ON THE INEVITABILITY OF THE CONSISTENCY OPERATOR

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ABSTRACT. We examine recursive monotonic functions on the Lindenbaum algebra of EA. We prove that no such function sends every consistent φ to a sentence with deductive strength strictly between φ and $(\varphi \wedge \mathsf{Con}(\varphi))$. We generalize this result to iterates of consistency into the effective transfinite. We then prove that for any recursive monotonic function f, if there is an iterate of Con that bounds f everywhere, then f must be somewhere equal to an iterate of Con.

1. INTRODUCTION

It is a well-known empirical phenomenon that natural axiomatic theories are well-ordered by their consistency strength. However, without a precise mathematical definition of "natural," it is difficult to explain this observation in a strictly mathematical way. One expression of this phenomenon comes from *ordinal analysis*, a research program whereby recursive ordinals are assigned to theories as a measurement of their consistency strength. One method for calculating the proof-theoretic ordinal of a theory T involves demonstrating that T can be approximated over a weak base theory by a class of formulas that are well understood. In particular, the Π^0_1 fragments of natural theories are often proof-theoretically equivalent to iterated consistency statements over a weak base theory, making these theories amenable to ordinal analysis. For discussion, see, e.g., Beklemishev [4, 5] and Joosten [10].

Why are the Π^0_1 fragments of natural theories proof-theoretically equivalent to iterated consistency statements? Our approach to this question is inspired by Martin's approach to another famous question from mathematical logic: why are natural Turing degrees well-ordered by Turing reducibility? Martin conjectured that (i) the non-constant degree invariant functions meeting a certain simplicity condition $(f \in L(\mathbb{R}))$[1] are pre-well-ordered by the relation "$f(a) \leqslant_T g(a)$ on a cone in the Turing degrees" and (ii) the successor for this well-ordering is induced by the Turing jump. Martin's conjecture is meant to capture the idea that the Turing jump and its iterates into the transfinite are the only natural non-trivial degree invariant functions.

In this paper we investigate analogous hypotheses concerning jumps on consistent axiomatic theories, namely, consistency statements. We fix elementary arithmetic EA as our base theory. EA is a subsystem of PA that is often used as a base theory in ordinal analysis and in which standard approaches to arithmetization of syntax can be carried out without substantial changes; see [6] for details. We write $[\varphi]$ to denote the equivalence class of φ modulo EA-provable equivalence. We write $\varphi \vdash \psi$

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[1] Martin's Conjecture is stated under the hypothesis $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$, which is satisfied by $L(\mathbb{R})$ assuming that there are ω many Woodin cardinals with a measurable above them all.

if $\mathsf{EA} \vdash \varphi \to \psi$ and say that φ *implies* ψ. If $\varphi \vdash \psi$ but $\psi \nvdash \varphi$ we say that φ *strictly implies* ψ. The *Lindenbaum algebra* of EA is the set of equivalence classes of sentences ordered by $\vdash$. We focus on recursive functions f that are *monotonic*, i.e.,

$$\text{if } \varphi \vdash \psi, \text{ then } f(\varphi) \vdash f(\psi).$$

We note that (i) a function f is monotonic just in case f preserves implication over EA and (ii) all monotonic functions induce functions on the Lindenbaum algebra of EA. We adopt the convention that all functions named "f" in this paper are recursive.

Our goal is to demonstrate that $\varphi \mapsto (\varphi \wedge \mathsf{Con}(\varphi))$ and its iterates into the transfinite are canonical among monotonic functions. Our first theorem to this end is the following.

Theorem 1.1. *Let f be monotonic. Suppose that for all consistent φ,*
(i) $\varphi \wedge \mathsf{Con}(\varphi)$ implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies φ.
Then for every true φ, there is a true ψ such that $\psi \vdash \varphi$ and $[f(\psi)] = [\psi \wedge \mathsf{Con}(\psi)]$.

Corollary 1.2. *There is no monotonic function f such that for all consistent φ,*
(i) $\varphi \wedge \mathsf{Con}(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies φ.

We note that this result depends essentially on the condition of monotonicity. Shavrukov and Visser [13] studied recursive functions f that are *extensional* over the Lindenbaum algebra of PA, i.e.,

$$\text{if } \mathsf{PA} \vdash (\varphi \leftrightarrow \psi), \text{ then } \mathsf{PA} \vdash (f(\varphi) \leftrightarrow f(\psi)),$$

and proved the following theorem.

Theorem 1.3. *(Shavrukov–Visser) There is a recursive extensional function f such that for all consistent φ,*
(i) $\varphi \wedge \mathsf{Con}(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies φ.

In particular, Shavrukov and Visser proved that for any consistent φ, the sentence

$$\varphi^\star := \varphi \wedge \forall x\big(\mathsf{Con}(I\Sigma_x + \varphi) \to \mathsf{Con}(I\Sigma_x + \varphi + \mathsf{Con}(I\Sigma_x + \varphi))\big)$$

has deductive strength strictly between φ and $\varphi \wedge \mathsf{Con}(\varphi)$, and that the map $\varphi \mapsto \varphi^\star$ is extensional. By a theorem of Kripke and Pour-El [11], the Lindenbaum algebras of PA and EA are effectively isomorphic, whence Theorem 1.3 also applies to EA. Thus, Corollary 1.2 cannot be strengthened by weakening the hypothesis of monotonicity to the hypothesis of extensionality.

We also note that Friedman, Rathjen, and Weiermann [8] introduced a notion of *slow consistency* with which they produced a Π^0_1 sentence $\mathsf{SlowCon}(\mathsf{PA})$ with deductive strength strictly between PA and $\mathsf{PA}+\mathsf{Con}(\mathsf{PA})$. In general, the statement $\mathsf{SlowCon}(\varphi)$ has the form

$$\forall x(F_{\epsilon_0}(x) \downarrow \to \mathsf{Con}(I\Sigma_x + \varphi))$$

where F_{ϵ_0} is a standard representation of a recursive function that is not provably total in PA. This is not in conflict with Corollary 1.2, however, since $\varphi \wedge \mathsf{Con}(\varphi)$ and $\varphi \wedge \mathsf{SlowCon}(\varphi)$ are provably equivalent for all φ such that $\varphi \vdash \forall x F_{\epsilon_0}(x) \downarrow$.

On the other hand, changing the definition of the $\mathsf{SlowCon}(\varphi)$ so that the function in the antecedent varies with the input φ results in a map that is not monotonic.

Theorem 1.1 generalizes to the iterates of Con into the effective transfinite. For an elementary presentation α of a recursive well-ordering (see Definition 3.1) and a sentence φ, we define sentences $\mathsf{Con}^\beta(\varphi)$ for every $\beta < \alpha$.

$$\mathsf{Con}^0(\varphi) := \top$$
$$\mathsf{Con}^{\beta+1}(\varphi) := \mathsf{Con}(\varphi \wedge \mathsf{Con}^\beta(\varphi))$$
$$\mathsf{Con}^\lambda(\varphi) := \forall \beta < \lambda(\mathsf{Con}^\beta(\varphi))$$

For a precise definition using Gödel's fixed point lemma, see Definition 3.2. Note that for every φ, $[\mathsf{Con}^1(\varphi)] = [\mathsf{Con}(\varphi)]$.

Remark 1.4. We warn the reader that there is some discrepancy between our notation and the notation used by other authors. Our iteration scheme $\mathsf{Con}^{\alpha+1}(\varphi) \equiv \mathsf{Con}(\varphi \wedge \mathsf{Con}^\alpha(\varphi))$ is sometimes denoted $\mathsf{Con}((\mathsf{EA}+\varphi)_\alpha)$, e.g., [2]. Moreoever, the notation $\mathsf{Con}^{\alpha+1}(\varphi)$ is sometimes used to denote $\mathsf{Con}(\mathsf{Con}^\alpha(\varphi))$, e.g., [3].

With each predicate Con^α we associate a function

$$\varphi \mapsto (\varphi \wedge \mathsf{Con}^\alpha(\varphi)).$$

Theorem 1.1 then generalizes into the effective transfinite as follows.

Theorem 1.5. *Let f be monotonic. Suppose that for all φ,*
(i) $\varphi \wedge \mathsf{Con}^\alpha(\varphi)$ implies $f(\varphi)$,
(ii) if $[f(\varphi)] \neq [\bot]$, then $f(\varphi)$ strictly implies $\varphi \wedge \mathsf{Con}^\beta(\varphi)$ for all $\beta < \alpha$.
Then for every true φ, there is a true ψ such that $\psi \vdash \varphi$ and $[f(\psi)] = [\psi \wedge \mathsf{Con}^\alpha(\psi)]$.

Corollary 1.6. *There is no monotonic f such that for all φ, if $[\varphi \wedge \mathsf{Con}^\alpha(\varphi)] \neq [\bot]$, then both*
(i) $\varphi \wedge \mathsf{Con}^\alpha(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies $\varphi \wedge \mathsf{Con}^\beta(\varphi)$ for all $\beta < \alpha$.

Thus, if the range of a monotonic function f is sufficiently constrained, then for some φ and some α,

$$[f(\varphi)] = [\varphi \wedge \mathsf{Con}^\alpha(\varphi)] \neq [\bot].$$

This property still holds even when these constraints on the range of f are relaxed considerably. More precisely, if a monotonic function is everywhere bounded by a finite iterate of Con, then it must be somewhere equivalent to an iterate of Con.

Theorem 1.7. *Let f be a monotonic function such that for every φ,*
(i) $\varphi \wedge \mathsf{Con}^n(\varphi)$ implies $f(\varphi)$ and
(ii) $f(\varphi)$ implies φ.
Then for some φ and some $k \leqslant n$, $[f(\varphi)] = [\varphi \wedge \mathsf{Con}^k(\varphi)] \neq [\bot]$.

To generalize this result into the effective transfinite, we focus on a particular class of monotonic functions that we call Π^0_1.

Definition 1.8. A function f is Π^0_1 if $f(\varphi) \in \Pi^0_1$ for all φ.

Our main theorem is the following: if a monotonic function is everywhere bounded by a transfinite iterate of Con, then it must be somewhere equivalent to an iterate of Con. This to say that the iterates of the consistency operator are *inevitable*; no monotonic function that is everywhere bounded by some iterate of Con can avoid all of the iterates of Con.

Theorem 1.9. *Let $\varphi \mapsto f(\varphi)$ be a monotonic Π^0_1 function Then either*
(i) for some $\beta \leqslant \alpha$ and some φ, $[\varphi \wedge f(\varphi)] = [\varphi \wedge \mathsf{Con}^\beta(\varphi)] \neq [\bot]$ or
(ii) for some φ, $(\varphi \wedge \mathsf{Con}^\alpha(\varphi)) \nvdash f(\varphi)$.

