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METAPLECTIC CATEGORIES, GAUGING AND PROPERTY F

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Abstract. N-Metaplectic categories, unitary modular categories with the same fusion rules as $SO(N)_2$, are prototypical examples of weakly integral modular categories generalizing the model for the Ising anyons, i.e. metaplectic anyons. A conjecture of the second author would imply that images of the braid group representations associated with metaplectic categories are finite groups, i.e. have property F. While it was recently shown that $SO(N)_2$ itself has property F, proving property F for the more general class of metaplectic modular categories is an open problem. We verify this conjecture for N-metaplectic modular categories when N is odd, exploiting their recent enumeration together with a characterization in terms of Galois conjugation and twisting. In another direction, we prove that when N is divisible by 8 the N-metaplectic categories have 3 non-trivial bosons, and the boson condensation procedure applied to 2 of these bosons yields $\frac{N}{4}$-metaplectic categories. Otherwise stated: any $8k$-metaplectic category is a $\mathbb{Z}_2$-gauging of a $2k$-metaplectic category, so that the N even metaplectic categories lie towers of $\mathbb{Z}_2$-gaugings commencing with $2k$- or $4k$-metaplectic categories with k odd.

1. Introduction. N-Metaplectic categories are a major source of examples of weakly integral modular categories. As natural generalizations of the Ising anyons [21] they are important examples in the study of topological phases of matter and their applications [22] to quantum computation. They are defined as unitary modular categories with the same fusion rules as those obtained from the semisimple quotients $SO(N)_2$ (This notation is borrowed from conformal field theory. A more suitable notation might be $Spin(N)_2$ since the objects analogous to the spinor representations are included.) of $\mathrm{Rep}(U_q\mathfrak{so}_N)$ where $q = e^{\pi i/N}$ for N even and $q = e^{\pi i/(2N)}$ for N odd (see [26] for details of that construction). In general an N-metaplectic category has dimension $4N$ and has simple objects of dimension $1, 2$ and $\sqrt{N}$ (N odd) or $\sqrt{\frac{N}{2}}$ (N even). When N is odd, N-metaplectic categories are relative centers of Tambara-Yamagami categories [20]. Recently, a complete classification and enumeration of N-metaplectic categories has been completed [1, 6, 7]. In addition, the N-metaplectic modular categories coming from quantum groups, i.e. $SO(N)_2$, have been shown to have finite braid group image [30], verifying the property F conjecture for this subset of metaplectic categories (see [26]).

In this article we advance our understanding of N-metaplectic modular categories in two ways. First we extend the proof of property F from $SO(N)_2$ with N odd to all odd N-metaplectic categories. This is achieved as follows. In [1] it is shown that for N odd there are

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precisely 2^{s+1} inequivalent N-metaplectic categories where s is the number of prime factors of N. We show that each of these may be obtained from $SO(N)_2$ by Galois conjugation and twisting, which then allows us to describe the images of all N-metaplectic $\mathcal{B}_n$-representations in terms of those obtained from $SO(N)_2$. Although we believe this technique should apply to the even N cases as well, there are some further technicalities that have not been worked out yet. On the other hand our second result shows that even N-metaplectic categories appear in towers of gaugings. More precisely we show that if $8 \mid N$ then any N-metaplectic modular category is a $\mathbb{Z}_2$-gauging of an $\frac{N}{4}$-metaplectic modular category. Thus for each odd k there are towers of even N-metaplectic categories starting with the $2k$- and $4k$-metaplectic categories.

2. Preliminaries. We assume the reader is familiar with the basic notions in the theory of fusion categories such as spherical and braiding structures and their properties. Good references for these details are: [14, 15, 2, 32].

2.1. Galois conjugation and twisting. It is well known that a fusion (or modular or ribbon) category $\mathcal{C}$ can be defined over a number field $\mathbb{F} = \mathbb{Q}(\alpha)$. That is, the data needed to construct $\mathcal{C}$ ($6j$-symbols, braiding isomorphisms, twists, mapping class group representations) all lie in a finite Galois extension of $\mathbb{Q}$. Moreover, if σ is a Galois automorphism of $\mathbb{F}$ then twisting all data by σ produces another category $\mathcal{C}^\sigma$. Now if $\mathcal{C}$ is a unitary category or (possibly more generally) has dimension function taking values in $\mathbb{R}^+$, then $\mathcal{C}^\sigma$ may not have this property. Indeed, a Galois conjugate of a pseudo-unitary category is not generally pseudo-unitary.

On the other hand, any Galois conjugate of a *weakly integral* fusion category is pseudo-unitary [15, Proposition 8.24]. Thus, by [15, Propositions 8.23] any weakly integral fusion category admits a unique spherical structure j_+ with respect to which each object has positive dimension. Moreover, if $\mathcal{B}$ is the braided fusion category underlying a weakly integral modular category $\mathcal{C}$ (i.e. forgetting the spherical structure) then $\mathcal{B}$ equipped with any other choice of spherical structure is again modular (see [8, Lemma 2.4]). In particular, with respect to the unique spherical structure j_+ giving $\mathcal{B}^\sigma$ positive dimensions, $\mathcal{B}_+^\sigma = (\mathcal{B}^\sigma, j_+)$ is modular. Note that while $\mathcal{C}^\sigma$ and $\mathcal{B}_+^\sigma$ have the same underlying braided fusion category $\mathcal{B}^\sigma$, their spherical structures (and therefore S and T-matrices) may differ.

These arguments prove the following useful proposition:

PROPOSITION 1. Let $\mathcal{C}$ be any weakly integral modular category, $\mathcal{B}$ its underlying braided fusion category, and $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ a Galois automorphism. Then there is a unique choice of a spherical structure j_+ with respect to which $\mathcal{B}_+^\sigma = (\mathcal{B}^\sigma, j_+)$ is a modular category with positive dimensions.

It is worth pointing out that distinct spherical structures on the braided fusion category $\mathcal{B}$ underlying any modular category $\mathcal{C}$ are in 1-1 correspondence with invertible self-dual objects of $\mathcal{C}$ (see e.g. [14]).

A motivation for this paper is the following:

CONJECTURE 1. The braid group representations associated with any object in a weakly integral braided fusion category have finite image.

An object $X \in \mathcal{B}$ for whcih the corresponding braid group representations all have finite image is called a *property F* object, and $\mathcal{B}$ has property F if all objects are property F objects. It is conjectured (see [26]) that $\dim(X)^2 \in \mathbb{Z}$ if and only if X has property F, so that $\mathcal{B}$ has property F if and only if $\mathcal{B}$ is weakly integral.

