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Abstract

In this paper we establish Springer correspondence for the symmetric pair $(\mathrm{SL}(N), \mathrm{SO}(N))$ using Fourier transform, parabolic induction functor, and a nearby cycle sheaf construction. As an application of our results we see that the cohomology of Hessenberg varieties can be expressed in terms of irreducible representations of Hecke algebras of symmetric groups at $q = -1$. Conversely, we see that the irreducible representations of Hecke algebras of symmetric groups at $q = -1$ arise in geometry.

1. Introduction

In this paper we consider the Springer correspondence in the case of symmetric spaces. We will concentrate on the split case of type A, i.e., the case of $\mathrm{SL}(n, \mathbb{R})$. The case of $\mathrm{SL}(n, \mathbb{H})$ was considered by Henderson in [Hen01], Grinberg in [Gri98] and Lusztig in [Lus11a], and the case of $\mathrm{U}(p, q)$ was considered by Lusztig in [Lus11b] where he treats the general case of semisimple inner automorphisms. In both of these cases Springer theory closely resembles the classical situation. This turns out not to be so in the split case we consider here. In [CVX15a, CVX15b] we have computed Fourier transforms of IC sheaves supported on certain nilpotent orbits using resolutions of singularities of nilpotent orbit closures. In this paper we study the problem in general in the split case of type A replacing the resolutions with a nearby cycle sheaf construction in [GVX18] based on earlier ideas of Grinberg [Gri01, Gri98].

Let us call an irreducible IC sheaf supported on a nilpotent orbit a nilpotent orbital complex. We show that the Fourier transform gives a bijection between nilpotent orbital complexes and certain representations of (extended) braid groups. We identify these representations of (extended) braid groups and construct them explicitly in terms of irreducible representations of Hecke algebras of symmetric groups at $q = -1$. This bijection can be viewed as Springer correspondence for the symmetric pair $(\mathrm{SL}(N), \mathrm{SO}(N))$. Let us note that the fact that representations of (affine) Hecke algebras at $q = -1$ arise in this situation was already observed by Grojnowski in his thesis [Gro92].

The proof of our main result, Theorem 4.1, makes use of a nearby cycle sheaf construction in [GVX18] and smallness property of maps associated to certain θ-stable parabolic subgroups. The nearby cycle sheaves produce nilpotent orbital complexes whose Fourier transforms have

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full support. Those IC sheaves behave like 'cuspidal sheaves' in the sense that they do not appear as direct summands of parabolic inductions. On the other hand, the smallness property mentioned above implies a simple description of the images of parabolic induction functors (Propositions 3.1, 3.2). Those results together with a counting lemma (Lemma 4.2) imply Theorem 4.1. As corollaries, we obtain criteria for nilpotent orbital complexes to have full support Fourier transforms (Corollaries 4.8, 4.9) and results on cohomology of Hessenberg varieties (Theorem 5.2).

Our method appears to be applicable to general symmetric pairs and, more generally, polar representations studied in [Gri98] and we hope to return to this in future work.

Let us mention that in [LY17], the authors show that one can obtain all nilpotent orbital complexes using spiral induction functors introduced in [LY17] (in fact, they consider more general setting of cyclically graded Lie algebras). Using their results and Theorem 4.1, we show that all irreducible representations of Hecke algebras of symmetric groups at $q = -1$ appear in the intersection cohomology of Hessenberg varieties, with coefficient in certain local systems (see Theorem 6.3). This gives geometric constructions of irreducible representations of Hecke algebras of symmetric groups at $q = -1$ and provides them with a Hodge structure.

The paper is organized as follows. In § 2 we recall some facts about symmetric pairs and introduce a class of representations of equivariant fundamental groups. Moreover, we recall the definition of the parabolic induction functor. In § 3 we study parabolic induction functors for certain θ-stable parabolic subgroups. In § 4, we prove Theorem 4.1: the Fourier transform defines a bijection between the set of nilpotent orbital complexes and the class of representations of equivariant fundamental groups introduced in § 2. In §§ 5 and 6, we discuss applications of our results to cohomology of Hessenberg varieties and representations of Hecke algebras of symmetric groups at $q = -1$. Finally, in § 7, we propose a conjecture that gives a more precise description of the bijection in Theorem 4.1.

2. Preliminaries

For convenience we work over $\mathbb{C}$. We adopt the usual convention of cohomological degrees for perverse sheaves by having them be symmetric around 0. We also use the convention that all functors are derived, so we write, for example, π_* instead of $R\pi_*$. If X is smooth we write $\mathbb{C}_X[-]$ for the constant sheaf placed in degree $-\dim X$ so that $\mathbb{C}_X[-]$ is perverse. If $U \subset X$ is a smooth open dense subset of a variety X and $\mathcal{L}$ is a local system on U, we write $\mathrm{IC}(X, \mathcal{L})$ for the IC-extension of $\mathcal{L}[-]$ to X; in particular, it is perverse. For simplicity of notation, when we have a pair $(\mathcal{O}, \mathcal{E})$, where $\mathcal{O}$ is an orbit and $\mathcal{E}$ is a local system on $\mathcal{O}$, we also write $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ instead of $\mathrm{IC}(\bar{\mathcal{O}}, \mathcal{E})$.

2.1 Notation

For $e \geqslant 2$, a partition λ of a positive integer k is called e-regular if the multiplicity of any part of λ is *less than* e. In particular, a partition is 2-regular if and only if it has distinct parts. Let us denote by $\mathcal{P}(k)$ the set of all partitions of k and by $\mathcal{P}_2(k)$ the set of all 2-regular partitions of k.

We denote by $\mathcal{H}_{k,-1}$ the Hecke algebra of the symmetric group S_k with parameter -1. More precisely, $\mathcal{H}_{k,-1}$ is the $\mathbb{C}$-algebra generated by $T_i,\ i = 1, \ldots, k-1$, with the following relations

$$\begin{gathered} T_iT_j = T_jT_i \quad \text{if } |i-j| \geqslant 2,\ i,j \in [1,k-1], \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad i \in [1,k-2], \\ T_i^2 = q + (q-1)T_i, \quad \text{where } q = -1,\ i \in [1,k-1]. \end{gathered} \tag{2.1}$$

It is shown in [DJ86] that the set of isomorphism classes of irreducible representations of $\mathcal{H}_{k,-1}$ is parametrized by $\mathcal{P}_2(k)$. For $\mu \in \mathcal{P}_2(k)$, we write D_μ for the irreducible representation of $\mathcal{H}_{k,-1}$ corresponding to μ.

For a real number a, we write $[a]$ for its integer part.

2.2 The split symmetric pair $(\mathbf{SL}(N), \mathbf{SO}(N))$

Let $G = \mathrm{SL}(N)$ and $\theta : G \to G$ an involution such that $K = G^\theta = \mathrm{SO}(N)$ and write $\mathfrak{g} = \mathrm{Lie}\, G$. We have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\theta|_{\mathfrak{g}_i} = (-1)^i$. The pair (G, K) is a split symmetric pair, i.e., there exists a maximal torus A of G that is θ-split, where θ-split means that for all $x \in A$, $\theta(x) = x^{-1}$. We also think of the pair (G, K) concretely as $(\mathrm{SL}(V), \mathrm{SO}(V))$, where V is a vector space of dimension N equipped with a non-degenerate quadratic form Q such that $\mathrm{SO}(V) = \mathrm{SO}(V, Q)$. We write the non-degenerate bilinear form associated to Q as $\langle\ ,\ \rangle$.

Let $\mathfrak{g}^{\mathrm{rs}}$ denote the set of regular semisimple elements in $\mathfrak{g}$ and let $\mathfrak{g}_1^{\mathrm{rs}} = \mathfrak{g}_1 \cap \mathfrak{g}^{\mathrm{rs}}$. Similarly, let $\mathfrak{g}^{\mathrm{reg}}$ denote the set of regular elements in $\mathfrak{g}$ and let $\mathfrak{g}_1^{\mathrm{reg}} = \mathfrak{g}_1 \cap \mathfrak{g}^{\mathrm{reg}}$.

Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$ and let $\mathcal{N}_1 = \mathcal{N} \cap \mathfrak{g}$. When N is odd, the set of K-orbits in $\mathcal{N}_1$ is parametrized by $\mathcal{P}(N)$. When N is even, the set of $\mathrm{O}(N)$-orbits in $\mathcal{N}_1$ is parametrized by $\mathcal{P}(N)$, moreover, each $\mathrm{O}(N)$-orbit remains one K-orbit if λ has at least one odd part, and splits into two K-orbits otherwise. For $\lambda \in \mathcal{P}(N)$, we write $\mathcal{O}_\lambda$ for the corresponding nilpotent K-orbit in $\mathcal{N}_1$ when λ has at least one odd part, and write $\mathcal{O}_\lambda^{\mathrm{I}}$ and $\mathcal{O}_\lambda^{\mathrm{II}}$ for the corresponding two nilpotent K-orbits in $\mathcal{N}_1$ when λ has only even parts. Suppose that $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s > 0)$. For $x \in \mathcal{O}_\lambda$ (or $\mathcal{O}_\lambda^\omega$, $\omega = \mathrm{I}, \mathrm{II}$), we have

$$\dim Z_K(x) = \sum_{i=1}^{s} (i-1)\lambda_i. \tag{2.2}$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{g}_1$, which is also a Cartan subspace of $\mathfrak{g}$. We have the 'little' Weyl group

$$W = N_K(\mathfrak{a})/Z_K(\mathfrak{a}) = S_N.$$

2.3 Equivariant fundamental group and its representations

As was discussed in [CVX15a], the equivariant fundamental group

$$\pi_1^K(\mathfrak{g}_1^{\mathrm{rs}}) \cong Z_K(\mathfrak{a}) \rtimes B_N \cong (\mathbb{Z}/2\mathbb{Z})^{N-1} \rtimes B_N,$$

where B_N is the braid group of N strands and it acts on

$$Z_K(\mathfrak{a}) \cong \left\{ (i_1, \ldots, i_N) \in (\mathbb{Z}/2\mathbb{Z})^N \;\middle|\; \sum_{k=1}^{N} i_k = 0 \right\} \cong (\mathbb{Z}/2\mathbb{Z})^{N-1}$$

via the natural map $B_N \to S_N$. For simplicity we write

$$\widetilde{B}_N = (\mathbb{Z}/2\mathbb{Z})^{N-1} \rtimes B_N \quad \text{and} \quad I_N = (\mathbb{Z}/2\mathbb{Z})^{N-1}.$$

It is easy to see that the action of B_N on $I_N^\vee$ has $[N/2] + 1$ orbits. We choose a set of representatives

$$\chi_m \in I_N^\vee, \quad 0 \leqslant m \leqslant [N/2],$$

of the B_N-orbits as follows. Let $\tau_i' \in (\mathbb{Z}/2\mathbb{Z})^N$ be the element with all entries 0 except the ith position. Then $\{\tau_i = \tau_i' + \tau_{i+1}',\ i = 1, \ldots, N-1\}$, is a set of generators for I_N. For $0 \leqslant m \leqslant [N/2]$, we define a character χ_m as follows:

$$\chi_m(\tau_m) = -1 \quad \text{and} \quad \chi_m(\tau_i) = 1 \quad \text{for } i \neq m. \tag{2.3}$$

For $\chi \in I_N^\vee$, we set

$$B_\chi = \mathrm{Stab}_{B_N} \chi.$$

Let s_i, $i = 1, \dots, N-1$, be the simple reflections in $W = S_N$. It is easy to check that

$$\begin{gathered} \mathrm{Stab}_{S_N}(\chi_m) = \langle s_i, i \neq m \rangle \cong S_m \times S_{N-m} \quad \text{if } m \neq N/2, \quad \text{and} \\ \mathrm{Stab}_{S_N}(\chi_m) \text{ contains } S_m \times S_m \text{ as an index-2 normal subgroup if } m = N/2. \end{gathered} \tag{2.4}$$

Let us define

$$B_{m,N-m} = \text{the inverse image of } S_m \times S_{N-m} \cong \langle s_i, i \neq m \rangle \text{ under the map } B_N \to S_N.$$

Then it follows from (2.4) that

$$\begin{gathered} B_{\chi_m} = B_{m,N-m}, \text{ when } m \neq N/2, \\ \text{and } B_{\chi_m} \text{ contains } B_{m,N-m} \text{ as an index-2 normal subgroup when } m = N/2. \end{gathered} \tag{2.5}$$

Let $\sigma_i, i = 1, \dots, N-1$, be the standard generators of B_N which are lifts of the s_i under the map $B_N \to S_N$. Then $B_{m,N-m}$ is generated by σ_i, $i \neq m$, and σ_m^2. We have a natural quotient map

$$\mathbb{C}[B_{m,N-m}] \twoheadrightarrow \mathcal{H}_{m,-1} \otimes \mathcal{H}_{N-m,-1} \cong \mathbb{C}[B_{m,N-m}]/\langle (\sigma_i - 1)^2, i \neq m, \sigma_m^2 - 1 \rangle. \tag{2.6}$$

Note that in the above formula the σ_i corresponds to $-T_i$ of the Hecke algebra in (2.1). For us the σ_i are more natural generators since they arise from geometry as unipotent monodromy operators.

Let us write

$$\mathcal{H}_{m,-1} \otimes \mathcal{H}_{N-m,-1} = \mathcal{H}_{\chi_m,-1}.$$

We consider a family of representations of $\widetilde{B}_N$ as follows. For $0 \leqslant m \leqslant [N/2]$, we define

$$L_{\chi_m} := \mathrm{Ind}_{\mathbb{C}[B_{m,N-m}]}^{\mathbb{C}[B_N]} \mathcal{H}_{\chi_m,-1} \cong \mathbb{C}[B_N] \otimes_{\mathbb{C}[B_{m,N-m}]} \mathcal{H}_{\chi_m,-1}, \tag{2.7}$$

where in the tensor product $\mathbb{C}[B_{m,N-m}]$ acts on $\mathcal{H}_{\chi_m,-1}$ via the quotient map (2.6) and on $\mathbb{C}[B_N]$ by right multiplication. The module L_{χ_m} has a natural $\widetilde{B}_N$-action defined as follows. We let B_N act on L_{χ_m} by left multiplication and we let I_N act on L_{χ_m} via $a.(b \otimes v) = ((b.\chi_m)(a))(b \otimes v)$ for $a \in I_N$, $b \in B_N$ and $v \in \mathcal{H}_{\chi_m,-1}$. We view L_{χ_m} as a representation of the equivariant fundamental group $\widetilde{B}_N$ in this manner.

