PAPER

Asymptotic symmetries and charges at spatial infinity in general relativity

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Asymptotic symmetries and charges at spatial infinity in general relativity

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Abstract

We analyze the asymptotic symmetries and their associated charges at spatial infinity in four-dimensional asymptotically-flat spacetimes. We use the covariant formalism of Ashtekar and Hansen where the asymptotic fields and symmetries live on the three-manifold of spatial directions at spatial infinity, represented by a timelike unit-hyperboloid (or de Sitter space). Using the covariant phase space formalism, we derive formulae for the charges corresponding to asymptotic supertranslations and Lorentz symmetries at spatial infinity. With the motivation of, eventually, proving that these charges match with those defined on null infinity—as has been conjectured by Strominger—we do not impose any restrictions on the choice of conformal factor in contrast to previous work on this problem. Since we work with a general conformal factor we expect that our charge expressions will be more suitable to prove the matching of the Lorentz charges at spatial infinity to those defined on null infinity, as has been recently shown for the supertranslation charges.

Keywords: general relativity, asymptotic symmetries, spatial infinity

1. Introduction

In general relativity, the asymptotic symmetries of asymptotically-flat spacetimes at both past and future null infinity are elements of the infinite-dimensional Bondi–Metzner–Sachs (BMS) group [1, 2] (see [3, 4] for recent reviews). It has been conjectured by Strominger [5] that the (*a priori* independent) BMS groups at past and future null infinity are related via an antipodal reflection near spatial infinity. This matching relation gives a *global* 'diagonal' asymptotic symmetry group for general relativity. If similar matching conditions

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relate the gravitational fields, then there exist infinitely many conservation laws in classical gravitational scattering between the incoming fluxes associated with the BMS group at past null infinity and the outgoing fluxes of the corresponding (antipodally identified) BMS group at future null infinity. These conservation laws are also related to soft graviton theorems [6–11], gravitational memory effects [6, 7, 12–16] and the black hole information paradox [17–19] (see [20] for a detailed review of recent developments and a complete list of references).

Such matching conditions on the asymptotic symmetries and fields have been shown in Maxwell theory on a background Minkowski spacetime [21] and in general asymptotically-flat spacetimes [22]. In the gravitational case, the matching of the supertranslation symmetries and supermomentum charges has also be proven for linearized perturbations on a Minkowski background [23] and in general asymptotically-flat spacetimes [24]. For the translation symmetries these reduce to the much older result of [25] which shows that the Bondi four-momentum on future and past null infinity matches the four-momentum at spatial infinity.

The main technique used in [21–24] to prove these matching conditions is to 'interpolate' between the symmetries and charges at past and future null infinities using the field equations and the asymptotic symmetries and charges defined near spatial infinity. In a background Minkowski spacetime this analysis can be done using asymptotic Bondi–Sachs coordinates near each null infinity and asymptotic Beig–Schmidt coordinates near spatial infinity. Using the explicit transformations between these coordinate systems the matching conditions can be shown to hold for Maxwell fields and linearized gravity on Minkowski spacetime [21, 23]. But in general asymptotically-flat spacetimes the transformations between the asymptotic coordinates is not known explicitly. In this case the covariant formulation of asymptotic-flatness given by Ashtekar and Hansen [26], which treats both null and spatial infinities in a unified spacetime-covariant manner, has proven fruitful to analyze the matching of the symmetries and charges [22, 24].

However, for the charges associated with the Lorentz symmetries such matching conditions between past and future null infinity have not yet been proven, except for the case of stationary spacetimes [27]. With an eye toward establishing these conjectured matching conditions for Lorentz symmetries and charges we revisit the formulation of the asymptotic symmetries and charges at spatial infinity.

The asymptotic behavior at spatial infinity can be studied using many different (but related) formalisms. Since our primary motivation is to, ultimately, make contact with null infinity it will be more useful to use a spacetime covariant formalism without using any $(3 + 1)$ decomposition of the spacetime by spacelike hypersurfaces [28–30]. Such a four-dimensional formulation of asymptotic-flatness at spatial infinity can be be given using suitable asymptotic coordinates as formulated by Beig and Schmidt [31]. The asymptotic symmetries and charges using the asymptotic expansion of the metric in these coordinates have been worked out in detail [31–33]. But as mentioned above, the relation between the Beig–Schmidt coordinates and the coordinates adapted to null infinity (like the Bondi–Sachs coordinates) is not known in general spacetimes. Thus, we will use the coordinate independent formalism of Ashtekar and Hansen [26, 34] (definition 2.1) to investigate the symmetries and their associated charges at spatial infinity[4].

The asymptotic behavior of the gravitational field for any asymptotically-flat spacetime is most conveniently described in a conformally-related unphysical spacetime, the Penrose

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[4] The relation between the Ashtekar–Hansen formalism and the Beig–Schmidt coordinates is summarized in appendix A.

conformal-completion. In the unphysical spacetime, null infinities $\mathscr{I}^{\pm}$ are smooth null boundaries while spatial infinity is a boundary point i^0 which is the vertex of 'the light cone at infinity' formed by $\mathscr{I}^{\pm}$. For Minkowski spacetime the unphysical spacetime is smooth (in fact, analytic) at i^0. However, in more general spacetimes, the unphysical metric is not even once-differentiable at spatial infinity unless the ADM mass of the spacetime vanishes [26], and the unphysical spacetime manifold *does not* have a smooth differential structure at i^0. Thus, in the Ashtekar–Hansen formalism, instead of working directly at the point i^0 where sufficiently smooth structure is unavailable, one works on a 'blowup'—the space of spatial directions at i^0—given by a timelike-unit-hyperboloid $\mathscr{H}$ in the tangent space at i^0. Suitably conformally rescaled fields, whose limits to i^0 depend on the direction of approach, induce *smooth* fields on $\mathscr{H}$ and we can study these smooth limiting fields using standard differential calculus on $\mathscr{H}$. For instance, in Maxwell theory the rescaled field tensor ΩF_{ab} and in general relativity the rescaled (unphysical) Weyl tensor $\Omega^{1/2}C_{abcd}$ (where Ω is the conformal factor used in the Penrose conformal completion) admit regular direction-dependent limits to i^0, and these fields induce smooth tensor fields on $\mathscr{H}$. Similarly, the Maxwell gauge transformations and vector fields in the physical spacetime (suitably rescaled) admit regular direction-dependent limits which generate the asymptotic symmetries at i^0 (see section 6).

The asymptotic symmetries in general relativity at spatial infinity have also been studied in detail in the Ashtekar–Hansen formalism [26, 34]. However in deriving the charges associated with these symmetries Ashtekar and Hansen reduced the asymptotic symmetry algebra from the infinite-dimensional 𝔰𝔭𝔦 algebra to the Poincaré algebra consisting only of translations and Lorentz transformations. This reduction was accomplished by demanding that the 'leading order' magnetic part of the Weyl tensor, given by a tensor $\boldsymbol{B}_{ab}$ on $\mathscr{H}$ (see equation (4.5)), vanish and additionally choosing the conformal factor near i^0 so that the tensor potential $\boldsymbol{K}_{ab}$ for $\boldsymbol{B}_{ab}$ also vanishes (see remark 6.3). This restriction was also imposed in [32, 35]. In the work of Compère and Dehouck in [33], the condition $\boldsymbol{B}_{ab} = 0$ was not imposed however, they also specialized to a conformal factor where the trace $\boldsymbol{h}^{ab}\boldsymbol{K}_{ab}$ (where $\boldsymbol{h}^{ab}$ denotes the inverse of the metric on $\mathscr{H}$) was set to vanish. As we will show below (see section 7.3) the charges of the Lorentz symmetries at spatial infinity are not conformally-invariant but shift by the charge of a supertranslation. This is entirely analogous to the supertranslation ambiguities in the Lorentz charges at null infinity. Thus, when matching the Lorentz charges at spatial infinity to those at past and future null infinity, one would need to perform this matching in the 'same' choice of conformal factor in all three regions. *A priori*, it is not clear what the special choices of conformal factor chosen in the above mentioned analyses imply at null infinity. Thus, we will not impose any such restrictions on the conformal factor and not impose any conditions on $\boldsymbol{K}_{ab}$ (apart from its equations of motion arising from the Einstein equation) in our analysis. As we will show, one peculiar consequence of keeping a completely unrestricted conformal factor will be that our charges will not be exactly conserved but will have a non-vanishing flux through regions of $\mathscr{H}$ (except for pure translations). Thus, these charges are not associated with the point i^0 at spatial infinity, but with cross-sections of the 'blowup' $\mathscr{H}$. This is not a serious drawback; as shown in [22, 24] for matching the symmetries and charges at null infinity, one only requires that the *total* flux of the charges through all of $\mathscr{H}$ vanish but there can be a non-vanishing flux through local regions of $\mathscr{H}$. Thus, our main goal in this work is to analyze the symmetries and charges in general relativity without imposing any restrictions on the choice of conformal factor near spatial infinity.

In our analysis of the asymptotic charges we will use the *covariant phase space* formalism described below. Since the relevant quantities in the covariant phase space are defined in

terms of the physical metric and their perturbations, we first analyze the conditions on the corresponding unphysical quantities so that they preserve the asymptotic-flatness conditions and the universal structure at i^0 (section 5). To derive the asymptotic symmetry algebra we then consider a physical metric perturbation $\pounds_\xi \hat{g}_{ab}$ generated by an infinitesimal diffeomorphism and demand that it preserve the asymptotic conditions in the unphysical spacetime in the limit to i^0. This will provide us with the following description of the asymptotic symmetries at i^0 (section 6). The asymptotic symmetry algebra $\mathfrak{spi}$ is parametrized by a pair (f, X^a) where f is any smooth function and X^a is a Killing field on $\mathscr{H}$. The function f parametrizes the supertranslations and X^a parametrize the Lorentz symmetries. The $\mathfrak{spi}$ algebra is then a semi-direct sum of the Lorentz algebra with the infinite-dimensional abelian subalgebra of supertranslations. Note that this is the same as the asymptotic symmetries derived in [26, 34]. The only difference in our analysis is that we obtain the symmetries by analyzing the conditions on diffeomorphisms in the physical spacetime instead of using the unphysical spacetime directly as in [26, 34].

To obtain the charges associated with these symmetries, the primary quantity of interest is the *symplectic current* derived from the Lagrangian of a theory (see, [36, 37] for details). The symplectic current $\omega(\hat{g}; \delta_1 \hat{g}, \delta_2 \hat{g})$, is a local and covariant three-form which is an antisymmetric bilinear in two metric perturbations, $\delta\hat{g}$ on the physical spacetime. It can be shown that when the second perturbation $\delta\hat{g}_{ab} = \pounds_\xi \hat{g}_{ab}$ is the perturbation corresponding to an infinitesimal diffeomorphism generated by a vector field ξ^a we have

$$\omega(\hat{g}; \delta\hat{g}, \pounds_\xi \hat{g}) = d[\delta Q_\xi - \xi \cdot \theta(\delta\hat{g})], \tag{1.1}$$

where we have assumed that $\hat{g}_{ab}$ satisfies the equations of motion and $\delta\hat{g}_{ab}$ satisfies the linearized equations of motion. The two-form Q_ξ is the *Noether charge* associated with the vector field ξ^a and the three-form $\theta(\delta\hat{g})$ is the *symplectic potential* [36, 37]. If we integrate equation (1.1) over a three-dimensional surface Σ with boundary $\partial\Sigma$ we get

$$\int_\Sigma \omega[\hat{g}; \delta\hat{g}, \pounds_\xi \hat{g}] = \int_{\partial\Sigma} \delta Q_\xi - \xi \cdot \theta(\delta\hat{g}). \tag{1.2}$$

To define the asymptotic charges at spatial infinity, we would like to evaluate equation (1.2) when the surface Σ extends to a suitably regular three-surface at i^0 in the unphysical spacetime. Given the low amount of differentiability at i^0 the appropriate condition is that Σ extends to a $C^{>1}$ surface at i^0. The limit of the boundary $\partial\Sigma$ to i^0 corresponds to a two-sphere cross-section S of the unit-hyperboloid $\mathscr{H}$ in the Ashtekar–Hansen formalism. Then, the limiting integral on the right-hand side of equation (1.2) (with the asymptotic conditions imposed on the metric perturbations as well as the symmetries) will define a perturbed charge on S associated with the asymptotic symmetry generated by ξ^a. However, even though the explicit expressions for the integrand on the right-hand side of equation (1.2) are well-known (see for instance [37]), computing this limiting integral is difficult. So we will use an alternative strategy described next.

We will show that with the appropriate asymptotic-flatness conditions at i^0, the symplectic current three-form $\omega \equiv \omega_{abc}$ is such that $\Omega^{3/2} \omega_{abc}$ has a direction-dependent limit to i^0. The pullback of this limit to $\mathscr{H}$, which we denote by $\underset{\leftarrow}{\omega}$, defines a symplectic current on $\mathscr{H}$. We show that when one of the perturbations in this symplectic current is generated by an asymptotic $\mathfrak{spi}$ symmetry (f, X^a), we have

$$\underset{\leftarrow}{\omega}(g; \delta g, \delta_{(f,X)} g) = -\varepsilon_3 D^a Q_a(g; \delta g, (f, X)), \tag{1.3}$$

where ε_3 and $\boldsymbol{D}$ are the volume element and covariant derivative on $\mathscr{H}$. The covector $\boldsymbol{Q}_a(g; \delta g, (\boldsymbol{f}, \boldsymbol{X}))$ is a local and covariant functional of the background fields corresponding to the asymptotic (unphysical) metric g_{ab}, and linear in the asymptotic (unphysical) metric perturbations δg_{ab} and the asymptotic symmetry parametrized by $(\boldsymbol{f}, X^a)$. Thus, we can write the symplectic current, with one perturbation generated by an asymptotic symmetry, as a total derivative on $\mathscr{H}$. Then, in analogy to equation (1.2), we define the perturbed charge on a cross-section S of $\mathscr{H}$ by the integral

$$\int_S \varepsilon_2 \boldsymbol{u}^a \boldsymbol{Q}_a (g; \delta g, (\boldsymbol{f}, \boldsymbol{X}), \tag{1.4}$$

where ε_2 is the area element and $\boldsymbol{u}^a$ is a unit-timelike normal to the cross-section S within $\mathscr{H}$. We then show that when the asymptotic symmetry is a supertranslation $\boldsymbol{f}$, the quantity $\boldsymbol{Q}_a(g; \delta g, \boldsymbol{f})$ is integrable, i.e, it can be written as the δ of some covector which is itself a local and covariant functional of the asymptotic fields and supertranslation symmetries. Then 'integrating' equation (1.4) in the space of asymptotic fields, we can define a charge associated with the supertranslations on any cross-section S of $\mathscr{H}$ (see section 7.1). When the asymptotic symmetry is a Lorentz symmetry parameterized by a Killing vector field X^a on $\mathscr{H}$, equation (1.4) *cannot* be written as the δ of some quantity (unless we restrict to the choice of conformal factor where $\boldsymbol{h}^{ab}\boldsymbol{K}_{ab} = 0$ as described above). In this case, we will adapt the prescription by Wald and Zoupas [37] to define an integrable charge for Lorentz symmetries (section 7.2). Then the change of these charges over a region $\Delta\mathscr{H}$ bounded by two cross-sections provides a flux formula for these charges. In general, these fluxes will be non-vanishing (except for translation symmetries) unless we again restrict to the conformal factor where $\boldsymbol{h}^{ab}\boldsymbol{K}_{ab} = 0$. However, as mentioned above, from the point of view of matching these charges to those on null infinity, the special conformal choices might not be convenient and it is not necessary to have exactly conserved charges on $\mathscr{H}$. Thus, we will not restrict the conformal factor in any way and work with charges which can have non-trivial fluxes through some region of $\mathscr{H}$.

The rest of this paper is organized as follows. In section 2 we recall the definition of asymptotic-flatness at spatial infinity in terms of an Ashtekar–Hansen structure. To illustrate our approach outlined above we first study the simpler case of Maxwell fields at spatial infinity, and derive the associated symmetries and charges in section 3. In section 4 we then consider the asymptotic gravitational fields and Einstein equations at spatial infinity. We also describe the universal structure, that is the structure that is common to all spacetimes which are asymptotically-flat at i^0, in section 4.1. In section 5 we analyze the conditions on metric perturbations which preserve asymptotic flatness and obtain the limiting form of the symplectic current of general relativity on the space of directions $\mathscr{H}$. In section 6, using the analysis of the preceding section, we derive the asymptotic symmetry algebra (the 𝔰𝔭𝔦 algebra) by considering infinitesimal metric perturbations generated by diffeomorphisms which preserve the asymptotic flatness conditions. In section 7 we derive the charges and fluxes corresponding to these 𝔰𝔭𝔦 symmetries. We end with a summary and describe possible future directions in section 8.

We collect some useful results and asides in the appendices. In appendix A we construct a useful coordinate system near i^0 using the asymptotic flatness conditions on the unphysical metric and relate it to the Beig–Schmidt coordinates in the physical spacetime. Appendix B collects useful results on the unit-hyperboloid $\mathscr{H}$ on Killing vector fields, symmetric tensor fields and a theorem by Wald showing that (with suitable conditions) closed differential forms are exact. Computations detailing the change in the Lorentz charge under conformal

transformations are presented in appendix C. In appendix D we show that our charges are unambiguously defined by the the symplectic current of vacuum general relativity. In appendix E we generalize the Lorentz charges derived in section 7.2 to include spacetimes where the 'leading order' magnetic part of the Weyl tensor $\boldsymbol{B}_{ab}$ is allowed to be non-vanishing.

We use an abstract index notation with indices $a, b, c, \ldots$ for tensor fields. Quantities defined on the physical spacetime will be denoted by a 'hat', while the ones on the conformally-completed unphysical spacetime are without the 'hat' e.g. $\hat{g}_{ab}$ is the physical metric while g_{ab} is the unphysical metric on the conformal-completion. We denote the spatial directions at i^0 by $\vec{\eta}$. Regular direction-dependent limits of tensor fields, which we will denote to be $C^{>-1}$, will be represented by a boldface symbol e.g. $\boldsymbol{C}_{abcd}(\vec{\eta})$ is the limit of the (rescaled) unphysical Weyl tensor along spatial directions at i^0. The rest of our conventions follow those of Wald [38].

2. Asymptotic-flatness at spatial infinity: Ashtekar–Hansen structure

We define spacetimes which are asymptotically-flat at null and spatial infinity using an Ashtekar–Hansen structure [26, 34]. We use the following the notation for causal structures from [39]: $J(i^0)$ is the causal future of a point i^0 in M, $\overline{J}(i^0)$ is its closure, $\dot{J}(i^0)$ is its boundary and $\mathscr{I} := \dot{J}(i^0) - i^0$. We also use the definition and notation for direction-dependent tensors from [40], see also appendix B of [24].

Definition 2.1 (Ashtekar–Hansen structure [34]). A *physical* spacetime $(\hat{M}, \hat{g}_{ab})$ has an *Ashtekar–Hansen structure* if there exists another *unphysical* spacetime (M, g_{ab}), such that

(a) M is C^∞ everywhere except at a point i^0 where it is $C^{>1}$,
(b) The metric g_{ab} is C^∞ on $M - i^0$, and C^0 at i^0 and $C^{>0}$ along spatial directions at i^0,
(c) There is an embedding of $\hat{M}$ into M such that $\overline{J}(i^0) = M - \hat{M}$,
(d) There exists a function Ω on M, which is C^∞ on $M - i^0$ and C^2 at i^0 so that $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$ and

    1. $\Omega = 0$ on $\dot{J}(i^0)$,
    2. $\nabla_a \Omega \neq 0$ on $\mathscr{I}$,
    3. At i^0, $\nabla_a \Omega = 0$, $\nabla_a \nabla_b \Omega = 2g_{ab}$.