Suppose that every object in a modular category $\mathcal{C}$ has property F. Then the same is true of $\mathcal{C}^\sigma$, since the relations defining a finite group are polynomials. Moreover, the braid group image only depends on the underlying braided fusion category $\mathcal{B}$, i.e. is independent of the spherical structure. Thus if a weakly integral modular category $\mathcal{C}$ has property F then for any Galois conjugation σ the underlying braided fusion category, $\mathcal{B}^\sigma$ equipped with the positive spherical structure $\mathcal{B}_+^\sigma$ also has property F.

Recently it was shown [30] that the integral modular categories $SO(N)_2$ obtained from quantum groups $U_q so_N$ at $q = e^{\pi i/N}$ (N even) and $q = e^{\pi i/(2N)}$ (N odd) have property F. The proof involves a detailed analysis of representations of these quantum groups, rather than categorical-level arguments. In particular the proof does not immediately imply that unitary modular categories with the same fusion rules as $SO(N)_2$, i.e. *metaplectic modular categories*, also have property F. On the other hand, metaplectic modular categories have now been classified and enumerated. This suggests that we can infer property F for those metaplectic modular categories with underlying braided fusion categories Galois conjugate to $SO(N)_2$. It turns out we need slightly more–we must also use a twisting procedure [5].

2.2. Boson Condensation and Gauging. Two processes that we employ in our analysis are gauging and de-gauging (sometimes called anyon condensation), which may be interpreted physically as phase transitions for anyon systems [9]. First let us introduce the basic construction we call de-gauging (which was first described in [29] and subsequently rediscovered and developed in [25, 4, 12] under various conditions and under different names). Let $\mathcal{C}$ be modular and $\operatorname{Rep}(G) \cong \mathcal{D} \subset \mathcal{C}$ a Tannakian subcategory (here a Tannakian category is a symmetric braided fusion category equivalent to $\operatorname{Rep}(G)$ for some finite group G). The G-de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ is a faithfully G-graded category (in fact, a braided G-crossed category) with modular trivial component $[\mathcal{C}_G]_e$ of dimension $\dim(\mathcal{C})/|G|^2$ and $[\mathcal{C}_G]_e$ is the *G-de-gauging* of $\mathcal{C}$ [12]. One does not need to understand the full G-de-equivariantization of $\mathcal{C}$ to obtain $[\mathcal{C}_G]_e$: in fact $[\mathcal{C}_G]_e = (\mathcal{D}')_G$, where

$$\mathcal{D}' = \{Y \in \mathcal{C} : c_{X,Y}c_{Y,X} = \mathrm{id}_{Y\otimes X} \text{ for all } X \in \mathcal{D}\}$$

is the Müger centralizer of $\mathcal{D} \subset \mathcal{C}$ [12].

The simplest case of de-gauging is *boson condensation*. Whenever a modular category $\mathcal{C}$ contains a *boson* b, i.e. a self-dual invertible object with twist $\theta_b = 1$, then the fusion subcategory $\langle b\rangle$ is equivalent to $\operatorname{Rep}(\mathbb{Z}_2)$. In this case, the de-equivariantization functor $F : \mathcal{C} \to \mathcal{C}_G$ is easier to understand. In particular, if $X \in \mathcal{C}$ is a simple object and $b \otimes X \ncong X$, then $F(X) \cong X^{(1)} \oplus X^{(2)}$ for simple objects $X^{(1)}, X^{(2)}$. On the other hand, if $b \otimes X \cong X$, then $F(X)$ is a simple object. There is a trichotomy among self-dual invertible objects in a ribbon category: they are either bosons as above, *semions* s with $\theta_s = \pm i$ in which case the subcategory $\langle s \rangle$ is modular or *fermions* f with $\theta_f = -1$ and $\langle f \rangle \cong \mathrm{sVec}$.

The reverse process, G-gauging, is more complicated [3, 10]. Here one starts with a modular category $\mathcal{B}$ and an action of a finite group G by braided tensor autoequivalences: $\rho : G \to \mathrm{Aut}_{\otimes}^{br}(\mathcal{B})$. A *$G$-gauging* of $\mathcal{B}$, when it exists, is a new modular category obtained by first constructing a G-graded fusion category $\mathcal{D}$ with trivial component $\mathcal{D}_e = \mathcal{B}$ and then equivariantizing to obtain a new modular category $\mathcal{D}^G$. There are obstructions to the existence of a gauging, and when the obstructions vanish there can be many G-gaugings (see [10]). A recent result of Natale [27] implies that any *weakly group-theoretical* modular category is a G-gauging of either a pointed modular category or a Deligne product of a pointed modular category and an Ising category. In [16, Question 2] they ask if every weakly integral modular category is weakly group-theoretical (the converse is known to be true). If the answer is "yes" (as many suspect) then to prove one direction of the property F conjecture it would be enough to prove that G-gauging preserves property F.

3. Metaplectic Categories.

We begin with the following definition:

DEFINITION 1. A metaplectic modular category is a unitary modular category with the same fusion rules as $SO(N)_2$ for some $N > 1$.

The structure and properties of $SO(N)_2$ were studied in some detail in [26], from which much of the results we outline are taken. The fusion rules for $SO(N)_2$ (and hence N-metaplectic modular categories) naturally split into three cases, depending on the value of N mod 4.

3.1. Fusion rules for odd N. The N-metaplectic modular categories for odd $N > 1$ have 2 simple objects X_1, X_2 of dimension $\sqrt{N}$, two simple objects $\mathbf{1}, Z$ of dimension 1, and $\frac{N-1}{2}$ objects Y_i, $1 \leq i \leq \frac{N-1}{2}$ of dimension 2. The fusion rules are [1]:

(1) $Z \otimes Y_i \cong Y_i$, $Z \otimes X_i \cong X_{i+1}$ (modulo 2), $Z^{\otimes 2} \cong \mathbf{1}$,
(2) $X_i^{\otimes 2} \cong \mathbf{1} \oplus \bigoplus_i Y_i$,
(3) $X_1 \otimes X_2 \cong Z \oplus \bigoplus_i Y_i$,
(4) $Y_i \otimes Y_j \cong Y_{\min\{i+j, N-i-j\}} \oplus Y_{|i-j|}$, for $i \neq j$ and $Y_i^{\otimes 2} = \mathbf{1} \oplus Z \oplus Y_{\min\{2i, N-2i\}}$.

It is shown in [1] that Z is always a boson, and N-metaplectic modular categories with N odd were classified and enumerated by condensing Z: there are precisely 2^{s+1} inequivalent such categories, where s is the number of distinct primes dividing N. The fusion rules for the (adjoint) subcategory generated by Y_1 with simple objects $\mathbf{1}, Z$ and all Y_i are precisely those of the dihedral group D_N of order $2N$, and, moreover this subcategory coincides the centralizer of the Tannakian $\langle Z \rangle \cong \mathrm{Rep}(\mathbb{Z}_2)$.

