MARTIN'S CONJECTURE: A CLASSIFICATION OF THE NATURALLY OCCURRING TURING DEGREES

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This paper is about naturally occurring objects in *computability theory*, the area inside mathematical logic that studies the complexity of infinite countable objects. It is about the structure that emerges when we look at almost-everywhere behavior with respect to a measure called Martin's measure. Posed in the 70s, *Martin's conjecture* was the first indication of this hidden structure. Martin's conjecture is currently one of the main open questions in computability theory and is of great foundational importance. Progress has been slow, and only recently we have started to appreciate the extent to which Martin's measure can be used to understand the behavior of naturally occurring objects in computability theory.

Computability Theory

Computability theory is the area of logic that studies the complexity of infinite countable objects. Computer science and set theory take care of the finite objects and uncountable objects respectively — well, not exactly, but more or less. The main notion in computability theory is that of computable function. A function f is *computable* if it can be calculated using a mechanical, step-by-step algorithm. The informal notion of algorithm was already known to the Greeks two millennia ago. But it was not until the 1930s that Gödel, Turing, and Church among others proposed the first formal definitions of computable functions. Conceived before the age of computers, those three equivalent definitions capture what we still call the class of computable functions: Let us use $\mathbb{N}^{\mathbb{N}}$ to denote the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Today, we say that a function $f \in \mathbb{N}^{\mathbb{N}}$ is *computable* if it can be calculated using a computer program in your favorite programing language.[†] Which programing language you use is not important. You always get the same class of functions as long as the language has some minimum basic functionality. Some languages might be faster or easier to use for certain types of functions, but the class of computable functions from $\mathbb{N}$ to $\mathbb{N}$ is independent of the language. The restriction to functions on the natural numbers is just for simplicity. It is not even a real restriction, as any finite object can be coded by a natural number

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[†]We impose no restriction on time spent or memory used so long as, on each input, the program gives an answer after finitely many steps, hence using only finitely much memory.

— that's what your computer does when it encodes everything you see on the screen into numbers written in binary. In our examples below, we will mention finite objects, like strings of characters or integer polynomials, and assume they are being encoded by natural numbers in some standard way without even mentioning it.

A subset $A \subseteq \mathbb{N}$ is *computable* if its characteristic function $\chi_A \colon \mathbb{N} \to \{0,1\}$ is computable. Here are some examples of non-computable sets.

***K*:** *The halting problem* is the problem of deciding whether an input-less computer program will eventually halt or run forever without halting, maybe because it enters an infinite loop or something. We represent this problem by the set K of input-less computer programs that eventually halt. (Here, programs are being viewed as strings of characters and coded by a single number.) To solve the halting problem, one could try to run the given program and see if it halts. The problem is that while a program is running and hasn't halted yet, we can't tell for sure if it is going to stop later or just keep on going.[‡]

***WP*:** *The word problem.* Consider finitely presented groups. These are groups that have finitely many generators $a_1, a_2, ..., a_k$ which satisfy a finite set of relations of the form $R_i = e$, where $R_1, ..., R_\ell$ are words on the letters $a_1, a_2, ..., a_k, a_1^{-1}, ..., a_k^{-1}$. The word problem is to decide, given such a group presentation and another word w, whether w equivalent to the identity e of the group or not. We define WP to be the set of tuples $((a_1, a_2, ..., a_k), (R_1, ..., R_\ell), w)$ for which the answer is yes.

***HTP*:** *Hilbert's tenth problem.* Let HTP be the set of polynomials in $\mathbb{Z}[y_0, y_1, y_2, ...]$ that have an integer solution. Question ten in Hilbert's famous 1900 list was whether there was an algorithm to decide membership in HTP. It was answered negatively in 1970 by Matiyasevich, building on work by Martin Davis, Hilary Putnam, and Julia Robinson.

***TF*:** Let TF be the set of finite presentations $((a_1, a_2, ..., a_k), (R_1, ..., R_\ell))$ of groups that are torsion-free.

***COF*:** Let COF be the set of polynomials in $\mathbb{Z}[x, y_0, y_1, y_2, ...]$ which have integer solutions for all but at most finitely many values of x.

***TA*:** Let TA be the set of 1st-order sentences in the language of arithmetic $\{0, 1, +, \times, \leq\}$ which are true about the natural numbers.