(e) There exists a neighborhood N of $\dot{J}(i^0)$ such that (N, g_{ab}) is strongly causal and time orientable, and in $N \cap \hat{M}$ the physical metric $\hat{g}_{ab}$ satisfies the vacuum Einstein equation $\hat{R}_{ab} = 0$,
(f) The space of integral curves of $n^a = g^{ab} \nabla_b \Omega$ on $\dot{J}(i^0)$ is diffeomorphic to the space of null directions at i^0,
(g) The vector field $\varpi^{-1} n^a$ is complete on $\mathscr{I}$ for any smooth function ϖ on $M - i^0$ such that $\varpi > 0$ on $\hat{M} \cup \mathscr{I}$ and $\nabla_a(\varpi^4 n^a) = 0$ on $\mathscr{I}$.

The physical role of the conditions in definition 2.1 is to ensure that the point i^0 is spacelike related to all points in the physical spacetime $\hat{M}$, and represents *spatial infinity*, and that null infinity $\mathscr{I} := \dot{J}(i^0) - i^0$ has the usual structure. Note that the metric g_{ab} is only $C^{>0}$ at i^0 along spatial directions, that is, the metric is continuous but the metric connection is allowed to have limits which depend on the direction of approach to i^0. This low

differentiability structure is essential to allow spacetimes with non-vanishing ADM mass [26, 34]. In the following we will only consider the behavior of the spacetime approaching i^0 along spatial directions, and we will not need the conditions corresponding to null infinity.

For spacetimes satisfying definition 2.1 we have the following limiting structures at i^0 when approached along spatial directions.

Along spatial directions $\eta_a := \nabla_a \Omega^{1/2}$ is $C^{>-1}$ at i^0 and

$$\boldsymbol{\eta}^a := \lim_{\to i^0} \nabla^a \Omega^{1/2}, \tag{2.1}$$

determines a $C^{>-1}$ spatial unit vector field at i^0 representing the spatial directions $\vec{\eta}$ at i^0. The space of directions $\vec{\eta}$ in Ti^0 is a unit-hyperboloid $\mathscr{H}$.

If $T^{a\cdots}{}_{b\cdots}$ is a $C^{>-1}$ tensor field at i^0 in spatial directions then, $\lim_{\to i^0} T^{a\cdots}{}_{b\cdots} = \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta})$ is a smooth tensor field on $\mathscr{H}$. Further, the derivatives of $\boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta})$ to all orders with respect to the direction $\vec{\eta}$ satisfy[5]

$$\boldsymbol{\partial}_c \cdots \boldsymbol{\partial}_d \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta}) = \lim_{\to i^0} \Omega^{1/2} \nabla_c \cdots \Omega^{1/2} \nabla_d T^{a\cdots}{}_{b\cdots}, \tag{2.2}$$

where $\boldsymbol{\partial}_a$ is the derivative with respect to the directions $\vec{\eta}$ defined by

$$\boldsymbol{v}^c \boldsymbol{\partial}_c \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta}) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta} + \epsilon \vec{v}) - \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta}) \right] \quad \text{for all } \boldsymbol{v}^a \in T\mathscr{H},$$

$$\boldsymbol{\eta}^c \boldsymbol{\partial}_c \boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta}) := 0. \tag{2.3}$$

The metric $\boldsymbol{h}_{ab}$ induced on $\mathscr{H}$ by the universal metric $\boldsymbol{g}_{ab}$ at i^0, satisfies

$$\boldsymbol{h}_{ab} := \boldsymbol{g}_{ab} - \boldsymbol{\eta}_a \boldsymbol{\eta}_b = \boldsymbol{\partial}_a \boldsymbol{\eta}_b. \tag{2.4}$$

Further, if $\boldsymbol{T}^{a\cdots}{}_{b\cdots}(\vec{\eta})$ is orthogonal to $\boldsymbol{\eta}^a$ in all its indices then it defines a tensor field $\boldsymbol{T}^{a\cdots}{}_{b\cdots}$ intrinsic to $\mathscr{H}$. In this case, it follows from equation (2.4) and $\boldsymbol{\partial}_c \boldsymbol{g}_{ab} = 0$ (since $\boldsymbol{g}_{ab}$ is direction-independent at i^0) that projecting *all* the indices in equation (2.2) using $\boldsymbol{h}_{ab}$ defines a derivative operator $\boldsymbol{D}_a$ intrinsic to $\mathscr{H}$ which is also the covariant derivative operator associated with $\boldsymbol{h}_{ab}$. We also define

$$\varepsilon_{abc} := -\boldsymbol{\eta}^d \varepsilon_{dabc}, \quad \varepsilon_{ab} := \boldsymbol{u}^c \varepsilon_{cab}, \tag{2.5}$$

where ε_{abcd} is volume element at i^0 corresponding to the metric $\boldsymbol{g}_{ab}$, ε_{abc} is the induced volume element on $\mathscr{H}$, and ε_{ab} is the induced area element on some cross-section S of $\mathscr{H}$ with a future-pointing timelike normal $\boldsymbol{u}^a$ such that $\boldsymbol{h}_{ab} \boldsymbol{u}^a \boldsymbol{u}^b = -1$.

Remark 2.1 (conformal freedom). It follows from the conditions in definition 2.1 that the allowed conformal freedom $\Omega \mapsto \omega \Omega$ is such that $\omega > 0$ is smooth in $M - i^0$, is $C^{>0}$ at i^0 and $\omega|_{i^0} = 1$. From these conditions it follows that

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[5] The factors of $\Omega^{1/2}$ on the right-hand side of equation (2.2) convert between ∇_a and the derivatives with respect to the directions; see [34, 41].

$$\omega = 1 + \Omega^{1/2}\alpha, \tag{2.6}$$

where α is $C^{>-1}$ at i^0. Let $\boldsymbol{\alpha}(\vec{\eta}) := \lim_{\to i^0} \alpha$, then from equation (2.6) we also get

$$\lim_{\to i^0} \nabla_a \omega = \boldsymbol{\alpha}\boldsymbol{\eta}_a + \boldsymbol{D}_a\boldsymbol{\alpha}. \tag{2.7}$$

Note in particular, that the unphysical metric $\boldsymbol{g}_{ab}$ at i^0 is invariant under conformal transformations. While

$$\eta^a \mapsto \omega^{-2}[\omega^{1/2}\eta^a + \frac{1}{2}\omega^{-1/2}\Omega^{1/2}\nabla^a\omega] \implies \boldsymbol{\eta}^a \mapsto \boldsymbol{\eta}^a. \tag{2.8}$$

Thus, unit spatial directions $\vec{\eta}$, the space of directions $\mathscr{H}$, and the induced metric on it $\boldsymbol{h}_{ab}$ are also invariant.

3. Maxwell fields: symmetries and charges at i^0

To illustrate our general strategy, we first consider the simpler case of Maxwell fields on any fixed background spacetime satisfying definition 2.1.

In the physical spacetime $\hat{M}$, let $\hat{F}_{ab}$ be the Maxwell field tensor satisfying the Maxwell equations

$$\hat{g}^{ac}\hat{g}^{bd}\hat{\nabla}_b\hat{F}_{dc} = 0, \quad \hat{\nabla}_{[a}\hat{F}_{bc]} = 0. \tag{3.1}$$

In the unphysical spacetime M with $F_{ab} := \hat{F}_{ab}$ we have

$$\nabla_b F^{ba} = 0, \quad \nabla_{[a}F_{bc]} = 0. \tag{3.2}$$

The Maxwell tensor F_{ab} is smooth everywhere in the unphysical spacetime except at i^0. Analyzing the behavior of F_{ab} in the simple case of a static point charge in Minkowski spacetime, it can be seen that F_{ab} diverges in the limit to i^0, but ΩF_{ab} admits a direction-dependent limit[6]. Hence we assume as our asymptotic condition that

$$\lim_{\to i^0} \Omega F_{ab} = \boldsymbol{F}_{ab}(\vec{\eta}) \text{ is } C^{>-1}. \tag{3.3}$$

The direction-dependent limit of the Maxwell tensor, $\boldsymbol{F}_{ab}$, induces smooth tensor fields on $\mathscr{H}$. These are given by the 'electric' and 'magnetic' parts of the Maxwell tensor defined by

$$\boldsymbol{E}_a(\vec{\eta}) = \boldsymbol{F}_{ab}(\vec{\eta})\boldsymbol{\eta}^b, \quad \boldsymbol{B}_a(\vec{\eta}) = *\boldsymbol{F}_{ab}(\vec{\eta})\boldsymbol{\eta}^b. \tag{3.4}$$

where $*\boldsymbol{F}_{ab}(\vec{\eta}) := \frac{1}{2}\varepsilon_{ab}{}^{cd}\boldsymbol{F}_{cd}(\vec{\eta})$ is the Hodge dual with respect to the unphysical volume element ε_{abcd} at i^0. The electric and magnetic fields are orthogonal to $\boldsymbol{\eta}^a$ and thus induce intrinsic fields $\boldsymbol{E}_a$ and $\boldsymbol{B}_a$ on $\mathscr{H}$. Note that $\boldsymbol{F}_{ab}$ can be reconstructed from $\boldsymbol{E}_a$ and $\boldsymbol{B}_a$ using

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[6] Note that this diverging behavior of F_{ab} refers to the tensor in the unphysical spacetime with the chosen $C^{>1}$ differential structure at i^0. In an asymptotically Cartesian coordinate system of the physical spacetime, this behavior reproduces the standard $1/r^2$ falloff for F_{ab} and $\boldsymbol{F}_{ab}(\vec{\eta})$ is the 'leading order' piece at $O(1/r^2)$.

$$\boldsymbol{F}_{ab} = 2\boldsymbol{E}_{[a}\boldsymbol{\eta}_{b]} + \varepsilon_{abcd}\boldsymbol{\eta}^c\boldsymbol{B}^d. \tag{3.5}$$

The asymptotic Maxwell equations are obtained by multiplying equation (3.2) by $\Omega^{3/2}$ and taking the limit to i^0 in spatial directions (see [26] for details)

$$\boldsymbol{D}^a\boldsymbol{E}_a = 0, \quad \boldsymbol{D}_{[a}\boldsymbol{E}_{b]} = 0,$$
$$\boldsymbol{D}^a\boldsymbol{B}_a = 0, \quad \boldsymbol{D}_{[a}\boldsymbol{B}_{b]} = 0. \tag{3.6}$$

To use the symplectic formalism for Maxwell theory, we will need to introduce the vector potential as the basic dynamical field. Let $\hat{A}_a$ be a vector potential for $\hat{F}_{ab}$ so that $\hat{F}_{ab} = 2\hat{\nabla}_{[a}\hat{A}_{b]}$ in the physical spacetime. Then, $A_a := \hat{A}_a$ is a vector potential for F_{ab} in the unphysical spacetime. We further assume that the vector potential A_a for F_{ab} is chosen so that $\Omega^{1/2}A_a$ is $C^{>-1}$ at i^0. Then define the asymptotic potentials

$$\boldsymbol{V}(\vec{\eta}) := \boldsymbol{\eta}^a \lim_{\to i^0} \Omega^{1/2}A_a, \quad \boldsymbol{A}_a(\vec{\eta}) := \boldsymbol{h}_a{}^b \lim_{\to i^0} \Omega^{1/2}A_b. \tag{3.7}$$

Then the corresponding smooth fields $\boldsymbol{V}$ and $\boldsymbol{A}_a$ induced on $\mathscr{H}$ act as potentials for the electric and magnetic field through

$$\boldsymbol{E}_a = \boldsymbol{D}_a\boldsymbol{V}, \quad \boldsymbol{B}_a = \frac{1}{2}\varepsilon_a{}^{bc}\boldsymbol{D}_b\boldsymbol{A}_c. \tag{3.8}$$

Even though we do not need this form, for completeness, we note that the Maxwell equations on $\mathscr{H}$ (equation (3.6)) can be written in terms of the potentials $\boldsymbol{V}$ and $\boldsymbol{A}_a$ as

$$\boldsymbol{D}^2\boldsymbol{V} = 0, \quad \boldsymbol{D}^2\boldsymbol{A}_a = \boldsymbol{D}_a\boldsymbol{D}^b\boldsymbol{A}_b + 2\boldsymbol{A}_a. \tag{3.9}$$

Now consider a gauge transformation of the vector potential

$$A_a \mapsto A_a + \nabla_a\lambda, \tag{3.10}$$

where λ is $C^{>-1}$ at i^0. Then with $\boldsymbol{\lambda}(\vec{\eta}) := \lim_{\to i^0} \lambda$, the gauge transformations of the asymptotic potentials (equation (3.7)) on $\mathscr{H}$ is given by

$$\boldsymbol{V} \mapsto \boldsymbol{V}, \quad \boldsymbol{A}_a \mapsto \boldsymbol{A}_a + \boldsymbol{D}_a\boldsymbol{\lambda}. \tag{3.11}$$

Thus, the asymptotic symmetries of Maxwell fields at i^0 are given by the functions $\boldsymbol{\lambda}$ on $\mathscr{H}$.

Remark 3.1 (special choices of gauge). The gauge freedom in the Maxwell vector potential can be used to impose further restrictions on the potential $\boldsymbol{A}_a$ on $\mathscr{H}$. We illustrate the following two gauge conditions which will have analogues in the gravitational case (see remark 6.3).

(a) Consider the Lorenz gauge condition $\hat{g}^{ab}\hat{\nabla}_a\hat{A}_b = 0$ on the physical vector potential $\hat{A}_a$ in the physical spacetime as used in [21, 42]. Multiplying this condition by Ω^{-1} and taking the limit to i^0, using equation (3.7) we get the asymptotic gauge condition

$$\boldsymbol{D}^a\boldsymbol{A}_a = 2\boldsymbol{V}. \tag{3.12}$$

Alternatively, from equation (3.11) we see that

$$\boldsymbol{D}^a\boldsymbol{A}_a \mapsto \boldsymbol{D}^a\boldsymbol{A}_a + \boldsymbol{D}^2\boldsymbol{\lambda}. \tag{3.13}$$

By solving a linear hyperbolic equation for $\boldsymbol{\lambda}$ we can choose a new gauge in which

$$\boldsymbol{D}^a\boldsymbol{A}_a = 0. \tag{3.14}$$

Both these gauge conditions reduce the allowed asymptotic symmetries to

$$\boldsymbol{D}^2\boldsymbol{\lambda} = 0. \tag{3.15}$$

(b) If we impose the restriction $\boldsymbol{B}_a = 0$ then $\boldsymbol{D}_{[a}\boldsymbol{A}_{b]} = 0$ and thus there exists a function $\boldsymbol{A}$ so that $\boldsymbol{A}_a = \boldsymbol{D}_a\boldsymbol{A}$.[7] Then using the transformation equation (3.11) we can set $\boldsymbol{A}_a = 0$. The remaining asymptotic symmetries are just the Coulomb symmetries $\boldsymbol{\lambda} =$ constant. This is analogous to the condition used by Ashtekar and Hansen in the gravitational case to reduce the asymptotic symmetries to the Poincaré algebra [26].

In what follows we will not need to impose any gauge condition on the potential $\boldsymbol{A}_a$ and our analysis will be completely gauge invariant.

Remark 3.2 (Logarithmic gauge transformations). Note that above we only considered gauge transformations equation (3.10) where the gauge parameter λ was $C^{>-1}$ at i^0. However, there is an additional ambiguity in the choice of gauge given by the *logarithmic gauge transformations* of the form

$$A_a \mapsto A_a + \nabla_a(\ln \Omega^{1/2}\Lambda), \tag{3.16}$$

where Λ is $C^{>0}$ at i^0. Under this gauge transformation $\Omega^{1/2}A_a$ is still $C^{>-1}$ at i^0, and from equation (3.7) we have the transformations

$$\boldsymbol{V} \mapsto \boldsymbol{V} + \boldsymbol{\Lambda}, \quad \boldsymbol{A}_a \mapsto \boldsymbol{A}_a, \tag{3.17}$$

where $\boldsymbol{\Lambda} := \lim_{\to i^0} \Lambda$ which is direction-independent at i^0 and induces a constant function on $\mathscr{H}$. From equation (3.8) we see that the fields $\boldsymbol{E}_a$ and $\boldsymbol{B}_a$ are invariant under this transformation. Since our charges and fluxes, derived below, will be expressed in terms of $\boldsymbol{E}_a$ we will not need to fix this logarithmic gauge ambiguity in the potentials for electromagnetism. However, there is an analogous logarithmic translation ambiguity in the gravitational case which we will need to fix (see remark 4.2). Thus we now illustrate how this logarithmic gauge ambiguity can be fixed even in electromagnetism.

Since the metric $\boldsymbol{g}_{ab}$ in the tangent space Ti^0 is universal and isometric to the Minkowski metric it is invariant under the reflection of the spatial directions $\vec{\eta} \mapsto -\vec{\eta}$. This gives rise to a reflection isometry of the metric $\boldsymbol{h}_{ab}$ on the space of directions $\mathscr{H}$. It was shown in [22] that the Maxwell fields on $\mathscr{H}$ which 'match' on to asymptotically-flat Maxwell fields on null infinity are the ones where the electric field $\boldsymbol{E}_a$ is reflection-odd i.e.

$$\boldsymbol{E}_a(\vec{\eta}) = -\boldsymbol{E}_a(-\vec{\eta}). \tag{3.18}$$

Further, since the logarithmic gauge parameter $\boldsymbol{\Lambda}$ is *direction-independent* we have that, $\boldsymbol{\Lambda}$ is reflection-even

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[7] This follows from the fact that every one-loop in $\mathscr{H}$ is contractible to a point and hence the first de Rahm cohomology group of $\mathscr{H}$ is trivial.

$$\mathbf{\Lambda}(\vec{\eta}) = \mathbf{\Lambda}(-\vec{\eta}). \tag{3.19}$$

Using a reflection-odd $\boldsymbol{E}_a$ in equation (3.8) we see that using a logarithmic gauge transformation we can demand that the potential $\boldsymbol{V}$ is also reflection-odd, so that

$$\boldsymbol{V}(\vec{\eta}) = -\boldsymbol{V}(-\vec{\eta}). \tag{3.20}$$

This fixes the logarithmic gauge ambiguity in the potentials.

Let us now analyze the charges and fluxes for this theory. To do this, we start by studying the symplectic current. In vacuum electromagnetism, this is given by:

$$\omega_{abc}(\delta_1 A, \delta_2 A) = \hat{\varepsilon}_{abcd} \left(\delta_1 \hat{F}^{de} \delta_2 \hat{A}_e - \delta_2 \hat{F}^{de} \delta_1 \hat{A}_e \right), \tag{3.21}$$

where the indices on $\delta \hat{F}_{ab}$ have been raised with the physical metric $\hat{g}^{ab}$. In terms of quantities in the unphysical spacetime we have

$$\omega_{abc}(\delta_1 A, \delta_2 A) = \varepsilon_{abcd} \left(\delta_1 F^{de} \delta_2 A_e - \delta_2 F^{de} \delta_1 A_e \right), \tag{3.22}$$

where we have used $\hat{\varepsilon}_{abcd} = \Omega^{-4} \varepsilon_{abcd}$, and $\hat{g}^{ab} = \Omega^2 g^{ab}$.

To obtain the limit to i^0 we rewrite this in terms of direction-dependent quantities from equations (3.3) and (3.7). We see that $\Omega^{3/2} \omega_{abc}$ is $C^{>-1}$ at i^0. The pullback of this direction-dependent limit to $\mathscr{H}$ is then given by

$$\underset{\leftarrow}{\boldsymbol{\omega}}(\delta_1 A, \delta_2 A) = -\varepsilon_3 \left(\delta_1 \boldsymbol{E}^a \delta_2 \boldsymbol{A}_a - \delta_2 \boldsymbol{E}^a \delta_1 \boldsymbol{A}_a \right), \tag{3.23}$$

where $\varepsilon_3 = \varepsilon_{abc}$ is the volume element on $\mathscr{H}$.