3.2. Fusion rules for $N \equiv 2 \pmod 4$. The N metaplectic modular categories for $N \equiv 2 \pmod 4$ have rank $k + 7$, where $k = N/2$ (an odd number). We will denote by $SO(2)_2$ the pointed modular category $\mathcal{C}(\mathbb{Z}_8, Q)$ with twists $e^{j^2\pi i/16}$ for uniformity of notation so that there are 4 inequivalent 2-metaplectic modular categories (since there are 4 inequivalent non-degenerate symmetric quadratic forms on $\mathbb{Z}_8$, see [33]). Generally, there are exactly 2^{s+1} inequivalent N-metaplectic modular categories in this case [6], where s is the number of prime divisors of N. The group of isomorphism classes of invertible objects for $N \geq 6$ is isomorphic to $\mathbb{Z}_4$. Let g be a generator of this group, so the (isomorphism classes of) invertible objects are g^j for $0 \leq j \leq 3$. There are $k-1$ self-dual simple objects, X_i and Y_i for $1 \leq i \leq \frac{k-1}{2}$, of dimension 2. The remaining four simple objects, V_i for $1 \leq i \leq 4$, have dimension $\sqrt{k}$. The following fusion rules hold [6]:

- $g \otimes X_a \cong Y_{\frac{k+1}{2}-a}$, and $g^2 \otimes X_a \cong X_a$, and $g^2 \otimes Y_a \cong Y_a$ for $1 \leq a \leq (k-1)/2$.
- $X_a \otimes X_a \cong \mathbf{1} \oplus g^2 \oplus X_{\min\{2a,k-2a\}}$; $X_a \otimes X_b \cong X_{\min\{a+b,k-a-b\}} \oplus X_{|a-b|}$ $(a \neq b)$.
- $V_1 \otimes V_1 \cong g \oplus \bigoplus_{a=1}^{\frac{k-1}{2}} Y_a$.
- $gV_1 = V_3$, $gV_3 \cong V_4$, $gV_2 \cong V_1$, $gV_4 \cong V_2$ and $g^3 V_a \cong V_a^*$, $V_2 \cong V_1^*$, $V_4 \cong V_3^*$.

Again adopting the same notion for simple objects in a general N-metaplectic category $\mathcal{C}$ with $N \equiv 2 \pmod 4$ one finds that g^2 is always a boson. In fact, the classification of N-metaplectic modular categories with $N \equiv 2 \pmod 4$ was obtained in [6] by condensing $\langle g^2 \rangle$ to obtain a pointed cyclic modular category. Indeed, the centralizer of $\langle g^2 \rangle \cong \operatorname{Rep}(\mathbb{Z}_2)$ has simple objects X_i, Y_i and the g^j i.e. all simple objects of dimension 1 or 2. The simple object Y_1 generates this subcategory, which has the same fusion rules as $\operatorname{Rep}(\mathbb{Z}_4 \ltimes \mathbb{Z}_k)$ (with the generator of $\mathbb{Z}_4$ acting by inversion on $\mathbb{Z}_k$) [26, Remark 4.4 and Theorem 4.8]. In this notation the $\mathbb{Z}_4$-grading on $\mathcal{C}$ has trivial component $\mathcal{C}_0$ with simple objects $\mathbf{1}, g^2, X_1, \ldots, X_{\frac{k-1}{2}}$, component $\mathcal{C}_2$ with simple objects $g, g^3, Y_1, \ldots, Y_{\frac{k-1}{2}}$ and the other two components with simple objects $\{V_1, V_3\}$ and $\{V_2, V_4\}$ respectively. Obviously there are labeling ambiguities associated with $g \leftrightarrow g^3$ and $\{V_1, V_3\} \leftrightarrow \{V_2, V_4\}$.

3.3. Fusion rules for $N \equiv 0 \pmod 4$. The N-metaplectic modular categories with $N \equiv 0 \pmod 4$ with $2k = N$ have rank $k+7$ and dimension $4N$ [26]. The simple objects have dimension $1, 2$ and $\sqrt{k}$ and are all self-dual. Setting $r = \frac{k}{2} - 1$, the $(2r+1 = k-1)$ simple objects X_i for $0 \leq i \leq r-1$ and Y_j for $0 \leq j \leq r$ have dimension 2 and the simple objects V_i, W_i have dimension $\sqrt{k}$. For $k > 2$ the key fusion rules are as follows [7]:

- $h^{\otimes 2} \cong g^{\otimes 2} \cong \mathbf{1}$, $h \otimes X_i \cong g \otimes X_i \cong X_{r-i-1}$ and $h \otimes Y_i \cong g \otimes Y_i \cong Y_{r-i}$
- $g \otimes V_1 \cong V_2, h \otimes V_1 \cong V_1$ and $h \otimes W_1 \cong W_2, g \otimes W_1 \cong W_1$
- $V_1^{\otimes 2} \cong \mathbf{1} \oplus h \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1^{\otimes 2} \cong \mathbf{1} \oplus g \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1 \otimes V_1 \cong \bigoplus_{i=0}^{r} Y_i$
- $X_i \otimes X_j \cong \begin{cases} X_{i+j+1} \oplus X_{j-i-1} & i < j \leq \frac{r-1}{2} \\ \mathbf{1} \oplus hg \oplus X_{2i+1} & i = j < \frac{r-1}{2} \\ \mathbf{1} \oplus h \oplus g \oplus hg & i = j = \frac{r-1}{2} < r-1 \end{cases}$
- $Y_i \otimes Y_j \cong \begin{cases} X_{i+j} \oplus X_{j-i-1} & i < j \leq \frac{r}{2} \\ \mathbf{1} \oplus hg \oplus X_{2i} & i = j \leq \frac{r-1}{2} \\ \mathbf{1} \oplus h \oplus g \oplus hg & i = j = \frac{r}{2}. \end{cases}$

Notice that all other fusion rules may be derived from the above by tensoring with h or g as needed. For example $V_1 \otimes V_2 \cong g \otimes V_1^{\otimes 2} \cong h \oplus hg \oplus \bigoplus_{i=0}^{r-1} X_i$. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading is clear from these rules, we denote the trivial component by $\mathcal{C}_{(0,0)}$ and the component with simple objects Y_j by $\mathcal{C}_{(1,1)}$. The classification of N-metaplectic modular categories with $4 \mid N$ was obtained in [7] by condensing hg, which is always a boson. It is shown in [7] that, for $N \geq 8$ there are $3 \cdot 2^{s+1}$ inequivalent N-metaplectic modular categories where s is the number of distinct primes dividing N. The degenerate case $N = 4$ is special: its fusion rules coincide with $\text{Ising}^{\nu} \boxtimes \text{Ising}^{\mu}$ for which there are 20 inequivalent metaplectic modular categories, rather than 12.

