)/\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$ has prime-to-ℓ order,

$$G[\ell] = \underline{G}(\mathbb{F}_\ell)[\ell] = \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)[\ell],$$

so by Theorems 1.5(c) and 2.1(iii), the Nori group of G equals $\underline{G}^{\mathrm{der}}$. As G acts semisimply on $\mathbb{F}_\ell^n$, its maximal normal ℓ-subgroup is trivial. The composition $\Gamma \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_\ell) \to \mathrm{GL}_n(\mathbb{F}_\ell)$ is a semisimple representation by Corollary 2.16, so by Theorem 1.5(d) and Brauer–Nesbitt, it is conjugate to ϕ.

We now suppose the theorem known in the case in which $\mathbf{G}$ and $\underline{G}$ are semisimple. We defined Γ' to be the topological group generated by commutators in Γ, but in fact every element of Γ' is a finite product of commutators. Indeed, the commutator morphism $\mathbf{G} \times \mathbf{G} \to \mathbf{G}^{\mathrm{der}}$ factors through $\mathbf{G}^{\mathrm{ss}} \times \mathbf{G}^{\mathrm{ss}}$. Now Γ^{ss} is a compact Zariski-dense subgroup of the $\mathbb{Q}_\ell$-points of a semisimple algebraic group, so it is open, by a theorem of Chevalley. As the generalized commutator morphism $\mathbf{G}^{\mathrm{ss}} \times \mathbf{G}^{\mathrm{ss}} \to \mathbf{G}^{\mathrm{der}}$ is dominant, the implicit function theorem implies that the set of commutators of elements of Γ^{ss} in $\mathbf{G}^{\mathrm{der}}(\mathbb{Q}_\ell)$ has nonempty interior. It follows that every element in Γ' can be written as a finite product of commutators.

If $\phi'\colon \Gamma' \to G$ denotes the restriction of ϕ to Γ', it follows that $\phi'(\Gamma') = G'$. Note that ϕ' is semisimple, since G' is a normal subgroup of G, and the restriction of a semisimple representation to a normal subgroup is again semisimple. Conditions 1.5(a)–1.5(d) for $(\mathbf{G}^{\mathrm{der}}, \underline{G}^{\mathrm{der}}, \Gamma', G', \phi')$ are immediate from the same conditions for $(\mathbf{G}, \underline{G}, \Gamma, G, \phi)$, whereas Theorem 1.5(f) is trivial. By Proposition 3.2, Theorem 1.5(g) for Γ' and G' follows from Theorem 1.5(g) for Γ and G.

For Theorem 1.5(e), we note that $G \cap \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$ is of index at most k in $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$. By assuming $\ell > k$, we can ensure that G contains all elements of $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)[\ell]$, so $G \supset \underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)^+$. The index of $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)^+$ in $\underline{G}^{\mathrm{der}}(\mathbb{F}_\ell)$ is bounded by 2^{n-1} by Theorem 2.1(ii). Therefore, at the cost of replacing the index k by 2^{n-1}, we may assume that all conditions 1.5(a)–1.5(g) hold for $(\mathbf{G}^{\mathrm{der}}, \underline{G}^{\mathrm{der}}, \Gamma', G', \phi')$, while $\mathbf{G}^{\mathrm{der}}$ and $\underline{G}^{\mathrm{der}}$ are semisimple.

Applying the theorem in the semisimple case, we conclude that the inverse image of Γ' in $\mathbf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is a hyperspecial maximal compact subgroup. The central isogeny $\mathbf{G}^{\mathrm{der}} \to \mathbf{G}^{\mathrm{ss}}$ maps Γ' to $(\Gamma^{\mathrm{ss}})'$, so the inverse image of Γ^{ss} in $\mathbf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$, which is compact, contains the inverse image of Γ', which is maximal compact. This implies the theorem in the reductive case.

Thus, we may assume without loss of generality that $\mathbf{G}$ and $\underline{G}$ are semisimple. By definition, Γ^{sc} is the inverse image of Γ in $\mathbf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$. By Corollary 2.5, the isogeny $\mathbf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell) \to \mathbf{G}(\mathbb{Q}_\ell)$ has bounded cokernel, so we may replace Γ with the image of $\Gamma^{\mathrm{sc}} \to \Gamma$ and G with the new $\phi(\Gamma)$ at the cost of increasing k by a bounded factor. This will not affect Theorem 1.5(g); the left-hand side of $(*)$ is unchanged since Γ is Zariski-dense in $\mathbf{G}$, and $\mathbf{G}$ is connected, while the right-hand side is unchanged by Proposition 3.1(iv).

Thus, we may assume Γ^{sc} maps onto Γ, and for Γ-representations, Γ^{sc}-invariance is the same as Γ-invariance. By Lemma 2.17, G is the quotient of Γ by its maximal normal pro-ℓ subgroup. Let

$$\Pi \subset \mathbf{G}(\mathbb{Q}_\ell) \tag{12}$$

be a maximal compact subgroup containing Γ. Then $\Pi^{\mathrm{sc}} \subset \mathbf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is a maximal compact subgroup containing Γ^{sc} and fixes some vertex x_0 in the Bruhat–Tits building $\mathcal{B}(\mathbf{G}^{\mathrm{sc}}, \mathbb{Q}_\ell)$. The vertex x_0 corresponds to a group scheme $\mathcal{H}/\mathbb{Z}_\ell$ with generic fiber isomorphic to $\mathbf{G}^{\mathrm{sc}}$. When ℓ is large enough depending on k, the total ℓ-rank of Γ is equal to $\operatorname{rank} \mathbf{G}^{\mathrm{sc}}$ by Theorems 1.5(b) and 1.5(e). Lemma 2.10 and Theorem 2.11(i) imply that the total ℓ-rank of Π (and hence $\Pi^{\mathrm{sc}} = \mathcal{H}(\mathbb{Z}_\ell)$) is equal to $\operatorname{rank} \mathbf{G}^{\mathrm{sc}}$ if ℓ is large enough depending on n. Then Theorem 2.11(iv) implies that there exist some finite totally ramified extension $F/\mathbb{Q}_\ell$ and a hyperspecial maximal compact subgroup Ω of $\mathbf{G}^{\mathrm{sc}}(F)$ corresponding to a semisimple group scheme $\mathcal{J}/O_F$ (Section 2.4.1) such that

$$\Gamma^{\mathrm{sc}} \subset \mathcal{H}(\mathbb{Z}_\ell) \subset \Omega = \mathcal{J}(O_F) \subset \mathbf{G}^{\mathrm{sc}}(F).$$