We will next identify the composition factors of the modules L_{χ_m}. Let $\mu^1 \in \mathcal{P}_2(m)$ and $\mu^2 \in \mathcal{P}_2(N-m)$, $m \in [0, [N/2]]$. Proceeding just as in the definition of L_{χ_m}, one obtains the following representation of $\widetilde{B}_N$:

$$V_{\mu^1,\mu^2} := \mathrm{Ind}_{\mathbb{C}[B_{m,N-m}]}^{\mathbb{C}[B_N]} (D_{\mu^1} \otimes D_{\mu^2}) \cong \mathbb{C}[B_N] \otimes_{\mathbb{C}[B_{m,N-m}]} (D_{\mu^1} \otimes D_{\mu^2}). \tag{2.8}$$

Using (2.5), one readily checks that V_{μ^1,μ^2} is an irreducible representation of $\widetilde{B}_N$ when $m \neq N/2$, or when $m = N/2$ and $\mu^1 \neq \mu^2$. When $m = N/2$ and $\mu^1 = \mu^2$, V_{μ^1,μ^2} breaks into the direct sum of two non-isomorphic irreducible representations of $\widetilde{B}_N$, which we denote by $V^{\mathrm{I}}_{\mu^1,\mu^2}$ and $V^{\mathrm{II}}_{\mu^1,\mu^2}$, i.e., we have

$$V_{\mu,\mu} \cong V^{\mathrm{I}}_{\mu,\mu} \oplus V^{\mathrm{II}}_{\mu,\mu}. \tag{2.9}$$

Moreover,

$$\begin{gathered} \text{when } m \neq N/2,\ V_{\mu^1,\mu^2} \cong V_{\nu^1,\nu^2} \quad \text{if and only if } (\mu^1,\mu^2) = (\nu^1,\nu^2); \\ \text{when } m = N/2,\ V_{\mu^1,\mu^2} \cong V_{\nu^1,\nu^2} \quad \text{if and only if} \\ \text{either } (\mu^1,\mu^2) = (\nu^1,\nu^2) \quad \text{or} \quad (\mu^1,\mu^2) = (\nu^2,\nu^1). \end{gathered}$$

As the $D_{\mu^1} \otimes D_{\mu^2}$ are the composition factors of $\mathcal{H}_{\chi_m,-1}$ we conclude the following.

LEMMA 2.1. *The composition factors of L_{χ_m} consist of the V_{μ^1,μ^2}, $\mu^1 \neq \mu^2$, $\mu^1 \in \mathcal{P}_2(m)$, $\mu^2 \in \mathcal{P}_2(N-m)$, and when $N = 2m$ we have two additional composition factors $V^{\mathrm{I}}_{\mu,\mu}$ and $V^{\mathrm{II}}_{\mu,\mu}$ for $\mu \in \mathcal{P}_2(m)$.*

2.4 Parabolic induction functor and Fourier transform

Let L be a θ-stable Levi subgroup contained in a θ-stable parabolic subgroup $P \subset G$. We write

$$\mathfrak{l} = \operatorname{Lie} L, \quad \mathfrak{p} = \operatorname{Lie} P, \quad L_K = L \cap K, \quad P_K = P \cap K, \quad \mathfrak{l}_1 = \mathfrak{l} \cap \mathfrak{g}_1, \quad \mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1.$$

We will make use of the parabolic induction functor

$$\operatorname{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1 \subset \mathfrak{p}_1} : D_{L_K}(\mathfrak{l}_1) \to D_K(\mathfrak{g}_1)$$

defined in [Hen01, Lus11b]. In the following we recall its definition. Let

$$\mathrm{pr} : \mathfrak{p}_1 = \mathfrak{l}_1 \oplus (\mathfrak{n}_P)_1 \to \mathfrak{l}_1$$

be the natural projection map, where $\mathfrak{n}_P$ is the nilpotent radical of $\mathfrak{p}$ and $(\mathfrak{n}_P)_1 = \mathfrak{n}_P \cap \mathfrak{g}_1$. Consider the diagram

$$\mathfrak{l}_1 \xleftarrow{\mathrm{pr}} \mathfrak{p}_1 \xleftarrow{p_1} K \times \mathfrak{p}_1 \xrightarrow{p_2} K \times^{P_K} \mathfrak{p}_1 \xrightarrow{\check{\pi}} \mathfrak{g}_1, \tag{2.10}$$

where p_1 and p_2 are natural projection maps and $\check{\pi} : (k, x) \mapsto \operatorname{Ad}(k)(x)$.

The maps in (2.10) are $K \times P_K$-equivariant, where K acts trivially on $\mathfrak{l}_1$ and $\mathfrak{p}_1$, by left multiplication on the K-factor on $K \times \mathfrak{p}_1$ and on $K \times^{P_K} \mathfrak{p}_1$, and by adjoint action on $\mathfrak{g}_1$, and P_K acts on $\mathfrak{l}_1$ by $a.l = \mathrm{pr}(\operatorname{Ad} a(l))$, by adjoint action on $\mathfrak{p}_1$, by $a.(k,p) = (ka^{-1}, \operatorname{Ad} a(p))$ on $K \times \mathfrak{p}_1$, trivially on $K \times^{P_K} \mathfrak{p}_1$ and $\mathfrak{g}_1$.

Let A be a complex in $D_{L_K}(\mathfrak{l}_1)$. Then $(\mathrm{pr} \circ p_1)^* A \cong p_2^* A'$ for a well-defined complex A' in $D_K(K \times^{P_K} \mathfrak{p}_1)$. Define

$$\operatorname{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1 \subset \mathfrak{p}_1} A = \check{\pi}_! A'[\dim P - \dim L].$$

Let $\mathfrak{F} : D_K(\mathfrak{g}_1) \to D_K(\mathfrak{g}_1)$ be the Fourier transform functor (we also use the same notation $\mathfrak{F}$ for the functor defined for $(L_K, \mathfrak{l}_1)$). Here we have identified $\mathfrak{g}_1$ with $\mathfrak{g}_1^*$ via a K-invariant non-degenerate bilinear form on $\mathfrak{g}_1$. It is shown in [Hen01, Lus11b] that the induction functor commutes with Fourier transform, i.e.,

$$\mathfrak{F}(\operatorname{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1 \subset \mathfrak{p}_1} A) \cong \operatorname{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}(\mathfrak{F}(A)). \tag{2.11}$$

3. Maximal θ-stable parabolic subgroups and parabolic induction

In this section, we study the parabolic induction functor with respect to a chosen family of $L^m \subset P^m$, $1 \leqslant m < N/2$, and two more pairs $L^{n,\omega} \subset P^{n,\omega}$, $\omega = \mathrm{I}, \mathrm{II}$, if $N = 2n$, where P^m (respectively $P^{n,\omega}$) is a maximal θ-stable parabolic subgroup and L^m (respectively $L^{n,\omega}$) is a θ-stable Levi subgroup of P^m (respectively $P^{n,\omega}$) defined as follows.

Fix a basis $\{e_i, 1 \leqslant i \leqslant N\}$ of V such that $\langle e_i, e_j \rangle = \delta_{i+j,N+1}$. For $1 \leqslant m < N/2$, we define P^m to be the parabolic subgroup of G that stabilizes the flag

$$0 \subset V^0_m \subset V^{0\perp}_m \subset \mathbb{C}^N,$$

where $V_m^0 = \mathrm{span}\{e_i, 1 \leqslant i \leqslant m\}$. We define L^m to be the θ-stable Levi subgroup of P^m which consists of diagonal block matrices of sizes $m, N-2m, m$. When $N = 2n$, for $\omega = \mathrm{I}, \mathrm{II}$, we define $P^{n,\omega}$ to be the parabolic subgroup of G that stabilizes the flag

$$0 \subset V_n^\omega \subset V_n^{\omega\perp} \subset \mathbb{C}^{2n},$$

where $V_n^{\mathrm{I}} = \mathrm{span}\{e_i, 1 \leqslant i \leqslant n\}$ and $V_n^{\mathrm{II}} = \mathrm{span}\{e_i, 1 \leqslant i \leqslant n-1, e_{n+1}\}$. Let $L^{n,\omega}$ be a θ-stable Levi subgroup of $P^{n,\omega}$. According to [BH00], every maximal θ-stable parabolic subgroup of G is K-conjugate to one of the above form.

Let $\mathfrak{p}^m = \mathrm{Lie}\, P^m$, $\mathfrak{p}_1^m = \mathfrak{p}^m \cap \mathfrak{g}_1$, and $(\mathfrak{n}_{P^m})_1 = \mathfrak{n}_{P^m} \cap \mathfrak{g}_1$, where $\mathfrak{n}_{P^m}$ is the nilpotent radical of $\mathfrak{p}^m$, etc.

Proposition 3.1. *We have the following.*

(i) *The map*

$$\pi_m^N : K \times^{P_K^m} (\mathfrak{n}_{P^m})_1 \to \mathcal{N}_1, \quad (k, x) \mapsto \mathrm{Ad}\, k(x)$$

is a small map onto its image, generically one-to-one.

(ii) *The map*

$$\check{\pi}_m^N : K \times^{P_K^m} \mathfrak{p}_1^m \to \mathfrak{g}_1, \quad (k, x) \mapsto \mathrm{Ad}\, k(x).$$

is a small map onto its image, generically one-to-one.

The same holds for the two maps $\pi_n^{2n,\omega}$ and $\check{\pi}_n^{2n,\omega}$ defined using $P^{n,\omega}$, $\omega = \mathrm{I}, \mathrm{II}$.

We define

$$\mathfrak{g}_1^m = \mathrm{Im}\, \check{\pi}_m^N, \quad 1 \leqslant m < N/2, \quad \mathfrak{g}_1^{n,\omega} = \mathrm{Im}\, \check{\pi}_n^{2n,\omega}, \quad \omega = \mathrm{I}, \mathrm{II}. \tag{3.1}$$

For $m < N/2$, $\mathfrak{g}_1^m$ consists of elements in $\mathfrak{g}_1$ with eigenvalues $a_1, a_1, \ldots, a_m, a_m, a_j, j \in [2m+1, N]$, where $\sum_{k=1}^m 2a_k + \sum_{j=2m+1}^N a_j = 0$. Let

$$Y_m^r = \big\{x \in \mathfrak{g}_1^{\mathrm{reg}} \mid x \text{ has eigenvalues } a_1, a_1, \ldots, a_m, a_m, a_j, j \in [2m+1, N], \text{ where } a_i \neq a_j \text{ for } i \neq j\big\}.$$

One checks readily that $\overline{Y_m^r} = \mathfrak{g}_1^m$.

Consider the case $m = N/2 = n$. For $\omega = \mathrm{I}, \mathrm{II}$, let

$$Y_n^{r,\omega} = \big\{x \in \mathfrak{g}_1^{\mathrm{reg}} \mid x \text{ has eigenvalues } a_1, a_1, \ldots, a_n, a_n, \text{ where } a_i \neq a_j \text{ for } i \neq j, \text{ and the nilpotent part of } x \text{ lies in the orbit } \mathcal{O}_{2^n}^\omega\big\},$$

where $\mathcal{O}_{2^n}^\omega$ is the nilpotent orbit given by the partition 2^m and defined by the equation $\mathrm{Im}\, \pi_n^{2n,\omega} = \bar{\mathcal{O}}_{2^n}^\omega$. Then $Y_n^{r,\omega}$ is an open dense subset in $\mathfrak{g}_1^{n,\omega}$.

Let $(\mathfrak{p}_1^m)^r = \mathfrak{p}_1^m \cap Y_m^r$ and $(\mathfrak{l}_1^m)^{\mathrm{rs}} = \mathfrak{l}_1^m \cap (\mathfrak{l}^m)^{\mathrm{rs}}$.

Proposition 3.2. (1) *Suppose that $1 \leqslant m < N/2$. There is a natural surjective map*

$$\pi_1^K(Y_m^r) \twoheadrightarrow \pi_1^{L_K^m}((\mathfrak{l}_1^m)^{\mathrm{rs}}) \cong B_m \times \widetilde{B}_{N-2m} \tag{3.2}$$

such that for an L_K^m-equivariant local system $\mathcal{T}$ on $(\mathfrak{l}_1^m)^{\mathrm{rs}}$ associated to a $\pi_1^{L_K^m}((\mathfrak{l}_1^m)^{\mathrm{rs}})$-representation E, we have

$$\mathrm{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathrm{IC}(\mathfrak{l}_1^m, \mathcal{T}) \cong \mathrm{IC}(\mathfrak{g}_1^m, \mathcal{T}'),$$

where $\mathcal{T}'$ is the K-equivariant local system on Y_m^r associated to the representation of $\pi_1^K(Y_m^r)$ which is obtained from E by pull-back under the map (3.2).

(2) *We have a natural surjective map*

$$\pi_1^K(Y_n^{r,\omega}) \twoheadrightarrow \pi_1^{L_K^{n,\omega}}((\mathfrak{l}_1^{n,\omega})^{\mathrm{rs}}) \cong B_n, \quad \omega = \mathrm{I}, \mathrm{II}, \tag{3.3}$$

such that for an $L_K^{n,\omega}$-equivariant local system $\mathcal{T}$ on $(\mathfrak{l}_1^{n,\omega})^{\mathrm{rs}}$ associated to a $\pi_1^{L_K^{n,\omega}}((L_1^{n,\omega})^{\mathrm{rs}})$-representation E, we have

$$\mathrm{Ind}_{\mathfrak{l}_1^{n,\omega}\subset\mathfrak{p}_1^{n,\omega}}^{\mathfrak{g}_1} \mathrm{IC}(\mathfrak{l}_1^{n,\omega}, \mathcal{T}) \cong \mathrm{IC}(\mathfrak{g}_1^{n,\omega}, \mathcal{T}'),$$

where $\mathcal{T}'$ is the K-equivariant local system on $Y_n^{r,\omega}$ associated to the representation of $\pi_1^K(Y_n^{r,\omega})$ which is obtained from E by pull-back under the map (3.3).