The centralizer of the pointed subcategory $\langle h, g\rangle$ is always the trivial component $\mathcal{C}_{(0,0)}$ with simple objects $\mathbf{1}, h, g, hg$, and all X_i, whereas $\langle hg\rangle'$ also includes the component $\mathcal{C}_{(1,1)}$ with simple objects Y_j and the component with simple objects V_i by $\mathcal{C}_{(1,0)}$ for concreteness. There is a slight further subtlety related to the value of $N \equiv 0, 4 \pmod 8$. The objects h, g are bosons precisely when $8 \mid N$, and are fermions otherwise. Moreover, when $8 \mid N$ one sees that h centralizes the trivial component as well as the component $\mathcal{C}_{(1,0)}$ containing V_1 and V_2, while g centralizes the W_i. When $8 \nmid N$ the opposite is true: g centralizes the V_i and h centralizes the W_i [7]. In [26] it is shown that the fusion subcategory generated by Y_0 (i.e. $\langle hg\rangle' = \mathcal{C}_{(0,0)} \oplus \mathcal{C}_{(1,1)}$) has the same fusion rules as the representation category $\text{Rep}(D_N)$ of the dihedral group of order N.

4. Property F for N-Metaplectic Categories with N odd. In [23] fusion categories with the same fusion rules as $SU(N)_k$ were characterized in terms of Galois conjugation and associativity twisting (see also [24] for the $SU(2)_k$ case). We use a similar characterization of N-metaplectic categories to obtain the following:

THEOREM 1. *If $\mathcal{C}$ is an N-metaplectic modular category with $N := 2r + 1$ odd, then $\mathcal{C}$ has property F.*

PROOF. Let $N = p_1^{a_1} \cdots p_s^{a_s}$ be the prime factorization of N. From [1], we know that there are precisely 2^{s+1} N-metaplectic modular categories. We will show that Galois conjugation and twisting [5] produce all of these categories.

A Galois conjugate $\mathbb{C}$ of the quantum group category $SO(N)_2$ is not necessarily unitary. However, it is pseudounitary, so there exists a unique choice of spherical structure on its underlying braided fusion category so that the objects have positive dimension and this new category $\mathbb{C}_+$ remains modular (see Proposition 1). This choice does not affect the braiding eigenvalues of the category. To ensure that $\mathcal{C}_+$ is metaplectic we must verify unitarity, which is a slightly delicate argument: By [1] $SO(N)_2$ is weakly group theoretical [11]. It follows that $[SO(N)_2]^{\sigma} = \mathcal{C}$ is also weakly group theoretical (one proof is to observe that $\mathcal{C}$ is the $\mathbb{Z}_2$-gauging of a pointed modular category). By [19, Theorem 5.20] *any* weakly group theoretical *fusion* category is monoidally equivalent to a *unique* unitary fusion category. By the main result of [18] *any* braiding on such a category is unitary, and, moreover, it admits a *unique* structure of a unitary ribbon (i.e. spherical braided) fusion category. By the uniqueness of the unitary and spherical structures and the freedom to keep the braiding we seek, $\mathcal{C}_+$ is metaplectic.

Let $\zeta = e^{\frac{2\pi i}{16r+8}}$. There exists a simple object $W \in SO(N)_2$ of dimension $\sqrt{N}$ such that the eigenvalues of the braiding $R_{W,W}$ are ζ^{n_j} for $n_j = (4r+2)((r-j)(r-j+1)-j)+(2r+1)r+2j^2$, and $0 \le j \le r$ [22]. The non-isomorphic simple object W' of dimesion $\sqrt{N}$ has braiding eigenvalues $-\zeta^{n_j}$ for $0 \le j \le r$.

The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/8N\mathbb{Z})^\times$ acts on the set of eigenvalue exponents $\{n_j : 0 \le j \le r\} \subset \mathbb{Z}/8N\mathbb{Z}$ by multiplication. By the Chinese Remainder theorem, the Galois group acts on each factor of $\mathbb{Z}/8N\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_s^{a_s}\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ independently.

We first observe that $n_j = 2j^2 \pmod N$. Since $2(-j)^2 = 2j^2$, we have $\{n_j \pmod N : 0 \le j \le r\} = \{2j^2 : j \in \mathbb{Z}/N\mathbb{Z}\}$ as sets. Hence, for any i, we have $X := \{n_j \pmod{p_i^{a_i}} : 0 \le j \le r\} = \{2j^2 : j \in \mathbb{Z}/p_i^{a_i}\mathbb{Z}\}$. The factor of the Galois group acting on $\mathbb{Z}/p_i^{a_i}\mathbb{Z}$ is $(\mathbb{Z}/p_i^{a_i})^\times$. Since $(\mathbb{Z}/p_i^{a_i}\mathbb{Z})^\times$ is cyclic, the stabilizer subgroup $\mathrm{Stab}_{(\mathbb{Z}/p_i^{a_i}\mathbb{Z})^\times}(X) = \{x^2 : x \in (\mathbb{Z}/p_i^{a_i}\mathbb{Z})^\times\}$ has index 2. Thus, we get two distinct sets of eigenvalues mod $p_i^{a_i}$ for each i.

Moreover, we have $n_j = r \pmod 8$ for all j. If r is relatively prime to 8, this gives 4 choices of Galois conjugates for $[n_j]_8$. If $r = 2$ or $r = 6$ mod 8, we have 2 choices. If $r = 0$ or $r = 4$ mod 8, there is only one choice. In all but the last ($r = 0$ or $r = 4$) case, we must divide by 2 to account for labelling ambiguity on the nonintegral objects. Thus, when r is relatively prime to 8, we get $(2^s)(4)/2 = 2^{s+1}$ distinct categories from Galois conjugation. When $r = 2$ or $r = 6$ mod 8, we get $(2^s)(2)/2 = 2^s$ distinct categories. When $r = 0$ or $r = 4$ mod 8, we get $(2^s)(1) = 2^s$ modular categories.

To construct the remaining metaplectic modular categories, we will use twisting in the sense of [5]. Let $\mathcal{D}$ be a modular category. Let $B \subset G(\mathcal{D})$ be a subgroup of the group of the invertibles of $\mathcal{D}$, and let $w \in Z^3(\widehat{B}, U(1))$ be a 3-cocycle. The twisted category $\mathcal{D}_{(1,w)}$ is a $\widehat{B}$-graded category with the same objects and tensor product as $\mathcal{D}$, but with an associator twisted by w. More explicitly, if $\sigma, \tau, \rho \in \widehat{B}$, then we have

$$\widehat{\alpha}_{X_\sigma, X_\tau, X_\rho} = w_{\sigma,\tau,\rho}\alpha_{X_\sigma, X_\tau, X_\rho},$$

where $\widehat{\alpha}$ and α are the associators of $\mathcal{D}_{(1,w)}$ and $\mathcal{D}$, respectively.