As $\mathcal{J}(O_F)$ is compact, it stabilizes some O_F-lattice $\Lambda \subset F^n$. Let $\psi\colon \Gamma^{\mathrm{sc}} \to \mathrm{GL}(\Lambda \otimes \mathbb{F}_\ell) \cong \mathrm{GL}_n(\mathbb{F}_\ell)$ denote the composition of the maps

$$\Gamma^{\mathrm{sc}} \hookrightarrow \mathcal{H}(\mathbb{Z}_\ell) \hookrightarrow \mathcal{J}(O_F) \to \mathrm{GL}_{O_F} \Lambda \to \mathrm{GL}_{\mathbb{F}_\ell}(\Lambda \otimes \mathbb{F}_\ell).$$

By Brauer–Nesbitt, ψ^{ss} is equivalent to ϕ, so by Corollary 2.16, ψ is equivalent to ϕ. As Γ^{sc} and $\mathcal{J}(O_F)$ are both Zariski-dense in $\mathbf{G}_F^{\mathrm{sc}}$,

$$\begin{aligned} \dim_F \big(\mathrm{End}_{\Gamma^{\mathrm{sc}}}(\Lambda \otimes_{O_F} F)\big) &= \dim_F \big(\mathrm{End}_{\mathcal{J}(O_F)}(\Lambda \otimes F)\big) \\ &\le \dim_{\mathbb{F}_\ell} \big(\mathrm{End}_{\mathcal{J}(O_F)}(\Lambda \otimes \mathbb{F}_\ell)\big). \end{aligned} \tag{13}$$

By Theorem 1.5(g) and $\Gamma^{\mathrm{sc}} \subset \mathcal{J}(O_F)$,

$$\begin{aligned} \dim_F \big(\mathrm{End}_{\Gamma^{\mathrm{sc}}}(\Lambda \otimes_{O_F} F)\big) &= \dim_{\mathbb{F}_\ell} \big(\mathrm{End}_{\Gamma^{\mathrm{sc}}}(\Lambda \otimes \mathbb{F}_\ell)\big) \\ &\ge \dim_{\mathbb{F}_\ell} \big(\mathrm{End}_{\mathcal{J}(O_F)}(\Lambda \otimes \mathbb{F}_\ell)\big). \end{aligned} \tag{14}$$

It follows that equality holds in both (13) and (14). By Lemma 2.14(iii), $\mathcal{J}(O_F)$ acts semisimply on $\Lambda \otimes \mathbb{F}_\ell \cong \mathbb{F}_\ell^n$.

By Lemma 2.17, the image I of $\mathcal{J}(O_F)$ in $\mathrm{GL}(\Lambda \otimes \mathbb{F}_\ell)$ is the quotient of $\mathcal{J}(O_F)$ by its maximal normal pro-ℓ subgroup, which is the quotient of $\mathcal{J}(\mathbb{F}_\ell)$ by a subgroup Z of its center. So I is a subgroup of bounded index of the $\mathbb{F}_\ell$-points of the semisimple group $\mathcal{J}_{\mathbb{F}_\ell}/Z$ (isogenous to $\mathcal{J}_{\mathbb{F}_\ell}$). As the image of ψ is contained in the image of $\mathcal{J}(O_F)$ in $\mathrm{GL}(\Lambda \otimes \mathbb{F}_\ell)$, we obtain an embedding of G in I, with

$$\dim_{\mathbb{F}_\ell}\bigl(\mathrm{End}_G(\Lambda \otimes \mathbb{F}_\ell)\bigr) = \dim_{\mathbb{F}_\ell}\bigl(\mathrm{End}_I(\Lambda \otimes \mathbb{F}_\ell)\bigr) = \dim_{\mathbb{Q}_\ell}\bigl(\mathrm{End}_{\Gamma^{\mathrm{sc}}}(\mathbb{Q}_\ell^n)\bigr).$$

The image H of $\mathcal{H}(\mathbb{Z}_\ell)$ in I satisfies $G \subset H \subset I$.

Let $\underline{H}$ and $\underline{I}$ denote the Nori groups of H and I, respectively, so $\underline{G} \subset \underline{H} \subset \underline{I}$. If ℓ is sufficiently large, $\underline{I}$ is semisimple by Proposition 2.3(i) and of rank equal to $\mathrm{rk}_\ell I = \mathrm{rank}(\mathcal{J}_F/Z) = \mathrm{rank}\,\mathcal{J}_F$ by Lemma 2.7 and Propositions 2.8 and 2.9. By Proposition 3.1(iv), the commutants of $\underline{G}$ and $\underline{I}$ in $\mathrm{End}(\Lambda \otimes \mathbb{F}_\ell)$ have the same dimension; they must therefore be the same. By hypothesis, $\underline{G}$ is semisimple, and we have equality of ranks:

$$\mathrm{rank}\,\underline{G} = \mathrm{rank}\,\mathbf{G} = \mathrm{rank}\,\mathcal{J}_F = \mathrm{rank}\,\underline{I}.$$

By the Borel–de Siebenthal theorem [16, Theorem 0.1], $\underline{G} = \underline{I}$, and it follows that $\underline{H} = \underline{G}$ is likewise semisimple. We have

$$\underline{G}(\mathbb{F}_\ell)^+ \subset G \subset H \subset I \subset \underline{G}(\mathbb{F}_\ell).$$