The main theorem bears a striking similarity to the following theorem of Slaman and Steel [14].

Theorem 1.10. *(Slaman–Steel) Suppose $f : 2^\omega \to 2^\omega$ is Borel, order-preserving with respect to $\leqslant_T$, and increasing on a cone. Then for any $\alpha < \omega_1$ either*
(i) for some $\beta \leqslant \alpha$, $f(x) \equiv_T x^{(\beta)}$ cofinally or
(ii) $(x^{(\alpha)} <_T f(x))$ cofinally.

There are two notable disanalogies between Theorem 1.9 and Theorem 1.10. First, Theorem 1.9 guarantees only that sufficiently constrained functions are *somewhere* equivalent to an iterate of Con, whereas Theorem 1.10 guarantees *cofinal* equivalence with an iterate of the Turing jump. Second, by assuming AD, Slaman and Steel inferred that this behavior happens not only cofinally but also *on a cone* in the Turing degrees. There is no obvious analogue of AD from which one can infer that if cofinally many Lindenbaum degrees have a property then every element in some non-trivial ideal of Lindenbaum degrees has that property.

We then turn our attention to a generalization of consistency, namely, 1-consistency. Recall that a theory T is *1-consistent* if T is consistent with the true Π^0_1 theory of arithmetic. Just as the Π^0_1 fragments of natural theories are often proof-theoretically equivalent to iterated consistency statements over a weak base theory, the Π^0_2 fragments of natural theories are often proof-theoretically equivalent to iterated 1-consistency statements over a weak base theory

Conservativity theorems relating 1-consistency and iterated consistency play an important role in the proof-theoretic analysis of arithmetic theories. For instance, it is a consequence of Beklemshev's *reduction principle* [6] that for any Π^0_1 φ,

$$\mathsf{EA} + 1\mathsf{Con}(\mathsf{EA}) \vdash \varphi \textit{ if and only if } \mathsf{EA} + \{\mathsf{Con}^k(\mathsf{EA}) : k < \omega\} \vdash \varphi.$$

This fact plays an integral role in Beklemishev's [5] consistency proof of PA. We show that this conservativity result is drastically violated *in the limit*. For functions f and g, we say that f *majorizes* g if there is a consistent φ such that for all ψ, if $\psi \vdash \varphi$ then $f(\psi) \vdash g(\psi)$; if in addition φ is true then we say that f *majorizes* g *on a true ideal.*

Proposition 1.11. *For any elementary presentation α of a recursive well-ordering,* $1\mathsf{Con}$ *majorizes* Con^α *on a true ideal.*

It is tempting to conjecture on the basis of this result that $1\mathsf{Con}$ is the weakest monotonic function majorizing each Con^α for α a recursive well-ordering. We prove that this is not the case.

Theorem 1.12. *There are infinitely many monotonic functions f such that for every recursive ordinal α, there is an elementary presentation a of α such that f majorizes* Con^a *on a true ideal but also* $1\mathsf{Con}$ *majorizes f on a true ideal.*

Theorem 1.1 demonstrates that for any monotonic f with a sufficiently constrained range, f must agree cofinally with Con. We would like to strengthen *cofinally* to *on a true ideal*. One strategy for establishing this claim would be to show that every set that is closed under EA provable equivalence and that contains cofinally many true sentences also contains every sentence in some true ideal. We show that this strategy fails.

Proposition 1.13. *There is a recursively enumerable set $\mathcal{A}$ that contains arbitrarily strong true sentences and that is closed under EA provable equivalence but does not contain any true ideals.*

It is not clear whether Theorem 1.1 can be strengthened in the desired manner.

2. No monotonic function is strictly between the identity and Con

In this section we prove that no monotonic function sends every consistent φ to a sentence with deductive strength strictly between φ and $(\varphi \wedge \mathsf{Con}(\varphi))$. Most of the work is contained in the proof of the following lemma.

Lemma 2.1. *Let f be a monotonic function such that for all consistent φ, $f(\varphi)$ strictly implies φ. Then for every true sentence φ there is a true sentence θ such that $\theta \vdash \varphi$ and $f(\theta) \vdash (\theta \wedge \mathsf{Con}(\theta))$.*

Proof. Let f be as in the statement of the theorem. By assumption the following statement is true.

$$\chi := \forall \zeta (\mathsf{Con}(\zeta) \to (\zeta \nvdash f(\zeta)))$$

Let φ be a true sentence. Then the sentence $\psi := \varphi \wedge \chi$ is true. Let

$$\theta := (\psi \wedge (f(\psi) \to \mathsf{Con}(\psi))).$$

Note that $\theta \vdash \varphi$.

Claim. $f(\theta) \vdash (\theta \wedge f(\psi))$.

Clearly $\theta \vdash \psi$. So $f(\theta) \vdash f(\psi)$ since f is monotonic. Also $f(\theta) \vdash \theta$ by assumption.

Claim. $(\theta \wedge f(\psi)) \vdash (\psi \wedge \mathsf{Con}(\psi))$.

Immediate from the definition of θ.

Claim. $(\psi \wedge \mathsf{Con}(\psi)) \vdash (\theta \wedge \mathsf{Con}(\theta))$.

Clearly $(\psi \wedge \mathsf{Con}(\psi)) \vdash \theta$. It suffices to show that

$$(\psi \wedge \mathsf{Con}(\psi)) \vdash \mathsf{Con}(\theta).$$

We reason as follows.

$$\begin{aligned}(\psi \wedge \mathsf{Con}(\psi)) &\vdash \forall \zeta (\mathsf{Con}(\zeta) \to (\zeta \nvdash f(\zeta))) \text{ by choice of } \psi.\\ &\vdash \mathsf{Con}(\psi) \to (\psi \nvdash f(\psi)) \text{ by instantiation.}\\ &\vdash (\psi \nvdash f(\psi)) \text{ by logic.}\\ &\vdash \mathsf{Con}(\psi \wedge \neg f(\psi)).\\ &\vdash \mathsf{Con}(\theta) \text{ by the definition of } \theta.\end{aligned}$$

It is immediate from the preceding claims that $f(\theta) \vdash (\theta \wedge \mathsf{Con}(\theta))$. ❑

A number of results follow immediately from the lemma.

Theorem 2.2. *Let f be monotonic. Suppose that for all consistent φ,*
(i) $\varphi \wedge \mathsf{Con}(\varphi)$ implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies φ.
Then for every true φ, there is a true ψ such that $\psi \vdash \varphi$ and $[f(\psi)] = [\psi \wedge \mathsf{Con}(\psi)]$.

Proof. By the lemma, for every true φ there is a true ψ such that $\psi \vdash \varphi$ and $f(\psi) \vdash (\psi \wedge \mathsf{Con}(\psi))$. Since we are assuming that $(\psi \wedge \mathsf{Con}(\psi)) \vdash f(\psi)$, it follows that $[f(\psi)] = [\psi \wedge \mathsf{Con}(\psi)]$. ❑

We note that this theorem applies to a number of previously studied operators. For instance, the theorem applies to the notion of *cut-free consistency*, i.e., consistency with respect to cut-free proofs. EA does not prove the cut-elimination theorem, which is equivalent to the totality of super-exponentiation (over EA), and does not prove the equivalence of cut-free consistency and consistency. Another such operator is the Friedman-Rathjen-Weiermann *slow consistency operator* discussed in §1. Theorem 2.2 implies that these operators exhibit the same behavior as the consistency operator "in the limit." Indeed, for any φ such that φ proves the cut-elimination theorem, $\varphi \wedge \mathsf{Con}(\varphi)$ and $\varphi \wedge \mathsf{Con}_{\mathsf{CF}}(\varphi)$ are EA-provably equivalent. Likewise, for any φ that proves the totality of F_{ϵ_0}, $\varphi \wedge \mathsf{Con}(\varphi)$ and $\varphi \wedge \mathsf{SlowCon}(\varphi)$ are EA-provably equivalent.

As a corollary of Theorem 2.2 we note that no monotonic function reliably produces sentences strictly between those produced by the identity and by Con.

Corollary 2.3. *There is no monotonic function f such that for all consistent φ,*
(i) $\varphi \wedge \mathsf{Con}(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies φ.

Shavrukov and Visser [13] studied functions over Lindenbaum algberas and discovered a recursive *extensional uniform density function g* for the Lindenbaum algebra of EA, i.e., (i) for any φ and ψ such that ψ strictly implies φ, $g(\langle \varphi, \psi \rangle)$ is a sentence with deductive strength strictly between φ and ψ and (ii) if $\mathsf{EA} \vdash (\varphi \leftrightarrow \psi)$ then, for any θ, $[g(\langle \varphi, \theta \rangle)] = [g(\langle \psi, \theta \rangle)]$ and $[g(\langle \theta, \varphi \rangle)] = [g(\langle \theta, \psi \rangle)]$. They asked whether this result could be strengthened by exhibiting a recursive uniform density function that is monotonic in both its coordinates. As a corollary of our theorem we answer their question negatively.

Corollary 2.4. *There is no monotonic uniform density function for the Lindenbaum algebra of* EA.

Proof. Suppose there were such a function g over the Lindenbaum algebra of EA. Then given any input of the form $\langle \varphi, (\varphi \wedge \mathsf{Con}(\varphi)) \rangle$, g would produce a sentence with deductive strength strictly between φ and $(\varphi \wedge \mathsf{Con}(\varphi))$. We then note that $f : \varphi \mapsto g(\langle \varphi, (\varphi \wedge \mathsf{Con}(\varphi)) \rangle)$ is monotonic, but that for every consistent φ, $\varphi \wedge \mathsf{Con}(\varphi)$ strictly implies $f(\varphi)$ and $f(\varphi)$ strictly implies φ, contradicting the previous theorem. ❑

Our negative answer to the question raised by Shavrukov and Visser makes use of a Π^0_2 sentence $\forall \zeta (\mathsf{Con}(\zeta) \rightarrow (\zeta \nvdash f(\zeta)))$. Shavrukov and Visser raised the following question in private communication.

Question 2.5. *Is there a recursive uniform density function for the lattice of Π^0_1 sentences over* EA *that is monotonic in both its coordinates?*

Remark 2.6. It is clear from the proof of the lemma that any monotonic f meeting the hypotheses of the theorem is not only cofinally equivalent to Con; for every true φ that implies

$$\chi := \forall\zeta(\mathsf{Con}(\zeta) \to (\zeta \nvdash f(\zeta))),$$

there is a true ψ such that $\psi \vdash \varphi$ and $[\varphi \wedge \mathsf{Con}(\varphi)] = [\psi \wedge \mathsf{Con}(\psi)] = [f(\psi)]$.