Let $B \subset G(B)$ be a subgroup such that the induced map $U(G) \to \widehat{G(B)} \to \widehat{B} \cong \mathbb{Z}_2$ corresponds to the GN-grading. Let $w \in Z^3(\mathbb{Z}_2, U(1))$ be the normalized 3-cocycle given by $w(1,1,1) = -1$. Let α and c denote the associator and braiding for some metaplectic modular category $\mathcal{D}$, and let $\widehat{\alpha}$ and $\widehat{c}$ denote the associator and braiding of the twisted category $\mathcal{D}_{(1,w)}$, respectively.

We claim that a solution to the hexagon equations is given by

$$\widehat{c}_{X_\sigma, X_\tau} = \varepsilon_{\sigma,\tau} c_{X_\sigma, X_\tau},$$

where $\varepsilon_{\sigma,\tau} = i$ if $\sigma = \tau = 1$, and $\varepsilon_{\sigma,\tau} = 1$ otherwise. Indeed, in diagrammatic composition order, we have

$$\begin{aligned} &\widehat{\alpha}_{X_\sigma,X_\tau,X_\rho} \circ \widehat{c}_{X_\sigma, X_\tau\otimes X_\rho} \circ \widehat{\alpha}_{X_\tau,X_\rho,X_\sigma} \\ &\quad = (w_{\sigma,\tau,\rho}\alpha_{X_\sigma,X_\tau,X_\rho}) \circ \varepsilon_{\sigma,\tau\rho} c_{X_\sigma,X_\tau\otimes X_\rho} \circ (w_{\tau,\rho,\sigma}\alpha_{X_\tau,X_\rho,X_\sigma}) \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_{\sigma,\tau\rho}\alpha_{X_\sigma,X_\tau,X_\rho} \circ c_{X_\sigma,X_\tau\otimes X_\rho} \circ \alpha_{X_\tau,X_\rho,X_\sigma} \\
&= \varepsilon_{\sigma,\tau\rho} \cdot (c_{X_\sigma,X_\tau} \otimes \mathrm{id}_{X_\rho}) \circ \alpha_{X_\tau,X_\sigma,X_\rho} \circ (\mathrm{id}_{X_\tau} \otimes c_{X_\sigma,X_\rho}) \\
&= \varepsilon_{\sigma,\tau\rho}\varepsilon^{-1}_{\sigma,\tau}\varepsilon^{-1}_{\sigma,\rho} w_{\tau,\sigma,\rho} \cdot (\widehat{c}_{X_\sigma,X_\tau} \otimes \mathrm{id}_{X_\rho}) \circ (\widehat{\alpha}_{X_\tau,X_\sigma,X_\rho}) \circ (\mathrm{id}_{X_\tau} \otimes \widehat{c}_{X_\sigma,X_\rho}) \\
&= (\widehat{c}_{X_\sigma,X_\tau} \otimes \mathrm{id}_{X_\rho}) \circ (\widehat{\alpha}_{X_\tau,X_\sigma,X_\rho}) \circ (\mathrm{id}_{X_\tau} \otimes \widehat{c}_{X_\sigma,X_\rho}),
\end{aligned}
$$

where the last equality follows from case analysis. The verification for the other hexagon equation is analogous.

The spherical structure on the twisted category $\mathcal{D}_{(1,w)}$ is the same as the spherical structure on $\mathcal{D}$. Since ε and w are $U(1)$-valued, the modular category $\mathcal{D}_{(1,w)}$ is also unitary.

Since any matrix in the twisted braid group representation differs from a matrix in the untwisted representation by a factor of the form i^n, this twisting preserves Property F. By examining the exponents of the braiding eigenvalues mod 8, we find that twisting accounts for another factor of 2 in our count when r is even, covering the remaining modular categories. □

REMARK 4.1. These results suggest that one can generalize this characterization to modular categories with the same fusion rules as $SO(N)_k$, akin to the results of [23]. As this is beyond our current scope we will leave this for a future publication.

We illustrate the classification used to prove Theorem 1 with the following tables of braiding eigenvalues for 3- and 5-metaplectic categories.

3-metaplectic categories. The following table gives the exponents of the relevant braiding eigenvalues of the Galois conjugates of $SO(3)_2$. More explicitly, given $\sigma \in (\mathbb{Z}/24\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ and $n \in \mathbb{Z}/24\mathbb{Z}$, we have the group action $\sigma(n) = \sigma \cdot n$. Letting $\zeta = e^{\frac{\pi i}{12}}$, the braiding eigenvalues of the σ-Galois conjugate of the first nonintegral object are $\sigma(R^i_{V_1,V_1}) = \zeta^{\sigma(n_i)}$. The braiding values of the other nonintegral object are given by $\sigma(R^i_{V_2,V_2}) = -\sigma(R^i_{V_1,V_1}) = \zeta^{\sigma(12+n_i)}$. Since $n_0 = 9$ and $n_1 = 1$, we have the following table of exponents of braiding eigenvalues of Galois conjugates.

σ	$\sigma(n_0)$	$\sigma(n_1)$	$\sigma(12+n_0)$	$\sigma(12+n_1)$
1	9	1	21	13
5	21	5	9	17
7	15	7	3	19
11	3	11	15	23

Since we know there are precisely four 3-metaplectic categories, this table illustrates the fact that all four 3-metaplectic categories lie in the same orbit under the Galois conjugation action, since they are distinguished by these eigenvalues.

5-Metaplectic categories. Here $\zeta = e^{\frac{\pi i}{20}}$. Similarly, we have the following table of exponents of braiding eigenvalues.

σ	$\sigma(n_0)$	$\sigma(n_1)$	$\sigma(n_2)$	$\sigma(20+n_0)$	$\sigma(20+n_1)$	$\sigma(20+n_2)$
1	10	18	2	30	38	22
3	30	14	6	10	34	26
7	30	6	14	10	26	34
9	10	2	18	30	22	38
11	30	38	22	10	18	2
13	10	34	26	30	14	6
17	10	26	34	30	6	14
19	30	22	38	10	2	18

Since $r = 2$, we only have two distinct sets of braiding eigenvalues in the table, so that Galois conjugation only provides two of the four 5-metaplectic categories. The other two categories are obtained by twisting: at the level of eigenvalues this is manifested by twisting by i, i.e. adding 10 to each exponent in a row of the table.

5. A Sequence of Gaugings. N-metaplectic modular categories with $4 \mid N$ have 4 self-dual invertible objects, are are therefore $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded. The $(0,0)$-graded component is the adjoint subcategory, and without loss of generality we may assume that the $(1,1)$-graded component contains all of the remaining 2-dimensional simple objects. The $(1,0)$- and $(0,1)$-graded components each contain two isomorphism classes of $\sqrt{N/2}$-dimensional simple objects.