As H acts semisimply, since the image of $\ker(\mathcal{H}(\mathbb{Z}_\ell) \to \mathcal{H}(\mathbb{F}_\ell))$ in H is a normal ℓ-subgroup, it must be trivial. Thus, H is a quotient of $\mathcal{H}(\mathbb{F}_\ell)$. If the vertex $x_0 \in \mathcal{B}(\mathbf{G}^{\mathrm{sc}}, \mathbb{Q}_\ell)$ associated to $\mathcal{H}$ is not hyperspecial, then the unipotent radical of $\mathcal{H}_{\mathbb{F}_\ell}$ is nontrivial. Since $\mathcal{H}$ is flat, the dimension of $\mathcal{H}_{\mathbb{F}_\ell}$ equals the dimension of $\mathbf{G}$, which is also the dimension of $\mathcal{J}_{\mathbb{F}_\ell}$ and therefore the dimension of $\underline{G} = \underline{H} = \underline{I}$. By Proposition 2.9(iii), we obtain

$$\dim_\ell H \le \dim_\ell\bigl(\mathcal{H}(\mathbb{F}_\ell)\bigr) = \dim \mathcal{H}_{\mathbb{F}_\ell}^{\mathrm{ss}} < \dim \mathcal{H}_{\mathbb{F}_\ell} = \dim \underline{G} = \dim_\ell G,$$

which is impossible by Lemma 2.10 since $G \subset H \subset \mathrm{GL}_n(\mathbb{F}_\ell)$. Thus, $\mathcal{H}(\mathbb{Z}_\ell)$ is a hyperspecial maximal compact subgroup, which means that $\mathcal{H}(\mathbb{F}_\ell)$ is the group of $\mathbb{F}_\ell$-points of a simply connected semisimple algebraic group over $\mathbb{F}_\ell$, and H is a quotient of $\mathcal{H}(\mathbb{F}_\ell)$ by a subgroup of its center. As G is of bounded index in H, the compact subgroup Γ^{sc} of $\mathcal{H}(\mathbb{Z}_\ell)$ maps onto a bounded index subgroup of $\mathcal{H}(\mathbb{F}_\ell)$, which has to be $\mathcal{H}(\mathbb{F}_\ell)$ itself when ℓ is sufficiently large. By [43, Theorem 1.3], this implies that $\Gamma^{\mathrm{sc}} = \mathcal{H}(\mathbb{Z}_\ell)$, as claimed.

Condition $(*)$ is used only to prove that the commutants of $\underline{G}$ and $\underline{I}$ have the same dimension. In any case, we have $\operatorname{rank} \underline{G} = \operatorname{rank} \underline{I} = \operatorname{rank} \mathbf{G}$ and, therefore, $\operatorname{rk}_\ell(\underline{G}(\mathbb{F}_\ell)) = \operatorname{rank} \mathbf{G}$. As G is of bounded index in $\underline{G}(\mathbb{F}_\ell)$, if ℓ is sufficiently large, $\operatorname{rk}_\ell G = \operatorname{rk}_\ell(\underline{G}(\mathbb{F}_\ell))$, and since G is a quotient of Γ, we obtain $\operatorname{rank} \mathbf{G} = \operatorname{rk}_\ell \Gamma \leq \operatorname{rk}_\ell \Pi$ by the construction (12) and Lemma 2.10. Theorems 2.11(i)–2.11(iii) now imply the remaining claims since Π is maximal compact in $\mathbf{G}(\mathbb{Q}_\ell)$. ☐

4. Maximality of Galois actions

4.1. Algebraic envelopes

Let $\{\rho_\ell\}_\ell$ be the system of ℓ-adic representations in Theorem 1.2. The monodromy group (resp., algebraic monodromy group) of ρ_ℓ is denoted by Γ_ℓ (resp., $\mathbf{G}_\ell$). The quotient of $\mathbf{G}_\ell$ by its unipotent radical is denoted by $\mathbf{G}_\ell^{\mathrm{red}}$. If X is a projective nonsingular variety over K, then for each ℓ, the image of $H^i(X_{\overline{K}}, \mathbb{Z}_\ell)$ in $H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^n$ is a $\mathbb{Z}_\ell$-lattice Λ_ℓ stabilized by ρ_ℓ, and

$$\bar{\rho}_\ell^{\mathrm{ss}} : \mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{F}_\ell) \tag{15}$$

denotes the semisimplification of the (mod ℓ) reduction of $\rho_\ell : \mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{Z}_\ell)$ (the action of Gal_K on this lattice). Denote by G_ℓ the image $\bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}_K)$ for all ℓ. In [20], we construct the algebraic envelope $\underline{G}_\ell$ (a connected reductive subgroup of $\mathrm{GL}_{n,\mathbb{F}_\ell}$) of G_ℓ to study the ℓ-independence of the total ℓ-rank and the $\mathfrak{g}$-type ℓ-rank of G_ℓ for all sufficiently large ℓ. The idea of constructing such a $\underline{G}_\ell$ is due to Serre in [38], who considered the Galois action on the ℓ-torsion points of abelian varieties without complex multiplication (see also [5]). The algebraic envelope $\underline{G}_\ell$ can be written as $\underline{S}_\ell \underline{Z}_\ell$, where $\underline{S}_\ell := \underline{G}_\ell^{\mathrm{der}}$ is also the Nori group of $G_\ell \subset \mathrm{GL}_n(\mathbb{F}_\ell)$ (by [20, Section 2.5]) and $\underline{Z}_\ell$ is the identity component of the center of $\underline{G}_\ell$. Theorems 4.1 and 4.2 below present the key properties of the algebraic envelopes $\underline{G}_\ell$.