We now take δ_2 to correspond to a gauge transformation as in equation (3.11) to get

$$\underset{\leftarrow}{\boldsymbol{\omega}}(\delta A, \delta_\lambda A) = -\varepsilon_3 \delta \boldsymbol{E}^a \boldsymbol{D}_a \boldsymbol{\lambda} = -\varepsilon_3 \boldsymbol{D}^a (\delta \boldsymbol{E}_a \boldsymbol{\lambda}). \tag{3.24}$$

where in the last step we have used the linearized Maxwell equation $\boldsymbol{D}_a \delta \boldsymbol{E}^a = 0$ (see equation (3.6)). That is, the symplectic current (with one of the perturbations being generated by a gauge transformation) can be written as a total derivative of $\delta \boldsymbol{E}_a \boldsymbol{\lambda}$. Thus we define the perturbed charge $\delta \mathcal{Q}[\boldsymbol{\lambda}; S]$ on a cross-section S of $\mathscr{H}$ by

$$\delta \mathcal{Q}[\boldsymbol{\lambda}; S] = \int_S \varepsilon_2 \boldsymbol{u}^a \delta \boldsymbol{E}_a \boldsymbol{\lambda}, \tag{3.25}$$

where $\varepsilon_2 \equiv \varepsilon_{ab}$ is the area element on S and $\boldsymbol{u}^a$ is the future-directed normal to it. Note that this expression is manifestly integrable and defines the unperturbed charge once we choose a reference solution on which $\mathcal{Q}[\boldsymbol{\lambda}; S] = 0$ for all $\boldsymbol{\lambda}$ and all S. For the reference solution we choose the trivial solution $F_{ab} = 0$ so that $\boldsymbol{E}_a = 0$. Then the unperturbed charge is given by

$$\mathcal{Q}[\boldsymbol{\lambda}; S] = \int_S \varepsilon_2 \boldsymbol{u}^a \boldsymbol{E}_a \boldsymbol{\lambda}, \tag{3.26}$$

Let $\Delta \mathscr{H}$ be any region of $\mathscr{H}$ bounded by the cross-sections S_2 and S_1 (with S_2 in the future of S_1), then the flux of the charge equation (3.26) through $\Delta \mathscr{H}$ is given by

$$\mathcal{F}[\boldsymbol{\lambda}; \Delta\mathcal{H}] = -\int_{\Delta\mathscr{H}} \varepsilon_3 \boldsymbol{E}_a \boldsymbol{D}^a \boldsymbol{\lambda}. \tag{3.27}$$

Note that the flux of the charge vanishes for $\boldsymbol{\lambda} = $ constant in which case equation (3.26) is the Coulomb charge. The charges associated with a general smooth $\boldsymbol{\lambda}$ are only associated with the blowup $\mathscr{H}$ and not to i^0 itself. These additional charges are nevertheless useful to relate the charges defined on past and future null infinity and derive the resulting conservation laws for their fluxes in a scattering process; see [22].

4. Gravitational fields and Einstein equations at i^0

Now we turn to a similar analysis of symmetries, charges and fluxes for general relativity. To set the stage in this section we analyze the consequences of Einstein equations and the universal structure common to all spacetimes satisfying definition 2.1.

Using the conformal transformation relating the unphysical Ricci tensor R_{ab} to the physical Ricci tensor $\hat{R}_{ab}$ (see appendix D of [38]), the vacuum Einstein equation $\hat{R}_{ab} = 0$ can be written as

$$S_{ab} = -2\Omega^{-1}\nabla_a\nabla_b\Omega + \Omega^{-2}\nabla^c\Omega\nabla_c\Omega g_{ab},$$
$$\Omega^{1/2}S_{ab} = -4\nabla_a\eta_b + 4\Omega^{-1/2}\left(g_{ab} - \frac{1}{\eta^2}\eta_a\eta_b\right)\eta_c\eta^c, \tag{4.1}$$

where, as before, $\eta_a = \nabla_a\Omega^{1/2}$, and S_{ab} is given by

$$S_{ab} := R_{ab} - \frac{1}{6}Rg_{ab}. \tag{4.2}$$

Further, the Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ on the unphysical Riemann tensor along with equation (4.1) gives the following equations for the unphysical Weyl tensor C_{abcd} (see [41] for details).

$$\nabla_{[e}(\Omega^{-1}C_{ab]cd}) = 0, \tag{4.3a}$$

$$\nabla^d C_{abcd} = -\nabla_{[a}S_{b]c}. \tag{4.3b}$$

Since the physical Ricci tensor $\hat{R}_{ab}$ vanishes, the gravitational field is completely described by the physical Weyl tensor $\hat{C}_{abcd}$. The unphysical Weyl tensor is then $C_{abcd} = \Omega^2\hat{C}_{abcd}$. Since the unphysical metric g_{ab} is $C^{>0}$ at i^0, $\Omega^{1/2}C_{abcd}$ is $C^{>-1}$ at i^0 [26], and let

$$\boldsymbol{C}_{abcd}(\vec{\eta}) := \lim_{\to i^0} \Omega^{1/2}C_{abcd}. \tag{4.4}$$

The *electric* and *magnetic* parts of $\boldsymbol{C}_{abcd}(\vec{\eta})$ are, respectively, defined by

$$\boldsymbol{E}_{ab}(\vec{\eta}) := \boldsymbol{C}_{acbd}(\vec{\eta})\boldsymbol{\eta}^c\boldsymbol{\eta}^d, \quad \boldsymbol{B}_{ab}(\vec{\eta}) := *\boldsymbol{C}_{abcd}(\vec{\eta})\boldsymbol{\eta}^c\boldsymbol{\eta}^d. \tag{4.5}$$

where $*\boldsymbol{C}_{abcd}(\vec{\eta}) := \frac{1}{2}\varepsilon_{ab}{}^{ef}\boldsymbol{C}_{efcd}(\vec{\eta})$. It follows from the symmetries of the Weyl tensor that both $\boldsymbol{E}_{ab}(\vec{\eta})$ and $\boldsymbol{B}_{ab}(\vec{\eta})$ are orthogonal to $\boldsymbol{\eta}^a$, symmetric and traceless with the respect to the metric

h_{ab} on $\mathscr{H}$, and thus define smooth tensor fields E_{ab} and B_{ab} on $\mathscr{H}$, respectively. The limiting Weyl tensor can be obtained from these fields using

$$C^{ab}{}_{cd}(\vec{\eta}) = 4\eta^{[a}\eta_{[c}E^{b]}{}_{d]} - 4h^{[a}{}_{[c}E^{b]}{}_{d]} + 2\varepsilon^{abe}\eta_{[c}B_{d]e} + 2\varepsilon_{cde}\eta^{[a}B^{b]e}. \tag{4.6}$$

Further, as shown in [26], multiplying equation (4.3a) by Ω and taking the limit to i^0 gives the equations of motion

$$D_{[a}E_{b]c} = 0, \quad D_{[a}B_{b]c} = 0. \tag{4.7}$$

These are the asymptotic Einstein equations at spatial infinity. Taking the trace over the indices a and c and using the fact that E_{ab} and B_{ab} are traceless, it also follows that

$$D^b E_{ab} = D^b B_{ab} = 0. \tag{4.8}$$

To apply the symplectic formalism to general relativity, we will need to consider metric perturbations instead of just perturbations of the Weyl tensor. As we will show below (equation (5.8)) suitably rescaled limits of the unphysical metric perturbations can be expressed in terms of perturbations of certain potentials for E_{ab} and B_{ab} provided by the tensor S_{ab} in equation (4.2). These potentials are obtained as follows: since g_{ab} is $C^{>0}$, $\Omega^{1/2}S_{ab}$ is $C^{>-1}$ and let $S_{ab}(\vec{\eta}) := \lim_{\to i^0} \Omega^{1/2}S_{ab}$. Define

$$E(\vec{\eta}) := S_{ab}(\vec{\eta})\eta^a\eta^b, \quad K_{ab}(\vec{\eta}) := h_a{}^c h_b{}^d S_{cd}(\vec{\eta}) - h_{ab}E(\vec{\eta}), \tag{4.9}$$

which induce the fields E and K_{ab} intrinsic to $\mathscr{H}$. Following [26], multiplying equation (4.3b) by Ω and taking the limit to i^0, along with equation (4.7) implies that

$$h_a{}^b\eta^c S_{bc}(\vec{\eta}) = D_a E, \tag{4.10}$$

and

$$E_{ab} = -\frac{1}{4}(D_a D_b E + h_{ab}E), \quad B_{ab} = -\frac{1}{4}\varepsilon_{cda}D^c K^d{}_b. \tag{4.11}$$

Thus, E is a scalar potential for E_{ab} while K_{ab} is a tensor potential for B_{ab}.[8]

The potentials E and K_{ab} are not free fields on $\mathscr{H}$. Suitably commuting the derivatives and using equation (B.1) one can verify that E_{ab} identically satisfies equation (4.7) when written in terms of the potential E while $h^{ab}E_{ab} = 0$ gives

$$D^2 E + 3E = 0. \tag{4.12}$$

On the other hand, since K_{ab} is symmetric the magnetic field B_{ab} in equation (4.11) is identically traceless. Since B_{ab} is symmetric and satisfies equation (4.7), we get that

$$\varepsilon_a{}^{bc}B_{bc} = 0 \implies D^b K_{ab} = D_a K, \tag{4.13a}$$

$$\varepsilon_a{}^{cd}D_c B_{db} = 0 \implies D^2 K_{ab} = D_a D_b K + 3K_{ab} - h_{ab}K, \tag{4.13b}$$

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[8] Since B_{ab} is curl-free (equation (4.7)), there also exists a scalar potential for B_{ab} (see lemma B.1). However this scalar potential cannot be obtained as the limit of some tensor field on spacetime.

where $K := h^{ab}K_{ab}$, and to get equation (4.13b) we have commuted derivatives using equation (B.1) and used equation (4.13a). Considering the potentials E and K_{ab} as the basic fields, the asymptotic Einstein equations are given by equations (4.12) and (4.13), while the Weyl tensors E_{ab} and B_{ab} are derived quantities through equation (4.11).

To define the charge for asymptotic Lorentz symmetries, e.g. angular momentum in section 7.2, we will need the 'subleading' part of the magnetic Weyl tensor. Following Ashtekar and Hansen [26], we will restrict to the class of spacetimes satisfying the additional condition $B_{ab} = 0$. We also require that the 'subleading' magnetic field defined by

$$\beta_{ab} := \lim_{\to i^0} *C_{abcd}\eta^c\eta^d, \tag{4.14}$$

exists as a $C^{>-1}$ tensor field at i^0. The condition $B_{ab} = 0$ is satisfied in any spacetime which is *either* stationary *or* axisymmetric [43]. In appendix E we show how one can define a 'subleading' magnetic Weyl tensor and the Lorentz charges even when $B_{ab} \neq 0$. Since those computations are more tedious we impose the above restriction in the main body of the paper.

The consequences of this restriction are as follows. Since $B_{ab} = 0$ from equation (4.11) the 'curl' of K_{ab} vanishes

$$D_{[a}K_{b]c} = 0. \tag{4.15}$$

It follows from lemma B.1 that there exists a scalar potential k such that

$$K_{ab} = D_aD_bk + h_{ab}k. \tag{4.16}$$

The scalar potential k is a free function on $\mathscr{H}$ since the equations of motion equation (4.13) are identically satisfied after using equation (4.16). Using the freedom in the conformal factor one can now set $K_{ab} = 0$ (see [26] and remark 6.3). Since, we do not wish to impose any restrictions on the conformal factor, we will *not* demand that K_{ab} vanishes.

Note that from equation (4.14) it follows that β_{ab} is symmetric, tangent to $\mathscr{H}$ and traceless. In the following we shall also need an equation of motion for β_{ab} which is obtained as follows: contract the indices e and d in equation (4.3a) and multiply by 3Ω to get

$$\nabla^d C_{abcd} = \Omega^{-1}C_{abcd}\nabla^d\Omega = 2\Omega^{-1/2}C_{abcd}\eta^d. \tag{4.17}$$

Using the Hodge dual of the above equation we obtain

$$\Omega^{1/2}\nabla^b(*C_{acbd}\eta^c\eta^d) = -2 * C_{acbd}\eta^b\eta^c\eta^d + 2\Omega^{1/2} * C_{acbd}\nabla^b(^c\eta^{d)}. \tag{4.18}$$

The first term on the right-hand side vanishes due to the symmetries of the Weyl tensor. In the second term on the right-hand side we substitute for the derivative of η_a using equation (4.1) to get

$$\Omega^{1/2}\nabla^b(*C_{acbd}\eta^c\eta^d) = -\frac{1}{4}(\Omega^{1/2} * C_{acbd})(\Omega^{1/2}S^{bc})\eta^d. \tag{4.19}$$

Taking the limit to i^0, writing the tensor S_{ab} in terms of the gravitational potentials through equations (4.9) and (4.10), and using $B_{ab} = 0$ along with equation (4.6), we get the equation of motion

$$D^b\beta_{ab} = \frac{1}{4}\varepsilon_{cda}E^c{}_bK^{bd}. \tag{4.20}$$

Remark 4.1 (conformal transformations of the asymptotic fields). Under changes of the conformal factor $\Omega \mapsto \omega\Omega$ we have

$$S_{ab} \mapsto S_{ab} - 2\omega^{-1}\nabla_a\nabla_b\omega + 4\omega^{-2}\nabla_a\omega\nabla_b\omega - \omega^{-2}g_{ab}\nabla^c\omega\nabla_c\omega,$$

$$C_{abcd} \mapsto \omega^2 C_{abcd}. \tag{4.21}$$

From the conditions in remark 2.1 it follows that $\boldsymbol{E}_{ab}$, $\boldsymbol{B}_{ab}$ and $\boldsymbol{E}$ are invariant while

$$\boldsymbol{K}_{ab} \mapsto \boldsymbol{K}_{ab} - 2(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{\alpha} + \boldsymbol{h}_{ab}\boldsymbol{\alpha}). \tag{4.22}$$

Further, when $\boldsymbol{B}_{ab} = 0$ we also have the transformation of the 'subleading' magnetic Weyl tensor $\boldsymbol{\beta}_{ab}$ given by

$$\boldsymbol{\beta}_{ab} \mapsto \boldsymbol{\beta}_{ab} - \varepsilon_{cd(a}\boldsymbol{E}^c{}_{b)}\boldsymbol{D}^d\boldsymbol{\alpha}. \tag{4.23}$$

4.1. The universal structure at i^0

In this section we summarize the *universal structure* at i^0, that is, the structure common to all spacetimes which are asymptotically-flat in the sense of definition 2.1 and thus is independent of the choice of the physical spacetime under consideration.

Consider any two unphysical spacetimes (M, g_{ab}, Ω) and (M', g'_{ab}, Ω') with their respective $C^{>1}$ differential structures at their spatial infinities corresponding to two different physical spacetimes. Using a C^1 diffeomorphism we can identify the points representing the spatial infinities and their tangent spaces without any loss of generality. Each of the metrics g_{ab} and g'_{ab} induces a metric in the tangent space Ti^0 which is isometric to the Minkowski metric. Thus, the metric $\boldsymbol{g}_{ab}$ at i^0 is also universal. This also implies that the spatial directions $\vec{\eta}$, the space of directions $\mathscr{H}$ and the induced metric $\boldsymbol{h}_{ab}$ are universal.

So far we have only used the C^1 differential structure. However since the differential structure at i^0 is slightly better, being $C^{>1}$, we can identify the spacetimes at the 'next order'. In [26] this structure was imposed by suitably identifying spacelike geodesics in the *physical* spacetimes. But as pointed out by [44] this identification cannot be performed except in very special cases. Below we argue that a similar identification of the spacetimes can be done using equivalence classes of $C^{>1}$ curves in the unphysical spacetimes. The proof is based on constructing a suitable $C^{>1}$ coordinate system at i^0 and is deferred to appendix A, we summarize the main construction below.

Consider the unphysical spacetime (M, g_{ab}, Ω), and a spacelike $C^{>1}$ curve Γ_v in M passing through i^0 with tangent v^a. Since the curve is $C^{>1}$ its tangent vector v^a is $C^{>0}$. Using the universal metric $\boldsymbol{g}_{ab}$ at i^0 we can then demand that v^a be unit-normalized at i^0 and thus along the curve Γ_v

$$\lim_{\to i^0} v^a = \boldsymbol{\eta}^a, \tag{4.24}$$

that is the curve Γ_v points in some spatial direction $\vec{\eta}$ at i^0. Further, since Γ_v is $C^{>1}$, $v^b\nabla_b v^a$ is a $C^{>-1}$ vector. Thus, define the *acceleration* of Γ_v at i^0 by the projection of this vector on to $\mathscr{H}$

$$\boldsymbol{A}^a[\Gamma_v] := \boldsymbol{h}^a{}_b \lim_{\to i^0} v^c\nabla_c v^b. \tag{4.25}$$

Now we define the curves Γ_v (with tangent v^a) and Γ_η (with tangent η^a) to be equivalent if their accelerations are equal at i^0. To see what this entails, note that since v^a is $C^{>0}$ and equals

$\boldsymbol{\eta}^a$ in the limit to i^0 we have that $v^a = \eta^a + \Omega^{1/2} w^a$ for some w^a which is $C^{>-1}$ at i^0. Then, from equation (4.25) we have

$$\boldsymbol{A}^a[\Gamma_v] = \boldsymbol{A}^a[\Gamma_\eta] \iff \boldsymbol{h}_{ab} \lim_{\to i^0} w^b = 0. \tag{4.26}$$

Thus, we have an equivalence class of curves through i^0 pointing in each direction $\vec{\eta}$ defined by[9]

$$\Gamma_v \sim \Gamma_\eta \iff \boldsymbol{h}_{ab} \lim_{\to i^0} \Omega^{-1/2}(v^b - \eta^b) = 0. \tag{4.27}$$

We will show in appendix A that using a $C^{>1}$ diffeomorphism one can identify these equivalence classes of curves between any any two spacetimes (M, g_{ab}, Ω) and (M', g'_{ab}, Ω'). Further, we show that the conformal factors Ω and Ω' can also be identified in a neighborhood of i^0.

Thus, the universal structure at i^0 consists of the point i^0, the tangent space Ti^0, the metric $\boldsymbol{g}_{ab}$ at i^0 and the equivalence classes of $C^{>1}$ curves given by equation (4.27). In addition, the conformal factor Ω can also be chosen to be universal.

Remark 4.2 (logarithmic translations). So far we have worked with a fixed $C^{>1}$ differential structure in the unphysical spacetime at i^0. But given a physical spacetime the unphysical spacetime is ambiguous up to a four-parameter family of *logarithmic translations* at i^0 which simultaneously change the $C^{>1}$ differential structure and the conformal factor at i^0; see [45] or remark B.1 of [24] for details. The logarithmic translations at i^0 are parameterized by a *direction-independent* vector $\boldsymbol{\Lambda}^a$ at i^0. Any such vector can be written as

$$\boldsymbol{\Lambda}^a = \boldsymbol{\Lambda}\eta^a + \boldsymbol{D}^a\boldsymbol{\Lambda}, \tag{4.28}$$

where $\boldsymbol{\Lambda}(\vec{\eta}) = \eta_a\boldsymbol{\Lambda}^a$ is a function on $\mathscr{H}$ satisfying

$$\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{\Lambda} + \boldsymbol{h}_{ab}\boldsymbol{\Lambda} = 0. \tag{4.29}$$

Under such logarithmic translations the potentials equation (4.9) transform as [45]

$$\boldsymbol{E} \mapsto \boldsymbol{E} + 4\boldsymbol{\Lambda}, \quad \boldsymbol{K}_{ab} \mapsto \boldsymbol{K}_{ab}, \tag{4.30}$$

while $\boldsymbol{E}_{ab}$ and $\boldsymbol{B}_{ab}$ are invariant. The presence of these logarithmic translations will lead to the following issue when we define the charges for supertranslations in section 7.1. For general supertranslations (which are not translations) our charges will depend on the potential $\boldsymbol{E}$ instead of just the electric field $\boldsymbol{E}_{ab}$. Thus, even if we take the physical spacetime to be the Minkowski spacetime our charges will not vanish due to the logarithmic translation ambiguity equation (4.30) in $\boldsymbol{E}$. Thus, now we will fix these logarithmic translations following the argument in [45].

