ON THE CASSELMAN-JACQUET FUNCTOR

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Dedicated to J. Bernstein

Abstract. We study the Casselman-Jacquet functor J, viewed as a functor from the (derived) category of $(\mathfrak{g}, K)$-modules to the (derived) category of $(\mathfrak{g}, N^-)$-modules, N^- is the negative maximal unipotent. We give a functorial definition of J as a certain right adjoint functor, and identify it as a composition of two averaging functors $\mathrm{Av}_!^{N^-} \circ \mathrm{Av}_*^N$. We show that it is also isomorphic to the composition $\mathrm{Av}_*^{N^-} \circ \mathrm{Av}_!^N$. Our key tool is the *pseudo-identity* functor that acts on the (derived) category of (twisted) D-modules on an algebraic stack.

Contents

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Introduction

0.1. **The Casselman-Jacquet functor.**

0.1.1. Let G be a real reductive algebraic group, $\mathfrak{g}$ the complexification of Lie(G), K the complexification of a maximal compact subgroup in $G(\mathbb{R})$, and $\mathfrak{n}, \mathfrak{n}^- \subset \mathfrak{g}$ the complexifications of the Lie algebras of the unipotent radicals of opposite minimal parabolics in G. Let $(\mathfrak{g}, K)$-mod denote the corresponding category of $(\mathfrak{g}, K)$-modules. Recall that a Harish-Chandra module is a $(\mathfrak{g}, K)$-module that is of finite length (equivalently, finitely generated and acted on locally finitely by the center of $U(\mathfrak{g})$).

In his work on representations of real reductive groups, W.A. Casselman introduced a remarkable functor on the category of Harish-Chandra modules: it is defined by the formula

$$\mathcal{M} \mapsto \widehat{J}(\mathcal{M}) := \underset{\underset{k}{\longleftarrow}}{\lim} \mathcal{M}/\mathfrak{n}^k \cdot M, \tag{0.1}$$

where $\mathfrak{n}$ is the unipotent radical of a minimal parabolic.

A key property of the functor $\widehat{J}$ is that it is *exact* and *conservative*; this provided a new tool for the study of the category of Harish-Chandra modules, leading to an array of powerful results.

0.1.2. The functor (0.1) has an algebraic cousin, denoted J, and defined as follows.

Pick a cocharacter $\mathbb{G}_m \to G$ that is dominant and regular in the split Cartan, and let $J(\mathcal{M})$ be the subset of $\widehat{J}(\mathcal{M})$ on which $\mathbb{A}^1 = \text{Lie}(\mathbb{G}_m)$ acts locally finitely, i.e., the direct sum of generalized eigenspaces with respect to the generator $t \in \mathbb{A}^1$:

$$J(\mathcal{M}) \simeq \underset{\lambda}{\oplus} J(\mathcal{M})_\lambda := \underset{\lambda}{\oplus} \widehat{J}(\mathcal{M})_\lambda.$$

One shows that the entire $\widehat{J}(\mathcal{M})$ can be recovered as

$$\underset{\lambda}{\Pi} J(\mathcal{M})_\lambda,$$

so the information contained in J is more or less equivalent to that possessed by $\widehat{J}$. In particular, the functor J is also exact.

We will refer to J as the *Casselman-Jacquet* functor.

0.1.3. An important feature of the functor J, and one relevant to this paper, is that it can be extracted from $\widehat{J}$ using the Lie algebra $\mathfrak{n}^-$ (the unipotent radical of the opposite parabolic) rather than the split Cartan.

Namely, an elementary argument shows that $J(\mathcal{M})$ can be identified with the subset of vectors in $\widehat{J}(\mathcal{M})$ on which $\mathfrak{n}^-$ acts locally nilpotently.

Thus, we can think of J as a functor

$$(\mathfrak{g}, K)\text{-mod}_\chi \to (\mathfrak{g}, M_K \cdot N^-)\text{-mod}_\chi,$$

where our notations are as follows:

- $(\mathfrak{g}, K)$-mod denotes the abelian category of $(\mathfrak{g}, K)$-modules (K is the algebraic group corresponding to the maximal compact);
- $(\mathfrak{g}, M_K \cdot N^-)$-mod denotes the abelian category of $(\mathfrak{g}, M_K \cdot N^-)$-modules ($N^-$ and M_K are the algebraic groups corresponding to the (opposite) maximal unipotent and compact part of the Levi, respectively);
- The subscript χ indicates that we are considering categories of modules with a fixed central character χ.

0.1.4. *Our goals.* The primary goals of the present paper are as follows:

–Extend the definitions of the functors $\widehat{J}$ and J from the abelian categories to the corresponding derived categories $\mathfrak{g}\text{-mod}_\chi^K$, $\mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-}$, etc., (and in particular, explain their functorial meaning);

–Express the functor J as a double-averaging functor, and thus reprove the corresponding result from the paper [CY], where it was obtained by interpreting J via nearby cycles using [ENV];

–Record a conjecture that states that the functor J is (up to some twist) the *right* adjoint of the functor of *averaging* with respect to K:

$$\mathrm{Av}_*^{K/M_K} : \mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-} \to \mathfrak{g}\text{-mod}_\chi^K,$$

and explain that this is analogous to Bernstein's "2nd adjointness theorem" for $\mathfrak{p}$-adic groups.

0.1.5. In the course of realizing these goals we will encounter another operation of interest: Drinfeld's pseudo-identity functor on the category of $M_K \cdot N$-equivariant (twisted) D-modules on the flag variety X.

This functor will be used in the proofs of the main results, and as such, may seem to be not more than a trick. However, in the sequel to this paper it will be explained that this functor plays a conceptual role at the categorical level.

0.2. **Functorial interpretation of the Casselman-Jacquet functor.**

0.2.1. We first give a functorial interpretation of the (derived version of the) functor $\widehat{J}$.

Namely, in Sects. 2.2 and 2.3, we show that (the derived) version of this functor identifies with the composition

$$\mathfrak{g}\text{-mod}_\chi \xrightarrow{\mathrm{Av}_*^N} \mathfrak{g}\text{-mod}_\chi^N \xrightarrow{(\mathrm{Av}_*^N)^R} \mathfrak{g}\text{-mod}_\chi.$$

Here Av_*^N is the functor of *-averaging with respect to N, i.e., the right adjoint to the forgetful functor

$$\mathbf{oblv}_N : \mathfrak{g}\text{-mod}_\chi^N \to \mathfrak{g}\text{-mod}_\chi, \tag{0.2}$$

and $(\mathrm{Av}_*^N)^R$ is the (a priori, discontinuous) *right* adjoint of Av_*^N.

In fact, we show this in a rather general situation when instead of $\mathfrak{g}\text{-mod}_\chi$ we consider the category A-mod, where A is an associative algebra, equipped with a *Harish-Chandra structure* with respect to N (see Sect. 1.1.2 for what this means).

0.2.2. Next, we consider the functor J, in the general setting of a category $\mathcal{C}$ equipped with an action of G (for example, for $\mathcal{C} = A$-mod for an associative algebra A, equipped with a Harish-Chandra structure with respect to all of G), see Sect. 1.1.1 for what this means.

We define the (derived version of the) functor J as the composite

$$J := \mathrm{Av}_*^{N^-} \circ (\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N : \mathcal{C}^{M_K} \to \mathcal{C}^{M_K \cdot N^-}.$$

Assume that the following property is satisfied (which is the case for $\mathcal{C} = \mathfrak{g}\text{-mod}_\chi$ or $\mathcal{C}$ being the category $\text{D-mod}_\lambda(X)$ of twisted D-modules on the flag variety):

(*) *The "long intertwining functor"*

$$\Upsilon := \mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-} : \mathcal{C}^{M_K \cdot N^-} \to \mathcal{C}^{M_K \cdot N}, \tag{0.3}$$

given by forgetting N^--equivariance and then averaging with respect to N, is an equivalence.

In this case we show (see Proposition 2.1.4 and its variant in Sect. 4.1.3) that we have a canonical isomorphism of functors

$$J \simeq \mathrm{Av}_!^{N^-} \circ \mathrm{Av}_*^N. \tag{0.4}$$

In the above formula, $\mathrm{Av}_!^{N^-}$ is the !-averaging functor with respect to N^-, i.e., the *left* adjoint to (0.2) (with N replaced by N^-).

The isomorphism (0.4) had been initially obtained in [CY]; we will comment on that in Sect. 0.2.4 below.