THEOREM 4.1 ([20, Theorem 2.0.5, Proof of Theorem 2.0.5(iii)])
After replacing K by a finite normal field extension L if necessary, for all sufficiently large ℓ, the algebraic envelope $\underline{G}_\ell \subseteq \mathrm{GL}_{n,\mathbb{F}_\ell}$ has the following properties:

(i) *G_ℓ is a subgroup of $\underline{G}_\ell(\mathbb{F}_\ell)$ whose index is bounded uniformly independent of ℓ;*
(ii) *$\underline{G}_\ell$ acts semisimply on the ambient space;*
(iii) *the representations $\{\underline{S}_\ell \to \mathrm{GL}_{n,\mathbb{F}_\ell}\}_{\ell \gg 0}$ and $\{\underline{Z}_\ell \to \mathrm{GL}_{n,\mathbb{F}_\ell}\}_{\ell \gg 0}$ have bounded formal characters, and in particular, $\{\underline{G}_\ell \to \mathrm{GL}_{n,\mathbb{F}_\ell}\}_{\ell \gg 0}$ has bounded formal characters.*

THEOREM 4.2 ([20, Theorem A, Theorem 3.1.1])
Let $\mathbf{G}_\ell$ be the algebraic monodromy group of ρ_ℓ. After replacing K by a finite normal field extension L if necessary, the following statements hold for all sufficiently large ℓ.

(i) *The formal character of $\underline{S}_\ell \to \mathrm{GL}_{n,\mathbb{F}_\ell}$ (resp., $\underline{G}_\ell \to \mathrm{GL}_{n,\mathbb{F}_\ell}$) is independent of ℓ and is equal to the formal character of $(\mathbf{G}_\ell^{\mathrm{red}})^{\mathrm{der}} \to \mathrm{GL}_{n,\mathbb{Q}_\ell}$ (resp., $\mathbf{G}_\ell^{\mathrm{red}} \to \mathrm{GL}_{n,\mathbb{Q}_\ell}$).*

(ii) *The non-abelian composition factors of G_ℓ and the non-abelian composition factors of $\underline{S}_\ell(\mathbb{F}_\ell)$ are in bijective correspondence. Thus, the composition factors of G_ℓ are finite simple groups of Lie type in characteristic ℓ and cyclic groups.*

Note that the implicit constants in Theorems 4.1 and 4.2 depend only on the system (15) of (mod ℓ) Galois representations.

Remark 4.3
The formal bi-character [21, Definition 2.3] of $\mathbf{G}_\ell^{\mathrm{red}} \to \mathrm{GL}_{n,\mathbb{Q}_\ell}$ is independent of ℓ (see [19, Theorem 3.19]).

PROPOSITION 4.4
The system of algebraic envelopes $\{\underline{G}_\ell \subset \mathrm{GL}_{n,\mathbb{F}_\ell}\}_{\ell\gg 0}$ is characterized by the conditions 4.1(i) and 4.1(iii) in the sense that if $\{\underline{H}_\ell \subset \mathrm{GL}_{n,\mathbb{F}_\ell}\}_{\ell\gg 0}$ is another system of connected reductive subgroups such that, for $\ell \gg 0$, G_ℓ is a subgroup of $\underline{H}_\ell(\mathbb{F}_\ell)$ whose index is uniformly bounded and the formal character of $\underline{H}_\ell$ is uniformly bounded, then $\underline{G}_\ell = \underline{H}_\ell$ for all sufficiently large ℓ.

Proof
This follows directly from Proposition 2.23. □

THEOREM 4.5
If the algebraic monodromy group $\mathbf{G}_\ell$ is connected for all ℓ, then $G_\ell \subset \underline{G}_\ell(\mathbb{F}_\ell)$ for all sufficiently large ℓ.

Proof
Let L be a finite normal extension of K in Theorem 4.1 such that $\bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}_L) \subset \underline{G}_\ell(\mathbb{F}_\ell)$ for $\ell \gg 0$. Let $\underline{N}_\ell$ be the normalizer of the Nori group $\underline{S}_\ell$ in $\mathrm{GL}_{n,\mathbb{F}_\ell}$. Then $G_\ell \subset \underline{N}_\ell(\mathbb{F}_\ell)$.

We claim that $G_\ell = \bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}_K)$ normalizes the connected reductive group $\underline{G}_\ell$ for $\ell \gg 0$. By construction (see [20, Proof of Theorem 2.0.5(i) and (ii)]), $\underline{G}_\ell$ is the preimage of a torus $\underline{I}_\ell \subset \mathrm{GL}_{W_\ell}$ under some morphism[3]

$$t_\ell : \underline{N}_\ell \twoheadrightarrow \underline{N}_\ell/\underline{S}_\ell \hookrightarrow \mathrm{GL}_{W_\ell},$$

where W_ℓ is some $\mathbb{F}_\ell$-vector space whose dimension is bounded independent of ℓ. Since the index $[\underline{I}_\ell(\mathbb{F}_\ell) : t_\ell(\bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}_L))]$ and the formal character of $\underline{I}_\ell \subset \mathrm{GL}_{W_\ell}$ are both bounded independent of ℓ (see [20, Theorem 2.4.2]), the normality of the field extension L/K and Proposition 2.23 imply that $t_\ell(G_\ell)$ normalizes $\underline{I}_\ell$ for all sufficiently large ℓ. Hence, the product $G_\ell\underline{G}_\ell$ is a subgroup of $\mathrm{GL}_{n,\mathbb{F}_\ell}$ with identity component $\underline{G}_\ell$ for $\ell \gg 0$.