Since the metric $\boldsymbol{g}_{ab}$ in the tangent space Ti^0 is universal and isometric to the Minkowski metric it is invariant under the reflection of the spatial directions $\vec{\eta} \mapsto -\vec{\eta}$. This gives rise to a reflection isometry of the metric $\boldsymbol{h}_{ab}$ on the space of directions $\mathscr{H}$. Now it was shown in [24] that the only spacetimes which are asymptotically-flat at spatial infinity and which 'match' on to asymptotically-flat spacetimes on null infinity are the ones where $\boldsymbol{E}_{ab}$ is reflection-even, i.e.

$$\boldsymbol{E}_{ab}(\vec{\eta}) = \boldsymbol{E}_{ab}(-\vec{\eta}). \tag{4.31}$$

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[9] These equivalence classes of curves form a principal bundle over $\mathscr{H}$, called *Spi* in [26].

Further, since $\mathbf{\Lambda} = \boldsymbol{\eta}_a \mathbf{\Lambda}^a$ for the *direction-independent* vector $\mathbf{\Lambda}^a$ we have that, $\mathbf{\Lambda}$ is reflection-odd

$$\mathbf{\Lambda}(\vec{\eta}) = -\mathbf{\Lambda}(-\vec{\eta}). \tag{4.32}$$

For a reflection-even $\boldsymbol{E}_{ab}$, from equations (4.11) and (4.29), it follows that using a logarithmic translation we can demand that the potential $\boldsymbol{E}$ is also reflection-even, so that

$$\boldsymbol{E}(\vec{\eta}) = \boldsymbol{E}(-\vec{\eta}). \tag{4.33}$$

Having fixed the logarithmic translations in this way, $\boldsymbol{E}_{ab} = 0$ then implies that $\boldsymbol{E} = 0$. In particular, for Minkowski spacetime we have

$$\boldsymbol{E} = 0, \quad \boldsymbol{B}_{ab} = 0, \quad \boldsymbol{\beta}_{ab} = 0 \quad (\text{on Minkowski spacetime}). \tag{4.34}$$

Note that when $\boldsymbol{E}_{ab} = 0$, $\boldsymbol{\beta}_{ab}$ is conformally-invariant (see equation (4.23)) and the conditions equation (4.34) do not depend on the conformal factor chosen for Minkowski spacetime. These conditions will ensure that our all our charges will vanish on Minkowski spacetime. Thus, from here on we will assume that the logarithmic translations have been fixed as above that is, we work the choice of $C^{>1}$ differential structure at i^0 where the parity condition equation (4.33) is satisfied.

5. Metric perturbations and symplectic current at i^0

Now consider a one-parameter family of asymptotically-flat physical metrics $\hat{g}_{ab}(\lambda)$ where $\hat{g}_{ab} = \hat{g}_{ab}(\lambda = 0)$ is some chosen background spacetime. Define the physical metric perturbation $\hat{\gamma}_{ab}$ around the background $\hat{g}_{ab}$ by

$$\hat{\gamma}_{ab} = \delta \hat{g}_{ab} := \frac{\mathrm{d}}{\mathrm{d}\lambda} \hat{g}_{ab}(\lambda)\bigg|_{\lambda=0}. \tag{5.1}$$

We will use 'δ' to denote perturbations of other quantities defined in a similar way.

As discussed above, the conformal factor Ω can be chosen universally, i.e., independently of the choice of the physical metric. The unphysical metric perturbation is

$$\delta g_{ab} = \gamma_{ab} = \Omega^2 \hat{\gamma}_{ab}, \tag{5.2}$$

and we also have

$$\delta \eta_a = \delta \nabla_a \Omega^{1/2} = 0, \quad \delta \eta^a = \delta(g^{ab}\eta_b) = -\gamma^{ab}\eta_b. \tag{5.3}$$

Now we investigate the conditions on the unphysical perturbation γ_{ab} which preserve asymptotic flatness and the universal structure at i^0 described in section 4.1. First recall that since the unphysical metric g_{ab} is $C^{>0}$ and universal at i^0, it follows that the unphysical metric perturbation γ_{ab} is $C^{>0}$ and $\gamma_{ab}|_{i^0} = 0$. Therefore

$$\boldsymbol{\gamma}_{ab}(\vec{\eta}) := \lim_{\to i^0} \Omega^{-1/2} \gamma_{ab} \text{ is } C^{>-1}, \tag{5.4}$$

With equations (5.3) and (5.4) we also see that $\delta \boldsymbol{\eta}^a = 0$. Thus, the metric perturbation also preserves the spatial directions $\vec{\eta}$ at i^0, the space of directions $\mathscr{H}$ and the metric $\boldsymbol{h}_{ab}$ on it.

Now consider the universal structure given by the equivalence classes of $C^{>1}$ curves through i^0 as described in section 4.1. Consider the equivalence class of a fixed curve Γ_v with tangent v^a. For this equivalence class to be preserved, the perturbation of equation (4.27) must vanish. Evaluating this condition using equations (5.3) and (5.4) we obtain the condition

$$\boldsymbol{h}_a{}^b\boldsymbol{\eta}^c\boldsymbol{\gamma}_{bc}(\vec{\eta}) = 0. \tag{5.5}$$

In summary, equations (5.4) and (5.5) are the asymptotic conditions on the unphysical metric perturbations which preserve the asymptotic flatness and the universal structure at i^0.

The metric perturbation $\boldsymbol{\gamma}_{ab}$ can be directly related to the perturbations of the gravitational potentials $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$ defined in equation (4.9). Perturbing equation (4.1) to evaluate $\Omega^{1/2}\delta S_{ab}$ and taking the limit to i^0 using equations (5.3) and (5.4) we get

$$\delta\boldsymbol{S}_{ab} = \lim_{\to i^0}\Omega^{1/2}\delta S_{ab} = 4\boldsymbol{\partial}_{(a}\boldsymbol{\gamma}_{b)c}\boldsymbol{\eta}^c + 4\boldsymbol{\eta}_{(a}\boldsymbol{\gamma}_{b)c}\boldsymbol{\eta}^c + 2\boldsymbol{\gamma}_{ab} - 4\boldsymbol{\gamma}_{cd}\boldsymbol{\eta}^c\boldsymbol{\eta}^d\boldsymbol{g}_{ab}. \tag{5.6}$$

Using the definition of the gravitational potentials equations (4.9) and (5.5) we obtain

$$\delta\boldsymbol{E} = 2\boldsymbol{\gamma}_{ab}\boldsymbol{\eta}^a\boldsymbol{\eta}^b, \tag{5.7a}$$

$$\delta\boldsymbol{K}_{ab} = -2\boldsymbol{h}_a{}^c\boldsymbol{h}_b{}^d\boldsymbol{\gamma}_{cd} - \boldsymbol{h}_{ab}\delta\boldsymbol{E}. \tag{5.7b}$$

Using equations (5.5) and (5.7) we can reconstruct the metric perturbation $\boldsymbol{\gamma}_{ab}(\vec{\eta})$ in terms of the perturbed gravitational potentials on $\mathscr{H}$ as

$$\boldsymbol{\gamma}_{ab}(\vec{\eta}) = \frac{1}{2}\left[\delta\boldsymbol{E}(\boldsymbol{\eta}_a\boldsymbol{\eta}_b - \boldsymbol{h}_{ab}) - \delta\boldsymbol{K}_{ab}\right]. \tag{5.8}$$

The linearized Einstein equations for $\boldsymbol{\gamma}_{ab}$ in the form equation (5.8) are then equivalent to the linearizations of equations (4.12) and (4.13).

Next we consider the behavior of the symplectic current of vacuum general relativity near i^0. The symplectic current is given by (see [37])

$$\omega_{abc} = -\frac{1}{16\pi}\hat{\varepsilon}_{abcd}\hat{w}^d \quad \text{with} \quad \hat{w}^a = \hat{P}^{abcdef}\hat{\gamma}_{2bc}\hat{\nabla}_d\hat{\gamma}_{1ef} - [1 \leftrightarrow 2], \tag{5.9}$$

where '$[1 \leftrightarrow 2]$' denotes the preceding expression with the 1 and 2, labeling the perturbations, interchanged and the tensor $\hat{P}^{abcdef}$ is given by

$$\hat{P}^{abcdef} = \hat{g}^{ae}\hat{g}^{fb}\hat{g}^{cd} - \frac{1}{2}\hat{g}^{ad}\hat{g}^{be}\hat{g}^{fc} - \frac{1}{2}\hat{g}^{ab}\hat{g}^{cd}\hat{g}^{ef} - \frac{1}{2}\hat{g}^{bc}\hat{g}^{ae}\hat{g}^{fd} + \frac{1}{2}\hat{g}^{bc}\hat{g}^{ad}\hat{g}^{ef}. \tag{5.10}$$

To analyze the behavior of the symplectic current in the limit to i^0 we first express it in terms of quantities in the unphysical spacetime using

$$\varepsilon_{abcd} = \Omega^4\hat{\varepsilon}_{abcd}, \quad P^{abcdef} = \Omega^{-6}\hat{P}^{abcdef}, \quad \gamma_{ab} = \Omega^2\hat{\gamma}_{ab}, \tag{5.11}$$

where P^{abcdef} is defined through the unphysical metric by the same expression as equation (5.10). Using these, and converting the physical derivative operator $\hat{\nabla}$ to the unphysical one ∇ as

$$\hat{\nabla}_d\hat{\gamma}_{1ef} = \nabla_d\hat{\gamma}_{1ef} + \Omega^{-1}[\hat{\nabla}_d\Omega\hat{\gamma}_{1ef} + \hat{\nabla}_e\Omega\hat{\gamma}_{1df} - g_{ed}\hat{\nabla}^a\Omega\hat{\gamma}_{1af} + (e \leftrightarrow f)], \tag{5.12}$$

we obtain

$$
\omega_{abc} = -\frac{1}{16\pi}\varepsilon_{abcd}w^d,
$$

$$
\text{with} \quad w^a = \Omega^{-2}P^{abcdef}\gamma_{2bc}\nabla_d\gamma_{1ef} + \Omega^{-3}\gamma_1^{ab}\nabla_b\Omega\gamma_{2c}{}^c - [1 \leftrightarrow 2].
$$

(5.13)

Converting to quantities which are direction-dependent at i^0 and using equation (5.4) we see that $\Omega^{3/2}\omega_{abc}$ is $C^{>-1}$. The pullback $\underset{\leftarrow}{\omega}$ to $\mathscr{H}$ of $\lim_{\to i^0}\Omega^{3/2}\omega_{abc}$ is given by

$$
\underset{\leftarrow}{\omega} = -\frac{1}{16\pi}\varepsilon_3\;\eta^a\left(2\eta^b\gamma_{2ab}\gamma_1 - \frac{1}{2}\gamma_{1ab}\partial^b\gamma_2 + \gamma_1^{bc}\partial_c\gamma_{2ab} - \frac{1}{2}\gamma_1\partial^b\gamma_{2ab}\right) - [1 \leftrightarrow 2].
$$

(5.14)

This expression can be considerably simplified by rewriting it in terms of the perturbed gravitational potentials δE and δK_{ab} using equation (5.8). An easy but long computation gives

$$
\underset{\leftarrow}{\omega} = \frac{1}{64\pi}\varepsilon_3(\delta_1 K \delta_2 E - \delta_2 K \delta_1 E),
$$

(5.15)

where, as before, $K := h^{ab}K_{ab}$.

6. Asymptotic symmetries at i^0: the 𝔰𝔭𝔦 algebra

In this section we analyze the asymptotic symmetries at i^0. We show that the diffeomorphisms of the physical spacetime which preserve the asymptotic flatness of the spacetime (defined by definition 2.1) generate an infinite-dimensional algebra 𝔰𝔭𝔦. This asymptotic symmetry algebra was obtained in [26, 34] by analyzing the infinitesimal diffeomorphisms which preserve the universal structure at i^0. Here we provide an alternative derivation by considering the physical perturbations generated by such infinitesimal diffeomorphisms and demanding that the corresponding unphysical perturbations satisfy the asymptotic conditions equations (5.4) and (5.5).

Consider an infinitesimal diffeomorphism generated by a vector field $\hat{\xi}^a$ in the physical spacetime, and let $\xi^a = \hat{\xi}^a$ be the corresponding vector field in the unphysical spacetime. For ξ^a to be a representative of an asymptotic symmetry at i^0 the infinitesimal diffeomorphism generated by ξ^a must preserve the universal structure at i^0. Firstly, the infinitesimal diffeomorphism must keep the the point i^0 fixed and preserve the $C^{>1}$ differential structure at i^0. Thus, ξ^a must be $C^{>0}$ at i^0 and $\xi^a|_{i^0} = 0$. This implies that $\Omega^{-1/2}\xi^a$ is $C^{>-1}$ at i^0 and let

$$
X^a(\vec{\eta}) := \lim_{\to i^0}\Omega^{-1/2}\xi^a.
$$

(6.1)

Now consider the physical metric perturbation $\hat{\gamma}_{ab}^{(\xi)} = \delta_\xi\hat{g}_{ab} := \pounds_\xi\hat{g}_{ab}$ corresponding to an infinitesimal diffeomorphism generated by ξ^a. The corresponding unphysical metric perturbation is given by

$$
\gamma_{ab}^{(\xi)} = \Omega^2\pounds_\xi\hat{g}_{ab} = \pounds_\xi g_{ab} - 4\Omega^{-1/2}\xi^c\eta_c g_{ab}.
$$

(6.2)

Since $\gamma_{ab}^{(\xi)}$ must satisfy the asymptotic conditions at i^0 in equations (5.4) and (5.5), we have that $\gamma_{ab}^{(\xi)}$ is $C^{>0}$ at i^0 and $\gamma_{ab}^{(\xi)}|_{i^0} = 0$. To see the implications of these conditions first evaluate the condition $\gamma_{ab}^{(\xi)}|_{i^0} = 0$ using equations (6.1) and (6.2) which gives

$$\eta_a X^a(\vec{\eta}) = 0, \quad D_{(a}X_{b)} = 0, \tag{6.3}$$

that is, the vector field X^a is tangent to $\mathscr{H}$ and is a Killing vector field on it. Thus, X^a is an element of the Lorentz algebra $\mathfrak{so}(1,3)$. Some useful properties of these Killing vectors and their relationship to infinitesimal Lorentz transformations in the tangent space Ti^0 are collected in appendix B.1.

Further, since both $\gamma_{ab}^{(\xi)}$ and $\pounds_\xi g_{ab}$ are $C^{>0}$ we must have that $\Omega^{-1/2}\xi^a\eta_a$ is also $C^{>0}$. Since $\Omega^{-1/2}\xi^a\eta_a|_{i^0} = 0$ (which follows from equations (6.1) and (6.3) we have that $\Omega^{-1}\xi^a\eta_a$ is $C^{>-1}$ at i^0 so define

$$f(\vec{\eta}) := \lim_{\to i^0} \Omega^{-1}\xi^a\eta_a. \tag{6.4}$$

The function f on $\mathscr{H}$ then parametrizes the *supertranslations*. The vector field generating a supertranslation can be obtained as follows. Consider ξ^a such that the corresponding X^a (equation (6.1)) vanishes and $\chi^a := \lim_{\to i^0} \Omega^{-1}\xi^a$ is $C^{>-1}$ so that $f = \chi^a\eta_a$. Now consider the metric perturbation equation (6.2) corresponding to such a vector field. From equation (5.5) we must have

$$h_a{}^b\eta^c\gamma_{bc}^{(\xi)} = 0, \tag{6.5}$$

where, as before, $\gamma_{ab}^{(\xi)} = \lim_{\to i^0} \Omega^{-1/2}\gamma_{ab}^{(\xi)}$. Evaluating this condition using equation (6.2) and $\chi^a = \lim_{\to i^0} \Omega^{-1}\xi^a$ we get

$$h_{ab}\chi^b = -D_a f. \tag{6.6}$$

Thus a pure supertranslation f is represented by a vector field ξ^a such that

$$\lim_{\to i^0} \Omega^{-1}\xi^a = f\eta^a - D^a f. \tag{6.7}$$

In summary, the asymptotic symmetries at i^0 are parameterized by a pair (f, X^a) where f is a smooth function and $X^a \in \mathfrak{so}(1,3)$ is a smooth Killing vector field on $\mathscr{H}$.

The Lie algebra structure of these symmetries can be obtained as follows. Let ξ_1^a and ξ_2^a be the vector fields representing the asymptotic Spi-symmetries (f_1, X_1^a) and (f_2, X_2^a) respectively. Then the Lie bracket $[\xi_1, \xi_2]^a = \xi_1^b\nabla_b\xi_2^a - \xi_2^b\nabla_b\xi_1^a$ of the representatives induces a Lie bracket on the Spi-symmetries. Using equations (6.1), (6.3) and (6.4) the induced Lie bracket on the Spi-symmetries can be computed to be

$$
\begin{aligned}
(f, X^a) &= [(f_1, X_1^a), (f_2, X_2^a)], \\
\text{with} \quad f &= X_1^b D_b f_2 - X_2^b D_b f_1, \\
X^a &= X_1^b D_b X_2^a - X_2^b D_b X_1^a.
\end{aligned} \tag{6.8}
$$

Thus, the Spi symmetries form a Lie algebra $\mathfrak{spi}$ with the above Lie bracket structure. Note that if $X_1^a = X_2^a = 0$ then $f = X^a = 0$—the supertranslations form an infinite-dimensional abelian subalgebra $\mathfrak{s}$. Further if $X_1^a = 0$ and $X_2^a \neq 0$ then $X^a = 0$, thus the supertranslations $\mathfrak{s}$ are a

Lie ideal in $\mathfrak{spi}$. The quotient algebra $\mathfrak{spi}/\mathfrak{s}$ is then isomorphic to the algebra of Killing fields on $\mathscr{H}$ i.e. the Lorentz algebra $\mathfrak{so}(1,3)$. Thus the Spi symmetry algebra has the structure of a semi-direct sum

$$\mathfrak{spi} \cong \mathfrak{so}(1,3) \ltimes \mathfrak{s}. \tag{6.9}$$

The $\mathfrak{spi}$ algebra also has a preferred four-dimensional subalgebra $\mathfrak{t}$ of *translations*. These are obtained as the supertranslations $\boldsymbol{f}$ satisfying the additional condition

$$\boldsymbol{D}_a \boldsymbol{D}_b \boldsymbol{f} + \boldsymbol{h}_{ab} \boldsymbol{f} = 0. \tag{6.10}$$

The space of solutions to the above condition is indeed four-dimensional—this can be seen from the argument in remark 6.1 below, or by solving the equation in a suitable coordinate system on $\mathscr{H}$; see equations D.204 and D.205 of [33] or equation C.12 of [24]. Further from equation (6.8) it can be verified that the Lie bracket of a translation with any other element of $\mathfrak{spi}$ is again a translation, that is, the translations $\mathfrak{t}$ are a four-dimensional Lie ideal of $\mathfrak{spi}$.

