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CENTERS OF SYLOW SUBGROUPS AND AUTOMORPHISMS

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ABSTRACT

Suppose that p is an odd prime and G is a finite group having no normal non-trivial p'-subgroup. We show that if a is an automorphism of G of p-power order centralizing a Sylow p-group of G, then a is inner.

1. Introduction

Let p be a prime. There has been quite a lot of interest in the problem of characterizing automorphisms of order a power of p centralizing a Sylow p-subgroup of a finite group G.

In particular, Question 14.1 of the Kourovka Notebook [14] which was posed in 1999 asked whether if G has no non-trivial normal odd order subgroups (i.e., $O_{2'}(G) = 1$), then for any such automorphism a with $p = 2$, a^2 is inner. This had already been answered in the affirmative in Glauberman [4, Corollary 8] in 1968. By taking $G = A_n$ with $n \geq 6$ with $n \equiv 2, 3 \bmod 4$ and $g \in S_n$ a transposition, one sees that it is not always true that a is inner.

If p is odd, then under the assumptions that $O_p(G) = O_{p'}(G) = 1$, Gross [10] showed that any such automorphism is inner. Gross used the classification of finite simple groups while Glauberman did not. In this note, we show how to extend the result of Gross allowing the possibility of nontrivial $O_p(G)$ (and using the classification of finite simple groups). This was conjectured by Gross in [10] and a partial result was obtained by Murai [16]. We complete the proof and show:

Theorem 1.1: *Let p be an odd prime and L a finite group with $O_{p'}(L) = 1$. Suppose that a is an automorphism of L whose order is a power of p and a centralizes a Sylow p-subgroup of L. Then a is an inner automorphism of L.*

It is not hard to see that Theorem 1.1 can fail if $O_{p'}(L) \neq 1$. For example, take $L = G \times O_{p'}(L)$ and choose a to be a non-inner automorphism of $O_{p'}(L)$, viewed as an automorphism of L acting trivially on G. A consequence of Theorem 1.1 is the following. Recall that $F^*(G)$ is the generalized Fitting subgroup of G.

Corollary 1.2: *Let p be an odd prime, and let G be a finite group with $O_{p'}(G) = 1$. Let P be a Sylow p-subgroup of G. Then*

$$Z(P) \leq F^*(G).$$

If $p = 2$, the analogous corollary is that if G is a finite group with no odd normal subgroups, then $Z(P)/(Z(P) \cap F^*(G))$ is of exponent 2 (and because of Glauberman's result, this does not depend upon the classification of finite simple groups). If $G = S_n$ with $n \equiv 2 \bmod 4$, then each transposition is central in some Sylow 2-subgroup P and so certainly it is not the case that $Z(P) \leq F^*(G)$.

We will give two proofs of Theorem 1.1. The first proof uses the Thompson subgroup and the Z_p^* theorem for p odd (the analogue of Glauberman's Z^* theorem for $p = 2$). In contrast to Glauberman's Z^* theorem, the Z_p^* theorem relies on the classification of finite simple groups.

The second proof reduces to the almost simple case (where we use Gross's result [10]) and to a more subtle result about automorphisms of quasi-simple groups. See Theorem 4.2.

We mention that there is a result directly analogous to Theorem 1.1 that applies to a centric linking system $\mathcal{L}$ associated to a saturated fusion system over a p-group S with p odd. Namely, any automorphism of $\mathcal{L}$ which restricts to the identity on S, appropriately defined, is "conjugation by" an element of $Z(S) \leq \mathrm{Aut}_{\mathcal{L}}(S)$. This follows by combining [2, Proposition III.5.12], which connects automorphisms of $\mathcal{L}$ with limits of the center functor, with [17, Theorem 3.4] or [6, Theorem 1.1], which show that the center functor is acyclic at odd primes. Likewise, it is shown in work to appear [7] that there is an analogue of [4, Corollary 8] for centric linking systems at the prime 2. These purely local results do not require an appeal to the Classification.

The paper is organized as follows. In the next section we introduce some notation and prove a few preliminary results. In the following section, we discuss the Z_p^* theorem and indicate some connections with Gross's result and then give our first proof. In the next section we prove Theorem 4.2 (which is stronger than the main theorem for quasi-simple groups) and then show how to deduce Theorem 1.1 from the results about simple and quasi-simple groups. Finally we deduce the corollary.