The number of conjugacy classes of elements of the finite group $\mathrm{Gal}(L/K)$ is bounded by $m := [L : K]$. Since $\mathbf{G}_\ell$ is connected for all ℓ and the strictly compatible system $\{\rho_\ell\}_\ell$ is pure of weight i, the method of Frobenius tori of Serre (see, e.g., [29], [21, Theorem 2.6, Corollary 2.7]) implies that there is a Dirichlet density 1 set of finite places v of K such that the *Frobenius torus* $\mathbf{T}_{\bar{v},\ell} \subset \mathrm{GL}_{n,\mathbb{Q}_\ell}$ is a maximal torus of $\mathbf{G}_\ell$ if $v \nmid \ell$ and $\bar{v}$ is some place on $\overline{K}$ extending v on K. Thus, for each conjugacy class c of $\mathrm{Gal}(L/K)$, we can fix a finite place v_c of K (unramified in L) mapping to c and a place $\bar{v}_c$ of $\overline{K}$ above v_c such that $\mathbf{T}_{\bar{v}_c,\ell}$ is a maximal torus of $\mathbf{G}_\ell$ for $\ell \gg 0$. To prove $G_\ell\underline{G}_\ell = \underline{G}_\ell$ for $\ell \gg 0$, it suffices to show, for each c and all sufficiently large ℓ, the semisimple part $\bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Fr}_{\bar{v}_c})_{\mathrm{ss}} \in \underline{G}_\ell(\mathbb{F}_\ell)$.

For each c, there exist a torus $\mathbf{T}_c \subset \mathrm{GL}_{n,\mathbb{Q}}$ and an element $\gamma_c \in \mathbf{T}_c(\mathbb{Q})$ such that, for all sufficiently large ℓ, the chain

$$\gamma_c \in \mathbf{T}_c \subset \mathrm{GL}_{n,\mathbb{Q}} \tag{16}$$

is conjugate to the chain

$$\rho_\ell(\mathrm{Fr}_{\bar{v}_c})_{\mathrm{ss}} \in \mathbf{T}_{\bar{v}_c,\ell} \subset \mathrm{GL}_{n,\mathbb{Q}_\ell} \tag{17}$$

by some element in $\mathrm{GL}_n(\mathbb{Q}_\ell)$. For $\ell \gg 0$, the reduction modulo ℓ

$$g_{c,\ell} \in \underline{T}_{c,\ell} \subset \mathrm{GL}_{n,\mathbb{F}_\ell} \tag{18}$$

of (16) can be well defined, and the two semisimple elements $g_{c,\ell}$ and $\bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Fr}_{\bar{v}_c})_{\mathrm{ss}}$ in $\mathrm{GL}_n(\mathbb{F}_\ell)$ are conjugate since they have the same characteristic polynomial. Without loss of generality, assume $g_{c,\ell} = \bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Fr}_{\bar{v}_c})_{\mathrm{ss}}$. The formal characters of $\underline{T}_{c,\ell}$ and $\underline{G}_\ell$ are equal and bounded by some $N \in \mathbb{N}$ independent of $\ell \gg 0$ by Theorem 4.2(i). Since the powers γ_c^m are Zariski-dense in $\mathbf{T}_c$, there exists $M \geq C_8(n, N)$ (in Section 2.6.2) such that for $\ell \gg 0$ the torus $\underline{T}_{c,\ell} \subset \mathrm{GL}_{n,\mathbb{F}_\ell}$ after diagonalization is the intersection of the kernels of some characters in $I_M^n \cap X^*(\mathbb{G}_m^n)$ and the element $g_{c,\ell}^m$ is M-regular in $\underline{T}_{c,\ell}$. Since we have

[3]The groups $\underline{I}_\ell, \underline{S}_\ell, \underline{N}_\ell, \underline{G}_\ell$ are denoted $\bar{\mathbf{I}}_\ell, \bar{\mathbf{S}}_\ell, \bar{\mathbf{N}}_\ell, \bar{\mathbf{G}}_\ell$ in [20].

$$g_{c,\ell}^m = \bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Fr}_{\bar{v}_c})_{\mathrm{ss}}^m \in \underline{G}_\ell(\mathbb{F}_\ell),$$

it follows by Lemma 2.19 and the M-regularity of $g_{c,\ell}^m \in \underline{T}_{c,\ell}$ that $\underline{T}_{c,\ell} \subset \underline{G}_\ell$ for $\ell \gg 0$. We are done since $\bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Fr}_{\bar{v}_c})_{\mathrm{ss}} = g_{c,\ell} \in \underline{T}_{c,\ell}$. □

4.2. Proof of Theorem 1.2

Proof of Theorem 1.2
After taking a finite extension L of K and the semisimplification of ρ_ℓ, we may assume that $\mathbf{G}_\ell$ is connected reductive for all ℓ and G_ℓ is a subgroup of $\underline{G}_\ell(\mathbb{F}_\ell)$ for $\ell \gg 0$ by Theorem 4.5. By the constructions of

$$\Gamma_\ell \subset \mathbf{G}_\ell(\mathbb{Q}_\ell) \subset \mathrm{GL}_n(\mathbb{Q}_\ell),$$
$$G_\ell \subset \underline{G}_\ell(\mathbb{F}_\ell) \subset \mathrm{GL}_n(\mathbb{F}_\ell)$$

and Theorem 4.2(i), we are in the setting of Theorem 1.5, and the conditions 1.5(a)–1.5(d) are verified. Moreover, Theorems 1.5(e) and 1.5(f) are verified by Theorems 4.1(i) and 4.1(iii). Thus, for $\ell \gg 0$, the condition $(*)$ implies the hyperspeciality of $\Gamma_\ell^{\mathrm{sc}}$ in $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ (which in turn implies the unramifiedness of $\mathbf{G}_\ell^{\mathrm{sc}}$, $\mathbf{G}_\ell^{\mathrm{der}}$, and $\mathbf{G}_\ell$ (Proposition 4.6(i))).