Remark 6.1 (translation vectors at i^0). Let $\boldsymbol{v}^a$ be a direction-independent vector at i^0, and $\boldsymbol{v}^a = \boldsymbol{f}\eta^a + \boldsymbol{f}^a$ where $\eta_a \boldsymbol{f}^a = 0$. Then, since $\boldsymbol{v}^a$ is direction-independent we have

$$0 = \partial_a \boldsymbol{v}_b = \boldsymbol{D}_a \boldsymbol{f}_b + \boldsymbol{h}_{ab}\boldsymbol{f} + \eta_b(\boldsymbol{D}_a\boldsymbol{f} - \boldsymbol{f}_a), \tag{6.11}$$

which then implies $\boldsymbol{f}_a = \boldsymbol{D}_a\boldsymbol{f}$ and that $\boldsymbol{f}$ satisfies equation (6.1). Thus, any vector $\boldsymbol{v}^a \in Ti^0$ gives rise to a Spi-translation in $\mathfrak{t}$. Conversely, given any translation $\boldsymbol{f} \in \mathfrak{t}$, the vector at i^0 defined by (note the sign difference in the hyperboloidal component relative to equation (6.7))

$$\boldsymbol{v}^a := \boldsymbol{f}\eta^a + \boldsymbol{D}^a\boldsymbol{f}, \tag{6.12}$$

is direction-independent i.e., $\boldsymbol{v}^a \in Ti^0$. Thus, the Spi-translations $\mathfrak{t}$ can be represented by vectors in Ti^0.

Remark 6.2 (conformal transformation of Spi symmetries). Let $(\boldsymbol{f}, X^a)$ be a Spi symmetry defined by a vector field ξ^a as above, i.e.,

$$X^a := \lim_{\to i^0} \Omega^{-1/2}\xi^a, \quad \boldsymbol{f} := \lim_{\to i^0} \Omega^{-1}\xi^a\eta_a. \tag{6.13}$$

For a fixed ξ^a, consider the change in the conformal factor $\Omega \mapsto \omega\Omega$. Then, from remark 2.1 we have the transformations

$$X^a \mapsto X^a, \quad \boldsymbol{f} \mapsto \boldsymbol{f} + \frac{1}{2}\pounds_X\boldsymbol{\alpha}. \tag{6.14}$$

Note that a pure supertranslation $(\boldsymbol{f}, X^a = 0)$ is conformally-invariant, while a 'pure Lorentz' symmetry $(\boldsymbol{f} = 0, X^a)$ is not invariant but shifts by a supertranslation given by $\frac{1}{2}\pounds_X\boldsymbol{\alpha}$. This further reflects the semi-direct structure of the $\mathfrak{spi}$ algebra given in equation (6.9).

To find the charge corresponding to the Spi-symmetries we need to evaluate the symplectic current equation (5.15) when the perturbation denoted by δ_2 is generated by a Spi-symmetry. So we now calculate the perturbations $\delta_{(f,X)}\boldsymbol{E}$ and $\delta_{(f,X)}\boldsymbol{K}$ in the gravitational potentials corresponding to the metric perturbation equation (6.2).

The potentials E and K_{ab} are defined in terms of (a rescaled) limit of S_{ab} by equation (4.9). Consider then the change in S_{ab} under the perturbation equation (6.2). The second term on the right-hand side of equation (6.2) is a linearized conformal transformation (see remark 2.1) with $\alpha = -2f$. Thus, the change in E and K_{ab} induced by this linearized conformal transformation is given by (see remark 4.1)

$$\delta_f E = 0, \quad \delta_f K_{ab} = 4(D_a D_b f + f h_{ab}). \tag{6.15}$$

The first term on the right-hand side of equation (6.2) is a linearized diffeomorphism and, since S_{ab} is a local and covariant functional of g_{ab} the corresponding perturbation in S_{ab} is $\pounds_\xi S_{ab}$. Explicitly computing the Lie derivative, using equations (6.1) and (6.3) gives

$$\delta_X S_{ab} = \lim_{\to i^0} \Omega^{1/2} \pounds_\xi S_{ab} = X^c \partial_c S_{ab} + 2S_{c(a}\eta_{b)}X^c + 2S_{c(a}\partial_{b)}X^c. \tag{6.16}$$

Then, from the definition of the gravitational potentials equation (4.9) we have

$$\delta_X E = \pounds_X E, \quad \delta_X K_{ab} = \pounds_X K_{ab}. \tag{6.17}$$

As a result, under a general Spi symmetry parametrized by (f, X^a) we have

$$\delta_{(f,X)} E = \pounds_X E, \quad \delta_{(f,X)} K_{ab} = \pounds_X K_{ab} + 4(D_a D_b f + h_{ab}f). \tag{6.18}$$

Note that our parity condition equation (4.33) does not place any further restrictions on these symmetries.

Remark 6.3 (special choices of conformal factor). The freedom in the conformal factor can be used to impose further restrictions on the potential K_{ab}. We note the following two conditions that have been used in prior work.

(a) From equation (4.22) we see that $K := h^{ab} K_{ab}$ transforms as

$$K \mapsto K - 2(D^2\alpha + 3\alpha). \tag{6.19}$$

Now given a choice of conformal factor so that $K \neq 0$ we can always solve a linear hyperbolic equation for α on $\mathscr{H}$ and choose a new conformal factor (as in remark 2.1) so that in the new conformal completion $K = 0$. This is the choice made in [23, 32, 33]. With this restriction on K we see from equation (6.18) that the allowed supertranslations are reduced to functions f which satisfy

$$D^2 f + 3f = 0. \tag{6.20}$$

(b) Consider the restricted class of spacetimes where $B_{ab} = 0$. Then, the tensor K_{ab} can be written in terms of a scalar potential k as in equation (4.16). Comparing equation (4.16) with equation (4.22) we see that we can choose $\alpha = \tfrac{1}{2}k$. Then, we can choose a new conformal factor (as in remark 2.1) so that in the new conformal completion $K_{ab} = 0$. This is the choice made in [26, 34]. With this restriction we see from equation (6.18) that the allowed supertranslations are reduced to the translation algebra (equation (6.1)), and the full asymptotic symmetry algebra reduces to the Poincaré algebra.

It is not clear, *a priori*, what such special choices of conformal factor imply at null infinity. From the point of view of matching the Spi symmetries and charges to the ones defined on null infinity such choices of conformal factors might not be convenient. So we will *not* impose any such conditions on the conformal factor in our analysis and work with the full $\mathfrak{spi}$ algebra.

However, we will argue that our results reduce to those of [26, 33] when the corresponding restrictions are imposed.

7. Spi-charges

In this section we now compute the charges associated with the Spi-symmetries. Following our strategy we consider the symplectic current $\underset{\leftarrow}{\omega}$ where one of the perturbations, δ_2, is a perturbation generated by an asymptotic Spi-symmetry represented by (f, X^a). Using equations (5.15) and (6.18) we have

$$\underset{\leftarrow}{\omega}(\delta g, \delta_{(f,X)} g) = \frac{1}{64\pi} \varepsilon_3 \left[\delta K \pounds_X E - \delta E \pounds_X K - 4 \delta E (D^2 f + 3f) \right]. \tag{7.1}$$

We show next that, under suitable conditions, the above expression can be written as a total derivative on $\mathscr{H}$ that is,

$$\underset{\leftarrow}{\omega}(\delta g, \delta_{(f,X)} g) = -\varepsilon_3 \, D^a Q_a(g; \delta g; (f, X)), \tag{7.2}$$

where Q_a is a local and covariant functional of its arguments on $\mathscr{H}$.

It will be convenient to do this separately for supertranslations and Lorentz symmetries. In section 7.1, we will find that for supertranslations the functional Q_a is integrable, and defines the *supermomentum* charges on cross-sections S of $\mathscr{H}$. Then we show in section 7.2 that for Lorentz symmetries Q_a is not integrable, in general. In this case we will adopt the prescription of Wald and Zoupas with suitable modifications to define an integrable charge for Lorentz symmetries. Finally, as noted in remark 6.2, a 'pure Lorentz' symmetry is not conformally-invariant but shifts by a supertranslation. Similarly, we show in section 7.3 that the Lorentz charge shifts by a supertranslation charge under conformal transformations, in accord with the semi-direct structure of the $\mathfrak{spi}$ algebra (equation (6.9)).

7.1. Charges for supertranslations: Spi-supermomentum

To define the charge for the supertranslations consider equation (7.1) for a pure supertranslation $(f, X^a = 0)$

$$\begin{aligned}
\underset{\leftarrow}{\omega}(\delta g, \delta_f g) &= -\frac{1}{16\pi} \varepsilon_3 \, \delta E (D^2 f + 3f), \\
&= -\frac{1}{16\pi} \varepsilon_3 D^a \delta (E D_a f - f D_a E),
\end{aligned} \tag{7.3}$$

where the second line uses equation (4.12). In this case, the symplectic current can be written in the form equation (7.2) where the Q_a is manifestly integrable. Thus, we define the Spi *supermomentum* charge at a cross-section S of $\mathscr{H}$ by

$$\mathcal{Q}[f; S] = \frac{1}{16\pi} \int_S \varepsilon_2 \, u^a (E D_a f - f D_a E). \tag{7.4}$$

Here we have chosen the charge to vanish on Minkowski spacetime where $E = 0$ (see equation (4.34)). The corresponding flux is given by (using equation (4.12))

$$\mathcal{F}[f; \Delta \mathscr{H}] := \mathcal{Q}[f; S_2] - \mathcal{Q}[f; S_1] = -\frac{1}{16\pi} \int_{\Delta \mathscr{H}} \varepsilon_3 \, E (D^2 f + 3f). \tag{7.5}$$

When $\boldsymbol{f} \in \mathfrak{t}$ is a Spi-translation the charge equation (7.4) can be written in an alternative form as follows: using equations (4.11) and (4.12) we have the identity

$$
\begin{aligned}
-\boldsymbol{f}\boldsymbol{D}_a\boldsymbol{E} + \boldsymbol{E}\boldsymbol{D}_a\boldsymbol{f} = {}& 2\boldsymbol{E}_{ab}\boldsymbol{D}^b\boldsymbol{f} + \boldsymbol{D}^b\left(\boldsymbol{D}_{[a}\boldsymbol{E}\boldsymbol{D}_{b]}\boldsymbol{f}\right) \\
& - \frac{1}{2}\left[\boldsymbol{D}_a\boldsymbol{E}(\boldsymbol{D}^2\boldsymbol{f} + 3\boldsymbol{f}) - \boldsymbol{D}^b\boldsymbol{E}(\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f})\right].
\end{aligned}
\tag{7.6}
$$

The second term on the right-hand side corresponds to an exact two-form and vanishes upon integrating on S, while the last line vanishes for translations due to equation (6.1). Hence, the charge for any translation $\boldsymbol{f} \in \mathfrak{t}$ can be written as

$$
\mathcal{Q}[\boldsymbol{f}; S] = \frac{1}{8\pi}\int\limits_S \varepsilon_2 \; \boldsymbol{u}^a\boldsymbol{E}_{ab}\boldsymbol{D}^b\boldsymbol{f},
\tag{7.7}
$$

which reproduces the charge for translations given in [26]. Using equation (6.10) the flux of translations vanishes across any region $\Delta\mathscr{H}$ and thus the translation charge is independent of the choice of cross-section S. Using the isomorphism between Spi-translations $\boldsymbol{f}$ and vectors $\boldsymbol{v}^a$ in Ti^0 (see remark 6.1), the translation charge in equation (7.7) defines a four-momentum vector $\boldsymbol{P}^a$ at i^0 such that

$$
\boldsymbol{P}^a\boldsymbol{v}_a = \mathcal{Q}[\boldsymbol{f}; S].
\tag{7.8}
$$

Note that this relation is well-defined at i^0 since the translation charge is independent of the cross-section S. The vector $\boldsymbol{P}^a$ is precisely the ADM four-momentum at i^0 [46] and also coincides with the limit to i^0 of the Bondi four-momentum on null infinity [25] (the corresponding result for all the supertranslation charges was proven in [24]).

The charge expression equation (7.4) agrees with the results of Compère and Dehouck [33]. Note that when the conformal factor is chosen so that $\boldsymbol{K} = 0$ the supertranslation algebra is reduced to the subalgebra satisfying equation (6.20) and the flux corresponding to such supertranslations vanishes across any region $\Delta\mathscr{H}$. As was shown in [24], to relate the supertranslation symmetries and charges at spatial infinity to the ones on null infinity, it is sufficient that the *total* flux of these charges vanishes on *all* of $\mathscr{H}$,[10] and the flux need not vanish across some local region $\Delta\mathscr{H}$. Thus the restriction on the conformal factor imposing $\boldsymbol{K} = 0$ is not necessary.

Note that in [24] the supermomentum charges at spatial infinity were related to those on null infinity using the Ashtekar–Hansen expression equation (7.7) for *all* supertranslations (even those which are not translations), instead of the expression equation (7.4). On $\mathscr{H}$, these charge expressions differ by the integral of last line of equation (7.6) over some cross-section S. However, the regularity conditions on $\boldsymbol{E}$ and $\boldsymbol{f}$ used in [24] as the spatial directions $\vec{\eta}$ limit to null directions at i^0 ensure that the additional terms vanish (see, for instance, appendix. D of [24]) and both expressions yield the same *finite* supermomenta in null directions which further equals the supermomenta at null infinity. Thus, the result of [24] can also be derived using the expression equation (7.4) for the supertranslation charges.

7.2. Lorentz charges with $B_{ab} = 0$

Next we will obtain a charge formula for the Lorentz symmetries. As emphasized in [26, 34], to obtain such a charge formula one needs to consider the 'subleading' piece of the magnetic

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[10] To make this rigorous it is necessary to additionally complete $\mathscr{H}$ to include the null directions at i^0. This construction is detailed in [22, 24].

part of the Weyl tensor. Thus, in the following we will make the additional assumption that $\boldsymbol{B}_{ab} = 0$ and that the 'subleading' magnetic part $\boldsymbol{\beta}_{ab}$ defined in equation (4.14) exists. However, in appendix E we show how the restriction that $\boldsymbol{B}_{ab}$ vanishes can be lifted to obtain a charge for the Lorentz symmetries.

For a 'pure Lorentz' symmetry ($\boldsymbol{f} = 0, \boldsymbol{X}^a$) we have from equation (7.1)

$$\underset{\leftarrow}{\boldsymbol{\omega}}(\delta g, \delta_X g) = \frac{1}{64\pi}\varepsilon_3(\pounds_X \boldsymbol{E}\delta\boldsymbol{K} - \pounds_X\boldsymbol{K}\delta\boldsymbol{E}). \tag{7.9}$$

We now want to write this as a total derivative of the form equation (7.2). To do so consider the following tensor

$$\boldsymbol{W}_{ab} := \boldsymbol{\beta}_{ab} + \frac{1}{8}\varepsilon_{cd(a}\boldsymbol{D}^c\boldsymbol{E}\boldsymbol{K}^d{}_{b)} - \frac{1}{16}\varepsilon_{abc}\boldsymbol{K}\boldsymbol{D}^c\boldsymbol{E}. \tag{7.10}$$

Using equations (4.13a), (4.15) and (4.20), we obtain

$$\boldsymbol{D}^a\boldsymbol{W}_{ab} = 0, \quad \boldsymbol{h}^{ab}\boldsymbol{W}_{ab} = 0. \tag{7.11}$$

Note that $\boldsymbol{W}_{ab}$ is not a symmetric tensor. Further using equations (7.10) and (B.3) we have

$$\boldsymbol{D}^a[\boldsymbol{W}_{ab}{}^\star\boldsymbol{X}^b] = \frac{1}{8}\boldsymbol{X}^a\boldsymbol{D}_a\boldsymbol{E}\boldsymbol{K}, \tag{7.12}$$

where $^\star\boldsymbol{X}^a := \frac{1}{2}\varepsilon^{abc}\boldsymbol{D}_b\boldsymbol{X}_c$ is the 'dual' Killing vector field to $\boldsymbol{X}^a$ (see equation (B.4)). Therefore, equation (7.9) can be written as

$$\underset{\leftarrow}{\boldsymbol{\omega}}(\delta g, \delta_X g) = \frac{1}{8\pi}\varepsilon_3\boldsymbol{D}^a\left[\delta\boldsymbol{W}_{ab}{}^\star\boldsymbol{X}^b - \frac{1}{8}\delta\boldsymbol{E}\boldsymbol{K}\boldsymbol{X}_a\right], \tag{7.13}$$

which is again of the form equation (7.2). However the functional $\boldsymbol{Q}_a$ in this case is not integrable, in general. To see this consider

$$\int_S \varepsilon_2\ \boldsymbol{u}^a\boldsymbol{Q}_a[\delta g; X] = -\frac{1}{8\pi}\int_S \varepsilon_2\ \boldsymbol{u}^a\left[\delta\boldsymbol{W}_{ab}{}^\star\boldsymbol{X}^b - \frac{1}{8}\delta\boldsymbol{E}\boldsymbol{K}\boldsymbol{X}_a\right], \tag{7.14}$$

and compute an antisymmetrized second variation to get

$$\int_S \varepsilon_2\boldsymbol{u}^a\left(\delta_1\boldsymbol{Q}_a[\delta_2 g; X] - \delta_2\boldsymbol{Q}_a[\delta_1 g; X]\right) = \frac{1}{64\pi}\int_S \varepsilon_2\boldsymbol{u}^a\boldsymbol{X}_a\left(\delta_1\boldsymbol{K}\delta_2\boldsymbol{E} - \delta_2\boldsymbol{K}\delta_1\boldsymbol{E}\right)$$

$$= -\int_S \boldsymbol{X}\cdot\underset{\leftarrow}{\boldsymbol{\omega}}(\delta_1 g, \delta_2 g). \tag{7.15}$$

If equation (7.14) were integrable then the above antisymmetrized second variation would vanish for all perturbations and all cross-sections S. However, since we allow arbitrary perturbations of both $\boldsymbol{E}$ and $\boldsymbol{K}_{ab}$ the expression on the right-hand side vanishes if and only if the Lorentz vector field happens to be tangent to the cross-section S. However a general Lorentz vector field is not tangent to any cross-section of $\mathscr{H}$, in particular Lorentz boosts do not preserve any cross-section of $\mathscr{H}$. Thus, the expression equation (7.14) is not integrable and cannot be used to define the charge of Lorentz symmetries.

To remedy this, note that equation (7.15) is similar to the integrability criterion derived by Wald and Zoupas (see equation (16) of [37]). Wald and Zoupas further developed a general prescription to define an integrable charge ('conserved quantity') which we now adapt to our case. Let $\Theta(g; \delta g)$ be a three-form on $\mathscr{H}$ which is a symplectic potential for the pullback of the symplectic current (equation (5.15)) to $\mathscr{H}$, that is,

$$\underset{\leftarrow}{\omega}(g; \delta_1 g, \delta_2 g) = \delta_1 \Theta(g; \delta_2 g) - \delta_2 \Theta(g; \delta_1 g), \tag{7.16}$$

for all backgrounds and all perturbations. We also require that the choice of Θ satisfy the following conditions

(a) Θ is locally and covariantly constructed out of the dynamical fields (E, K_{ab}), their perturbations, and finitely many of their derivatives, along with the 'universal background structure' h_{ab} present on $\mathscr{H}$.

(b) Θ is independent of any arbitrary choices made in specifying the background structure, in particular, Θ is conformally-invariant.

(c) $\Theta(g; \delta g) = 0$ for Minkowski spacetime for *all* perturbations δg.

In analogy to the Wald–Zoupas prescription we define the charge $\mathcal{Q}[X^a; S]$ associated with a Lorentz symmetry through

$$\delta \mathcal{Q}[X^a; S] := \int_S \varepsilon_2 u^a Q_a(\delta g; X^a) + \int_S X \cdot \Theta(\delta g). \tag{7.17}$$

From equations (7.15) and (7.16) it follows that the above defining relation is integrable and thus defines a charge $\mathcal{Q}[X^a; S]$ once we pick a reference solution where the charge vanishes.

For the three-form Θ we choose

$$\Theta(g; \delta g) := -\frac{1}{64\pi} \varepsilon_3 E \delta K. \tag{7.18}$$

It can be verified that this choice satisfies all the criteria listed below equation (7.16). In particular Θ is conformally-invariant, and for Minkowski spacetime $E = 0$ (equation (4.34)) and so $\Theta = 0$ on Minkowski spacetime for *all* perturbations. This choice for Θ is not unique, but we will argue in appendix D that the ambiguity in the the choice of Θ does not affect our final charge expression.