We will need to use detailed properties of automorphism groups of simple groups. Most of these results were first obtained by Steinberg. We refer the reader to the reference [9, 2.5]. In particular, if $p \geq 3$, then the only quasi-simple groups with nontrivial outer automorphisms of order a power of p are groups of Lie type. Moreover if $p \geq 5$, then only field automorphisms are possible unless we are in type A, in which case there may be diagonal automorphisms of p-power order. If $p = 3$, there are more cases with diagonal automorphisms and triality needs to be dealt with as well.

In the final section, we prove Theorem 6.1 on permutation groups, and show that it is equivalent to the Z_p^* theorem.

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2. Notation and preliminary results

Let G be a finite group and p a prime. We recall some notation (see [13] for more details). The maximal normal p-subgroup of G is denoted by $O_p(G)$ and $O_{p'}(G)$ is the maximal normal p'-subgroup of G. The Fitting subgroup, $F(G)$, is the maximal normal nilpotent subgroup of G and is the direct product of the subgroups $O_r(G)$ where r ranges over all prime divisors of $|G|$.

A quasi-simple group is a group Q such that Q is perfect (i.e., $Q = [Q, Q]$) and $Q/Z(Q)$ is a non-abelian simple group. A component of G is a subnormal quasi-simple subgroup. Then $E(G)$ is the subgroup of G generated by all components of G (and is the central product of all the components of G). The generalized Fitting subgroup $F^*(G)$ of G is the central product $E(G)F(G)$. It has the very important property that $C_G(F^*(G)) = Z(F^*(G)) = Z(F(G))$.

Lemma 2.1: *Suppose that a finite group L acts via automorphisms on a finite abelian group M. Let N be an L-invariant subgroup of M such that L acts trivially on the p-group M/N. Assume that $M = NC_M(T)$ for some Sylow p-subgroup T of L. Then $M = NC_M(L)$.*

Proof. Let $m \in C_M(T)$. Thus, $X := \{m^\ell \mid \ell \in L\}$ is an L-invariant subset of M of cardinality e with e prime to p. Let

$$m' = \prod_{x \in X} x.$$

Clearly, $m' \in C_M(L)$. Using that L acts trivially on M/N, we have that $m'N = m^e N$. Then $m^e \in C_M(L)N$. Since e is prime to p, we have that the map $x \mapsto x^e$ is a bijection $C_M(T)/C_N(T) \to C_M(T)/C_N(T)$. Therefore $C_M(T) \subseteq C_M(L)N$, and we deduce that $M = C_M(L)N$. ∎

The following result handles an easy case of the theorem and gives a reduction to the case that $L = E(L)T$ where T is a Sylow p-subgroup of L. The next two results hold without assuming that a has order a power of p, but that is the important case and the only case we use.

Lemma 2.2: *Let L be a finite group, p prime and $O_{p'}(L) = 1$. If a is an automorphism of L of order a power of p which centralizes $E(L)T$ with T a Sylow p-subgroup of L, then a induces an inner automorphism of L.*

Proof. Let $G = L\langle a\rangle$ be the semidirect product. Notice that if Q is a component of G, since Q is perfect and $Q/(Q \cap L)$ is cyclic, we have that $Q \subseteq L$. Thus the components of G are the components of L. Hence, we have that $F^*(G) = E(L)S$ where $S = F(G)$ is a p-group.

By hypothesis, a is in the center of the Sylow p-subgroup $T\langle a\rangle$ of G and a centralizes $E(L)T$. Since $F(G) \le T\langle a\rangle$, we have that a centralizes

$$F^*(G) = E(G)F(G) = E(L)F(G).$$

So a lies in $Z(F^*(G)) = Z(F(G)) = Z(S)$.

Note that $M := Z(S)$ is an abelian p-group and M/N is centralized by L where $N = M \cap L$. Now apply the previous lemma to conclude that there exists $z \in N \le L$ so that az centralizes L and so a induces conjugation by z^{-1}. ∎

A special case of the previous result is the following:

Corollary 2.3: *Let L be a finite group with $F^*(L) = O_p(L)$. If a is a nontrivial automorphism of L of order a power of p which centralizes a Sylow p-subgroup P of L, then a is induced by conjugation by an element of $Z(P)$.*

The following result will be used in studying quasi-simple groups. If $g \in L$, then g^L denotes the conjugacy class of g in L.

