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HESSENBERG VARIETIES, INTERSECTIONS OF QUADRICS, AND THE SPRINGER CORRESPONDENCE

TSAO-HSIEN CHEN, KARI VILONEN, AND TING XUE

Abstract. In this paper we introduce a certain class of families of Hessenberg varieties arising from Springer theory for symmetric spaces. We study the geometry of those Hessenberg varieties and investigate their monodromy representations in detail using the geometry of complete intersections of quadrics. We obtain decompositions of these monodromy representations into irreducibles and compute the Fourier transforms of the IC complexes associated to these irreducible representations. The results of the paper refine (part of) the Springer correspondece for the split symmetric pair $(SL(N), SO(N))$ in [Compos. Math. 154 (2018), pp. 2403–2425].

Contents

1. Introduction

In this paper we study the geometry of Hessenberg varieties of [GKM] arising from Springer theory for symmetric spaces [CVX1, CVX2]. Let us recall the set-up. Let G be a reductive group and let θ be an involution of G. We write $K = (G^\theta)^0$ for the connected component of the fixed point set. This gives rise to a symmetric pair (G, K). We also have the corresponding decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is the fixed point set and $\mathfrak{g}_1$ is the (-1)-eigenspace of θ, respectively. We write $\mathcal{N}_1 = \mathcal{N} \cap \mathfrak{g}_1$ where $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$. Let us call an irreducible K-equivaraint IC-sheaf[1] supported on the nilpotent cone $\mathcal{N}_1$ a *nilpotent*

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[1]IC=intersection cohomology.

orbital complex. We address the following question which can be regarded as an analogue of the classical Springer correspondence: what are the Fourier transforms of nilpotent orbital complexes?

We work in the context of the split symmetric pair $(G, K) = (SL(N), SO(N))$ where $K = G^\theta$ is given by an involution $\theta : G \to G$ and N is odd. Recall that the K-orbits in $\mathcal{N}_1$ are parametrized by partitions of N. In [CVX2], we show that the Fourier transform gives a bijection between nilpotent orbital complexes and certain representations of braid groups. We identify these representations of braid groups and constructed them explicitly using representations of Hecke algebra of symmetric groups at $q = -1$. This bijection can be viewed as Springer correspondence for the symmetric pair $(SL(N), SO(N))$. In the course of the proof we give a construction of nilpotent orbital complexes with full support Fourier transforms, following Lusztig we call them *thick* nilpotent orbital complexes, using Grinberg's nearby cycle sheaves.

The goal of this paper is to use the geometry of Hessenberg varieties to reach a better understanding of Fourier transforms of nilpotent orbital complexes and Springer correspondence for the symmetric pair $(SL(N), SO(N))$. We concentrate on the thick nilpotent orbital complexes because in [CVX2, Corollary 4.8] we show that one can obtain all nilpotent orbital complexes by induction from thick nilpotent orbital complexes of smaller groups.

To this end we proposed in [CVX1] a general method of analyzing Fourier transforms of nilpotent orbital complexes. We replace the Springer resolution and the Grothendieck simultaneous resolution of the classical Springer correspondence by (several) pairs of families of Hessenberg varieties $\mathcal{X}$ and $\check{\mathcal{X}}$ and obtain the following picture:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \check{\mathcal{X}} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\check{\pi}} \\ \mathcal{N}_1 & \longrightarrow & \mathfrak{g}_1. \end{array} \tag{1.1}$$

The image of π is a nilpotent orbit closure $\bar{\mathcal{O}}$ but neither π nor $\check{\pi}$ are semi-small in general. In fact, their generic fibers are not just points in general but they form smooth families of varieties. The key to analyzing the Fourier transforms of nilpotent orbital complexes in this manner is that the constant sheaf on $\check{\mathcal{X}}$ is the Fourier transform of the constant sheaf on $\mathcal{X}$. Thus, at least as a first approximation, we are reduced to decomposing the push-forwards $\pi_*\mathbb{C}_{\mathcal{X}}$ and $\check{\pi}_*\mathbb{C}_{\check{\mathcal{X}}}$ into direct sums of IC-sheaves; this is possible by the decomposition theorem. In [CVX1] we study the case when the thick nilpotent orbital complexes are supported on nilpotent orbits of order 2, i.e., orbits which correspond to partitions that only involve 2's and 1's, and we show that the relevant families of Hessenberg varieties are closely related to Jacobians of hyperelliptic curves (see [CVX1, §3]). In this paper we treat the case of orbits of order 3. It turns out that the relevant Hessenberg varieties are closely related to complete intersection of quadrics.

We now describe our results in more detail. Let us recall the definition of Hessenberg varieties in our setting following [GKM]. Let $x \in \mathfrak{g}_1$, let P be a parabolic subgroup of K, and let $\Sigma \subset \mathfrak{g}_1$ be a P-invariant subspace. The Hessenberg variety associated to the triple (x, P, Σ), denoted by $\mathrm{Hess}_x(K/P, \mathfrak{g}_1, \Sigma)$, is by definition the following variety:

$$\mathrm{Hess}_x(K/P, \mathfrak{g}_1, \Sigma) := \{g \in K/P \mid g^{-1}x \in \Sigma\}.$$

As x varies over $\mathfrak{g}_1$, we get a family of Hessenberg varieties $\mathrm{Hess}(K/P, \mathfrak{g}_1, \Sigma) \to \mathfrak{g}_1$.

The particular pairs of families of Hessenberg varieties we study here have the following properties. One of the families in the pair, when restricted to the regular semi-simple locus, is isomorphic to a family of complete intersections of quadrics (see Theorem 2.5); this is the family $\check{\pi} : \check{\mathcal{X}} \to \mathfrak{g}_1$ in (1.1). The other family, corresponding to $\pi : \mathcal{X} \to \mathcal{N}_1$ in (1.1), is supported on the locus of nilpotent elements of order at most 3 and the fibers of this family admit affine pavings (see §2.3).

The main results in this paper are in §§4 and 5. When restricted to the locus of regular semi-simple elements $\mathfrak{g}_1^{rs}$ our particular families of Hessenberg varieties can be interpreted as families of intersections of quadrics in projective spaces. In §4, we study the monodromy representations arising from the primitive cohomology of these families (and their natural double covers). This is accomplished by establishing a relative version of results of T. Terasoma [T] in §3. The resulting monodromy representations of $\pi_1^K(\mathfrak{g}_1^{rs})$ can be expressed in terms of monodromy representations of certain families of hyperelliptic curves over $\mathfrak{g}_1^{rs}$. In particular, we see that the cohomology of those Hessenberg varieties can be expressed in terms of cohomology of hyperelliptic curves. In Theorems 4.1 and 4.2 we describe these monodromy representations completely by decomposing them into irreducible pieces which we call E_{ij}^N and $\widetilde{E}_{ij}^N$, respectively. In §5, we study Fourier transforms of the IC complexes arising from the local systems E_{ij}^N and $\widetilde{E}_{ij}^N$. Recall that these were obtained from the primitive cohomology of the particular families of Hessenberg varieties and their double covers. We show that their Fourier transforms are supported on the closed sub-variety $\mathcal{N}_1^3 \subset \mathcal{N}_1$ consisting of nilpotent elements of order less than or equal to 3. In particular, we obtain various examples of thick nilpotent orbital complexes. Let $\{(\mathcal{O}, \mathcal{E})\}_{\leq 3}$ be the set of all pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O}$ is a K-orbit in $\mathcal{N}_1^3$ and $\mathcal{E}$ is an irreducible K-equivariant local system on $\mathcal{O}$ (up to isomorphism). In this manner we obtain an injective map

$$\{E_{ij}^N\} \cup \{\widetilde{E}_{ij}^N\} \hookrightarrow \{(\mathcal{O}, \mathcal{E})\}_{\leq 3}\,. \tag{1.2}$$

This map refines (part of) the Springer correspondence in [CVX2, Theorem 4.1]. We would like to emphasize that such a refinement of Springer correspondence is crucial for applications, for example, computing cohomologies of Fano varieties in [CVX3].

As an interesting corollary (see Example 5.5), we show that the Fourier transform of the IC complex for the unique non-trivial irreducible K-equivariant local system on the minimal nilpotent orbit has full support and the corresponding local system is given by the monodromy representation of the universal family of hyperelliptic curves of genus n, where $2n + 1 = N$.

The paper is organized as follows. In §2, we introduce certain pairs of families of Hessenberg varieties and prove basic facts about them. In §§3 and 4, we establish a relative version of the results of Terasoma [T]. We utilize these results to obtain a decomposition of the monodromy representations into irreducibles. In §5, using the results in previous sections, we show that Fourier transforms of the IC complexes for the local systems arising from our families of Hessenberg varieties and their double covers are supported on the closed sub-variety $\mathcal{N}_1^3 \subset \mathcal{N}_1$ consisting of nilpotent elements of order at most 3. In §6 we give a conjectural (explicit) description of the map in (1.2) (see Conjectures 6.1 and 6.3) for E_{ij}^N and we verify the conjectures in various examples.

2. Hessenberg varieties

In this section we introduce certain families of Hessenberg varieties which naturally arise when computing the Fourier transforms of IC complexes supported on nilpotent orbits of order less than or equal to 3, i.e., orbits of the form $\mathcal{O}_{3^i2^j1^k}$. Our main theorem (see Theorem 2.5) says that, generically, these families of Hessenberg varieties are isomorphic to families of complete intersections of quadrics.

2.1. **Hessenberg varieties.** Let G be a reductive group and let V be a representation of G. Let $P \subset G$ be a parabolic subgroup and let $\Sigma \subset V$ be a P-invariant subspace. Consider the vector bundle $G \times^P \Sigma$ and write

$$G \times^P \Sigma = \mathrm{Hess}(G/P, V, \Sigma) \to G/P. \tag{2.1}$$

The natural projection to V gives us a projective morphism

$$\mathrm{Hess}(G/P, V, \Sigma) \to V\,. \tag{2.2}$$

The fibers of this morphism are called *Hessenberg varieties*; the fiber over v is given by

$$\mathrm{Hess}_v(G/P, V, \Sigma) = \{gP \mid g^{-1}v \in \Sigma\}. \tag{2.3}$$

Consider now the situation when we have a connected reductive group G and an involution $\theta : G \to G$. Let $K = G^\theta$. The involution θ induces a grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ on the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid d\theta(x) = (-1)^i x\}$. The group K acts on $\mathfrak{g}_1$ by adjoint action. An element x_0 in $\mathfrak{g}_1$ is said to be regular if $\dim Z_K(x) \geq \dim Z_K(x_0)$ for all $x \in \mathfrak{g}_1$. We write $\mathfrak{g}_1^{rs}$ for the set of regular semi-simple elements in $\mathfrak{g}_1$.

Let T_K be a maximal torus of K and consider a co-character $\lambda : \mathbb{G}_m \to T_K$. We write $P = P(\lambda)$ for the parabolic subgroup of K associated to λ, $\mathfrak{p}$ for the Lie algebra of P, and $\mathfrak{g}_1 = \bigoplus \mathfrak{g}_{1,j}$ for the grading induced by λ. For any $i \in \mathbb{Z}$ we define $\mathfrak{g}_{1,\geq i} = \bigoplus_{j \geq i} \mathfrak{g}_{1,j}$. Let $\Sigma \subset \mathfrak{g}_1$ be a P-invariant subspace. The Hessenberg varieties that we are concerned with are of the form $\mathrm{Hess}_v(K/P, \mathfrak{g}_1, \Sigma)$. We have the following.

Lemma 2.1 ([GKM]). *Suppose $\Sigma \supset \mathfrak{g}_{1,\geq i}$ for some $i \leq 0$. Then the projective morphism $\mathrm{Hess}(K/P, \mathfrak{g}_1, \Sigma) \to \mathfrak{g}_1$ is smooth over $\mathfrak{g}_1^{rs}$, the set of regular semi-simple elements in $\mathfrak{g}_1$.*

Proof. This is proved in [GKM, §2.5]. For the reader's convenience, we recall the argument here. Observe that the Zariski tangent space to $\mathrm{Hess}_v(K/P, \mathfrak{g}_1, \Sigma) \subset K/P$ at a point $x = kP \in \mathrm{Hess}_v(K/P, \mathfrak{g}_1, \Sigma)$ can be identified with the kernel of

$$[v, -] : T(K/P)|_x \cong K \times^P (\mathfrak{g}_0/\mathfrak{p})|_x \to K \times^P (\mathfrak{g}_1/\Sigma)|_x,\ (k, w) \mapsto (k, [k^{-1}v, w]).$$

So it suffices to show that the map above is surjective on the fibers at each point $kP \in \mathrm{Hess}_v(K/P, \mathfrak{g}_1, \Sigma)$ if $v \in \mathfrak{g}_1^{rs}$. For this we show that any $v^* \in \mathfrak{g}_1^*$ that annihilates both $[k^{-1}v, \mathfrak{g}_0]$ and Σ is zero. Since v^* annihilates Σ and $\Sigma \supset \mathfrak{g}_{1,\geq i}$ for some $i \leq 0$, there exists $\delta > 0$ such that $v^* \in \mathfrak{g}^*_{1,\geq\delta}$, that is, v^* is K-unstable. Since $k^{-1}v \in \mathfrak{g}_1^{rs}$, there is no non-zero K-unstable vector $v^* \in \mathfrak{g}_1^*$ that annihilates the subspace $[k^{-1}v, \mathfrak{g}_0] \subset \mathfrak{g}_1$, we have $v^* = 0$. The lemma is proved. ∎

2.2. **Families of Hessenberg varieties.** From now on we concentrate on the following symmetric pair. Let $G = SL(N, \mathbb{C})$ and let $\theta : G \to G$ be the involution such that $K := G^\theta = SO(N, \mathbb{C})$. The pair (G, K) is called a split symmetric pair. As in [CVX1], we will assume, starting with §3.2, that $N = 2n + 1$ is odd, mainly for simplicity.

Let us write $(G, K) = (SL(V), SO(V, Q))$, where Q is a non-degenerate quadratic form on V. Denote by $\langle , \rangle_Q$ the non-degenerate bilinear form associated to Q. For a subspace $U \subset V$, we write $U^\perp = \{v \in V \mid \langle v, U\rangle_Q = 0\}$.

Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$ and let $\mathcal{N}_1 = \mathfrak{g}_1 \cap \mathcal{N}$. It is known that the number of K-orbits in $\mathcal{N}_1$ is finite (see [KR]). Moreover, the K-orbits in $\mathcal{N}_1$ are parametrized as follows (see [S]). For N odd (resp., even), each partition of N corresponds to one K-orbit in $\mathcal{N}_1$ (resp., except that each partition with only even parts corresponds to two K-orbits). In this paper we do not distinguish the two orbits corresponding to the same partition when N is even; thus we write $\mathcal{O}_\lambda$ for an orbit corresponding to λ.

Let $\{e_i,\, i = 1, \dots, N\}$ be a basis of V such that $\langle e_i, e_j\rangle_Q = \delta_{i+j,N+1}$. For any $l \le \frac{N}{2}$, let P_l be the parabolic subgroup of K that stabilizes the partial flag

$$0 \subset V^0_{l-1} \subset V^0_l \subset V^{0,\perp}_l \subset V^{0,\perp}_{l-1} \subset V = \mathbb{C}^N,$$

where $V^0_i = \operatorname{span}\{e_1, \dots, e_i\}$. Consider the following two subspaces of $\mathfrak{g}_1$:

$$E_l = \{x \in \mathfrak{g}_1 \mid xV^0_l = 0,\, xV^{0\perp}_l \subset V^0_{l-1}\} \text{ and } O_l = \{x \in \mathfrak{g}_1 \mid xV^0_l = 0,\, xV^{0\perp}_{l-1} \subset V^0_{l-1}\}.$$

Note that both E_l and O_l are P_l-invariant. We form the corresponding families of Hessenberg varieties

$$\tau^N_l : \mathrm{Hess}^E_l := \mathrm{Hess}(K/P_l, \mathfrak{g}_1, E_l) \to \mathfrak{g}_1\,, \tag{2.4}$$

$$\sigma^N_l : \mathrm{Hess}^O_l := \mathrm{Hess}(K/P_l, \mathfrak{g}_1, O_l) \to \mathfrak{g}_1\,. \tag{2.5}$$

A direct calculation shows that

$$\text{(2.6a)}\quad \mathrm{Im}\, \tau^N_l = \bar{\mathcal{O}}_{3^{l-1}2^1 1^{N+1-3l}} \text{ if } 3l \le N+1,\ \ \mathrm{Im}\, \tau^N_l = \bar{\mathcal{O}}_{3^{N-2l}2^{3l-N}} \text{ if } 3l > N+1,$$

and

$$\text{(2.6b)}\quad \mathrm{Im}\, \sigma^N_l = \bar{\mathcal{O}}_{3^{l-1}1^{N+3-3l}} \text{ if } 3l \le N+1,\ \ \mathrm{Im}\, \sigma^N_l = \bar{\mathcal{O}}_{3^{N-2l}2^{3l-N}} \text{ if } 3l > N+1.$$

Remark 2.2. When $3l \le N+1$, τ^N_l coincides with Reeder's resolution of $\bar{\mathcal{O}}_{3^{l-1}2^1 1^{N+1-3l}}$ [R].

Let $E^\perp_l$ and $O^\perp_l$ be the orthogonal complements of E_l and O_l in $\mathfrak{g}_1$ with respect to the non-degenerate trace form, respectively. Let us now consider the following families of Hessenberg varieties:

$$\check{\tau}^N_l : \mathrm{Hess}^{E,\perp}_l := \mathrm{Hess}(K/P_l, \mathfrak{g}_1, E^\perp_l) \to \mathfrak{g}_1\,, \tag{2.7}$$

$$\check{\sigma}^N_l : \mathrm{Hess}^{O,\perp}_l := \mathrm{Hess}(K/P_l, \mathfrak{g}_1, O^\perp_l) \to \mathfrak{g}_1\,. \tag{2.8}$$

Concretely, we have

$$E^\perp_l = \{x \in \mathfrak{g}_1 \mid xV^0_{l-1} \subset V^0_l,\, xV^0_l \subset V^{0\perp}_l\}, \quad O^\perp_l = \{x \in \mathfrak{g}_1 \mid xV^0_{l-1} \subset V^0_l\};$$

and then

$$\mathrm{Hess}^{O,\perp}_l \simeq \{(x, 0 \subset V_{l-1} \subset V_l \subset V^\perp_l \subset V^\perp_{l-1} \subset \mathbb{C}^N) \mid x \in \mathfrak{g}_1,\, xV_{l-1} \subset V_l\},$$

$$\mathrm{Hess}^{E,\perp}_l \simeq \{(x, 0 \subset V_{l-1} \subset V_l \subset V^\perp_l \subset V^\perp_{l-1} \subset \mathbb{C}^N) \mid x \in \mathfrak{g}_1,\, xV_{l-1} \subset V_l,\, xV_l \subset V^\perp_l\}.$$

Finally, note that, as the notation indicates, the bundle $\mathrm{Hess}_l^{E,\perp} \to K/P_l$ is the orthogonal complement of the bundle $\mathrm{Hess}_l^E \to K/P_n$ in the trivial bundle $\mathfrak{g}_1 \times K/P_l$ and similarly for $\mathrm{Hess}_l^{O,\perp}$ and Hess_l^O. Hence, by functoriality of the Fourier transform, we have:

$$\mathfrak{F}((\sigma_l^N)_*\mathbb{C}[-]) \cong (\check{\sigma}_l^N)_*\mathbb{C}[-] \quad \text{and} \quad \mathfrak{F}((\tau_l^N)_*\mathbb{C}[-]) \cong (\check{\tau}_l^N)_*\mathbb{C}[-]. \tag{2.9}$$

2.3. **Affine pavings.** In this subsection we show that

$$\begin{gathered}\text{the fibers of } \tau_m^N : \mathrm{Hess}_m^E \to \mathcal{N}_1 \text{ and } \sigma_m^N : \mathrm{Hess}_m^O \to \mathcal{N}_1 \text{ have} \\ \text{a paving by affine spaces.}\end{gathered} \tag{2.10}$$

Lemma 2.3. *Let $x \in \mathcal{O}_{3^i2^j1^{N-3i-2j}} \subset \mathrm{Im}\,\tau_m^N$ (resp., $\mathrm{Im}\,\sigma_m^N$) and $x_0 \in \mathcal{O}_{2^j1^{N-3i-2j}}$. We have*

$$(\tau_m^N)^{-1}(x) \cong (\tau_{m-i}^{N-3i})^{-1}(x_0) \ (\textit{resp.}, \ (\sigma_m^N)^{-1}(x) \cong (\sigma_{m-i}^{N-3i})^{-1}(x_0)).$$

Proof. We prove the lemma for τ_m^N. The argument for σ_m^N is entirely similar and omitted. We have

$$(\tau_m^N)^{-1}(x) \cong \{0 \subset V_{m-1} \subset V_m \subset V_m^\perp \subset V_{m-1}^\perp \subset \mathbb{C}^N \mid xV_m = 0,\, xV_m^\perp \subset V_{m-1}\}.$$

Let $(V_{m-1} \subset V_m) \in (\tau_m^N)^{-1}(x)$. We have that $\mathrm{Im}\ x \subset (\ker x)^\perp \subset V_m^\perp$. Thus $\mathrm{Im}\ x^2 \subset V_{m-1}$.