***WF*:** Let WF be the set of programs $\mathtt{p}$ for which there exists no sequence $a_0, a_1, a_2, ...$ of natural numbers such that $\mathtt{p}(a_{i+1}) = a_i$ for all $i \in \mathbb{N}$.[§]

To prove that any of these examples cannot be decided by a computer program, one starts with the halting problem. For that, one uses a Cantor-style diagonalization argument to define, under the assumption that K is computable,

[‡]The 'K' is for Stephen Kleene.

[§]The notation $\mathtt{p}(a) = b$ means that the program $\mathtt{p}$ halts on input a and outputs b.

a computable function that is different from all computable functions. For the other examples, the idea is to reduce them to the halting problem. For instance, to prove that the word problem is non-computable, we produce an algorithm that decides membership in the halting problem using information about which words in which finitely presented groups become trivial.[¶]

Let us focus on this idea of reducing one problem to another. Consider a function $g \in \mathbb{N}^{\mathbb{N}}$. We say that a function $f \in \mathbb{N}^{\mathbb{N}}$ is *g-computable*, or *computable in* g, and write $f \leq_T g$, if the values of f can be calculated using a computer program that is allowed to consult values of g during its computation. For example, this program could contain an instruction of the form: "`let x:=17; if` g`(x)>27 then do this, else do that`." The function g is called the *oracle* of the computation, as we are not specifying how we are getting the values of g — they are just given to us. The relation $\leq_T$ is transitive and reflexive, and hence a pre-ordering on $\mathbb{N}^{\mathbb{N}}$. As any pre-ordering does, $\leq_T$ naturally induces an equivalence relation, $f \equiv_T g$ if $f \leq_T g$ and $g \leq_T f$, and a partial ordering on the quotient $\mathbb{N}^{\mathbb{N}}/\equiv_T$. The equivalence classes are called Turing degrees, and this partial ordering $(\mathbb{N}^{\mathbb{N}}/\equiv_T; \leq_T)$ is denoted by $\mathcal{D}$. Each Turing degree corresponds to a level of complexity, and $\mathcal{D}$ maps out all complexity levels. The properties of this partial ordering have been widely studied since the 50s. Here are some of them. The first four are quite simple, while the last two require extremely elaborate proofs.

- $\mathcal{D}$ has a least element, 0, that consists of the equivalence class of the computable functions.
- Every two degrees have a least upper bound given by $f \oplus g$, where $f \oplus g$ is the function h such that $h(2n) = f(n)$ and $h(2n + 1) = g(n)$.
- $\mathcal{D}$ has the countable predecessor property, i.e., for every g, there are only countably many f's that are $\leq_T g$. This is because there are only countably many programs one can write.
- In particular, every degree is countable, and hence $\mathcal{D}$ has size continuum.
- $\mathcal{D}$ is fat: There are continuum-size sets of degrees that are pairwise incomparable.
- $\mathcal{D}$ is very rich: Every countable partial ordering embeds in $\mathcal{D}$ as an initial segment.

There are many more results showing that $\mathcal{D}$ is extremely complex. An example of a question that has eluded an answer for decades is whether $\mathcal{D}$ has a non-trivial automorphism.

[¶]This isn't easy at all. One has to encode the nuts and bolts of a Turing machine into a presentation of a group in such a way that a certain word is equivalent to the identity if and only if the given Turing machine halts.

The linearity phenomenon

We wanted to find a ruler that we can use to measure complexity, a hierarchy of complexity levels that we can use to classify problems. Instead, we've got an ordering, $\leq_T$, that gives us a way of saying that certain problems are more complex than others and that some are incomparable. The world turned out messier than we wanted it to be. The examples we gave above are ordered as follows under Turing reduction:

$$0 <_T K \equiv_T WP \equiv_T HTP <_T TF <_T COF <_T TA <_T WF.$$

They do seem to form a hierarchy. There are many more examples one can get from elsewhere in mathematics and many more from computability theory that are still very natural, even if they may look slightly weird to outsiders. All the examples we know are ordered in a line.[‖] Furthermore, we know of no natural example strictly in between 0 and K, and all natural examples strictly in between 0 and COF are Turing equivalent to either K or TF. We know many natural examples between COF and TA, though they are nicely ordered in a line like the ordering of the natural numbers. We know many natural examples between TA and WF, and they are nicely ordered in a well-ordered line. The hierarchy we were looking for seems to exist, but $\mathcal{D}$ seems too chaotic to help us find it. The contrast between the general behavior in $\mathcal{D}$ and the behavior of the naturally occurring objects is so stark that there must a deep reason behind it. We need to dig deeper.