When $8 \mid N$ the N-metaplectic modular categories of have 3 bosons hg, h, g, i.e. invertible, self-centralizing objects with trivial twists. The centralizer of each of these bosons consists of the $(0,0)$-graded (adjoint) component and one of the three other components. It was shown in [7] that condensing the boson hg that centralizes the (integral) $(1,1)$-graded component yields a cyclic modular category of the form $\mathcal{C}(\mathbb{Z}_N, q)$, for some non-degenerate symmetric quadratic form q on $\mathbb{Z}_N$. Except for the degenerate case $N = 8$, the bosons h, g are uniquely determined (up to the labeling ambiguity $h \leftrightarrow g$) by the condition that they centralize a simple object of dimension $\sqrt{\frac{N}{2}}$. For $N = 8$ all non-invertible simple objects have dimension 2, and the labels of all 3 bosons are ambiguous, i.e. one cannot distinguish them by any of their properties. The follow shows that condensing either of the two bosons h, g yields another metaplectic modular category.

THEOREM 2. *Let $\mathcal{C}$ be an N-metaplectic modular category with $8 \mid N$, and let $\mathcal{D}$ be the unitary modular category given by condensing the boson $h \in \mathcal{C}$ (or g) in the notation of Subsection 3.3. Then $\mathcal{D}$ is an $\frac{N}{4}$-metaplectic modular category.*

PROOF. For the moment, assume that $N \geq 16$. It is relatively straightforward to verify that $\mathcal{D}$ has the right rank and dimensions of simple objects. Set $N = 2k$ and $r = \frac{k}{2} - 1$. $\mathcal{C}$ has rank $k + 7$, with $k - 1 = 2r + 1$ objects of dimension 2: $X_0, \ldots, X_{r-1}$ and $Y_0, \ldots, Y_r$. By definition $\mathcal{D} = (\langle h\rangle')_{\mathbb{Z}_2}$ where $\mathrm{Rep}(\mathbb{Z}_2) = \langle h\rangle$. From the discussion in Section 2 we see that $\langle h\rangle' = \mathcal{C}_{(0,0)} \oplus \mathcal{C}_{(1,0)}$ with simple objects

$$\{\mathbf{1}, h, hg, g, X_0, X_1, \dots, X_{r-1}, V_1, V_2\}.$$

Let $F : \langle h\rangle' \longrightarrow (\langle h\rangle')_{\mathbb{Z}_2}$ be the de-equivariantization functor. As $h \otimes V_i \cong V_i$ and $h \otimes X_i \cong X_{r-i-1}$ we have the following, where we set $t = \frac{r-1}{2} = \frac{N}{8} - 1$:

(1) $F(V_i) = V_i^{(0)} \oplus V_i^{(1)}$, where $V_i^{(j)}$ are $\sqrt{\frac{N}{8}}$-dimensional objects.
(2) $\tilde{Y}_i := F(X_{2i}) \cong F(X_{r-2i-1})$ are simple objects of dimension 2, for $0 \le i < t/2$ (provided $N \ge 16$, otherwise there are no $\tilde{Y}_i$).
(3) $\tilde{X}_j := F(X_{2j+1}) \cong F(X_{r-2j-2})$ are simple objects of dimension 2, for $0 \le j < t/2$ (provided $N \ge 24$, otherwise there are no $\tilde{X}_j$).
(4) $F(X_t) = g_1 \oplus g_2$ with g_1, g_2 invertible.
(5) $F(h) = F(\mathbf{1}) = \mathbf{1}_{\mathcal{D}}$.
(6) $F(hg) = F(g) = Z$ an invertible object.

In particular, the modular category $\mathcal{D} = F(\langle h\rangle')$ has the same dimensions (1 with multiplicity 4, 2 with multiplicity $t = \frac{N}{8} - 1$ and $\sqrt{\frac{N}{8}}$ with multiplicity 4), global dimension (N) and rank ($\frac{N}{8} + 7$) as an $\frac{N}{4}$-metaplectic modular category. It is important to point out that when $16 \mid N$ we have $\frac{N}{4} \equiv 0 \pmod 4$ so that t is odd, while $\frac{N}{4} \equiv 2 \pmod 4$ so that t is even otherwise, so these cases correspond to either the self-dual fusion rules of Subsection 3.3 or the non-self-dual fusion rules of Subsection 3.2. Here are a few useful observations that can be deduced from the fusion rules of $\mathcal{C}$:

- $\mathcal{D}$ is graded by a group of order 4, with each component of dimension $\frac{N}{4}$.
- If $16 \mid N$ (so that t is odd) the trivial component $\mathcal{D}_0$ contains all 1-dimensional simple objects and $\frac{t-1}{2}$ simple objects of dimension 2, otherwise (i.e. t is even) the trivial component contains $\mathbf{1}_{\mathcal{D}}$ and Z but not g_1 or g_2.
- The object $\tilde{Y}_0$ generates the subcategory with simple objects $\mathbf{1}_{\mathcal{D}}, Z, g_1, g_2$ and all $\tilde{X}_j, \tilde{Y}_i$.
- The objects $Z, \tilde{Y}_i, \tilde{X}_j$ are self-dual.
- The subcategory generated by $\tilde{X}_0$ is the adjoint subcategory $\mathcal{D}_0$. In particular no $\tilde{Y}_i$ lie in the adjoint subcategory.
- The 4 objects $V_i^{(j)}$ appear in two distinct graded components, in pairs.

One may directly show that $\mathcal{D}$ has the same fusion rules as $SO(\frac{N}{4})_2$ using standard techniques, however this is a somewhat tedious task. We will instead make use of [26, Theorem 4.2 and Remark 4.4] and the descriptions in Section 3 to derive the result. The first step is to verify that the fusion rules for the $\frac{N}{2}$-dimensional subcategory $\langle \tilde{Y}_0\rangle$ with simple objects of dimensions 1 and 2 has the fusion rules of either $\mathrm{Rep}(D_{\frac{N}{4}})$ for t odd or $\mathrm{Rep}(\mathbb{Z}_4 \ltimes \mathbb{Z}_{\frac{k}{4}})$ for t even. Then we must verify the fusion rules involving the $V_i^{(j)}$ are also as expected. We will do these tasks simultaneously.