With the choice equations (7.18) and (7.14) and (7.17), we have

$$\delta \mathcal{Q}[X^a; S] = -\frac{1}{8\pi} \int_S \varepsilon_2 \, u^a \delta[W_{ab}{}^\star X^b - \frac{1}{8} KEX_a], \tag{7.19}$$

We define the unperturbed charge by picking the reference solution to be Minkowski spacetime which satisfies $E = 0$ and $\beta_{ab} = 0$ (equation (4.34)). Thus, we have the charge

$$\mathcal{Q}[X^a; S] = -\frac{1}{8\pi} \int_S \varepsilon_2 \, u^a [W_{ab}{}^\star X^b - \frac{1}{8} KEX_a], \tag{7.20}$$

The corresponding flux of the Lorentz charges is given by

$$\mathcal{F}[X^a, \Delta\mathscr{H}] = -\frac{1}{64\pi} \int_{\Delta\mathscr{H}} \varepsilon_3 \, E \mathcal{L}_X K. \tag{7.21}$$

Note that the flux is essentially given by $\mathcal{F}[X^a, \Delta\mathcal{H}] = \int_{\Delta\mathcal{H}} \Theta(g; \delta_X g)$ in analogy to the Wald–Zoupas prescription (see equation (32) of [37]).

When the conformal factor is chosen so that $K_{ab} = 0$ then the Lorentz charge reduces to

$$\mathcal{Q}[X^a; S] = -\frac{1}{8\pi} \int\limits_S \varepsilon_2 \, u^a \beta_{ab}{}^\star X^b, \tag{7.22}$$

which is the expression given by [26]. Note that when the conformal factor is chosen such that $K = 0$, the expression equation (7.14) is manifestly integrable and our 'correction term' Θ (equation (7.18)) vanishes. In both these cases, the flux of the Lorentz charges vanishes across any region $\Delta\mathcal{H}$, i.e., the Lorentz charges are identically conserved. Further, since the vector fields X^a correspond precisely to infinitesimal Lorentz transformations Λ_{ab} in Ti^0 (see equation (B.6)), the charge defines an 'angular momentum' tensor J^{ab} at i^0 through

$$J^{ab}\Lambda_{ab} = \mathcal{Q}[X^a; S], \tag{7.23}$$

where the right-hand side is independent of the cross-section since the charge is conserved.

7.3. Transformation of charges under conformal changes

We now consider the transformation of the charges and fluxes for a Spi symmetry under changes of the choice of conformal factor as discussed in remark 2.1.

Consider a pure supertranslation symmetry $(f, X^a = 0)$. As shown in remark 6.2, a pure supertranslation is conformally-invariant. Further from remark 4.1 the potential E is also conformally-invariant. Thus, the charge and flux of supertranslations in equations (7.4) and (7.5) is also conformally-invariant.

However a 'pure Lorentz' symmetry $(f = 0, X^a)$ is not conformally-invariant (see remark 6.2), and hence we expect that the charge and flux of a Lorentz symmetry must transform nontrivially under changes of the conformal factor. Consider first the flux of Lorentz charges given by equation (7.21). Using the transformation of K_{ab} (equation (4.22)) we see that this flux expression transforms as

$$\mathcal{F}[X^a; \Delta\mathcal{H}] \mapsto \mathcal{F}[X^a; \Delta\mathcal{H}] + \frac{1}{32\pi} \int\limits_{\Delta\mathcal{H}} \varepsilon_3 E(D^2 \pounds_X \alpha + 3\pounds_X \alpha). \tag{7.24}$$

Comparing the second term on the right-hand side to equation (7.5), we see that it is precisely the flux of a supertranslation given by $(-1/2\pounds_X\alpha)$. Thus, under a change of conformal factor the Lorentz flux shifts by the flux of a supertranslation

$$\mathcal{F}[X^a; \Delta\mathcal{H}] \mapsto \mathcal{F}[X^a; \Delta\mathcal{H}] + \mathcal{F}[-1/2\pounds_X\alpha; \Delta\mathcal{H}]. \tag{7.25}$$

One can similarly verify that the Lorentz charge equation (7.20) also shifts by the charge of a supertranslation. The explicit computation is a bit tedious and is presented in appendix C. However, we can derive the transformation of the Lorentz charge by a more general argument which we present below. This argument also holds in the more general case when $B_{ab} \neq 0$ considered in appendix E below.

From the transformation of the flux equation (7.25), we can deduce that the Lorentz charge expression equation (7.20) must transform as

$$\mathcal{Q}[X^a; S] \mapsto \mathcal{Q}[X^a; S] + \mathcal{Q}[-1/2\pounds_X\alpha; S] + \int\limits_S \varepsilon_2 u^a \mu_a[\alpha], \tag{7.26}$$

where the second term on the right-hand side is the charge of a supertranslation $(-\frac{1}{2}\pounds_X\alpha)$ and the third term is a possible additional term determined by a covector $\boldsymbol{\mu}_a$ which depends linearly on α and is divergence-free, $\boldsymbol{D}^a\boldsymbol{\mu}_a[\alpha] = 0$ for *all* α. Since α is a free function on $\mathscr{H}$ we can apply theorem 1 with α as the 'dynamical field'. Thus, from equation (B.17) we conclude that the final integral above vanishes, and that the Lorentz charge shifts by the charge of a supertranslation $(-\frac{1}{2}\pounds_X\alpha)$.

$$\mathcal{Q}[X^a; S] \mapsto \mathcal{Q}[X^a; S] + \mathcal{Q}[-1/2\pounds_X\alpha; S]. \tag{7.27}$$

If we restrict to the choice of conformal factor where $\boldsymbol{K}_{ab} = 0$, so that the asymptotic symmetries are reduced to the Poincaré algebra and α is a Spi-translation satisfying equation (6.10), then equation (7.27) reproduces the transformation law given in equation (29) of [26] and equation (6.8) of [34].

Consider the charge of any Spi-symmetry represented by (f, X^a), then under a conformal transformation the same Spi-symmetry is now represented by $(f + \frac{1}{2}\pounds_X\alpha, X^a)$ (see remark 6.2). The total charge of the Spi-symmetry transforms as

$$\mathcal{Q}[f; S] + \mathcal{Q}[X^a; S] \mapsto \mathcal{Q}[f + 1/2\pounds_X\alpha; S] + \mathcal{Q}[X^a; S] + \mathcal{Q}[-1/2\pounds_X\alpha; S],$$
$$= \mathcal{Q}[f; S] + \mathcal{Q}[X^a; S], \tag{7.28}$$

that is, the charge of any Spi-symmetry is independent of the choice of conformal factor—the change in the function f representing the symmetry is exactly compensated by the change in the Lorentz charge given in equation (7.27).

8. Discussion

In this paper, we analyzed the asymptotic symmetries and the corresponding charges for asymptotically-flat spacetimes at spatial infinity i^0 using the Ashtekar–Hansen formalism, without any restrictions on the choice of the conformal factor at spatial infinity, which were imposed in previous analyses. Using the covariant phase space, we considered the direction-dependent limit of symplectic current of vacuum general relativity to spatial infinity. Using the pullback of this limit of the symplectic current to the space of spatial directions $\mathscr{H}$ at spatial infinity, we obtained expressions for charges corresponding to all asymptotic symmetries. We rederived the known expressions for supertranslation charges but more a general expression for the Lorentz charge when conformal factor is completely unrestricted. In this case, we used a Wald–Zoupas type correction to make the Lorentz charge integrable, which also ensures that this charge transforms correctly under the action of a supertranslation, or equivalently, that the charge of a general Spi-symmetry is conformally-invariant.

The main motivation behind our analysis is to eventually relate the Lorentz charges at spatial infinity to the ones defined on null infinity. In this context, the Lorentz charge expressions would have to be matched in the 'same' choice of conformal factor at both null infinity and spatial infinity, and it is not clear what the restrictions on the conformal factor at spatial infinity placed in previous works imply at null infinity. Thus, we hope that our more general expression for the Lorentz charge at spatial infinity will be more useful to repeat the matching analysis for the case of Lorentz symmetries that was done previously for Maxwell theory [22] and supertranslations in general relativity [24]. If this works out as expected, this would imply that the full BMS group at past null infinity is matched to the full BMS group at future null infinity and moreover, that the incoming fluxes of all BMS symmetries through past null infinity are equal to the outgoing fluxes of the anitpodally identified BMS symmetries through future null

infinity. This would then prove the existence of infinitely many conservation laws, one for each generator of the BMS group, in classical gravitational scattering in asymptotically-flat spacetimes, as anticipated by Strominger [5].

Another avenue for future investigation would be to quantize the asymptotic fields on $\mathscr{H}$ in the spirit of the asymptotic quantization program on null infinity [47], see also [48]. This could lead to the possibility of relating the asymptotic 'in-states' on past null infinity to the 'out-states' on future null infinity, similar to the matching conditions in the classical theory, and provide further insight into the structure of quantum scattering.

We also note that the asymptotic fields at spatial infinity in both Maxwell theory and general relativity are described by smooth tensor fields living on a unit-hyperboloid $\mathscr{H}$. As is well-known $\mathscr{H}$ is precisely the three-dimensional de Sitter spacetime. To prove the matching conditions for Maxwell and gravitational fields on $\mathscr{H}$ with those on null infinity, $\mathscr{H}$ was conformally-completed into a cylinder in the analysis of [22, 24]. It would be interesting to see if insights from the de Sitter/CFT correspondence [49] can be applied to develop a holographic understanding of electromagnetism and general relativity in asymptotically-flat spacetimes at spatial infinity, perhaps similar to [50].

Acknowledgments

Appendix A. Coordinates, universal structure and asymptotic expansions near i^0

In this appendix we construct a suitable asymptotic coordinate system near spatial infinity. Using these coordinates we explicitly demonstrate the universal structure near i^0 described in section 4.1. We also describe the asymptotic expansion of the unphysical and physical metrics in these coordinates, thus making contact with the expansions used in previous works [31–33].

Consider the unphysical spacetime (M, g_{ab}) obtained from some physical spacetime satisfying definition 2.1. The unphysical metric g_{ab} at i^0 induces a metric which is isometric to the Minkowski metric in the tangent space Ti^0. Thus we can introduce asymptotically Cartesian coordinates (t, x, y, z) so that i^0 is at the origin of this coordinate system and

$$\boldsymbol{g}_{ab} \equiv -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{A.1}$$

Note that $x^i = (t, x, y, z)$ define a C^1 coordinate system at i^0. To define a $C^{>1}$ differential structure we allow any other coordinate chart $x'^i(x)$ such that

$$\frac{\partial^2 x'^i(x)}{\partial x^j \partial x^k} \quad \text{and} \quad \frac{\partial^2 x^i(x')}{\partial x'^j \partial x'^k} \text{ are } C^{>-1} \text{ at } i^0. \tag{A.2}$$

A collection of all coordinate charts related by equation (A.2) defines a choice of $C^{>1}$-structure on M at i^0, see [40] and appendix A of [22] for details.

It is more convenient to use coordinates which are adapted to the space of unit spacelike directions $\mathscr{H}$. Thus define (ρ, τ) by

$$\rho^2 := -t^2 + x^2 + y^2 + z^2, \quad \tanh \tau := \frac{t}{\sqrt{x^2 + y^2 + z^2}}. \tag{A.3}$$

In these coordinates the metric in Ti^0 takes the form

$$\boldsymbol{g}_{ab} \equiv d\rho^2 + \rho^2 \left(-d\tau^2 + \cosh^2 \tau s_{AB} d\theta^A d\theta^B\right), \tag{A.4}$$

where s_{AB} is the unit metric on $\mathbb{S}^2$ in some coordinates θ^A, say the usual (θ, ϕ) coordinates. Note that the coordinates (ρ, τ, θ^A) are *not* $C^{>1}$ coordinates—the bases $(d\rho, \rho d\tau, \rho d\theta^A)$ are not continuous but are direction-dependent at i^0.

The unit spatial directions $\vec{\eta}$ then correspond to the unit vectors ∂_ρ in Ti^0 which are parameterized by (τ, θ^A). The space of directions $\mathscr{H}$ is then the surface $\rho = 1$ in Ti^0 with the induced metric

$$\boldsymbol{h}_{ab} \equiv -d\tau^2 + \cosh^2 \tau s_{AB} d\theta^A d\theta^B. \tag{A.5}$$

The reflection of the directions $\vec{\eta} \mapsto -\vec{\eta}$ then induces the reflection isometry

$$(\tau, \theta^A) \mapsto (-\tau, -\theta^A), \tag{A.6}$$

on $\mathscr{H}$, where $\theta^A \mapsto -\theta^A$ is the antipodal reflection on $\mathbb{S}^2$.

So far we have only considered the structure at i^0, now we extend the metric away from i^0. Since the unphysical metric g_{ab} is $C^{>0}$ and limits to $\boldsymbol{g}_{ab}$ at i^0 (where $\rho = 0$), it can be verified that g_{ab} admits an expansion in ρ of the form

$$g_{ab} \equiv [1 + \sigma\rho + o(\rho)]^2 d\rho^2 + 2[\rho A_a + o(\rho)] d\rho(\rho dy^a)$$
$$+ \left[h_{ab}^{(0)} + \rho h_{ab}^{(1)} + o(\rho)\right](\rho dy^a)(\rho dy^b), \tag{A.7}$$

where $y^a = (\tau, \theta^A)$ are coordinates on the unit hyperboloid, and $h_{ab}^{(0)} \equiv \boldsymbol{h}_{ab}$ is the unit hyperboloid metric. The expansion coefficients σ, A_a and $h_{ab}^{(1)}$ can be considered as tensor fields on $\mathscr{H}$. The $o(\rho)$ denotes terms which falloff faster than ρ in the limit to i^0, that is, $\lim_{\rho \to 0} \rho^{-1} o(\rho) = 0$.

For the conformal factor, one can choose

$$\Omega = \rho^2, \tag{A.8}$$

which can be verified to satisfy all the conditions in definition 2.1, that is, in the limit $\rho \to 0$, $\Omega = 0$, $\nabla_a \Omega = 0$ and $\nabla_a \nabla_b \Omega = 2g_{ab}$. Before considering the physical metric lets analyze the universal structure at i^0.

From the above discussion it is clear that the metric $\boldsymbol{g}_{ab}$ and the space of directions $\mathscr{H}$ is universal, that is, independent of which unphysical metric is chosen. What is the structure corresponding to the equivalence classes of $C^{>1}$ curves described in section 4.1? Consider the $C^{>1}$ curves Γ_v through i^0 with tangents $v^a \equiv \partial_\rho$ in these coordinates. Further, with the choice of conformal factor in equation (A.8) we have

$$\eta^a = \nabla^a \Omega^{1/2} \equiv (1 - 2\rho\sigma)\frac{\partial}{\partial\rho} + \rho h^{(0)ab} A_b \frac{\partial}{\rho\partial y^a} + o(\rho). \tag{A.9}$$

From equation (4.27) we see that the curves Γ_v (with tangent $v^a \equiv \partial_\rho$) will be equivalent to the curves Γ_η (with tangent η^a) for all spacetimes if we can always choose A_a to vanish. This can be accomplished using the freedom in the choice of the hyperboloid coordinates y^a at 'next order' in ρ. Consider the coordinate transformation[11]

$$\rho \mapsto \rho, \quad y^a \mapsto y^a + \rho h^{(0)ab} A_b. \tag{A.10}$$

By rewriting this in terms of the Cartesian coordinates $x^i = (t, x, y, z)$, it can be verified that the transformation equation (A.10) is a $C^{>1}$ coordinate transformation (equation (A.2)). It can be also be verified that using this transformation the $\mathrm{d}\rho \mathrm{d}y^a$ term in the metric, i.e. A_a, vanishes in the new coordinates. Thus, the curves Γ_v and Γ_η can always be chosen to be equivalent. Further, this choice can always be made in any choice of the physical spacetime. Thus, the equivalence classes of $C^{>1}$ curves through i^0 is also universal.

Having made this choice the unphysical metric takes the form

$$g_{ab} \equiv [1 + \sigma\rho + o(\rho)]^2 \mathrm{d}\rho^2 + \rho\, o\,(\rho)\, \mathrm{d}\rho\mathrm{d}y^a + \rho^2 \left[h^{(0)}_{ab} + \rho h^{(1)}_{ab} + o(\rho) \right] \mathrm{d}y^a\, \mathrm{d}y^b. \tag{A.11}$$

To get the form of the physical metric $\hat{g}_{ab} = \Omega^{-2} g_{ab}$ we use equation (A.8) and define the Beig–Schmidt coordinate $\rho_{(BS)} := 1/\rho$ to obtain

$$\begin{aligned}
\hat{g}_{ab} \equiv & \left[1 + \frac{\sigma}{\rho_{(BS)}} + o(1/\rho_{(BS)}) \right]^2 \mathrm{d}\rho^2_{(BS)} + \rho_{(BS)} o(1/\rho_{(BS)}) \mathrm{d}\rho_{(BS)} \mathrm{d}y^a \\
& + \rho^2_{(BS)} \left[h^{(0)}_{ab} + \frac{h^{(1)}_{ab}}{\rho_{(BS)}} + o(1/\rho_{(BS)}) \right] \mathrm{d}y^a \mathrm{d}y^b,
\end{aligned} \tag{A.12}$$

This is the form of the physical metric assumed by Beig and Schmidt [31].

The asymptotic potentials equation (4.9) are related to the metric coefficients in the above expansion by

$$\boldsymbol{E} \equiv 4\sigma, \quad \boldsymbol{K}_{ab} \equiv -2(h^{(1)}_{ab} + 2\sigma h^{(0)}_{ab}). \tag{A.13}$$

From these the asymptotic Weyl tensors can be computed using equation (4.11). Note that the parity condition equation (4.33) imposed on $\boldsymbol{E}$ to eliminate the logarithmic translation ambiguity then corresponds to

$$\sigma(\tau, \theta^A) = \sigma(-\tau, -\theta^A). \tag{A.14}$$

From equation (A.13) it straightforward to see that our charges for supertranslations equation (7.4) matches the expression obtained by Compère and Dehouck, equation (4.88) of [33].

For the 'subleading' magnetic Weyl tensor $\boldsymbol{\beta}_{ab}$ (defined by equation (4.14) when $\boldsymbol{B}_{ab} = 0$) to exist, we need additional regularity conditions on the metric expansion equation (A.7). Thus, to define $\boldsymbol{\beta}_{ab}$ we assume the 'next order' expansion

$$\begin{aligned}
g_{ab} \equiv & [1 + \sigma\rho + o(\rho)]^2 \mathrm{d}\rho^2 + \rho\, o(\rho) \mathrm{d}\rho\mathrm{d}y^a \\
& + \rho^2 \left[h^{(0)}_{ab} + \rho h^{(1)}_{ab} + \rho^2 h^{(2)}_{ab} + o(\rho^2) \right] \mathrm{d}y^a \mathrm{d}y^b,
\end{aligned} \tag{A.15}$$

---

[11] This is essentially the unphysical spacetime version of the coordinate transformations consider in lemma 2.2 of [31].

where $h_{ab}^{(2)}$ is a smooth tensor on $\mathscr{H}$. Then, we have (using $\boldsymbol{B}_{ab} = 0$)

$$\beta_{ab} = \varepsilon_{cd(a}\boldsymbol{D}^c h_{b)}^{(2)d} - \frac{1}{8}\varepsilon_{cd(a}\boldsymbol{D}^c\boldsymbol{E}\boldsymbol{K}^d{}_{b)} - \frac{1}{16}\varepsilon_{cd(a}\boldsymbol{D}_{b)}\boldsymbol{K}^{ce}\boldsymbol{K}^d{}_e. \tag{A.16}$$

When the conformal factor is chosen so that $\boldsymbol{K}_{ab} = 0$, the above expression simplifies considerably. In this case, our Lorentz charge matches the one found by Compère, Dehouck and Virmani [32]. We discuss the case when $\boldsymbol{B}_{ab} \neq 0$ in appendix E.