0.2.3. Finally, in the particular case of $\mathcal{C} = \mathfrak{g}\text{-mod}_\chi$ we will show (see Theorem 2.4.2 and its variant in Sect. 4.1.3) that J is canonically isomorphic to its Verdier dual functor

$$J \simeq \mathrm{Av}_*^{N^-} \circ \mathrm{Av}_!^N, \tag{0.5}$$

when applied to objects in $\mathfrak{g}\text{-mod}_\chi^{M_K}$ whose cohomologies are finitely generated over $\mathfrak{n}$ (or are direct limits of such).

We will deduce (0.5) from a similar statement for $\mathcal{C} = \text{D-mod}_\lambda(X)$ (see Theorem 2.4.3 and its variant in Sect. 4.1.3), where the isomorphism in question holds for objects on $\text{D-mod}_\lambda(X)^{M_K}$ that are ULA with respect to the projection $X \to N\backslash X$ (see Sect. 3.2.1 for what this means).

The isomorphism (0.5) implies that the functor J is t-exact when applied to $\mathfrak{g}\text{-mod}_\chi^K$. In particular, it shows that the procedure in Sect. 0.1.3 *does not omit higher cohomologies* (in principle, the functor of taking $\mathfrak{n}^-$-locally nilpotent vectors should be derived).

0.2.4. The isomorphism (0.5), applied to objects from $\mathfrak{g}\text{-mod}_\chi^K$, had been obtained in [CY] as a combination of the following two results:

–One is the main theorem of [ENV] that shows that under the localization equivalence (for this one assumes that χ is regular), the functor J (defined as in Sect. 0.1.3) corresponds to a certain nearby cycles functor

$$\Psi : \text{D-mod}_\lambda(X)^K \to \text{D-mod}_\lambda(X)^{M_K \cdot N^-}.$$

This was done by explicitly analyzing the V-filtration on the corresponding D-module.

–The other is the key result from the paper [CY] itself, which establishes an isomorphism

$$\Psi \simeq \mathrm{Av}_*^{N^-} \circ \mathrm{Av}_!^N,$$

again on $\text{D-mod}_\lambda(X)^K$;

The isomorphism (0.4) was then deduced from (0.5) using the Verdier self-duality property of the nearby cycles functor Ψ, which implies that

$$\mathrm{Av}_*^{N^-} \circ \mathrm{Av}_!^N \simeq \Psi \simeq \mathrm{Av}_!^{N^-} \circ \mathrm{Av}_*^N.$$

So, the present paper gives another, in a sense more direct proof of (0.4) and (0.5), which does not appeal to the nearby cycles functor (however, we do *not* to imply that the latter is irrelevant: see Sect. 0.4.3).

0.3. **The pseudo-identity functor and the ULA condition.**

0.3.1. The pseudo-identity functor is a certain canonical endofunctor of the category of (twisted) D-modules on any algebraic stack, denoted

$$\text{Ps-Id}_{\mathcal{Y}} : \text{D-mod}_\lambda(\mathcal{Y}) \to \text{D-mod}_\lambda(\mathcal{Y}),$$

see Sect. 3.1. Its definition was suggested by V. Drinfeld and was recorded in [Ga1].

This functor is *uninteresting* (equals to the identity functor up to a shift) when $\mathcal{Y}$ is a smooth separated scheme, but has some very interesting properties on certain algebraic stacks that appear in geometric representation theory, see, e.g., [Ga3].

0.3.2. In this paper we apply this functor to the stack $\mathcal{Y}$ equal to $H\backslash X$, where X a proper scheme acted on by an algebraic group H (in our applications, X will be the flag variety of G and $H = M_K \cdot N$).

We prove the following result (see Theorem 3.2.6):

Theorem 0.3.3. *Let f denote the projection $X \to H\backslash X$. Then the functors*

$$f_![2\dim(H)] \text{ and } \text{Ps-Id}_{H\backslash X} \circ f_*[2\dim(X)]$$

are canonically isomorphic when evaluated on objects of $\text{D-mod}_\lambda(X)$ *that are ULA with respect to f.*

In other words, this theorem says that the functor $\text{Ps-Id}_{H\backslash X}$ intertwines the !- and *- direct images along f.

0.3.4. From Theorem 0.3.3 we deduce:

Corollary 0.3.5. *For X being the flag variety of G, the functor* $\text{Ps-Id}_{M_K\cdot N\backslash X}$ *induces a self-equivalence of* $\text{D-mod}_\lambda(M_K \cdot N\backslash X)$*; moreover, this self-equivalence is canonically isomorphic (up to a shift) to the composition*

$$\text{D-mod}_\lambda(M_K \cdot N\backslash X) \xrightarrow{\Upsilon^{-1}} \text{D-mod}_\lambda(M_K \cdot N^-\backslash X) \xrightarrow{(\Upsilon^-)^{-1}} \text{D-mod}_\lambda(M_K \cdot N\backslash X).$$

where Υ is the long intertwining functor of (0.3)*, and Υ^- is the analogous functor where the roles of N and N^- are swapped.*

0.3.6. The application of the functor $\text{Ps-Id}_{M_K\cdot N\backslash X}$ in this paper is the following one:

Combining Corollary 0.3.5 and Theorem 0.3.3 we obtain an isomorphism of functors

$$\mathrm{Av}_*^{N^-} \circ \mathrm{Av}_!^N \text{ and } \mathrm{Av}_!^{N^-} \circ \mathrm{Av}_*^N : \text{D-mod}_\lambda(M_K\backslash X) \rightrightarrows \text{D-mod}_\lambda(M_K \cdot N^-\backslash X), \tag{0.6}$$

on the subcategory objects of $\text{D-mod}_\lambda(M_K\backslash X)$ that are ULA with respect to the projection $M_K\backslash X \to M_K \cdot N\backslash X$ (or are direct limits of such).

0.3.7. Finally, let us comment on the relationship between the isomorphism of functors (0.6) and the isomorphism

$$\mathrm{Av}_*^{N^-} \circ \mathrm{Av}_!^N \text{ and } \mathrm{Av}_!^{N^-} \circ \mathrm{Av}_*^N : \mathfrak{g}\text{-mod}_\chi^{M_K} \rightrightarrows \mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-}, \tag{0.7}$$

on the subcategory consisting of modules which are finitely generated over $\mathfrak{n}$ (or are direct limits of such).

The point is that the isomorphisms (0.6) and (0.7) are logically equivalent, using the following observation (Proposition 2.4.5):

The functors

$$\mathrm{Loc} : \mathfrak{g}\text{-mod}_\chi \rightleftarrows \text{D-mod}_\lambda(X) : \Gamma(X, -)$$

map the corresponding subcategories to one-another.

0.4. **The "2nd adjointness" conjecture.**

0.4.1. Let us consider the "principal series" functor in the context of $(\mathfrak{g}, K)$-modules. We stipulate this to be the functor

$$\mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-} \xrightarrow{\mathbf{oblv}_{N^-}} \mathfrak{g}\text{-mod}_\chi^{M_K} \xrightarrow{\mathrm{Av}_*^{K/M_K}} \mathfrak{g}\text{-mod}_\chi^K.$$

Tautologically, this functor is the *right* adjoint of the functor

$$\mathfrak{g}\text{-mod}_\chi^K \xrightarrow{\mathbf{oblv}_*^{K/M_K}} \mathfrak{g}\text{-mod}_\chi^{M_K} \xrightarrow{\mathrm{Av}_!^{N^-}} \mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-}.$$

A priori, it is not clear clear that $\mathrm{Av}_*^{K/M_K} \circ \mathbf{oblv}_{N^-}$ itself should admit a *right* adjoint given by a nice formula. However, based on the analogy with Bernstein's 2nd adjointness theorem (see Sect. 4.4) we propose the following conjecture:

Conjecture 0.4.2. *The functor $J \circ \mathbf{oblv}_{K/M_K}$ (up to a shift) provides a right adjoint to the principal series functor $\mathrm{Av}_*^{K/M_K} \circ \mathbf{oblv}_{N^-}$.*

In the sequel to this paper further evidence towards the validity of Conjecture 0.4.2 will be provided, and the logical equivalence between Conjecture 0.4.2 and [Yo, Conjectures 9.1.4, 9.1.6] will be explained.

In addition to Conjecture 0.4.2, we make a similar conjecture when the category $\mathfrak{g}\text{-mod}_\chi$ is replaced by $\text{D-mod}_\lambda(X)$.