This observation points the way toward a corollary of our theorem; namely that any monotonic function strictly meeting the hypotheses of the theorem must have the same range as $\varphi \mapsto (\varphi \wedge \mathsf{Con}(\varphi))$ in the limit. To prove this, we first prove a version of jump inversion—Con inversion—for Lindenbaum algebras. This is to say that the range of Con contains a true ideal in the Lindenbaum algebra. A similar result is established for true Π^0_2 sentences in [1].

Proposition 2.7. *Suppose $\varphi \vdash \mathsf{Con}(\top)$. Then for some ψ, $[\varphi] = [(\psi \wedge \mathsf{Con}(\psi))]$.*

Proof. Let $\psi := \mathsf{Con}(\top) \to \varphi$.

Claim. $\varphi \vdash (\psi \wedge \mathsf{Con}(\psi))$.

Trivially, $\varphi \vdash \psi$. Since $\varphi \vdash \mathsf{Con}(\top)$, it follows that from the formalized second incompleteness theorem, i.e., $\mathsf{Con}(\top) \vdash \mathsf{Con}(\neg\mathsf{Con}(\top))$, that $\varphi \vdash \mathsf{Con}(\neg\mathsf{Con}(\top))$. But $\neg\mathsf{Con}(\top)$ is the first disjunct of ψ, so $\varphi \vdash \mathsf{Con}(\psi)$.

Claim. $(\psi \wedge \mathsf{Con}(\psi)) \vdash \varphi$.

Note that $\mathsf{Con}(\psi) \vdash \mathsf{Con}(\top)$. The claim then follows since clearly $(\psi \wedge \mathsf{Con}(\top)) \vdash \varphi$. ❑

Corollary 2.8. *Let f be monotonic. Suppose that for all consistent φ,*
(i) $\varphi \wedge \mathsf{Con}(\varphi)$ implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies φ.
Then the intersection of the ranges of f and Con in the Lindenbaum algebra contains a true ideal.

Proof. Let φ be a sentence such that $\varphi \vdash \mathsf{Con}(\top)$ and

$$\varphi \vdash \forall\zeta(\mathsf{Con}(\zeta) \to (\zeta \nvdash f(\zeta))).$$

Note that both of these sentences are true, and hence φ is in an element of a true ideal. By the previous theorem, there is a ψ such that $[\psi \wedge \mathsf{Con}(\psi)] = [\varphi]$. By Remark 2.6 there is a θ such that $[f(\theta)] = [\psi \wedge \mathsf{Con}(\psi)] = [\theta \wedge \mathsf{Con}(\theta)]$. ❑

3. Iterating Con into the transfinite

By analogy with Martin's Conjecture, we would like to show that there is a natural well-ordered hierarchy of monotonic functions and that the successor for this well-ordering is induced by Con. Thus, we define the iterates of Con along elementary presentations of well-orderings.

Definition 3.1. By an *elementary presentation* of a recursive well-ordering we mean a pair $(\mathcal{D}, <)$ of elementary formulas, such that (i) the relation $<$ well-orders $\mathcal{D}$ in the standard model of arithmetic and (ii) EA proves that $<$ linearly orders the elements satisfying $\mathcal{D}$, and (iii) it is elementarily calculable whether an element represents zero or a successor or a limit.

Definition 3.2. Given an elementary presentation $\langle \alpha, < \rangle$ of a recursive well-ordering and a sentence φ, we use Gödel's fixed point lemma to define sentences $\mathbf{Con}^\star(\varphi, \beta)$ for $\beta < \alpha$ as follows.

$$\mathsf{EA} \vdash \mathbf{Con}^\star(\varphi, \beta) \leftrightarrow \forall \gamma < \beta, \mathsf{Con}(\varphi \wedge \mathbf{Con}^\star(\varphi, \gamma)).$$

We use the notation $\mathsf{Con}^\beta(\varphi)$ for $\mathbf{Con}^\star(\varphi, \beta)$.

Remark 3.3. Note that, since it is elementarily calculable whether a number represents zero or a successor or a limit, the following clauses are provable in EA.

- $\mathsf{Con}^0(\varphi) \leftrightarrow \top$
- $\mathsf{Con}^{\gamma+1}(\varphi) \leftrightarrow \mathsf{Con}(\varphi \wedge \mathsf{Con}^\gamma(\varphi))$
- $\mathsf{Con}^\lambda(\varphi) \leftrightarrow \forall \gamma < \lambda, \mathsf{Con}^\gamma(\varphi)$ for λ a limit.

Note that this hierarchy is proper for true φ by Gödel's second incompleteness theorem. We need to prove that for transfinite α, Con^α is monotonic over the Lindenbaum algebra of EA. Before proving this claim we recall Schmerl's [12] technique of *reflexive transfinite induction*. Note that "$\mathsf{Pr}(\varphi)$" means that φ is provable in EA.

Proposition 3.4. *(Schmerl) Suppose that* $<$ *is an elementary linear order and that* $\mathsf{EA} \vdash \forall \alpha (\mathsf{Pr}(\forall \beta < \alpha, A(\beta)) \to A(\alpha))$. *Then* $\mathsf{EA} \vdash \forall \alpha A(\alpha)$.

Proof. From $\mathsf{EA} \vdash \forall \alpha (\mathsf{Pr}(\forall \beta < \alpha, A(\beta)) \to A(\alpha))$ we infer

$$\begin{aligned} \mathsf{EA} \vdash \mathsf{Pr}(\forall \alpha A(\alpha)) &\to \forall \alpha \mathsf{Pr}(\forall \beta < \alpha, A(\beta)) \\ &\to \forall \alpha A(\alpha). \end{aligned}$$

Löb's theorem, i.e.,

$$\text{if } \mathsf{EA} \vdash \mathsf{Pr}(\zeta) \to \zeta, \text{ then } \mathsf{EA} \vdash \zeta,$$

then yields $\mathsf{EA} \vdash \forall \alpha A(\alpha)$. ❑

Proposition 3.5. *If* $\varphi \vdash \psi$, *then* $\mathsf{Con}^\alpha(\varphi) \vdash \mathsf{Con}^\alpha(\psi)$.

Proof. Let $\mathcal{A}(\beta)$ denote the claim that $\mathsf{Con}^\beta(\varphi) \vdash \mathsf{Con}^\beta(\psi)$.

We want to prove that $\mathcal{A}(\alpha)$, without placing any restrictions on α. We prove the equivalent claim that $\mathsf{EA} \vdash \mathcal{A}(\alpha)$. By Proposition 3.4, it suffices to show that

$$\mathsf{EA} \vdash \forall \alpha (\mathsf{Pr}(\forall \beta < \alpha, \mathcal{A}(\beta)) \to \mathcal{A}(\alpha)).$$

Reason within EA**.** Suppose that $\mathsf{Pr}(\forall \beta < \alpha, \mathcal{A}(\beta))$, which is to say that

$$\mathsf{Pr}(\forall \beta < \alpha, (\mathsf{Con}^\beta(\varphi) \vdash \mathsf{Con}^\beta(\psi))).$$

Since $\mathsf{Con}^\alpha(\varphi)$ contains EA, we infer that

$$\mathsf{Con}^\alpha(\varphi) \vdash \forall \beta < \alpha (\mathsf{Con}^\beta(\varphi) \vdash \mathsf{Con}^\beta(\psi)),$$

which is just to say that

$$\mathsf{Con}^\alpha(\varphi) \vdash \big(\forall \beta < \alpha, \mathsf{EA} \vdash (\mathsf{Con}^\beta(\varphi) \to \mathsf{Con}^\beta(\psi))\big).$$

Since $\mathsf{Con}^\alpha(\varphi)$ proves that for all $\beta < \alpha$, $\mathsf{EA} \nvdash \neg\mathsf{Con}^\beta(\varphi)$ we infer that

$$\mathsf{Con}^\alpha(\varphi) \vdash \forall \beta < \alpha (\mathsf{EA} \nvdash \neg\mathsf{Con}^\beta(\psi)).$$

EA proves its own Σ^0_1 completeness, i.e., EA proves that if EA does not prove a Σ^0_1 statement ζ, then ζ is false. Thus,

$$\mathsf{Con}^\alpha(\varphi) \vdash \forall \beta < \alpha (\mathsf{Con}^\beta(\psi)).$$

This concluses the proof of the proposition. ❑

Thus, for each predicate Con^α the function

$$\varphi \mapsto (\varphi \wedge \mathsf{Con}^\alpha(\varphi))$$

is monotonic over the Lindenbaum algebra of EA.

In this section we show that the functions given by iterated consistency are minimal with respect to each other. We fix an elementary presentation α of a recursive well-ordering. We assume that f is a monotonic function such that for every consistent φ, $f(\varphi)$ strictly implies $\varphi \wedge \mathsf{Con}^\beta(\varphi)$ for all $\beta < \alpha$. We would like to relativize the proof of Lemma 2.1 to Con^β. However, the proof of Lemma 2.1 relied on the truth of the principle

$$\forall\zeta(\mathsf{Con}(\zeta) \to (\zeta \nvdash f(\zeta))), i.e.,$$

$$\forall\zeta(\mathsf{Con}(\zeta) \to \mathsf{Con}(\zeta \wedge \neg f(\zeta))).$$

It is not in general clear that $\mathsf{Con}^\alpha(\varphi)$ implies $\mathsf{Con}^\alpha(\varphi \wedge \neg f(\varphi))$. To solve this problem, we define a sequence of true sentences $(\theta_\beta)_{\beta\leqslant\alpha}$ such that for every sentence φ, if $\varphi \vdash \theta_\beta$ then $\mathsf{Con}^\beta(\varphi)$ implies $\mathsf{Con}^\beta(\varphi \wedge \neg f(\varphi))$. Thus, we are able to relativize the proof of Lemma 2.1 for Con^β to sentences that imply θ_β.