Appendix B. Some useful relations on $\mathscr{H}$

In this appendix we collect some relations on the unit-hyperboloid $\mathscr{H}$ which are useful in the main paper.

The Riemann tensor of $\mathscr{H}$ is given by

$$\mathcal{R}_{abcd} = \boldsymbol{h}_{ac}\boldsymbol{h}_{bd} - \boldsymbol{h}_{ad}\boldsymbol{h}_{bc}. \tag{B.1}$$

Using the above it is easy to derive simple expressions for commuting derivatives on tensor fields on $\mathscr{H}$, see appendix A of [31].

B.1. Killing vector fields

Let X^a be a Killing vector field on $\mathscr{H}$, so that $\boldsymbol{D}_{(a}X_{b)} = 0$. For any Killing vector field using equation C.3.6 of [38] and equation (B.1) we have

$$\boldsymbol{D}_a\boldsymbol{D}_bX_c = \mathcal{R}_{cbad}X^d = \boldsymbol{h}_{ac}X_b - \boldsymbol{h}_{ab}X_c. \tag{B.2}$$

Contracting the indices a and b we get

$$\boldsymbol{D}^2X_a + 2X_a = 0. \tag{B.3}$$

Define the 'dual' vector field $^\star X^a$ on $\mathscr{H}$ for any Killing vector field X^a by

$$^\star X^a := \frac{1}{2}\varepsilon^{abc}\boldsymbol{D}_bX_c. \tag{B.4}$$

Then, using equation (B.2) we have

$$\boldsymbol{D}_a{}^\star X^b = \varepsilon_{abc}X^c, \quad X^a = -\frac{1}{2}\varepsilon^{abc}\boldsymbol{D}_b{}^\star X^c = -^\star(^\star X)^a, \quad \boldsymbol{D}_aX_b = -\varepsilon_{abc}{}^\star X^c. \tag{B.5}$$

In particular $\boldsymbol{D}_{(a}{}^\star X_{b)} = 0$ so $^\star X^a$ is also a Killing vector field on $\mathscr{H}$. In a suitable choice of coordinates on $\mathscr{H}$ this relation maps Lorentz rotations and Lorentz boosts into each other, see appendix B of [32].

The relationship between the Killing vector fields on $\mathscr{H}$ and Lorentz transformations in the tangent space Ti^0 is as follows. Let Λ_{ab} be a *direction-independent* antisymmetric tensor at i^0 corresponding to an infinitesimal Lorentz transformation in Ti^0. Then the *direction-dependent* vector field defined by[12]

$$X^a(\vec{\eta}) := \Lambda^{ab}\eta_b, \tag{B.6}$$

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[12] The relation equation (B.6) is the 'dual' of the relation used below equation (27) of [26].

is tangent to $\mathscr{H}$. Further, since $\mathbf{\Lambda}_{ab}$ is direction-independent, $\partial_c\mathbf{\Lambda}_{ab} = 0$. Projecting the indices of $\partial_c\mathbf{\Lambda}_{ab} = 0$ tangent and normal to $\mathscr{H}$ in all possible ways it follows that X^a is a Killing vector field on $\mathscr{H}$ and

$$\mathbf{\Lambda}_{ab} = -\mathbf{D}_a X_b - 2\boldsymbol{\eta}_{[a}X_{b]} = \varepsilon_{abc}{}^\star X^c + \boldsymbol{\eta}_{[a}\varepsilon_{b]cd}\mathbf{D}^{c\,\star}X^d, \tag{B.7}$$

where the last equality uses equation (B.5). Similarly, it can be shown that if X^a is the Killing vector field on $\mathscr{H}$ corresponding to $\mathbf{\Lambda}_{ab}$ through equation (B.6), then $(-^\star X^a)$ is the Killing vector field on $\mathscr{H}$ corresponding to the 'dual' Lorentz transformation $*\mathbf{\Lambda}_{ab} := \frac{1}{2}\varepsilon_{ab}{}^{cd}\mathbf{\Lambda}_{cd}$.

B.2. Symmetric tensors

Let $\boldsymbol{T}_{ab}$ be any symmetric tensor on $\mathscr{H}$. Then $\boldsymbol{T}_{ab}$, its curl and divergence are related by the identity

$$-2\boldsymbol{T}_{ab}{}^\star X^b + 2\varepsilon_{cd(a}\boldsymbol{D}^c\boldsymbol{T}^d{}_{b)}X^b - \boldsymbol{D}_c\boldsymbol{T}^{cb}\boldsymbol{D}_a{}^\star X^b = \boldsymbol{D}^b\left(\varepsilon_{abc}\boldsymbol{T}^c{}_d X^d + 2\boldsymbol{T}^c{}_{[a}\boldsymbol{D}_{b]}{}^\star X^c\right), \tag{B.8}$$

where X^a is any Killing vector on $\mathscr{H}$ and $^\star X^a$ is the corresponding 'dual' Killing vector (equation (B.4)). This identity can be verified by expanding out the right-hand side and using equations (B.1), (B.4), and (B.5). Note that the right-hand side of equation (B.8) corresponds to an exact two-form on $\mathscr{H}$, and thus vanishes when integrated over any cross-section S of $\mathscr{H}$. This gives the following useful integral identity on any cross-section S

$$\int_S \varepsilon_2 \; \boldsymbol{u}^a\boldsymbol{T}_{ab}{}^\star X^b = \int_S \varepsilon_2 \; \boldsymbol{u}^a\left[\varepsilon_{cd(a}\boldsymbol{D}^c\boldsymbol{T}^d{}_{b)}X^b - \frac{1}{2}\boldsymbol{D}_c\boldsymbol{T}^{cb}\boldsymbol{D}_a{}^\star X^b\right]. \tag{B.9}$$

In the following lemma we show that any symmetric, curl-free tensor on $\mathscr{H}$ admits a scalar potential. A proof using a choice of coordinates on $\mathscr{H}$ can be found in appendix A of [32]. Our proof below is adapted from similar arguments for a two-sphere in appendix A.4 of [4].

Lemma B.1. *Let $\boldsymbol{T}_{ab}$ be a symmetric tensor on $\mathscr{H}$ with vanishing curl, i.e, $\boldsymbol{D}_{[c}\boldsymbol{T}_{a]b} = 0$ then there exists a function $\boldsymbol{t}$ on $\mathscr{H}$ such that*

$$\boldsymbol{T}_{ab} = \boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{t} + \boldsymbol{h}_{ab}\boldsymbol{t}. \tag{B.10}$$

Proof. Let $\boldsymbol{f} \in \mathfrak{t}$ be a Spi-translation so that[13]

$$\boldsymbol{D}_a\boldsymbol{D}_b\boldsymbol{f} + \boldsymbol{h}_{ab}\boldsymbol{f} = 0. \tag{B.11}$$

Note that the vector field $\boldsymbol{Y}^a := \boldsymbol{D}^a\boldsymbol{f}$ is a conformal Killing field on $\mathscr{H}$. Any conformal Killing field is completely determined by its *conformal Killing data* specified at some chosen point $p \in \mathscr{H}$ [55], which in this case is given by

$$(\boldsymbol{Y}^a, \boldsymbol{D}_{[a}\boldsymbol{Y}_{b]}, \boldsymbol{D}_a\boldsymbol{Y}^a, \boldsymbol{D}^a\boldsymbol{D}_b\boldsymbol{Y}^b)\big|_p = (\boldsymbol{D}^a\boldsymbol{f}, 0, -3\boldsymbol{f}, -3\boldsymbol{D}^a\boldsymbol{f})\big|_p. \tag{B.12}$$

Thus, there is an isomorphism between the vector space of $\boldsymbol{f} \in \mathfrak{t}$ and the vector space of the conformal Killing data $\boldsymbol{f}|_p$ and $\boldsymbol{D}^a\boldsymbol{f}|_p$ at any chosen point p.

---

[13] As shown in remark 6.1 Spi-translations can also be represented as vectors in the tangent space at i^0.

Since T_{ab} is symmetric and curl-free, using equation (B.11) we have $D_{[c}(T_{a]b}D^b f) = 0$. Thus, $T_{ab}D^b f$ is a closed one-form on $\mathscr{H}$ and thus exact,[14] that is, there exists a function H such that

$$T_{ab}D^b f = D_a H. \tag{B.13}$$

Thus, T_{ab} can be viewed as a linear map from the vector space of Spi-translations to functions on $\mathscr{H}$. Since the vector space of Spi-translations is isomorphic to the space of conformal Killing data equation (B.12) specified at any point on $\mathscr{H}$, there exists a function t and a covector field t_a on $\mathscr{H}$ such that

$$H = tf + t_a D^a f. \tag{B.14}$$

Inserting this into equation (B.13) and using equation (B.11) we get

$$T_{ab}D^b f = (D_a t_b + h_{ab}t)D^b f + f(D_a t - t_a). \tag{B.15}$$

Since the conformal Killing data $f|_p$ and $D^a f|_p$ can be freely specified at any point it follows that $t_a = D_a t$ and

$$T_{ab} = D_a D_b t + h_{ab}t. \tag{B.16}$$

$$\square$$

Note that the potential t is not uniquely determined, since one is free to add solutions of equation (B.11) to t without affecting the tensor T_{ab}. Further, the potential is *not* locally and covariantly determined by T_{ab} and finitely many of its derivatives. In particular, even if T_{ab} is the (direction-dependent) limit to i^0 of some tensor field on spacetime, there may not exist any tensor on spacetime whose limit gives the potential t.

B.3. Closed and exact forms

For some results in the main paper we need to argue that certain two-forms on $\mathscr{H}$ which are closed are also exact, so that their integral on cross-sections of $\mathscr{H}$ vanishes. In general, not all closed two-forms on $\mathscr{H}$ are exact since the topology of $\mathscr{H}$ is $\mathbb{S}^2 \times \mathbb{R}$ and the second de Rahm cohomology group is nontrivial. However, when the closed two-forms considered are local and covariant functionals of suitable fields (as described below) then they can be shown to be exact by a general theorem of Wald [56].

In the theorem stated below, the differential forms $\mu[\phi, \psi]$ under consideration will be functionals of two types of fields. The 'dynamical fields', denoted by ϕ, are arbitrary cross-sections of some vector bundle, and we require that $d\mu = 0$ for every cross-section ϕ. The form μ also can depend on some 'background fields', denoted by ψ. The 'background fields' ψ need not have a linear structure and are allowed to satisfy (possibly nonlinear) differential equations. Now we can state the theorem from [56].

Theorem 1 ([56]). *Let $\mu[\phi, \psi]$ be a p-form on a d-dimensional manifold M with $p < d$, which is a local and covariant functional of a collection of two sets of fields (ϕ, ψ) (as described above) and finitely many of their derivatives on M. Then, if for any 'background fields' ψ*

---

[14] This follows from the fact that every one-loop in $\mathscr{H}$ is contractible to a point and hence the first de Rahm cohomology group of $\mathscr{H}$ is trivial.

(a) $\mathrm{d}\mu[\phi, \psi] = 0$ *for all cross-sections of the vector bundle of 'dynamical fields' ϕ and*

(b) $\mu[\phi, \psi] = 0$ *for the zero cross-section $\phi = 0$*

　　then there exists a $(p-1)$-form $\nu[\phi, \psi]$ which is a local and covariant functional of (ϕ, ψ) and finitely many of their derivatives such that $\mu[\phi, \psi] = \mathrm{d}\nu[\phi, \psi]$. That is the closed p-form μ is also exact.

　　Note that it is essential for this theorem that the 'dynamical fields' have a linear structure as the cross-sections of some vector bundle and further, the p-form μ must be closed for all possible cross-sections of this vector bundle, i.e., one must be able to freely specify the 'dynamical fields' and all of their derivatives at any point of M. In contrast, the 'background fields' ψ, need not have a linear structure and are allowed to satisfy differential equations, and in fact the set of 'background fields' can also be empty. Further, the proof in [56] also provides a constructive procedure for finding the $(p-1)$-form ν though we will not need to use this construction.

　　For our applications of this theorem we will be concerned with closed two-forms on $\mathscr{H}$. Using the volume element ε_{abc} on $\mathscr{H}$, we will write this two-form in terms of a covector $\boldsymbol{\mu}_a$ such that $\boldsymbol{D}^a \boldsymbol{\mu}_a = 0$. Then, from theorem 1 we conclude that this two-form is exact and thus

$$\boldsymbol{D}^a \boldsymbol{\mu}_a[\phi, \psi] = 0 \implies \int_S \varepsilon_2 \, \boldsymbol{u}^a \boldsymbol{\mu}_a[\phi, \psi] = 0, \tag{B.17}$$

for any cross-section S of $\mathscr{H}$ with ε_2 and $\boldsymbol{u}^a$ being the area element and normal to S. The choice of the 'dynamical fields' ϕ depends on the particular case. Since the fields $\boldsymbol{E}$, $\boldsymbol{K}_{ab}$ and $\boldsymbol{\beta}_{ab}$ satisfy differential equations of motion (equations (4.12), (4.13), and (4.20)) they cannot be used as the 'dynamical fields'. Similarly, the Lorentz vector fields $\boldsymbol{X}^a$ form a six-dimensional vector space and cannot be arbitrary sections of some vector bundle and also cannot be used as the 'dynamical fields'. Thus, these fields, along with the metric and volume form on $\mathscr{H}$, will always be in the collection of 'background fields' ψ.

　　However, the supertranslation symmetries $\boldsymbol{f}$, the freedom in the conformal factor $\boldsymbol{\alpha}$ (remark 2.1) and the scalar potential $\boldsymbol{k}$ for $\boldsymbol{K}_{ab}$ (when $\boldsymbol{B}_{ab} = 0$) are free functions on $\mathscr{H}$ and will be used as 'dynamical fields' in our applications of this theorem.

Appendix C. Conformal transformation of the Lorentz charges

In section 7.3 we argued that under conformal transformations the Lorentz charge shifts by the charge of a supertranslation (equation (7.27)). In this appendix we collect the explicit computation of this transformation.

　　Using equations (4.22) and (4.23), and that $\boldsymbol{E}$ is conformally-invariant, we have the following transformation for the tensor $\boldsymbol{W}_{ab}$ defined in equation (7.10) under changes of the conformal factor

$$\boldsymbol{W}_{ab} \mapsto \boldsymbol{W}_{ab} + \frac{1}{4}\varepsilon_{cd(a}\boldsymbol{D}^c \left[\boldsymbol{D}_{b)}\boldsymbol{E}\boldsymbol{D}^d\boldsymbol{\alpha} + \boldsymbol{D}^d\boldsymbol{E}\boldsymbol{D}_{b)}\boldsymbol{\alpha} \right] + \frac{1}{8}\varepsilon_{abc}\boldsymbol{D}^c\boldsymbol{E}(\boldsymbol{D}^2\boldsymbol{\alpha} + 3\boldsymbol{\alpha}). \tag{C.1}$$

Thus, we have (note that the Lorentz vector does not transform under changes of the conformal factor remark 6.2)

$$\boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b \mapsto \boldsymbol{W}_{ab}{}^{\star}\boldsymbol{X}^b + \frac{1}{4}\varepsilon_{cd(a}\boldsymbol{D}^c\boldsymbol{T}^d{}_{b)}{}^{\star}\boldsymbol{X}^b + \frac{1}{8}(\boldsymbol{D}^2\boldsymbol{\alpha} + 3\boldsymbol{\alpha})\boldsymbol{D}^b\boldsymbol{E}\boldsymbol{D}_a\boldsymbol{X}_b, \tag{C.2}$$

where we have defined the shorthand $T_{ab} := D_a E D_b \alpha + D_b E D_a \alpha$ and used the last identity in equation (B.5). Now using the identity equation (B.9) (with X^a replaced by $^\star X^a$) we have

$$\int_S \varepsilon_2 \, u^a \varepsilon_{cd(a} D^c T^d{}_{b)}{}^\star X^b = -\int_S \varepsilon_2 \, u^a \left[\frac{1}{2} D_c T^{cb} D_a X_b + T_{ab} X^b \right]. \tag{C.3}$$

A straightforward but tedious computation using the definition of T_{ab}, equations (4.12), (B.1), and (B.3) gives

$$\int_S \varepsilon_2 \, u^a \varepsilon_{cd(a} D^c T^d{}_{b)}{}^\star X^b = \int_S \varepsilon_2 \, u^a \left[-\frac{1}{2} (D^2 \alpha + 3\alpha)(2 E X_a + D^b E D_a X_b) \right.$$

$$\left. + (E D_a \pounds_X \alpha - D_a E \pounds_X \alpha) \right], \tag{C.4}$$

where we have dropped terms that integrate to zero on S. Using the above in equation (C.2) we get

$$\int_S \varepsilon_2 \, u^a W_{ab}{}^\star X^b \mapsto \int_S \varepsilon_2 \, u^a W_{ab}{}^\star X^b + \frac{1}{4} \int_S \varepsilon_2 \, u^a \left[(E D_a \pounds_X \alpha - D_a E \pounds_X \alpha) \right.$$

$$\left. - (D^2 \alpha + 3\alpha) E X_a \right]. \tag{C.5}$$

Further, from equation (4.22) we also have

$$-\frac{1}{8} K E X_a \mapsto -\frac{1}{8} K E X_a + \frac{1}{4} (D^2 \alpha + 3\alpha) E X_a. \tag{C.6}$$

Thus,

$$\int_S \varepsilon_2 \, u^a \left[W_{ab}{}^\star X^b - \frac{1}{8} K E X_a \right] \mapsto \int_S \varepsilon_2 \, u^a \left[W_{ab}{}^\star X^b - \frac{1}{8} K E X_a \right]$$

$$+ \frac{1}{4} \int_S \varepsilon_2 \, u^a \left(E D_a \pounds_X \alpha - D_a E \pounds_X \alpha \right). \tag{C.7}$$

The Lorentz charge equation (7.20) then transforms as

$$\mathcal{Q}[X^a; S] \mapsto \mathcal{Q}[X^a; S] - \frac{1}{16\pi} \int_S \varepsilon_2 \, u^a \frac{1}{2} \left(E D_a \pounds_X \alpha - D_a E \pounds_X \alpha \right). \tag{C.8}$$

Comparing to equation (7.4), we recognize the last integral above as the charge of the supertranslation $(-^1/_2 \pounds_X \alpha)$. Thus, the Lorentz charge shifts by the charge of a supertranslation under changes of the conformal factor as argued in section 7.3.

Appendix D. Ambiguities in the Spi-charges

In this section we analyze the ambiguities in our procedure to define the Spi charges. We show our Spi charges are unambiguously defined by the choice of the symplectic current for general relativity in equations (5.14) and (5.15).

Recall that our charges on a cross-section S of $\mathscr{H}$ are defined by

$$\delta\mathcal{Q}[(\boldsymbol{f},X^a);S] := \int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{Q}_a(\delta g;(\boldsymbol{f},X^a)) + \int_S X \cdot \boldsymbol{\Theta}(\delta g), \tag{D.1}$$

with $\mathcal{Q} = 0$ on Minkowski spacetime as the reference solution. The covector $\boldsymbol{Q}_a$ is a local and covariant functional of its arguments and linear in the metric perturbations and the asymptotic symmetry satisfying equation (7.2). While the three-form $\boldsymbol{\Theta}$ is a symplectic potential for $\underset{\leftarrow}{\omega}$ satisfying equation (7.16).