Choose a basis $\{x^k u_l,\, k\in[0,2],\, l\in[1,i],\ v_k,\, xv_k,\, k\in[1,j],\ w_l,\, l\in[1,N-3i-2j]\}$ of V as in [CVX1, Lemma 5.6]. Let

$$\begin{aligned} U^0 &= \mathrm{span}\{x^l u_k, l \in [0,2], k \in [1,i]\}, \\ V^0 &= \mathrm{span}\{v_k, xv_k, k \in [1,j], w_l,\ l \in [1, N-3i-2j]\}. \end{aligned}$$

Then $Q|_{U^0}$, $Q|_{V^0}$ are non-degenerate and $V^0 = (U^0)^\perp$. We have

$$V_m = \mathrm{Im}\ x^2 \oplus W_{m-i} \quad \text{and} \quad V_{m-1} = \mathrm{Im}\ x^2 \oplus W_{m-i-1},$$

where $W_{m-i} = V_m \cap V^0 \supset W_{m-i-1} = V_{m-1} \cap V^0$. We have $\dim W_{m-i} = m-i$ and $\dim W_{m-i-1} = m-i-1$. Let $x_0 = x|_{V^0}$. Then $x_0 \in \mathcal{O}_{2^j1^{N-3i-2j}}$. Note that $xV_m = 0$ if and only if $x_0 W_{m-i} = 0$. Now

$$V_m^\perp = \mathrm{span}\{x^l u_k, l = 1,2, k \in [1,i]\} \oplus W_{m-i}^{\perp_0},$$

where $W_{m-i}^{\perp_0}$ denotes the orthogonal complement of W_{m-i} in V^0 with respect to $Q|_{V^0}$. Thus $xV_m^\perp \subset V_{m-1}$ if and only if $x_0 W_{m-i}^{\perp_0} \subset W_{m-i-1}$. This gives us the desired isomorphism

$$(\tau_m^N)^{-1}(x) \cong (\tau_{m-i}^{N-3i})^{-1}(x_0),\ (V_{m-1}, V_m) \mapsto (\mathrm{pr}_{V^0}(V_{m-1}), \mathrm{pr}_{V^0}(V_m)),$$

where pr_{V^0} is the projection from V to V^0 with respect to $V = U^0 \oplus V^0$. ◻

Let $\mathrm{OGr}(k,N)$ denote the orthogonal Grassmannian variety of k-dimensional isotropic subspaces in $\mathbb{C}^N$ with respect to a non-degenerate bilinear form on $\mathbb{C}^N$ and let $\mathrm{Gr}(k,N)$ denote the Grassmannian variety of k-dimensional subspaces in $\mathbb{C}^N$.

By Lemma 2.3, to describe the fibers $(\tau_m^N)^{-1}(x)$ and $(\sigma_m^N)^{-1}(x)$, it suffices to consider the case when $x \in \mathcal{O}_{2^j1^{N-2j}}$. We first introduce some notation. Let $x \in \mathcal{O}_{2^j1^{N-2j}}$. We write

$$\Sigma := \ker x / \mathrm{Im}\, x \quad \text{and} \quad \bar{U} = U/(U \cap \mathrm{Im}\, x) \text{ for } U \subset \ker x.$$

Define a bilinear from $(,)$ on $\operatorname{Im} x$ by

$$(xv, xw) := \langle v, xw\rangle_Q. \tag{2.11}$$

Using $(\operatorname{Im} x)^\perp = \ker x$, we see that $(,)$ is non-degenerate. For $U \subset \operatorname{Im} x$, we define

$$U^{\perp_{(,)}} = \{u \in \operatorname{Im} x \mid (u, U) = 0\}.$$

Let us denote

$$\Upsilon^N_{m,j} := (\tau^N_m)^{-1}(x),\quad \Gamma^N_{m,j} := (\sigma^N_m)^{-1}(x),\quad x \in \mathcal{O}_{2^j1^{N-2j}}.$$

We partition $\Upsilon^N_{m,j}$ into pieces indexed by the dimension of $V_m \cap \operatorname{Im} x$ by letting

$$\Upsilon^{N,k}_{m,j} = \{(0 \subset V_{m-1} \subset V_m \subset V_m^\perp \subset V_{m-1}^\perp \subset \mathbb{C}^N) \in \Upsilon^N_{m,j} \mid \dim(V_m \cap \operatorname{Im} x) = k\}.$$

We now describe the pieces $\Upsilon^{N,k}_{m,j}$. To this end, let

$$\begin{aligned}\Theta^{N,k}_{m,j} = \{0 \subset V_m \subset V_m^\perp \subset \mathbb{C}^N \mid \dim(V_m \cap \operatorname{Im} x) = k,\\ xV_m = 0,\, xV_m^\perp \subset V_m,\, \dim(xV_m^\perp) \le m-1\}.\end{aligned}$$

Consider the following map:

$$\eta : \Theta^{N,k}_{m,j} \to \operatorname{Gr}(j-k, \operatorname{Im}\, x) \times \operatorname{Gr}(m-k, \Sigma),\ (V_m) \mapsto ((V_m \cap \operatorname{Im}\, x)^{\perp_{(,)}}, \bar{V}_m).$$

We claim that

$$\operatorname{Im} \eta \cong \operatorname{OGr}(j-k, \operatorname{Im}\, x) \times \operatorname{OGr}(m-k, \Sigma),$$

where Im x is equipped with the non-degenerate bilinear form $(,)$ (see (2.11)), and Σ is equipped with the non-degenerate bilinear form induced by $\langle,\rangle_Q$. It is clear that $\bar{V}_m \subset \operatorname{OGr}(m-k, \Sigma)$ as $\langle,\rangle_Q|_{V_m} = 0$. It is easy to check that $x(V_m^\perp) \subset (V_m \cap \operatorname{Im}\, x)^{\perp_{(,)}}$ and $\dim x(V_m^\perp) = \dim (V_m \cap \operatorname{Im}\, x)^{\perp_{(,)}} = j-k$. Thus $x(V_m^\perp) = (V_m \cap \operatorname{Im}\, x)^{\perp_{(,)}}$. Therefore the condition $xV_m^\perp \subset V_m$ is equivalent to $(V_m \cap \operatorname{Im}\, x)^{\perp_{(,)}} \subset V_m \cap \operatorname{Im}\, x$, i.e., $(V_m \cap \operatorname{Im}\, x)^{\perp_{(,)}} \in \operatorname{OGr}(j-k, \operatorname{Im}\, x)$. This proves the claim.

Thus we obtain a surjective map

$$\eta : \Theta^{N,k}_{m,j} \to \operatorname{OGr}(j-k, \operatorname{Im}\, x) \times \operatorname{OGr}(m-k, \Sigma) \tag{2.12}$$

and it is easy to see that the fibers of η are affine spaces $\mathbb{A}^{(j-k)(m-k)}$. Note that the fiber of the natural projection map

$$\Upsilon^{N,k}_{m,j} \to \Theta^{N,k}_{m,j} : (V_{m-1}, V_m) \mapsto V_m \tag{2.13}$$

at V_m is the projective space $\mathbb{P}(V_m/(xV_m^\perp)) \cong \mathbb{P}^{m-j+k-1}$. It is easy to check using the above maps that each piece $\Upsilon^{N,k}_{m,j}$ has an affine paving. Therefore $\Upsilon^N_{m,j}$ also has an affine paving.

We can similarly partition $\Gamma^N_{m,j}$ into pieces indexed by the dimension of $V_{m-1} \cap \operatorname{Im}\, x$. Let

$$\Gamma^{N,k}_{m,j} = \{(0 \subset V_{m-1} \subset V_m \subset V_m^\perp \subset V_{m-1}^\perp \subset \mathbb{C}^N) \in \Gamma^N_{m,j} \mid \dim(V_{m-1} \cap \operatorname{Im}\, x) = k\}$$

$$\begin{aligned}\Lambda^{N,k}_{m,j} =& \{(0 \subset V_{m-1} \subset V_{m-1}^\perp \subset \mathbb{C}^N) \mid \dim(V_{m-1} \cap \operatorname{Im}\, x)\\ &= k,\ xV_{m-1} = 0,\ xV_{m-1}^\perp \subset V_{m-1}\}.\end{aligned}$$

We have a surjective map

$$\begin{aligned}\eta' : \Lambda^{N,k}_{m,j} \to \operatorname{OGr}(j-k, \operatorname{Im}\, x) \times \operatorname{OGr}(m-k-1, \Sigma),\\ (V_{m-1}) \mapsto ((V_{m-1} \cap \operatorname{Im}\, x)^{\perp_{(,)}}, \bar{V}_{m-1}).\end{aligned}$$

The fibers of η' are affine spaces $\mathbb{A}^{(j-k)(m-k-1)}$. The fiber of the natural projection map

$$\Gamma^{N,k}_{m,j} \to \Lambda^{N,k}_{m,j} : \ (V_{m-1}, V_m) \mapsto V_{m-1}$$

at a given V_{m-1} is the variety of isotropic lines in $(V_{m-1}^{\perp} \cap \ker x)/V_{m-1}$ with respect to the quadratic form induced by Q. The same argument as before shows that $\Gamma^N_{m,j}$ is paved by affines.

In particular, we see from the above discussion that

$$\begin{aligned} \Upsilon^{N,k}_{m,j} \neq \emptyset &\Leftrightarrow \max\{m+j-N/2, j/2, j+1-m\} \leq k \leq \min\{j,m\}; \\ \Gamma^{N,k}_{m,j} \neq \emptyset &\Leftrightarrow \max\{m+j-N/2-1, j/2, j+1-m\} \leq k \leq \min\{j,m-1\}. \end{aligned} \tag{2.14}$$

Finally, in [CVX1, Proof of Proposition 4.3] we have used the following fact.

Lemma 2.4. *For $x_i \in \mathcal{O}_{3^i 2^{2m-1-2i} 1^{2n-4m+3+i}}$ we have*

$$(\tau_m^{2n+1})^{-1}(x_i) \cong \mathrm{OGr}(m-1-i, 2m-1-2i)\,.$$

This can be deduced from the results in this subsection as follows. Using Lemma 2.3 and (2.14) we see that $\tau_m^{2n+1}(x_i) \cong \Upsilon^{2n+1-3i,\, m-i}_{m-i,\, 2m-1-2i}$. The conclusion follows by considering the maps in (2.12) and (2.13).

2.4. **Families of complete intersections of quadrics and their identification with Hessenberg varieties.** Let $m \in [1, N-1]$ be an integer. For any $s \in \mathfrak{g}_1^{rs}$, let

$$X_{m,s} \subset \mathbb{P}(V) \simeq \mathbb{P}^{N-1}$$

be the complete intersection of m quadrics

$$\langle s^i -, - \rangle_Q = 0, \ i = 0, \ldots, m-1$$

in $\mathbb{P}(V)$. As s varies over $\mathfrak{g}_1^{rs}$, we get a family

$$\pi_m : X_m \to \mathfrak{g}_1^{rs}$$

of complete intersections of m quadrics in $\mathbb{P}(V)$.

The families of Hessenberg varieties $\mathrm{Hess}_l^{O,\perp}$ and $\mathrm{Hess}_l^{E,\perp}$ over $\mathfrak{g}_1^{rs}$ are identified with X_m's as follows.

Theorem 2.5. *Assume that $k \leq \frac{N-1}{2}$. Then we have*

(1) *There is a K-equivariant isomorphism $\mathrm{Hess}_k^{O,\perp}|_{\mathfrak{g}_1^{rs}} \simeq X_{2k-1}$ of varieties over $\mathfrak{g}_1^{rs}$.*
(2) *There is a K-equivariant isomorphism $\mathrm{Hess}_k^{E,\perp}|_{\mathfrak{g}_1^{rs}} \simeq X_{2k}$ of varieties over $\mathfrak{g}_1^{rs}$.*

We begin with the following simple observation.

(2.15)
Let $s \in \mathfrak{g}_1^{rs}$. For any isotropic subspace $0 \neq U \subset V$, $\dim(sU \cap U) < \dim(sU)$.

This follows from the fact that s has no isotropic eigenspaces.

Proof of Theorem 2.5. We first define a map from X_{2k-1} to $\mathrm{Hess}_n^{O,\perp}$. Let $(s,l) \in X_{2k-1}$, where $s \in \mathfrak{g}_1^{rs}$ and l is in the complete intersection of $2k-1$ quadrics $\langle s^i -, - \rangle_Q = 0$, $i = 0, \ldots, 2k-2$, in $\mathbb{P}(V)$. Let $0 \neq v \in l$. For $1 \leq i \leq k$, consider the subspaces

$$V_i = \mathrm{span}\{v, sv, \ldots, s^{i-1}v\}.$$

Note that V_i is isotropic. We show that $\dim V_i = i$. We have $V_i = V_{i-1} + sV_{i-1}$. Thus $\dim V_i = \dim V_{i-1} + \dim sV_{i-1} - \dim(sV_{i-1} \cap V_{i-1}) > \dim V_{i-1}$, where in the last inequality we use (2.15). By induction we see that $\dim V_i = i$. Hence the assignment $(s, l) \mapsto (s, V_{k-1} \subset V_k)$ defines a map

$$\iota : X_{2k-1} \to \mathrm{Hess}_k^{O,\perp}|_{\mathfrak{g}_1^{rs}}.$$

One checks readily that ι is K-equivariant. We prove that ι is an isomorphism by constructing an explicit inverse map. Let $(s, V'_{k-1} \subset V'_k) \in \mathrm{Hess}_k^{O,\perp}$ with $s \in \mathfrak{g}_1^{rs}$. We define a sequence of subspaces $0 \subset V'_1 \subset V'_2 \subset \cdots \subset V'_{k-2}$ recursively. Let us first define V'_{k-2}. Consider the map $\bar{s} : V'_{k-1} \xrightarrow{s} V'_k \to V'_k/V'_{k-1}$. Note that by (2.15), the map $\bar{s}$ is non-zero, hence surjective as $\dim V'_k/V'_{k-1} = 1$. Let $V'_{k-2} = \ker \bar{s}$. We have $\dim V'_{k-2} = k-2$ and $V'_{k-1} = V'_{k-2} \cup sV'_{k-2}$. By induction we can assume that we have defined V'_i such that $\dim V'_i = i$ and $V'_{i+1} = V'_i \cup sV'_i$. Let

$$V'_{i-1} = \ker(\bar{s} : V'_i \xrightarrow{s} V'_{i+1} \to V'_{i+1}/V'_i).$$

The same argument as before shows that $\dim V'_{i-1} = i-1$ and $V'_i = V'_{i-1} \cup sV'_{i-1}$. Thus in particular we obtain that $\dim V'_1 = 1$, and it is easy to see that the map

$$\mathrm{Hess}_k^{O,\perp}|_{\mathfrak{g}_1^{rs}} \to X_{2k-1}, (s, V'_{k-1} \subset V'_k) \mapsto (s, V'_1)$$

defines an inverse of ι. This finishes the proof of (1).

For (2), we observe that, under the isomorphism $\iota : X_{2k-1} \simeq \mathrm{Hess}_k^{O,\perp}$, the equation $\langle s^{2k-1}v, v\rangle_Q = 0$ for the divisor $X_{2k} \subset X_{2k-1}$ becomes $\langle sV'_k, V'_k\rangle_Q = 0$, which is the equation for the divisor $\mathrm{Hess}_k^{E,\perp} \subset \mathrm{Hess}_k^{O,\perp}$. Thus (2) follows. $\square$

3. Complete intersections of quadrics and their double covers

In §2.4 we have introduced the families $X_m \to \mathfrak{g}_1^{rs}$ of complete intersections of quadrics, which we have identified with families of Hessenberg varieties $\mathrm{Hess}_n^{E,\perp}|_{\mathfrak{g}_1^{rs}}$, $\mathrm{Hess}_n^{O,\perp}|_{\mathfrak{g}_1^{rs}}$. In order to study the monodromy representations of the equivariant fundamental group $\pi_1^K(\mathfrak{g}_1^{rs})$ associated with the above families of Hessenberg varieties, we introduce families $\mathcal{Y}_m$ of branched covers of $\mathbb{P}^{N-m-1}$ and relate them with X_m. We also introduce a family of branched double covers of X_m, denoted by $\tilde{X}_m$, and relate them to families $\tilde{\mathcal{Y}}_m$ of branched covers of $\mathbb{P}^{N-m-1}$ which we introduce in §3.5. Our construction can be regarded as a relative version of the construction in [T].

3.1. **Some notation.** In this section we choose a Cartan subspace $\mathfrak{a} \subset \mathfrak{g}_1$ that consists of diagonal matrices. Let $\mathfrak{a}^{rs} = \mathfrak{a} \cap \mathfrak{g}^{rs}$. We write an element $a \in \mathfrak{a}$ with diagonal entries $a_1, \ldots, a_N$ as $a = (a_1, \ldots, a_N)$ (note that we have diagonalized the elements in $\mathfrak{a}$ with respect to a standard basis f_i of V, where $\langle f_i, f_j\rangle_Q = \delta_{i,j}$, rather than the basis chosen in §2.2). Thus $a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}$ if and only if $a_i \neq a_j$ for $i \neq j$.

Define

$$I_N := (\mathbb{Z}/2\mathbb{Z})^N/(\mathbb{Z}/2\mathbb{Z})$$

where we regard $\mathbb{Z}/2\mathbb{Z}$ as a subgroup of $(\mathbb{Z}/2\mathbb{Z})^N$ via the diagonal embedding. For any $\chi \in I_N^\vee = \mathrm{Hom}(I_N, \mathbb{G}_m)$, we define

$$\mathrm{supp}(\chi) = \{i \in [1, N] \mid \chi(\xi_i) = -1\} \quad \text{and} \quad |\chi| = \#\,\mathrm{supp}(\chi),$$

where ξ_i is the image of $(0, \ldots, 1, \ldots, 0) \in (\mathbb{Z}/2\mathbb{Z})^N$ in I_N. Note that $|\chi|$ is even.

If we identify the centralizer $Z_K(a)$ of $a \in \mathfrak{a}^{rs}$ with the kernel of the map $(\mathbb{Z}/2\mathbb{Z})^N \to \mathbb{Z}/2\mathbb{Z}$, $(b_1,\ldots,b_N) \mapsto \sum b_i$, we obtain a natural map

$$Z_K(a) \to (\mathbb{Z}/2\mathbb{Z})^N \xrightarrow{pr} I_N. \tag{3.1}$$

Note when N is odd which we will assume from this point on, the map (3.1) is an isomorphism. Therefore, in what follows, we often make the canonical identification $Z_K(a) \cong I_N$.

To emphasize we have the following.

Assumption. From now on we assume that N is odd.

3.2. **Family of curves.** In this subsection we introduce certain families of curves which will be used to construct the families $\mathcal{Y}_m$ and $\tilde{\mathcal{Y}}_m$ of branched covers of projective spaces.