Lemma 2.4: *Let L be a finite group with Z a central p-subgroup. Let $g \in L$ such that $|L : C_L(g)|$ is not divisible by p. Then $g^L \cap gZ = \{g\}$.*

Proof. Suppose that $g^u = gz$ for some $z \in Z$ and $u \in L$. Thus $[g, u] = z$ and so $[g, u^r] = z^r$ for every integer r.

Note also that u normalizes the centralizer of g. If $H = N_L(C_L(g))$, then the hypotheses of the lemma are satisfied in H, and working by induction on $|L|$, we may assume that $H = L$. Now, $L/C_L(g)$ is a p'-group. Let r be the p'-part of the order of u, and let $v = u^r$. Then v is in $C_L(g)$ and $g^v = gz^r$. Hence, $z^r = 1$. Since Z is a p-group, $z = 1$. ∎

3. The Z_p^* theorem and a proof

Let G be a group and $x \in H$ for some subgroup H of G. We say x is isolated or weakly closed in H with respect to G if $x^G \cap H = \{x\}$.

Theorem 3.1: *Let G be a finite group with a Sylow p-subgroup S. If $x \in S$ is isolated with respect to G, then x is central modulo $O_{p'}(G)$.*

The result is usually stated for elements of order p but this implies the result for elements of order p^a by Lemma 3.2 below. We also note that it is easy to see that if x is isolated in a Sylow p-subgroup, then x is isolated in its centralizer (i.e., it does not commute with any distinct conjugate).

If $p = 2$ and x is an involution, then this is Glauberman's Z^* theorem [3] and does not depend on the classification of finite simple groups. So assume that p is odd.

It was observed by many that this follows from the classification of finite simple groups (with some effort) and always stated with the extra assumption that x has order p. It was proved in [1, Thm. 1], [11, Thm. 4.1] and follows easily by [9, 7.8.2, 7.8.3]. Interestingly, the first two proofs both used [10] to reduce from the almost simple case (i.e., $F^*(G)$ is simple) to the simple case. The last proof uses a different result about weakly closed subgroups of order p. The result for any p-element x follows by the result for elements of order p and Lemma 3.2 below.

We give a quick sketch to indicate the connection with Gross's result. There is no loss in assuming $O_{p'}(G) = 1$ and then showing $x \in Z(G)$. The result is clear if $x \in O_p(G)$; so assume this is not the case. By induction we assume that G is the normal closure $\langle x^G \rangle$ of x because otherwise we obtain $x \in Z(\langle x^G \rangle) \leq O_p(G)$. If $E(G) = 1$, then $F^*(G) = O_p(G)$ and $x \in C_G(F^*(G)) = Z(O_p(G))$ and the result holds. Let Q be a component of G. Thus $|Q/Z(Q)|$ has order divisible by p, whence x normalizes each component and therefore so does G. Note that x must act non-trivially on $Q/Z(Q)$ (as G is the normal closure of x).

Clearly $xZ(Q)$ is isolated in $G/Z(Q)$. First assume that $Z(Q) = 1$ and so we may assume that G is almost simple. At this point, we can invoke [10] to conclude that x induces an inner automorphism and so reduce to the simple case. In order to avoid that, we use [9, 7.8.2, 7.8.3] (see also [8, 4.250]) to reduce to the simple case and to a short list of possibilities. In all these cases, there is an involution inverting x and the result follows.

More generally this shows that $xZ(Q)$ is central in $G/Z(Q)$ whence $x \in O_p(G)$, which contradicts the fact that $G = \langle x^G \rangle$.