Martin's conjecture is a formal statement trying to capture the essence of this hierarchy within the Turing degrees. For now, let us say that Martin's conjecture gives a formal mathematical understanding of the following empirical observation:

> While the infinite sequences in $\mathbb{N}^{\mathbb{N}}$ are not linearly ordered by Turing computability, the naturally occurring sequences are.

What will allow us to formalize this statement is the observation that, in this setting, there are two properties natural objects have that general objects do not: they *relativize* (to be defined below) and they are *constructible*. We can formally define what a constructible relativizable object is, and we can then write a formal mathematical statement saying they are nicely well-ordered by complexity. That is Martin's conjecture. Posed in the seventies, Martin's conjecture is considered the most important open question in computability theory and of great foundational importance within mathematical logic. We will describe its statement in detail after we give more background and context.

[‖] A linear ordering is a partial ordering where every two elements are comparable. They are also called total orderings.

Similar behavior in proof theory

Another well-known expression of this linearity phenomenon is with axiomatizations of fragments of mathematics: the hundreds of axiomatizations for mathematics that logicians have looked at, starting from elementary arithmetic, going up through Peano arithmetic, Zermelo–Fraenkel set theory, and up through the large-cardinal hypotheses, are linearly ordered under consistency strength, giving rise to what is called the *Gödel hierarchy*. One can cook up ad hoc theories which are incomparable under consistency strength, but the natural ones are always comparable. Again, we can't even state this in a precise way, as we don't know what makes a theory *natural*. Our recent paper with James Walsh [MW] takes a step toward that problem, proving that the transfinite iterates of the consistency operator form a spine of theories that is canonical in some sense. This must be somewhat connected to the linearity phenomenon in computability theory, but we are far from understanding precisely how they are connected.

Martin's measure

Let us recall that we've got a pre-ordering $\leq_T$ on $\mathbb{N}^\mathbb{N}$ given by $g \leq_T f$ if g can be computed from f. The equivalence classes it induces are called Turing degrees, and $\leq_T$ partially orders the Turing degrees. This partial ordering is an upper-semi-lattice, meaning that every two elements have a least upper bound. Also, every countable subset has an upper bound — though maybe not a least one. Other than that, this partial ordering is extremely messy and very hard to describe. The following peculiarity is the first indication it is not all chaos. We need two definitions first: A subset $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$ is $\equiv_T$*-invariant* if whenever $f \equiv_T g$, we have that $f \in \mathcal{A} \iff g \in \mathcal{A}$. The *cone* above $f \in \mathbb{N}^\mathbb{N}$ is the set $\{g \in \mathbb{N}^\mathbb{N} : g \geq_T f\}$.

Theorem (Martin's Turing determinacy)**.** *Every $\equiv_T$-invariant Borel subset of $\mathbb{N}^\mathbb{N}$ either contains a cone or is disjoint from a cone.*

The topology on $\mathbb{N}^\mathbb{N}$ we are using to define the Borel sets is the product of the discrete topology of $\mathbb{N}$.** It is important to point out that the theorem above is not about Borel sets; It is, essentially, about any set you can define without invoking the black magic of the axiom of choice. I stated it using Borel sets because that's the most we can prove in ZFC, the standard axiomatization for all of mathematics. Let me just say that if we add a few large-cardinal assumptions, assumptions that set theorists believe to be true (whatever that means), then the theorem is true for a much larger class of sets called $L(\mathbb{R})$, the class of all sets that are constructible over the reals. For a mathematician who isn't a set theorist, the only way to build a subset of $\mathbb{N}^\mathbb{N}$ outside $L(\mathbb{R})$ is using the axiom of choice. The theorem thus

**Thus the Borel subsets of $\mathbb{N}^\mathbb{N}$ are the smallest class of sets that is closed under complements, countable unions and countable intersections, and contain the basic open sets $\mathcal{V}_{n,m} = \{f \in \mathbb{N}^\mathbb{N} : f(n) = m\}$.

says that for every $\equiv_T$-invariant constructible set $\mathcal{A}$, there is an $f \in \mathbb{N}^\mathbb{N}$ such that either every $g \geq_T f$ is in $\mathcal{A}$ or none is.