3.1 Proof of Proposition 3.1

We begin with the proof of (1). Consider the following projection

$$\tau_m^N : \{(x, 0 \subset V_m \subset V_m^\perp \subset V = \mathbb{C}^N) \,|\, x \in \mathfrak{g}_1,\ xV_m = 0,\ xV_m^\perp \subset V_m\} \to \mathcal{N}_1.$$

When $m \neq N/2$, the map τ_m^N can be identified with the map π_m^N. When $N = 2m$, the image of the map τ_m^{2m} has two irreducible components, i.e., closures of the two orbits $\mathcal{O}_{2m}^{\mathrm{I}}$ and $\mathcal{O}_{2m}^{\mathrm{II}}$. The two maps $\pi_m^{N,\mathrm{I}}$ and $\pi_m^{N,\mathrm{II}}$ can be identified with the map τ_m^{2m} restricted to $\overline{(\tau_m^{2m})^{-1}(\mathcal{O}_{2m}^{\mathrm{I}})}$ and $\overline{(\tau_m^{2m})^{-1}(\mathcal{O}_{2m}^{\mathrm{II}})}$ respectively. Thus it suffices to show that

$$\text{the map } \tau_m^N \text{ is small over its image and generically one-to-one.} \tag{3.4}$$

When $m \neq N/2$, one can check that the image of τ_m^N is as follows

$$\mathrm{Im}\,\tau_m^N = \bar{\mathcal{O}}_{3^m1^{N-3m}} \quad \text{if } N \geqslant 3m, \quad \mathrm{Im}\,\tau_m^N = \bar{\mathcal{O}}_{3^{N-2m}2^{3m-N}} \quad \text{if } N < 3m.$$

Assume that $N \geqslant 3m$ and $x \in \mathcal{O}_{3^m1^{N-3m}}$. Then $(\tau_m^N)^{-1}(x)$ consists of the flag $0 \subset V_m \subset V_m^\perp \subset V$, where $V_m = \mathrm{Im}\, x^2$. Assume that $N < 3m$ and $x \in \mathcal{O}_{3^{N-2m}2^{3m-N}}$. Then $(\tau_m^N)^{-1}(x)$ consists of the flag $0 \subset V_m \subset V_m^\perp \subset V$, where $V_m = \ker x$. This proves that τ_m^N is generically one-to-one.

Let $x \in \mathcal{O}_{3^i2^j1^{N-3i-2j}} \subset \mathrm{Im}\,\tau_m^N$. We assume that $3^i2^j1^{N-3i-2j} \neq 3^m1^{N-3m}$ if $N \geqslant 3m$, and $3^i2^j1^{N-3i-2j} \neq 3^{N-2m}2^{3m-N}$ if $N < 3m$. It suffices to show that

$$\dim(\tau_m^N)^{-1}(x) < \mathrm{codim}_{\mathrm{Im}\,\tau_m^N} \mathcal{O}_{3^i2^j1^{N-3i-2j}}/2.$$

Let $x_0 \in \mathcal{O}_{2^j1^{N-3i-2j}} \subset \mathrm{Im}\,\tau_{m-i}^{N-3i}$. (Note that τ_{m-i}^{N-3i} is defined since $m - i \leqslant (N - 3i)/2$.) One checks readily that (using (2.2) for the second equality)

$$(\tau_m^N)^{-1}(x) \cong (\tau_{m-i}^{N-3i})^{-1}(x_0) \quad \text{and} \quad \mathrm{codim}_{\mathrm{Im}\,\tau_m^N} \mathcal{O}_{3^i2^j1^{N-3i-2j}} = \mathrm{codim}_{\mathrm{Im}\,\tau_{m-i}^{N-3i}} \mathcal{O}_{2^j1^{N-3i-2j}}.$$

Thus it suffices to show that

$$\dim(\tau_{m-i}^{N-3i})^{-1}(x_0) < \mathrm{codim}_{\mathrm{Im}\,\tau_{m-i}^{N-3i}} \mathcal{O}_{2^j1^{N-3i-2j}}/2.$$

Let us write

$$\Omega_{m,j}^N = (\tau_m^N)^{-1}(\zeta_j) \quad \text{for } \zeta_j \in \mathcal{O}_{2^j1^{N-2j}} \subset \mathrm{Im}\,\tau_m^N$$

and

$$a_{m,j}^N = \mathrm{codim}_{\mathrm{Im}\,\pi_m^N} \mathcal{O}_{2^j1^{N-2j}} = m(2N - 3m) - j(N - j).$$

To prove that the map τ_m^N is small, we are reduced to proving that

$$\dim \Omega_{m,j}^N < \frac{a_{m,j}^N}{2}. \tag{3.5}$$

To prove this we recall the partitioning of $\Omega_{m,j}^N$ into $\Omega_{m,j}^{N,k}$ given in [CVX15b, § 2] as follows:

$$\Omega_{m,j}^{N,k} = \{(0 \subset V_m \subset V_m^\perp \subset V = \mathbb{C}^N) \mid \dim(V_m \cap \zeta_j V) = k\}.$$

We have

$$\Omega_{m,j}^{N,k} \neq \emptyset \Leftrightarrow \max\{m + j - N/2, j/2\} \leqslant k \leqslant \min\{j, m\}.$$

Recall that we have a surjective map $\Omega_{m,j}^{N,k} \to \mathrm{OGr}(j-k, j) \times \mathrm{OGr}(m-k, N-2j)$ with fibers being affine spaces $\mathbb{A}^{(m-k)(j-k)}$. We have

$$\dim \overline{\Omega_{m,j}^{N,k}} = -2k^2 + (-N + 3j + 2m + 1)\, k + mN - mj - \frac{j^2 + 3m^2 + j + m}{2}.$$

One checks that

$$\text{if } j \geqslant N - 2m, \quad \dim \overline{\Omega_{m,j}^{N,k}} \text{ is maximal when } k = m + j - \left[\frac{N}{2}\right],$$
$$\text{if } j < N - 2m, \quad \dim \overline{\Omega_{m,j}^{N,k}} \text{ is maximal when } k = \left[\frac{j+1}{2}\right].$$

Thus a direct calculation shows that

$$\dim(\pi_m^N)^{-1}(\zeta_j) = \begin{cases} \dfrac{a_{m,j}^N}{2} + \dfrac{j + m - N}{2} & \text{if } j \geqslant N - 2m \text{ and } N \text{ even, or } j < N - 2m \text{ and } j \text{ odd,} \\ \dfrac{a_{m,j}^N}{2} - \dfrac{m}{2} & \text{if } j \geqslant N - 2m \text{ and } N \text{ odd, or } j < N - 2m \text{ and } j \text{ even.} \end{cases}$$

This proves (3.5) (note that $m + j < N$). The proof of (3.4) is complete. This finishes the proof of the claim (1) in the proposition.

It then follows that we have

$$(\pi_m^N)_* \mathbb{C}[-] \cong \mathrm{IC}(\mathcal{O}_\lambda, \mathbb{C}) \quad (\text{respectively } ((\pi_{N/2}^N)^\omega)_* \mathbb{C}[-] \cong \mathrm{IC}(\mathcal{O}_\lambda^\omega, \mathbb{C}),\ \omega = \mathrm{I}, \mathrm{II}), \tag{3.6}$$

where

$$\lambda = 3^m 1^{N-3m} \quad \text{if } N \geqslant 3m, \quad \lambda = 3^{N-2m} 2^{3m-N} \quad \text{if } N < 3m.$$

Note that $K \times^{P_K^m} \mathfrak{p}_1^m$ is the orthogonal complement of $K \times^{P_K^m} (\mathfrak{n}_{P^m})_1$ in the trivial bundle $K \times \mathfrak{g}_1$ over K/P_K^m. By the functoriality of Fourier transform, we have that

$$\mathfrak{F}((\pi_m^N)_* \mathbb{C}[-]) \cong (\check{\pi}_m^N)_* \mathbb{C}[-]. \tag{3.7}$$

Since Fourier transform sends simple perverse sheaves to simple perverse sheaves, we can conclude from (3.6) and (3.7) that

$$(\check{\pi}_m^N)_* \mathbb{C}[-] \cong \mathrm{IC}(\mathrm{Im}\, \check{\pi}_m^N, \mathbb{C}).$$

This proves the claim (2) of the proposition. The argument for $(\check{\pi}_n^{2n})^\omega$, $\omega = \mathrm{I}, \mathrm{II}$, is the same. The proof of the proposition is complete.

3.2 Proof of Proposition 3.2

Note that we have that

$$L_K^m \cong \mathrm{GL}(m) \times \mathrm{SO}(N-2m) \quad \text{and} \quad (\mathfrak{l}^m)_1 \cong \mathfrak{gl}(m) \oplus \mathfrak{sl}(N-2m)_1. \tag{3.8}$$

To ease notation, let us now write that $L = L^m$, $P = P^m$, and $\check{\pi} = \check{\pi}_m^N$ etc.

We first show the following.

$$\begin{aligned}&\text{The map } \check{\pi} \text{ (respectively } \check{\pi}_n^\omega), \text{ when restricted to } \check{\pi}^{-1}(Y^r) \text{ (respectively } \check{\pi}^{-1}(Y_n^{r,\omega})),\\ &\quad \text{is one-to-one.}\end{aligned} \tag{3.9}$$

Each element in Y^r is K-conjugate to an element $x_0 \in \mathfrak{p}_1$ (see [KR71, Theorem 7]), where

$$\begin{gathered} x_0 e_i = a_i e_i, \quad x_0 e_{N+1-i} = e_i + a_i e_{N+1-i} \quad \text{for } i \in [1,m],\\ x_0 e_j = b_j e_j + c_j e_{N+1-j}, \quad x_0 e_{N+1-j} = c_j e_j + b_j e_{N+1-j} \quad \text{for } j \in [m+1, [N/2]]\\ x_0 e_{(N+1)/2} = b_{(N+1)/2} e_{(N+1)/2} \quad \text{if } N \text{ is odd}\end{gathered} \tag{3.10}$$

and the numbers $a_i, i = 1, \dots, m, b_j + c_j, b_j - c_j, j = m+1, \dots, [N/2], b_{(N+1)/2}$ are distinct.

Thus it suffices to show that $\check{\pi}^{-1}(x_0)$ consists of one point. Note that $\check{\pi}^{-1}(x_0)$ consists of x_0-stable m-dimensional isotropic subspaces of V. Assume that $U_m \in \check{\pi}^{-1}(x_0)$. Then U_m is a direct sum of generalized eigenspaces of x_0. The generalized eigenspace of x_0 with eigenvalue a_i is $\mathrm{span}\{e_i, e_{N+1-i}\}$, $i = 1, \dots, m$, and the eigenspace of x_0 with eigenvalue $b_j + c_j$, $b_j - c_j$, $b_{(N+1)/2}$ is $\mathrm{span}\{e_j e_{N+1-j}\}$, $\mathrm{span}\{e_j + e_{N+1-j}\}$, $\mathrm{span}\{e_{(N+1)/2}\}$, respectively. Since U_m is isotropic, U_m has to equal $\mathrm{span}\{e_i, i = 1, \dots, m\}$. This proves (3.9) for $\check{\pi}_m$, $m < N/2$. The proof for $\check{\pi}_n^\omega$ is entirely similar and omitted.

Now we show the following.

$$\text{The image of } \mathfrak{p}_1^r \text{ under the map } \mathrm{pr} : \mathfrak{p}_1 \to \mathfrak{l}_1 \text{ is } \mathfrak{l}_1^{\mathrm{rs}}. \tag{3.11}$$

Let $x \in \mathfrak{p}_1^r$. By the above proof of (3.9) we can assume that $\mathrm{Ad}(k)x = x_0$ for some $k \in K$, where x_0 is as in (3.10). Thus $(k, x) \in \check{\pi}^{-1}(x_0)$. It follows from (3.9) that $(k, x) = (1, x_0) \in K \times^{P_K} \mathfrak{p}_1$. Hence $k \in P_K$. Assume that $k = lu$ where $l \in L_K$ and $u \in U_K = U \cap K$ (U is the unipotent radical of P). Then we have $\mathrm{pr}(x) = \mathrm{pr}(\mathrm{Ad}(u^{-1}l^{-1})x_0) = \mathrm{pr}(\mathrm{Ad}(l^{-1})x_0) = \mathrm{Ad}(l^{-1})\,\mathrm{pr}(x_0)$. It is clear that $\mathrm{pr}(x_0) \in \mathfrak{l}^{\mathrm{rs}}$. Thus (3.11) follows.

By (3.9) and (3.11), we have the following diagram, when restricting (2.10) to Y^r,

$$\mathfrak{l}_1^{\mathrm{rs}} \xleftarrow{\mathrm{pr}} \mathfrak{p}_1^r \xleftarrow{p_1} K \times \mathfrak{p}_1^r \xrightarrow{p_2} K \times^{P_K} \mathfrak{p}_1^r \xrightarrow{\check{\pi}} Y^r.$$

Using (3.9), we see that

$$\pi_1^K(Y^r) \cong \pi_1^{K \times P_K}(Y^r) \cong \pi_1^{K \times P_K}(K \times^{P_K} \mathfrak{p}_1^r) \cong \pi_1^{K \times P_K}(K \times \mathfrak{p}_1^r) \cong \pi_1^{P_K}(\mathfrak{p}_1^r).$$

Finally, the canonical map $\pi_1^{P_K}(\mathfrak{p}_1^r) \to \pi_1^{P_K}(\mathfrak{l}_1^{\mathrm{rs}}) \cong \pi_1^{L_K}(\mathfrak{l}_1^{\mathrm{rs}})$ is surjective. We see this as follows. First, the canonical map above can be identified with the canonical map $\pi_1^{P_K}(\mathfrak{p}_1^r) \to \pi_1^{P_K}(\mathrm{pr}^{-1}(\mathfrak{l}_1^{\mathrm{rs}}))$. Now, because $\mathfrak{p}_1^r$ is an open subset in $\mathrm{pr}^{-1}(\mathfrak{l}_1^{\mathrm{rs}})$, which is smooth, the map $\pi_1(\mathfrak{p}_1^r) \to \pi_1(\mathrm{pr}^{-1}(\mathfrak{l}_1^{\mathrm{rs}}))$ is a surjection. To conclude that this property persists when we pass to the equivariant fundamental group it suffices to remark that the equivariant fundamental group is always a quotient of the ordinary fundamental group as long as the group is connected. We now conclude the argument making use of Proposition 3.1.