Definition 3.6. Given an elementary presentation α of a recursive well-ordering, we use effective transfinite recursion to define a sequence of sentences $(\theta_\beta)_{\beta\leqslant\alpha}$.

$$\theta_1 := \forall\zeta(\mathsf{Con}(\zeta) \to \mathsf{Con}(\zeta \wedge \neg f(\zeta)))$$
$$\theta_\beta := \forall\gamma < \beta(\mathsf{True}_{\Pi_3}(\theta_\gamma)) \wedge \forall\zeta\big(\big(\forall\gamma < \beta(\zeta \vdash \theta_\gamma)\big) \to \big(\mathsf{Con}^\beta(\zeta) \to \mathsf{Con}^\beta(\zeta \wedge \neg f(\zeta))\big)\big).$$

Remark 3.7. Note that every sentence in the sequence $(\theta_\beta)_{\beta\leqslant\alpha}$ has complexity Π^0_3. Note moreover that for a successor $\beta + 1$, $\theta_{\beta+1}$ is equivalent to

$$\theta_\beta \wedge \forall\zeta\big((\zeta \vdash \theta_\beta) \to \big(\mathsf{Con}^{\beta+1}(\zeta) \to \mathsf{Con}^{\beta+1}(\zeta \wedge \neg f(\zeta))\big)\big).$$

Lemma 3.8. *Let f be monotonic such that, for all φ,*
(i) $\varphi \wedge \mathsf{Con}^\alpha(\varphi)$ implies $f(\varphi)$,
(ii) if $[f(\varphi)] \neq [\bot]$, then $f(\varphi)$ strictly implies $\varphi \wedge \mathsf{Con}^\beta(\varphi)$ for all $\beta < \alpha$.
Then for each $\beta \leqslant \alpha$, the sentence θ_β is true.

Proof. Let f be as in the statement of the lemma. We prove the claim by induction on $\beta \leqslant \alpha$. The **base case** $\beta = 1$ is trivial.

For the **successor case** we assume that $\beta < \alpha$ and that θ_β is true; we want to show that $\theta_{\beta+1}$ is true. So let ζ be a sentence such that $\zeta \vdash \theta_\beta$. We want to show that $\mathsf{Con}^{\beta+1}(\zeta)$ implies $\mathsf{Con}^{\beta+1}(\zeta \wedge \neg f(\zeta))$. We prove the contrapositive, that $\neg\mathsf{Con}^{\beta+1}(\zeta \wedge \neg f(\zeta))$ implies $\neg\mathsf{Con}^{\beta+1}(\zeta)$. So suppose $\neg\mathsf{Con}^{\beta+1}(\zeta \wedge \neg f(\zeta))$, i.e.,

$$(\dagger) \qquad \zeta \wedge \neg f(\zeta) \vdash \neg\mathsf{Con}^\beta(\zeta \wedge \neg f(\zeta)).$$

We reason as follows.

Since $\zeta \vdash \theta_\beta$, $\zeta \vdash \forall\gamma < \beta, \mathsf{True}_{\Pi_3}(\theta_\gamma)$. From this we infer

$$(\star) \qquad \zeta \vdash (\zeta \vdash \forall\gamma < \beta, \mathsf{True}_{\Pi_3}(\theta_\gamma))$$

by Σ^0_1 completeness. Moreover, since $\zeta \vdash \theta_\beta$,

$$\zeta \vdash \forall\varphi\big((\forall\gamma < \beta(\varphi \vdash \theta_\gamma)) \to (\mathsf{Con}^\beta(\varphi) \to \mathsf{Con}^\beta(\varphi \wedge \neg f(\varphi)))\big) \text{ by the definition of } \theta_\beta.$$

$$\vdash \forall\gamma < \beta(\zeta \vdash \theta_\gamma) \to (\mathsf{Con}^\beta(\zeta) \to \mathsf{Con}^\beta(\zeta \wedge \neg f(\zeta))) \text{ by instantiation.}$$

$$\vdash \mathsf{Con}^\beta\zeta \to \mathsf{Con}^\beta(\zeta \wedge \neg f(\zeta)) \text{ by } (\star).$$

$$\zeta \wedge \neg f(\zeta) \vdash \neg\mathsf{Con}^\beta(\zeta \wedge \neg f(\zeta)) \text{ by } (\dagger).$$

$$\vdash \neg\mathsf{Con}^\beta(\zeta) \text{ by logic.}$$

$$\zeta \vdash \mathsf{Con}^\beta(\zeta) \to f(\zeta) \text{ by logic.}$$

Thus, $(\zeta \wedge \mathsf{Con}^\beta(\zeta)) \vdash f(\zeta)$. Since $f(\varphi)$ always strictly implies $\varphi \wedge \mathsf{Con}^\beta(\varphi)$, we infer that

$$[\zeta \wedge \mathsf{Con}^\beta(\zeta)] = [\bot].$$

This is to say that $\neg\mathsf{Con}^{\beta+1}(\zeta)$.

For the **limit case** we let β be a limit ordinal and assume that for every $\gamma < \beta$, θ_γ is true. We want to show that θ_β is true. Let ζ be a sentence such that for every $\gamma < \beta$, $\zeta \vdash \theta_\gamma$. We want to show that $\mathsf{Con}^\beta(\zeta)$ implies $\mathsf{Con}^\beta(\zeta \wedge \neg f(\zeta))$. So assume that $\mathsf{Con}^\beta(\zeta)$, i.e., for every $\gamma < \beta, \mathsf{Con}^\gamma(\zeta)$. Let $\gamma < \beta$. Since β is a limit ordinal, $\gamma + 1 < \beta$. So by the inductive hypothesis $\theta_{\gamma+1}$ is true. That is, by the definition of $\theta_{\gamma+1}$,

$$\forall\varphi\big((\varphi \vdash \theta_\gamma) \to (\mathsf{Con}^\gamma(\varphi) \to \mathsf{Con}^\gamma(\varphi \wedge \neg f(\varphi)))\big).$$

By instantiation, we infer that

$$(\zeta \vdash \theta_\gamma) \to (\mathsf{Con}^\gamma(\zeta) \to \mathsf{Con}^\gamma(\zeta \wedge \neg f(\zeta))).$$

Since $\zeta \vdash \theta_\gamma$ and $\mathsf{Con}^\gamma(\zeta)$, this means that $\mathsf{Con}^\gamma(\zeta \wedge \neg f(\zeta))$. Since γ was a generic ordinal less than β, we get that

$$\forall\gamma < \beta, \mathsf{Con}^\gamma(\zeta \wedge \neg f(\zeta)),$$

i.e., $\mathsf{Con}^\beta(\zeta)$. This completes the proof of the lemma. ❑

Theorem 3.9. *Let f be monotonic. Suppose that for all φ,*
(i) $\varphi \wedge \mathsf{Con}^\alpha(\varphi)$ implies $f(\varphi)$,
(ii) if $[f(\varphi)] \neq [\bot]$, then $f(\varphi)$ strictly implies $\varphi \wedge \mathsf{Con}^\beta(\varphi)$ for all $\beta < \alpha$.
Then for every true χ, there is a true ψ such that $\psi \vdash \chi$ and $[f(\psi)] = [\psi \wedge \mathsf{Con}^\alpha(\psi)]$.

Proof. Let χ be a true sentence. By the lemma, θ_α is true. So

$$\varphi := \chi \wedge \theta_\alpha$$

is true. We let

$$\psi := \varphi \wedge (f(\varphi) \to \mathsf{Con}^\alpha(\varphi)).$$

Note that $\psi \vdash \chi$. We now show that $[\psi \wedge \mathsf{Con}^\alpha(\psi)] = [f(\psi)]$.

Claim. $f(\psi) \vdash (\psi \wedge f(\varphi))$.

Since f is monotonic.

Claim. $(\psi \wedge f(\varphi)) \vdash (\varphi \wedge \mathsf{Con}^\alpha(\varphi))$.

By the definition of ψ.

Claim. $(\varphi \wedge \mathsf{Con}^\alpha(\varphi)) \vdash (\psi \wedge \mathsf{Con}^\alpha(\psi))$.

It is clear from the definition of ψ that $(\varphi \wedge \mathsf{Con}^\alpha(\varphi)) \vdash \psi$. So it suffices to show that $(\varphi \wedge \mathsf{Con}^\alpha(\varphi)) \vdash \mathsf{Con}^\alpha(\psi)$.

$$\begin{aligned}\varphi \wedge \mathsf{Con}^\alpha(\varphi) &\vdash \forall\zeta\big(\big(\forall\beta < \alpha(\zeta \vdash \theta_\beta)\big) \rightarrow \big(\mathsf{Con}^\alpha(\zeta) \rightarrow \mathsf{Con}^\alpha(\zeta \wedge \neg f(\zeta))\big)\big) \text{ by choice of } \varphi.\\ &\vdash \forall\beta < \alpha(\varphi \vdash \theta_\beta) \rightarrow \big(\mathsf{Con}^\alpha(\varphi) \rightarrow \mathsf{Con}^\alpha(\varphi \wedge \neg f(\varphi))\big) \text{ by instantiation.}\\ &\vdash \forall\beta < \alpha(\varphi \vdash \theta_\beta) \rightarrow \mathsf{Con}^\alpha(\varphi \wedge \neg f(\varphi)) \text{ by logic.}\end{aligned}$$

Since $\mathsf{Con}^\alpha(\varphi \wedge \neg f(\varphi)) \vdash \mathsf{Con}^\alpha(\psi)$, to prove the desired claim it suffices to show that

$$\varphi \wedge \mathsf{Con}^\alpha(\varphi) \vdash \forall\beta < \alpha(\varphi \vdash \theta_\beta).$$

We reason as follows.

$$\begin{aligned}\varphi &\vdash \theta_\alpha \text{ by choice of } \varphi.\\ &\vdash \forall\beta < \alpha(\mathsf{True}_{\Pi_3}\theta_\beta) \text{ by definition of } \theta_\alpha.\\ &\vdash (\varphi \vdash \forall\beta < \alpha(\mathsf{True}_{\Pi_3}\theta_\beta)) \text{ by } \Sigma^0_1 \text{ completeness.}\\ &\vdash \forall\beta < \alpha(\varphi \vdash \mathsf{True}_{\Pi_3}\theta_\beta)\\ &\vdash \forall\beta < \alpha(\varphi \vdash \theta_\beta)\end{aligned}$$

It is immediate from the preceding claims that $f(\psi) \vdash \psi \wedge \mathsf{Con}^\alpha(\psi)$. By assumption, $\psi + \mathsf{Con}^\alpha(\psi) \vdash f(\psi)$, so it follows that $[f(\psi)] = [\psi \wedge \mathsf{Con}^\alpha(\psi)]$. ❑

Corollary 3.10. *There is no monotonic f such that for all φ, if $[\varphi \wedge \mathsf{Con}^\alpha(\varphi)] \neq [\bot]$, then both*
(i) $\varphi \wedge \mathsf{Con}^\alpha(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies $\varphi \wedge \mathsf{Con}^\beta(\varphi)$ for all $\beta < \alpha$.