Thinking of cones as *large* sets, we can define a probability measure on all Borel $\equiv_T$-invariant subsets $\mathcal{A}$ of $\mathbb{N}^\mathbb{N}$: $m(A) = 1$ if $\mathcal{A}$ contains a cone, and $m(A) = 0$ if $\mathcal{A}$ is disjoint from a cone. It is not hard to see that the intersection of countably many cones contains a cone.[††] Thus, this measure is thus σ-additive. The theorem above says that this measure is $\equiv_T$-ergodic. A corollary of the theorem is that every Borel $\equiv_T$-to-$=$ invariant function must be constant on a set of Martin's measure 1, where $\mathcal{F}\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is $\equiv_T$*-to-*$=$ *invariant* if $\mathcal{F}(g) = \mathcal{F}(f)$ whenever $g \equiv_T f$. This resembles the ergodicity of Lebesgue measure with respect to Vitali's equivalence relation: If $\mathcal{F}\colon [0,1] \to \mathbb{R}$ is a Borel function such that, whenever $x - y \in \mathbb{Q}$, $\mathcal{F}(x) = \mathcal{F}(y)$, then $\mathcal{F}$ is constant on a set of Lebesgue measure 1.

The reader might be wondering about examples of $\equiv_T$-invariant sets where one could apply the theorem: The main tool to produce $\equiv_T$-invariant sets is called relativization.

Relativization

Given $f \in \mathbb{N}^\mathbb{N}$, the class of f-computable functions satisfies pretty much all the same properties as the class of computable functions: the partial f-computable functions are closed under composition, recursion, and minimization; they contain the basic functions namely the projections, the zero function and the plus-one function; and there is a universal partial f-computable function. These imply pretty much all the other properties of the computable functions that are used in almost all the proofs in computability theory. Only if one gets to the inner workings of a Turing machine, can one see a difference between these classes. Except for very few occasions, such as in the in proof of undecidability of the word problem, it is extremely rare to find a proof that deals with the nuts and bolts of a Turing machine. Since this is so rare, pretty much all the results in computability theory could be restated using the notion of f-computability instead of plain computability, and these new statements would still be true by pretty much the same proofs. This process is call *relativization*.

When we relativize the halting problem to f, we obtain the set of all programs which are allowed to call f during their computation and that eventually halt. We denote this set by f'. This is an extremely important operation — it's called the Turing jump. It is a monotone and order-preserving operation. That is, for all $f, g \in \mathbb{N}^\mathbb{N}$,

- $f <_T f'$, and
- $f \leq g \implies f' \leq g'$.

[††] The intersection of the cones above f_n for $n \in \mathbb{N}$ contains the cone above f for any upper bound f of all the f_n's, as for instance the function $f(\langle n,m\rangle) = f_n(m)$ where $\langle n,m\rangle$ is a number coding the pair (n,m), say $2^n 3^m$.

In particular, the Turing jump is $\equiv_T$-to-$\equiv_T$-invariant, i.e., $f \equiv_T g$ implies $f' \equiv_T g'$. Let us use 0 to denote the constant-zero function $n \mapsto 0\colon \mathbb{N} \to \mathbb{N}$. Then the halting problem K has degree $0'$, and one can show that so do *WP* and *HTP*. If we apply the Turing jump again, we get $0''$. It turns out that *TF* is Turing equivalent to $0''$. If we take another jump, we get to $0'''$, which happens to be Turing equivalent to *COF*. We can then keep on iterating, but we will not get to *TA* in any finite number of steps. However, *TA* is Turing equivalent to $0^{(\omega)}$, the set of pairs $\{(n,m) : n \in 0^{(m)}\}$, where $0^{(m)}$ is the mth iterate of the Turing jump, and ω denotes the first ordinal number that comes after all the natural numbers. The last example in our list, *WF*, is beyond any iteration of the Turing jump along any computable ordinal.

Before getting deeper into the iterates of the jump, let us take a side step and see an example of how relativization works in conjunction with Martin's measure.

Computable Categoricity

To see how relativization works, let's look at an example from computable structure theory. The reader may skip this section if the reader is too eager to learn about Martin's conjecture in the next section. Computable structure theory is the sub-field of computability theory where we study the complexity of countable structures. These are mathematical structures like groups, fields, linear orderings, or graphs. The objective is to understand the forms complexity takes in this setting and find connections between complexity and algebraic properties (see [Mon]).