For t odd, the observations above reduce the verification of the hypotheses of Theorem 4.2 of [26] to showing that the g_i are self-dual, from which we can conclude that $\langle \tilde{Y}_0\rangle$ has the same fusion rules as $\mathrm{Rep}(D_{\frac{N}{4}})$. For t even, we must show that $g_1 \cong g_2^*$ to verify the hypotheses of Remark 4.4 of [26] to conclude that $\langle\tilde{Y}_0\rangle$ has the same fusion rules as $\mathrm{Rep}(\mathbb{Z}_4 \ltimes \mathbb{Z}_{\frac{k}{4}})$. We calculate:

$$F(V_1^{\otimes 2}) = \left(V_1^{(0)} \oplus V_1^{(1)}\right)^{\otimes 2} \cong g_1 \oplus g_2 \oplus 2\left(\mathbf{1}_{\mathcal{D}} \bigoplus_{i=0}^{\frac{t-1}{2}} \tilde{Y}_i \oplus \bigoplus_{j=0}^{\frac{t-3}{2}} \tilde{X}_j\right). \tag{1}$$

Since

$$\left(V_1^{(0)} \oplus V_1^{(1)}\right)^{\otimes 2} \cong \left(V_1^{(1)}\right)^{\otimes 2} \oplus \left(V_1^{(2)}\right)^{\otimes 2} \oplus 2(V_1^{(1)} \otimes V_1^{(2)})$$

it is clear that the g_1, g_2 cannot be subobjects of $(V_1^{(1)} \otimes V_1^{(2)})$ and $V_1^{(j)}$ for $j = 0, 1$ are either self-dual or dual to each other. As we have a labeling choice we may assume $g_j \subset \left(V_1^{(j)}\right)^{\otimes 2}$ for $j = 0, 1$.

Now observe that $\left(V_1^{(j)}\right)^{\otimes 2}$ is odd-dimensional when t is even so that in this case $\mathbf{1}_{\mathcal{D}}$ is not a subobject of $\left(V_1^{(j)}\right)^{\otimes 2}$ and hence the $V_1^{(j)}$ are non-self-dual, i.e. are dual to each other for $j = 0, 1$. Moreover, the g_i are not in the trivially graded component for t even so that we can conclude that the grading is by $\mathbb{Z}_4$ in this case, so that the group of (isomorphism classes of) invertible objects is isomorphic to $\mathbb{Z}_4$ and hence $g_1 \cong g_2^*$. Thus we can conclude that the fusion rules are the same as those of $\mathrm{Rep}(\mathbb{Z}_4 \ltimes \mathbb{Z}_k)$. Since the adjoint subcategory $\mathcal{D}_0$ contains only simple objects $\mathbf{1}_{\mathcal{D}}, Z$ and all $\tilde{X}_j$, the fusion rules involving $V_1^{(j)}$ (and similarly $V_2^{(j)}$) are completely determined.

When t is odd $\left(V_1^{(j)}\right)^{\otimes 2}$ is even-dimensional so we must have both $\mathbf{1}_{\mathcal{D}}$ and g_j as subojects. In particular, the grading is by $\mathbb{Z}_2 \times \mathbb{Z}_2$ so that the g_i are self-dual. Now we can conclude that the fusion rules of the subcategory $\langle\tilde{Y}_0\rangle$ are the same as $\mathrm{Rep}(D_{\frac{N}{2}})$ and the fusion rules involving $V_1^{(j)}$ (and similarly $V_2^{(j)}$) are determined from equation (1).

Condensing the boson h in an 8-metaplectic modular category produces a pointed category of dimension 8, with the same fusion rules as $\mathbb{Z}_8$, which we have conveniently identified with a 2-metaplectic modular category. □

6. Conclusions and Speculations. We have obtained two results on metaplectic modular categories. For odd N, we extend the results of [30] proving property F for $SO(N)_2$ to all N-metaplectic modular categories. This provides some insight into the relationships among (certain) braided fusion categories with the same fusion rules. A recent paper of Nikshych [28] explores the different braidings that a fixed fusion category may have. One consequence (see [28, Remark 4.2]) is that if a *modular* category has property F then any braiding on the underlying fusion category has property F as well (whether the braiding in non-degenerate or not). This is not useful for metaplectic categories since the underlying fusion categories usually differ. On the other hand, it seems to be often the case that all (finitely many, by Ocneanu rigidity [15]) fusion categories with a fixed set of fusion rules are related to each other by some type of twisting of associativity constraints (see [23, 31], for example). One conceptual step towards proving property F would be to extend the results of [28] to prove that braided fusion categories with a fixed set of fusion rules either all have property F or all do not.

In a related direction, we have shown that the N-metaplectic modular categories for $8 \mid N$ are obtained from $2k$- and $4k$-metaplectic modular categories (with $k \geq 1$ odd) by iteratively gauging by a non-trivial $\mathbb{Z}_2$-action. Physically, this can be interpreted to mean that the systems modeled by $2^t k$-metaplectic modular categories for all $t \geq 1$ of the same parity are just different phases of the same topological order [9]. It is interesting to note that the number of $2^t k$-metaplectic modular categories stabilizes for $t \geq 2$, so that the choices in the $\mathbb{Z}_2$-gauging process are eventually unique. Of course it is already known that any N-metaplectic modular category is a $\mathbb{Z}_2$-gauging of a pointed category ([1, 6, 7]), but this result provides an infinite sequence of categories with non-trivial Picard group (see [17]), i.e. non-trivial braided tensor autoequivalences. Notice this is in contrast to the odd N-metaplectic modular categories: for example $N = 3$ we see that 3-metaplectic modular categories admit no non-trivial braided tensor autoequivalences. This can be deduced from [13]: the Brauer-Picard group of $SO(3)_2 = SU(2)_4$ is $\mathbb{Z}_2$, with the non-trivial element corresponding to interchanging the two $\sqrt{3}$-dimensional objects. Since the twists of these two objects are distinct, this action does not give a braided tensor autoequivalence. Of course, failing to have a non-trivial Picard group does not preclude a category from having non-trivial (i.e. not a Deligne product) gaugings: the pointed category Sem has trivial Picard group and yet prime modular categories of the form $\mathcal{C}(\mathbb{Z}_8, Q)$ can be obtained as $\mathbb{Z}_2$-gaugings of Sem [3].

In the special case when $N = 2^k$ we encounter degenerate (in the sense of Lie algebras) categories: an 8-metaplectic modular category with the fusion rules of $SO(8)_2$ has 3 non-trivial bosons, but they cannot be distinguished. Condensing any of them yields a category with the fusion rules of $\mathbb{Z}_8$, i.e. a 2-metaplectic category. If we condense the boson in any of the four $\mathcal{C}(\mathbb{Z}_8, Q)$ theories we obtain either Sem or $\overline{\text{Sem}}$, which we could call $\frac{1}{2}$-metaplectic. It is worth pointing out that $SO(8)_2$ should have an S_3 action by braided tensor autoequivalences.