4. Fourier transform of nilpotent orbital complexes for $(\mathbf{SL}(N), \mathbf{SO}(N))$

Consider the symmetric pair $(G, K) = (\mathrm{SL}(N), \mathrm{SO}(N))$. Let us write $\mathcal{A}_N$ for the set of all simple K-equivariant perverse sheaves on $\mathcal{N}_1$ (up to isomorphism), that is, the set of IC complexes $\mathrm{IC}(\mathcal{O}, \mathcal{E})$, where $\mathcal{O}$ is a K-orbit in $\mathcal{N}_1$ and $\mathcal{E}$ is an irreducible K-equivariant local system on $\mathcal{O}$ (up to isomorphism). An IC complex in $\mathcal{A}_N$ is called a nilpotent orbital complex.

Let $n = [N/2]$. We set

$$\Sigma_N = \{(\nu; \mu^1, \mu^2) \mid 0 \leqslant m \leqslant n,\ \nu \in \mathcal{P}(m), \\ 0 \leqslant k \leqslant n-m,\ \mu^1 \in \mathcal{P}_2(k),\ \mu^2 \in \mathcal{P}_2(N-2m-k)\}.$$

In the case when N is even, we identify the triple $(\nu; \mu^1, \mu^2)$ with $(\nu; \mu^2, \mu^1)$ if $|\mu^1| = |\mu^2|$ and $\mu^1 \neq \mu^2$, and the triples $(\nu; \mu, \mu)$ attain two labels I and II.

Given a triple $(\nu; \mu^1, \mu^2) \in \Sigma_N$ (respectively $(\nu; \mu, \mu)^\omega \in \Sigma_N$, $\omega = \mathrm{I}, \mathrm{II}$), where $|\nu| = m < N/2$, we define an irreducible K-equivariant local system $\mathcal{T}(\nu; \mu^1, \mu^2)$ (respectively $\mathcal{T}(\nu; \mu, \mu)^\omega$) on Y_m^r (here we write $Y_0^r = \mathfrak{g}_1^{\mathrm{rs}}$) as follows. We obtain a map

$$\tau : \pi_1^K(Y_m^r) \to B_m \times \widetilde{B}_{N-2m} \to S_m \times \widetilde{B}_{N-2m}$$

by composing the map in (3.2) with the natural map $B_m \times \widetilde{B}_{N-2m} \to S_m \times \widetilde{B}_{N-2m}$. Note that the map τ is surjective. Then $\mathcal{T}(\nu; \mu^1, \mu^2)$ (respectively $\mathcal{T}(\nu; \mu, \mu)^\omega$) is the irreducible local system associated to the irreducible representation of $\pi_1^K(Y_m^r)$ given by pulling back the irreducible representation $\rho_\nu \boxtimes V_{\mu^1, \mu^2}$ (respectively $\rho_\nu \boxtimes V_{\mu,\mu}^\omega$) via the map τ; here $\rho_\nu \in S_m^\vee$ is the irreducible representation of S_m corresponding to $\nu \in \mathcal{P}(m)$ and V_{μ^1, μ^2} (respectively $V_{\mu,\mu}^\omega$) is the irreducible representation of $\widetilde{B}_{N-2m}$ defined in (2.7) (respectively (2.9)).

Assume now that $N = 2n$. Given a triple $(\nu; \emptyset, \emptyset)^\omega \in \Sigma_N$, $\omega = \mathrm{I}, \mathrm{II}$, we define the irreducible K-equivariant local system $\mathcal{T}(\nu; \emptyset, \emptyset)^\omega$ on $(Y_n^r)^\omega$ as the local system associated to the representation of $\pi_1^K((Y_n^r)^\omega)$ obtained by pulling back the representation $\rho_\nu \in S_n^\vee$ corresponding to $\nu \in \mathcal{P}(n)$ under that map

$$\pi_1^K((Y_n^r)^\omega) \twoheadrightarrow B_n \twoheadrightarrow S_n.$$

Now we are ready to formulate our main result.

Theorem 4.1. *The Fourier transform $\mathfrak{F} : \mathrm{Perv}_K(\mathfrak{g}_1) \to \mathrm{Perv}_K(\mathfrak{g}_1)$ induces a bijection*

$$\begin{aligned}\mathfrak{F} : \mathcal{A}_N \xrightarrow{\sim} &\{\mathrm{IC}(\mathfrak{g}_1^m, \mathcal{T}(\nu; \mu^1, \mu^2)) \mid (\nu; \mu^1, \mu^2) \in \Sigma_N, \mu^1 \neq \mu^2,\ |\nu| = m < N/2\} \\ &\cup \{\mathrm{IC}(\mathfrak{g}_1^m, \mathcal{T}(\nu; \mu, \mu)^\omega) \mid (\nu; \mu, \mu)^\omega \in \Sigma_N, \omega = \mathrm{I}, \mathrm{II},\ |\nu| = m < N/2\} \quad (\textit{if } N \textit{ is even}), \\ &\cup \{\mathrm{IC}(\mathfrak{g}_1^{n,\omega}, \mathcal{T}(\nu; \emptyset, \emptyset)^\omega) \mid (\nu; \emptyset, \emptyset)^\omega \in \Sigma_N, \omega = \mathrm{I}, \mathrm{II},\ |\nu| = n = N/2\} \quad (\textit{if } N \textit{ is even}),\end{aligned}$$

where $\mathfrak{g}_1^0 = \mathfrak{g}_1$, $\mathfrak{g}_1^m$ and $\mathfrak{g}_1^{n,\omega}$ are defined in (3.1).

4.1 Proof of Theorem 4.1

Let $p(k)$ denote the number of partitions of k and let $q(k)$ denote the number of 2-regular partitions of k. We write $p(0) = q(0) = 1$. Let us define

$$d(k) = \sum_{s=0}^{k} q(s) q(2k+1-s), \tag{4.1}$$

$$e(k) = \sum_{s=0}^{k-1} q(s)\, q(2k-s) + \frac{q(k)^2 + 3q(k)}{2}. \tag{4.2}$$

LEMMA 4.2. *We have*

$$|\mathcal{A}_{2n+1}| = \sum_{k=0}^{n} p(n-k)d(k) = |\Sigma_{2n+1}|, \tag{4.3}$$

$$|\mathcal{A}_{2n}| = \sum_{k=0}^{n} p(n-k)e(k) = |\Sigma_{2n}|. \tag{4.4}$$

Proof. Note that

$$\sum_{k\geqslant 0} p(k)x^k = \prod_{s\geqslant 1} \frac{1}{1-x^s} \quad \text{and} \quad \sum_{k\geqslant 0} q(k)x^k = \prod_{s\geqslant 1}(1+x^s). \tag{4.5}$$

Let $p(l,k)$ denote the number of partitions of l into (not necessarily distinct) parts of exactly k different sizes. We have (see for example [Wil83])

$$\sum_{l,k\geqslant 0} p(l,k)x^l y^k = \prod_{s\geqslant 1}\left(1+\frac{yx^s}{1-x^s}\right). \tag{4.6}$$

Assume first that $N = 2n+1$. Note that if λ is a partition of N with parts of k different sizes, then the component group $A_K(x)$ of the centralizer $Z_K(x)$ for $x \in \mathcal{O}_\lambda$ is $(\mathbb{Z}/2\mathbb{Z})^{k-1}$. Thus there are 2^{k-1} irreducible K-equivariant local systems on $\mathcal{O}_\lambda$ (up to isomorphism). Hence using (4.6), we see that

$$|\mathcal{A}_{2n+1}| = \sum_{k\geqslant 0} p(2n+1,k)2^{k-1} = \text{Coefficient of } x^{2n+1} \text{ in } \frac{1}{2}\prod_{s\geqslant 1}\left(\frac{1+x^s}{1-x^s}\right).$$

Using (4.5), we see that

$$\prod_{s\geqslant 1}\left(\frac{1+x^s}{1-x^s}\right) = \left(\sum_{k\geqslant 0} p(k)x^{2k}\right)\left(\sum_{k\geqslant 0} q(k)x^k\right)^2. \tag{4.7}$$

It then follows that $|\mathcal{A}_{2n+1}|$ is the desired number. The fact that $|\Sigma_{2n+1}|$ equals the same number is clear from the definition. Thus (4.3) holds.

Assume now that $N = 2n$. Suppose that λ is a partition of N with parts of exactly k different sizes. If λ has at least one odd part, then there are 2^{k-1} irreducible K-equivariant local systems on $\mathcal{O}_\lambda$ (up to isomorphism). If λ has only even parts, then there are 2^k irreducible K-equivariant local systems on each $\mathcal{O}_\lambda^\omega$ (up to isomorphism), $\omega = \mathrm{I}, \mathrm{II}$.

Thus we have that

$$\begin{aligned}
|\mathcal{A}_{2n}| &= \sum_{k\geqslant 0} p(2n,k)\, 2^{k-1} + \sum_{k\geqslant 0} p(n,k)\, 3\cdot 2^{k-1} \\
&= \text{Coefficient of } x^{2n} \text{ in } \frac{1}{2}\prod_{s\geqslant 1}\left(\frac{1+x^s}{1-x^s}\right) + \text{Coefficient of } x^n \text{ in } \frac{3}{2}\prod_{s\geqslant 1}\left(\frac{1+x^s}{1-x^s}\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{n} p(n-k)\left(2\sum_{s=0}^{k-1} q(s)q(2k-s) + q(k)^2\right)\right) + \frac{3}{2}\sum_{k=0}^{n} p(n-k)q(k) = \sum_{k=0}^{n} p(n-k)e(k).
\end{aligned}$$

Here we have used (4.7) and the following equation

$$\prod_{s\geqslant 1}\left(\frac{1+x^s}{1-x^s}\right) = \left(\sum_{k\geqslant 0} p(k)x^k\right)\left(\sum_{k\geqslant 0} q(k)x^k\right).$$

Again the fact that $|\Sigma_{2n}|$ equals the desired number is clear from the definition. □

Note that the IC sheaves appearing on the right-hand side of the Fourier transform map $\mathfrak{F}$ in Theorem 4.1 are pairwise non-isomorphic. Thus, in view of Lemma 4.2, Theorem 4.1 follows from the following.

Proposition 4.3. *Let* $(\nu;\mu^1,\mu^2)\in\Sigma_N$ *(respectively* $(\nu;\mu,\mu)^\omega\in\Sigma_N$, $\omega=\mathrm{I},\mathrm{II}$*) and write* $m=|\nu|$. *The Fourier transform of* $\mathrm{IC}(\mathfrak{g}_1^m,\mathcal{T}(\nu;\mu^1,\mu^2))$ *(respectively* $\mathrm{IC}(\mathfrak{g}_1^m,\mathcal{T}(\nu;\mu,\mu)^\omega)$, $\mathrm{IC}(\mathfrak{g}_1^{n,\omega},\mathcal{T}(\nu;\emptyset,\emptyset)^\omega)$*) is supported on a* K*-orbit in* $\mathcal{N}_1$.

Proof. Let $n=[N/2]$. We begin the proof by showing the following for $(\emptyset;\mu^1,\mu^2)\in\Sigma_N$ (respectively $(\emptyset;\mu,\mu)^\omega\in\Sigma_N$, $\omega=\mathrm{I},\mathrm{II}$).

$$\begin{aligned}&\text{The Fourier transform of } \mathrm{IC}(\mathfrak{g}_1,\mathcal{T}(\emptyset;\mu^1,\mu^2))(\text{respectively } \mathrm{IC}(\mathfrak{g}_1,\mathcal{T}(\emptyset;\mu,\mu)^\omega))\\ &\text{is supported on a } K\text{-orbit in } \mathcal{N}_1.\end{aligned}\tag{4.8}$$

Recall that $\mathcal{T}(\emptyset;\mu^1,\mu^2)$ (respectively $\mathcal{T}(\emptyset;\mu,\mu)^\omega$) is the irreducible K-equivariant local system on $\mathfrak{g}_1^{\mathrm{rs}}$ corresponding to V_{μ^1,μ^2} (respectively $V^\omega_{\mu,\mu}$).

We will now appeal to the nearby cycle construction in [GVX18]. Let us recall the characters $\chi_m\in I_N^\vee$, $0\leqslant m\leqslant n$ of (2.3). In [GVX18] we apply a nearby cycle construction to local systems associated to the χ_m and obtain a K-equivariant perverse sheaf $\mathcal{P}_{\chi_m}$ on the nilpotent cone $\mathcal{N}_1$. More precisely, for each character χ_m, let us write $W_{\chi_m}=\mathrm{Stab}_W(\chi_m)$. Consider the following base-change diagram of the adjoint quotient map.

$$\begin{array}{ccc}\mathfrak{g}_{1,\chi_m} & \longrightarrow & \mathfrak{g}_1\\ {\scriptstyle f_{\chi_m}}\downarrow & & \downarrow{\scriptstyle f}\\ \mathfrak{a}/W_{\chi_m} & \longrightarrow & \mathfrak{a}/W\end{array}\tag{4.9}$$

Let us write $\mathfrak{g}^{\mathrm{rs}}_{1,\chi_m}$ for the base change of the regular semisimple locus $\mathfrak{g}_1^{\mathrm{rs}}$. Denote by $\mathcal{F}_{\chi_m}$ the K-equivariant local system on $\mathfrak{g}^{\mathrm{rs}}_{1,\chi_m}$ corresponding to the representation of $\pi_1^K(\mathfrak{g}^{\mathrm{rs}}_{1,\chi_m})=I_N\rtimes B_{\chi_m}$, where I_N acts via the character χ_m and B_{χ_m} acts trivially (recall that $B_{\chi_m}=\mathrm{Stab}_{B_N}(\chi_m)$). We form the nearby cycle sheaf $\mathcal{P}_{\chi_m}=\psi_{f_{\chi_m}}\mathcal{F}_{\chi_m}$, appropriately shifted, so that $\mathcal{P}_{\chi_m}\in\mathrm{Perv}_K(\mathcal{N}_1)$.