For the sake of simplicity, let us describe our example only for the case of groups, even though it works for structures in general. A *representation* of a countable group $\mathcal{G} = (G; e, *)$ is an isomorphic copy of $\mathcal{G}$ with domain $\mathbb{N}$. That is, it is a group $\mathcal{A} = (\mathbb{N}; e_{\mathcal{A}}, *_{\mathcal{A}}) \cong \mathcal{G}$, where $e_{\mathcal{A}} \in \mathbb{N}$ and $*_{\mathcal{A}} : \mathbb{N}^2 \to \mathbb{N}$. The point of making the domain of the group $\mathbb{N}$ is that we can now use tools from computability theory. Such a representation $\mathcal{A}$ is said to be *computable* if the group operation $*_{\mathcal{A}} : \mathbb{N}^2 \to \mathbb{N}$ is computable. A group $\mathcal{G}$ may have many representations, as there are many bijections between $\mathbb{N}$ and G, some of which might be computable and some not. There may also be many different computable representations with different computational properties. Structures whose computable representations all have the same computational properties are said to be computably categorial:

Definition 0.1. A group $\mathcal{G}$ is said to be *computably categorical* if any two computable representations of $\mathcal{G}$ are computably isomorphic. That is, if $\mathcal{A} = (\mathbb{N}; e_{\mathcal{A}}, *_{\mathcal{A}})$ and $\mathcal{B} = (\mathbb{N}; e_{\mathcal{B}}, *_{\mathcal{B}})$ are isomorphic to $\mathcal{G}$, and $*_{\mathcal{A}}$ and $*_{\mathcal{B}}$ are computable functions, then there is a computable bijection $p\colon \mathbb{N} \to \mathbb{N}$ such that $p(e_{\mathcal{A}}) = e_{\mathcal{B}}$ and $p(n *_{\mathcal{A}} m) = p(n) *_{\mathcal{B}} p(m)$.

This is the case, for instance, of $(\mathbb{Q}; 0, +)$, because given a representation $\mathcal{A} = (\mathbb{N}; 0_{\mathcal{A}}, \oplus_{\mathcal{A}})$ of $(\mathbb{Q}; 0, +)$, once we fix an element $1^{\mathcal{A}}$ that we are going to map to $1^{\mathbb{Q}}$, for any other $a \in \mathcal{A}$, all we need to do is search for $n, m \in \mathbb{N}$ such that

$$\text{either} \quad \underbrace{a \oplus_{A} a \oplus_{A} \cdots \oplus_{A} a}_{m} = \underbrace{1^A \oplus_{A} 1^A \oplus_{A} \cdots \oplus_{A} 1^A}_{n}$$
$$\text{or} \quad \underbrace{a \oplus_{A} a \oplus_{A} \cdots \oplus_{A} a}_{m} \oplus_{A} \underbrace{1^A \oplus_{A} \cdots \oplus_{A} 1^A}_{n} = 0_{A},$$

and then map a to n/m or $-n/m$.

This is not the case for $(\mathbb{Q}^{+}; 1, \times)$: On the standard representation of $(\mathbb{Q}^{+}; 1, \times)$, the *power relation* $\{(a, b) \in {\mathbb{Q}^{+}}^{2} : (\exists n \in \mathbb{N}) a^n = b\}$ can be easily computed using prime decompositions of the numerators and denominators. However, in an arbitrary representation, deciding if b is a power of a requires going through all powers of a and checking if any of them equals b. One can build a computable representation $\mathcal{A} = (\mathbb{N}; e_{A}, *_{A})$ of $(\mathbb{Q}^{+}; 1, \times)$ where the *power relation*

$$\{(a, b) \in \mathbb{N}^2 : (\exists n \in \mathbb{N}) \underbrace{a *_{A} a *_{A} \cdots *_{A} a}_{n} = b\}$$

is not computable. In other words, the fact that we can compute the group operation doesn't mean we can compute the power relation. Since the power relation is computable in the standard representation of $(\mathbb{Q}^{+}; 1, \times)$, these two representations cannot be computably isomorphic.