For $N = 16$ if we condense either of the two bosons that centralize a simple object of dimension $\sqrt{8}$ we obtain a 4-metaplectic modular category of the form $\text{Ising}^\nu \boxtimes \text{Ising}^\mu$, e.g. $SO(4)_2$. It is known ([7]) that there are 12 inequivalent 16-metaplectic modular categories, whereas there are 20 with the same fusion rules as $\text{Ising} \boxtimes \text{Ising}$. Which of the 20 can appear in this way? In this case we find that only the 12 that are $\mathbb{Z}_2$-gaugings of the 4 pointed categories $\mathcal{C}(\mathbb{Z}_4, Q_s)$ have the correct central charge $e^{(2s+1)\pi i/4}$. We could call these $\mathcal{C}(\mathbb{Z}_4, Q_s)$ theories 1-metaplectic categories as they are obtained by condensing a boson in $SO(4)_2$. More generally, let $k \geq 3$ be an odd number with precisely s distinct prime factors. Then there are 2^{s+2} inequivalent $2k$-metaplectic categories [6] and $3 \cdot 2^{s+2}$ inequivalent $2^a k$-metaplectic categories for $a \geq 2$ [7]. In particular we find that the cohomological choices in the gauging process from a $2^a k$-metaplectic category to a $2^{a+2} k$-metaplectic category does not increase the number of such categories, rather their cardinality stabilizes.

REFERENCES

[1] E. ARDONNE, M. CHENG, E. ROWELL AND Z. WANG, Classification of metaplectic modular categories, J. Algebra 466 (2016), 141–146.

[2] B. BAKALOV AND A. KIRILLOV JR., Lectures on tensor categories and modular functors, University Lecture Series, American Mathematical Society, 2001.

[3] M. BARKESHLI, P. BONDERSON, M. CHENG AND Z. WANG, Symmetry, Defects, and Gauging of Topological Phases, preprint arXiv:1410.4540.

[4] A. BRUGUIERES, Categories premodulaires, modularisations et invariants des varietes de dimension 3, Math. Ann. 316 (2000), 215–236.

[5] P. BRUILLARD, C. GALINDO, T. HAGGE, S.-H. NG, J. PLAVNIK, E. ROWELL AND Z. WANG, Fermionic Modular Categories and the 16-fold Way, J. Math. Phys. 58 (2017), 041704, 31 pp.

[6] P. BRUILLARD, J. Y. PLAVNIK AND E. C. ROWELL, Modular categories of dimension p^3m with m square-free, Proc. Amer. Math. Soc. 147 (2019), no. 1, 21–34.

[7] P. BRUILLARD, P. GUSTAFSON, J. PLAVNIK AND E. C. ROWELL, Dimension as a quantum statistic and the classification of metaplectic categories, 89–113, Contemp. Math. 747, Amer. Math. Soc., Providence, RI, 2020.

[8] P. BRUILLARD, S.-H. NG, E. C. ROWELL AND Z. WANG, Rank-finiteness for modular categories, J. Amer. Math. Soc. 29 (2016), no. 3, 857–881.

[9] F. BURNELL, Anyon condensation and its applications, Ann. Rev. Condensed Matter Phys. 9 (2018), 307–327.

[10] S. X. CUI, C. GALINDO, J. Y. PLAVNIK AND Z. WANG, On gauging symmetry of modular categories, Comm. Math. Phys. 348 (2016), no. 3, 1043–1064.

[11] V. DRINFELD, S. GELAKI, D. NIKSHYCH AND V. OSTRIK, On braided fusion categories I, Selecta Math. 16 (2010), 1–119.

[12] A. DAVYDOV, M. MÜGER, D. NIKSHYCH AND V. OSTRIK, The Witt group of non-degenerate braided fusion categories, J. Reine Angew. Math. 677 (2013), 135–177.

[13] C. EDIE-MICHELL, The Brauer-Picard groups of fusion categories coming from the ADE subfactors, Internat. J. Math. 29 (2018), no. 5, 1850036, 43 pp.

[14] P. ETINGOF, S. GELAKI, D. NIKSHYCH AND V. OSTRIK, Tensor categories, Mathematical Surveys and Monographs 205, American Mathematical Society, 2015.

[15] P. ETINGOF, D. NIKSHYCH AND V. OSTRIK, On fusion categories, Ann. of Math. 162 (2005), no. 2, 581–642.

[16] P. ETINGOF, D. NIKSHYCH AND V. OSTRIK, Weakly group-theoretical and solvable fusion categories, Adv. Math. 226 (2011), no. 1, 176–205.

[17] P. ETINGOF, D. NIKSHYCH AND V. OSTRIK, Fusion categories and homotopy theory, With an appendix by Ehud Meir, Quantum Topol. 1 (2010), no. 3, 209–273.

[18] C. GALINDO, On braided and ribbon unitary fusion categories, Canad. Math. Bull. 57 (2014), 506–510.

[19] C. GALINDO, S.-M. HONG AND E. C. ROWELL, Generalized and quasi-localization of braid group representations, Int. Math. Res. Not. (2013), no. 3, 693–731.

[20] S. GELAKI, D. NAIDU AND D. NIKSHYCH, Centers of graded fusion categories, Algebra Number Theory 3 (2009), no. 8, 959–990.

[21] M. B. HASTINGS, C. NAYAK AND Z. WANG, Metaplectic anyons, Majorana zero modes, and their computational power, Phys. Rev. B 87 (2013), 165421.

[22] M. B. HASTINGS, C. NAYAK AND Z. WANG, On metaplectic modular categories and their applications, Comm. Math. Phys. 330 (2014), no. 1, 45–68.

[23] D. KAZHDAN AND H. WENZL, Reconstructing monoidal categories, I. M. Gelfand Seminar, 111–136, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993.

[24] J. FRÖHLICH AND T. KERLER, Quantum groups, quantum categories and quantum field theory. Lecture Notes in Mathematics, 1542. Springer-Verlag, Berlin, 1993.

[25] M. MÜGER, Galois extensions of braided tensor categories and braided crossed G-categories, J. Algebra 277 (2004), no. 1, 256–281.

[26] D. NAIDU AND E. C. ROWELL, A finiteness property for braided fusion categories, Algebr. Represent. Theory 14 (2011), no. 5, 837–855.

[27] S. NATALE, The core of a weakly group-theoretical braided fusion category, Internat. J. Math. 29 (2018), no. 2, 1850012, 23 pp.

[28] D. NIKSHYCH, Classifying braidings on fusion categories, Tensor categories and Hopf algebras, 155–167, Contemp. Math., 728, Amer. Math. Soc., Providence, RI, 2019.

[29] B. PAREIGIS, On braiding and dyslexia, J. Algebra 171 (1995), 413–425.

[30] E. C. ROWELL AND H. WENZL, $SO(N)_2$ Braid Group Representations are Gaussian, Quantum Topol. 8 (2017), no. 1, 1–33.

[31] I. TUBA AND H. WENZL, On braided tensor categories of type BCD, J. Reine Angew. Math. 581 (2005), 31–69.

[32] V. G. TURAEV, Quantum invariants of knots and manifolds, Second revised edition, De Gruyter Studies in Mathematics, 18, Walter de Gruyter & Co., Berlin, 2010.

[33] C. T. C. WALL, Quadratic Forms on Finite Groups, and related Topics, Topology 2 (1963), 281–298.

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