For any $a = (a_1,\ldots,a_N) \in \mathfrak{a}^{rs}$, there are natural isomorphisms

$$\pi_1^{ab}(\mathbb{P}^1 - \{a_1,\ldots,a_N\}) \otimes \mathbb{Z}/2\mathbb{Z} \simeq I_N,$$
$$\pi_1^{ab}(\mathbb{P}^1 - \{a_1,\ldots,a_N, a_{N+1} = \infty\}) \otimes \mathbb{Z}/2\mathbb{Z} \simeq I_{N+1}.$$

The isomorphisms are given by assigning to a small loop around each a_i the element in I_N (resp., I_{N+1}) with only a non-trivial coordinate in position i. Let

$$C_a \to \mathbb{P}^1 \text{ (resp., } \tilde{C}_a \to \mathbb{P}^1)$$

be the abelian covering of $\mathbb{P}^1$ ramified at $\{a_1,\ldots,a_N\}$ (resp., $\{a_1,\ldots,a_{N+1} = \infty\}$) with Galois groups given by I_N (resp., I_{N+1}). Concretely, C_a (resp., $\widetilde{C}_a$) is the smooth projective curve corresponding to the function field

$$\mathbb{C}(t)\Big(\big(\frac{t-a_i}{t-a_1}\big)^{1/2}_{i=2,\ldots,N}\Big) \text{ (resp., } \mathbb{C}(t)\big((t-a_i)^{1/2}_{i=1,\ldots,N}\big)).$$

The group I_N (resp., I_{N+1}) acts on C_a (resp., $\tilde{C}_a$). For any $\chi \in I_N^\vee$ (resp., $\chi \in I_{N+1}^\vee$) we define

$$C_{a,\chi} = C_a/\ker\chi \text{ (resp., } \tilde{C}_{a,\chi} = \tilde{C}_a/\ker\chi),$$

which is a branched double cover of $\mathbb{P}^1$ with branch locus $\{a_i \,|\, i \in \operatorname{supp}(\chi)\}$. Concretely, $C_{a,\chi}$ (resp., $\widetilde{C}_{a,\chi}$) is isomorphic to the smooth projective hyperelliptic curve with affine equation

$$y^2 = \prod_{i\in\operatorname{supp}\chi}(x-a_i) \text{ (resp., } y^2 = \prod_{i\in\operatorname{supp}\chi,\ i\neq N+1}(x-a_i)).$$

We have $\dim H^1(C_\chi,\mathbb{C}) = |\chi| - 2$ (resp., $\dim H^1(\tilde{C}_\chi,\mathbb{C}) = |\chi| - 2$).

As a varies over $\mathfrak{a}^{rs}$, we obtain a family of curves

$$C \to \mathfrak{a}^{rs} \text{ (resp., } \tilde{C} \to \mathfrak{a}^{rs})$$

and we similarly obtain families of hyperelliptic curves

$$C_\chi \to \mathfrak{a}^{rs} \text{ (resp., } \tilde{C}_\chi \to \mathfrak{a}^{rs}) \text{ for any } \chi.$$

We also note that the Weyl group $W = S_N$ acts naturally on C (resp., $\widetilde{C}$) making the projection $C \to \mathfrak{a}^{rs}$ (resp., $\widetilde{C} \to \mathfrak{a}^{rs}$) a W-equivariant map.

We will now associate monodromy representations to these families. Let us fix $a \in \mathfrak{a}^{rs}$ and a character $\chi \in I_N^\vee$ (resp., $\chi \in I_{N+1}^\vee$) and we recall that $\pi_1(\mathfrak{a}^{rs}, a)$ is the pure braid group P_N. The monodromy representation of the family $C_\chi \to \mathfrak{a}^{rs}$ factors through the symplectic group and we denote it by $\rho_{C_\chi} : P_N \to Sp(H^1(C_{a,\chi}, \mathbb{C})) \simeq Sp(2m-2)$ where the $m = \frac{|\chi|}{2}$. Similarly we obtain a monodromy representation $\rho_{\tilde{C}_\chi} : P_N \to Sp(H^1(\tilde{C}_{a,\chi}, \mathbb{C})) \simeq Sp(2m-2)$.

We claim:

$$\begin{gathered}\text{The images of the representations } \rho_{C_\chi} \text{ and } \rho_{\tilde{C}_\chi} \text{ are Zariski dense}\\ \text{in } Sp(H^1(C_{a,\chi}, \mathbb{C})) \text{ and } Sp(H^1(\tilde{C}_{a,\chi}, \mathbb{C})), \text{ respectively}.\\ \text{In particular, the representations } \rho_{C_\chi} \text{ and } \rho_{\tilde{C}_\chi} \text{ are irreducible}.\end{gathered} \tag{3.2}$$

We see this as follows. Consider the following subvariety $\mathfrak{a}^{rs}(a,\chi) \subset \mathfrak{a}^{rs}$

$$\mathfrak{a}^{rs}(a,\chi) = \{a' \in \mathfrak{a}^{rs} \mid a'_i = a_i \text{ if } i \notin \operatorname{supp}(\chi)\}. \tag{3.3}$$

It suffices to show that the monodromy representation of the restriction of $C_\chi \to \mathfrak{a}^{rs}$ to $\mathfrak{a}^{rs}(a,\chi)$ has Zariski dense image. Now, $\mathfrak{a}^{rs}(a,\chi)$ is an open subset of the space $\mathcal{M}_{2m}$ of $2m$ distinct marked points in $\mathbb{C}$ and the family $C_\chi \times_{\mathfrak{a}^{rs}} \mathfrak{a}^{rs}(a,\chi)$ is the restriction of the universal family of hyperelliptic curves parametrized by $\mathcal{M}_{2m}$. Note further, that $\mathcal{M}_{2m}$ itself is an open subset of the space $\widetilde{\mathcal{M}}_{2m}$ of $2m$ distinct marked points in $\mathbb{P}^1$ carrying its own family of hyperelliptic curves. Now, by [A] (see also [KS, Theorem 10.1.18.3]), the monodromy representation on $\widetilde{\mathcal{M}}_{2m}$ is irreducible and has Zariski dense image. Therefore ρ_{C_χ}, as a restriction to an open subset, has the same property. The argument in the case $\tilde{C}_\chi \to \mathfrak{a}^{rs}$ is completely analogous except one has to take into account that $a_{N+1} = \infty$.

Finally, there is a unique character $\chi_0 \in I_{N+1}^\vee$ with $|\chi_0| = N+1$ (here we use the assumption that N is odd). The character χ_0 is invariant under the Weyl group action. Thus we can pass to a quotient of $\widetilde{C}_{\chi_0} \to \mathfrak{a}^{rs}$ under the W action and in this way obtain a family $\overline{C}_{\chi_0} \to \mathfrak{c}^{rs} = \mathfrak{a}^{rs}/W$. The family $\overline{C}_{\chi_0} \to \mathfrak{c}^{rs} = \mathfrak{a}^{rs}/W$ is the universal family of hyperelliptic curves $y^2 = \prod_{i=1}^N (x - a_i)$ and $\widetilde{C}_{\chi_0} \to \mathfrak{a}^{rs}$ is a similar universal family with marked ramification points.

3.3. Branched cover $\mathcal{Y}_m$ of projective spaces and X_m.

Define

$$\bar{I}_N^{N-m-1} = \ker(sum : I_N^{N-m-1} \to I_N),$$

where sum is the summation map. Fix $a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}$. Let $C_a \to \mathbb{P}^1$ be the curve introduced in §3.2. The semi-direct product $\bar{I}_N^{N-m-1} \rtimes S_{N-m-1}$ acts naturally on C_a^{N-m-1} and we define

$$\mathcal{Y}_{m,a} = C_a^{N-m-1}/\bar{I}_N^{N-m-1} \rtimes S_{N-m-1}.$$

We have a natural map

$$\iota_a : \mathcal{Y}_{m,a} \to C_a^{N-m-1}/I_N^{N-m-1} \rtimes S_{N-m-1} \simeq \mathbb{P}^{N-m-1}.$$

According to [T, Proposition 2.4.4], for a suitable choice of homogeneous coordinates $[x_1, \ldots, x_{N-m}]$ of $\mathbb{P}^{N-m-1}$, each ramification point a_i defines a hyperplane

$$H_{a,i} = x_1 + a_i x_2 + \cdots + a_i^{N-m-1} x_{N-m} = 0 \tag{3.4}$$

in $\mathbb{P}^{N-m-1}$ and the map ι_a is an I_N-branched cover of $\mathbb{P}^{N-m-1}$ with branch locus $\{H_{a,i} = 0\}_{i=1,\ldots,N}$. As a varies over $\mathfrak{a}^{rs}$, we get an $\mathfrak{a}^{rs}$-family of I_N-branched covers of $\mathbb{P}^{N-m-1}$

$$\begin{tikzcd} \mathcal{Y}_m \arrow[rr, "\iota"] \arrow[dr] & & \mathbb{P}^{N-m-1}_{\mathfrak{a}^{rs}} \arrow[dl] \\ & \mathfrak{a}^{rs} & \end{tikzcd}$$

where

$$\mathcal{Y}_m = C^{N-m-1}/\bar{I}_N^{N-m-1} \rtimes S_{N-m-1}$$

and the base change of ι to a is equal to ι_a. Observe that the W-action on C induces a W-action on $\mathcal{Y}_m$ making the projection $\mathcal{Y}_m \to \mathfrak{a}^{rs}$ a W-equivariant map.

Let $X_m \to \mathfrak{g}_1^{rs}$ be the family of complete intersections of quadrics introduced in §2.4. For $a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}$ the equation of $X_{m,a}$ is given by

$$a_1^i v_1^2 + \cdots + a_N^i v_N^2 = 0, \quad i = 0, \ldots, m-1.$$

Consider the map

$$s : \mathbb{P}(V) \to \mathbb{P}(V),\ [v_1, \ldots, v_N] \mapsto [v_1^2, \ldots, v_N^2].$$

The image $s(X_{m,a})$ is equal to $\mathbb{P}(V_{m,a})$, where

$$V_{m,a} = \{v \in V \mid a_1^i v_1 + \cdots + a_N^i v_N = 0,\ i = 0, \ldots, m-1\} \subset V. \tag{3.5}$$

The resulting map

$$s_a : X_{m,a} \to \mathbb{P}(V_{m,a})$$

is an I_N-branched cover with branch locus $\{v_i = 0\}_{i=1,\ldots,N}$. As a varies over $\mathfrak{a}^{rs}$, we obtain

$$\begin{tikzcd} X_m|_{\mathfrak{a}^{rs}} \arrow[rr, "s"] \arrow[dr] & & \mathbb{P}(V_m) \arrow[dl] \\ & \mathfrak{a}^{rs} & \end{tikzcd}$$

Here $V_m \to \mathfrak{a}^{rs}$ is the vector bundle over $\mathfrak{a}^{rs}$ whose fiber over a is $V_{m,a}$ and $\mathbb{P}(V_m)$ is the associated projective bundle.

The two families $X_m|_{\mathfrak{a}^{rs}}$ and $\mathcal{Y}_m$ are related as follows. Let

$$(\tilde{\mathfrak{a}}^{rs})' = \{(a,c) \mid a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}, c = (c_1, \ldots, c_N) \in \mathbb{C}^N,\ c_i^2 = d_i := \prod_{j \neq i}(a_j - a_i)\}.$$

The projection $(\tilde{\mathfrak{a}}^{rs})' \to \mathfrak{a}^{rs}$, $(a_1, \ldots, a_N, c_1, \ldots, c_N) \mapsto (a_1, \ldots, a_N)$ realizes $(\tilde{\mathfrak{a}}^{rs})'$ as a $(\mathbb{Z}/2\mathbb{Z})^N$-torsor over $\mathfrak{a}^{rs}$. Consider the following I_N-torsor over $\mathfrak{a}^{rs}$:

$$\tilde{\mathfrak{a}}^{rs} := (\tilde{\mathfrak{a}}^{rs})'/(\mathbb{Z}/2\mathbb{Z}) \to \mathfrak{a}^{rs}; \tag{3.6}$$

here we view $\mathbb{Z}/2\mathbb{Z}$ as a subgroup of $(\mathbb{Z}/2\mathbb{Z})^N$ via the diagonal embedding. The Weyl group W acts naturally on $\tilde{\mathfrak{a}}^{rs}$ making the projection to $\mathfrak{a}^{rs}$ a W-equivariant map. We observe that the I_N-torsor $\tilde{\mathfrak{a}}^{rs}$ of (3.6) gives rise to the following canonical map:

$$\rho : \pi_1(\mathfrak{a}^{rs}, a) \cong P_N \to I_N. \tag{3.7}$$

Proposition 3.1. *We have an $I_N \rtimes W$-equivariant isomorphism*

$$X_m|_{\mathfrak{a}^{rs}} \simeq (\mathcal{Y}_m \times_{\mathfrak{a}^{rs}} \tilde{\mathfrak{a}}^{rs})/I_N := \mathcal{Y}_m^t, \tag{3.8}$$

where W (resp., I_N) acts on $\mathcal{Y}_m^t$ by the diagonal action (resp., on the first factor).

Proof. Following [T, §5], we consider the family

$$X'_m|_{\mathfrak{a}^{rs}} \to \mathfrak{a}^{rs}$$

whose fiber over $a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}$ is the complete intersection of m quadrics in $\mathbb{P}(V)$ given by

$$\frac{a_1^i}{d_1} v_1^2 + \cdots + \frac{a_N^i}{d_N} v_N^2 = 0, \quad i = 0, \ldots, m-1,$$

where $d_i := \prod_{j \neq i}(a_j - a_i)$. One can think of $X_m|_{\mathfrak{a}^{rs}}$ as a *twist* of $X'_m|_{\mathfrak{a}^{rs}}$. More precisely, we have a natural map

$$\tilde{\mathfrak{a}}^{rs} \times_{\mathfrak{a}^{rs}} X'_m|_{\mathfrak{a}^{rs}} \to X_m|_{\mathfrak{a}^{rs}},\ (a, c, [v_1, \ldots, v_N]) \mapsto (a, [v_1/c_1, \ldots, v_N/c_N])$$

and it is not hard to see that it descends to a canonical I_N-equivariant isomorphism

$$X_m|_{\mathfrak{a}^{rs}} \simeq (\tilde{\mathfrak{a}}^{rs} \times_{\mathfrak{a}^{rs}} X'_m|_{\mathfrak{a}^{rs}})/I_N. \tag{3.9}$$

Here I_N acts on the product via the diagonal action. The Weyl group $W = S_N$ acts naturally on $X_m|_{\mathfrak{a}^{rs}}$, $X'_m|_{\mathfrak{a}^{rs}}$, $\tilde{\mathfrak{a}}^{rs}$, and $(\tilde{\mathfrak{a}}^{rs} \times_{\mathfrak{a}^{rs}} X'_m|_{\mathfrak{a}^{rs}})/I_N$, making the projections to $\mathfrak{a}^{rs}$ equivariant maps under the W-actions. Moreover, the isomorphism in (3.9) is also W-equivariant. In §3.6 (see Proposition 3.3), we show that there is an $I_N \rtimes W$-equivariant isomorphism

$$X'_m|_{\mathfrak{a}^{rs}} \simeq \mathcal{Y}_m. \tag{3.10}$$

Combining (3.9) with (3.10) we obtain (3.8). ☐

3.4. **Branched double covers $\tilde{X}_m$ of complete intersections of quadrics.** We introduce a branched double cover of X_m as follows. Let $\widetilde{V} = V \oplus \mathbb{C}$. For any $s \in \mathfrak{g}_1^{rs}$, consider the following quadrics in $\mathbb{P}(\widetilde{V})$:

$$\widetilde{Q}_i(v, \epsilon) = \langle s^i v, v \rangle_Q = 0,\ i = 0, \ldots, m-1,$$
$$\widetilde{Q}_m(v, \epsilon) = \langle s^m v, v \rangle_Q - \epsilon^2 = 0.$$

We define $\widetilde{X}_{m,s}$ to be the complete intersection of $m+1$ quadrics $\widetilde{Q}_i = 0$, $i = 0, \ldots, m$. As s varies over $\mathfrak{g}_1^{rs}$, we get a family

$$\tilde{\pi}_m : \widetilde{X}_m \to \mathfrak{g}_1^{rs}$$

of complete intersections of $m+1$ quadrics in $\mathbb{P}(\widetilde{V})$. The projection $\widetilde{V} = V \oplus \mathbb{C} \to V$, $(v, \epsilon) \mapsto v$, induces a map $p_m : \widetilde{X}_m \to X_m$ which is a branched double cover with branch locus $X_{m+1} \subset X_m$.

The map $\widetilde{V} = V \oplus \mathbb{C} \to \widetilde{V}$ given by $(v, \epsilon) \mapsto (v, -\epsilon)$ defines an involution on $\widetilde{X}_m$. We denote this involution by σ.

The group $K = SO(V, Q)$ acts naturally on both X_m and $\widetilde{X}_m$. The maps $\pi_m : X_m \to \mathfrak{g}_1^{rs}$ and $\tilde{\pi}_m : \widetilde{X}_m \to \mathfrak{g}_1^{rs}$ are K-equivariant. In particular, the centralizer $Z_K(s)$ acts on the fibers $X_{m,s}$ and $\widetilde{X}_{m,s}$.

3.5. **Branched cover $\tilde{\mathcal{Y}}_m$ of projective spaces and $\widetilde{X}_m$.** In this subsection we generalize Proposition 3.1 to the branched double cover $\widetilde{X}_m$ introduced in §3.4.

For $a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}$ the equations of $\widetilde{X}_{m,a} \subset \mathbb{P}^{N-m-1}(\widetilde{V})$ (recall that $\widetilde{V} = V \oplus \mathbb{C}$) are given by

$$a_1^i v_1^2 + \cdots + a_N^i v_N^2 = 0, \quad i = 0, \ldots, m-1, \quad a_1^m v_1^2 + \cdots + a_N^m v_N^2 - \epsilon^2 = 0.$$

Consider the map

$$\tilde{s} : \mathbb{P}(\widetilde{V}) \to \mathbb{P}(\widetilde{V}), \ [v_1, \ldots, v_N, \epsilon = v_{N+1}] \mapsto [v_1^2, \ldots, v_N^2, v_{N+1}^2]. \tag{3.11}$$

We have $\tilde{s}(\widetilde{X}_{m,a}) \simeq \mathbb{P}(\widetilde{V}_{m,a})$, where $\widetilde{V}_{m,a} \subset \widetilde{V}$ is the subspace defined by the equations

$$a_1^i v_1 + \cdots + a_N^i v_N = 0, \quad i = 0, \ldots, m-1, \quad a_1^m v_1 + \cdots + a_N^m v_N - v_{N+1} = 0.$$

The map

$$\tilde{s}_a : \widetilde{X}_{m,a} \to \mathbb{P}(\widetilde{V}_{m,a})$$

is an I_{N+1}-branched cover with branch locus $\{v_i = 0\}_{i=1,\ldots,N+1}$. As a varies over $\mathfrak{a}^{rs}$, we obtain

$$\begin{tikzcd} \widetilde{X}_m|_{\mathfrak{a}^{rs}} \arrow[rr, "\tilde{s}"] \arrow[dr] & & \mathbb{P}(\widetilde{V}_m) \arrow[dl] \\ & \mathfrak{a}^{rs} & \end{tikzcd}$$

Here $\widetilde{V}_m$ is the vector subbundle of the trivial bundle $\widetilde{V} \times \mathfrak{a}^{rs}$ whose fiber over a is the subspace $\widetilde{V}_{m,a}$, and $\mathbb{P}(\widetilde{V}_m)$ is the associated projective bundle.

We now introduce another family $\tilde{\mathcal{Y}}_m$ of branched covers of $\mathbb{P}^{N-m-1}$. Let $sum : I_{N+1}^{N-m-1} \to I_{N+1}$ be the summation map and define $\bar{I}_{N+1}^{N-m-1} = \ker(sum)$. For any $a \in \mathfrak{a}^{rs}$ let $\tilde{C}_a \to \mathbb{P}^1$ be the I_{N+1}-branched cover of $\mathbb{P}^1$ introduced in §3.2. The semi-direct product $\bar{I}_{N+1}^{N-m-1} \rtimes S_{N-m-1}$ acts naturally on $(\tilde{C}_a)^{N-m-1}$. We define

$$\tilde{\mathcal{Y}}_{m,a} = (\tilde{C}_a)^{N-m-1} / \bar{I}_{N+1}^{N-m-1} \rtimes S_{N-m-1}.$$

Similar to the case of $\mathcal{Y}_{m,a}$, the natural map

$$\tilde{\iota}_a : \tilde{\mathcal{Y}}_{m,a} \to (\tilde{C}_a)^{N-m-1} / I_{N+1}^{N-m-1} \rtimes S_{N-m-1} \simeq \mathbb{P}^{N-m-1}$$

is an I_{N+1}-branched cover of $\mathbb{P}^{N-m-1}$ with branch locus $\{H_{a,i} = 0\}_{i=1,\ldots,N+1}$. Here $H_{a,i} = 0$ for $i = 1, \ldots, N$ are the hyperplanes as before (see (3.4)) and $H_{a,N+1} := x_{N-m} = 0$ is the hyperplane corresponding to the ramification point $a_{N+1} = \infty$.