4. Finite iterates of Con are inevitable

In this section and the next section we prove that the iterates of Con are, in a sense, inevitable. First we show that, for every natural number n, if a monotonic function f is always bounded by Con^n, then it is somewhere equivalent to Con^k for some $k \leqslant n$. In §5, we turn to generalizations of this result into the effective transfinite.

Theorem 4.1. *Let f be a monotonic function such that for every φ,*
(i) $\varphi \wedge \mathsf{Con}^n(\varphi)$ implies $f(\varphi)$ and
(ii) $f(\varphi)$ implies φ.
Then for some φ and some $k \leqslant n$, $[f(\varphi)] = [\varphi \wedge \mathsf{Con}^k(\varphi)] \neq [\bot]$.

Proof. We suppose, towards a contradiction, that there is no ψ and no $k \leqslant n$ such that $[f(\psi)] = [\psi \wedge \mathsf{Con}^k(\psi)] \neq [\bot]$. We then let φ_1 be a true statement such that

$$\begin{aligned}&\varphi_1 \vdash \forall\zeta(\mathsf{Con}(\zeta) \rightarrow \mathsf{Con}(\zeta \wedge \neg f(\zeta)))\\ &\varphi_1 \vdash \forall k\forall\zeta\big(\mathsf{Con}^{k+1}(\zeta) \rightarrow \neg\mathsf{Pr}\big((\zeta \wedge \mathsf{Con}^k(\zeta)) \leftrightarrow f(\zeta)\big)\big).\end{aligned}$$

The first condition is that φ_1 proves that for every consistent φ, $f(\varphi)$ strictly implies φ. The second condition is that φ_1 proves that $f(\zeta)$ never coincides with $\zeta \wedge \mathsf{Con}^k(\zeta)$, unless $[\zeta \wedge \mathsf{Con}^k(\zeta)] = [\bot]$.

We define a sequence of statements, starting with φ_1, as follows:

$$\varphi_{k+1} := \varphi_k \wedge (f(\varphi_k) \rightarrow \mathsf{Con}^k(\varphi_k)).$$

Note that each sentence of the form φ_k. We will use our assumption to show that, for all k, $\varphi_k \wedge \mathsf{Con}^k(\varphi_k) \vdash \mathsf{Con}^k(\varphi_{k+1})$. From this we will deduce that $[f(\varphi_{n+1})] = [\varphi_{n+1} \wedge \mathsf{Con}^n(\varphi_{n+1})] \neq [\bot]$, contradicting the assumption that f and Con^n never coincide. Most of the work is contained in the proof of the following lemma.

Lemma 4.2. *For all k, for all $j \geqslant k$, $[\varphi_k \wedge \mathsf{Con}^k(\varphi_k)] \vdash [\mathsf{Con}^k(\varphi_j)]$.*

Proof. We prove the claim by a double induction. The primary induction is on k. For the **base case** $k = 1$, we prove the claim by induction on j. The *base case* $j = 1$ follows trivially. For the *inductive step* we assume that $[\varphi_1 \wedge \mathsf{Con}(\varphi_1)] \vdash [\mathsf{Con}(\varphi_j)]$ and show that $[\varphi_1 \wedge \mathsf{Con}(\varphi_1)] \vdash [\mathsf{Con}(\varphi_{j+1})]$.

$$\begin{aligned} \varphi_1 \wedge \mathsf{Con}(\varphi_1) &\vdash \forall\zeta(\mathsf{Con}(\zeta) \to \mathsf{Con}(\zeta \wedge \neg f(\zeta))) \text{ by choice of } \varphi_1. \\ &\vdash \mathsf{Con}(\varphi_j) \to \mathsf{Con}(\varphi_j \wedge \neg f(\varphi_j)) \text{ by instantiation.} \\ \varphi_1 \wedge \mathsf{Con}(\varphi_1) &\vdash \mathsf{Con}(\varphi_j) \text{ by the inductive hypothesis.} \\ &\vdash \mathsf{Con}(\varphi_j \wedge \neg f(\varphi_j)) \text{ by logic.} \\ &\vdash \mathsf{Con}(\varphi_{j+1}) \text{ by definition of } \varphi_{j+1}. \end{aligned}$$

For the **inductive step** we assume that the claim is true of $k-1$, i.e.,

$$\forall j \geqslant k-1\big(\big(\varphi_{k-1} \wedge \mathsf{Con}^{k-1}(\varphi_{k-1})\big) \vdash \big(\mathsf{Con}^{k-1}(\varphi_j)\big)\big).$$

We prove the claim for k. Once again, we prove the claim by induction on j. The *base case* $j = k$ follows trivially. For the *inductive step* we assume that $\varphi_k \wedge \mathsf{Con}^k(\varphi_k) \vdash \mathsf{Con}^k(\varphi_j)$. We want to prove that $\varphi_k \wedge \mathsf{Con}^k(\varphi_k) \vdash \mathsf{Con}^k(\varphi_{j+1})$.

$$\begin{aligned} \varphi_k \wedge \mathsf{Con}^k(\varphi_k) &\vdash \forall x \forall\zeta\big(\mathsf{Con}^{x+1}(\zeta) \to \neg\mathsf{Pr}\big((\zeta \wedge \mathsf{Con}^x(\zeta)) \leftrightarrow f(\zeta)\big)\big) \text{ by choice of } \varphi_1. \\ &\vdash \mathsf{Con}^k(\varphi_j) \to \neg\mathsf{Pr}\big((\varphi_j \wedge \mathsf{Con}^{k-1}(\varphi_j)) \leftrightarrow f(\varphi_j)\big) \text{ by instantiation.} \\ \varphi_k \wedge \mathsf{Con}^k(\varphi_k) &\vdash \mathsf{Con}^k(\varphi_j) \text{ by the inner inductive hypothesis.} \\ &\vdash \neg\mathsf{Pr}\big((\varphi_j \wedge \mathsf{Con}^{k-1}(\varphi_j)) \leftrightarrow f(\varphi_j)\big) \text{ by logic.} \end{aligned}$$

Thus, $\varphi_k \wedge \mathsf{Con}^k(\varphi_k)$ proves that one of the following cases holds.

$$(\varphi_j \wedge \mathsf{Con}^{k-1}(\varphi_j)) \nvdash f(\varphi_j)$$
$$f(\varphi_j) \nvdash (\varphi_j \wedge \mathsf{Con}^{k-1}\varphi_j)$$

We now show that $\varphi_k \wedge \mathsf{Con}^k(\varphi_k)$ refutes the second option.

Claim. $\varphi_k \wedge \mathsf{Con}^k(\varphi_k) \vdash \big(f(\varphi_j) \vdash (\varphi_j \wedge \mathsf{Con}^{k-1}\varphi_j)\big)$.

By the outer inductive hypothesis, EA proves the following conditional:

$$\theta := \big((\varphi_{j-1} \wedge \mathsf{Con}^{k-1}(\varphi_{j-1})) \to (\mathsf{Con}^{k-1}(\varphi_j))\big).$$

Thus, $f(\varphi_j)$ (which contains EA) also proves θ. We now show that $f(\varphi_j) \vdash \mathsf{Con}^{k-1}(\varphi_j)$.

$$\begin{aligned} f(\varphi_j) &\vdash \varphi_j \wedge f(\varphi_{j-1}) \text{ since } f \text{ is monotonic.} \\ &\vdash (\varphi_{j-1} \wedge (f(\varphi_{j-1}) \to \mathsf{Con}^{j-1}(\varphi_{j-1}))) \wedge f(\varphi_{j-1}) \text{ by the definition of } \varphi_j. \\ &\vdash \varphi_{j-1} \wedge \mathsf{Con}^{j-1}(\varphi_{j-1}) \text{ by logic.} \\ &\vdash \varphi_{j-1} \wedge \mathsf{Con}^{k-1}(\varphi_{j-1}) \text{ since } j \geqslant k. \\ &\vdash \mathsf{Con}^{k-1}(\varphi_j) \text{ since } f(\varphi_j) \text{ proves } \theta. \end{aligned}$$

By Σ^0_1 completeness, $(\varphi_k \wedge \mathsf{Con}^k(\varphi_k)) \vdash \big(f(\varphi_j) \vdash \mathsf{Con}^{k-1}(\varphi_j)\big)$.

Claim. $(\varphi_k \wedge \mathsf{Con}^k(\varphi_k)) \vdash \mathsf{Con}^k(\varphi_{j+1})$.

We reason as follows.

$$\begin{aligned}(\varphi_k \wedge \mathsf{Con}^k(\varphi_k)) &\vdash \big((\varphi_j \wedge \mathsf{Con}^{k-1}(\varphi_j)) \nvdash f(\varphi_j)\big) \text{ by the previous claim.}\\ &\vdash \mathsf{Con}(\varphi_j \wedge \neg f(\varphi_j) \wedge \mathsf{Con}^{k-1}(\varphi_j)).\\ &\vdash \mathsf{Con}(\varphi_{j+1} \wedge \mathsf{Con}^{k-1}(\varphi_j)) \text{ by the definition of } \varphi_{j+1}.\\ &\vdash \mathsf{Con}(\varphi_{j+1} \wedge \mathsf{Con}^{k-1}(\varphi_{j+1})) \text{ by the outer inductive hypothesis.}\\ &\vdash \mathsf{Con}^k(\varphi_{j+1}) \text{ by definition of } \mathsf{Con}^k.\end{aligned}$$

This conclude the proof of the lemma. ❑

As an instance of the lemma, we get that $(\varphi_n \wedge \mathsf{Con}^n(\varphi_n)) \vdash \mathsf{Con}^n(\varphi_{n+1})$. We reason as follows.

$$\begin{aligned}f(\varphi_{n+1}) &\vdash \varphi_n \wedge (f(\varphi_n) \to \mathsf{Con}^n(\varphi_n)) \text{ by the definition of } \varphi_{n+1}.\\ f(\varphi_{n+1}) &\vdash f(\varphi_n) \text{ since } f \text{ is monotonic.}\\ &\vdash \mathsf{Con}^n(\varphi_n) \text{ by logic.}\\ &\vdash \mathsf{Con}^n(\varphi_{n+1}) \text{ by the lemma.}\end{aligned}$$

On the other hand, $\varphi_{n+1} \wedge \mathsf{Con}^n(\varphi_{n+1}) \vdash f(\varphi_{n+1})$ since f is everywhere bounded by Con^n. Thus, $[f(\varphi_{n+1})] = [\varphi_{n+1} \wedge \mathsf{Con}^n(\varphi_{n+1})]$, contradicting the assumption that there is no ψ and no $k \leqslant n$ such that $[f(\psi)] = [\psi \wedge \mathsf{Con}^k(\psi)] \neq [\bot]$. ❑

5. Transfinite iterates of Con are inevitable.

Generalizing the proof of Theorem 4.1 into the transfinite poses the following difficulty. Recall that the proof of Theorem 4.1 makes use of a sequence of sentences starting with φ_1 where

$$\varphi_{k+1} := \varphi_k \wedge (f(\varphi_k) \to \mathsf{Con}^k(\varphi_k)).$$

It is not clear what the ωth sentence in the sequence should be. A natural idea is that for a limit ordinal λ the corresponding "limit sentence" should quantify over the sentences in the sequence beneath it and express, roughly,

$$\forall \gamma < \lambda \big(\mathsf{True}(\varphi_\gamma) \wedge (\mathsf{True}(f(\varphi_\gamma)) \to \mathsf{Con}^\gamma(\varphi_\gamma))\big).$$

However, if the sentences in the sequence $(\varphi_\gamma)_{\gamma<\lambda}$ have unbounded syntactic complexity, then we are not guaranteed to have a truth-predicate with which we can quantify over them.