Applying [GVX18, Theorem 3.2], we obtain that

$$\mathfrak{F}(\mathcal{P}_{\chi_m})=\mathrm{IC}(\mathfrak{g}_1,\mathcal{M}_{\chi_m}),$$

where $\mathcal{M}_{\chi_m}$ is the K-equivariant local system on $\mathfrak{g}_1^{\mathrm{rs}}$ corresponding to the $\widetilde{B}_N$-representation

$$M_{\chi_m}=\mathbb{C}[\widetilde{B}_N]\otimes_{\mathbb{C}[\widetilde{B}^0_{\chi m}]}(\mathbb{C}_{\chi_m}\otimes\mathcal{H}_{W^0_{\chi_m}}).$$

Let us explain the notation in the above formula in our setting. Let $W^0_{\chi_m}$ be the Coxeter subgroup of W generated by s_α with $\chi_m(\check{\alpha}(-1))=1$, $\alpha\in\Phi(\mathfrak{g},\mathfrak{a})$, where $\Phi(\mathfrak{g},\mathfrak{a})$ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Note that $\check{\alpha}(-1)\in I_N$. Then $\mathcal{H}_{W^0_{\chi_m}}$ is the Hecke algebra associated to the Coxeter group $W^0_{\chi_m}$ with parameter -1. Let $B^0_{\chi_m}\subset B_N$ be the inverse image of $W^0_{\chi_m}\subset W$ under the natural map $B_N\to W=S_N$. Then $\widetilde{B}^0_{\chi_m}=I_N\rtimes B^0_{\chi_m}$. In our setting, one readily checks that $W^0_{\chi_m}=\langle s_i, i\neq m\rangle$, $\mathcal{H}_{W^0_{\chi_m}}=\mathcal{H}_{\chi_m,-1}$ and $B^0_{\chi_m}=B_{m,N-m}$ (here we use the notation in § 2.3). The action of $\widetilde{B}^0_{\chi_m}$ on $(\mathbb{C}_{\chi_m}\otimes\mathcal{H}_{W^0_{\chi_m}})$ is given by I_N acting via the character χ_m and $B^0_{\chi_m}$ acting via the quotient map $\mathbb{C}[B^0_{\chi_m}]\to\mathcal{H}_{W^0_{\chi_m}}$. Thus we conclude

$$\mathfrak{F}(\mathcal{P}_{\chi_m})\cong\mathrm{IC}(\mathfrak{g}_1,\mathcal{L}_{\chi_m}),\tag{4.10}$$

where $\mathcal{L}_{\chi_m}$ is the K-equivariant local system on $\mathfrak{g}_1^{\mathrm{rs}}$ corresponding to the representations L_{χ_m} of $\pi_1^K(\mathfrak{g}_1^{\mathrm{rs}}) = \widetilde{B}_N$ defined in (2.8).

By Lemma 2.1 the IC sheaves $\mathrm{IC}(\mathfrak{g}_1, \mathcal{T}(\emptyset; \mu^1, \mu^2))$ and $\mathrm{IC}(\mathfrak{g}_1, \mathcal{T}(\emptyset; \mu, \mu)^\omega)$ are composition factors of the $\mathrm{IC}(\mathfrak{g}_1, \mathcal{L}_{\chi_m})$. Hence (4.8) follows from (4.10).

Now let $(\nu; \mu^1, \mu^2) \in \Sigma_N$ with $|\nu| = m > 0$. Let

$$\mathcal{K}(\rho_\nu \boxtimes V_{\mu^1,\mu^2})$$

denote the irreducible L_K-equivariant local system on $\mathfrak{l}_1^{\mathrm{rs}}$ associated to the irreducible representation of $\pi_1^{L_K}(\mathfrak{l}_1^{\mathrm{rs}})$ obtained as a pullback of $\rho_\nu \boxtimes V_{\mu^1,\mu^2}$ via the map $\pi_1^{L_K}(\mathfrak{l}_1^{\mathrm{rs}}) \cong B_m \times \widetilde{B}_{N-2m} \twoheadrightarrow S_m \times \widetilde{B}_{N-2m}$.

By Proposition 3.2, we have that

$$\mathrm{IC}(\mathfrak{g}_1^m, \mathcal{T}(\nu; \mu^1, \mu^2)) = \mathrm{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathrm{IC}(\mathfrak{l}_1, \mathcal{K}(\rho_\nu \boxtimes V_{\mu^1,\mu^2})). \tag{4.11}$$

Since Fourier transform commutes with induction (see (2.11)), it suffices to show that the Fourier transform of $\mathrm{IC}(\mathfrak{l}_1, \mathcal{K}(\rho_\nu \boxtimes V_{\mu^1,\mu^2}))$ is supported on an L_K-nilpotent orbit in $\mathfrak{l}_1$. This follows from the classical Springer correspondence for $\mathfrak{gl}(m)$ and (4.8) applied to the symmetric pair $(\mathrm{SL}(N-2m), \mathrm{SO}(N-2m))$ (see (3.8)).

The proof for $\mathrm{IC}(\mathfrak{g}_1^m, \mathcal{T}(\nu; \mu, \mu)^\omega)$, $\mathrm{IC}(\mathfrak{g}_1^{n,\omega}, \mathcal{T}(\nu; \emptyset, \emptyset)^\omega)$ proceeds in the same manner; in the latter case one uses the corresponding θ-stable Levi and parabolic subgroups. We omit the details. □

4.2 More on induction

Let $(\nu; \mu^1, \mu^2) \in \Sigma_N$. Assume that $|\nu| = m > 0$. Let $L^m \subset P^m$ be as in § 3. Recall that $L_K^m \cong \mathrm{GL}(m) \times \mathrm{SO}(N-2m)$ and $\mathfrak{l}_1^m \cong \mathfrak{gl}(m) \oplus \mathfrak{sl}(N-2m)_1$.

A nilpotent L_K^m-orbit in $\mathfrak{l}_1^m$ is given by a nilpotent orbit in $\mathfrak{gl}(m)$ and a nilpotent $\mathrm{SO}(N-2m)$-orbit in $\mathfrak{sl}(N-2m)_1$. Thus the nilpotent L_K^m-orbits in $\mathfrak{l}_1^m$ are parametrized by $\mathcal{P}(m) \times \mathcal{P}(N-2m)$, with extra labels I and II for partitions in $\mathcal{P}(N-2m)$ with all parts even. For $\alpha \in \mathcal{P}(m)$ and $\beta \in \mathcal{P}(N-2m)$, we denote by $\mathcal{O}_{\alpha,\beta}$ (or $\mathcal{O}^\omega_{\alpha,\beta}$) the nilpotent L_K^m-orbit in $\mathfrak{l}_1^m$ given by the nilpotent orbit $\mathcal{O}_\alpha$ in $\mathfrak{gl}(m)$ and the nilpotent $\mathrm{SO}(N-2m)$-orbit $\mathcal{O}_\beta$ (or $\mathcal{O}^\omega_\beta$) in $\mathfrak{sl}(N-2m)_1$.

In the following we will omit the labels I and II with the understanding that everything should have corresponding labels, for example, $\mathcal{O}^\omega_\lambda = \mathrm{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathcal{O}^\omega_{\alpha,\beta}$ etc.

PROPOSITION 4.4. *Let $\alpha \in \mathcal{P}(m)$ and $\beta \in \mathcal{P}(N-2m)$. Let $\mathcal{O}_\lambda = \mathrm{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathcal{O}_{\alpha,\beta}$, i.e., $\lambda_i = \beta_i + 2\alpha_i$. Assume that $u \in \mathcal{O}_{\alpha,\beta}$ and $v \in \mathcal{O}_\lambda \cap (u + (\mathfrak{n}_{P^m})_1)$. We have a natural surjective map*

$$\psi : A_K(v) \twoheadrightarrow A_{L_K^m}(u).$$

Moreover, let $\mathbb{C} \boxtimes \mathcal{E}$ be an L_K^m-equivariant irreducible local system on $\mathcal{O}_{\alpha,\beta}$ and let $\tilde{\mathcal{E}}$ be the K-equivariant local system on $\mathcal{O}_\lambda$ obtained from $\mathbb{C} \boxtimes \mathcal{E}$ via the map ψ above. Then $\mathrm{IC}(\mathcal{O}_\lambda, \tilde{\mathcal{E}})$ is a direct summand of $\mathrm{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathrm{IC}(\mathcal{O}_{\nu,\mu}, \mathbb{C} \boxtimes \mathcal{E})$.

COROLLARY 4.5. *If moreover $(\mathcal{O}_\mu, \mathcal{E}) \in \mathcal{A}_{N-2m}$ is a pair such that $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_\mu, \mathcal{E}))$ has full support, then we have*

$$\mathrm{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathrm{IC}(\mathcal{O}_{\nu,\mu}, \mathbb{C} \boxtimes \mathcal{E}) \cong \mathrm{IC}(\mathcal{O}_\lambda, \tilde{\mathcal{E}}).$$

As before let us now write $L = L^m$ and $P = P^m$ etc. We begin the proof of the above proposition with the following lemma.

Lemma 4.6. *The map*

$$\gamma : K \times^{P_K} (\bar{\mathcal{O}}_{\alpha,\beta} + (\mathfrak{n}_P)_1) \to \bar{\mathcal{O}}_\lambda$$

is generically one-to-one.

Proof. Let $x_0 \in \mathcal{O}_\lambda$. We can and will assume that $x_0 \in \mathcal{O}_{\alpha,\beta} + (\mathfrak{n}_P)_1$. We show that $\gamma^{-1}(x_0)$ is a point. Assume that $\gamma(k, x) = x_0$, i.e., $\operatorname{Ad} k(x) = x_0$. Then $x \in \mathcal{O}_{\alpha,\beta} + (\mathfrak{n}_P)_1$. Let $\widetilde{\mathcal{O}}_\lambda$ (respectively $\widetilde{\mathcal{O}}_{\alpha,\beta}$) be the (unique) G-orbit (respectively L-orbit) in $\mathfrak{g}$ (respectively $\mathfrak{l}$) that contains $\mathcal{O}_\lambda$ (respectively $\mathcal{O}_{\alpha,\beta}$). We have that

$$\widetilde{\mathcal{O}}_\lambda = \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \widetilde{\mathcal{O}}_{\alpha,\beta}$$

in the notation of Lusztig and Spaltenstein [LS79]. By [LS79, Theorem 1.3], we have $Z_G^0(x_0) \subset P$. In fact, we have that $Z_G(x_0) \subset P$. This can be seen by enlarging the group G to $\mathrm{GL}(N)$ and using the fact that $Z_{\mathrm{GL}(N)}(x_0)$ is connected. Thus $Z_K(x_0) \subset P_K$. Furthermore, $\widetilde{\mathcal{O}}_\lambda \cap (\widetilde{\mathcal{O}}_{\alpha,\beta} + \mathfrak{n}_P)$ is a single orbit under P. Thus there exists $p \in P$ such that $\operatorname{Ad} p(x) = x_0$. It follows that $k^{-1}p \in Z_G(x_0) \subset P$. Thus $k \in P \cap K = P_K$. Now we have that $(k, x) = (1, \operatorname{Ad} k(x)) = (1, x_0)$. □

Proof of Proposition 4.4. Note that the proof of the above lemma shows that $Z_G(v) = Z_P(v)$. We have $Z_P(v) \subset Z_L(u)U_P$. Thus $Z_K(v) = Z_{P_K}(v) \subset Z_{L_K}(u)(U_P \cap K)$. It follows that we have a natural projection map

$$Z_K(v)/Z_K^0(v) = Z_{P_K}(v)/Z_{P_K}^0(v) \to Z_{L_K}(u)/Z_{L_K}^0(u).$$

We show that this gives us the desired map ψ. Following [LS79], we have that $Z_{L_K}(u)(U_P \cap K)$ has a dense orbit, i.e., the orbit of v, in the irreducible variety $u + (\mathfrak{n}_P)_1$. Thus $Z_{P_K}(v) = Z_K(v)$ meets all the irreducible components of $Z_{L_K}(u)(U_P \cap K)$, which implies that ψ is surjective.

It is easy to see that

$$\operatorname{supp}(\operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathrm{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E})) = \bar{\mathcal{O}}_\lambda. \tag{4.12}$$

The proposition follows from the definition of parabolic induction functor and Lemma 4.6. □

Remark 4.7. The proof of Lemma 4.6 and the existence and surjectivity of the map ψ in Proposition 4.4 works for any θ-stable Levi contained in a θ-stable parabolic subgroup.

Proof of Corollary 4.5. Note that the assumption implies that $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E}))$ has full support, i.e., $\mathrm{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E}) = \mathrm{IC}(\mathfrak{l}_1, \mathcal{G})$ for some irreducible L_K-equivariant local system $\mathcal{G}$ on $\mathfrak{l}_1^{\mathrm{rs}}$. We have that

$$\mathfrak{F}(\operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathrm{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E})) = \operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathfrak{F}(\mathrm{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E})) = \operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathrm{IC}(\mathfrak{l}_1, \mathcal{G}).$$

It suffices to show that $\operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathrm{IC}(\mathfrak{l}_1, \mathcal{G})$ is irreducible. This follows from the definition of the induction functor and Proposition 3.1. □

Corollary 4.8. *The Fourier transform of a nilpotent orbital complex* $\mathrm{IC}(\mathcal{O}, \mathcal{E}) \in \mathcal{A}_N$ *has full support, i.e.,* $\operatorname{supp} \mathfrak{F}(\mathrm{IC}(\mathcal{O}, \mathcal{E})) = \mathfrak{g}_1$, *if and only if it is not of the form* $\operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathrm{IC}(\mathcal{O}', \mathcal{E}')$ *where* $\operatorname{supp} \mathfrak{F}(\mathrm{IC}(\mathcal{O}', \mathcal{E}')) = \mathfrak{l}_1$, *and* $L \subset P$ *is a pair chosen as in* § 3.