0.4.3. At the moment, it is not clear to the authors how to write down either the unit or the counit for the conjectural adjunction between $J \circ \mathbf{oblv}_{K/M_K}$ or $\mathrm{Av}_*^{K/M_K} \circ \mathbf{oblv}_{N^-}$, either in the context of $\mathfrak{g}\text{-mod}_\chi$ or in that of $\text{D-mod}_\lambda(X)$.

The following, however, seems very tempting:

In the paper [BK] it is explained that the in the context of $\mathfrak{p}$-adic groups, Bernstein's 2nd adjointness can be obtained by analyzing the *wonderful degeneration* of G, i.e., the geometry of the *wonderful compactification* near the stratum of the boundary corresponding to the given parabolic.

Now, as was mentioned above, one of the main results of [CY] says that the functor J for $\text{D-mod}_\lambda(X)$ is isomorphic to the nearby cycles functor along the same wonderful degeneration.

So it would be very nice to adapt the ideas of [BK] to prove Conjecture 0.4.2. However, so far, we do not know how to carry this out.

0.5. **Organization of the paper.**

0.5.1. The main body of the paper starts with Sect. 1 where we recall (but also reprove and supply proofs that we could not find in the literature) the following topics:

–The notion of action of an algebraic group on a (DG) category; the associated notions of equivariance and the *- and !-averaging functors;

–The Beilinson-Bernstein localization theory;

–Translation functors;

–The long intertwining functor between N- and N^--equivariant categories (either for $\mathfrak{g}$-modules, or D-modules on the flag variety).

The reader may consider skipping this section on the first pass, and return to it when necessary.

0.5.2. In Sect. 2 we initiate the study of the Casselman-Jacquet functor. However, in order to simplify the exposition, in this section instead of working with a real reductive group (or the corresponding symmetric pair), we work in a completely algebraic situation.

I.e., in this section we take N to be a maximal unipotent subgroup in a reductive group G, and consider the Casselman-Jacquet functor J as a functor

$$\mathfrak{g}\text{-mod}_\chi \to \mathfrak{g}\text{-mod}_\chi^N.$$

(Analogous results in the case of symmetric pairs require very minor modifications, which will be explained in Sect. 4.1.3).

–We define the functor J (in the context of a category $\mathcal{C}$ acted on by G) as the composition

$$\text{Av}_*^{N^-} \circ (\text{Av}_*^N)^R \circ \text{Av}_*^N : \mathcal{C} \to \mathcal{C}^{N^-}.$$

–We show that for $\mathcal{C} = A\text{-mod}$ (for an associative algebra A equipped with a Harish–Chandra structure with respect to G), the functor

$$(\text{Av}_*^N)^R \circ \text{Av}_*^N : A\text{-mod} \to A\text{-mod}$$

is given by $\mathfrak{n}$-adic completion.

–We show that if the functor

$$\Upsilon := \text{Av}_*^N \circ \mathbf{oblv}_{N^-} : \mathcal{C}^{N^-} \to \mathcal{C}^N$$

is an equivalence, then J identifies canonically with

$$\text{Av}_!^{N^-} \circ \text{Av}_*^N.$$

–We state that for $\mathcal{C} = \text{D-mod}_\lambda(X)$, the functor J is canonically isomorphic to $\text{Av}_*^{N^-} \circ \text{Av}_!^N$, when evaluated on objects that are ULA with respect to $X \to N\backslash X$.

–From here we deduce the corresponding isomorphism for $\mathfrak{g}\text{-mod}_\chi$ (on objects that are finitely generated with respect to $\mathfrak{n}$).

–Finally, we show the equivalence between the ULA and $\mathfrak{n}$-f.g. conditions under the localization functor

$$\text{Loc} : \mathfrak{g}\text{-mod}_\chi \to \text{D-mod}_\lambda(X).$$

0.5.3. In Sect. 3, our ostensible goal is to prove the isomorphism

$$\mathrm{Av}_!^{N^-} \circ \mathrm{Av}_*^N \simeq \mathrm{Av}_*^{N^-} \circ \mathrm{Av}_!^N \tag{0.8}$$

on objects of $\text{D-mod}_\lambda(X)$ that are ULA with respect to $X \to N\backslash X$.

In order to do this we introduce the pseudo-identity functor $\text{Ps-Id}_{\mathcal{Y}}$, which is an endofunctor on the category of twisted D-modules on an algebraic stack $\mathcal{Y}$.

We deduce (0.8) from a key geometric result, Theorem 3.2.6.

0.5.4. In Sect. 4 we adapt the results of the preceding sections to the context of a symmetric pair, and thereby deduce the results announced earlier in the introduction.

Finally, we state our version of the 2nd adjointness conjecture and explain the analogy with the corresponding assertion (which is a theorem of J. Bernstein) in the case of $\mathfrak{p}$-adic groups.

0.6. **Conventions and notation.**

0.6.1. Throughout the paper we will be working over an algebraically closed field k of characteristic 0, and we let G be a connected reductive group over k. Throughout the paper, X will denote the flag variety of G.

In Sects. 1-3 we let N be the unipotent radical of a Borel subgroup of G, and by N^- the unipotent radical of an opposite Borel.

In Sect. 4 we will change the context, and assume that G is endowed with involution θ; we let $K := G^\theta$. Let P be a minimal parabolic compatible with θ; in particular $P^- := \theta(P)$ is an opposite parabolic. For the duration of Sect. 4, we let N be the unipotent radical of P and N^- the unipotent radical of P^-.

0.6.2. This paper will make a (mild) use of higher algebra, in that we will be working with DG categories rather than with triangulated categories (the reluctant reader can avoid this, see Sect. 0.7). See [DrGa1, Sect. 0.6] for a concise summary of the theory of DG categories.

Unless specified otherwise, our DG categories will be assumed *cocomplete*, i.e., contain infinite direct sums. Similarly, unless specified otherwise, functors between DG categories will be assumed *continuous*, i.e., preserving infinite direct sums.

We denote by Vect the DG category of chain complexes of vector spaces.

For a DG category $\mathcal{C}$ and $\mathbf{c}_0, \mathbf{c}_1 \in \mathcal{C}$ we will denote by $\mathcal{H}om_{\mathcal{C}}(\mathbf{c}_0, \mathbf{c}_1) \in \text{Vect}$ the Hom complex between them.

For a DG category $\mathcal{C}$ we will denote by $\mathcal{C}^c$ the full (but not cocomplete) subcategory consisting of compact objects.

0.6.3. If $\mathcal{C}$ is endowed with a t-structure, we will denote by $\mathcal{C}^{\leq 0}$ (resp., $\mathcal{C}^{\geq 0}$) the subcategory of connective (resp., coconnective) objects, and by $\mathcal{C}^\heartsuit = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ its heart.

We will say that a functor between DG categories $\mathcal{C}_1$ and $\mathcal{C}_2$, each endowed with a t-structure, is left t-exact (resp., right t-exact, t-exact) if it sends $\mathcal{C}_1^{\geq 0}$ to $\mathcal{C}_2^{\geq 0}$ (resp., $\mathcal{C}_1^{\leq 0}$ to $\mathcal{C}_2^{\leq 0}$, both of the above).

0.6.4. For an associative algebra A we will denote by A-mod the corresponding DG category of A-modules (and *not* the abelian category). The same applies to $\mathfrak{g}$-mod for a Lie algebra $\mathfrak{g}$.

For a smooth scheme Y, equipped with a twisting λ, we let $\text{D-mod}_\lambda(Y)$ denote the DG category of twisted D-modules on Y.

For an algebraic group H, we denote by $\text{Rep}(H)$ the DG category of H-representations.

0.7. **How to get rid of DG categories?**

0.7.1. Unlike its sequel, in this paper we can make do by working with triangulated categories, rather than derived ones.

In general, the necessity to use DG categories arises for two reasons:

–We perform operations on DG categories (e.g., tensor two DG categories over a monoidal DG category acting on them).

–We take limits/colimits in a given DG category.

Both operations are actually present in this paper, but the general procedures can be replaced by *ad hoc* constructions.

0.7.2. We will be working with the notion of DG category acted on by an algebraic group H; if $\mathcal{C}$ is such a category, we will be considering the corresponding category $\mathcal{C}^H$ of H-equivariant objects, equipped with the forgetful functor $\mathbf{oblv}_H : \mathcal{C}^H \to \mathcal{C}$. The passage

$$\mathcal{C} \rightsquigarrow (\mathcal{C}^H, \mathbf{oblv}_H)$$

cannot be intrinsically defined within the world of triangulated categories, and that is why we need DG categories.