Nevertheless, we show that Theorem 4.1 generalizes into the transfinite given an additional assumption on complexity. Note that $\varphi \mapsto (\varphi \wedge \mathsf{Con}(\varphi))$ can be factored into two functions—the identity and $\varphi \mapsto \mathsf{Con}(\varphi)$—the latter of which always produces a Π^0_1 sentence. For the rest of this section, we will focus on monotonic functions $\varphi \mapsto \varphi \wedge f(\varphi)$ where f is monotonic and also $f(\varphi) \in \Pi^0_1$ for all φ.

Definition 5.1. A function f is Π^0_1 if $f(\varphi) \in \Pi^0_1$ for all φ.

For the next theorem we fix an elementary presentation Γ of a recursive well-ordering. In the statement of the theorem and throughout the proof α, β, γ, δ, etc. are names of ordinals from the notation system Γ.

Theorem 5.2. *Let f be a monotonic Π^0_1 function. Then either*
(i) for some $\beta \leqslant \alpha$ and some φ, $[\varphi \wedge f(\varphi)] = [\varphi \wedge \mathsf{Con}^{\beta}(\varphi)] \neq [\bot]$ or
(ii) for some φ, $(\varphi \wedge \mathsf{Con}^{\alpha}(\varphi)) \nvdash f(\varphi)$.

Proof. Let f be a monotonic Π^0_1 function such that for every φ,

$$(\varphi \wedge \mathsf{Con}^{\alpha}(\varphi)) \vdash (\varphi \wedge f(\varphi)).$$

We assume, for the sake of contradiction, that there is no sentence ζ and no $\beta \leqslant \alpha$ such that $[\zeta \wedge \mathsf{Con}^{\beta}(\zeta)] = [\zeta \wedge f(\zeta)] \neq [\bot]$. We then let φ be the conjunction of the following four sentences.

$$\forall \zeta (\mathsf{Con}(\zeta) \rightarrow \mathsf{Con}(\zeta \wedge \neg f(\zeta)))$$
$$\forall \beta \leqslant \alpha \forall \zeta \big(\mathsf{Con}^{\beta}(\zeta) \rightarrow \forall \delta < \beta, \neg \mathsf{Pr}\big((\zeta \wedge \mathsf{Con}^{\delta}(\zeta)) \leftrightarrow (\zeta \wedge f(\zeta))\big)\big)$$
$$\forall \zeta \forall \eta ((\zeta \vdash \eta) \rightarrow (f(\zeta) \vdash f(\eta)))$$
$$\forall x \big(\mathsf{Pr}(\mathsf{True}_{\Pi^0_2}(x)) \rightarrow \mathsf{True}_{\Pi^0_2}(x)\big)$$

The first expresses that for every consistent φ, $f(\varphi)$ strictly implies φ. The second sentence expresses that if $\beta < \alpha$, then $f(\zeta)$ and $\zeta \wedge \mathsf{Con}^{\beta}(\zeta)$ never coincide, unless $[\zeta \wedge \mathsf{Con}^{\beta}(\zeta)] = [\bot]$. The third sentence expresses the monotonicity of f. The fourth sentence expresses the Π^0_2 soundness of EA. Note that each of these sentences is true, so their conjunction φ is also true. Each of the four sentences is Π^0_2, whence so is φ.

We are interested in the following sequence $(\varphi_\beta)_{\beta \leqslant \Gamma}$. Note that the sentences in the sequence $(\varphi_\beta)_{\beta \leqslant \Gamma}$ all have complexity Π^0_2. Note moreover that since φ_1 is true, so is φ_β for every β.

$$\varphi_1 := \varphi.$$
$$\varphi_\gamma := \varphi_1 \wedge \forall \delta < \gamma \big(\mathsf{True}_{\Pi_1}(f(\varphi_\delta)) \rightarrow \mathsf{Con}^{\delta}(\varphi_\delta)\big) \text{ for } \gamma > 1.$$

Remark 5.3. We may assume that the ordinal notation system Γ is provably linear in EA. Thus, $\mathsf{EA} \vdash \forall \beta \leqslant \alpha, \forall \gamma < \beta (\mathsf{True}_{\Pi_2}(\varphi_\beta) \rightarrow \mathsf{True}_{\Pi_2}(\varphi_\gamma))$.

Our goal is to show that

$$[\varphi_{\alpha+1} \wedge \mathsf{Con}^{\alpha}(\varphi_{\alpha+1})] = [\varphi_{\alpha+1} \wedge f(\varphi_{\alpha+1})]$$

contradicting the assumption that f and Con^{α} never coincide. The main lemmas needed to prove this result are the following.

Lemma 5.4. $\mathsf{EA} \vdash \forall \gamma \leqslant \alpha \big((\varphi_\gamma \wedge \neg f(\varphi_\gamma)) \vdash \varphi_\alpha\big)$.

Lemma 5.5. $\mathsf{EA} \vdash \forall \beta \leqslant \alpha \forall \gamma \leqslant \beta \big(\varphi_\beta + \mathsf{Con}^{\gamma}(\varphi_\beta) \vdash \mathsf{Con}^{\gamma}(\varphi_\beta \wedge \neg f(\varphi_\beta))\big)$.

Lemma 5.4 is needed to derive Lemma 5.5. We now show how we use Lemma 5.5 to derive Theorem 5.2. As an instance of Lemma 5.5, letting $\alpha = \beta = \gamma$, we infer that

$$\mathsf{EA} \vdash \big(\varphi_\alpha + \mathsf{Con}^{\alpha}(\varphi_\alpha) \vdash \mathsf{Con}^{\alpha}(\varphi_\alpha \wedge \neg f(\varphi_\alpha))\big).$$

From the soundness of EA, we infer that

$$(\mp) \qquad \varphi_\alpha + \mathsf{Con}^{\alpha}(\varphi_\alpha) \vdash \mathsf{Con}^{\alpha}(\varphi_\alpha \wedge \neg f(\varphi_\alpha)).$$

We then reason as follows.

$$\begin{aligned}\varphi_{\alpha+1} &\vdash \varphi_\alpha \wedge (f(\varphi_\alpha) \to \mathsf{Con}^\alpha(\varphi_\alpha)) \text{ by the definition of } \varphi_{\alpha+1}.\\ f(\varphi_{\alpha+1}) &\vdash f(\varphi_\alpha) \text{ since } f \text{ is monotonic.}\\ \varphi_{\alpha+1} + f(\varphi_{\alpha+1}) &\vdash \varphi_\alpha \wedge \mathsf{Con}^\alpha(\varphi_\alpha) \text{ by logic.}\\ &\vdash \mathsf{Con}^\alpha(\varphi_{\alpha+1}) \text{ by } \mp.\end{aligned}$$

On the other hand, $\varphi_{\alpha+1} + \mathsf{Con}^\alpha(\varphi_{\alpha+1}) \vdash f(\varphi_{\alpha+1})$ since f is everywhere bounded by Con^α. Since φ_1 is true, so too is $\varphi_{\alpha+1}$, whence we infer that

$$[\varphi_{\alpha+1} \wedge \mathsf{Con}^\alpha(\varphi_{\alpha+1})] = [\varphi_{\alpha+1} \wedge f(\varphi_{\alpha+1})] \neq [\bot],$$

contradicting the claim that there is no sentence ζ and no $\beta \leqslant \alpha$ such that $[\zeta \wedge \mathsf{Con}^\beta(\zeta)] = [\zeta \wedge f(\zeta)] \neq [\bot]$. ❑

It remains to prove Lemma 5.4 and Lemma 5.5. We devote one subsection to each.

5.1. Proof of Lemma 5.4.

In this subsection we prove Lemma 5.4. First we recall the statement of the lemma.

Lemma 5.6. $\mathsf{EA} \vdash \forall \gamma \leqslant \alpha\big((\varphi_\gamma \wedge \neg f(\varphi_\gamma)) \vdash \varphi_\alpha\big)$.

Proof. We reason in EA. Let $\gamma \leqslant \alpha$. We assume that

$$(\eta) \qquad \mathsf{True}_{\Pi_2}(\varphi_\gamma) \wedge \neg\mathsf{True}_{\Pi_1}(f(\varphi_\gamma)).$$

We we want to derive φ_α, i.e.

$$\varphi_1 \wedge \forall \sigma < \alpha(\mathsf{True}_{\Pi_1}(f(\varphi_\sigma)) \to \mathsf{Con}^\sigma(\varphi_\sigma)).$$

The first conjunct follows trivially from the assumption that $\mathsf{True}_{\Pi_2}(\varphi_\gamma)$. We now prove the second conjunct of φ_α in two parts, first for all σ such that $\alpha > \sigma \geqslant \gamma$ and then for all $\sigma < \gamma$.

$\alpha > \sigma \geqslant \gamma$: From the assumption that $\mathsf{True}_{\Pi^0_2}(\varphi_\gamma)$ we infer that φ_1, whence we infer that f is monotonic. Thus, for all $\delta \geqslant \gamma$, $f(\varphi_\delta) \vdash f(\varphi_\gamma)$, i.e., $\mathsf{EA} \vdash \big(f(\varphi_\delta) \to f(\varphi_\gamma)\big)$. From φ_1 we also infer that EA is Π^0_2 sound, and so we infer that for all $\delta \geqslant \gamma$, $\mathsf{True}_{\Pi_1}(f(\varphi_\delta)) \to \mathsf{True}_{\Pi_1}(f(\varphi_\gamma))$. From the assumption that $\neg\mathsf{True}_{\Pi_1}(f(\varphi_\gamma))$ we then infer that for all $\delta \geqslant \gamma$, $\neg\mathsf{True}_{\Pi_1}(f(\varphi_\delta))$, whence for all $\delta \geqslant \gamma$, $\mathsf{True}_{\Pi_1}(f(\varphi_\delta)) \to \mathsf{Con}^\delta(\varphi_\delta)$.