One can also prove the result more directly after reducing to the case of simple groups. If G is alternating, one sees that $N_G(\langle x \rangle) \neq C_G(\langle x \rangle)$ for any nontrivial p-element x, whence x is not isolated. One checks the sporadic groups directly. If G is a finite group of Lie type in characteristic p, the center consists of root elements (or products of commuting short and long root elements in a few cases) and it is easy to check. If G is a finite group of Lie type in characteristic $r \neq p$, then in most groups, we have -1 in the Weyl group and every semisimple element of odd order is real, whence not isolated. This leaves the cases $\mathrm{PSL}_n(q)$, $n > 2$, $\mathrm{PSU}_n(q)$, $n > 2$ and orthogonal groups in dimension $2m$ with $m > 3$ odd, and E_6 and ${}^2E_6(q)$. The argument for the classical groups is an easy linear algebra argument and the group of type E_6 follows by inspection of normalizers of maximal tori. See [12] for similar arguments (proving a somewhat different result).

Lemma 3.2: *Let G be a finite group with Sylow p-subgroup S. If $a \in S$ is weakly closed in S with respect to G, then any power of a has the same property.*

Proof. The hypothesis implies that $a \in Z(S)$. Let b be a power of a, and let $g \in G$ with $b_1 := b^{g^{-1}} \in S$. Note that a commutes with b_1 and so a^g commutes with b. So a^g is a p-element in $C_G(b)$ and S is a Sylow p-subgroup of $C_G(b)$. Thus, $a^{gh} \in S$ for some $h \in C_G(b)$. Since a is weakly closed in S, $a^{gh} = a$ and so also $b^{gh} = b$. Thus, $b^g = b$ whence $b_1 = b$ and so b is isolated in S. ∎

We can now prove the main theorem.

Theorem 3.3: *Let p be an odd prime, and let G be a finite group with Sylow p-subgroup S. Let a be an automorphism of G of p-power order which centralizes S. If $O_{p'}(G) = 1$, then a is inner.*

Proof. The argument is similar in part to the proof of [6, Lemma 8.2]. We induct on the order of a. Let $\widehat{G} = G\langle a \rangle$ be the semidirect product, and let $\widehat{S} = S\langle a \rangle$. For each subgroup X of $\widehat{S}$, denote by $J(X)$ the Thompson subgroup of X generated by the abelian subgroups of X of maximum order. Set $\widehat{D} = Z(J(\widehat{S}))$ and $D = \widehat{D} \cap S$ for short. Then $\widehat{S}$ is Sylow in $\widehat{G}$ and $\langle a \rangle \leq Z(\widehat{S}) \leq \widehat{D}$. So $\widehat{X} = X \times \langle a \rangle$ for $X \in \{S, D\}$.

Let $\widehat{H} = N_{\widehat{G}}(J(\widehat{S}))$ in which $\widehat{S}$ is Sylow, and let n be the index of $\widehat{S}$ in $\widehat{H}$. As $\widehat{G}/G$ is abelian and $\widehat{D}$ is normal in $\widehat{H}$,

$$[\widehat{D}, \widehat{H}] \leq G \cap \widehat{D} = D,$$

so

$$\widehat{H} \text{ centralizes } \widehat{D}/D. \tag{3.1}$$

Consider the norm/transfer/trace map

$$\mathbf{N} = \mathbf{N}_{\widehat{S}}^{\widehat{H}} : C_{\widehat{D}}(\widehat{S}) \to C_{\widehat{D}}(\widehat{H})$$

defined by setting

$$\mathbf{N}(d) = \prod_{h \in [\widehat{H}/\widehat{S}]} d^h.$$

By (3.1),

$$\mathbf{N}(a) \equiv a^n \pmod{D}.$$

Since $|a|$ is coprime to n, the restriction $\mathbf{N}_{\langle a \rangle}$ is injective, and we may choose $m \geq 1$ with $\mathbf{N}(a^m) \equiv a \pmod{D}$. Thus we may find $z \in D$ such that

$$az = \mathbf{N}(a^m) \in C_{\widehat{D}}(\widehat{H}).$$

Since $az \in C_{\widehat{D}}(\widehat{H}) \leq Z(\widehat{S})$ and $a \in Z(\widehat{S})$, we see that

$$z \in Z(\widehat{S}) \cap G = Z(S).$$

Set $a_1 = az$; it has order $|a|$. By construction a_1 is weakly closed in $\widehat{S}$ with respect to $\widehat{H}$. Since p is odd, a_1 is weakly closed in $\widehat{S}$ with respect to $\widehat{G}$ by [5, Theorem 14.4].