However, in our particular situation, H will be unipotent, and $\mathcal{C}^H$ can be defined as the *full subcategory* of $\mathcal{C}$, consisting of H-invariant objects. This does make sense at the triangulated level, where we regard $\mathcal{C}$ as a triangulated category equipped with the action of the monoidal triangulated category $\text{D-mod}(H)$.

0.7.3. In Sect. 4 the DG categories $\mathcal{C}$ that we consider will themselves arise in the form $\mathcal{C} = \mathcal{C}_0^H$ for a *non-unipotent* H. However, this will only occur in the following examples:

(a) $\mathcal{C}_0$ is the category of (twisted) D-modules on a scheme Y acted on by H;

(b) $\mathcal{C}_0$ is the category $\mathfrak{g}\text{-mod}_\chi$, where $\mathfrak{g}$ is a Lie algebra and χ is its central character, and $(\mathfrak{g}, H)$ is a Harish-Chandra pair.

In both these examples, there are several ways to define the corresponding category $\mathcal{C} = \mathcal{C}_0^H$ "by hand".

Note, however, that, typically, in neither of these cases will $\mathcal{C}$ be the derived category of the heart of its natural t-structure.

0.7.4. Finally, the only limits and colimits procedures that we consider will be indexed by filtered sets (in fact, by $\mathbb{N}$), and they will consist of objects inside the heart of a t-structure. So the limit/colimit objects will stay in the heart.

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1. Recollections

In this section we recall some facts and constructions pertaining to the notion of action of a group on a DG category, to the Beilinson-Bernstein localization theory, translations functors, and the long intertwining functor.

1.1. Groups acting on categories: a reminder.

1.1.1. In this paper we will extensively use the notion of (strong) action of an algebraic group H on a DG category $\mathcal{C}$; for the definition see [Ga2, Sect. 10.2] (in the terminology of *loc.cit.*, these are categories acted on by H_{dR}).

One of the possible definitions is that such a data is equivalent to that of co-action of the co-monoidal DG category D-mod(H) on $\mathcal{C}$, where the co-monoidal structure on D-mod(H) is given by !-pullback with respect to the product operation on H:

$$\mathcal{C} \overset{\text{co-act}}{\longrightarrow} \text{D-mod}(H) \otimes \mathcal{C}. \tag{1.1}$$

We can regard D-mod(H) also as a *monoidal* category, with respect to the operation of *convolution*, i.e., *-pushforward with respect to the product operation on H. If H acts on $\mathcal{C}$, we obtain also a monoidal action of D-mod(H) on $\mathcal{C}$ by the formula

$$\text{D-mod}(H) \otimes \mathcal{C} \overset{\text{Id} \boxtimes \text{co-act}}{\longrightarrow} \text{D-mod}(H) \otimes \text{D-mod}(H) \otimes \mathcal{C} \overset{\overset{!}{\otimes} \boxtimes \text{Id}}{\longrightarrow} \text{D-mod}(H) \otimes \mathcal{C} \overset{p_* \otimes \text{Id}}{\longrightarrow} \text{Vect} \otimes \mathcal{C} \simeq \mathcal{C},$$

where p_* denotes the pushforward functor D-mod$(H) \to$ D-mod(pt) = Vect.

We denote the corresponding monoidal operation by

$$\mathcal{F} \in \text{D-mod}(H),\, \mathbf{c} \in \mathcal{C} \mapsto \mathcal{F} \star \mathbf{c}.$$

1.1.2. Here are some examples of groups acting on categories that we will use:

(i) Let H act on a scheme/algebraic stack Y. Then H acts on D-mod(Y).

(i') Suppose that Y is equipped with a twisting λ (see [GR, Sect. 6] for what this means) that is H-equivariant (the latter means that the twisting descends to one on the quotient stack $H\backslash Y$). Then H acts on the category D-mod$_\lambda(Y)$.

(ii) H acts on the category $\mathfrak{h}$-mod of modules over its own Lie algebra.

(ii') Let χ be the character of the center $Z(\mathfrak{h}) = U(\mathfrak{h})^{\mathrm{Ad}_H} \subset Z(U(\mathfrak{h}))$. Then H acts on the category $\mathfrak{h}$-mod$_\chi$, the latter being the category of $\mathfrak{h}$-modules on which $Z(\mathfrak{h})$ acts via χ.

(iii) Let A be a (classical) associative algebra, equipped with a Harish-Chandra structure with respect to H. I.e., we are given an action of H on A by automorphisms, and a map of Lie algebras $\phi : \mathfrak{h} \to A$ such that

- ϕ is H-equivariant;

- The adjoint action of $\mathfrak{h}$ on A (coming from ϕ) equals the derivative of the given H-action on A.

Then A-mod is acted on by H. This example contains examples (ii) and (ii') (and also (i) and (i') for Y affine) as particular cases.

An example of this situation is when $A = U(\mathfrak{g})$, where $(\mathfrak{g}, H)$ is a Harish-Chandra pair, or A is the quotient of $U(\mathfrak{g})$ by a central character.

1.1.3. If $\mathcal{C}$ is acted on by H, there is a well-defined category $\mathcal{C}^H$ of H-equivariant objects in $\mathcal{C}$, equipped with a pair of adjoint functors

$$\mathbf{oblv}_H : \mathcal{C}^H \rightleftarrows \mathcal{C} : \mathrm{Av}_*^H.$$

One way to define $\mathcal{C}^H$ is as the totalization of the co-Bar co-simplicial category

$$\mathcal{C} \rightrightarrows \text{D-mod}(H) \otimes \mathcal{C}...$$

associated with the co-action of D-mod(H) on $\mathcal{C}$. Under this identification, $\mathbf{oblv}_H$ is given by evaluation on 0-simplices.

Equivalently, we can define $\mathcal{C}^H$ as the category of co-modules over the co-monad

$$\mathbf{oblv}_H \circ \mathrm{Av}_*^H := k_H \star -$$

acting on $\mathcal{C}$, where $k_H \in \text{D-mod}(H)$ is the *constant sheaf* D-module.

1.1.4. Note that the functor $\mathbf{oblv}_H$ is not necessarily fully faithful. In fact, the composition

$$\mathrm{Av}_*^H \circ \mathbf{oblv}_H : \mathcal{C}^H \to \mathcal{C}^H$$

is given by tensor product with $\mathrm{C}^*_{\mathrm{dR}}(H)$ (de Rham cochains on H), where the unit of the adjunction $\mathrm{Id} \to \mathrm{Av}_*^H \circ \mathbf{oblv}_H$ corresponds to the canonical map $k \to \mathrm{C}^*_{\mathrm{dR}}(H)$.

The above implies, among the rest, that $\mathbf{oblv}_H$ *is* fully faithful if H is unipotent.

1.1.5. The functor $\mathbf{oblv}_H$ does not necessarily admit a left adjoint. Its partially defined left adjoint[1] will be denoted by $\mathrm{Av}_!^H$. Concretely, the means that $\mathrm{Av}_!^H$ is defined on the full subcategory of $\mathcal{C}$ consisting of objects c for which the functor

$$\mathcal{C}^H \to \mathrm{Vect}, \quad c' \mapsto \mathcal{H}om_{\mathcal{C}}(c, \mathbf{oblv}_H(c'))$$

is co-representable.

1.1.6. Given two subgroups $H_1 \subset H_2$ we will denote by

$$\mathbf{oblv}_{H_2/H_1} : \mathcal{C}^{H_2} \rightleftarrows \mathcal{C}^{H_1} : \mathrm{Av}_*^{H_2/H_1}$$

the corresponding adjoint pair of functors.

Similarly, will denote by $\mathrm{Av}_!^{H_2/H_1}$ the partially defined left adjoint to $\mathbf{oblv}_{H_2/H_1}$.

[1]For the terminology of partially defined left adjoints etc. see, for example, appendix A of [DrGa2].

1.1.7. When $\mathcal{C} = \text{D-mod}_\lambda(Y)$ we have a canonical identification

$$\mathcal{C}^H = \text{D-mod}_\lambda(H\backslash Y),$$

where Av_*^H is given by the *-pushforward functor

$$f_* : \text{D-mod}_\lambda(Y) \to \text{D-mod}_\lambda(H\backslash Y),$$

and hence $\mathbf{oblv}_H$ is given by the *-pullback functor f^*. Note that the functor f^* is well-defined on all D-modules (and not just holonomic ones) because the morphism f is smooth.