Studying the difference between different representations of the same structures and the properties that are independent of representations is central to computable structure theory, and that's why the notion of computable categoricity becomes so important. Unfortunately, as proved by Downey, Kach, Lempp, Lewis, Montalbán and Turetsky [DKL+15], there is no structural way to characterize the groups that are computably categorical. This is unsatisfying. The picture gets brighter, though, if we relativize. Given $f \in \mathbb{N}^{\mathbb{N}}$, we say that a group is *f-computably categorical* if any two f-computable representations are f-computably isomorphic. Now, a given group $\mathcal{G}$ might be f-computably categorical for some f's and not for others. From Martin's theorem above, we get that a group $\mathcal{G}$ is either computably categorical relative to Martin's-almost-all f's, or computably categorical relative to Martin's-almost-no f's. The meaning of "Martin's-almost-all" comes, of course, from Martin's measure. The interesting point here is that we can structurally characterize the groups which are computably categorical relative to Martin's-almost-all f's. We include the theorem just to give the reader a taste of the type of structural conditions we deal with:

Definition 0.2. A group $\mathcal{G}$ is $\exists$-*atomic* over a tuple $\bar{a} \in G^{<\mathbb{N}}$ if, for every tuple $\bar{p} \in G^{<\mathbb{N}}$, there exist words $v_1, ..., v_k, w_1, ..., w_\ell$ on the letters $\bar{a}, \bar{x}, \bar{y}$ and their inverses that determine the automorphism orbit of $\bar{p}$ in the sense that: a tuple $\bar{q}$ is automorphic to $\bar{p}$ if and only if $\bar{q}$ satisfies $v_i[\bar{x} \mapsto \bar{q}, \bar{y} \mapsto \bar{b}] = e$ and $w_j[\bar{x} \mapsto \bar{q}, \bar{y} \mapsto \bar{b}] \neq e$ for all $i \leq k$ and $j \leq \ell$ and some $\bar{b} \in G^{<\mathbb{N}}$.

Theorem. [Mon16] *A group $\mathcal{G}$ is computably categorical relative to Martin's-almost-all f's if and only if it is $\exists$-atomic over some tuple.*

The use of Martin's measure is necessary here. There exist groups that are computable categorical and not $\exists$-atomic, and groups that are $\exists$-atomic but not computable categorical. Such groups need to be constructed in rather ad hoc ways, making sure certain objects aren't computable by diagonalizing against each computable function. However, if $\mathcal{G}$ is a naturally defined group, then any proof that shows that it is or isn't computable categorical must relativize. So it would be either computably categorical relative to all f or to no f. In that case, the usage of "almost all" in the theorem above is not really relevant, and we don't even need to relativize the notion of computable categoricity. Thus, for naturally occurring $\mathcal{G}$, $\mathcal{G}$ is computably categorical if and only if it is $\exists$-atomic over some tuple. Since we do not know how to formally speak about naturally occurring groups, the theorem above is the best we can do.

Martin's conjecture

Martin's conjecture is about $\equiv_T$-to-$\equiv_T$-invariant functions. Whenever we have a natural object defined in computability theory, we can relativize it. For example, if *FIN* is the set of programs which halt on a finite number of inputs, then FIN^f is the set of programs with oracle f that halt on a finite number of inputs. Whenever we define such an operator, we get a $\equiv_T$-to-$\equiv_T$-invariant function. That is, if $f \equiv_T g$, then $\mathit{FIN}^f \equiv_T \mathit{FIN}^g$.[†] Thus, one way of analyzing the naturally occurring objects is to look at the relativizable ones by studying $\equiv_T$-to-$\equiv_T$-invariant functions. If we have two such functions, $\mathcal{F}$ and $\mathcal{G}$, then the set of X such that $\mathcal{F}(X) \equiv_T \mathcal{G}(X)$ is either large or small. We say that $\mathcal{F}$ is *equivalent to $\mathcal{G}$ on a cone*, and write $\mathcal{F} \equiv_T^\triangledown \mathcal{G}$, if $\mathcal{F}(X) \equiv_T \mathcal{G}(X)$ for Martin's-almost-all X. Again, for naturally defined $\mathcal{F}$ and $\mathcal{G}$, this is either true for all X or for no X, so it is not necessary to consider the cones. But to prove general statements, we need to compare on cones, as otherwise one can easily build strange functions which satisfy $\mathcal{F}(X) \leq_T \mathcal{G}(X)$ for some X's and not others.

We say that a $\equiv_T$-to-$\equiv_T$-invariant function is *constant on a cone* if, for all g on some cone, $\mathcal{F}(g)$ has the same Turing degree.