$\sigma < \gamma$: By Remark 5.3, η implies that

$$\forall \sigma < \gamma(\mathsf{True}_{\Pi_1}\big(f(\varphi_\sigma)) \to \mathsf{Con}^\sigma(\varphi_\sigma)\big).$$

This completes the proof of Lemma 5.4. ❑

5.2. Proof of Lemma 5.5.

In this subsection we prove Lemma 5.5. We recall the statement of Lemma 5.5.

Lemma 5.7. $\mathsf{EA} \vdash \forall \beta \leqslant \alpha \forall \gamma \leqslant \beta\big(\varphi_\beta + \mathsf{Con}^\gamma(\varphi_\beta) \vdash \mathsf{Con}^\gamma(\varphi_\beta \wedge \neg f(\varphi_\beta))\big)$.

The proof of this lemma is importantly different from the proof of Lemma 4.2. In particular, to push the induction through limit stages we need to know not only that the inductive hypothesis is true but also that it is provable in EA. We resolve this issue by using Schmerl's technique of *reflexive transfinite induction* (see Proposition 3.4).

In the proof of the lemma, we let $\mathcal{C}(\gamma,\delta)$ abbreviate the claim that

$$\varphi_\delta + \mathsf{Con}^\gamma(\varphi_\delta) \vdash \mathsf{Con}^\gamma(\varphi_\delta \wedge \neg f(\varphi_\delta)).$$

Proof. We want to show that

$$\mathsf{EA} \vdash \forall \beta \leqslant \alpha (\forall \gamma \leqslant \beta(\mathcal{C}(\gamma,\beta))).$$

By Proposition 3.4 it suffices to show that

$$\mathsf{EA} \vdash \forall \alpha (\mathsf{Pr}(\forall \beta \leqslant \alpha \forall \gamma \leqslant \beta \mathcal{C}(\gamma,\beta)) \to \forall \gamma \leqslant \alpha \mathcal{C}(\gamma,\alpha)).^2$$

Thus, we **reason in** EA and fix α. We assume that

$$(\triangle) \qquad \mathsf{Pr}(\forall \beta \leqslant \alpha, \forall \gamma \leqslant \beta, \mathcal{C}(\gamma,\beta)).$$

We let $\gamma \leqslant \alpha$ and we want to show that $\mathcal{C}(\gamma,\alpha)$.

Since $\varphi_\alpha \vdash \varphi$ we infer that

$$(\sharp) \qquad \varphi_\alpha + \mathsf{Con}^\gamma(\varphi_\alpha) \vdash \forall \delta < \gamma, \neg \mathsf{Pr}\big((\varphi_\alpha \wedge \mathsf{Con}^\delta(\varphi_\alpha)) \leftrightarrow (\varphi_\alpha \wedge f(\varphi_\alpha))\big).$$

We first note that both

$$\varphi_\alpha \vdash \forall \delta < \gamma(\mathsf{True}_{\Pi_1}(f(\varphi_\delta)) \to \mathsf{Con}^\delta(\varphi_\delta)) \text{ by the definition of } \varphi_\alpha \text{ and also}$$

$$\begin{aligned}
\varphi_\alpha + f(\varphi_\alpha) &\vdash \forall \delta < \gamma(f(\varphi_\alpha) \vdash f(\varphi_\delta)) \text{ since } \varphi_1 \text{ proves the monotonicity of } f. \\
&\vdash \forall \delta < \gamma(\mathsf{EA} \vdash (f(\varphi_\alpha) \to f(\varphi_\delta))). \\
&\vdash \forall \delta < \gamma(f(\varphi_\alpha) \to \mathsf{True}_{\Pi_1}(f(\varphi_\delta))) \text{ since } \varphi_1 \text{ proves the } \Pi^0_2 \text{ soundness of } \mathsf{EA}. \\
&\vdash \forall \delta < \gamma, \mathsf{True}_{\Pi_1}(f(\varphi_\delta)) \text{ by logic.}
\end{aligned}$$

Thus, we may reason as follows.

$$\begin{aligned}
\varphi_\alpha + f(\varphi_\alpha) &\vdash \forall \delta < \gamma, \mathsf{Con}^\delta(\varphi_\delta) \\
&\vdash \forall \delta < \gamma, \mathsf{Con}^\delta(\varphi_\delta \wedge \neg f(\varphi_\delta))) \text{ since } (\triangle) \text{ delivers } \mathcal{C}(\delta,\delta). \\
&\vdash \forall \delta < \gamma, \mathsf{Con}^\delta(\varphi_\alpha) \text{ by Lemma 5.4.}
\end{aligned}$$

Thus, by Σ^0_1 completeness,

$$\mathsf{EA} \vdash \forall \delta < \gamma\big(\varphi_\alpha \wedge f(\varphi_\alpha) \vdash \mathsf{Con}^\delta(\varphi_\alpha)\big).$$

Combined with ($\sharp$), this delivers

$$\begin{aligned}
\varphi_\alpha + \mathsf{Con}^\gamma(\varphi_\alpha) &\vdash \forall \delta < \gamma\big(\varphi_\alpha + \mathsf{Con}^\delta(\varphi_\alpha) \nvdash f(\varphi_\alpha)\big). \\
&\vdash \forall \delta < \gamma, \mathsf{Con}(\varphi_\alpha \wedge \neg f(\varphi_\alpha) \wedge \mathsf{Con}^\delta(\varphi_\alpha)). \\
&\vdash \forall \delta < \gamma, \mathsf{Con}\big(\varphi_\alpha \wedge \neg f(\varphi_\alpha) \wedge \mathsf{Con}^\delta(\varphi_\alpha \wedge \neg f(\varphi_\alpha))\big) \text{ since } (\triangle) \text{ delivers } \mathcal{C}(\delta,\alpha). \\
&\vdash \mathsf{Con}^\gamma(\varphi_\alpha \wedge \neg f(\varphi_\alpha)).
\end{aligned}$$

This completes the proof of Lemma 5.5. ❑

Theorem 5.2 shows the inevitability of the consistency operator. For a sufficiently constrained monotonic function f, f and must coincide with an iterate of Con on some non-trivial sentence. However, it is not clear from the proofs of Theorem 4.1 or Theorem 5.2 that f must coincide with Con on a *true* sentence.

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[2] The reader might expect that we need to write "$\beta < \alpha$" instead of "$\beta \leqslant \alpha$" in the antecedent for this to match the statement of Proposition 3.4. However, it is clear from the proof of Proposition 3.4 that this suffices.

Question 5.8. *Let f be a monotonic Π^0_1 function. Suppose that for every φ,*

$$(\varphi \wedge \mathsf{Con}^{\alpha}(\varphi)) \vdash f(\varphi).$$

*Must there be some $\beta \leqslant \alpha$ and some **true** φ such that*

$$[\varphi \wedge f(\varphi)] = [\varphi \wedge \mathsf{Con}^{\beta}(\varphi)]?$$

6. 1-consistency and iterated consistency

Just as the Π^0_1 fragments of natural theories can often be approximated by iterated consistency statements, the Π^0_2 fragments of natural theories can often be approximated by iterated 1-consistency statements. A theory T is *1-consistent* if $T + \mathrm{Th}_{\Pi^0_1}(\mathbb{N})$ is consistent. The 1-consistency of $\mathsf{EA} + \varphi$ can be expressed by the following Π^0_2 sentence, $1\mathsf{Con}(\varphi)$:

$$\forall x(\mathsf{True}_{\Pi^0_1}(x) \to \mathsf{Con}(\varphi \wedge \mathsf{True}_{\Pi^0_1}(x))).$$

In this section, we investigate the relationship between 1-consistency and iterated consistency. First, we show that $1\mathsf{Con}$ majorizes every iterate of Con^{α}.

Proposition 6.1. *For any elementary presentation α of a recursive well ordering, there is a true sentence φ such that for every ψ, if $\psi \vdash \varphi$, then $(\psi \wedge 1\mathsf{Con}(\psi))$ implies $(\psi \wedge \mathsf{Con}^{\alpha}(\psi))$. Moreover, if $[\psi \wedge \mathsf{Con}^{\alpha}(\psi)] \neq [\bot]$ then $(\psi \wedge 1\mathsf{Con}(\psi))$ strictly implies $(\psi \wedge \mathsf{Con}^{\alpha}(\psi))$.*

Proof. Let α be an elementary presentation of a recursive well-ordering. Let φ be a true sentence such that $\varphi \vdash \mathsf{TI}^{\alpha}_{\Pi^0_1}$, i.e., φ implies the validity of transfinite induction along α for Π^0_1 predicates. We prove that

$$(\varphi \wedge 1\mathsf{Con}(\varphi)) \vdash \mathsf{Con}^{\alpha+1}(\varphi).$$

Since $\varphi \wedge 1\mathsf{Con}(\varphi) \vdash \mathsf{TI}^{\alpha}_{\Pi^0_1}$, it suffices to show that:
Base case: $(\varphi \wedge 1\mathsf{Con}(\varphi)) \vdash \mathsf{Con}(\varphi)$
Successor case: $(\varphi \wedge 1\mathsf{Con}(\varphi)) \vdash \forall \beta < \alpha(\mathsf{Con}^{\beta}(\varphi) \to \mathsf{Con}^{\beta+1}(\varphi))$
Limit case: $(\varphi \wedge 1\mathsf{Con}(\varphi)) \vdash \forall \lambda\Big(lim(\lambda) \to \big((\forall \beta < \lambda \mathsf{Con}^{\beta}(\varphi)) \to \mathsf{Con}^{\lambda}(\varphi)\big)\Big)$

The **base case** and the **limit case** are both trivial. For the **successor case** we first note that by the definition of $1\mathsf{Con}(\varphi)$,

$$1\mathsf{Con}(\varphi) \vdash \forall x(\mathsf{True}_{\Pi^0_1}(x) \to \mathsf{Con}(\varphi \wedge \mathsf{True}_{\Pi^0_1}(x))),$$

and so by substituting $\mathsf{Con}^{\beta}(\varphi)$ in for x,

$$(\oplus) \qquad 1\mathsf{Con}(\varphi) \vdash \mathsf{True}_{\Pi^0_1}(\mathsf{Con}^{\beta}(\varphi)) \to \mathsf{Con}(\varphi \wedge \mathsf{True}_{\Pi^0_1}(\mathsf{Con}^{\beta}(\varphi))).$$