Let us be in Example (iii) above with $A = U(\mathfrak{g})$, where $(\mathfrak{g}, H)$ is a Harish-Chandra pair. Then the corresponding category $\mathfrak{g}\text{-mod}^H$ is by definition the (derived) category of $(\mathfrak{g}, H)$-modules. For $\mathfrak{g} = \mathfrak{h}$ we have

$$\mathfrak{g}\text{-mod}^H = \text{Rep}(H),$$

the category of H-representations.

1.1.8. Suppose that $\mathcal{C}$ is equipped with a t-structure, so that the co-action functor (1.1) is t-exact, where $\text{D-mod}(H)$ is taken with respect to the *left* D-module t-structure.

Then the co-monad

$$\mathbf{c} \mapsto k_H \star \mathbf{c}$$

is left t-exact. This implies that the category $\mathcal{C}^H$ carries a t-structure, uniquely characterized by the property that the forgetful functor $\mathbf{oblv}_H$ is t-exact.

In this case, the functor Av_*^H, being the right adjoint of a t-exact functor, is left t-exact.

We will need the following technical assertion:

Lemma 1.1.9. *Assume that the t-structure on $\mathcal{C}$ is left-complete, i.e., for an object $\mathbf{c} \in \mathcal{C}$, the map*

$$\mathbf{c} \to \lim_n \tau^{\geq -n}(\mathbf{c})$$

is an isomorphism. Then:

(a) *The t-structure on $\mathcal{C}^H$ is also left-complete and for $\mathbf{c} \in \mathbf{C}$, the natural map*

$$\text{Av}_*^H(\mathbf{c}) \to \lim_n \text{Av}_*^H(\tau^{\geq -n}(\mathbf{c}))$$

is an isomorphism.

(b) *If for an object $\mathbf{c} \in \mathbf{C}$, the partially defined functor $\text{Av}_!^H$ is defined on every $\tau^{\geq -n}(\mathbf{c})$, then it is defined on $\mathbf{c}$ itself, and the natural map*

$$\text{Av}_!^H(\mathbf{c}) \to \lim_n \text{Av}_!^H(\tau^{\geq -n}(\mathbf{c}))$$

is an isomorphism.

(b') *If the t-structure on $\mathcal{C}$ is compatible with filtered colimits, and for an object $\mathbf{c} \in \mathbf{C}$, the partially defined functor $\text{Av}_!^H$ is defined on every $H^n(\mathbf{c})$, then it is defined on $\mathbf{c}$ itself.*

1.2. **Localization theory.**

1.2.1. Let G be an algebraic group acting on a smooth variety X, equipped with a G-equivariant twisting λ. Let A be an associative algebra equipped with a Harish-Chandra structure with respect to G, and let us be given a map

$$A \to \Gamma(X, \mathrm{D}_\lambda), \tag{1.2}$$

as associative algebras equipped with Harish-Chandra structures with respect to G.

Then the functor

$$\Gamma(X, -) : \text{D-mod}_\lambda(X) \to \mathrm{Vect}$$

naturally factors as

$$\text{D-mod}_\lambda(X) \to \Gamma(X, \mathrm{D}_\lambda)\text{-mod} \to A\text{-mod} \to \mathrm{Vect}\,.$$

By a slight abuse of notation, we will denote the resulting functor

$$\text{D-mod}_\lambda(X) \to A\text{-mod}$$

by the same symbol Γ. It admits a left adjoint, denoted Loc. Both these functors are compatible with the G-actions.

Note that the functor Loc is fully faithful if and only if the map (1.2) is an isomorphism.

1.2.2. The example of this situation of interest for us is, of course, when G is reductive and X is the flag variety of G. In this case G-equivariant twistings on X are in bijection with elements of $\mathfrak{t}^*$ (the dual vector space of the abstract Cartan $\mathfrak{t}$), where we take the ρ-shift into account.

Given $\lambda \in \mathfrak{t}^*$, we take

$$A = U(\mathfrak{g})_\chi := U(\mathfrak{g}) \underset{Z(\mathfrak{g})}{\otimes} k,$$

where $Z(\mathfrak{g}) \to k$ is the homomorphism χ corresponding to λ via the Harish-Chandra map $Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{t})$.

It is a theorem of Kostant that in this case the corresponding map

$$U(\mathfrak{g})_\chi \to \Gamma(X, \mathrm{D}_\lambda)$$

is an isomorphism, i.e., the functor Loc is fully faithful.

The following is the Localization Theorem of [BB1] (amplified by [BB2]):

Theorem 1.2.3. *Consider* D-mod$_\lambda(X)$ *as equipped with the* left *D-module t-structure.*

(a) *Let λ be such that $\check{\alpha}(\lambda) \neq 0$ for any coroot $\check{\alpha}$ of G (we call such λ "regular"). Then the functors Γ and* Loc *are mutually inverse equivalences.*

(b) *Let λ be such that $\check{\alpha}(\lambda) \notin \mathbb{Z}^{<0}$ for any coroot $\check{\alpha}$ of G (we call such λ "dominant"). Then The functor Γ is t-exact.*

(c) *Let λ be regular and dominant. Then the functors Γ and* Loc *are mutually inverse equivalences, compatible with the t-structures.*

1.2.4. *A warning.* When χ is *irregular*, the algebra $U(\mathfrak{g})_\chi$ has an infinite cohomological dimension. This implies, in particular, that the natural t-structure on $\mathfrak{g}\text{-mod}_\chi$ does *not* descend to one on $\mathfrak{g}\text{-mod}_\chi^c$.

Let $\mathfrak{g}\text{-mod}_\chi^{\text{f.g.}} \subset \mathfrak{g}\text{-mod}_\chi$ be the full subcategory consisting of objects with finitely many non-vanishing cohomology groups, and with each cohomology finitely generated as a $U(\mathfrak{g})$-module. We have

$$\mathfrak{g}\text{-mod}_\chi^c \subset \mathfrak{g}\text{-mod}_\chi^{\text{f.g.}},$$

but this inclusion is *not* an equality (the latter would be equivalent to $U(\mathfrak{g})_\chi$ having a finite cohomological dimension).

The functor $\Gamma(X,-)$ sends compact objects in $\text{D-mod}_\lambda(X)$ to $\mathfrak{g}\text{-mod}_\chi^{\text{f.g.}}$, but not necessarily to $\mathfrak{g}\text{-mod}_\chi^c$.

A related fact is that in this case, the functor Loc has an unbounded cohomological amplitude on the left (it is right t-exact, being the left adjoint of a left t-exact functor $\Gamma(X,-)$).

1.2.5. In what follows we will need the following observation:

Proposition 1.2.6. *Let $H_1 \subset H_2$ be a pair of subgroups of G, and let $\mathcal{M}$ be an object of $\mathfrak{g}\text{-mod}_\chi^{H_1}$. Assume that the functor $\text{Av}_!^{H_2/H_1}$ is defined on $\text{Loc}(\mathcal{M})$. Then $\text{Av}_!^{H_2/H_1}$ is defined on $\mathcal{M}$ and we have*

$$\text{Loc}(\text{Av}_!^{H_2/H_1}(\mathcal{M})) \simeq \text{Av}_!^{H_2/H_1}(\text{Loc}(\mathcal{M})) \text{ and } \text{Av}_!^{H_2/H_1}(\mathcal{M}) \simeq \Gamma \circ \text{Av}_!^{H_2/H_1} \circ \text{Loc}(\mathcal{M}).$$

We will prove the following abstract version of Proposition 1.2.6. Let

$$\begin{array}{ccc} \mathcal{C}_2' & \xrightarrow{i_2} & \mathcal{C}_2 \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{C}_1' & \xrightarrow{i_1} & \mathcal{C}_1 \end{array}$$

be a commutative diagram of categories, and assume that the diagram

$$\begin{array}{ccc} \mathcal{C}_2' & \xrightarrow{i_2} & \mathcal{C}_2 \\ {\scriptstyle F'}\uparrow & & \uparrow{\scriptstyle F} \\ \mathcal{C}_1' & \xrightarrow{i_1} & \mathcal{C}_1, \end{array}$$

where F (resp., F') is the left adjoint of G (resp., G'), also commutes.

Let $\mathbf{c}_1 \in \mathcal{C}_1$ be an object, and assume that the partially defined left adjoint $(i_2)^L$ is defined on $F(\mathbf{c}_1)$.