As a varies over $\mathfrak{a}^{rs}$, we get an $\mathfrak{a}^{rs}$-family of an I_{N+1}-branched cover of $\mathbb{P}^{N-m-1}$

$$\begin{tikzcd} \tilde{\mathcal{Y}}_m \arrow[rr, "\tilde{\iota}"] \arrow[dr] & & \mathbb{P}^{N-m-1}_{\mathfrak{a}^{rs}} \arrow[dl] \\ & \mathfrak{a}^{rs} & \end{tikzcd}$$

We will again make use of $\tilde{\mathfrak{a}}^{rs}$ of (3.6) to relate the two families $\widetilde{X}_m|_{\mathfrak{a}^{rs}}$ and $\tilde{\mathcal{Y}}_m$. The Weyl group W acts naturally on $\widetilde{X}_m|_{\mathfrak{a}^{rs}}$ and $\tilde{\mathcal{Y}}_m$, making the projections to $\mathfrak{a}^{rs}$ equivariant with respect to these W-actions. We let I_N act on $\tilde{\mathcal{Y}}_m$ via the map

$$\kappa : I_N \cong Z_K(a) \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^{N+1} \xrightarrow{pr} I_{N+1}, \tag{3.12}$$

where the first arrow is given by $(\zeta_1,\dots,\zeta_N)\mapsto(\zeta_1,\dots,\zeta_N,0)$. We also let W act on I_{N+1} by permuting the first N coordinates and we use this convention to form the semi-direct product $I_{N+1}\rtimes W$.

Proposition 3.2. *There is an $I_{N+1}\rtimes W$-equivaraint isomorphism*

$$\widetilde{X}_m|_{\mathfrak{a}^{rs}}\simeq\tilde{\mathcal{Y}}_m^t:=(\tilde{\mathcal{Y}}_m\times_{\mathfrak{a}^{rs}}\tilde{\mathfrak{a}}^{rs})/I_N, \tag{3.13}$$

where W (resp., I_{N+1}) acts on $\tilde{\mathcal{Y}}_m^t$ by the diagonal action (resp., on the first factor).

Proof. Let us consider the following *twist* of $\widetilde{X}_m|_{\mathfrak{a}^{rs}}$:

$$\widetilde{X}'_m|_{\mathfrak{a}^{rs}}\to\mathfrak{a}^{rs}$$

whose fiber over $a=(a_1,\dots,a_N)$ is the complete intersection of quadrics given by

$$\frac{a_1^i}{d_1}v_1^2+\cdots+\frac{a_N^i}{d_N}v_N^2=0,\quad i=0,\dots,m-1,\quad \frac{a_1^m}{d_1}v_1^2+\cdots+\frac{a_N^m}{d_N}v_N^2-\epsilon^2=0,$$

where d_i is defined as before, i.e., $d_i=\prod_{j\neq i}(a_j-a_i)$. Similar to the case of X_m, we have a canonical isomorphism

$$\widetilde{X}_m|_{\mathfrak{a}^{rs}}\simeq(\tilde{\mathfrak{a}}^{rs}\times_{\mathfrak{a}^{rs}}\widetilde{X}'_m|_{\mathfrak{a}^{rs}})/I_N. \tag{3.14}$$

Here the I_N-action on $\widetilde{X}'_m|_{\mathfrak{a}^{rs}}$ is defined as the composition of $I_N\to I_{N+1}$ in (3.12) with the natural action of I_{N+1} on $\widetilde{X}_m$. The Weyl group W acts naturally on $\widetilde{X}_m$, $\widetilde{X}'_m$, and $\tilde{\mathcal{Y}}_m$, making the projections to $\mathfrak{a}^{rs}$ equivariant maps under the W-actions. Thus we obtain $I_{N+1}\rtimes W$-actions on $\widetilde{X}_m$, $\widetilde{X}'_m$, and $\tilde{\mathcal{Y}}_m$ and the projections to $\mathfrak{a}^{rs}$ are $I_{N+1}\rtimes W$-equivariant. Moreover, the isomorphism in (3.14) is $I_{N+1}\rtimes W$-equivariant. In §3.6 (see Proposition 3.3), we show that there is an $I_{N+1}\rtimes W$-equivariant isomorphism $\widetilde{X}'_m|_{\mathfrak{a}^{rs}}\simeq\tilde{\mathcal{Y}}_m$. Combining this with (3.14) we obtain (3.13). ∎

3.6. **The families $\widetilde{X}'_m$ and $\tilde{\mathcal{Y}}_m$.** In this subsection we state and prove the following proposition which was used in the previous subsections.

Proposition 3.3. *We have an $I_{N+1}\rtimes W$-equivariant isomorphism $\widetilde{X}'_m|_{\mathfrak{a}^{rs}}\simeq\tilde{\mathcal{Y}}_m$. In particular, it induces an $I_N\rtimes W$-equivariant isomorphism on the quotient*

$$X'_m|_{\mathfrak{a}^{rs}}\simeq\widetilde{X}'_m|_{\mathfrak{a}^{rs}}/(\mathbb{Z}/2\mathbb{Z})\simeq\tilde{\mathcal{Y}}_m/(\mathbb{Z}/2\mathbb{Z})\simeq\mathcal{Y}_m.$$

Here $\mathbb{Z}/2\mathbb{Z}$ acts on $\widetilde{X}'_m$ and $\tilde{\mathcal{Y}}_m$ via the map $\mathbb{Z}/2\mathbb{Z}\to I_{N+1}$ given by $1\mapsto(0,\dots,0,1)$.

We follow closely the argument in [T, §2]. We begin by introducing some auxiliary spaces and maps. Let $\widetilde{V}'_m\subset\widetilde{V}\times\mathfrak{a}^{rs}$ be the vector subbundle whose fiber over $a\in\mathfrak{a}^{rs}$ is the subspace $\widetilde{V}'_{m,a}\subset\widetilde{V}$ defined by the equations

$$\frac{a_1^i}{d_1}v_1+\cdots+\frac{a_N^i}{d_N}v_N=0,\ \ i=0,\dots,m-1,\ \ \frac{a_1^m}{d_1}v_1+\cdots+\frac{a_N^m}{d_N}v_N-v_{N+1}=0. \tag{3.15}$$

The map $\tilde{s}:\mathbb{P}(\widetilde{V})\times\mathfrak{a}^{rs}\to\mathbb{P}(\widetilde{V})\times\mathfrak{a}^{rs}$ (see (3.11)) maps $\widetilde{X}'_m|_{\mathfrak{a}^{rs}}$ to $\mathbb{P}(\widetilde{V}'_m)$ and the resulting map

$$\tilde{s}':\widetilde{X}'_m|_{\mathfrak{a}^{rs}}\to\mathbb{P}(\widetilde{V}'_m)$$

is an I_{N+1}-branched cover with branch locus $\{v_i=0,\ i=1,\dots,N+1\}$.

Let $\mathbb{C}(\mathbb{P}(\widetilde{V}'_m))$ be the function field of $\mathbb{P}(\widetilde{V}'_m)$. Then the function field of $\widetilde{X}'_m|_{\mathfrak{a}^{rs}}$ is given by the following field extension:

$$F := \mathbb{C}(\mathbb{P}(\widetilde{V}'_m))((\frac{v_i}{v_{N+1}})^{1/2}_{i=1,\ldots,N}) \supset \mathbb{C}(\mathbb{P}(\widetilde{V}'_m)).$$

Since $\widetilde{X}'_m|_{\mathfrak{a}^{rs}}$ is smooth and $\tilde{s}'$ is finite, it follows that

$$\widetilde{X}'_m|_{\mathfrak{a}^{rs}} \text{ is the normalization of } \mathbb{P}(\widetilde{V}'_m) \text{ in } F.^2$$

The group I_{N+1} acts on F by $\zeta : v_i^{1/2} \mapsto (-1)^{\zeta_i}\, v_i^{1/2},\ \zeta = (\zeta_1, ..., \zeta_{N+1}) \in I_{N+1}$ and W acts on F by $w : (\frac{v_i}{v_{N+1}})^{1/2} \mapsto (\frac{v_{w(i)}}{v_{N+1}})^{1/2}$.

Similarly, let k_η be the function field of $\mathfrak{a}^{rs}$ and let $\eta = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}(k_\eta)$ be the corresponding generic point. Then the function field of $\tilde{\mathcal{Y}}_m$ is given by the following field extension:

$$F' = \mathbb{C}(\mathbb{P}^{N-m-1}_{\mathfrak{a}^{rs}})((\frac{H_{\eta,i}}{H_{\eta,N+1}})^{1/2}_{i=1,\ldots,N}) \supset \mathbb{C}(\mathbb{P}^{N-m-1}_{\mathfrak{a}^{rs}}).$$

Here, $H_{\eta,i}$, $i = 1, \ldots, N$ are the hyperplanes associated to $\eta \in \mathfrak{a}^{rs}$ in (3.4), $H_{\eta,N+1} = x_{N-m}$, and $\frac{H_{\eta,i}}{H_{\eta,N+1}}$ are rational functions on $\mathbb{P}^{N-m-1}_\eta$, regraded as elements in $\mathbb{C}(\mathbb{P}^{N-m-1}_\eta) = \mathbb{C}(\mathbb{P}^{N-m-1}_{\mathfrak{a}^{rs}})$. Since $\tilde{\mathcal{Y}}_m$ is smooth and $\tilde{\iota} : \tilde{\mathcal{Y}}_m \to \mathbb{P}^{N-m-1}_{\mathfrak{a}^{rs}}$ is finite, it follows that

$$\tilde{\mathcal{Y}}_m \text{ is the normalization of } \mathbb{P}^{N-m-1}_{\mathfrak{a}^{rs}} \text{ in } F'.$$

The group I_{N+1} acts on F' by $\zeta : H_{\eta,i}^{1/2} \mapsto (-1)^{\zeta_i}\, H_{\eta,i}^{1/2},\ \zeta = (\zeta_1, \ldots, \zeta_{N+1}) \in I_{N+1}$ and W acts on F by $w : (\frac{H_{\eta,i}}{H_{\eta,N+1}})^{1/2} \mapsto (\frac{H_{\eta,w(i)}}{H_{\eta,N+1}})^{1/2}$.

By the discussion above, to prove Proposition 3.3, it is enough to prove the following statement:

$$\text{(3.16)}\quad \begin{array}{c}\text{The two } \mathfrak{a}^{rs}\text{-families of configurations}\\ (\mathbb{P}(\widetilde{V}'_{m,a}), \{v_i\}_{i=1,\ldots N+1}) \text{ and } (\mathbb{P}^{N-m-1}_{\mathfrak{a}^{rs}}, \{H_{a,i}\}_{i=1,\ldots,N+1}) \text{ are equivalent.}\end{array}$$

That is, there is an isomorphism (or trivialization) of vector bundles $\phi : \mathbb{C}^{N-m} \times \mathfrak{a}^{rs} \simeq \widetilde{V}'_m$ over $\mathfrak{a}^{rs}$ such that for any k-point $a \in \mathfrak{a}^{rs}(k)$, k a field, the induced map on the dual fibers $\phi_a^* : (\widetilde{V}'_{m,a})^* \simeq k^{N-m}$ satisfies $\phi_a^*(v_i) = H_{a,i}$ for $i = 1, \ldots, N+1$.

To prove (3.16), we need to construct, for each S-point $a \in \mathfrak{a}^{rs}(S)$, a functorial isomorphism $\phi_a : \mathbb{C}^{N-m} \times S \simeq \widetilde{V}'_{m,a}$ satisfying the desired property. For notational simplicity we construct such isomorphisms on the level of k-points. The argument for general S-points is the same.

Consider the following map:

$$\psi_a : \widetilde{V} \otimes k \xrightarrow{pr} V \otimes k \overset{\times d}{\simeq} V \otimes k,$$

where $pr : \widetilde{V} \otimes k = (V \otimes k) \oplus k \to V \otimes k$ is the projection map and the second isomorphism is given by multiplying the diagonal matrix $d = \mathrm{diag}(d_1^{-1}, \ldots, d_N^{-1}) \in GL(V \otimes k)$ (recall for $a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}$, $d_i = \prod_{j\neq i}(a_j - a_i)$). One can check that ψ_a maps $\widetilde{V}'_{m,a}$ isomorphically onto $V_{m,a}$ (see (3.15) and (3.5) for the definitions

[2]Recall for any irreducible variety X and K a finite extension of the function field $\mathbb{C}(X)$, there exists a unique normal variety Y and a finite morphism $f : Y \to X$ such that the induced map $\mathbb{C}(X) \to \mathbb{C}(Y) = K$ is the given field extension. We call Y the *normalization* of X in K.

of $\widetilde{V}'_{m,a}$ and $V_{m,a}$, respectively) and the resulting isomorphism $\psi_a : \widetilde{V}'_{m,a} \simeq V_{m,a}$ satisfies

$$\psi_a^*(d_i \cdot v_i) = v_i \text{ for } i = 1, \ldots, N \quad \text{and} \quad \psi_a^*(H_\infty) = v_{N+1},$$

where $H_\infty := a_1^m v_1 + \cdots + a_N^m v_N$. Thus we are reduced to show that

$$(\mathbb{P}(V_{m,a}), \{d_i \cdot v_i\}_{i=1,\ldots,N} \cup H_\infty) \text{ and } (\mathbb{P}(k^{N-m}), \{H_{a,i}\}_{i=1,\ldots N+1}) \text{ are equivalent,}$$

that is, there is an isomorphism

$$\gamma_a : k^{N-m} \simeq V_{m,a}$$

such that $\gamma_a^*(d_i \cdot v_i) = H_{a,i}$, $i = 1, \ldots, N$ and $\gamma_a^*(H_\infty) = H_{a,N+1}$.

Consider the basis $u_i = (a_1^i, \ldots, a_N^i)$, $i = 0, \ldots, N-1$ of $V \otimes k$. Then the isomorphism $V \otimes k \simeq (V \otimes k)^*$, given by the pairing $\langle (v_i), (w_i) \rangle = \sum v_i w_i$, induces the following isomorphism:

$$f_1 : V_{m,a}^* \simeq V \otimes k / k\langle u_0, \ldots, u_{m-1} \rangle \simeq k\langle u_m, \ldots, u_{N-1} \rangle.$$

Let s_i be the elementary symmetric polynomial in $a_1, \ldots, a_N$ of degree i and let $A = (a_{ij}) \in GL_{N-m}(k)$ be the matrix with entries $a_{ij} = (-1)^{i-1} s_{j-i}$ if $j \geq i$ and $a_{ij} = 0$ otherwise. Consider the following isomorphism:

$$f : V_{m,a}^* \overset{f_1}{\simeq} k\langle u_m, \ldots, u_{N-1} \rangle \overset{f_2}{\simeq} k^{N-m} \overset{f_3}{\simeq} k^{N-m}.$$

Here $f_2 : k\langle u_m, \ldots, u_{N-1} \rangle \simeq k^{N-m}$ is the isomorphism given by $u_{N-i} \mapsto (-1)^{i-1} x_i$[3] and f_3 is the isomorphism given by right multiplication by A^{-1}. We claim that the dual

$$\gamma_a := f^* : k^{N-m} \simeq V_{m,a}$$

is the desired isomorphism, i.e., we have $f(d_i \cdot v_i) = H_{a,i}$ for $i = 1, \ldots, N$, and $f(H_\infty) = H_{a,N+1}$. Note that the configuration $(\mathbb{P}(V_{m,a}), \{d_i \cdot v_i\}_{i=1,\ldots,N})$ (resp., $(\mathbb{P}(k^{N-m}), \{H_{a,i}\}_{i=1,\ldots,N})$) is equal to the configuration $(P, H_1, \ldots, H_N)$ (resp., $(P', H'_1, \ldots, H'_N)$) in [T, §2.1]. Moreover, the map f is the one used in [T, Proof of Theorem 2.1.1] to show that the above configurations are equivalent. Thus according to [T, Proof of Theorem 2.1.1] we have

$$f(d_i \cdot v_i) = H_{a,i} \text{ for } i = 1, \ldots, N.$$

So it remains to show that $f(H_\infty) = H_{a,N+1}$. For this we observe that $f_1(H_\infty) = u_m$. Hence

$$f(H_\infty) = f_3 \circ f_2 \circ f_1(H_\infty) = f_3 \circ f_2(u_m) = (-1)^{N-m-1} \cdot f_3(x_{N-m})$$

$$= (-1)^{N-m-1} \cdot x_{N-m} \cdot A = x_{N-m} = H_{a,N+1}.$$

This proves (3.16).

The proof of Proposition 3.3 is complete.

[3] Here we regard x_i as the ith coordinate vector of k^{N-m}.

4. Monodromy of families of Hessenberg varieties

In this section we study the monodromy representation of $\pi_1^K(\mathfrak{g}_1^{rs},a)=Z_K(a)\rtimes B_N$ on the primitive cohomology of complete intersections of quadrics X_m (and on the primitive cohomology of their branched double covers $\widetilde{X}_m$). By Theorem 2.5 this gives us a complete description of the monodromy representations of $\pi_1^K(\mathfrak{g}_1^{rs})$ associated to the families of Hessenberg varieties $\mathrm{Hess}_n^{O,\perp}$ and $\mathrm{Hess}_n^{E,\perp}$.

To state the result, let us recall, from §3.2, the monodromy representations $\rho_{C_\chi}: P_N\to Sp(H^1(C_{a,\chi},\mathbb{C}))\simeq Sp(2i-2)$ and $\rho_{\tilde{C}_\chi}: P_N\to Sp(H^1(\tilde{C}_{a,\chi},\mathbb{C}))\simeq Sp(2i-2)$ where $i=\frac{|\chi|}{2}$. Recall further that, by (3.2), these representations are irreducible with Zariski dense image. Let us consider the irreducible representation of $Sp(2i-2)$ associated to the fundamental weight ω_j. Composing ρ_{C_χ} and $\rho_{\tilde{C}_\chi}$ with this fundamental representation we obtain irreducible representations P_χ^j and $\tilde{\mathrm{P}}_\chi^j$ of the pure braid group P_N.

For a character χ of an abelian group we write V_χ for the corresponding one dimensional representation. Recall that the group $Z_K(a)$ can be naturally identified with I_N as explained in (3.1). We also relate the characters of I_N and I_{N+1} using the map κ defined in (3.12). From these considerations we conclude that

$$Z_K(a)^\vee=I_N^\vee \qquad \text{and we have a map} \qquad \check{\kappa}: I_{N+1}^\vee\to I_N^\vee\,. \tag{4.1}$$

In particular, characters of I_N and I_{N+1} can be regarded as characters of $Z_K(a)$. To state the main theorems of this section we define two $Z_K(a)\rtimes P_N$-representations as follows:

$$E_{ij}^N\simeq\bigoplus_{\chi\in I_N^\vee,\,|\chi|=2i}\mathrm{P}_\chi^j\otimes V_\chi \qquad\text{and}\qquad \widetilde{E}_{ij}^N=\bigoplus_{\substack{\chi\in I_{N+1}^\vee,\,|\chi|=2i,\\ N+1\in\operatorname{supp}\chi}}\tilde{\mathrm{P}}_\chi^j\otimes V_\chi,$$

where the I_N acts on $\widetilde{E}_{ij}^N$ via the map $\check{\kappa}: I_{N+1}^\vee\to I_N^\vee$ of (4.1), and P_N acts on V_χ via the map $\rho: P_N\to I_N$ of (3.7). Lemmas 4.4 and 4.6 show that the $Z_K(a)\rtimes P_N$ actions on E_{ij}^N and $\widetilde{E}_{ij}^N$ extend naturally to $Z_K(a)\rtimes B_N$-actions.

The main results of this section are the following.

Theorem 4.1. *For $1\le m\le N-1$, the monodromy representation of $\pi_1^K(\mathfrak{g}_1^{rs},a)$ on $P(X_m):=H_{prim}^{N-m-1}(X_{m,a},\mathbb{C})$ decomposes into irreducible representations in the following manner:*

$$P(X_m)\simeq\bigoplus_i\bigoplus_{\substack{j\equiv N-m-1 \bmod 2\\ j\in[0,l]}}E_{ij}^N,$$

with $N-m+1\le 2i\le N$, $l=\min\{N-m-1,-N+m+2i-1\}$.