Next, we prove the converse. Let $\mathscr{G}_\ell'$ be the Zariski closure of the derived group Γ_ℓ' in $\mathrm{GL}_{\Lambda_\ell}$ (Λ_ℓ being the lattice in $\mathbb{Q}_\ell^n$) endowed with the unique structure of reduced closed subscheme. For $\ell \gg 0$, the group scheme $\mathscr{G}_\ell'$ is smooth with constant rank over $\mathbb{Z}_\ell$ (see [6, Theorem 9.1, Section 9.2.1]) and has generic fiber $\mathbf{G}_\ell^{\mathrm{der}}$. We first show that $\mathscr{G}_\ell'$ is semisimple for $\ell \gg 0$. Suppose that $\Gamma_\ell^{\mathrm{sc}}$ is a hyperspecial maximal compact subgroup of $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$. When ℓ is sufficiently large, there exists a semisimple group scheme $\mathscr{H}/\mathbb{Z}_\ell$ whose generic fiber is $\mathbf{G}_\ell^{\mathrm{sc}}$, satisfying $\mathscr{H}(\mathbb{Z}_\ell) = \Gamma_\ell^{\mathrm{sc}}$. Since $\Gamma_\ell^{\mathrm{sc}}$ is perfect if ℓ is large enough depending on n (see [23, Theorem 3.4]), it maps into the commutator subgroup $\Gamma_\ell' \subset \mathscr{G}_\ell'(\mathbb{Z}_\ell)$. Consider

$$\bar{\rho}_\ell : \Gamma_\ell^{\mathrm{sc}} \to \Gamma_\ell' \hookrightarrow \mathscr{G}_\ell'(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}(\Lambda_\ell) \to \mathrm{GL}(\Lambda_\ell \otimes \mathbb{F}_\ell), \tag{19}$$

and let $\underline{R}_\ell \subset \mathrm{GL}_{\Lambda_\ell \otimes \mathbb{F}_\ell}$ be the Nori group of $\bar{\rho}_\ell(\Gamma_\ell^{\mathrm{sc}})$. For ℓ large enough depending on n, the groups

$$\mathscr{H}(\mathbb{F}_\ell), \Gamma_\ell^{\mathrm{sc}}, \bar{\rho}_\ell(\Gamma_\ell^{\mathrm{sc}}), \underline{R}_\ell(\mathbb{F}_\ell)$$

have the same ℓ-dimension by Theorems 2.1 and 2.2 and the remarks of Section 2.3.2. Then it follows by Proposition 2.9(iii) that

$$\dim \mathbf{G}_\ell^{\mathrm{der}} = \dim \mathscr{H}_{\mathbb{F}_\ell} = \dim_\ell\big(\mathscr{H}(\mathbb{F}_\ell)\big) = \dim_\ell\big(\underline{R}_\ell(\mathbb{F}_\ell)\big) = \dim \underline{R}_\ell^{\mathrm{ss}}. \tag{20}$$

Since $\dim \mathbf{G}_\ell^{\mathrm{der}} \geq \dim \underline{R}_\ell$ by [28, Theorem 7] for ℓ large enough depending on n, it follows that the Nori group $\underline{R}_\ell$ is semisimple and the action of $\underline{R}_\ell(\mathbb{F}_\ell)$

(resp., $\underline{R}_\ell(\mathbb{F}_\ell)^+ = \bar{\rho}_\ell(\Gamma_\ell^{\mathrm{sc}})^+$) on $\Lambda_\ell \otimes \mathbb{F}_\ell$ is also semisimple by [27, Theorem 3.5]. Since $\bar{\rho}_\ell(\Gamma_\ell^{\mathrm{sc}})^+$ is normal in $\bar{\rho}_\ell(\Gamma_\ell^{\mathrm{sc}})$ of prime-to-ℓ index, (19) is semisimple.

Since the equality $\dim \mathbf{G}_\ell^{\mathrm{der}} = \dim \underline{R}_\ell$ holds, it follows by [28, Theorem 7(3)] that $\bar{\rho}_\ell(\Gamma_\ell^{\mathrm{sc}})$ is a subgroup of $\mathcal{G}'_\ell(\mathbb{F}_\ell)$ of index bounded by a constant depending only on n. Hence, if the unipotent radical of the special fiber of $\mathcal{G}'_\ell$ is nontrivial for some large enough ℓ, then $\bar{\rho}_\ell(\Gamma_\ell^{\mathrm{sc}})$ has a nontrivial normal subgroup of unipotent elements, which contradicts the semisimplicity of (19). Thus, the group scheme $\mathcal{G}'_\ell$ is semisimple over $\mathbb{Z}_\ell$.

The weights appearing in the natural n-dimensional representation of the generic fiber $\mathcal{G}'_{\ell,\overline{\mathbb{Q}}_\ell}$ remain bounded as ℓ varies. By [40, Corollary 4.3], if ℓ is sufficiently large, the (mod ℓ) reduction of every irreducible factor in this representation is again irreducible, and the (mod ℓ) reductions of distinct irreducible factors are distinct. Thus, the composition of the special fiber $\mathcal{G}'_{\ell,\overline{\mathbb{F}}_\ell} \to \mathrm{GL}_{\Lambda_\ell \otimes \overline{\mathbb{F}}_\ell}$ with the adjoint representation of $\mathrm{GL}_{\Lambda_\ell \otimes \overline{\mathbb{F}}_\ell}$ is semisimple, its irreducible factors have bounded highest weights, and they are in one-to-one correspondence with the irreducible factors of the composition of $\mathcal{G}'_{\ell,\overline{\mathbb{Q}}_\ell} \to \mathrm{GL}_{n,\overline{\mathbb{Q}}_\ell}$ with its adjoint representation. So we obtain

$$\dim_{\mathbb{F}_\ell} \mathrm{End}_{\mathcal{G}'_{\ell,\mathbb{F}_\ell}}(\Lambda_\ell \otimes \mathbb{F}_\ell) = \dim_{\mathbb{Q}_\ell} \mathrm{End}_{\mathcal{G}'_{\ell,\mathbb{Q}_\ell}}(\mathbb{Q}_\ell^n) = \dim_{\mathbb{Q}_\ell} \mathrm{End}_{\Gamma'_\ell}(\mathbb{Q}_\ell^n).$$