Proposition 1.2.7. *Assume that the functors F and F' are fully faithful. Then the partially defined left adjoint $(i_1)^L$ is defined on $\mathbf{c}_1$ and $F'((i_1)^L(\mathbf{c}_1)) \simeq (i_2)^L(F(\mathbf{c}_1))$.*

Proof. Denote $\mathbf{c}_2 := F(\mathbf{c}_1)$ and $\mathbf{c}_2' := (i_2)^L(\mathbf{c}_2)$. We claim that the map

$$F' \circ G'(\mathbf{c}_2') \to \mathbf{c}_2' \tag{1.3}$$

is an isomorphism. (If this is this case, then it is easy to see that the object $\mathbf{c}_1' := G(\mathbf{c}_2')$ is the value of $(i_1)^L$ on $\mathbf{c}_1$).

Let $\widetilde{\mathbf{c}}_2'$ denote the cone of (1.3). Then $G'(\widetilde{\mathbf{c}}_2') = 0$. We claim that this implies that $\widetilde{\mathbf{c}}_2' = 0$, which is equivalent to the map $\mathbf{c}_2' \to \widetilde{\mathbf{c}}_2'$ being zero.

Indeed, for *any* $\widetilde{\mathbf{c}}'_2 \in \mathcal{C}'_2$, we have

$$\mathcal{H}om_{\mathcal{C}'_2}(\mathbf{c}'_2, \widetilde{\mathbf{c}}'_2) \simeq \mathcal{H}om_{\mathcal{C}_2}(\mathbf{c}_2, i_2(\widetilde{\mathbf{c}}'_2)) \simeq \mathcal{H}om_{\mathcal{C}_1}(\mathbf{c}_1, G \circ i_2(\widetilde{\mathbf{c}}'_2)) \simeq \mathcal{H}om_{\mathcal{C}_1}(\mathbf{c}_1, i_1 \circ G'(\widetilde{\mathbf{c}}'_2)).$$

Hence, if $G'(\widetilde{\mathbf{c}}'_2) = 0$, then $\mathcal{H}om_{\mathcal{C}'_2}(\mathbf{c}'_2, \widetilde{\mathbf{c}}'_2) = 0$, as desired.

□

1.3. **Translation functors.** In this subsection we take G to be a reductive group and X its flag variety.

1.3.1. Let λ be a dominant weight and let μ be a dominant integral weight. Set $\lambda' = \lambda + \mu$, and let χ and χ' be the corresponding characters of $Z(\mathfrak{g})$.

Let V^μ be the irreducible G-module with highest weight λ.

1.3.2. Let us view $U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*$ as a $(U(\mathfrak{g}), U(\mathfrak{g})_{\chi'})$-bimodule, where $U(\mathfrak{g})_{\chi'}$ acts on the $U(\mathfrak{g})_{\chi'}$-factor in $U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*$ by right multiplication and $U(\mathfrak{g})$ acts diagonally, with the action on the $U(\mathfrak{g})_{\chi'}$-factor being that by left multiplication.

It is easy to show that when viewed as a $U(\mathfrak{g})$-module, $U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*$ splits as a direct sum

$$(U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_\chi \oplus (U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_{\neq\chi},$$

where the set-theoretic support of $(U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_\chi$ over $\mathrm{Spec}(Z(\mathfrak{g}))$ is $\{\chi\}$ and the set-theoretic support of $(U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_{\neq\chi}$ over $\mathrm{Spec}(Z(\mathfrak{g}))$ is finite and disjoint from χ.

This decomposition automatically respects the right $U(\mathfrak{g})_{\chi'}$-action.

The key observation is that the assumptions on λ and λ' imply that $(U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_\chi$ is *scheme-theoretically* supported at $\chi \in \mathrm{Spec}(Z(\mathfrak{g}))$, i.e., $(U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_\chi$ is well-defined as an object of $(\mathfrak{g}\text{-mod}_\chi)^\heartsuit$.

1.3.3. We define the functor

$$T_{\chi' \to \chi} : \mathfrak{g}\text{-mod}_{\chi'} \to \mathfrak{g}\text{-mod}_\chi$$

to be given by

$$\mathcal{M} \mapsto (U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_\chi \underset{U(\mathfrak{g})_{\chi'}}{\otimes} \mathcal{M}.$$

This functor is t-exact, since $(U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_\chi$ is projective as a right $U(\mathfrak{g})_{\chi'}$-module (being a direct summand of such). At the level of abelian categories it is described as follows: for $\mathcal{M} \in \mathfrak{g}\text{-mod}_{\chi'}$, the tensor product $\mathcal{M} \otimes (V^\mu)^*$ splits as a direct sum

$$(\mathcal{M} \otimes (V^\mu)^*)_\chi \oplus (\mathcal{M} \otimes (V^\mu)^*)_{\neq\chi},$$

where $(\mathcal{M} \otimes (V^\mu)^*)_\chi \in \mathfrak{g}\text{-mod}_\chi$ and the set-theoretic support of $(\mathcal{M} \otimes (V^\mu)^*)_{\neq\chi}$ over $\mathrm{Spec}(Z(\mathfrak{g}))$ is finite and disjoint from χ. Then

$$T_{\chi' \to \chi}(\mathcal{M}) = (\mathcal{M} \otimes (V^\mu)^*)_\chi. \tag{1.4}$$

Remark 1.3.4. At the level of derived categories, we had to define the functor $T_{\chi' \to \chi}$ using the bimodule $(U(\mathfrak{g})_{\chi'} \otimes (V^\mu)^*)_\chi$, rather than by the formula (1.4), because for an object of $\mathfrak{g}$-mod, to belong to $\mathfrak{g}\text{-mod}_\chi$ is not a property, but extra structure.

1.3.5. The basic property of the translation functor $T_{\chi\to\chi'}$ is that it makes the following diagram commute:

$$\begin{array}{ccc} \text{D-mod}_{\lambda'}(X) & \xrightarrow{-\otimes \mathcal{O}(-\mu)} & \text{D-mod}_{\lambda}(X) \\ \Gamma\big\downarrow & & \big\downarrow\Gamma \\ \mathfrak{g}\text{-mod}_{\chi'} & \xrightarrow{T_{\chi'\to\chi}} & \mathfrak{g}\text{-mod}_{\chi}, \end{array}$$

where the G-equivariant line bundle $\mathcal{O}(\mu)$ on X is normalized so that it is ample and $\Gamma(X, \mathcal{O}(\mu)) \simeq V^\mu$.

1.3.6. We let

$$T_{\chi\to\chi'} : \mathfrak{g}\text{-mod}_\chi \to \mathfrak{g}\text{-mod}_{\chi'}$$

be the left adjoint functor of $T_{\chi'\to\chi}$.

Tautologically, it makes the following diagram commute:

$$\begin{array}{ccc} \text{D-mod}_{\lambda}(X) & \xrightarrow{-\otimes \mathcal{O}(\mu)} & \text{D-mod}_{\lambda'}(X) \\ \text{Loc}\big\uparrow & & \big\uparrow\text{Loc} \\ \mathfrak{g}\text{-mod}_{\chi} & \xrightarrow{T_{\chi\to\chi'}} & \mathfrak{g}\text{-mod}_{\chi'}. \end{array}$$

Consider the object

$$T_{\chi\to\chi'}(U(\mathfrak{g})_\chi) \in \mathfrak{g}\text{-mod}_{\chi'}.$$

The functor $T_{\chi\to\chi'}$ is given by

$$\mathcal{M} \mapsto T_{\chi\to\chi'}(U(\mathfrak{g})_\chi) \underset{U(\mathfrak{g})_\chi}{\otimes} \mathcal{M},$$

where we regard $T_{\chi\to\chi'}(U(\mathfrak{g})_\chi)$ as a $(U(\mathfrak{g})_{\chi'}, U(\mathfrak{g})_\chi)$-bimodule.

For what follows, we will need to describe the above object $T_{\chi\to\chi'}(U(\mathfrak{g})_\chi)$ more explicitly.

1.3.7. Consider the tensor product

$$U(\mathfrak{g})_\chi \otimes V^\mu$$

as a $(U(\mathfrak{g}), U(\mathfrak{g})_\chi)$-bimodule as in Sect. 1.3.2, and consider the corresponding decomposition

$$(U(\mathfrak{g})_\chi \otimes V^\mu)_{\chi'} \oplus (U(\mathfrak{g})_\chi \otimes V^\mu)_{\neq\chi'}$$

according to set-theoretic support over $\text{Spec}(Z(\mathfrak{g}))$.