Proof. The only if part follows from the facts that Fourier transform commutes with parabolic induction and that $\operatorname{supp} \operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} A \subsetneq \mathfrak{g}_1$. The if part follows from (4.11), Corollary 4.5 and Theorem 4.1. □

COROLLARY 4.9. *Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots) \in \mathcal{P}(N)$.*

(1) *If $\lambda_i - \lambda_{i+1} \geqslant 3$ for some i, then $\operatorname{supp} \mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda, \mathcal{E})) \neq \mathfrak{g}_1$ for any K-equivariant local system $\mathcal{E}$ on $\mathcal{O}_\lambda$. The same holds for $\mathcal{O}_\lambda^\omega$ if λ has only even parts.*

(2) *Suppose that $\lambda_i - \lambda_{i+1} \leqslant 2$ for all i. Let f_λ be the number of different sizes of parts of λ, and g_λ the number of i such that $\lambda_i - \lambda_{i+1} = 2$.*

   (a) *If at least one part of λ is odd, then there are $2^{f_\lambda - 1 - g_\lambda}$ irreducible K-equivariant local systems $\mathcal{E}$ on $\mathcal{O}_\lambda$ such that $\operatorname{supp} \mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda, \mathcal{E})) = \mathfrak{g}_1$.*

   (b) *If all parts of λ are even, then there is exactly one irreducible K-equivariant local system $\mathcal{E}^\omega$ on each orbit $\mathcal{O}_\lambda^\omega$, $\omega = \mathrm{I}, \mathrm{II}$, such that $\operatorname{supp} \mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda^\omega, \mathcal{E}^\omega)) = \mathfrak{g}_1$.*

In particular, if $\lambda_i - \lambda_{i+1} \leqslant 1$ for all i, then $\operatorname{supp} \mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda, \mathcal{E})) = \mathfrak{g}_1$ for any K-equivariant local system $\mathcal{E}$ on $\mathcal{O}_\lambda$.

Proof. (1) Assume that $\lambda_{i_0} - \lambda_{i_0+1} \geqslant 3$. Let $m = i_0$, $\alpha = 1^{i_0}$, $\beta = (\lambda_1 - 2, \ldots, \lambda_{i_0} - 2, \lambda_{i_0+1}, \ldots)$. Then $\mathcal{O}_\lambda = \operatorname{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathcal{O}_{\alpha,\beta}$. Let $u \in \mathcal{O}_{\alpha,\beta}$ and $v \in \mathcal{O}_\lambda \cap (u + (\mathfrak{n}_{P^m})_1)$. Note that $A_K(v) \cong A_{L_K^m}(u)$. It then follows from Proposition 4.4 that for each irreducible K-equivariant local system $\mathcal{E}$ on $\mathcal{O}_\lambda$, $\mathrm{IC}(\mathcal{O}_\lambda, \mathcal{E})$ is a direct summand of $\operatorname{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathrm{IC}(\mathcal{O}_{\alpha,\beta}, \mathcal{E}_0)$ for some irreducible L_K-equivariant local system $\mathcal{E}_0$ on $\mathcal{O}_{\alpha,\beta}$. As before, this shows that $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda, \mathcal{E}))$ has smaller support.

In the case when λ has only even parts, we let $\mathcal{O}_\lambda^\omega = \operatorname{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathcal{O}_{\alpha,\beta}^\omega$, if $m < N/2$, and we let $\mathcal{O}_\lambda^\omega = \operatorname{Ind}_{\mathfrak{l}_1^{n,\omega} \subset \mathfrak{p}_1^{n,\omega}}^{\mathfrak{g}_1} \mathcal{O}_{\alpha,\beta}$, if $m = N/2 = n$, where $\omega = \mathrm{I}, \mathrm{II}$. The proof for $\mathcal{O}_\lambda^\omega$ then proceeds in the same way.

(2) We argue by induction on g_λ. If $g_\lambda = 0$, then (2) follows from (4.12) and Corollary 4.8. Assume by induction hypothesis that (2) holds for all μ with $g_\mu < g_\lambda$.

Assume first that λ has at least one odd part. Suppose that $i_1, \ldots, i_k$ are such that $\lambda_{i_j} - \lambda_{i_j+1} = 2$, where $k = g_\lambda$.

Let $a = (a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant 0)$ be a partition such that $a \neq \emptyset$, $a_k \leqslant 1$, and $a_l \leqslant a_{l+1} - 1$. Note that the number of such partitions is $2^k - 1$. Consider a partition $\mu(a)$ such that $\mu_l = \lambda_l - 2a_j$ for $l \in [i_{j-1} + 1, i_j]$. Then $\mu(a)$ satisfies that $\mu(a)_i - \mu(a)_{i+1} \leqslant 2$ and $g_{\mu(a)} < g_\lambda$. Moreover, μ has at least one odd part, and $f_\lambda - g_\lambda = f_{\mu(a)} - g_{\mu(a)}$. Let $m = \sum_{j=1}^k i_j$. We have that

$$\operatorname{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathcal{O}_{a,\mu(a)} = \mathcal{O}_\lambda.$$

By induction hypothesis, there are $2^{f_\lambda - g_\lambda - 1}$ irreducible K-equivariant local systems $\mathcal{E}$ on $\mathcal{O}_{a,\mu(a)}$ such that $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_{a,\mu(a)}, \mathcal{E})$ has full support. By Corollary 4.5, we have that

$$\operatorname{Ind}_{\mathfrak{l}_1^m \subset \mathfrak{p}_1^m}^{\mathfrak{g}_1} \mathrm{IC}(\mathcal{O}_{a,\mu(a)}, \mathcal{E}) = \mathrm{IC}(\mathcal{O}_\lambda, \tilde{\mathcal{E}}).$$

This gives rise to $(2^k - 1) \cdot 2^{f_\lambda - g_\lambda - 1} = 2^{f_\lambda - 1} - 2^{f_\lambda - g_\lambda - 1}$ irreducible K-equivariant local systems $\tilde{\mathcal{E}}$ on $\mathcal{O}_\lambda$ such that $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda, \tilde{\mathcal{E}})$ has smaller support (with a varying).

The case when all parts of λ even can be argued in the same way. Note that in this case $g_\lambda = f_\lambda$.

Let us write m_λ (respectively m_λ^ω, $\omega = \mathrm{I}, \mathrm{II}$) for the number of irreducible K-equivariant local systems $\tilde{\mathcal{E}}$ on $\mathcal{O}_\lambda$ (respectively $\mathcal{O}_\lambda^\omega$) such that $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda, \tilde{\mathcal{E}})$ (respectively $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda^\omega, \tilde{\mathcal{E}})$) has full support when at least one part of λ is odd (respectively when all parts of λ are even).

We conclude from the discussion above that

$$\begin{aligned} &m_\lambda \leqslant 2^{f_\lambda - g_\lambda - 1} \text{ if } \lambda \text{ has at least one odd part,} \\ &\text{respectively } m_\lambda^\omega \leqslant 1 \text{ if all parts of } \lambda \text{ are even.} \end{aligned} \tag{4.13}$$

Theorem 4.1 implies that the number of pairs $\mathrm{IC}(\mathcal{O},\mathcal{E}) \in \mathcal{A}_N$ such that $\operatorname{supp}\mathfrak{F}(\mathrm{IC}(\mathcal{O},\mathcal{E})) = \mathfrak{g}_1$ is $d(n)$ (see (4.1)), when $N = 2n+1$, and $e(n)$ (see (4.2)), when $N = 2n$. In view of (4.13) and claim (1) of the corollary, it suffices to show that

$$\sum_{\substack{\lambda\in\mathcal{P}(2n+1)\\ \lambda_i-\lambda_{i+1}\leqslant 2}} 2^{f_\lambda-g_\lambda-1} = d(n), \qquad \sum_{\substack{\lambda\in\mathcal{P}(2n),\lambda_i-\lambda_{i+1}\leqslant 2,\\ \text{not all parts of } \lambda \text{ even}}} 2^{f_\lambda-g_\lambda-1} + 2q(n) = e(n). \tag{4.14}$$

This can be seen as follows. Note that when N is even, the number of orbits of the form $\mathcal{O}^\omega_\lambda$, where all parts of λ are even and $\lambda_i - \lambda_{i+1} \leqslant 2$, is $2q(n)$. We know that

$$d(n) = \text{Coefficient of } x^{2n+1} \text{ in } \frac{1}{2}\prod_{s\geqslant 1}(1+x^s)^2,$$

$$e(n) = \frac{3}{2}q(n) + \text{Coefficient of } x^{2n} \text{ in } \frac{1}{2}\prod_{s\geqslant 1}(1+x^s)^2.$$

A partition λ satisfies that $\lambda_i - \lambda_{i+1} \leqslant 2$ if and only if each part of the transpose partition λ' has multiplicity at most 2. We have $f_\lambda = f_{\lambda'}$ and g_λ equals the number of parts in λ' with multiplicity 2. It is easy to see that each λ' whose parts have multiplicity at most 2 appears in $\prod_{s\geqslant 1}(1+x^s)^2$ exactly $2^{f_\lambda-g_\lambda}$ times. Hence (4.14) follows. □

Remark 4.10. In [CVX15a, Conjecture 1.2], we conjectured that one can obtain all nilpotent orbital complexes by induction from those of smaller groups whose Fourier transforms have full support. This conjecture follows from Corollary 4.8.

5. Cohomology of Hessenberg varieties

Hessenberg varieties, defined generally in [GKM06], arise naturally in our setting (for details, see [CVX15b]). In particular, they arise as fibers of maps π and $\check{\pi}$ in the following diagram

$$\begin{array}{ccccc} & & K/P_K\times\mathfrak{g}_1 & & \\ & \nearrow & & \nwarrow & \\ K\times^{P_K}E & & & & K\times^{P_K}E^\perp \\ \downarrow{\scriptstyle\pi} & & & & \downarrow{\scriptstyle\check{\pi}} \\ \mathcal{N}_1 & & & & \mathfrak{g}_1 \end{array}$$

where $P_K = P \cap K$ for a θ-stable parabolic subgroup P of G, E is a P_K-stable subspace of $\mathfrak{g}_1$ consisting of nilpotent elements, and $E^\perp$ is the orthogonal complement of E in $\mathfrak{g}_1$ via a K-invariant non-degenerate form on $\mathfrak{g}_1$. The generic fibers of maps $\check{\pi}$ are Hessenberg varieties.

In this section we discuss an application of our result to cohomology of Hessenberg varieties. Let us fix $s \in \mathfrak{g}_1^{\mathrm{rs}}$ and consider the corresponding Hessenberg variety

$$\mathrm{Hess} := \check{\pi}^{-1}(s) = \{gP_K \in K/P_K \mid g^{-1}sg \in E^\perp\}.$$

The centralizer $Z_K(s)$ acts naturally on Hess and it induces an action of the component group $\pi_0(Z_K(s)) \cong I_N$ on the cohomology groups $H^*(\mathrm{Hess},\mathbb{C})$. Let

$$\mathrm{H}^*(\mathrm{Hess},\mathbb{C}) = \bigoplus_{\chi\in I_N^\vee} \mathrm{H}^*(\mathrm{Hess},\mathbb{C})_\chi$$

be the eigenspace decomposition with respect to the action of I_N.

DEFINITION 5.1. The stable part $\mathrm{H}^*(\mathrm{Hess},\mathbb{C})_{\mathrm{st}}$ of $\mathrm{H}^*(\mathrm{Hess},\mathbb{C})$ is the direct summand $\mathrm{H}^*(\mathrm{Hess},\mathbb{C})_{\chi_{\mathrm{triv}}}$ where $\chi_{\mathrm{triv}} \in I_N^\vee$ is the trivial character.

For simplicity we now assume $\check{\pi}$ is onto. In this case $\check{\pi}$ is smooth over $\mathfrak{g}_1^{\mathrm{rs}}$ (e.g. see [CVX15b, Lemma 2.1]) and the equivariant fundamental group $\pi_1^K(\mathfrak{g}_1^{\mathrm{rs}}, s) \cong I_N \rtimes B_N$ acts on $H^*(\mathrm{Hess},\mathbb{C})$ by the monodromy action. Recall that for $\chi \in I_N^\vee$, B_χ stands for the stabilizer of χ in B_N. Clearly, each summand $H^*(\mathrm{Hess},\mathbb{C})_\chi$ is stable under the action of B_χ. Let $\chi_m \in I_N^\vee$, B_{χ_m}, and $B_{m,N-m}$ be as in § 2.3. Assume that χ is in the B_N-orbit of χ_m. Then for any $b \in B_N$ with $b.\chi = \chi_m$ we have an isomorphism $\iota_b : B_\chi \cong B_{\chi_m}, u \to bub^{-1}$. Note that $\chi_{\mathrm{triv}} = \chi_0$ and $B_{\chi_m} = B_{m,N-m}$ except when N is even and $m = N/2$. In that case, $B_{m,N-m}$ is an index-2 subgroup of B_{χ_m}.

Recall the algebra $\mathcal{H}_{\chi_m,-1} = \mathcal{H}_{m,-1} \otimes \mathcal{H}_{N-m,-1}$ and their representations $D_{\mu^1} \otimes D_{\mu^2}$ introduced in § 2.3. Each $\mathcal{H}_{\chi_m,-1}$ is a quotient of the group algebra $\mathbb{C}[B_{m,N-m}]$ and $\mathcal{H}_{\chi_0,-1} = \mathcal{H}_{\chi_{\mathrm{triv}},-1} = \mathcal{H}_{N,-1}$ is the Hecke algebra of S_N at $q = -1$.