To state the second main result, we set

$$P(\widetilde{X}_m):=H_{prim}^{N-m-1}(\widetilde{X}_{m,a},\mathbb{C}).$$

Recall that there is an involution action σ on $\tilde{X}_m$ and the projection map $p_m:\tilde{X}_m\to X_m$ is a branched double cover with Galois group $\langle\sigma\rangle\simeq\mathbb{Z}/2\mathbb{Z}$ (see §3.4). Then $P(\widetilde{X}_m)=P(\widetilde{X}_m)^{\sigma=\mathrm{id}}\oplus P(\widetilde{X}_m)^{\sigma=-\mathrm{id}}$ and we have $P(\widetilde{X}_m)^{\sigma=\mathrm{id}}=P(X_m)$. The next theorem describes $P(\widetilde{X}_m)^{\sigma=-\mathrm{id}}$.

Theorem 4.2. *For $1 \leq m \leq N-1$, the monodromy representation of $\pi_1^K(\mathfrak{g}_1^{rs}, a)$ on $P(\widetilde{X}_m)^{\sigma=-\mathrm{id}}$ decomposes into irreducible representations in the following manner:*

$$P(\widetilde{X}_m)^{\sigma=-\mathrm{id}} \simeq \bigoplus_i \bigoplus_{\substack{j \equiv N-m-1 \bmod 2 \\ j \in [0,l]}} \widetilde{E}_{ij}^N,$$

with $N-m+1 \leq 2i \leq N+1$, $l = \min\{N-m-1, -N+m+2i-1\}$.

4.1. **Proof of Theorem 4.1.** Let us start with the following proposition which is a consequence of Proposition 3.1.

Proposition 4.3. *There is an isomorphism of representations of $\pi_1^K(\mathfrak{g}_1^{rs}, a) \simeq I_N \rtimes B_N$*

$$H^i(X_{m,a}, \mathbb{C}) \simeq \bigoplus_{\chi \in I_N^\vee} H^i(\mathcal{Y}_{m,a}, \mathbb{C})_\chi \otimes V_\chi.$$

The group I_N acts on the summand $H^i(\mathcal{Y}_{m,a}, \mathbb{C})_\chi \otimes V_\chi$ via the character $\chi \in I_N^\vee$.

Proof. Observe that the families $X_m|_{\mathfrak{a}^{rs}} \to \mathfrak{a}^{rs}$, $\mathcal{Y}_m \to \mathfrak{a}^{rs}$, and $\tilde{\mathfrak{a}}^{rs} \to \mathfrak{a}^{rs}$ are all W-equivariant. Hence their cohomology groups $H^i(X_{m,a}, \mathbb{C})$, $H^i(\mathcal{Y}_{m,a}, \mathbb{C})$, $H^0((\tilde{\mathfrak{a}}^{rs})_a, \mathbb{C})$ carry an action of the braid group $B_N \simeq \pi_1^W(\mathfrak{a}^{rs}, a)$. Let

$$H^i(X_{m,a}, \mathbb{C}) = \bigoplus_{\chi \in I_N^\vee} H^i(X_{m,a}, \mathbb{C})_\chi, \qquad H^i(\mathcal{Y}_{m,a}, \mathbb{C}) = \bigoplus_{\chi \in I_N^\vee} H^i(\mathcal{Y}_{m,a}, \mathbb{C})_\chi,$$

$$H^0((\tilde{\mathfrak{a}}^{rs})_a, \mathbb{C}) = \bigoplus_{\chi \in I_N^\vee} V_\chi,$$

be the decompositions with respect to the action of I_N; for the last identity we recall that $\tilde{\mathfrak{a}}^{rs} \to \mathfrak{a}^{rs}$ is an I_N-torsor. For $\chi \in I_N^\vee$ and $b \in B_N$, we write $b \cdot \chi$ for the action of b on χ. Then the braid group action on $H^i(\mathcal{Y}_{m,a}, \mathbb{C})$ is described as follows:

$$b \in B_N : H^i(\mathcal{Y}_{m,a}, \mathbb{C})_\chi \mapsto H^i(\mathcal{Y}_{m,a}, \mathbb{C})_{b \cdot \chi}.$$

The B_N-actions on $H^i(X_{m,a}, \mathbb{C})$ and $H^0((\tilde{\mathfrak{a}}^{rs})_a, \mathbb{C})$ are described in the same manner.

By the Künneth formula, the cohomology of the fiber of $\mathcal{Y}_m^t = (\mathcal{Y}_m \times_{\mathfrak{a}^{rs}} \tilde{\mathfrak{a}}^{rs})/I_N$ over $a \in \mathfrak{a}^{rs}$ is canonically isomorphic to

$$H^i(\mathcal{Y}_{m,a}^t, \mathbb{C}) \simeq \bigoplus_{\chi \in I_N^\vee} H^i(\mathcal{Y}_{m,a}, \mathbb{C})_\chi \otimes V_\chi.$$

Thus by (3.8) we obtain the desired $\pi_1^K(\mathfrak{g}_1^{rs}, a) \simeq I_N \rtimes B_N$-equivariant isomorphism. □

The isomorphism in Proposition 4.3 implies the following isomorphism of monodromy representations:

$$P(X_m) \simeq \bigoplus_{\chi \in I_N^\vee} P(\mathcal{Y}_m)_\chi \otimes V_\chi, \tag{4.2}$$

where

$$P(\mathcal{Y}_m) := H_{prim}^{N-m-1}(\mathcal{Y}_{m,a}, \mathbb{C}), \quad P(\mathcal{Y}_m)_\chi := H^{N-m-1}(\mathcal{Y}_m, \mathbb{C})_\chi \cap P(\mathcal{Y}_m).$$

Our goal is to decompose the representation above into irreducible representations. Observe that each summand $P(\mathcal{Y}_m)_\chi$ is invariant under the action of the pure braid group P_N. According to [T, Theorem 2.5.1], there is an isomorphism of representations of P_N

$$P(\mathcal{Y}_m)_\chi \simeq \bigwedge^{N-m-1} H^1(C_a,\mathbb{C})_\chi \simeq \bigwedge^{N-m-1} H^1(C_{a,\chi},\mathbb{C}),$$

where P_N acts on $H^1(C_{a,\chi},\mathbb{C})$ via the map $\rho_{C_\chi}: P_N \to Sp(2i-2)$, $i=|\chi|/2$.

As an $Sp(2i-2)$-representation $\bigwedge^{N-m-1} H^1(C_{a,\chi},\mathbb{C})$ decomposes into a direct sum of fundamental representations in a well-known manner. This implies the following decomposition of $P(\mathcal{Y}_m)_\chi$ into irreducible representations of P_N:

$$P(\mathcal{Y}_m)_\chi \simeq \bigwedge^{N-m-1} H^1(C_{a,\chi},\mathbb{C}) = \bigoplus_{j\equiv N-m-1 \bmod 2,\ j\in[0,l]} \mathrm{P}_\chi^j, \tag{4.3}$$

where $l=\min\{N-m-1,-N+m+|\chi|-1\}$.

Combining (4.2) with (4.3), we obtain the following decomposition:

$$P(X_m) \simeq \bigoplus_{\chi\in I_N^\vee} P(\mathcal{Y}_m)_\chi \otimes V_\chi \simeq \bigoplus_{\chi\in I_N^\vee}\bigoplus_j \mathrm{P}_\chi^j \otimes V_\chi. \tag{4.4}$$

Using the notation from the beginning of this section the decomposition (4.4) can be rewritten as

$$P(X_m) \simeq \bigoplus_i \bigoplus_{j\equiv N-m-1 \bmod 2,\ j\in[0,l]} E_{ij}^N\ , \tag{4.5}$$

where $N-m+1\le 2i\le N$ and $l=\min\{N-m-1,-N+m+2i-1\}$.

We have the following.

Lemma 4.4. (1) *Each E_{ij}^N is an irreducible representation of $\pi_1^K(\mathfrak{g}_1^{rs},a)$. We denote by $\rho_{ij}^N:\pi_1^K(\mathfrak{g}_1^{rs},a)\to GL(E_{ij}^N)$ the corresponding map.*

(2) *Suppose $j>0$. Let $H:=\overline{\rho_{ij}^N(P_N)}\subset GL(E_{ij}^N)$ be the Zariski closure of $\rho_{ij}^N(P_N)$ in $GL(E_{ij}^N)$ (recall $P_N\subset\pi_1^K(\mathfrak{g}_1^{rs},a)$ is the pure braid group). Then we have $\operatorname{Lie} H\simeq\mathfrak{sp}(2i-2)$. In particular, the image $\rho_{ij}^N(\pi_1^K(\mathfrak{g}_1^{rs},a))$ is infinite.*

Proof. We begin with the proof of (1). We first show that E_{ij}^N is a $\pi_1^K(\mathfrak{g}_1^{rs},a)$-invariant subspace of $P(X_m)$. For this, we observe that the decomposition in (4.4) is compatible with the action of B_N, that is, for $b\in B_N$,

$$b:\mathrm{P}_\chi^j\otimes V_\chi \mapsto \mathrm{P}_{b\cdot\chi}^j\otimes V_{b\cdot\chi}.$$

Since the braid group B_N acts transitively on the set $\{\chi\in I_N^\vee \mid |\chi|=2i\}$, it follows that the subspace

$$E_{ij}^N = \bigoplus_{\chi\in I_N^\vee,|\chi|=2i} \mathrm{P}_\chi^j\otimes V_\chi$$

is stable under the action of $\pi_1^K(\mathfrak{g}_1^{rs},a)$. Now since each summand $\mathrm{P}_\chi^j\otimes V_\chi$ is irreducible as a representation of P_N, it follows that each E_{ij}^N is an irreducible representation of $\pi_1^K(\mathfrak{g}_1^{rs},a)$.

We prove (2). For each $\chi\in I_N^\vee$, we define $\rho_\chi: P_N\xrightarrow{\rho} I_N\xrightarrow{\chi}\mu_2$. Here ρ is the map in (3.7). Define

$$\psi_1 := (\rho_{C_\chi}, \bigoplus_{\chi,|\chi|=2i}\rho_\chi): P_N\to Sp(2i-2)\times\mu_2^{\binom{N}{2i}}.$$

Let V_{ij} denote the irreducible representation of $Sp(2i-2)$ associated to the fundamental weight ω_j. Then the restriction of $\rho_{ij}^N : \pi_1^K(\mathfrak{g}_1^{rs}, a) \to GL(E_{ij}^N)$ to P_N can be identified with

$$\psi : P_N \xrightarrow{\psi_1} Sp(2i-2) \times \mu_2^{\binom{N}{2i}} \xrightarrow{\psi_2} GL(V_{ij})^{\times \binom{N}{2i}},$$

where ψ_2 maps $Sp(2i-2)$ diagonally into $GL(V_{ij})^{\times\binom{N}{2i}}$ and ψ_2 maps $\mu_2 = \{\pm 1\}$ to $\pm\,\mathrm{id} \in GL(V_{ij})$. Since $\overline{\rho_{C_\chi}(P_N)} = Sp(2i-2)$, it implies that the connected component $\overline{\psi_1(P_N)}^0 = Sp(2i-2)$. So to prove (2), it suffices to show that $\mathrm{Lie}(\mathrm{Im}(\psi_2)) \simeq \mathfrak{s}p(2i-2)$ for $j>0$. This follows from the fact that the induced map $d\psi_2 : \mathfrak{s}p(2i-2) \to \bigoplus \mathfrak{gl}(V_{ij})$ on the Lie algebras is injective. $\square$

It follows from the lemma above that (4.5) is the decomposition of the monodromy representation $P(X_m)$ into irreducible subrepresentations. This completes the proof of Theorem 4.1.

4.2. **Proof of Theorem 4.2.** The proof is similar to the case of X_m. First using the isomorphism (3.13) and the same argument as in the case of X_m, we obtain the following proppsition.

Proposition 4.5. *There is an isomorphism of $I_{N+1} \rtimes B_N$-representations*

$$H^i(\widetilde{X}_{m,a}, \mathbb{C}) \simeq \bigoplus_{\chi \in I_{N+1}^\vee} H^i(\tilde{\mathcal{Y}}_{m,a}, \mathbb{C})_\chi \otimes V_\chi,$$

where for V_χ, we regard χ as an element in $I_N^\vee$ via the map $\check{\kappa} : I_{N+1}^\vee \to I_N^\vee$ in (4.1), and the group I_{N+1} acts on the summand $H^i(\tilde{\mathcal{Y}}_{m,a}, \mathbb{C})_\chi \otimes V_\chi$ via the character $\chi \in I_{N+1}^\vee$.

Set

$$P(\tilde{\mathcal{Y}}_m) := H_{prim}^{N-m-1}(\tilde{\mathcal{Y}}_{m,a}, \mathbb{C}).$$

By Proposition 4.5, there is an isomorphism of $I_{N+1} \rtimes B_N$-representations

$$P(\widetilde{X}_m) \simeq \bigoplus_{\chi \in I_{N+1}^\vee} P(\tilde{\mathcal{Y}}_m)_\chi \otimes V_\chi. \tag{4.6}$$

For any $\chi \in I_{N+1}^\vee$ with $|\chi| = 2i$, let $\widetilde{C}_{a,\chi}$ be the hyperelliptic curve defined in §3.2 and let $\rho_{\widetilde{C}_\chi} : P_N \to Sp(H^1(\widetilde{C}_{a,\chi}, \mathbb{C})) \simeq Sp(2i-2)$ denote the monodromy representation for the family $\widetilde{C}_\chi \to \mathfrak{a}^{rs}$. Again by [T], we have an isomorphism of P_N-representations

$$P(\tilde{\mathcal{Y}}_m)_\chi \simeq \wedge^{N-m-1} H^1(\widetilde{C}_a, \mathbb{C})_\chi \simeq \wedge^{N-m-1} H^1(\widetilde{C}_{a,\chi}, \mathbb{C})). \tag{4.7}$$

Combining (4.6) with (4.7) we obtain the following decomposition:

$$P(\widetilde{X}_m) \simeq \bigoplus_{\chi \in I_{N+1}^\vee} \wedge^{N-m-1} H^1(\widetilde{C}_{a,\chi}, \mathbb{C}) \otimes V_\chi. \tag{4.8}$$

We describe the monodromy representation $P(\widetilde{X}_m)^{\sigma = -\mathrm{id}}$ (recall that σ is the involution on $\tilde{X}_m$). For this, we first observe that the involution action of $\langle \sigma \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ on $\widetilde{X}_m$ is equal to the composition of

$$i_\infty : \mathbb{Z}/2\mathbb{Z} \to I_{N+1},\ 1 \mapsto (0, \ldots, 0, 1),$$

with the action of I_{N+1} on $\widetilde{X}_m$. Hence by (4.8) we have

$$(4.9)\quad P(\widetilde{X}_m)^{\sigma=-id} = \bigoplus_{\substack{\chi\in I_{N+1}^\vee \\ N+1\in\operatorname{supp}\chi}} P(\widetilde{X}_m)_\chi \simeq \bigoplus_{\substack{\chi\in I_{N+1}^\vee \\ N+1\in\operatorname{supp}\chi}} \wedge^{N-m-1}H^1(\widetilde{C}_{a,\chi},\mathbb{C})\otimes V_\chi.$$

Again, since $\rho_{\widetilde{C}_\chi}(P_N)$ is Zariski dense in $Sp(H^1(\widetilde{C}_\chi,\mathbb{C}))$, we have the following decomposition:

$$\wedge^{N-m-1}H^1(\widetilde{C}_{a,\chi},\mathbb{C}) = \bigoplus_{j\equiv N-m-1 \bmod 2,\ j\in[0,l]} \tilde{\mathrm{P}}_\chi^j,$$

where $l=\min\{N-m-1,-N+m+|\chi|-1\}$.

Using the notation from the beginning of this section the decomposition (4.9) can be rewritten as

$$P(\widetilde{X}_m)^{\sigma=-\mathrm{id}} \simeq \bigoplus_i \bigoplus_{\substack{j\equiv N-m-1 \bmod 2 \\ j\in[0,l]}} \widetilde{E}_{ij}^N,$$

where $N-m+1\le 2i\le N+1$, $l=\min\{N-m-1,-N+m+2i-1\}$.

The same argument as in the proof of Lemma 4.4 shows the following.

Lemma 4.6. (1) *$\widetilde{E}_{ij}^N$ is an irreducible representation of $\pi_1^K(\mathfrak{g}_1^{rs},a)$. We denote by $\tilde{\rho}_{ij}^N:\pi_1^K(\mathfrak{g}_1^{rs},a)\to GL(E_{ij}^N)$ the corresponding map.*

(2) *Suppose $j>0$. Let $H:=\overline{\tilde{\rho}_{ij}^N(P_N)}\subset GL(\widetilde{E}_{ij}^N)$ be the Zariski closure of $\tilde{\rho}_{ij}^N(P_N)$ in $GL(\widetilde{E}_{ij}^N)$. Then we have* $\operatorname{Lie}H\simeq\mathfrak{sp}(2i-2)$. *In particular, the image $\tilde{\rho}_{ij}^N(\pi_1^K(\mathfrak{g}_1^{rs},a))$ is infinite.*

This completes the proof of Theorem 4.2.

4.3. **The local systems E_{ij}^N and $\widetilde{E}_{ij}^N$.** In this subsection, we show that from the constructions in previous sections, we have obtained the following set consisting of pairwise non-isomorphic irreducible K-equivariant local systems on $\mathfrak{g}_1^{rs}$

$$(4.10)\quad \left\{E_{ij}^{2n+1},\ i\in[1,n],\ j\in[0,i-1];\ \widetilde{E}_{ij}^{2n+1},\ i\in[1,n+1],\ j\in[1,i-1],\ \widetilde{E}_{n+1,0}^{2n+1}\cong\mathbb{C}\right\}.$$

For this, we first observe that

$$E_{ij}^N\simeq E_{i'j'}^N \quad\text{and}\quad \widetilde{E}_{ij}^N\simeq\widetilde{E}_{i'j'}^N \quad\text{if and only if } i=i',j=j'.$$

In fact, assume that $E_{ij}^N\simeq E_{i'j'}^N$. Then we must have $i=i'$, otherwise, the centralizer $Z_K(a)\cong I_N$ would act differently on E_{ij}^N and $E_{i'j'}^N$. Now regarding E_{ij}^N and $E_{ij'}^N$ as P_N-representations we see that $j=j'$. Similar argument applies to $\widetilde{E}_{ij}^N$.

It remains to prove the following.

Lemma 4.7. *We have $E_{i,j}^N\cong\widetilde{E}_{i',j'}^N$ if and only if $i+i'=(N+1)/2$ and $j=j'=0$.*

Proof. Recall

$$E_{ij}^N=\bigoplus_{\chi\in I_N^\vee,|\chi|=2i}\mathrm{P}_\chi^j\otimes V_\chi,\ \ \widetilde{E}_{i'j'}^N=\bigoplus_{\substack{\chi'\in I_{N+1}^\vee,\,|\chi|=2i', \\ N+1\in\operatorname{supp}\chi}}\tilde{\mathrm{P}}_{\chi'}^{j'}\otimes V_{\chi'}.$$

For $V_{\chi'}$ we regard χ' as an element in $I_N^\vee$ via the map $\check{\kappa}: I_{N+1}^\vee \to I_N^\vee$ in (4.1). Observe that for $\chi' \in I_{N+1}^\vee$ and $N+1 \in \operatorname{supp}\chi'$, we have

$$\check{\kappa}(\chi') = \chi \text{ if and only if } \operatorname{supp}\chi = \{1,\dots,N+1\}\backslash \operatorname{supp}\chi'. \tag{4.11}$$

Thus the map $\check{\kappa}$ maps the subset $\{\chi' \in I_{N+1}^\vee, |\chi'| = 2i',\ N+1 \in \operatorname{supp}\chi'\}$ bijectively to the subset $\{\chi \in I_N^\vee, |\chi| = 2i\}$ where $2i + 2i' = N+1$. Hence we have

$$\bigoplus_{\chi\in I_N^\vee, |\chi|=2i} V_\chi \cong \bigoplus_{\substack{\chi\in I_{N+1}^\vee, |\chi'|=2i',\\ N+1\in \operatorname{supp}\chi}} V_{\chi'} \text{ if and only if } i+i' = (N+1)/2. \tag{4.12}$$

This implies $E_{i,0}^N \cong \widetilde{E}_{i',0}^N$ for $i+i' = (N+1)/2$.