Let b_1 be a power of a_1 having order p. Lemma 3.2 gives that b_1 is also weakly closed in $\widehat{S}$ with respect to $\widehat{G}$. So the assumption $O_{p'}(G) = 1$ and the Z_p^* theorem yield $b_1 \in Z(\widehat{G})$. This shows that conjugation by a_1 induces an automorphism of G of order at most $|a|/p$ centralizing S, and so conjugation by a_1 is inner by induction. It follows that a is inner. ∎

4. Almost simple groups

The following two theorems about simple and quasi-simple groups provide the key to give a second proof of our main results. The first is a result of Gross [10], while the other is new and may be of independent interest. Both results depend upon the classification of finite simple groups.

THEOREM 4.1: *Let p be an odd prime, and suppose L is a finite non-abelian simple group with order divisible by p. Also, let a be an automorphism of L that has p-power order, and assume that a centralizes a Sylow p-subgroup of L. Then a is an inner automorphism of L.*

THEOREM 4.2: *Let p be an odd prime and suppose that Q is a finite quasi-simple group with center Z. Let P be a Sylow p-subgroup of Q and let $x \in Z(P)$. Let H be the largest subgroup of $\mathrm{Aut}(Q)$ with $[H, x] \leq Z$ (i.e., $H = C_{\mathrm{Aut}(Q)}(xZ/Z)$). Then there exists $y \in xZ$ a p-element such that H centralizes y.*

We prove Theorem 4.2 below. It gives the following corollary, which is used in the second proof of our main result. The corollary is an immediate consequence of the theorem by noting that an automorphism σ of Q commutes with conjugation by x if and only $[\sigma, x] \in Z$.

COROLLARY 4.3: *Let Q be a quasi-simple group with center Z a p-group. Suppose that $x \in Z(P)$ with P a Sylow p-subgroup of Q. There exists $y \in xZ$ such that if σ is an automorphism of Q that commutes with conjugation by x, then σ fixes y.*

The remainder of this section is devoted to proving Theorem 4.2. We are assuming that Theorem 4.1 holds.

We first note:

LEMMA 4.4: *Let Q be a quasi-simple group. If p does not divide $|Z(Q)|$ or p does not divide $|\mathrm{Out}(Q)|$, then Theorem 4.2 holds for Q.*

Proof. Suppose that $Z = Z(Q)$ is a p'-group. Then $\langle x \rangle$ is the Sylow p-subgroup of $\langle x, Z \rangle$ and so $[H, x] \leq Z$ implies that $[H, x] = 1$. Indeed, the same argument shows that by passing to $Q/O_{p'}(Z)$, we may assume that Z is a nontrivial p-group.

By Lemma 2.1, it suffices to prove the result for a Sylow p-subgroup R of H. If p does not divide $|\mathrm{Out}(Q)|$, then R induces inner automorphisms on Q. By Lemma 2.4, it follows that R centralizes x. ∎

The previous result shows that Theorem 4.2 holds when Q/Z is an alternating or sporadic group (since the outer automorphism group has order $1, 2$ or 4 [9, 5.2.1, Table 5.3]). Thus we may assume that $L := Q/Z$ is a finite group of Lie type and moreover that Z is a nontrivial p-group.

If the characteristic of L is prime to p, then almost always the Schur multiplier has order prime to p. Using [9] it is straightforward to check in the few cases where p does divide the order of the Schur multiplier, p does not divide the order of the outer automorphism group, whence Theorem 4.2 holds.

Thus, we may assume that L is a finite group of Lie type in characteristic $r \neq p$. By [9], the only L such that p divides both the order of the Schur multiplier and the outer automorphism group are:

(i) $L = \mathrm{PSL}(d, q)$ and p divides $(d, q-1)$; or
(ii) $L = \mathrm{PSU}(d, q)$ and p divides $(d, q+1)$; or
(iii) $p = 3$ and $L = E_6(q)$ and 3 divides $q-1$ or $L = {}^2E_6(q)$ and 3 divides $q+1$.

Note that in all cases the Schur multiplier of L is cyclic and so Z is cyclic. In the last case above, $Z(P) = Z$ by [15] whence the theorem holds in that case.