Since the index of Γ'_ℓ in $\mathcal{G}'_\ell(\mathbb{F}_\ell)$ and the formal character of $\mathcal{G}'_{\ell,\mathbb{F}_\ell} \subset \mathrm{GL}_{\Lambda_\ell \otimes \mathbb{F}_\ell}$ are uniformly bounded independent of ℓ, it follows by Propositions 3.1(i) and 3.1(iv) that for $\ell \gg 0$ the image of Γ'_ℓ in $\mathrm{GL}(\Lambda_\ell \otimes \mathbb{F}_\ell)$ in (19) can be identified with the semisimple action $G'_\ell \to \mathrm{GL}(\mathbb{F}_\ell^n)$ and

$$\dim_{\mathbb{F}_\ell} \mathrm{End}_{G'_\ell}(\mathbb{F}_\ell^n) = \dim_{\mathbb{F}_\ell} \mathrm{End}_{\mathcal{G}'_{\ell,\mathbb{F}_\ell}}(\Lambda_\ell \otimes \mathbb{F}_\ell)$$

holds. Hence, we deduce ($*'$) for $\ell \gg 0$.

Now, Γ_ℓ/Γ'_ℓ is a Zariski-dense subset of the torus $\mathbf{G}_\ell/\mathbf{G}_\ell^{\mathrm{der}}$, which acts on the space of Γ'_ℓ-invariants or, equivalently, the space of $\mathbf{G}_\ell^{\mathrm{der}}$-invariants, in the adjoint representation of $\mathrm{GL}_{n,\mathbb{Q}_\ell}$. Thus, any Zariski-dense subset of $\mathbf{G}_\ell(\mathbb{Q}_\ell)$ contains an element γ with the property that any vector in the adjoint representation of $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ which is fixed by Γ'_ℓ and by γ is fixed by Γ_ℓ.

By Theorem 4.1, G_ℓ is of bounded index in $\underline{G}_\ell(\mathbb{F}_\ell)$, so by Proposition 2.22, if ℓ is sufficiently large, then there exist a maximal torus $\underline{T}_\ell$ of $\underline{G}_\ell$ and an element $\bar{\gamma} \in G_\ell \cap \underline{T}_\ell(\mathbb{F}_\ell)$ such that $\bar{\gamma}$ is not in the kernel of any nontrivial character of $\underline{T}_\ell$ acting in the restriction to $\underline{G}_\ell$ of the adjoint representation of $\mathrm{GL}_{n,\mathbb{F}_\ell}$. We want to apply Proposition 3.3 with $\Gamma := \Gamma_\ell$ and $\Delta := \Gamma'_\ell$. The elements $\gamma \in \Gamma_\ell$ which reduce (mod ℓ) to $\bar{\gamma}$ are Zariski-dense in $\mathbf{G}_\ell(\mathbb{Q}_\ell)$, so we may choose γ to satisfy properties (a) and (b). By Theorem 4.2(i), the formal characters of $\underline{G}_\ell \to \mathrm{GL}_{n,\mathbb{F}_\ell}$ and $\mathbf{G}_\ell \to \mathrm{GL}_{n,\mathbb{Q}_\ell}$ are the same, so in particular, two eigenvalues λ_1 and λ_2 of γ are equal if $\lambda_1 - \lambda_2$ is not a unit. We can therefore apply the proposition, and $(*)$ follows. □

Without $(*)$, we can prove much less. Nevertheless, we still have the following.

PROPOSITION 4.6

Let $\{\rho_\ell\}_\ell$ be the system of ℓ-adic representations arising from the ith ℓ-adic cohomology of a proper smooth variety X defined over a number field K which is sufficiently large. Then for sufficiently large ℓ,

(i) *the reductive group $\mathbf{G}_\ell^{\mathrm{red}}$ splits over some finite unramified extension of $\mathbb{Q}_\ell$,*

(ii) *the reductive group $\mathbf{G}_\ell^{\mathrm{red}}$ is unramified over every degree 12 totally ramified extension of $\mathbb{Q}_\ell$.*

Recall that, according to standard terminology, a connected reductive group over a local field which splits over an unramified extension of that field need not be unramified, since it need not have a rational Borel subgroup.

Proof

The first assertion follows immediately by the method of Frobenius tori (see [9], [29], [37]). The second assertion follows from the first and the last statements in Theorem 1.5. □

4.3. Proof of Theorem 1.3(a)

Proof of Theorem 1.3(a)

Let X be an abelian variety defined over a subfield K of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$. Since the ℓ-adic representation ρ_ℓ arising from $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is semisimple by Faltings (see [14]), the algebraic monodromy group $\mathbf{G}_\ell$ is reductive.

We first treat the case $i = 1$. By taking a finite extension of K, we may assume $\mathbf{G}_\ell$ is connected for all ℓ (see, e.g., [5, Section 2.3]). There exists an abelian scheme $f : \mathcal{X} \to \mathcal{S}$ defined over some number field whose generic fiber is $X \to \operatorname{Spec} K$. Let s be a closed point of $\mathcal{S}$, and let $R^1 f_* \mathbb{Q}_\ell$ be the lisse sheaf on $\mathcal{S}$. Then by the proper-smooth base change theorem, ρ_ℓ factors through the ℓ-adic representation

$$\psi_\ell : \pi_1^{\text{ét}}(\mathcal{S}, \bar{s}) \to \mathrm{GL}(R^1 f_* \mathbb{Q}_\ell|_{\bar{s}})$$

for all ℓ. Hence, we may assume that Γ_ℓ (resp., $\mathbf{G}_\ell$) is the monodromy group (resp., algebraic monodromy group) of ψ_ℓ. Moreover, the composition $\psi_{\ell,s} := \psi_\ell \circ (\pi_1^{\text{ét}}(s, \bar{s}) \to \pi_1^{\text{ét}}(\mathcal{S}, \bar{s}))$ is isomorphic to the ℓ-adic representation $H^1(\mathcal{X}_{\bar{s}}, \mathbb{Q}_\ell)$ of the abelian variety $\mathcal{X}_s$ (the fiber over s) defined over the residue field of s (some number field) for all ℓ. Let $\Gamma_{\ell,s}$ (resp., $\mathbf{G}_{\ell,s}$) be the monodromy group (resp., algebraic monodromy group) of $\psi_{\ell,s}$. We may identify $\Gamma_{\ell,s}$ (resp., $\mathbf{G}_{\ell,s}$) as a subgroup of Γ_ℓ (resp., $\mathbf{G}_\ell$).