It is no longer true that $(U(\mathfrak{g})_\chi \otimes V^\mu)_{\chi'}$ is scheme-theoretically supported at $\chi' \in \text{Spec}(Z(\mathfrak{g}))$. Let $(U(\mathfrak{g})_\chi \otimes V^\mu)'_{\chi'}$ be the maximal quotient of $(U(\mathfrak{g})_\chi \otimes V^\mu)_{\chi'}$ with scheme-theoretic support at χ', i.e., the maximal quotient on which $U(\mathfrak{g})$ acts via $U(\mathfrak{g})_{\chi'}$.

Proposition 1.3.8. *We have a canonical isomorphism of* $(U(\mathfrak{g})_{\chi'}, U(\mathfrak{g})_\chi)$*-bimodules*

$$T_{\chi\to\chi'}(U(\mathfrak{g})_\chi) \simeq (U(\mathfrak{g})_\chi \otimes V^\mu)'_{\chi'};$$

in particular, $T_{\chi\to\chi'}(U(\mathfrak{g})_\chi)$ *lies in* $\mathfrak{g}\text{-mod}^\heartsuit_{\chi'}$.

Proof. Since $T_{\chi\to\chi'}$ is the left adjoint of a t-exact functor and since $U(\mathfrak{g})_\chi \in \mathfrak{g}\text{-mod}_\chi^\heartsuit$ is projective, we obtain that $T_{\chi\to\chi'}(U(\mathfrak{g})_\chi)$ is a projective object in $\mathfrak{g}\text{-mod}_{\chi'}^\heartsuit$.

The functor

$$\mathcal{M} \mapsto H^0\left(T_{\chi\to\chi'}(U(\mathfrak{g})_\chi) \underset{U(\mathfrak{g})_\chi}{\otimes} \mathcal{M}\right), \quad \mathfrak{g}\text{-mod}_\chi^\heartsuit \to \mathfrak{g}\text{-mod}_{\chi'}^\heartsuit$$

provides a left adjoint to the functor

$$T_{\chi'\to\chi} : \mathfrak{g}\text{-mod}_{\chi'}^\heartsuit \to \mathfrak{g}\text{-mod}_\chi^\heartsuit.$$

However, it is easy to see that the above left adjoint is given by

$$\mathcal{M} \mapsto (\mathcal{M} \otimes V^\mu)'_{\chi'},$$

where:

- $\mathcal{M} \otimes V^\mu \simeq (\mathcal{M} \otimes V^\mu)_{\chi'} \oplus (\mathcal{M} \otimes V^\mu)_{\neq\chi'}$ is the decomposition of $\mathcal{M} \otimes V^\mu$ according to set-theoretic support over $\operatorname{Spec}(Z(\mathfrak{g}))$;
- $(\mathcal{M} \otimes V^\mu)'_{\chi'}$ is the maximal quotient of $(\mathcal{M} \otimes V^\mu)_{\chi'}$ with scheme-theoretic support at $\chi' \in \operatorname{Spec}(Z(\mathfrak{g}))$.

Hence, we obtain

$$T_{\chi\to\chi'}(U(\mathfrak{g})_\chi) \simeq H^0\left(T_{\chi\to\chi'}(U(\mathfrak{g})_\chi)\right) \simeq (U(\mathfrak{g})_\chi \otimes V^\mu)'_{\chi'},$$

as desired.

□

From here, and using Noetherianness, we obtain:

Corollary 1.3.9. *The object $T_{\chi\to\chi'}(U(\mathfrak{g})_\chi)$, viewed as a $(U(\mathfrak{g}), U(\mathfrak{g})_\chi)$-bimodule admits a resolution with terms of the form $U(\mathfrak{g})_\chi \otimes V$, where V is a finite-dimensional representation of G.*

1.4. **The long intertwining operator.**

1.4.1. In this subsection we take G to be a reductive group and X its flag variety. For a given $\lambda \in \mathfrak{t}^*$, we consider the corresponding categories $\text{D-mod}_\lambda(X)^N$ and $\text{D-mod}_\lambda(X)^{N^-}$.

The goal of this subsection is to supply a (possibly new) proof of the following well-known statement (see also [CY, Theorem 5.2] for a similar statement in a more general setting of the Matsuki correspondence):

Proposition 1.4.2. *The partially defined functor $\text{Av}_!^{N^-}$ is defined on the essential image of $\mathbf{oblv}_N$, and the composition $\text{Av}_!^{N^-} \circ \mathbf{oblv}_N$ provides an inverse to the functor*

$$\Upsilon := \text{Av}_*^N \circ \mathbf{oblv}_{N^-}, \quad \text{D-mod}_\lambda(X)^{N^-} \to \text{D-mod}_\lambda(X)^N.$$

As a consequence we will now deduce:

Proposition 1.4.3. *Consider the action of G on the category $\mathfrak{g}\text{-mod}_\chi$ for a given central character χ. Then the partially defined functor $\text{Av}_!^{N^-}$ is defined on the essential image of*

$$\mathbf{oblv}_N : \mathfrak{g}\text{-mod}_\chi^N \to \mathfrak{g}\text{-mod}_\chi,$$

and the composition $\text{Av}_!^{N^-} \circ \mathbf{oblv}_N$ provides an inverse to the functor

$$\Upsilon := \text{Av}_*^N \circ \mathbf{oblv}_{N^-}, \quad \mathfrak{g}\text{-mod}_\chi^{N^-} \to \mathfrak{g}\text{-mod}_\chi^N.$$

Proof of Proposition 1.4.3. Choose a twisting λ that gives rise to χ via the Harish-Chandra homomorphism. The fact that $\mathrm{Av}_!^{N^-}$ is defined on $\mathfrak{g}\text{-mod}_\chi^N$ follows from the corresponding fact for $\text{D-mod}_\lambda(X)^N$ using Proposition 1.2.6. Moreover, for $\mathcal{M} \in \mathfrak{g}\text{-mod}_\chi^N$, we have:

$$\mathrm{Loc} \circ \mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N(\mathcal{M}) \simeq \mathrm{Av}_!^{N^-} \circ \mathrm{Loc} \circ \mathbf{oblv}_N(\mathcal{M})$$

Further, we have

$$(\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N) \simeq \Gamma \circ \mathrm{Loc} \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N) \simeq$$
$$\Gamma \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \circ (\mathrm{Loc} \circ \mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N) \simeq \Gamma \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \circ (\mathrm{Av}_!^{N^-} \circ \mathrm{Loc} \circ \mathbf{oblv}_N) \simeq$$
$$\simeq \Gamma \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N) \circ \mathrm{Loc} \simeq \Gamma \circ \mathrm{Loc} \simeq \mathrm{Id},$$

where the fifth isomorphism follows from Proposition 1.4.2.

Similarly,

$$(\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N) \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \simeq \Gamma \circ \mathrm{Loc} \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N) \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \simeq$$
$$\simeq \Gamma \circ \mathrm{Av}_!^{N^-} \circ \mathrm{Loc} \circ \mathbf{oblv}_N \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \simeq \Gamma \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N) \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \circ \mathrm{Loc} \simeq$$
$$\simeq \Gamma \circ \mathrm{Loc} \simeq \mathrm{Id}\,.$$

□

1.4.4. The rest of this subsection is devoted to the proof of Proposition 1.4.2. The key idea is to use the theorem of T. Braden's (see, e.g., [DrGa2]).

Choose a regular dominant coweight $\mathbb{G}_m \to T$, and consider the resulting $\mathbb{G}_m$-action on X. Note that the orbits of N (resp., N^-) are $\mathbb{G}_m$-invariant. In particular, every N- or N^--equivariant D-module on X is $\mathbb{G}_m$-monodromic.

Moreover, the disjoint union of N-orbits, to be denoted X^+ (resp., the disjoint union of N^--orbits, to be denoted X^-) is the attracting (resp., repelling) locus for the above action. Denote $X^0 = X^{\mathbb{G}_m}$ and denote by

$$i^+ : X^0 \rightleftarrows X^+ : q^+, \quad p^+ : X^+ \to X$$

and

$$i^- : X^0 \rightleftarrows X^- : q^-, \quad p^- : X^- \to X$$

the corresponding maps.

We will consider the functor of *hyperbolic restriction* from the full subcategory

$$\text{D-mod}_\lambda(X)^{\text{mon}} \subset \text{D-mod}_\lambda(X)$$

consisting of $\mathbb{G}_m$-monodromic objects to $\text{D-mod}_\lambda(X^0)$. Braden's theorem insures that this functor is well-defined:

$$(i^-)^! \circ (p^-)^* \simeq (q^-)_! \circ (p^-)^* =: \mathsf{H}^- \underset{\sim}{\overset{\Phi}{\leftarrow}} \mathsf{H}^+ := (q^+)_* \circ (p^+)^! \simeq (i^+)^* \circ (p^+)^!,$$

where the isomorphisms on the sides are known as the *contraction principle*.