We next prove an elementary result that is the key to proving Theorem 4.2. The statement of the result is almost as long as the proof.

LEMMA 4.5: *Let c and d be positive integers. Let p be an odd prime and let C be a cyclic group of order p^c. Let e be a positive integer with $e \equiv 1 \pmod{p}$. Set $M = C^d$ (the direct sum of d copies of C) and view M as a module for $S_d \times \langle\sigma\rangle$ where S_d acts on M by permuting the coordinates of M and $\sigma(x) = ex$ for all $x \in M$. Let $\epsilon : M \to C$ be the augmentation map (i.e., the sum of the coordinates) and $M_0 = \ker(\epsilon)$. Let Z be the group of fixed points of S_d on M and set $Z_0 = Z \cap M_0$. Let Q be a Sylow p-subgroup of S_d. Let*

$$M_1 = \{x \in M_0 | [x, Q] \leq Z_0\}.$$

Then

$$\{x \in M_1 | [x, \sigma] \in Z_0\} = Z_0 + C_{M_1}(\sigma).$$

Proof. Let $q = p^b$ be the largest power of p dividing d. If $q = 1$, then $M = M_0 \oplus Z$ and the result is clear. So assume that $q > 1$.

Note that if $[x, \sigma] \in Z$, then $x = (x_1, \ldots, x_d)$ where $x_i = w + s_i$ with $w, s_i \in C$ and $(e-1)s_i = 0$.

First suppose that $q \neq d$. Thus, Q has more than one orbit and so we see that M_1 consists of those elements in M_0 in which the coordinates are constant on each orbit of Q. Thus, if $x \in M_1$ and $[\sigma, x] \in Z_0$, it follows that $x = (x_1, \ldots, x_d)$ where $x_i = w + s_i$ and each s_i occurs a multiple of q times (since the coordinate is constant on each orbit of Q). Thus $dw = -qt$ where t is in the subgroup generated by the s_i and in particular $(e-1)t = 0$. Suppose that $dw \neq 0$. Then since q is the largest power of p dividing d, it follows that w and t generate the same subgroup of C and so $(e-1)w = 0$ and w and x are centralized by σ and the result holds. If $dw = 0$, then

$$x = (w, \ldots, w) + (s_1, \ldots, s_d) \in Z_0 + C_{M_1}(\sigma).$$

Finally suppose that $q = d$. Then Q permutes p blocks of imprimitivity (possibly $q = p$) of size q/p (we take the blocks to consist of consecutive integers). Let Q_1 be the subgroup of index p fixing each block. Let $x \in M$. Then $[x, Q_1] \leq Z$ implies that all coordinates of x on each block are constant. So write $x = (x_1, \ldots, x_p)$ where x_i is a constant vector corresponding to the ith block. Let $\rho \in Q$ be of order p permuting the blocks. We assume that ρ takes $(x_1, \ldots, x_p)$ to $(x_2, \ldots, x_p, x_1)$. Then $[x, \rho] \in Z$ implies that

$$x = (y, \ldots, y) + (0, u, 2u, \ldots, (p-1)u)$$

with $pu = 0$. Let $W = \{x \in M | [x, Q] \leq Z\}$. So we have shown that $W = Z \oplus Z'$ where Z' is the subgroup of order p generated by $(0, u, 2u, \ldots, (p-1)u)$ with u any element of order p. Note that $|Z'| = p$ and moreover $Z' \leq M_0$ (since p is odd) and is centralized by σ. Thus

$$M_1 = W \cap M_0 = Z_0 \oplus Z' = Z_0 + C_{M_1}(\sigma)$$

and the result follows. ∎

We now give the proof of Theorem 4.2 in the case that $L = \mathrm{PSL}(d, q)$ with p dividing $(d, q-1)$ and $q = r^e$. The proof is identical for $L = \mathrm{PSU}(d, q)$ (with p dividing $(d, q+1)$—using the fact that p is odd). The idea is to reduce to working in the normalizer of a maximal torus and then the result essentially follows by Lemma 4.5.

Let $R = \mathrm{SL}(d, q)$ with center Z_2. Then we can take $Q = R/Z_1$ for some subgroup $Z_1 \leq Z_2$ such that $Z_2/Z_1 = Z$ is a nontrivial p-group.