Martin's Conjecture. *Every Borel $\equiv_T$-to-$\equiv_T$-invariant function is either constant on a cone or $\equiv_T$-equivalent to an iterate (maybe transfinite) of the Turing jump on a cone.*

[†]One can generalize this example and consider any class $\mathcal{C} \subseteq \mathbb{N}^{\subseteq\mathbb{N}}$ of partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$ and let $\mathcal{C}^f$ be the set of programs that, when run with oracle f, produce a partial function in $\mathcal{C}$. The operator $f \mapsto \mathcal{C}^f$ is then uniformly $\equiv_T$-to-$\equiv_T$-invariant.

It implies, in particular, that the Borel $\equiv_T$-to-$\equiv_T$-invariant functions are linearly ordered when compared on a cone.[‡]

The original conjecture is not only for Borel functions, but for all functions, although working not over ZFC, but over over a system denoted by ZF+AD+DC which doesn't include the full axiom of choice, where AD denotes the axiom of determinacy of infinite games and DC a weaker version of the axiom of choice called dependent choice. In that case, we don't get that all functions are iterates of the Turing jump, but that they are well-ordered under comparability on a cone, and that the successor operator is the jump. If the reader prefers to assume the axiom of choice, but is willing to accept some generally accepted large-cardinal hypothesis, the statement would be about functions in $L[\mathbb{R}]$ which are constructible without the use of the axiom of choice. In any case, this conjecture is about much more than Borel functions.

The conjecture is still open. An important version of it was proved by Ted Slaman and John Steel in [Ste82, SS88], a version that could already be used to claim that the only naturally occurring Turing degrees are, in a sense, the iterates of the jump. What Slaman and Steel proved is that the conjecture holds for *uniformly* $\equiv_T$-to-$\equiv_T$-invariant functions. These are functions for which, if we know which Turing programs make f and g equivalent, we can figure out which programs make $\mathcal{F}(f)$ and $\mathcal{F}(g)$ equivalent. The whole argument we made about how natural objects induce $\equiv_T$-to-$\equiv_T$-invariant functions still holds for uniformly $\equiv_T$-to-$\equiv_T$-invariant functions. Thus, Slaman and Steel's result already gets us what we wanted.

Recent connections with the Wadge Hierarchy

Let us finish this article by mentioning some recent work related to Martin's conjecture and Martin's measure.

Takayuki Kihara and the author have recently discovered a finer version of the uniform Martin's conjecture [KM]. They looked at the relation of *many-one reducibility*, which is a refinement of Turing reducibility that has also been studied extensively over the last seventy years: It reduces a decision problem into another by directly converting instances of the former to instances of the latter. Formally, a set $A \subseteq \mathbb{N}$ is *m-reducible* to a set $B \subset \mathbb{N}$, written $A \leq_m B$, if there is a computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that $n \in A \iff f(n) \in B$ for all $n \in \mathbb{N}$. The structure of the m-degrees is as chaotic as that of the Turing degrees. Kihara and Montalbán proved that the uniformly $\equiv_T$-to-$\equiv_m$-invariant functions are almost well-ordered under comparability on-a-cone, and form a much finer hierarchy than the $\equiv_T$-to-$\equiv_T$-invariant functions used in Martin's conjecture. By the same argument made throughout this paper, our result gives a full description of the natural many-one degrees. Another interesting outcome of our result is that

[‡]Recall that a linear ordering is a partial ordering where every two elements are comparable.

it provides a concrete link between Martin's conjecture and the Wadge hierarchy, giving us a better understanding of where Martin's conjecture comes from.

The Wadge reducibility compares the complexities of sets of sequences in $\mathbb{N}^{\mathbb{N}}$ instead of sets of numbers, and, in contrast to Turing and many-one reducibilities, has the property that all degrees are natural, and hence there is no distinction between the general behavior and that of naturally occurring objects. It is defined as follows: Given two sets of sequences, $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}}$, we say that $\mathcal{A}$ is *Wadge reducible* to $\mathcal{B}$, and write $\mathcal{A} \leq_w \mathcal{B}$, if there is a continuous function $F\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $h \in \mathcal{A} \iff F(h) \in \mathcal{B}$ for all $h \in \mathbb{N}^{\mathbb{N}}$.[§] Wadge [Wad83] showed that the Borel subsets of $\mathbb{N}^{\mathbb{N}}$ are linearly ordered by $\leq_w$ modulo taking complements: More precisely, he showed that for any two Borel sets $\mathcal{A}$ and $\mathcal{B}$, either $\mathcal{A} \leq_w \mathcal{B}$ or $\mathcal{B} \leq_w \bar{\mathcal{A}}$, where $\bar{\mathcal{A}}$ is the complement of $\mathcal{A}$. Martin then showed that the Borel Wadge degrees are well-founded (i.e., have no infinite descending chains) and hence well-ordered if one ignores complements. Thus there is no chaotic behavior on the Borel Wadge degrees.[¶]