Conversely, we observe that $E_{i,j}^N \cong \widetilde{E}_{i',j'}^N$ implies $\mathrm{P}_\chi^j \otimes V_\chi \cong \tilde{\mathrm{P}}_{\chi'}^{j'} \otimes V_{\chi'}$ (as representations of $I_N \rtimes P_N$) for some $\chi \in I_N^\vee$ and $\chi' \in I_{N+1}^\vee$ with $N+1 \in \operatorname{supp}\chi'$. This implies that $\check{\kappa}(\chi') = \chi$ and it follows from (4.11) that $\operatorname{supp}\chi \cap \operatorname{supp}\chi' = \phi$. Therefore the monodromy representation of the restriction of $C_\chi \to \mathfrak{a}^{rs}$ (resp., $\widetilde{C}_{\chi'} \to \mathfrak{a}^{rs}$) to the subvariety $\mathfrak{a}^{rs}(a,\chi')$ (resp., $\mathfrak{a}^{rs}(a,\chi)$) in (3.3) is trivial. On the other hand, the monodromy representation of the restriction of $\widetilde{C}_{\chi'} \to \mathfrak{a}^{rs}$ (resp., $C_\chi \to \mathfrak{a}^{rs}$) to $\mathfrak{a}^{rs}(a,\chi')$ (resp., $\mathfrak{a}^{rs}(a,\chi)$) has Zariski dense image (see (3.2)). This forces $j = j' = 0$ and the desired claim follows again from (4.12). □

4.4. **The local systems E_{i0}^{2n+1}, $\widetilde{E}_{n+1,j}^{2n+1}$ and the $\mathcal{L}_i$'s, $\mathcal{F}_i$'s in** [CVX1]**.** Recall that in [CVX1, §2.3], we have defined the local systems $\mathcal{L}_i$ and $\mathcal{F}_i$ on $\mathfrak{g}_1^{rs}$. We have the following.

Lemma 4.8. *We have*

$$E_{i,0}^{2n+1} \cong \mathcal{L}_{2i} \quad \text{if} \quad 1 \le 2i \le n, \quad E_{i,0}^{2n+1} \cong \mathcal{L}_{2n-2i+1} \quad \text{if} \quad n+1 \le 2i \le 2n, \tag{4.13}$$

$$\widetilde{E}_{n+1,j}^{2n+1} \cong \mathcal{F}_j \text{ for } 1 \le j \le n. \tag{4.14}$$

Proof. We begin with the proof of (4.13). Recall from [CVX1] that we have

$$(\check{\pi}_{2^n1})_*\mathbb{C}|_{\mathfrak{g}_1^{rs}} \cong \bigoplus_{i=0}^n \mathcal{L}_i \quad \text{and} \quad \dim \mathcal{L}_i = \binom{2n+1}{i}, \tag{4.15}$$

where

$\check{\pi}_{2^n1}: K \times^{P_K} [\mathfrak{n}_P, \mathfrak{n}_P]_1^\perp := \{(x, 0 \subset V_n \subset V_n^\perp \subset \mathbb{C}^{2n+1}) \mid x \in \mathfrak{g}_1,\ xV_n \subset V_n^\perp\} \to \mathfrak{g}_1.$

On the other hand, recall the I_N-torsor over $\mathfrak{a}^{rs}$ in §3.3:

$$\begin{aligned}\tilde{\pi}: \tilde{\mathfrak{a}}^{rs} &= \{(a,c) \mid a = (a_1,\dots,a_N) \in \mathfrak{a}^{rs},\ c = (c_1,\dots,c_N),\ c_i^2 \\ &= \prod_{j\neq i}(a_j - a_i)\}/(\mathbb{Z}/2\mathbb{Z}) \to \mathfrak{a}^{rs}.\end{aligned}$$

We have

$$\tilde{\pi}_*\mathbb{C}|_{\mathfrak{g}_1^{rs}} \cong \mathbb{C} \oplus \bigoplus_{i=1}^n E_{i,0}^{2n+1} \quad \text{and} \quad \dim E_{i,0}^{2n+1} = \binom{2n+1}{2i}. \tag{4.16}$$

We show that there is an $I_N \rtimes W$-equivariant isomorphism

$$\tilde{\mathfrak{a}}^{rs} \cong K \times^{P_K} [\mathfrak{n}_P, \mathfrak{n}_P]_1^\perp|_{\mathfrak{a}^{rs}}. \tag{4.17}$$

Then (4.13) follows from (4.15), (4.16), and dimension considerations of the representations.

Using the identities $\sum_{i=1}^{2n+1} a_i^k\, c_i^{-2} = 0,\ k=0,\dots,2n-1$, it is easy to check that the map

$$\tilde{\mathfrak{a}}^{rs} \to X_{2n}|_{\mathfrak{a}^{rs}},\ (a,c)\in\tilde{\mathfrak{a}}^{rs} \mapsto [c_1^{-1},\dots,c_{2n+1}^{-1}]$$

defines an $I_N \rtimes W$-equivaraint isomorphism

$$\tilde{\mathfrak{a}}^{rs} \cong X_{2n}|_{\mathfrak{a}^{rs}}.$$

On the other hand, by the description of the Hessenberg varieties $\mathrm{Hess}_n^{E,\perp}$ in §2.2, we have a natural map

$$\mathrm{Hess}_n^{E,\perp} \to K\times^{P_K}[\mathfrak{n}_P,\mathfrak{n}_P]_1^{\perp},\ (x, V_{n-1}\subset V_n)\mapsto (x,V_n).$$

It is easy to check that the map above is a K-equivariant isomorphism over $\mathfrak{g}_1^{rs}$. The desired isomorphism (4.17) follows from the following compositions of isomorphisms:

$$\tilde{\mathfrak{a}}^{rs} \cong X_{2n}|_{\mathfrak{a}^{rs}} \overset{\text{Thm 2.5}}{\cong} \mathrm{Hess}_n^{E,\perp}|_{\mathfrak{a}^{rs}} \cong K\times^{P_K}[\mathfrak{n}_P,\mathfrak{n}_P]^{\perp}|_{\mathfrak{a}^{rs}}.$$

This completes the proof of (4.13).

To prove (4.14), we observe that

$$\widetilde{E}_{n+1,j}^{2n+1} \cong \tilde{P}_{\chi_0}^j \otimes V_{\chi_0} \cong (\bigwedge^j H^1(\widetilde{C}_{a,\chi_0},\mathbb{C}))_{prim}\otimes V_{\chi_0},$$

where $\chi_0\in I_{N+1}^\vee$ is the unique character such that $|\chi_0| = 2n+2$, and $\widetilde{C}_{a,\chi_0}$ is the hyperelliptic curve of genus n with affine equation $y^2=\prod_{i=1}^{2n+1}(x-a_i)$. By (4.11), χ_0, when regarded as an element in $I_N^\vee$ (see (4.1)), is trivial. Hence I_N acts trivially on $\widetilde{E}_{n+1,j}^{2n+1}$ and V_{χ_0}, i.e.,

$$\widetilde{E}_{n+1,j}^{2n+1} \cong (\bigwedge^j H^1(\widetilde{C}_{a,\chi_0},\mathbb{C}))_{prim} \cong \mathcal{F}_j, \tag{4.18}$$

where the last isomorphism follows from the discussion above and the definition of $\mathcal{F}_j$'s in [CVX1]. ☐

Remark 4.9. In [CVX1, Proof of Proposition 4.3] we used the fact that among the $\mathrm{IC}(\mathfrak{g}_1,\mathcal{L}_i)$'s ($i\geq 1$), only $\mathrm{IC}(\mathfrak{g}_1,\mathcal{L}_{2j-1})$, $1\leq j\leq m$, appear in the decomposition of $(\check{\tau}_m)_*\mathbb{C}[-]$, where $\check{\tau}_m=\check{\tau}_m^{2n+1}$ and $2m\leq n+1$. To prove this fact, it suffices to show that in the decomposition of the monodromy representation $P(X_{2m})$, only the above-mentioned local systems appear. Applying Theorem 4.1 to $P(X_{2m})$ with $N=2n+1$ we see that among the E_{i0}'s only those with $n-m+1\leq i\leq n$ appear. The desired conclusion follows from (4.13) and the fact that $2m\leq n+1$.

5. Computation of the Fourier transforms

Let $\mathfrak{F}: D_K(\mathfrak{g}_1)\to D_K(\mathfrak{g}_1)$ denote the Fourier transform, where we identify $\mathfrak{g}_1$ and $\mathfrak{g}_1^*$ via a K-invariant non-degenerate bilinear form on $\mathfrak{g}_1$. The Fourier transform $\mathfrak{F}$ induces an equivalence of categories $\mathfrak{F}:\mathrm{Perv}_K(\mathfrak{g}_1)\to\mathrm{Perv}_K(\mathfrak{g}_1)$.

In this section we study the Fourier transforms of $\mathrm{IC}(\mathfrak{g}_1,E_{ij}^N)$ and $\mathrm{IC}(\mathfrak{g}_1,\widetilde{E}_{ij}^N)$. We show that they are supported on $\mathcal{N}_1$, more precisely, on $\mathcal{N}_1^3\subset\mathcal{N}_1$, the closed subvariety consisting of nilpotent elements of order less than or equal to 3. Thus we obtain many more examples of IC complexes supported on nilpotent orbits whose Fourier transforms have both full support and infinite monodromy (see also [CVX1]). As an interesting corollary (see Example 5.5), we show that the Fourier transform of the IC extension of the unique non-trivial irreducible K-equivariant local system on the minimal nilpotent orbit has full support and its monodromy is given by a universal family of hyperelliptic curves.

The main result of this section is the following theorem.

Theorem 5.1. *Let $\mathcal{N}_1^3 \subset \mathcal{N}_1$ be the closed subvariety consisting of nilpotent elements of order less than or equal to* 3. *Then $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{ij}^N))$ and $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, \widetilde{E}_{ij}^N))$ are supported on $\mathcal{N}_1^3$.*

We first argue the case $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{ij}^N))$. For $m \leq \frac{N-1}{2}$, consider the families of Hessenberg varieties

$$\sigma_m^N : \mathrm{Hess}_m^O \to \mathfrak{g}_1,\ \tau_m^N : \mathrm{Hess}_m^E \to \mathfrak{g}_1$$

and

$$\check{\sigma}_m^N : \mathrm{Hess}_m^{O,\perp} \to \mathfrak{g}_1,\ \check{\tau}_m^N : \mathrm{Hess}_m^{E,\perp} \to \mathfrak{g}_1$$

defined in §2.2. We have (see (2.9))

$$\mathfrak{F}((\check{\sigma}_m^N)_*\mathbb{C}[-]) = (\sigma_m^N)_*\mathbb{C}[-],\ \ \mathfrak{F}((\check{\tau}_m^N)_*\mathbb{C}[-]) = (\tau_m^N)_*\mathbb{C}[-].$$

By Theorem 2.5, over $\mathfrak{g}_1^{rs}$, we have $\mathrm{Hess}_m^{O,\perp} \simeq X_{2m-1}$, $\mathrm{Hess}_m^E \simeq X_{2m}$. Hence the decomposition theorem implies that

$$\mathrm{IC}(\mathfrak{g}_1, P(X_{2m-1})) \text{ is a direct summand of } (\check{\sigma}_m^N)_*\mathbb{C}[-].$$

$$\mathrm{IC}(\mathfrak{g}_1, P(X_{2m}))) \text{ is a direct summand of } (\check{\tau}_m^N)_*\mathbb{C}[-]\,.$$

Therefore the Fourier transforms $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, P(X_{2m-1})))$ and $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, P(X_{2m})))$ appear as direct summands of $(\sigma_m^N)_*\mathbb{C}[-]$ and $(\tau_m^N)_*\mathbb{C}[-]$. Now in view of (2.6a) and (2.6b), we see that $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, P(X_{2m-1})))$ and $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, P(X_{2m})))$ are supported on $\mathcal{N}_1^3$. Since each local system E_{ij}^N appears in $P(X_m)$ for some m (see Theorem 4.1), we conclude that $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{ij}^N))$ is supported on $\mathcal{N}_1^3$.

It remains to consider the case $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, \widetilde{E}_{ij}^N))$. Since each local system $\widetilde{E}_{ij}^N$ appears in $P(\widetilde{X}_m)^{\sigma=-id}$ for some m, we are reduced to proving the following proposition.

Proposition 5.2. $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, P(\widetilde{X}_m)^{\sigma=-id}))$ *is supported on* $\mathcal{N}_1^3$.

The proof of this proposition occupies the remainder of this section.

5.1. **Proof of Proposition 5.2 when m is odd.** Recall that in §2.2 we have introduced the families of Hessenberg varieties

$$\mathrm{Hess}_k^{O,\perp} := \{(x, 0 \subset V_{k-1} \subset V_k \subset V_k^\perp \subset V_{k-1}^\perp \subset V = \mathbb{C}^N) \,|\, x \in \mathfrak{g}_1,\ xV_{k-1} \subset V_k\}$$

$$\mathrm{Hess}_k^{E,\perp} = \{(x, 0 \subset V_{k-1} \subset V_k \subset V_k^\perp \subset V_{k-1}^\perp \subset V = \mathbb{C}^N) \,|\, x \in \mathfrak{g}_1,\ xV_{k-1} \subset V_k, xV_k \subset V_k^\perp\}$$

and the natural projection maps $\check{\sigma}_k^N : \mathrm{Hess}_k^{O,\perp} \to \mathfrak{g}_1$, $\check{\tau}_k^N : \mathrm{Hess}_k^{E,\perp} \to \mathfrak{g}_1$.

Our first goal is to show that $\mathrm{IC}(\mathfrak{g}_1, P(\widetilde{X}_{2k-1}))$ appears as a direct summand in the push forward of a certain intersection cohomology complex on $\mathrm{Hess}_k^{O,\perp}$ along $\check{\sigma}_k^N$.

Let $[\mathbb{G}_a/\mathbb{G}_m^{[2]}]$ be the stack quotient, where $\mathbb{G}_m^{[2]} \cong \mathbb{G}_m$ acts on $\mathbb{G}_a$ via the square map, i.e., for $t \in \mathbb{G}_m$ and $x \in \mathbb{G}_a$, $t : x \mapsto t^2 x$. We first introduce a map

$$\alpha : \mathrm{Hess}_k^{O,\perp} \to [\mathbb{G}_a/\mathbb{G}_m^{[2]}].$$

Recall that such a map is equivalent to a pair $(\widetilde{\mathrm{Hess}_k}^{O,\perp}, \phi)$, where $\widetilde{\mathrm{Hess}_k}^{O,\perp}$ is a $\mathbb{G}_m$-torsor over $\mathrm{Hess}_n^{O,\perp}$ and $\phi : \widetilde{\mathrm{Hess}_k}^{O,\perp} \to \mathbb{G}_a$ is a map such that

$$\phi(t \cdot v) = t^2 \phi(v) \text{ for } v \in \widetilde{\mathrm{Hess}_k}^{O,\perp} \text{ and } t \in \mathbb{G}_m. \tag{5.1}$$

To construct such a pair, we set

$$\widetilde{\mathrm{Hess}_k}^{O,\perp} := \{(x, V_{k-1} \subset V_k, l) \,|\, (x, V_{k-1} \subset V_k) \in \mathrm{Hess}_k^{O,\perp},\ 0 \neq l \in V_k/V_{k-1} \simeq \mathbb{C}\},$$

where the action of $\mathbb{G}_m$ on $\widetilde{\mathrm{Hess}_k}^{O,\perp}$ is given by $t \cdot (x, V_{k-1} \subset V_k, l) = (x, V_{k-1} \subset V_k, tl)$ for $t \in \mathbb{G}_m$. Define

$$\phi : \widetilde{\mathrm{Hess}_k}^{O,\perp} \to \mathbb{G}_a,\ (x, V_{k-1} \subset V_k, l) \mapsto \langle xl, l \rangle_Q.$$

Note that the above pairing is well-defined since $xV_{k-1} \subset V_k$ and $xV_k \subset V_{k-1}^\perp$. One checks easily that ϕ satisfies (5.1). This finishes the construction of $(\widetilde{\mathrm{Hess}_k}^{O,\perp}, \phi)$, hence that of the map $\alpha : \mathrm{Hess}_k^{O,\perp} \to [\mathbb{G}_a/\mathbb{G}_m^{[2]}]$. By construction, the map α is K-equivaraint (where K acts trivially on $[\mathbb{G}_a/\mathbb{G}_m^{[2]}]$), moreover it factors through $\mathrm{Hess}_k^{E,\perp}$, i.e.,

$$\alpha : \mathrm{Hess}_k^{O,\perp} \to \mathrm{Hess}_k^{O,\perp} / \mathrm{Hess}_k^{E,\perp} \xrightarrow{\bar{\alpha}} [\mathbb{G}_a/\mathbb{G}_m^{[2]}]. \tag{5.2}$$

There is a unique non-trivial irreducible local system $\mathcal{L}$ on $[\mathbb{G}_m/\mathbb{G}_m^{[2]}] \subset [\mathbb{G}_a/\mathbb{G}_m^{[2]}]$. We denote by $\mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L})$ the corresponding intersection cohomology complex on $[\mathbb{G}_a/\mathbb{G}_m^{[2]}]$. Let

$$\mathcal{K} := (\check{\sigma}_k^N)_* \alpha^* \mathrm{IC}([\mathbb{G}_m/\mathbb{G}_m^{[2]}], \mathcal{L}) \in D_K(\mathfrak{g}_1).$$

The factorization in (5.2) and the functorial properties of Fourier transform (see [KaS, Proposition 3.7.14]) imply the following:

$$\mathfrak{F}(\mathcal{K}) \text{ is supported on } \mathrm{Im}\,(\tau_k^N) \subset \mathcal{N}_1^3. \tag{5.3}$$

Thus to show that $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, P(\widetilde{X}_{2k-1})))$ is supported on $\mathcal{N}_1^3$, it suffices to show that

$$\text{the complex } \mathcal{K} \text{ contains } \mathrm{IC}(\mathfrak{g}_1, P(\widetilde{X}_{2k-1})^{\sigma=-\mathrm{id}}) \text{ as a direct summand.} \tag{5.4}$$

Let

$$\tilde{\pi}_{2k-1} : \widetilde{X}_{2k-1} \xrightarrow{p_{2k-1}} X_{2k-1} \xrightarrow{\pi_{2k-1}} \mathfrak{g}_1^{rs}$$

be the branched double cover of X_{2k-1} and let σ be the involution on $\widetilde{X}_{2k-1}$ defined in §3.4. We have that $((\tilde{\pi}_{2k-1})_*\mathbb{C})^{\sigma=-\mathrm{id}}$ contains $P(\widetilde{X}_{2k-1})^{\sigma=-\mathrm{id}}$ as a direct summand. The statement (5.4) follows from the following claim:

$$\mathcal{K}|_{\mathfrak{g}_1^{rs}} \simeq ((\tilde{\pi}_{2k-1})_*\mathbb{C})^{\sigma=-\mathrm{id}}.$$

To prove the claim, let $s : [\mathbb{G}_a/\mathbb{G}_m] \to [\mathbb{G}_a/\mathbb{G}_m^{[2]}]$ be the descent of the map $\mathbb{G}_a \to \mathbb{G}_a$, $t \mapsto t^2$. Then from the definitions of $\widetilde{X}_{2k-1}$ and the map α, one can check that, under the isomorphism $X_{2k-1} \simeq \mathrm{Hess}_k^{O,\perp}|_{\mathfrak{g}_1^{rs}}$ in Theorem 2.5, the branched double cover $\widetilde{X}_{2k-1}$ can be identified with the following fiber product:

$$\begin{tikzcd} \widetilde{X}_{2k-1} \arrow[r] \arrow[d, "p_{2k-1}"] & {[\mathbb{G}_a/\mathbb{G}_m]} \arrow[d, "s"] \\ X_{2k-1} \simeq \mathrm{Hess}_k^{O,\perp}|_{\mathfrak{g}_1^{rs}} \arrow[r, "\alpha|_{\mathfrak{g}_1^{rs}}"] & {[\mathbb{G}_a/\mathbb{G}_m^{[2]}].} \end{tikzcd}$$

Since

$$s_*\mathbb{C} = (s_*\mathbb{C})^{\sigma=\mathrm{id}} \oplus (s_*\mathbb{C})^{\sigma=-\mathrm{id}} = \mathbb{C} \oplus \mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L}),$$

by proper base change we have

$$(\alpha|_{\mathfrak{g}_1^{rs}})^*\mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L}) \simeq ((p_{2k-1})_*\mathbb{C})^{\sigma=-\mathrm{id}}.$$

This implies that

$$\mathcal{K}|_{\mathfrak{g}_1^{rs}} \simeq (\pi_{2k-1})_*(\alpha|_{\mathfrak{g}_1^{rs}})^*\mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L}) \simeq ((\tilde{\pi}_{2k-1})_*\mathbb{C})^{\sigma=-\mathrm{id}}.$$

This proves (5.4).