We actually prove a bit more than we require by working in R rather than R/Z_1. Let T be the diagonal subgroup of R and let P be a Sylow p-subgroup of R contained in the normalizer of T. Note that the normalizer of T is just TS_d and we can take $P \leq TW_1$ with W_1 a Sylow p-subgroup of A_d. Let $x \in P$ with $[P, x] \leq Z_2$. It is straightforward to see that $x \in T$.

Let H be the subgroup of $\mathrm{Aut}(R)$ such $[H, x] \leq Z_2$ and let W be a Sylow p-subgroup of H. Note that we can take W so that $W = S(W_1 \times \langle \sigma \rangle)$ where S is the Sylow p-subgroup of T in $\mathrm{PGL}_d(q)$ (i.e., the corresponding split torus in PGL, which in particular centralizes T), W_1 is as above and σ is a standard Frobenius automorphism (of p-power order).

By Lemma 4.5, we can find $y \in xZ_2$ so that σ centralizes y and so replacing x by y we may assume that σ centralizes x. Thus, $[P, x] = [W, x]$. In particular if xZ_1 is central in P/Z_1, then W centralizes xZ_1, whence by an averaging argument (Lemma 2.1), H centralizes xZ_1 as required.

This completes the proof of Theorem 4.2.

5. Second proof of the theorem and proof of the corollary

Theorem 5.1: *Suppose that p is an odd prime and L is a finite group with $O_{p'}(L) = 1$. Suppose that $a \in \mathrm{Aut}(L)$ has order a power of p and a centralizes a Sylow p-subgroup of L. Then a acts as an inner automorphism of L.*

Proof. By Lemma 2.2 we may assume that $L = E(L)S$. Thus, the L-orbits of a component are precisely the S-orbits. Let Q be a component of L. Let t be the number of conjugates of Q in G. Since $O_{p'}(G) = 1$, $S \cap Q$ is not contained in $Z(Q)$ and since a centralizes S, a normalizes Q.

Then a induces an inner automorphism on $Q/Z(Q)$ by Theorem 4.1 and so on Q since it is perfect. Thus, by Corollary 4.3, there exists $q \in Q$ such that aq centralizes Q and moreover q centralizes any automorphism of Q that centralizes $qZ(Q)/Z(Q)$ in $Q/Z(Q)$. Since S centralizes a, it follows that $N_S(Q)$ centralizes $qZ(Q)/Z(Q)$ whence $N_S(Q)$ centralizes q. Thus, the set of S-conjugates of q consists of t elements with one in each of the t conjugates of Q. In particular, these conjugates commute and their product b is thus centralized by S. Moreover, since qa centralizes Q and S centralizes a, it follows that ab centralizes the central product of the conjugates of Q as well as S.

Repeating this for each orbit of components of L, we see that there is a p-element $c \in E(L)$ such that ac centralizes $SE(L) = L$, whence a induces conjugation by c on L and the result follows. ∎

Finally we deduce Corollary 1.2 from Theorem 1.1.

So let G be a finite group with $O_{p'}(G) = 1$ and let P be a Sylow p-subgroup of G, where p is odd. Let $x \in Z(P)$. Then conjugation by x induces an automorphism of $F^*(G)$ and centralizes a Sylow p-subgroup of $F^*(G)$. Thus, x is inner on $F^*(G)$. Thus,

$$x \in F^*(G)C_G(F^*(G)) = F^*(G)$$

and the result follows.

6. A permutation version of the Z_p^* theorem

Finally we prove a permutation group result that is essentially equivalent to the Z_p^* theorem. In particular, for $p = 2$, the proof does not require the classification of finite simple groups.