Fix a prime p, one can find a closed point s of $\mathcal{S}$ such that $\mathbf{G}_{p,s} = \mathbf{G}_p$ (see [37]). By the main theorem of [18], we have $\mathbf{G}_{\ell,s} = \mathbf{G}_\ell$ for all ℓ. Therefore, it suffices to deal with the case when K is a number field. Since the condition $(*)$ holds by the Tate conjecture for abelian varieties proved by Faltings in the strong form given in [15, Theorem 4.2], we are done by Theorem 1.2.

Since we have $H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \cong \bigwedge^i H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ as Gal_K-representations, the general case follows from the lemma below. □

LEMMA 4.7
Let $\mu : \mathbf{G} \to \mathbf{H}$ be a surjective morphism between connected reductive algebraic groups defined over $\mathbb{Q}_\ell$, and let Γ be a compact subgroup of $\mathbf{G}(\mathbb{Q}_\ell)$. If Γ^{sc} is a hyperspecial maximal compact subgroup of $\mathbf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$, then $\mu(\Gamma)^{\mathrm{sc}}$ is a hyperspecial maximal compact subgroup of $\mathbf{H}^{\mathrm{sc}}(\mathbb{Q}_\ell)$.

Proof
The surjective $\mathbb{Q}_\ell$-morphism $\mu : \mathbf{G} \to \mathbf{H}$ induces a surjective $\mathbb{Q}_\ell$-morphism $\mu^{\mathrm{sc}} : \mathbf{G}^{\mathrm{sc}} \to \mathbf{H}^{\mathrm{sc}}$ mapping Γ^{sc} into $\mu(\Gamma)^{\mathrm{sc}}$. Since both $\mathbf{G}^{\mathrm{sc}}$ and $\mathbf{H}^{\mathrm{sc}}$ are simply connected, $\mathbf{H}^{\mathrm{sc}}$ can be identified as a direct factor of $\mathbf{G}^{\mathrm{sc}}$ and μ^{sc} can be identified as the projection to the factor. It follows that $\mu^{\mathrm{sc}}(\Gamma^{\mathrm{sc}})$ is also a hyperspecial maximal compact subgroup of $\mathbf{H}^{\mathrm{sc}}(\mathbb{Q}_\ell)$. Since $\mu^{\mathrm{sc}}(\Gamma^{\mathrm{sc}}) \subset \mu(\Gamma)^{\mathrm{sc}}$ holds, the compact subgroup $\mu(\Gamma)^{\mathrm{sc}}$ is equal to $\mu^{\mathrm{sc}}(\Gamma^{\mathrm{sc}})$ and we are done. □

4.4. Proof of Theorem 1.3(b)

Proof of Theorem 1.3(b)
Let X be a hyper-Kähler variety defined over a subfield K of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$, and let ρ_ℓ the ℓ-adic representation arising from $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$. If the dimension n of the representation is less than 4, then $\mathbf{G}^{\mathrm{sc}}$ is trivial or of type A for all ℓ. Hence, Theorem 1.3(b) follows by [22, Theorem 15].

Suppose that $n \geq 4$, and write $X_{\mathbb{C}} := X \times_K \mathbb{C}$. By the Kuga–Satake construction, there are a complex abelian variety $A_{\mathbb{C}}$ and a surjective morphism

$$H^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes H^1(A_{\mathbb{C}}, \mathbb{Q}) \to H^2(X_{\mathbb{C}}, \mathbb{Q}) \tag{21}$$

of pure Hodge structures, that is, there is a Hodge cycle of

$$H^1(A_{\mathbb{C}}, \mathbb{Q})^* \otimes H^1(A_{\mathbb{C}}, \mathbb{Q})^* \otimes H^2(X_{\mathbb{C}}, \mathbb{Q})$$

giving the correspondence (21). Assume that $A_{\mathbb{C}}$ has a model A_L defined over a subfield L of $\mathbb{C}$; that is finitely generated over K. Since the Hodge cycle on the product $X_{\mathbb{C}} \times A_{\mathbb{C}}$ corresponding to (21) is motivated (see [1, Corollary 1.5.3]) and motivated cycles are absolute Hodge (see [2, Proposition 2.5.1]) in the sense of Deligne (see [13, 2.10]), we obtain for each ℓ a surjective morphism

$$H^1(A_{\overline{L}}, \mathbb{Q}_\ell) \otimes H^1(A_{\overline{L}}, \mathbb{Q}_\ell) \to H^2(X_{\overline{L}}, \mathbb{Q}_\ell)$$

of Gal_L-representations (after replacing L by a finite extension if necessary; [13, Proposition 2.9(b)]). Since A_L is an abelian variety defined over a subfield L of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$, the assertion of Theorem 1.3 holds for $X_L := X \times_K L$ by Theorem 1.3(a) and Lemma 4.7. By induction, it suffices to show that Theorem 1.3 for X/K also holds when L is a finite extension of K or $L = K(t)$, where t is transcendental over K. The former case is obvious, and the latter case can be done by the fact that $\pi_1^{\text{ét}}(\operatorname{Spec} \overline{K}[t])$ is trivial. □

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