Remark 5.3. The construction of the map α was inspired by discussions with Zhiwei Yun. In particular, the idea of making use of the local system $\mathcal{L}$ on $[\mathbb{G}_a/\mathbb{G}_m^{[2]}]$ was explained to one of us by him.

5.2. **Proof of Proposition 5.2 when m is even.** Let us consider the following family of Hessenberg varieties:

$$\begin{aligned} H = \{&(x, 0 \subset V_{k-1} \subset V_k \subset V_{k+1} \subset V_{k+1}^\perp \subset V_k^\perp \subset V_{k-1}^\perp \subset V = \mathbb{C}^N) \\ &\mid x \in \mathfrak{g}_1,\ xV_{k-1} \subset V_k, xV_k \subset V_k^\perp\}. \end{aligned}$$

Note that the natural map

$$p : H \to \mathrm{Hess}_k^{E,\perp},\ (x, V_{k-1} \subset V_k \subset V_{k+1}) \mapsto (x, V_{k-1} \subset V_k)$$

realizes H as a quadric bundle over $\mathrm{Hess}_k^{E,\perp}$.

We first construct a map $\beta : H \to [\mathbb{G}_a/\mathbb{G}_m^{[2]}]$. The construction is very similar to that of the map α in §5.1 and we use the notation there. Set

$$\widetilde{H} := \{(x, V_{k-1} \subset V_k \subset V_{k+1}, l) \mid (x, V_{k-1} \subset V_k \subset V_{k+1}) \in H,\ 0 \neq l \in V_k/V_{k-1} \simeq \mathbb{C}\},$$

where the action of $\mathbb{G}_m$ on $\widetilde{H}$ is given by $t \cdot (x, V_{k-1} \subset V_k \subset V_{k+1}, l) = (x, V_{k-1} \subset V_k \subset V_{k+1}, tl)$. Define

$$\phi : \widetilde{H} \to \mathbb{G}_a,\ (x, V_{k-1} \subset V_k \subset V_{k+1}, l) \mapsto \langle xl, xl\rangle_Q.$$

Note that the above pairing is well-defined since $xV_{k-1} \subset V_k$ and $xV_k \subset V_k^\perp$. One checks that ϕ satisfies (5.1). This finishes the construction of $(\widetilde{H}, \phi)$. Hence we obtain a map $\beta : H \to [\mathbb{G}_a/\mathbb{G}_m^{[2]}]$.

Let $f : H \to \mathfrak{g}_1$ be the natural projection map. Define

$$\mathcal{F} := f_*\beta^*\mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L}) \in D_K(\mathfrak{g}_1).$$

We show that

$$\mathfrak{F}(\mathcal{F}) \text{ is supported on } \mathcal{N}_1^3, \text{ and} \tag{5.5}$$

$$\text{the complex } \mathcal{F} \text{ contains } \mathrm{IC}(\mathfrak{g}_1, P(\widetilde{X}_{2k})^{\sigma=-\mathrm{id}}) \text{ as a direct summand.} \tag{5.6}$$

The proposition then follows from (5.5) and (5.6).

To prove (5.5), let

$$H' = \{(x, 0 \subset V_{k-1} \subset V_k \subset V_{k+1} \subset V_{k+1}^{\perp} \subset V_k^{\perp} \subset V_{k-1}^{\perp} \subset V = \mathbb{C}^N) \\ \mid x \in \mathfrak{g}_1,\ xV_{k-1} \subset V_k, xV_k \subset V_{k+1}\}.$$

Note that $H' \subset H$ is a subbundle. By construction, the map β factors through H', i.e.,

$$\beta : H \to H/H' \xrightarrow{\beta'} [\mathbb{G}_a/\mathbb{G}_m^{[2]}].$$

Let $\check{f}'$ be the natural projection map

$$\check{f}' : (H')^{\perp} := \{(x, 0 \subset V_{k-1} \subset V_k \subset V_{k+1} \subset V_{k+1}^{\perp} \subset V_k^{\perp} \subset V_{k-1}^{\perp} \subset \mathbb{C}^N) \\ \mid x \in \mathfrak{g}_1,\ xV_k = 0,\ xV_{k+1} \subset V_{k-1},\ xV_k^{\perp} \subset V_k\} \to \mathcal{N}_1.$$

A direct calculation shows that

$$\operatorname{Im} \check{f}' = \bar{\mathcal{O}}_{3^k 1^{N-3k}} \text{ if } 3k \leq N \text{ and } \operatorname{Im} \check{f}' = \bar{\mathcal{O}}_{3^{N-2k} 2^{3k-N}} \text{ if } 3k \geq N+1. \tag{5.7}$$

The standard properties of Fourier transform imply that

$$\mathfrak{F}(\mathcal{F}) \text{ is supported on } \operatorname{Im}(\check{f}') \subset \mathcal{N}_1^3.$$

This proves (5.5).

It remains to prove (5.6). Notice that the map β factors as $\beta : H \xrightarrow{p} \mathrm{Hess}_k^{E,\perp} \xrightarrow{\bar{\beta}} [\mathbb{G}_a/\mathbb{G}_m^{[2]}]$. Consider the following diagram:

$$\begin{tikzcd} \beta : H \arrow[r, "p"] \arrow[dr, "f"'] & \mathrm{Hess}_k^{E,\perp} \arrow[r, "\bar{\beta}"] \arrow[d, "\check{\tau}_k^N"] & {[\mathbb{G}_a/\mathbb{G}_m^{[2]}]} \\ & \mathfrak{g}_1 & \end{tikzcd}.$$

We have

$$\mathcal{F} := f_* \beta^* \mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}]) \simeq (\check{\tau}_k^N)_* p_* p^* \bar{\beta}^* (\mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L}))$$

which is isomorphic to $(\check{\tau}_k^N)_*(\bar{\beta}^*(\mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L})) \otimes p_*\mathbb{C})$. Since $\mathbb{C}$ is a direct summand of $p_*\mathbb{C}$, it implies that $(\check{\tau}_k^N)_*(\bar{\beta}^*(\mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L}))$ is a direct summand of $\mathcal{F}$. So it is enough to show that

$$\mathrm{IC}(\mathfrak{g}_1, P(\widetilde{X}_{2k})^{\sigma=-\mathrm{id}}) \text{ is a direct summand of } (\check{\tau}_k^N)_*(\bar{\beta}^*(\mathrm{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L})).$$

This follows from the same argument as in the proof of (5.4), replacing X_{2k-1} (resp., $\widetilde{X}_{2k-1}$) there by X_{2k} (resp., $\widetilde{X}_{2k}$). Thus the proof of the proposition is complete.

5.3. **Matching for** $\mathrm{IC}(\bar{\mathcal{O}}_{2^i 1^{2n+1-2i}}, \mathcal{E}_i)$**,** i **odd.** Here we complete the proof of [CVX1, Theorem 2.3] by treating the case of odd i. In [CVX1] we treated the even case of the proposition below and showed that there exists a permutation s of the set $\{2j+1 \mid 1 \leq 2j+1 \leq n\}$, such that $\mathfrak{F}(\mathrm{IC}(\bar{\mathcal{O}}_{2^i 1^{2n+1-2i}}, \mathcal{E}_i)) = \mathrm{IC}(\mathfrak{g}_1, \mathcal{F}_{s(i)})$ (see Proposition 3.2 and Theorem 2.3 in [CVX1]).

Proposition 5.4. *We have that*

$$\mathfrak{F}(\mathrm{IC}(\bar{\mathcal{O}}_{2^i 1^{2n+1-2i}}, \mathcal{E}_i)) = \mathrm{IC}(\mathfrak{g}_1, \mathcal{F}_i),$$

where $\mathcal{E}_i$ *denotes the unique non-trivial irreducible* K*-equivariant local system on* $\mathcal{O}_{2^i 1^{2n+1-2i}}$*.*

Proof. It remains to prove the proposition for odd i. Assume that $2m \leq n+1$. By (5.3) and (5.4), we see that the Fourier transform of $\mathrm{IC}(\mathfrak{g}_1, P(\tilde{X}_{2m-1})^{\sigma=-\mathrm{id}})$ is supported on $\operatorname{Im}\tau_m^N = \bar{\mathcal{O}}_{3^{m-1}2^1 1^{2n+2-3m}}$ (see (2.6a)). Using Theorem 4.2 and (4.14) we obtain that

$$\mathrm{IC}(\mathfrak{g}_1, \mathcal{F}_i) \text{ is a direct summand of } \mathrm{IC}(\mathfrak{g}_1, P(\tilde{X}_{2m-1})^{\sigma=-\mathrm{id}}) \text{ if and only if } i \text{ is odd and } 1 \leq i \leq 2m-1.$$

This implies that the Fourier transform of $\mathrm{IC}(\mathfrak{g}_1, \mathcal{F}_{2j-1})$, $1 \leq j \leq m$, is supported on $\bar{\mathcal{O}}_{3^{m-1}2^1 1^{2n+2-3m}}$. Now it is easy to check that $\mathcal{O}_{2^i 1^{2n+1-2i}} \subset \bar{\mathcal{O}}_{3^{m-1}2^1 1^{2n+2-3m}}$ if and only if $i \leq 2m-1$. In view of [CVX1, Proposition 3.2 and Theorem 2.3], the proposition follows by induction on m. ☐

Example 5.5. Let $\mathcal{O}_{min} = \mathcal{O}_{2^1 1^{2n-1}}$. By the above proposition, we have

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, \mathcal{F}_1)) \simeq \mathrm{IC}(\bar{\mathcal{O}}_{\min}, \mathcal{E}_1),$$

where

$$\mathcal{F}_1 \simeq \widetilde{E}^{2n+1}_{n+1,1} \simeq H^1(\widetilde{C}_{a,\chi_0}, \mathbb{C}) \text{ (see (4.18))}$$

is isomorphic to the monodromy representation associated with $\bar{C}_{\chi_0} \to \mathfrak{c}^{rs}$, the universal family of hyperelliptic curves in §3.2.

6. Conjectures and examples

Let $N = 2n+1$ and let E^{2n+1}_{ij} (resp., $\widetilde{E}^{2n+1}_{ij}$) be the monodromy representations of $\pi_1^K(\mathfrak{g}_1^{rs})$ constructed from the families of complete intersections of quadrics in $\mathbb{P}^{2n}$ (resp., their double covers); see §4. Let $\{(\mathcal{O}, \mathcal{E})\}_{\leq 3}$ denote the set of pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O}$ is a K-orbit in $\mathcal{N}_1^3$ and $\mathcal{E}$ is an irreducible K-equivariant local system on $\mathcal{O}$ (up to isomorphism). Using Theorem 5.1, we establish an injective map

$$\mathcal{S} : \left\{ \begin{array}{c} E^{2n+1}_{ij},\ i \in [1,n],\ j \in [0, i-1]; \\ \widetilde{E}^{2n+1}_{ij},\ i \in [1, n+1],\ j \in [1, i-1],\ \widetilde{E}^{2n+1}_{n+1,0} \cong \mathbb{C} \end{array} \right\} \hookrightarrow \{(\mathcal{O}, \mathcal{E})\}_{\leq 3}, \tag{6.1}$$

where $\mathcal{S}(E^{2n+1}_{ij}) = (\mathcal{O}, \mathcal{E})$ if and only if $\mathfrak{F}(\mathfrak{g}_1, E^{2n+1}_{ij}) = \mathrm{IC}(\bar{\mathcal{O}}, \mathcal{E})$, similarly for $\widetilde{E}^{2n+1}_{ij}$. Here the K-equivariant local systems on $\mathfrak{g}_1^{rs}$ in the left hand side of (6.1) are pairwise non-isomorphic; see (4.10).

In this section we state two conjectures (Conjectures 6.1 and 6.3) that describe the map $\mathcal{S}$ in (6.1) in the case of $\{E^{2n+1}_{ij}\}$ explicitly. We verify our conjectures in several examples by studying various families of Hessenberg varieties.

In what follows we make use of the following observation.

(6.2) an orbit $\mathcal{O}_{3^k 2^l 1^{2n+1-3k-2l}} \subset \mathcal{N}_1^3$ is odd dimensional $\Leftrightarrow$ k is odd and l is even.

This follows from the fact that $\dim \mathcal{O}_{3^k 2^l 1^{2n+1-3k-2l}} = 2(k+2kn+ln) - l(l-1) - 3k(k+l)$, which one readily deduces from the formula $\dim Z_K(x) = \sum (i-1)\lambda_i$ ([S]), for x in a nilpotent orbit corresponding to the partition $\lambda_1 \geq \lambda_2 \geq \cdots$.

6.1. Complete intersections of even number of quadrics and conjectural matching.

Recall that the local systems $E^{2n+1}_{i,2j}$, where $i \in [1,n]$ and $2j \in [0, i-1]$, are constructed from families of complete intersections X_{2m} of an even number of quadrics in $\mathbb{P}^{2n}$ for $m \in [1,n]$.

We first show that

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{i,2j})) \text{ is supported on an } \textit{even} \text{ dimensional } K\text{-orbit in } \mathcal{N}_1^3. \tag{6.3}$$

To this end we first note that each $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{i,2j}^{2n+1}))$ is a direct summand of $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, P(X_{2m})))$ for some m, which in turn is a direct summand of $(\tau_m^N)_*\mathbb{C}[-]$. One readily checks that

$$\dim \mathrm{Hess}_m^E = m(4n-3m+5)-2n-2, \text{ which is even.}$$

Note also that $\dim X_{2m,a}$ is even. Now (6.3) follows from the decomposition theorem and the fact that the fibers of τ_m^N have non-vanishing cohomology only in even degrees (see §2.3).

Thus we have that (6.2) puts a restriction on nilpotent orbits which can support $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{i,2j}^{2n+1}))$. Our first conjecture is the following.

Conjecture 6.1. *We have that*

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{i,2j}^{2n+1})) \cong \mathrm{IC}(\bar{\mathcal{O}}_{3^{2(n-i)+1}2^{2(i+j-n)-1}1^{2i-4j}}, \mathbb{C}) \text{ if } i+j \geq n+1,$$
$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{i,2j}^{2n+1})) \cong \mathrm{IC}(\bar{\mathcal{O}}_{3^{2j}2^{2(n-i-j)+1}1^{4i-2n-2j-1}}, \mathbb{C}) \text{ if } i+j \leq n \text{ and } 2i-j \geq n+1,$$
$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{i,2j}^{2n+1})) \cong \mathrm{IC}(\bar{\mathcal{O}}_{3^{2j}2^{2i-4j}1^{2n-4i+2j+1}}, \mathbb{C}) \text{ if } i+j \leq n \text{ and } 2i-j \leq n.$$

Remark 6.2. The nilpotent orbits appearing in the conjecture above exhaust all the non-zero *even* dimensional orbits of the form $\mathcal{O}_{3^i2^j1^k}$, where the partition $3^i2^j1^k$ has no gaps.

Note that the conjecture above holds for $E_{i,0}^{2n+1}$. This follows from (4.13) and [CVX1, Theorem 2.2], i.e., we have

$$\begin{aligned} \mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{i,0}^{2n+1})) &= \mathrm{IC}(\bar{\mathcal{O}}_{2^{2i}1^{2n-4i+1}}, \mathbb{C}) \quad \text{if} \quad 2i \leq n, \\ \mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{i,0}^{2n+1})) &= \mathrm{IC}(\bar{\mathcal{O}}_{2^{2n-2i+1}1^{4i-2n-1}}, \mathbb{C}) \quad \text{if} \quad 2i \geq n+1. \end{aligned} \tag{6.4}$$

Below we verify the conjecture in a simple case that involves nilpotent orbits of order 3.

6.2. **Complete intersection of** 4 **quadrics,** $n \geq 3$**.** In this subsection we show that

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{n,2}^{2n+1})) = \mathrm{IC}(\bar{\mathcal{O}}_{3^12^11^{2n-4}}, \mathbb{C}). \tag{6.5}$$

Let us write

$$\tau = \tau_2^{2n+1} : \mathrm{Hess}_2^E \to \bar{\mathcal{O}}_{3\,2\,1^{2n-4}}, \quad \check{\tau} = \check{\tau}_2^{2n+1} : \mathrm{Hess}_2^{E,\perp} \to \mathfrak{g}_1.$$

We have $\mathfrak{F}(\tau_*\mathbb{C}[-]) \cong \check{\tau}_*\mathbb{C}[-]$ and

$$\check{\tau}_*\mathbb{C}[-] = \mathrm{IC}(\mathfrak{g}_1, E_{n,2}^{2n+1} \oplus E_{n,0}^{2n+1} \oplus E_{n-1,0}^{2n+1}) \oplus \bigoplus_{a=0}^{2n-4} \mathrm{IC}(\mathfrak{g}_1, \mathbb{C})[2n-4-2a] \oplus \cdots,$$

where $\cdots$ is a direct sum of IC complexes with smaller support. We have

$$\bar{\mathcal{O}}_{3\,2\,1^{2n-4}} = \mathcal{O}_{3\,2\,1^{2n-4}} \cup \mathcal{O}_{3^11^{2n-2}} \bigcup_{0 \leq i \leq 3} \mathcal{O}_{2^i1^{2n-2i+1}}.$$

In view of Proposition 5.4, Lemma 4.7, (4.14), and (6.4), we conclude that $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{n,2}^{2n+1}))$ is not supported on $\bar{\mathcal{O}}_{2^i1^{2n+1-2i}}$'s. Now it follows from (6.2) and (6.3) that

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E_{n,2}^{2n+1})) \text{ is supported on } \bar{\mathcal{O}}_{3\,2\,1^{2n-4}}.$$

Thus (6.5) follows the fact that the only IC complex supported on $\mathcal{O}_{3\,2\,1^{2n-4}}$ appearing in $\tau_*\mathbb{C}[-]$ is $\mathrm{IC}(\bar{\mathcal{O}}_{3\,2\,1^{2n-4}}, \mathbb{C})$ as τ is a resolution of $\bar{\mathcal{O}}_{3\,2\,1^{2n-4}}$.

6.3. **Complete intersections of odd number of quadrics and a conjectural matching.** Recall that the local systems $E^{2n+1}_{i,2j-1}$, where $i \in [1,n]$ and $2j \in [2,i]$, are constructed from complete intersections X_{2m-1} of odd number of quadrics in $\mathbb{P}^{2n}$, $m \in [1,n]$.

Using that $\dim \mathrm{Hess}^{O}_{m} = m(2n-3m+5)-2n-3$, which is odd, and arguing as in (6.3), we obtain that

(6.6) $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{i,2j-1}))$ is supported on an *odd* dimensional K-orbit in $\mathcal{N}^3_1$.

Let $\mathcal{O} \subset \mathcal{N}^3_1$ be an odd dimensional K-orbit. To describe our second conjecture, let us first label the non-trivial irreducible K-equivariant local systems on $\mathcal{O}$ as follows. By (6.2), we can assume that $\mathcal{O} = \mathcal{O}_{3^{2k-1}2^{2l}1^{2n+4-6k-4l}}$.

Let $x \in \mathcal{O}_{3^{2k-1}2^{2l}1^{2n+4-6k-4l}}$, $k \geq 1$. We first define representatives for the component group $A_K(x) = Z_K(x)/Z_K(x)^0$. Take a basis

$$x^i u_j, i \in [0,2], j \in [1,2k-1], x^i v_j,\ i \in [0,1],$$
$$j \in [1,2l] \quad \text{and} \quad w_i,\ i \in [1,2n+4-6k-4l]$$

of V as in [CVX1, Lemma 5.6]. Define $\gamma_i \in Z_K(x)$, $i = 1,2$ as follows:

$$\gamma_1(w_1) = w_2,\ \gamma_1(w_2) = w_1,\ \gamma_1(x^i u_j) = -x^i u_j,\ i \in [0,2],\ j \in [1,2k-1],$$
$$\gamma_2(x^j v_1) = x^j v_2,\ \gamma_2(x^j v_2) = x^j v_1,\ j \in [0,1],$$
and γ_1(resp., γ_2) acts as identity on all other basis vectors.