Thus, we reason as follows.

$$\begin{aligned} 1\mathsf{Con}(\varphi) \vdash \mathsf{Con}^{\beta}(\varphi) &\to \mathsf{Con}(\varphi \wedge \mathsf{True}_{\Pi^0_1}(\mathsf{Con}^{\beta}(\varphi))) \text{ by } (\oplus). \\ &\to \mathsf{Con}(\varphi \wedge \mathsf{Con}^{\beta}(\varphi)). \\ &\to \mathsf{Con}^{\beta+1}(\varphi) \text{ by the definition of } \mathsf{Con}^{\beta+1}. \end{aligned}$$

It is clear that the implication $\varphi \wedge 1\mathsf{Con}(\varphi) \vdash \varphi \wedge \mathsf{Con}^{\alpha}(\varphi)$ is strict as long as $[\varphi \wedge \mathsf{Con}^{\alpha}(\varphi)] \neq [\bot]$. This completes the proof of the proposition. ❑

In light of the previous proposition, one might conjecture that $1\mathsf{Con}$ is the weakest monotonic function majorizing every function of the form Con^α for some recursive well-ordering α on true sentences. However, this is not so. To demonstrate this, we use a recursive linear order that has no hyperarithmetic infinite descending sequences. Harrison [9] introduced such an ordering with order-type $\omega_1^{CK} \times (1+\mathbb{Q})$; see also Feferman and Spector [7] who consider such orderings in the context of iterated reflection principles. We use a variant $\mathcal{H}$ of Harrison's ordering such that it is elementarily calculable whether an element of $\mathcal{H}$ is zero or a successor or a limit. We note that since $\mathcal{H}$ has no hyperarithmetic descending sequences, transfinite induction along $\mathcal{H}$ for Π^0_1 properties is valid. Our idea is to produce a function stronger than each Con^α but weaker than $1\mathsf{Con}$ by iterating Con along the Harrison linear order.

Theorem 6.2. *There are infinitely many monotonic functions f such that for every recursive ordinal α, there is an elementary presentation a of α such that f majorizes Con^a on a true ideal but also $1\mathsf{Con}$ majorizes f on a true ideal.*

Proof. In Definition 3.2, we used Gödel's fixed point lemma to produce iterates of Con along an elementary well-ordering. We similarly use Gödel's fixed point lemma to define sentences $\mathbf{Con}^\star(\varphi, \beta)$ for $\beta \in \mathcal{H}$ as follows.

$$\mathsf{EA} \vdash \mathbf{Con}^\star(\varphi, \beta) \leftrightarrow \forall \gamma <_{\mathcal{H}} \beta, \mathsf{Con}(\varphi \wedge \mathbf{Con}^\star(\varphi, \gamma)).$$

We use the notation $\mathsf{Con}^\beta(\varphi)$ for $\mathbf{Con}^\star(\varphi, \beta)$. Recall that we are assuming that it is elementarily calculable whether an element of $\mathcal{H}$ is zero or a successor or a limit. Thus, the following clauses are provable in EA.

- $\mathsf{Con}^0(\varphi) \leftrightarrow \top$
- $\mathsf{Con}^{\gamma+1}(\varphi) \leftrightarrow \mathsf{Con}(\varphi \wedge \mathsf{Con}^\gamma(\varphi))$
- $\mathsf{Con}^\lambda(\varphi) \leftrightarrow \forall \gamma <_{\mathcal{H}} \lambda, \mathsf{Con}^\gamma(\varphi)$ for λ a limit.

Claim. *For $\gamma \in \mathcal{H}$, the function $\varphi \mapsto Con^\gamma(\varphi)$ is monotonic.*

This follows immediately from Proposition 3.5. Note that in the statement of Lemma 3.4 we assume only that $<$ is an elementary *linear* ordering, not a well-ordering.

Claim. *There are infinitely many monotonic functions f such that for every recursive well-ordering α, there is an elementary presentation a of α such that f majorizes Con^a on true sentences.*

If $x <_{\mathcal{H}} y$ then $\mathsf{Con}^y(\varphi)$ strictly implies $\mathsf{Con}^x(\varphi)$ for every φ such that $\mathsf{Con}^x(\varphi) \neq [\bot]$. Given the order type of $\mathcal{H}$, this means that for infinitely many γ, for every recursive well-ordering α, Con^γ majorizes Con^a where a represents α in $\mathcal{H}$.

Claim. $1\mathsf{Con}$ *majorizes* Con^a *on true sentences for each $a \in \mathcal{H}$.*

Since every Π^0_1 definable subset of ω has an $\mathcal{H}$-least element, the sentence $\mathsf{TI}^{\mathcal{H}}_{\Pi^0_1}$, which expresses the validity of transfinite induction along $\mathcal{H}$ for Π^0_1 predicates, is true. But then if $\varphi \vdash \mathsf{TI}^{\mathcal{H}}_{\Pi^0_1}$, then for any $\gamma \in \mathcal{H}$, $(\varphi \wedge 1\mathsf{Con}(\varphi))$ strictly implies $(\varphi \wedge \mathsf{Con}^\gamma(\varphi))$ as long as $[(\varphi \wedge \mathsf{Con}^\gamma(\varphi))] \neq [\bot]$, as in Proposition 6.1. ❑

7. An unbounded recursively enumerable set that contains no true ideals

In this section we prove a limitative result. Theorem 2.2 demonstrates that if f is monotonic and that for all consistent φ, (i) $\varphi \wedge \mathsf{Con}(\varphi)$ implies $f(\varphi)$ and (ii) $f(\varphi)$ strictly implies φ, then for cofinally many true φ, $[f(\varphi)] = [\varphi \wedge \mathsf{Con}(\varphi)]$. It is natural to conjecture that cofinal equivalence with Con be strengthened to equivalence to Con *in the limit*, i.e., on a true ideal. One strategy to strengthen Theorem 2.2 in this way would be to show that every recursively enumerable set that contains arbitrarily strong true sentences and that is closed under provable equivalence contains a true ideal.

We now show that the aforementioned strategy fails. To this end, we define a recursively enumerable set $\mathcal{A}$ that contains arbitrarily strong true sentences and that is closed under provable equivalence but does not contain any true ideals. We are grateful to Matthew Harrison-Trainor for simplifying the proof of the following proposition.

Proposition 7.1. *There is a recursively enumerable set $\mathcal{A}$ that contains arbitrarily strong true sentences and that is closed under* EA *provable equivalence but does not contain any true ideals.*

Proof. Let $\{\varphi_0, \varphi_1, ...\}$ be an effective Gödel numbering of the language of arithmetic. We describe the construction of $\mathcal{A}$ in stages. During a stage n we may *activate* a sentence ψ, in which case we say that ψ is *active* until it is *deactivated* at some later stage $n + k$. After describing the construction of $\mathcal{A}$ we verify that A has the desired properties.

Stage 0: Numerate φ_0 and $\neg\varphi_0$ into $\mathcal{A}$. Activate the sentences $(\varphi_0 \wedge \mathsf{Con}(\varphi_0))$ and $(\neg\varphi_0 \wedge \mathsf{Con}(\neg\varphi_0))$.

Stage n+1: There are finitely many active sentences. For each such sentence ψ, numerate $\theta_0 := (\psi \wedge \varphi_{n+1})$ and $\theta_1 := (\psi \wedge \neg\varphi_{n+1})$ into $\mathcal{A}$. Deactivate the sentence ψ and activate the sentences $(\theta_0 \wedge \mathsf{Con}(\theta_0))$ and $(\theta_1 \wedge \mathsf{Con}(\theta_1))$.

We dovetail the construction with a search through EA proofs. If we ever see that $\mathsf{EA} \vdash \varphi \leftrightarrow \psi$ for some φ that we have already numerated into $\mathcal{A}$, then we numerate ψ into $\mathcal{A}$.

Now we check that $\mathcal{A}$ has the desired properties. It is clear that $\mathcal{A}$ is recursively enumerable and that $\mathcal{A}$ is closed under EA provable equivalence.

Claim. *$\mathcal{A}$ contains arbitrarily strong true sentences. That is, for each true sentence φ, there is a true sentence ψ such that $\psi \vdash \varphi$ and $\psi \in \mathcal{A}$.*

At any stage in the construction of $\mathcal{A}$, there are finitely many active sentences, ψ_0, ..., ψ_k. An easy induction shows that exactly one of $\psi_0, ..., \psi_k$ is true. Indeed, exactly one of φ_0 or $\neg\varphi_0$ is true, and hence so is exactly one of $\varphi_0 \wedge \mathsf{Con}(\varphi_0)$ and $\neg\varphi_0 \wedge \mathsf{Con}(\neg\varphi_0)$. And if θ is true, then so is exactly one of $\zeta_0 := \theta \wedge \varphi_k$ and $\zeta_1 := \theta \wedge \neg\varphi_k$, and hence so too is exactly one of $\zeta_0 \wedge \mathsf{Con}(\zeta_0)$ and $\zeta_1 \wedge \mathsf{Con}(\zeta_1)$.

Let φ_k be a true sentence. At stage k in the construction of $\mathcal{A}$ there are only finitely many active sentences $\psi_0, ..., \psi_n$. We have already seen that exactly one of ψ_i is true. But then $\varphi_k \wedge \psi_i$ is true, $(\varphi_k \wedge \psi_i \vdash \varphi_k)$, and $(\varphi_k \wedge \psi_i)$ is numerated into $\mathcal{A}$.

Claim. *$\mathcal{A}$ contains no true ideals.*

An easy induction shows that if ψ_0 and ψ_1 are both active at the same stage, then for any θ, if θ implies both ψ_0 and ψ_1 then $\theta \in [\bot]$.

Let φ be a true sentence in $\mathcal{A}$. By the previous remark, the only sentences in $\mathcal{A}$ that strictly imply φ are (i) EA refutable sentences and (ii) sentences that imply $\varphi \wedge \mathsf{Con}(\varphi)$. Since the Lindenbaum algebra of EA is dense, this means there is some ψ such that $(\varphi \wedge \mathsf{Con}(\varphi))$ strictly implies ψ strictly implies φ but $\psi \notin \mathcal{A}$. ❑

The following questions remain.

Question 7.2. *Is the relation of cofinal agreement on true sentences an equivalence relation on recursive monotonic operators?*

Question 7.3. *Let f be monotonic. Suppose that for all consistent φ,*
(i) $\varphi \wedge \mathsf{Con}(\varphi)$ implies $f(\varphi)$ and
(ii) $f(\varphi)$ implies φ.
Must f be equivalent to the identity or to Con on a true ideal?

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