THEOREM 5.2. (i) *Let $\chi_m \in I_N^\vee$ be the representatives of B_N-orbits in § 2.3. To every $\chi \in I_N^\vee$ in the orbit of χ_m and an element $b \in B_N$ satisfying $b(\chi) = \chi_m$, the monodromy action of b on* $\mathrm{H}^*(\mathrm{Hess},\mathbb{C})$ *induces an isomorphism* $\mathrm{H}^*(\mathrm{Hess},\mathbb{C})_\chi \cong \mathrm{H}^*(\mathrm{Hess},\mathbb{C})_{\chi_m}$ *compatible with the actions of $B_\chi \overset{\iota_b}{\cong} B_{\chi_m}$ on both sides.*

(ii) *The action of $\mathbb{C}[B_{m,N-m}]$ on* $\mathrm{H}^*(\mathrm{Hess},\mathbb{C})_{\chi_m}$ *factors through the algebra $\mathcal{H}_{\chi_m,-1}$ and the resulting representation is a direct sum of $D_{\mu^1} \otimes D_{\mu^2}$, $\mu^1 \in \mathcal{P}_2(m)$, $\mu^2 \in \mathcal{P}_2(N-m)$. In particular, the stable part* $\mathrm{H}^*(\mathrm{Hess},\mathbb{C})_{\mathrm{st}}$ *is generated by irreducible representations of the Hecke algebra of S_N at $q = -1$.*

Proof. Part (1) is clear. To prove part (2) we proceed as follows. By the decomposition theorem $\pi_*\mathbb{C}$ is a direct sum of shifts of nilpotent orbital complexes. Since $\mathfrak{F}(\pi_*\mathbb{C}) \cong \check{\pi}_*\mathbb{C}$ (up to shift), Theorem 4.1 implies that a generic stalk of $\check{\pi}_*\mathbb{C}$, which is isomorphic to $H^*(\mathrm{Hess},\mathbb{C})$, is a direct sum of the local systems $V_{\mu^1,\mu^2} = \mathrm{Ind}_{\mathbb{C}[B_{m,N-m}]}^{\mathbb{C}[B_N]} D_{\mu_1} \otimes D_{\mu_2}$ introduced in (2.8). Since I_N acts on V_{μ^1,μ^2} by the formula $a.(b \otimes v) = ((b.\chi_m)(a))(b \otimes v)$ for $a \in I_N$, $b \in B_N$ and $v \in D_{\mu_1} \otimes D_{\mu_2}$, we have $(V_{\mu^1,\mu^2})_\chi \cong D_{\mu_1} \otimes D_{\mu_2}$. The theorem follows. □

Example 5.3. Let C be the hyper-elliptic curve with affine equation $y^2 = \prod_{j=1}^N (x - a_j)$ (here $a_i \neq a_j$ for $i \neq j$). Assume $N = 2n+2$ is even. Then according to [CVX17, § 2.3] the Jacobian $\mathrm{Jac}(C)$ is an example of Hessenberg variety and the monodromy action of $\pi_1(\mathfrak{g}_1^{\mathrm{rs}}, s)$ factors through B_N, that is, $\mathrm{H}^*(\mathrm{Jac}(C),\mathbb{C}) = \mathrm{H}^*(\mathrm{Jac}(C),\mathbb{C})_{\mathrm{st}}$. Let $\mu_k = (N-k,k) \in \mathcal{P}_2(N)$ and D_{μ_k} be the corresponding representation of $\mathcal{H}_{N,-1}$. Using [A'Ca79], one can check that the induced action of the group algebra $\mathbb{C}[B_N]$ on $\mathrm{H}^i(\mathrm{Jac}(C),\mathbb{C})$ factors through $\mathcal{H}_{N,-1}$ and for $i \leqslant n$ the resulting representation of $\mathcal{H}_{N,-1}$ is isomorphic to

$$\mathrm{H}^i(\mathrm{Jac}(C),\mathbb{C}) \cong \bigoplus_{j=0}^{[i/2]} D_{\mu_{i-2j}}$$

with the primitive part $\mathrm{H}^i(\mathrm{Jac}(C),\mathbb{C})_{\mathrm{prim}} \cong D_{\mu_i}$.

Remark 5.4. It would be nice to have an explicit decomposition of $H^*(\mathrm{Hess},\mathbb{C})_{\chi_m}$ into irreducible representations of $\mathcal{H}_{\chi_m,-1}$. For this one needs finer information for the bijection in Theorem 4.1 (see § 7). In [CVX15a, CVX17], we establish an explicit bijection for certain nilpotent orbital complexes and we work out an explicit decomposition for the cohomology of the Hessenberg varieties that are isomorphic to Fano varieties of k-planes in smooth complete intersections of two quadrics in projective space.

6. Representations of $\mathcal{H}_{N,-1}$

In this section we show that *all* irreducible representations of the Hecke algebra $\mathcal{H}_{N,-1}$ come from geometry. Indeed they all appear in intersection cohomology of a Hessenberg variety with coefficient in a local system. In particular, this shows that all irreducible representations of $\mathcal{H}_{N,-1}$ carry a Hodge structure. In particular, the irreducible representations of $\mathcal{H}_{N,-1}$ can be viewed as variations of Hodge structure.

Let $\mathcal{O}$ be a nilpotent K-orbit on $\mathfrak{g}_1$ and $\mathcal{L}$ an irreducible K-equivariant local system on $\mathcal{O}$. We call $(\mathcal{O},\mathcal{L})$ a nilpotent pair. Following [LY17], we associate to each nilpotent pair $(\mathcal{O},\mathcal{L})$ two families of Hessenberg varieties $\mathrm{Hess}_{\mathcal{L},\pm1}\to\mathfrak{g}_1$ together with local systems $\hat{\mathcal{L}}_{\pm1}$ on open subsets $\overset{\circ}{\mathrm{Hess}}_{\mathcal{L},\pm1}\subset\mathrm{Hess}_{\mathcal{L},\pm1}$.

Let $x\in\mathfrak{g}_1$ be a nilpotent element in $\mathcal{O}$. Choose a normal sl_2-triple $\{x,h,y\}$ and let

$$\mathfrak{g}(i)=\{v\in\mathfrak{g}\mid[h,v]=iv\},\quad \mathfrak{g}_0(i)=\mathfrak{g}(i)\cap\mathfrak{g}_0,\quad\text{and}\quad \mathfrak{g}_1(i)=\mathfrak{g}(i)\cap\mathfrak{g}_1.$$

For any $N\in\mathbb{Z}$ we write $\underline{N}\in\{0,1\}$ for its image in $\mathbb{Z}/2\mathbb{Z}$. Define

$$\mathfrak{p}_N^x=\bigoplus_{k\geqslant 2N}\mathfrak{g}_{\underline{N}}(k),\quad \mathfrak{l}_N^x=\mathfrak{g}_{\underline{N}}(2N),\quad\text{and}\quad \mathfrak{l}^x=\bigoplus_{N\in\mathbb{Z}}\mathfrak{l}_N^x.$$

One can check that $\mathfrak{l}^x\subset\mathfrak{g}$ is a graded Lie subalgebra of $\mathfrak{g}$ and $x\in\mathfrak{l}_1^x=\mathfrak{g}_1(2)$. Let $L_0^x\subset K$ be the reductive subgroup with Lie algebra $\mathfrak{l}_0^x=\mathfrak{g}_0(0)$. By [LY17, 2.9(c)], the restriction

$$\mathcal{L}_1':=\mathcal{L}|_{\mathfrak{l}_1^x}$$

is an irreducible L_0^x-equivariant local system on the unique open L_0^x-orbit $\overset{\circ}{\mathfrak{l}}{}_1^x$ on $\mathfrak{l}_1^x$.

According to [Lus95], there exists a graded parabolic subalgebra $\mathfrak{q}=\bigoplus_{N\in\mathbb{Z}}\mathfrak{q}_N$ of $\mathfrak{l}^x$, a Levi subalgebra $\mathfrak{m}=\bigoplus_{N\in\mathbb{Z}}\mathfrak{m}_N$ of $\mathfrak{q}$, and a cuspidal local system $\mathcal{L}_1$ on the open M_0-orbit $\overset{\circ}{\mathfrak{m}}_1$ of $\mathfrak{m}_1$ (here M_0 is the reductive subgroup of L_0^x with Lie algebra $\mathfrak{m}_0$) such that

$$\text{some shift of the IC-complex } \mathrm{IC}(\mathfrak{l}_1^x,\mathcal{L}_1') \text{ is a direct summand of } \mathrm{Ind}_{\mathfrak{m}_1\subset\mathfrak{q}_1}^{\mathfrak{l}_1^x}\mathrm{IC}(\mathfrak{m}_1,\mathcal{L}_1).$$

In addition, we have

$$\mathfrak{F}(\mathrm{IC}(\mathfrak{m}_1,\mathcal{L}_1))\cong\mathrm{IC}(\mathfrak{m}_{-1},\mathcal{L}_{-1}),$$

where $\mathcal{L}_{-1}$ is a cuspidal local system on the unique open orbit $\overset{\circ}{\mathfrak{m}}_{-1}\subset\mathfrak{m}_{-1}$.

Define $\hat{\mathfrak{q}}_N$ to be the pre-image of $\mathfrak{q}_N$ under the projection map $\mathfrak{p}_N^x\to\mathfrak{l}_N^x$. Let $Q_K\subset K$ be the parabolic subgroup with Lie algebra $\hat{\mathfrak{q}}_0$. Denote by $\overset{\circ}{\hat{\mathfrak{q}}}_{\pm1}$ the preimage of $\overset{\circ}{\mathfrak{m}}_{\pm1}$ under the projection map $\hat{\mathfrak{q}}_{\pm1}\to\mathfrak{q}_{\pm1}\to\mathfrak{m}_{\pm1}$. The group Q_K acts naturally on $\hat{\mathfrak{q}}_{\pm1}$ and $\overset{\circ}{\hat{\mathfrak{q}}}_{\pm1}$ and we define

$$\mathrm{Hess}_{\mathcal{L},\pm1}:=K\times^{Q_K}\hat{\mathfrak{q}}_{\pm1},\quad \overset{\circ}{\mathrm{Hess}}_{\mathcal{L},\pm1}:=K\times^{Q_K}\overset{\circ}{\hat{\mathfrak{q}}}_{\pm1}.$$

Let

$$\pi_{\mathcal{L},\pm1}:\mathrm{Hess}_{\mathcal{L},\pm1}\to\mathfrak{g}_1,(x,v)\to xvx^{-1}$$

and let $\overset{\circ}{\pi}_{\mathcal{L},\pm1}$ be its restriction to $\overset{\circ}{\mathrm{Hess}}_{\mathcal{L},\pm1}$. For any $s\in\mathfrak{g}_1$, we denote by $\mathrm{Hess}_{\mathcal{L},\pm1,s}$ and $\overset{\circ}{\mathrm{Hess}}_{\mathcal{L},\pm1,s}$ the fiber of $\pi_{\mathcal{L},\pm1}$ and $\overset{\circ}{\pi}_{\mathcal{L},\pm1}$ over s, respectively.

There are natural maps

$$h_{\mathcal{L},\pm1} : \mathrm{Hess}_{\mathcal{L},\pm1} \to [\mathfrak{m}_{\pm1}/M_0], \quad \overset{\circ}{h}_{\mathcal{L},\pm1} : \overset{\circ}{\mathrm{Hess}}_{\mathcal{L},\pm1} \to [\overset{\circ}{\mathfrak{m}}_{\pm1}/M_0]$$

sending $(k, v) \in \mathrm{Hess}_{\mathcal{L},\pm1} = K \times^{Q_K} \hat{\mathfrak{q}}_{\pm1}$ to $\bar{v}$, the image of $v \in \hat{\mathfrak{q}}_{\pm1}$ under the map $\hat{\mathfrak{q}}_{\pm1} \to \mathfrak{m}_{\pm1} \to [\mathfrak{m}_{\pm1}/M_0]$. We define the following local system

$$\hat{\mathcal{L}}_{\pm1} := (\overset{\circ}{h}_{\mathcal{L},\pm1})^* \mathcal{L}_{\pm1}$$

on $\overset{\circ}{\mathrm{Hess}}_{\mathcal{L},\pm1}$. Here we view the M_0-local systems $\mathcal{L}_{\pm1}$ as sheaves on $[\overset{\circ}{\mathfrak{m}}_{\pm1}/M_0]$.

Example 6.1. Consider the nilpotent pair $(\mathcal{O}, \mathcal{L} = \mathcal{L}_{\mathrm{triv}})$ where $\mathcal{L}_{\mathrm{triv}}$ is the trivial local system on $\mathcal{O}$. Using [Lus95, Proposition 7.3] one can check that in this case $\mathfrak{q} = \bigoplus_{N\in\mathbb{Z}} \mathfrak{q}_N$ is a Borel subalgebra of $\mathfrak{l}^x$ and $\mathfrak{m} = \bigoplus_{N\in\mathbb{Z}} \mathfrak{m}_N$ is a Cartan subalgebra. Moreover the grading on $\mathfrak{m}$ is concentrated in degree zero, i.e., $\mathfrak{m} = \mathfrak{m}_0$, and the cuspidal local system $\mathcal{L}_{\pm1}$ is the skyscraper sheaf supported on $\mathfrak{m}_{\pm1} = \{0\}$. It follows that in this case $\mathrm{Hess}_{\mathcal{L}_{\mathrm{triv}},\pm1} = \overset{\circ}{\mathrm{Hess}}_{\mathcal{L}_{\mathrm{triv}},\pm1}$ and $\hat{\mathcal{L}}_{\pm1}$ is the constant local system.

In [LY17, § 7], the authors prove the following:

$$(\pi_{\mathcal{L},-1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1}) \text{is the Fourier transform of } (\pi_{\mathcal{L},1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},1}, \hat{\mathcal{L}}_1). \tag{6.1}$$

$$\text{Some shift of } \mathrm{IC}(\mathcal{O}, \mathcal{L}) \text{ (respectively the Fourier transform of } \mathrm{IC}(\mathcal{O}, \mathcal{L})) \text{ appears in } (\pi_{\mathcal{L},1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},1}, \hat{\mathcal{L}}_1) \text{ (respectively } (\pi_{\mathcal{L},-1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1})) \text{ as a direct summand.} \tag{6.2}$$

Assume from now on that $\pi_{\mathcal{L},-1} : \mathrm{Hess}_{\mathcal{L},-1} \to \mathfrak{g}_1$ is surjective. Then the sheaf $(\pi_{\mathcal{L},-1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1})$ is smooth over $\mathfrak{g}_1^{\mathrm{rs}}$. One sees this as follows. According to the first statement of (6.1) the characteristic variety of $(\pi_{\mathcal{L},-1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1})$ coincides with that of $(\pi_{\mathcal{L},1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},1}, \hat{\mathcal{L}}_1)$ as they are Fourier transforms of each other. But $(\pi_{\mathcal{L},1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},1}, \hat{\mathcal{L}}_1)$ is K-equivariant and supported on the nilpotent cone. A straightforward calculation then shows the smoothness of $(\pi_{\mathcal{L},-1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1})$ on $\mathfrak{g}_1^{\mathrm{rs}}$. Thus, by the decomposition theorem, we conclude that

$$(\pi_{\mathcal{L},-1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1})|_{\mathfrak{g}_1^{\mathrm{rs}}} \text{ is a direct sum of shifts of irreducible local systems.} \tag{6.3}$$

In addition, the $\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1})$ and hence $(\pi_{\mathcal{L},-1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1})$ has a canonical structure as a Hodge module and thus the direct summands are IC-extensions of irreducible variations of pure Hodge structure, see, [Sai88].