Given a fixed choice of the symplectic current, from equations (7.2) and (7.16) the ambiguities in the choice of $\boldsymbol{Q}_a$ and the $\boldsymbol{\Theta}$ are given by

$$\boldsymbol{Q}_a(g;\delta g;(\boldsymbol{f},X)) \mapsto \boldsymbol{Q}_a(g;\delta g;(\boldsymbol{f},X)) + \boldsymbol{\mu}_a(g;\delta g;(\boldsymbol{f},X)),$$
$$\boldsymbol{\Theta}(\delta g) \mapsto \boldsymbol{\Theta}(\delta g) + \boldsymbol{\varepsilon}_3 \delta\boldsymbol{\Xi}(g), \tag{D.2}$$

where the covector $\boldsymbol{\mu}_a(g;\delta g;(\boldsymbol{f},X))$ is a local and covariant functional of its arguments and linear in the metric perturbations and the asymptotic symmetry, and further satisfies

$$\boldsymbol{D}^a \boldsymbol{\mu}_a(g;\delta g;(\boldsymbol{f},X)) = 0, \tag{D.3}$$

for all background spacetimes and perturbations (satisfying the background and linearized equations of motion respectively) and all asymptotic symmetries. While the function $\boldsymbol{\Xi}$ is any local and covariant function of the background spacetime fields on $\mathscr{H}$.

Under these ambiguities the definition of $\delta\mathcal{Q}$ (equation (D.1)) changes by

$$\delta\mathcal{Q}[(\boldsymbol{f},X^a);S] \mapsto \delta\mathcal{Q}[(\boldsymbol{f},X^a);S] + \int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a \boldsymbol{\mu}_a(g;\delta g;(\boldsymbol{f},X)) - \delta\int_S \boldsymbol{\varepsilon}_2 \boldsymbol{u}^a X_a \boldsymbol{\Xi}(g). \tag{D.4}$$

Since the integrated charge $\mathcal{Q}$ is fixed by the requirement that it vanish on Minkowski spacetime (where $\boldsymbol{E} = \boldsymbol{\beta}_{ab} = 0$), we only need to analyze the ambiguities in $\delta\mathcal{Q}$.

We now argue that the last two integrals above must vanish under the following assumptions

(a) $\boldsymbol{\mu}_a$ and $\boldsymbol{\Xi}$ are local and covariant functionals of their arguments as mentioned above with $\boldsymbol{\mu}_a$ satisfying equation (D.3).
(b) The Lorentz charge $\mathcal{Q}[(\boldsymbol{f}=0,X^a);S]$ must match the Ashtekar–Hansen expression when the conformal factor is chosen such that $\boldsymbol{K}_{ab} = 0$.
(c) The total charge $\mathcal{Q}[(\boldsymbol{f},X^a);S]$ of any Spi symmetry is conformally-invariant.

Consider first the $\boldsymbol{\mu}_a$-ambiguity and the case of a pure supertranslation $(\boldsymbol{f},X^a = 0)$. Since the ambiguity $\boldsymbol{\mu}_a$ is linear in $\boldsymbol{f}$ we have $\boldsymbol{\mu}_a(g;\delta g;\boldsymbol{f}=0) = 0$. Further since $\boldsymbol{\mu}_a$ is divergence-free (equation (D.3)), we can use theorem 1 in the form equation (B.17) with $\boldsymbol{f}$ as 'dynamical field' to conclude that the second integral on the right-hand side of equation (D.4) vanishes on any cross-section S for a supertranslation.

Next consider the $\boldsymbol{\mu}_a$-ambiguity with a Lorentz transformation $(\boldsymbol{f}=0,X^a)$. Since the Lorentz vector fields X^a form a six-dimensional vector space and are not allowed to be arbitrary cross-sections of a vector bundle, we cannot use X^a as the 'dynamical' fields in theorem 1. So instead, we proceed another in another way. Consider the scalar potential $\boldsymbol{k}$ for the tensor $\boldsymbol{K}_{ab}$ (equation (4.16)). Since $\boldsymbol{k}$ is a completely free function on $\mathscr{H}$ it is allowed to be an arbitrary cross-section of a vector bundle on $\mathscr{H}$. Further, whenever $\boldsymbol{k} = 0$ we have $\boldsymbol{K}_{ab} = 0$ and by our assumption the Lorentz charge must the one found by Ashtekar and Hansen. Thus, the ambiguity $\boldsymbol{\mu}_a = 0$ whenever $\boldsymbol{k} = 0$ for all background spacetimes and all Lorentz vector

fields X^a. Now using k as the 'dynamical field', from theorem 1 in the form equation (B.17), we conclude again that the second integral on the right-hand side of equation (D.4) vanishes on any cross-section S for a Lorentz symmetry. Thus, the μ_a-ambiguity does not affect $\delta\mathcal{Q}$.

Finally, consider the Ξ-ambiguity in the choice of Θ. In section 7.3 we showed that the total charge $\mathcal{Q}$ for any Spi-symmetry (f, X^a) is invariant under conformal transformations with our choice of Θ (equation (7.18)) which implies that the charge of a 'pure Lorentz' symmetry must shift by a charge of a supertranslation under changes of the conformal factor (see equation (7.27)). It follows that for the redefined Lorentz charge to transform correctly the integral contributed by Ξ in equation (D.4) must be conformally-invariant. Further, for the redefined Lorentz charge to match the one found by Ashtekar and Hansen the integral contributed by Ξ in equation (D.4) must vanish whenever $K_{ab} = 0$. Since K_{ab} can be chosen to vanish by a choice of conformal factor (see remark 6.3) this implies the Ξ-ambiguity does not affect $\delta\mathcal{Q}$.

In summary, our charges are unambiguously determined by the pullback of the symplectic current equation (5.15).

Here we remark that the symplectic current three-form itself is *not* uniquely determined by the Lagrangian of the theory but is ambiguous up to

$$\omega(g; \delta_1 g, \delta_2 g) \mapsto \omega(g; \delta_1 g, \delta_2 g) + \mathrm{d}\left[\delta_1 \nu(g; \delta_2 g) - \delta_2 \nu(g; \delta_1 g)\right], \tag{D.5}$$

where $\nu(g; \delta g)$ is a local and covariant two-form and is linear in the perturbation δg. We have not analyzed the effect of this ambiguity on our charges.

Appendix E. Lorentz charges with $B_{ab} \neq 0$

In section 7.2 to define the Lorentz charges at i^0 we imposed the condition $B_{ab} = 0$ to gain access to the 'subleading' magnetic part β_{ab} of the asymptotic Weyl tensor (see equation (4.14)). In this section we show how we can define a 'subleading' magnetic Weyl tensor and the Lorentz charges even when $B_{ab} \neq 0$.

If B_{ab} does not vanish, then the 'subleading' piece as defined by equation (4.14) does not exist in the limit. However, consider the derivative of the magnetic part of the Weyl tensor along η^a:

$$\lim_{\to i^0} \Omega^{1/2} \eta^e \nabla_e (\Omega^{1/2} * C_{abcd} \eta^c \eta^d) = \eta^e \partial_e B_{ab} = 0. \tag{E.1}$$

Since the limit of the above quantity vanishes we can now demand that its 'next order' part exist, that is,

$$H_{ab}(\vec{\eta}) := \lim_{\to i^0} \eta^e \nabla_e (\Omega^{1/2} * C_{abcd} \eta^c \eta^d) \text{is } C^{>-1}. \tag{E.2}$$

The tensor field $H_{ab}(\vec{\eta})$ is not tangential to $\mathscr{H}$. We can compute

$$H_{ab}(\vec{\eta})\eta^b = \lim_{\to i^0} \eta^b \eta^e \nabla_e (\Omega^{1/2} * C_{abcd} \eta^c \eta^d) = -\lim_{\to i^0} \eta^e \nabla_e \eta^b (\Omega^{1/2} * C_{abcd} \eta^c \eta^d)$$

$$= \frac{1}{4} B_{ab} D^b E, \tag{E.3}$$

where in the first line we have used the fact that $*C_{abcd}$ is antisymmetric in the last two indices and to get the second line we replaced the derivative of η^a using the Einstein equation

equation (4.1), and used equations (4.5) and (4.10). Note that $\boldsymbol{H}_{ab}(\vec{\eta})\boldsymbol{\eta}^a\boldsymbol{\eta}^b = 0$, and thus the only remaining part of $\boldsymbol{H}_{ab}$ is its projection to $\mathscr{H}$ on both indices. We use this projection to define the 'subleading' magnetic part of the Weyl tensor, that is, instead of equation (4.14) we now use

$$\boldsymbol{\beta}_{ab} := \boldsymbol{h}_a{}^c \boldsymbol{h}_b{}^d \boldsymbol{H}_{cd}(\vec{\eta}). \tag{E.4}$$

As before $\boldsymbol{\beta}_{ab}$ is a symmetric and traceless tensor field on $\mathscr{H}$. Note that when $\boldsymbol{B}_{ab} = 0$, this new definition is completely equivalent to the previous one in equation (4.14) (see also [26]).

The generalization of the equation of motion equation (4.20) is rather tedious to obtain. We want to compute

$$
\begin{aligned}
\partial^b \boldsymbol{H}_{ab} &= \lim_{\to i^0} \Omega^{1/2} \nabla^b \left[\eta^e \nabla_e (\Omega^{1/2} * \boldsymbol{C}_{abcd} \eta^c \eta^d) \right] \\
&= \lim_{\to i^0} \left[(\nabla^b \eta^e) \Omega^{1/2} \nabla_e (\Omega^{1/2} * \boldsymbol{C}_{abcd} \eta^c \eta^d) + \Omega^{1/2} \eta^e \nabla^b \nabla_e (\Omega^{1/2} * \boldsymbol{C}_{abcd} \eta^c \eta^d) \right].
\end{aligned}
\tag{E.5}
$$

In the first term we substitute the derivative of η^a using equation (4.1) and then evaluate the limit of the expression using equations (4.6), (4.9), (4.10) and (E.3). For the second term on the right-hand side, we first commute the derivatives and introduce terms involving the the Riemann tensor of the unphysical spacetime. The term with the derivatives ∇^b and ∇_e interchanged vanishes in the limit while the Riemann tensor terms can be computed by decomposing the Riemann tensor in terms of the Weyl tensor C_{abcd} and S_{ab} (equation (4.2)). Then we can evaluate the limit using equations (4.6), (4.9), and (4.10). The final limit gives the equation

$$
\begin{aligned}
\partial^b \boldsymbol{H}_{ab} = &-\frac{1}{4}\partial_c \boldsymbol{B}_{ab}\boldsymbol{K}^{bc} - \frac{1}{4}\partial^b \boldsymbol{B}_{ab}\boldsymbol{E} + \frac{5}{4}\boldsymbol{B}_{ab}\boldsymbol{D}^b\boldsymbol{E} + \frac{1}{4}\varepsilon_{cda}\boldsymbol{E}^c{}_b\boldsymbol{K}^{db} - \frac{1}{4}\boldsymbol{\eta}_a \boldsymbol{B}_{bc}\boldsymbol{K}^{bc} \\
&- \boldsymbol{\eta}_a \boldsymbol{B}_{bc}\boldsymbol{E}^{bc}.
\end{aligned}
\tag{E.6}
$$

Using equation (E.3) and the equation of motion equation (4.7) it can be verified that the contraction of the above equation with $\boldsymbol{\eta}^a$ is trivial. Projecting the index a on to $\mathscr{H}$ we then get the equation of motion for $\boldsymbol{\beta}_{ab}$ as

$$\boldsymbol{D}^b \boldsymbol{\beta}_{ab} = \frac{1}{4}\varepsilon_{cda}\boldsymbol{E}^c{}_b\boldsymbol{K}^{bd} + \frac{5}{4}\boldsymbol{B}_{ab}\boldsymbol{D}^b\boldsymbol{E} - \frac{1}{4}\boldsymbol{D}_a \boldsymbol{B}_{bc}\boldsymbol{K}^{bc}, \tag{E.7}$$

which reduces to equation (4.20) when $\boldsymbol{B}_{ab} = 0$.

To define the Lorentz charge we now construct the generalization of the tensor $\boldsymbol{W}_{ab}$ (equation (7.10)). Note that the only essential properties of $\boldsymbol{W}_{ab}$ used to obtain equation (7.13) are that $\boldsymbol{W}_{[ab]} = -\frac{1}{16}\varepsilon_{abc}\boldsymbol{K}\boldsymbol{D}^c\boldsymbol{E}$ and $\boldsymbol{D}^a\boldsymbol{W}_{ab} = 0$ using the equation of motion for $\boldsymbol{\beta}_{ab}$. We will further require that $\boldsymbol{W}_{ab}$ is also traceless.

To find such a $\boldsymbol{W}_{ab}$, first note that the last term in equation (E.7) can be written as the divergence of a symmetric tensor using equations (4.8), (4.11), and (4.13)

$$-\frac{1}{4}\boldsymbol{D}_b \boldsymbol{B}_{ac}\boldsymbol{K}^{ac} = -\frac{1}{16}\boldsymbol{D}^a \left[-2\boldsymbol{B}_{ab}\boldsymbol{K} + 2\boldsymbol{h}_{ab}\boldsymbol{B}_{cd}\boldsymbol{K}^{cd} - \varepsilon_{cd(a}\boldsymbol{K}^c{}_{b)}\boldsymbol{D}^d\boldsymbol{K} - \varepsilon_{cd(a}\boldsymbol{D}_{b)}\boldsymbol{K}^{ce}\boldsymbol{K}^d{}_e \right]. \tag{E.8}$$

Note that the tensor in the square brackets is *not* traceless. However, we can add to it the following symmetric tensor

$$-\frac{5}{8}\left[2\boldsymbol{B}_{c(a}\boldsymbol{K}_{b)}^{c} - h_{ab}\boldsymbol{B}_{cd}\boldsymbol{K}^{cd} - \boldsymbol{B}_{ab}\boldsymbol{K}\right],\tag{E.9}$$

which has vanishing divergence and thus does not affect the left-hand side. With this we define

$$\begin{aligned}\boldsymbol{W}_{ab} := \boldsymbol{\beta}_{ab} &+ \frac{1}{8}\varepsilon_{cd(a}\boldsymbol{D}^{c}\boldsymbol{E}\boldsymbol{K}^{d}{}_{b)} - \frac{1}{16}\varepsilon_{abc}\boldsymbol{K}\boldsymbol{D}^{c}\boldsymbol{E}\\ &- \frac{3}{2}\boldsymbol{B}_{ab}\boldsymbol{E} + \frac{5}{4}\boldsymbol{B}_{c(a}\boldsymbol{K}_{b)}^{c} - \frac{1}{2}h_{ab}\boldsymbol{B}_{cd}\boldsymbol{K}^{cd} - \frac{3}{4}\boldsymbol{B}_{ab}\boldsymbol{K}\\ &- \frac{1}{16}\varepsilon_{cd(a}\boldsymbol{D}_{b)}\boldsymbol{K}^{ce}\boldsymbol{K}^{d}{}_{e} - \frac{1}{16}\varepsilon_{cd(a}\boldsymbol{K}^{c}{}_{b)}\boldsymbol{D}^{d}\boldsymbol{K},\end{aligned}\tag{E.10}$$

which satisfies

$$\boldsymbol{W}_{[ab]} = -\frac{1}{16}\varepsilon_{abc}\boldsymbol{K}\boldsymbol{D}^{c}\boldsymbol{E}\,,\quad \boldsymbol{D}^{a}\boldsymbol{W}_{ab} = 0\,,\quad h^{ab}\boldsymbol{W}_{ab} = 0.\tag{E.11}$$

Then the Lorentz charge formula takes the same form as in equation (7.20) with $\boldsymbol{W}_{ab}$ now defined as in equation (E.10). The flux of this charge is still given by the expression equation (7.21).

Note that when $\boldsymbol{B}_{ab} = 0$, the second line in equation (E.10) vanishes, but the terms in the third line are nonvanishing in general; denote these terms by a symmetric tensor $\boldsymbol{T}_{ab}$. It follows from equation (E.8) that $\boldsymbol{T}_{ab}$ is divergence-free when $\boldsymbol{B}_{ab} = 0$. Thus $\boldsymbol{D}^{a}(\boldsymbol{T}_{ab}^{\star}\boldsymbol{X}^{b}) = 0$ and $\boldsymbol{T}_{ab}^{\star}\boldsymbol{X}^{b} = 0$ when the scalar potential $\boldsymbol{k}$ for $\boldsymbol{K}_{ab}$ (equation (4.16)) vanishes. Using the scalar potential $\boldsymbol{k}$ as the 'dynamical field' in theorem 1 it follows from equation (B.17) that these terms do not contribute to the Lorentz charge expression. Thus, when $\boldsymbol{B}_{ab} = 0$ the Lorentz charge defined using equation (E.10) coincides with the one defined previously in section 7.2.

Under conformal transformations we can show that

$$\boldsymbol{\beta}_{ab} \mapsto \boldsymbol{\beta}_{ab} - \varepsilon_{cd(a}\boldsymbol{E}^{c}{}_{b)}\boldsymbol{D}^{d}\boldsymbol{\alpha} - \frac{3}{2}\boldsymbol{B}_{ab}\boldsymbol{\alpha} + \frac{1}{2}\boldsymbol{D}_{c}\boldsymbol{B}_{ab}\boldsymbol{D}^{c}\boldsymbol{\alpha},\tag{E.12}$$

and that equation (E.7) is invariant. The explicit computation of the transformation of the Lorentz charge presented in appendix C now becomes much more complicated. However, the general argument presented in section 7.3 still holds. Thus, even without the assumption $\boldsymbol{B}_{ab} = 0$ we have a satisfactory definition of Lorentz charges at spatial infinity.

The Lorentz charges for $\boldsymbol{B}_{ab} \neq 0$ case were also derived by Compère and Dehouck [33] (with $\boldsymbol{K} = 0$) using an asymptotic expansion in Beig–Schmidt coordinates which in the unphysical spacetime coordinates used in appendix A reads

$$\begin{aligned}g_{ab} \equiv &[1 + \sigma\rho + o(\rho)]^{2}\mathrm{d}\rho^{2} + \rho\,o(\rho)\mathrm{d}\rho\mathrm{d}y^{a}\\ &+ \rho^{2}\left[h_{ab}^{(0)} + \rho h_{ab}^{(1)} - \rho^{2}\,\ln\,\rho i_{ab} + \rho^{2}h_{ab}^{(2)} + o(\rho^{2})\right]\mathrm{d}y^{a}\mathrm{d}y^{b}.\end{aligned}\tag{E.13}$$

For $\boldsymbol{\beta}_{ab}$, as defined by equation (E.4), to exist we set the logarithmic term $i_{ab} = 0$. With this condition the $\boldsymbol{\beta}_{ab}$ is related to the curl of the metric coefficient $h_{ab}^{(2)}$ with additional terms whose

form is rather complicated (as compared to equation (A.16) when $\boldsymbol{B}_{ab} = 0$). Note that with $\boldsymbol{K} = 0$, our $\boldsymbol{W}_{ab}$ is a symmetric, divergence-free and traceless tensor and thus we expect that our charge expression in this case matches with the one derived in [33] in terms of $h_{ab}^{(2)}$, but we have not shown this explicitly.

When the logarithmic term i_{ab} does not vanish, our definition equation (E.4) cannot be used for the 'subleading' magnetic part of the Weyl tensor. We have not explored this case in detail but we expect the following strategy to be useful. We can assume that

$$\Omega^{1/2} * \boldsymbol{C}_{abcd}\eta^c\eta^d = B_{ab} + \Omega^{1/2}\ \ln\ \Omega^{1/2}b_{ab} + \Omega^{1/2}\beta_{ab} + o(\Omega^{1/2}), \qquad \text{(E.14)}$$

where each of the tensors B_{ab}, b_{ab} and β_{ab} are symmetric and orthogonal to η^a and admit a $C^{>-1}$ limit to i^0. Using such an expansion in the Hodge dual of equation (4.3a) we can derive the equations of motion for the limits of B_{ab}, b_{ab} and β_{ab}. Since the expression for the symplectic current equation (7.9) is unchanged, we can use these equations of motion to define an analogue of the tensor $\boldsymbol{W}_{ab}$ and the Lorentz charges. From the point of view of matching these charges to those on null infinity, we expect that the spacetimes with such a logarithmic behavior at spatial infinity would correspond to the polyhomogenous spacetimes at null infinity defined in [57].

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