The beauty of the Wadge hierarchy was present to Martin when he postulated his conjecture, though there was no clear connection between the two. That connection is now made explicit by the Kihara-Montalbán result, which proves that there is an order-preserving one-to-one correspondence between the uniformly $\equiv_T$-to-$\equiv_m$-invariant functions and the Wadge degrees.

The simplest non-trivial uniformly $\equiv_T$-to-$\equiv_m$-invariant function is the Turing jump operator that maps a sequence $f \in \mathbb{N}^{\mathbb{N}}$ to its jump f'. Under the correspondence in the Kihara-Montalbán result, this operator corresponds to the Wadge degree of the non-closed open sets.[‖] There are many more uniformly $\equiv_T$-to-$\equiv_m$-invariant functions that are Turing equivalent to the Turing jump but not many-one equivalent to it — there are uncountably many, actually, ordered as the ordinal ω_1, the first uncountable ordinal, if we ignore complements. The connection between the original Martin's conjecture and the many-one version is given by the Turing jump: If $F\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is uniformly $\equiv_T$-to-$\equiv_T$-invariant, then its composition with the Turing jump, $X \mapsto F(X)'$, is uniformly $\equiv_T$-to-$\equiv_m$-invariant. Hence the hierarchy of uniformly $\equiv_T$-to-$\equiv_T$-invariant functions can be studied by analyzing the uniformly $\equiv_T$-to-$\equiv_m$-invariant functions that are of the form $X \mapsto F(X)'$.

[§] Recall that the topology we use on $\mathbb{N}^{\mathbb{N}}$ is the product topology of the discrete topology on $\mathbb{N}$, getting a space homeomorphic to that of the irrational real numbers.

[¶] As before, we state these results in terms of Borel sets because that is how much we can prove in ZFC, but they are not really about Borel sets. All of this holds for all constructible sets in $L(\mathbb{R})$ if one assumes large-cardinal hypothesis, and for all sets if one assumes the axiom of determinacy (AD) and forgets about the axiom of choice.

[‖] All the non-closed open sets are Wadge equivalent to each other.

Other recent work

The remaining step to prove the full Martin's conjecture is called Steel's conjecture. It states that all Borel $\equiv_T$-to-$\equiv_T$-invariant functions are Turing equivalent on-a-cone to a uniformly $\equiv_T$-to-$\equiv_T$-invariant function. Recent work by Andrew Marks shows how this statement is completely orthogonal to the uniform Martin's conjecture, that it is a whole world on its own, and that it is related to other subjects like Borel combinatorics and ergodic theory. Adam Day and Andrew Marks have given a way to understand the step from the uniform to the non-uniform case of Martin's conjecture as a co-cycle superrigidity result: There is a way to associate to every Turing-invariant function a co-cycle of the free group on countably many generators so that Martin's conjecture is true if and only if these co-cycles are superrigid on a cone. In particular, they showed that Steel's conjecture would follow from the following statement: Every Borel co-cycle of the shift action of F_2, the free group on two generators, on the free part of 2^{F_2} is a conjugate of a homomorphism of F_2 on a set of Martin's measure 1. Here a co-cycle is a function $\alpha\colon F_2 \times 2^{F_2} \to F_2$ such that $\alpha(hg, X) = \alpha(h, gX)\alpha(g, X)$, and a conjugate of a homomorphism is a co-cycle of the form $\alpha(g, X) = b(gX)h(g)b^{-1}(X)$, where $b\colon 2^{F_2} \to F_2$ is the conjugating function and $h\colon F_2 \to F_2$ is a homomorphism.

The extent to which Martin's measure can be used to study the properties of natural objects in computability other than just Turing or many-one degrees is starting to be appreciated by recent work by the author and Matthew Harrison-Trainor, among others [Mon10, Mon12, Mon13, Mon15, HT, CHT]. Most of this new work is in computable structure theory, in the vein of the section on computable categoricity above, where by analyzing computational properties of structures by their Martin's-almost-everywhere behavior, one can obtain structural results that capture the behavior of naturally occurring structures.

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