We fix a generic $s \in \mathfrak{g}_1^{\mathrm{rs}}$ and then

$$H^*((\pi_{\mathcal{L},-1})_* \,\mathrm{IC}(\mathrm{Hess}_{\mathcal{L},-1}, \hat{\mathcal{L}}_{-1}))_s = \mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s}, \hat{\mathcal{L}}_{-1}). \tag{6.4}$$

Thus we obtain an action of the fundamental group $\pi_1^K(\mathfrak{g}_1^{\mathrm{rs}}, s)$ on $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s}, \hat{\mathcal{L}}_{-1})$ and by the discussion above this action breaks into a direct sum of irreducible representations which are also variations of Hodge structure.

The component group $\pi_0(Z_K(s)) \cong I_N$ acts on $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s}, \hat{\mathcal{L}}_{-1})$ and we write

$$\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s}, \hat{\mathcal{L}}_{-1}) = \bigoplus_{\chi\in I_N^\vee} \mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s}, \hat{\mathcal{L}}_{-1})_\chi$$

for the corresponding eigenspace decomposition.

Definition 6.2. The stable part $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s},\hat{\mathcal{L}}_{-1})_{\mathrm{st}}$ of $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s},\hat{\mathcal{L}}_{-1})$ is the direct summand $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s},\hat{\mathcal{L}}_{-1})_{\chi_{\mathrm{triv}}}$ where $\chi_{\mathrm{triv}} \in I_N^\vee$ is the trivial character.

Observe that $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L},-1,s},\hat{\mathcal{L}}_{-1})_{\mathrm{st}}$ is stable under the monodromy action of $\pi_1^K(\mathfrak{g}_1^{\mathrm{rs}},s)$. Moreover, the action factors through the braid group B_N via the quotient map $\pi_1^K(\mathfrak{g}_1^{\mathrm{rs}},s) \to B_N$.

For every irreducible representation D_μ of $\mathcal{H}_{N,-1}$, let V_μ be the local system on $\mathfrak{g}_1^{\mathrm{rs}}$ associated to D_μ. By Theorem 4.1, there exists a unique nilpotent pair $(\mathcal{O}_\mu,\mathcal{L}_\mu)$ such that $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_\mu,\mathcal{L}_\mu)) \cong \mathrm{IC}(\mathfrak{g}_1,V_\mu)$.

Theorem 6.3. *Let D_μ be an irreducible representation of $\mathcal{H}_{N,-1}$ and let $(\mathcal{O}_\mu,\mathcal{L}_\mu)$ be the associated nilpotent pair as above. We have the following.*

(i) *The map $\pi_{\mathcal{L}_\mu,-1}$ is onto, the action of the braid group B_N on $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L}_\mu,-1,s},\hat{\mathcal{L}}_{\mu,-1})_{\mathrm{st}}$ factors through the Hecke algebra $\mathcal{H}_{N,-1}$ and $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L}_\mu,-1,s},\hat{\mathcal{L}}_{\mu,-1})_{\mathrm{st}}$ is a direct sum of irreducible representations of $\mathcal{H}_{N,-1}$.*

(ii) *D_μ appears in $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L}_\mu,-1,s},\hat{\mathcal{L}}_{\mu,-1})_{\mathrm{st}}$ with non-zero multiplicity.*

Proof. Since for every irreducible subrepresentation W of $\mathrm{IH}^*(\mathrm{Hess}_{\mathcal{L}_\mu,-1,s},\hat{\mathcal{L}}_{\mu,-1})_{\mathrm{st}}$ the corresponding Fourier transform $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1,\mathcal{W}))$ is supported on the nilpotent cone (here $\mathcal{W}$ is the local system on $\mathfrak{g}_1^{\mathrm{rs}}$ associated to W), the same argument as in the proof of Theorem 5.2 implies part (1). Part (2) follows from (6.1), (6.2), and (6.4). □

7. Conjecture on more precise matching

In Theorem 4.1 we show that the Fourier transform establishes a bijection between two sets of intersection cohomology sheaves. In this section we formulate a conjecture which refines the bijection in Theorem 4.1. We also relate the conjecture to our earlier conjectures in [CVX15b]. Our conjecture is not strong enough to produce an exact matching. The exact description of the bijection is crucial for applications, for example, computing cohomologies of Hessenberg varieties as explained in § 5.

We begin with associating to each nilpotent orbit $\mathcal{O}_\lambda$ (respectively $\mathcal{O}_\lambda^\omega$, $\omega = \mathrm{I},\mathrm{II}$) a subset $\Sigma_\lambda \subset \Sigma_N$ (respectively $\Sigma_\lambda^\omega \subset \Sigma_N$), if $\lambda \in \mathcal{P}(N)$ has at least one odd part (respectively has only even parts). Let λ be a partition of N and let λ' be the transpose partition of λ. Suppose that

$$\lambda' = (\lambda'_1)^{2m_1}\cdots(\lambda'_l)^{2m_l}(\lambda'_{l+1})^{2m_{l+1}-1}\cdots(\lambda'_k)^{2m_k-1}, \tag{7.1}$$

where $m_i \geqslant 1$, $i = 1,\dots,k$. Here and in what follows we write the parts in a partition in the order which is most convenient for us. In particular, in (7.1) we place the parts with even multiplicity before the parts with odd multiplicity.

Let $\delta_i \in \{0,1\}$ for $i \in [1,l]$ and let

$$\nu(\delta_1,\dots,\delta_l) = (\lambda'_1)^{m_1-\delta_1}\cdots(\lambda'_l)^{m_l-\delta_l}(\lambda'_{l+1})^{m_{l+1}-1}\cdots(\lambda'_k)^{m_k-1},$$

$$\mu(\delta_1,\dots,\delta_l) = (\lambda'_1)^{2\delta_1}\cdots(\lambda'_l)^{2\delta_l}(\lambda'_{l+1})\cdots(\lambda'_k).$$

Note that $2|\nu(\delta_1,\dots,\delta_l)| + |\mu(\delta_1,\dots,\delta_l)| = N$. Let

$$J \subset J_0 := \{l+1,\dots,k\} \text{ such that } \sum_{j\in J}\lambda'_j < \sum_{j\in J_0-J}\lambda'_j.$$

We define

$$\mu^1(\delta_1,\dots,\delta_l;J)=(\lambda'_1)^{\delta_1}\cdots(\lambda'_l)^{\delta_l}(\lambda'_{j_1})\cdots(\lambda'_{j_s}),\quad J=\{j_1,\dots,j_s\}.$$

$$\mu^2(\delta_1,\dots,\delta_l;J)=(\lambda'_1)^{\delta_1}\cdots(\lambda'_l)^{\delta_l}(\lambda'_{i_1})\cdots(\lambda'_{i_{k-l-s}}),\quad J_0-J=\{i_1,\dots,i_{k-l-s}\}.$$

Note that $\lambda'_{l+1}=0$ if and only if all parts of λ are even. In this case, $J_0=\emptyset=J$ and $\mu^1(\delta_1,\dots,\delta_l;J)=\mu^2(\delta_1,\dots,\delta_l;J)$ and we write $\mu(\delta_1,\dots,\delta_l)=\mu^i(\delta_1,\dots,\delta_l;J)$, $i=1,2$.

If λ has at least one odd part, then let

$$\Sigma_\lambda:=\Big\{(\nu(\delta_1,\dots,\delta_l);\mu^1(\delta_1,\dots,\delta_l;J),\mu^2(\delta_1,\dots,\delta_l;J))\mid \delta_i\in\{0,1\},i=1,\dots,l,\\ J\subset\{l+1,\dots,k\},\ \text{such that}\ \sum_{j\in J}\lambda'_j<\sum_{j\in J_0-J}\lambda'_j\Big\}.$$

If all parts of λ are even (in which case $\lambda'_{l+1}=0$), then let

$$\Sigma^\omega_\lambda=\{(\nu(\delta_1,\dots,\delta_l);\mu(\delta_1,\dots,\delta_l),\mu(\delta_1,\dots,\delta_l))^\omega\mid \delta_i\in\{0,1\},i=1,\dots,l\},\quad \omega=\mathrm{I},\mathrm{II}.$$

We have $|\Sigma_\lambda|=2^{k-1}$ (respectively $|\Sigma^\omega_\lambda|=2^l$), which equals the number of non-isomorphic irreducible K-equivariant local systems on $\mathcal{O}_\lambda$ (respectively $\mathcal{O}^\omega_\lambda$).

CONJECTURE 7.1. Let λ be a partition of N.

(1) If λ has at least one odd part, then the Fourier transform $\mathfrak{F}$ induces the following bijection:

$$\mathfrak{F}:\{\mathrm{IC}(\mathcal{O}_\lambda,\mathcal{E})\mid\mathcal{E}\text{ irreducible }K\text{-equivariant local system on }\mathcal{O}_\lambda(\text{up to isomorphism})\}\\ \xrightarrow{\sim}\{\mathrm{IC}(\mathfrak{g}_1^{|\nu|},\mathcal{T}(\nu;\mu^1,\mu^2))\mid(\nu;\mu^1,\mu^2)\in\Sigma_\lambda\}.$$

Moreover,

$$\mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda,\mathbb{C}))=\mathrm{IC}(\mathfrak{g}_1^{|\nu_0|},\mathcal{T}(\nu_0;\mu_0^1,\mu_0^2))$$

where $(\nu_0;\mu_0^1,\mu_0^2)\in\Sigma_\lambda$ is the unique triple such that $|\nu_0|=\max\{|\nu|,(\nu,\mu^1,\mu^2)\in\Sigma_\lambda\}$ and the parts of μ_0^1 and the parts of μ_0^2 have the opposite parity (in particular, all parts of μ_0^i have the same parity).

(2) If all parts of λ are even, then the Fourier transform induces the following bijection

$$\mathfrak{F}:\{\mathrm{IC}(\mathcal{O}^\omega_\lambda,\mathcal{E})\mid\omega=\mathrm{I},\mathrm{II},\ \mathcal{E}\text{ irreducible }K\text{-equivariant local system on }\mathcal{O}^\omega_\lambda(\text{up to isom})\}\\ \xrightarrow{\sim}\{\mathrm{IC}(\mathfrak{g}_1^{|\nu|},\mathcal{T}(\nu;\mu,\mu)^\omega)\mid\omega=\mathrm{I},\mathrm{II},\ (\nu;\mu,\mu)^\omega\in\Sigma^\omega_\lambda,\ \mu\neq\emptyset\}\\ \cup\{\mathrm{IC}(\mathfrak{g}_1^{n,\omega},\mathcal{T}(\nu;\emptyset,\emptyset))\mid\omega=\mathrm{I},\mathrm{II},\ (\nu;\emptyset,\emptyset)\in\Sigma^\omega_\lambda\}.$$

Moreover,

$$\mathfrak{F}(\mathrm{IC}(\mathcal{O}^\omega_\lambda,\mathbb{C}))=\mathrm{IC}(\mathfrak{g}_1^{n,\omega},\mathcal{T}(\nu_0;\emptyset,\emptyset)),$$

where $|\nu_0|=n$ and $(\nu_0;\emptyset,\emptyset)\in\Sigma_\lambda$.

Note that $\mathfrak{F}(\mathrm{IC}(\mathcal{O}_\lambda,\mathcal{E}))$ has full support if and only if $\nu(\delta_1,\dots,\delta_l)=\emptyset$. Thus we see that the conjecture is compatible with Corollary 4.9. We also remark that special cases of the conjecture are verified by [CVX15a, Theorems 4.1 and 4.3].

Let us relate the conjecture above to our previous conjectures in [CVX15b]. In [CVX15b] we constructed local systems $E^{2n+1}_{i,j}$ and $\widetilde{E}^{2n+1}_{i,j}$ on $\mathfrak{g}_1^{\mathrm{rs}}$. In terms of the parametrization introduced

in this paper, we have

$$E^{2n+1}_{i,j} = \mathcal{T}(\emptyset; (2i-j,\, j), (2n+1-2i))$$
$$\widetilde{E}^{2n+1}_{i,j} = \mathcal{T}(\emptyset; (2i-1-j,\, j), (2n+2-2i)).$$

Thus we see that Conjecture 7.1 applied to $E^{2n+1}_{i,j}$ agrees with [CVX15b, Conjectures 6.1 and 6.3]. Applied to $\widetilde{E}^{2n+1}_{i,j}$, Conjecture 7.1 implies that the supports of $\mathfrak{F}(\mathrm{IC}(\mathfrak{g}_1, \widetilde{E}^{2n+1}_{i,j}))$ are as follows:

$$\mathcal{O}_{3^j 2^{2i-2j-1} 1^{2n+3-4i+j}} \quad \text{if } 4i-j \leqslant 2n+3,$$
$$\mathcal{O}_{3^j 2^{2n+2-2i-j} 1^{4i-j-2n-3}} \quad \text{if } 2i+j \leqslant 2n+2 \text{ and } 4i-j \geqslant 2n+3,$$
$$\mathcal{O}_{3^{2n-2i+2} 2^{2i+j-2n-2} 1^{2i-2j-1}} \quad \text{if } 2i+j \geqslant 2n+2.$$

Note that the above orbits are all of even dimension and each of the even-dimensional orbits appears twice there.

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