Assume that $l \geq 1$ and $2n+4-6k-4l \neq 0$. Then $A_K(x) \cong \{1, \gamma_1, \gamma_2, \gamma_1\gamma_2\} \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let

$$\mathcal{E}^1_{k,l} \text{ (resp., } \mathcal{E}^2_{k,l},\ \mathcal{E}^3_{k,l})$$

denote the irreducible K-equivariant local system on $\mathcal{O}_{3^{2k-1}2^{2l}1^{2n+4-6k-4l}}$ corresponding to the irreducible character of $A_K(x)$

$$\chi_1 \text{ (resp., } \chi_2,\ \chi_3) \text{ with } \chi(\gamma_1) = -1 \text{ (resp., } -1, 1) \text{ and } \chi(\gamma_2) = 1 \text{ (resp., } -1, -1).$$

Assume that $l = 0$ and $2n+4-6k \neq 0$. Then $A_K(x) \cong \{1, \gamma_1\} \cong \mathbb{Z}/2\mathbb{Z}$. We denote by $\mathcal{E}^1_{k,0}$ the irreducible K-equivariant local system on $\mathcal{O}_{3^{2k-1}1^{2n+4-6k}}$ corresponding to the irreducible character χ of $A_K(x)$ with $\chi(\gamma_1) = -1$.

Assume that $l \geq 1$ and $2n+4-6k-4l \neq 0$. Then $A_K(x) \cong \{1, \gamma_2\} \cong \mathbb{Z}/2\mathbb{Z}$. We denote by $\mathcal{E}^3_{k,l}$ the irreducible K-equivariant local system on $\mathcal{O}_{3^{2k-1}2^{n+2-3k}}$ corresponding to the irreducible character χ of $A_K(x)$ with $\chi(\gamma_2) = -1$.

We will simply write $\mathcal{E}^i$, $i = 1,2,3$, when the supports of these local systems are clear.

Our second conjecture is the following.

Conjecture 6.3. *We have that*

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{i,2j-1})) \cong \mathrm{IC}(\bar{\mathcal{O}}_{3^{2(n-i)+1}2^{2(i+j-n-1)}1^{2i-4j+2}}, \mathcal{E}^1) \text{ if } i+j \geq n+1,$$
$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{i,2j-1})) \cong \mathrm{IC}(\bar{\mathcal{O}}_{3^{2j-1}2^{2(n-i-j+1)}1^{4i-2j-2n}}, \mathcal{E}^2) \text{ if } i+j \leq n \text{ and } 2i-j \geq n+1,$$
$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{i,2j-1})) \cong \mathrm{IC}(\bar{\mathcal{O}}_{3^{2j-1}2^{2(i-2j+1)}1^{2n-4i+2j}}, \mathcal{E}^3) \text{ if } i+j \leq n \text{ and } 2i-j \leq n.$$

Remark 6.4. In particular, the conjecture above implies that the set of all Fourier transforms $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{i,2j-1}))$ coincides with the set of all IC complexes supported on *odd* dimensional orbits in $\mathcal{N}^3_1$, with *non-trivial* local systems.

In the following subsections we verify the conjecture above in two simple examples; see (6.7) and (6.8). We also prove a lemma (Lemma 6.7) that is compatible with our conjecture.

6.4. **Complete intersection of 3 quadrics, $n \geq 2$.** In this subsection we show that

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{n,1})) = \mathrm{IC}(\bar{\mathcal{O}}_{3^11^{2n-2}}, \mathcal{E}^1). \tag{6.7}$$

Let us write

$$\sigma = \sigma^{2n+1}_2 : \mathrm{Hess}^O_2 \to \bar{\mathcal{O}}_{3^11^{2n-2}}, \quad \check{\sigma} = \check{\sigma}^{2n+1}_2 : \mathrm{Hess}^{O,\perp}_2 \to \mathfrak{g}_1.$$

The fiber $\sigma^{-1}(x)$ at $x \in \mathcal{O}_{3^11^{2n-2}}$ is a non-singular quadric in $\mathbb{P}^{2n-3}$. Thus in the decomposition of $\sigma_*\mathbb{C}[-]$, we have the following direct summands:

$$\bigoplus_{a=0}^{2n-4} \mathrm{IC}(\bar{\mathcal{O}}_{3^11^{2n-2}}, \mathbb{C})[2n-4-2a] \oplus \mathrm{IC}(\bar{\mathcal{O}}_{3^11^{2n-2}}, \mathcal{E}^1).$$

We have $\mathfrak{F}(\sigma_*\mathbb{C}[-]) \cong \check{\sigma}_*\mathbb{C}[-]$ and

$$\check{\sigma}_*\mathbb{C}[-] \cong \mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{n,1}) \oplus \cdots.$$

Note that $\mathcal{O}_{3^11^{2n-2}}$ is the only odd dimensional orbit contained in $\bar{\mathcal{O}}_{3^11^{2n-2}}$ and there is a unique non-trivial irreducible K-equivariant local system on $\mathcal{O}_{3^11^{2n-2}}$, denoted by $\mathcal{E}^1$. In view of (6.6), the equation (6.7) follows from the fact that the support of $\mathfrak{F}(\mathrm{IC}(\bar{\mathcal{O}}_{3^11^{2n-2}}, \mathbb{C}))$ is a proper subset of $\mathfrak{g}_1$ (see [CVX2, Proposition 4.4]).

Remark 6.5. Here we see that Fourier trxansform of IC complexes supported on nilpotent orbits $\mathcal{O}_\lambda$, where λ has *gaps*, with non-trivial local systems can have full support (compare with [CVX2, Corollary 4.9]).

6.5. **Complete intersection of 5 quadrics, $n \geq 4$.** In this subsection, we show that

(6.8)

$$\mathfrak{F}(\mathrm{IC}(\bar{\mathcal{O}}_{3^12^21^{2n-6}}, \mathcal{E}^1)) = \mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{n,3}), \quad \mathfrak{F}(\mathrm{IC}(\bar{\mathcal{O}}_{3^12^21^{2n-6}}, \mathcal{E}^2)) = \mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{n-1,1}),$$
$$\mathfrak{F}(\mathrm{IC}(\bar{\mathcal{O}}_{3^12^21^{2n-6}}, \mathcal{E}^3)) = \mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{?,1}).$$

Let us write

$$\sigma = \sigma^{2n+1}_3 : \mathrm{Hess}^O_3 \to \bar{\mathcal{O}}_{3^21^{2n-5}}, \ \check{\sigma} = \check{\sigma}^{2n+1}_3 : \mathrm{Hess}^{O,\perp}_3 \to \mathfrak{g}_1.$$

We have $\mathfrak{F}(\sigma_*\mathbb{C}[-]) \cong \check{\sigma}_*\mathbb{C}[-]$ and

$$\check{\sigma}_*\mathbb{C}[-] \cong \mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{n,1} \oplus E^{2n+1}_{n,3} \oplus E^{2n+1}_{n-1,1}) \oplus \cdots.$$

The odd dimensional orbits contained in $\mathrm{Im}\,\sigma = \bar{\mathcal{O}}_{3^21^{2n-5}}$ are $\mathcal{O}_{3^12^21^{2n-6}}$ and $\mathcal{O}_{3^11^{2n-2}}$. In view of (6.7), the equation (6.8) follows from Lemma 6.7 (see §6.6) and the following statement:

(6.9)

The IC complexes supported on $\mathcal{O}_{3^12^21^{2n-6}}$, that appear in the decomposition of $\sigma_*\mathbb{C}[-]$, are $\mathrm{IC}(\bar{\mathcal{O}}_{3^12^21^{2n-6}}, \mathcal{E}^1 \oplus \mathcal{E}^2)$.

It remains to prove (6.9). Note that there is no orbit $\mathcal{O}$ such that $\mathcal{O}_{3^12^21^{2n-6}} < \mathcal{O} < \mathcal{O}_{3^21^{2n-5}}$. The fiber $\sigma^{-1}(x_2)$ at $x_2 \in \mathcal{O}_{3^21^{2n-5}}$ is a non-singular quadric in $\mathbb{P}^{2n-6}$.

Thus in the decomposition of $\sigma_*\mathbb{C}[-]$,

$$\text{the IC complexes supported on } \mathcal{O}_{3^21^{2n-5}} \text{ are } \bigoplus_{a=0}^{2n-7} \mathrm{IC}(\bar{\mathcal{O}}_{3^21^{2n-5}},\mathbb{C})[2n-7-2a].$$

The fiber at $x_1 \in \mathcal{O}_{3^12^21^{2n-6}}$ is a quadric bundle over $\pi_0^{-1}(x_1)$ with fibers being a quadric Q of rank $2n-8$ in $\mathbb{P}^{2n-6}$. Here π_0 is Reeder's resolution of $\bar{\mathcal{O}}_{3^21^{2n-5}}$, i.e.,

$$\pi_0 : \{(x, 0 \subset V_2 \subset V_2^\perp \subset V) \mid x \in \mathfrak{g}_1,\, xV_2 = 0, xV_2^\perp \subset V_2\} \to \bar{\mathcal{O}}_{3^21^{2n-5}}.$$

It is easy to check that the map π_0 is small. Thus we have

$$\mathcal{H}^{k-8(n-1)}_{x_1}\mathrm{IC}(\bar{\mathcal{O}}_{3^21^{2n-5}},\mathbb{C}) = H^k(\pi_0^{-1}(x_1),\mathbb{C}).$$

Note that $H^{\mathrm{odd}}(\pi_0^{-1}(x_1),\mathbb{C}) = 0$, $H^{\mathrm{odd}}(\sigma^{-1}(x_1),\mathbb{C}) = 0$, and

$$\begin{aligned} H^{2k}(\sigma^{-1}(x_1),\mathbb{C}) &= \textstyle\bigoplus_{a=0}^{2n-7} H^{2a}(Q,\mathbb{C}) \otimes H^{2k-2a}(\pi_0^{-1}(x_1),\mathbb{C}) \\ &\cong \textstyle\bigoplus_{a=0}^{2n-7} H^{2k-2a}(\pi_0^{-1}(x_1),\mathbb{C}) \oplus (H^{2n-6}_{prim}(Q,\mathbb{C}) \otimes H^{2k-2n+6}(\pi_0^{-1}(x_1),\mathbb{C})).\end{aligned}$$

We have $\operatorname{codim}_{\mathrm{Hess}_3^O} \mathcal{O}_{3^12^21^{2n-6}} = 2n-6$ and $\pi_0^{-1}(x_1)$ consists of two points. Moreover, $A_K(x_1)$ acts on $H^{2n-6}_{prim}(Q,\mathbb{C}) \otimes H^{2k-2n+6}(\pi_0^{-1}(x_1),\mathbb{C})$ as $\chi_1(1\oplus\chi_3) = \chi_1 \oplus \chi_2$. The equation (6.9) follows. This finishes the proof of (6.8).

Remark 6.6. Note that (6.8) shows that all three IC complexes supported on $\mathcal{O}_{3^12^21^{2n-6}}$ with *non-trivial* local systems correspond to the monodromy representations constructed from complete intersections of *odd* number of quadrics.

6.6. **The case of a curve.** In this subsection we prove the following lemma by considering the family X_{2n-1} of complete intersections of quadrics in $\mathbb{P}^{2n}$.

Lemma 6.7. *For each $i \in [1, n-1]$, there exists some $1 \le j \le [\frac{n-1}{2}]$ and a non-trivial local system $\mathcal{E}_j^s$ ($s = 2$ or 3) on $\mathcal{O}_{3^12^{2j}1^{2n-2j-2}}$ such that*

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{i,1})) \cong \mathrm{IC}(\bar{\mathcal{O}}_{3^12^{2j}1^{2n-4j-2}}, \mathcal{E}_j^s).$$

Let us write

$$\sigma = \sigma_n^{2n+1} : \mathrm{Hess}_n^O \to \bar{\mathcal{O}}_{3^12^{n-1}} \quad \text{and } \check{\sigma} = \check{\sigma}_n^{2n+1} : \mathrm{Hess}_n^{O,\perp} \to \mathfrak{g}_1.$$

Assume that $n \ge 3$. We have $\dim \mathrm{Hess}_n^O = n^2+3n-3$. We show that

$$\begin{aligned} \sigma_*\mathbb{C}[-] \cong & \bigoplus_{a=0}^{n-3} \mathrm{IC}(\bar{\mathcal{O}}_{3^12^{n-1}},\mathbb{C})[n-3-2a] \bigoplus_{2j=n-1} \mathrm{IC}(\bar{\mathcal{O}}_{3^12^{2j}},\mathcal{E}^3) \\ & \oplus \bigoplus_{2\le 2j\le n-2} \mathrm{IC}(\bar{\mathcal{O}}_{3^12^{2j}1^{2n-4j-2}},\mathcal{E}^2\oplus\mathcal{E}^3) \oplus \mathrm{IC}(\bar{\mathcal{O}}_{3^11^{2n-2}},\mathbb{C}\oplus\mathcal{E}^1) \\ & \oplus \mathrm{IC}(\bar{\mathcal{O}}_{1^{2n+1}},\mathbb{C})[1] \oplus \mathrm{IC}(\bar{\mathcal{O}}_{1^{2n+1}},\mathbb{C})[-1]. \end{aligned} \tag{6.10}$$

The lemma follows from the decomposition above, the equations $\mathfrak{F}(\check{\sigma}_*\mathbb{C}[-]) \cong \sigma_*\mathbb{C}[-]$, $\check{\sigma}_*\mathbb{C}[-] \cong \bigoplus_i \mathrm{IC}(\mathfrak{g}_1, E^{2n+1}_{i,1}) \oplus \cdots$, and (6.7).

In the remainder of this subsection we prove (6.10). Consider first Reeder's resolution of $\bar{\mathcal{O}}_{3^12^{n-1}}$ given by

$$\rho : \{(x, 0 \subset V_1 \subset V_n \subset V_n^\perp \subset V_1^\perp \subset \mathbb{C}^{2n+1}) \mid x \in \mathfrak{g}_1,\, xV_n = 0, xV_n^\perp \subset V_1\} \to \bar{\mathcal{O}}_{3^12^{n-1}}.$$

It is easy to check that ρ is a small map. Thus for $x_j \in \mathcal{O}_{3^12^j1^{2n-2j-2}}$, we have

$$\mathcal{H}^{k-n^2-2n}_{x_j}\mathrm{IC}(\bar{\mathcal{O}}_{3^12^{n-1}},\mathbb{C}) \cong H^k(\rho^{-1}(x_j),\mathbb{C}). \tag{6.11}$$

Now we study the map σ and the decomposition of $\sigma_*\mathbb{C}[-]$. The fiber $\sigma^{-1}(x_{n-1})$ at $x_{n-1} \in \mathcal{O}_{3^1 2^{n-1}}$ is a non-singular quadric in $\mathbb{P}^{n-2}$ and $\operatorname{codim}_{\mathrm{Hess}_n^O} \mathcal{O}_{3^1 2^{n-1}} = n-3$. Thus in the decomposition of $\sigma_*\mathbb{C}[-]$, the following IC complexes supported on $\mathcal{O}_{3^1 2^{n-1}}$ appear,

$$\bigoplus_{a=0}^{n-3} \mathrm{IC}(\bar{\mathcal{O}}_{3^1 2^{n-1}}, \mathbb{C})[n-3-2a] \text{ for all } n \text{ and } \mathrm{IC}(\bar{\mathcal{O}}_{3^1 2^{n-1}}, \mathcal{E})[-] \text{ if } n \text{ is odd}, \tag{6.12}$$

where $\mathcal{E}$ is the unique non-trivial irreducible K-equivariant local system on $\mathcal{O}_{3^1 2^{n-1}}$.

We have that $\sigma^{-1}(x_j)$ $(x_j \in \mathcal{O}_{3^1 2^j 1^{2n-2j-2}})$ is a Q_j-bundle over $\rho^{-1}(x_j)$ for $j \geq 1$, where Q_j is a quadric $\sum_{k=1}^{j} a_k^2 = 0$ in $\mathbb{P}^{n-2} := \{[a_1, \cdots, a_{n-1}]\}$.

For j odd, or $j \geq 2$ even and $2k > \operatorname{codim}_{\mathrm{Hess}_n^O} \mathcal{O}_{x_j}$, we have

$$H^{2k}(\sigma^{-1}(x_j), \mathbb{C}) \cong \bigoplus_{a=0}^{n-3} H^{2k-2a}(\rho^{-1}(x_j), \mathbb{C}) \cong \mathcal{H}^{2k}_{x_j} \bigoplus_{a=0}^{n-3} \mathrm{IC}(\bar{\mathcal{O}}_{3^1 2^{n-1}}, \mathbb{C})[-n^2-2n-2a]$$

where in the second isomorphism we use (6.11). Thus in view of (6.12) IC complexes supported on $\mathcal{O}_{3^1 2^j 1^{2n-2j-2}}$, for odd $j < n-1$, do not appear in the decomposition of $\sigma_*\mathbb{C}$.

For $j \geq 2$ even, and $2k = \operatorname{codim}_{\mathrm{Hess}_n^O} \mathcal{O}_{x_j}$, we have

$$\begin{aligned} H^{2k}(\sigma^{-1}(x_j), \mathbb{C}) &\cong \mathcal{H}^{2k-n^2-2n}_{x_j} \bigoplus_{a=0}^{n-3} \mathrm{IC}(\bar{\mathcal{O}}_{3^1 2^{n-1}}, \mathbb{C})[-2a] \\ &\oplus (H^{2n+j-4}_{\mathrm{prim}}(Q_j, \mathbb{C}) \otimes H^{2\dim \rho^{-1}(x_j)}(\rho^{-1}(x_j), \mathbb{C})). \end{aligned}$$

Note that $\rho^{-1}(x_j)$ has two irreducible components. Moreover $A_K(x_j)$ acts on $H^{2n+j-4}_{prim}(Q_j, \mathbb{C})$ via the character χ_3, and acts on $H^{2\dim \rho^{-1}(x_j)}(\rho^{-1}(x_j), \mathbb{C})$ via $1 \oplus \chi_1$. In view of (6.12), we conclude that IC complexes $\mathrm{IC}(\bar{\mathcal{O}}_{3^1 2^j 1^{2n-2j-2}}, \mathcal{E}^2)$ and $\mathrm{IC}(\bar{\mathcal{O}}_{3^1 2^j 1^{2n-2j-2}}, \mathcal{E}^3)$, for j even, appear in the decomposition of $\sigma_*\mathbb{C}[-]$.

For $j = 0$ and $2k = \operatorname{codim}_{\mathrm{Hess}_n^O} \mathcal{O}_{x_0} = n^2 - n - 2$, since $2k - 2a > 2\dim \rho^{-1}(x_0)$ for all $0 \leq a \leq n-3$, we have

$$\bigoplus_{a=0}^{n-3} \mathcal{H}^{2k-2a-n^2-2n}_{x_0} \mathrm{IC}(\bar{\mathcal{O}}_{3^1 2^{n-1}}, \mathbb{C}) = 0.$$

We have $2\dim \sigma^{-1}(x_0) = \operatorname{codim}_{\mathrm{Hess}_n^O} \mathcal{O}_{x_0}$ and $\sigma^{-1}(x_0) \cong \{0 \subset W_{n-2} \subset W_{n-1} \subset W_{n-1}^{\perp} \subset W_{n-2}^{\perp} \subset \mathbb{C}^{2n-2}\}$. Note that $\sigma^{-1}(x_0)$ has two connected components and $A_k(x_0)$ permutes them. We conclude that the IC complexes supported on $\mathcal{O}_{x_0}$ appearing in $\sigma_*\mathbb{C}[-]$ are $\mathrm{IC}(\bar{\mathcal{O}}_{3^1 1^{2n-2}}, \mathbb{C}) \oplus \mathrm{IC}(\bar{\mathcal{O}}_{3^1 1^{2n-2}}, \mathcal{E}^1)$.

The decomposition (6.10) follows from the above discussion and the fact that none of the IC complexes supported on $\mathcal{O}_{2^i 1^{2n+1-2i}}$, $i \geq 1$ can appear in the decomposition of $\sigma_*\mathbb{C}[-]$. The proof of Lemma 6.7 is complete.

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Department of Mathematics, University of Chicago, Chicago, Illinois 60637

Current address: School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Email address: chenth@umn.edu

School of Mathematics and Statistics, University of Melbourne, Australia; and Department of Mathematics and Statistics, University of Helsinki, Finland

Email address: kari.vilonen@unimelb.edu.au

School of Mathematics and Statistics, University of Melbourne, Australia; and Department of Mathematics and Statistics, University of Helsinki, Finland

Email address: ting.xue@unimelb.edu.au