Theorem 6.1: *Suppose p is a prime, G is a transitive subgroup of S_n, and G possesses a p-element g that has a unique fixed point w and is central in G_w, the stabilizer of w. Then $N := O_{p'}(G)$ is transitive, $G_w = C_G(g)$ and $G = NC_G(g)$.*

Proof. Since g has a unique fixed point w, $N_G(\langle g\rangle)$ also fixes w. By assumption, $G_w \leq C_G(g)$, whence $C_G(g) = G_w$. Let $x \in G$, and assume $g^x \in C_G(g) = G_w$. Then g^x fixes the unique point w^x, but as $g^x \in G_w$, it also fixes w. Thus,

$$x \in G_w = C_G(g)$$

by uniqueness, so that $g^x = g$. This shows that g is isolated in $C_G(g)$. Since p divides $n-1$, G_w contains a Sylow p-subgroup P of G. Thus g is isolated in P. By the Z_p^* theorem, it follows that g is central modulo N.

In particular, $M := \langle N, g\rangle$ is normal in G and $\langle g\rangle$ is a Sylow p-subgroup of M. By the Frattini argument, $G = MN_G(\langle g\rangle) = NC_G(g) = NG_w$ and the theorem follows. ∎

Let us note the previous theorem implies the Z_p^* theorem. By the usual reductions (as described after Theorem 3.1), it suffices to prove this when G is almost simple and its socle has order divisible by p. Suppose that $g \in G$ is a nontrivial p-element and g is isolated in a Sylow p-subgroup P. Then g is also isolated in $C_G(g)$ (for if $g^a \in C_G(g)$, then g and g^a are in a Sylow p-subgroup P^b of $C_G(g)$ and so $g^{ab^{-1}}$ are both in P, whence $g^a = g^b = g$). Let G act on the left cosets of $C_G(g)$. Since G is almost simple, the action is faithful. Since g is isolated in $C_G(g)$, g has a unique fixed point in this action. By the theorem, this implies that $O_{p'}(G)$ is transitive and trivial, whence g is central.

References

[1] O. D. Artemovich, *Isolated elements of prime order in finite groups*, Ukrainskiĭ Matematicheskiĭ Zhurnal **40** (1988), 397–400.

[2] M. Aschbacher, R. Kessar and B. Oliver, *Fusion Systems in Algebra and Topology*, London Mathematical Society Lecture Note Series, Vol. 391, Cambridge University Press, Cambridge, 2011.

[3] G. Glauberman, *Central elements in core-free groups*, Journal of Algebra **4** (1966), 403–420.

[4] G. Glauberman, *Weakly closed elements in Sylow subgroups*, Mathematische Zeitschrift **107** (1968), 1–20.

[5] G. Glauberman, *Global and local properties of finite groups*, in *Finite Simple Groups (Proc. Instructional Conf., Oxford, 1969)*, Academic Press, London, 1971, pp. 1–64.

[6] G. Glauberman and J. Lynd, *Control of fixed points and existence and uniqueness of centric linking systems*, Inventiones Mathematicae **206** (2016), 441–484.

[7] G. Glauberman and J. Lynd, *Rigid automorphisms of linking systems*, `https://arxiv.org/abs/1909.13370`.

[8] D. Gorenstein, *Finite Simple Groups*, University Series in Mathematics, Plenum, New York, 1982.

[9] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, Vol. 40.3, American Mathematical Society, Providence, RI, 1998.

[10] F. Gross, *Automorphisms which centralize a Sylow p-subgroup*, Journal of Algebra **77** (1982), 202–233.

[11] R. Guralnick and G. Robinson, *On extensions of the Baer–Suzuki theorem*, Israel Journal of Mathematics **82** (1993), 281–297.

[12] R. Guralnick, G. Malle and G. Navarro, *Self-normalizing Sylow subgroups*, Proceedings of the American Mathematical Society **132** (2004), 973–979.

[13] I. M. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics, Vol. 92, American Mathematical Society, Providence, RI, 2008.

[14] E. I. Khukhro and V. D. Mazurov, *Unsolved Problems in Group Theory. The Kourovka Notebook*, `https://arxiv.org/abs/1401.0300`.

[15] F. Lübeck, *Centralizers and numbers of semisimple classes in exceptional groups of Lie type*,
`http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/CentSSClasses/index.html`

[16] M. Murai, *Blocks of normal subgroups, automorphisms of groups, and the Alperin–McKay conjecture*, Kyoto Journal of Mathematics **54** (2014), 199–238.

[17] B. Oliver, *Existence and uniqueness of linking systems: Chermak's proof via obstruction theory*, Acta Mathematica **211** (2013), 141–175.