1.4.5. The fact that $\mathrm{Av}_!^{N^-}$ is defined on the essential image of $\mathbf{oblv}_N$ follows from the holonomicity. Here is, however, an alternative argument:

We claim that $\mathrm{Av}_!^{N^-}$ is defined on all of $\text{D-mod}_\lambda(X)^{\mathrm{mon}}$. Indeed, for an object $\mathcal{F} \in \text{D-mod}_\lambda(X)$, the object $\mathrm{Av}_!^{N^-}(\mathcal{F})$ is defined if

$$(p^-)^* \circ \mathrm{Av}_!^{N^-}(\mathcal{F}) \in \text{D-mod}_\lambda(X^-)^{N^-}$$

is defined, and that is if and only if

$$(i^-)^! \circ \mathbf{oblv}_{N^-} \circ (p^-)^* \circ \mathrm{Av}_!^{N^-}(\mathcal{F}) \in \text{D-mod}_\lambda(X^0)$$

is defined.

We rewrite

$$(1.5)\quad (i^-)^! \circ \mathbf{oblv}_{N^-} \circ (p^-)^* \circ \mathrm{Av}_!^{N^-}(\mathcal{F}) \simeq (q^-)_! \circ \mathbf{oblv}_{N^-} \circ (p^-)^* \circ \mathrm{Av}_!^{N^-}(\mathcal{F}) \simeq \\ \simeq (q^-)_! \circ \mathbf{oblv}_{N^-} \circ \mathrm{Av}_!^{N^-} \circ (p^-)^*(\mathcal{F}) \simeq (q^-)_! \circ (p^-)^*(\mathcal{F}) \simeq \mathsf{H}^-(\mathcal{F}),$$

where the first isomorphism is given by the contraction principle, and the third isomorphism is due to the fact that the projection q^- is N^--invariant.

1.4.6. From (1.5) we obtain a canonical isomorphism

$$(1.6)\qquad \mathsf{H}^- \circ \mathbf{oblv}_{N^-} \circ \mathrm{Av}_!^{N^-}(\mathcal{F}) \simeq \mathsf{H}^-(\mathcal{F}), \quad \mathcal{F} \in \text{D-mod}_\lambda(X)^{\mathrm{mon}}.$$

Similarly, we have:

$$(i^+)^* \circ \mathbf{oblv}_N \circ (p^+)^! \circ \mathrm{Av}_*^N(\mathcal{F}) \simeq (q^+)_* \circ \mathbf{oblv}_N \circ (p^+)^! \circ \mathrm{Av}_*^N(\mathcal{F}) \simeq \\ \simeq (q^+)_* \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N \circ (p^+)^!(\mathcal{F}) \simeq (q^+)_* \circ (p^+)^!(\mathcal{F}) \simeq \mathsf{H}^+(\mathcal{F}),$$

i.e.,

$$(1.7)\qquad \mathsf{H}^+ \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N(\mathcal{F}) \simeq \mathsf{H}^+(\mathcal{F}), \quad \mathcal{F} \in \text{D-mod}_\lambda(X)^{\mathrm{mon}}.$$

1.4.7. A priori, the functor $\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N$ is the left adjoint of the functor $\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}$. Let us prove that the unit of the adjunction

$$(1.8)\qquad \mathcal{F} \to (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N)(\mathcal{F})$$

is an isomorphism.

To show that (1.8) is an isomorphism, it is enough to show that it induces an isomorphism after applying the functor $\mathsf{H}^+ \circ \mathbf{oblv}_N$. The resulting map is

$$\mathsf{H}^+ \circ \mathbf{oblv}_N(\mathcal{F}) \to (\mathsf{H}^+ \circ \mathbf{oblv}_N) \circ (\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}) \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N)(\mathcal{F}) \overset{(1.7)}{\simeq} \\ \simeq \mathsf{H}^+ \circ \mathbf{oblv}_{N^-} \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N)(\mathcal{F}) \overset{\Phi}{\to} \mathsf{H}^- \circ \mathbf{oblv}_{N^-} \circ (\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N)(\mathcal{F}) \overset{(1.6)}{\simeq} \\ \simeq \mathsf{H}^- \circ \mathbf{oblv}_N(\mathcal{F}).$$

By unwinding the definition of the natural transformation $\Phi : \mathsf{H}^+ \to \mathsf{H}^-$ (see [DrGa2, Equation (3.5)]), we obtain that the above map is induced by the map

$$\mathsf{H}^+ \circ \mathbf{oblv}_N(\mathcal{F}) \overset{\Phi}{\to} \mathsf{H}^- \circ \mathbf{oblv}_N(\mathcal{F}),$$

and hence is an isomorphism by Braden's theorem.

1.4.8. It remains to show that the functor $\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}$ is conservative. But this follows from the fact that its composition with $\mathsf{H}^+ \circ \mathbf{oblv}_N$ is conservative:

$$\mathsf{H}^+ \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-} \overset{(1.7)}{\simeq} \mathsf{H}^+ \circ \mathbf{oblv}_{N^-} \overset{\Phi}{\simeq} \mathsf{H}^- \circ \mathbf{oblv}_{N^-},$$

while the latter is evidently conservative on $\text{D-mod}_\lambda(X)^{N^-}$.

2. Casselman-Jacquet functor as averaging

In this section we introduce the main character of this paper – the Casselman-Jacquet functor, and state the main results.

Throughout this section G is a reductive group, X is its flag variety. We let N (resp., N^-) be the unipotent radical of a Borel subgroup (resp., an opposite Borel) in G.

2.1. Casselman-Jacquet functor in the abstract setting.

2.1.1. Let $\mathcal{C}$ be a category acted on by N. Consider the (fully faithful) forgetful functor

$$\mathbf{oblv}_N : \mathcal{C}^N \to \mathcal{C}.$$

As was mentioned in Sect. 1.1.3, it admits a right adjoint functor Av_*^N, so that we have an adjoint pair $(\mathbf{oblv}_N, \mathrm{Av}_*^N)$.

We will now consider the (usually, *discontinuous*) right adjoint of Av_*^N, denoted

$$(\mathrm{Av}_*^N)^R : \mathcal{C}^N \to \mathcal{C}.$$

We will also consider the monad $(\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N$ acting on $\mathcal{C}$.

Remark 2.1.2. As we shall see in Sect. 2.3, the monad $(\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N$ is not as bizarre as one could initially think: in the case when $\mathcal{C}$ is the category of modules over an associative algebra, equipped with a Harish-Chandra structure with respect to N, the functor $(\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N$ is that of $\mathfrak{n}$-adic completion.

2.1.3. Suppose now that the N-action on $\mathcal{C}$ comes by restriction from a G-action. We define the functor

$$J : \mathcal{C} \to \mathcal{C}^{N^-}$$

to be the composition

$$\mathrm{Av}_*^{N^-} \circ (\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N.$$

By construction, the functor J is the right adjoint of the functor

$$\mathbf{oblv}_N \circ \mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-} : \mathcal{C}^{N^-} \to \mathcal{C}.$$

From here we obtain:

Proposition 2.1.4. *Let $\mathcal{C}$ be such that the functor*

$$\Upsilon := \mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-} : \mathcal{C}^{N^-} \to \mathcal{C}^N$$

is an equivalence. Then the functor J identifies with

$$\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N.$$

Proof. If Υ is an equivalence, then the functor $\mathrm{Av}_!^{N^-}$ is defined on the essential image of $\mathbf{oblv}_N$, and

$$\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N : \mathcal{C}^N \to \mathcal{C}^{N^-},$$

which is the (a priori partially defined) left adjoint of Υ, is well-defined and is the inverse of Υ.

In particular, $\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N$ is *right* adjoint to Υ, so

$$\mathrm{Av}_*^{N^-} \circ (\mathrm{Av}_*^N)^R \simeq \Upsilon^R \simeq \mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N.$$

Composing, we obtain

$$J = \mathrm{Av}_*^{N^-} \circ (\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N \simeq \mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N,$$

as desired.

□

Remark 2.1.5. According to Sect. 1.4, the functor Υ is an equivalence in the following cases of interest:

$$\mathcal{C} = \text{D-mod}_\lambda(X) \text{ and } \mathcal{C} = \mathfrak{g}\text{-mod}_\chi.$$

2.2. **Casselman-Jacquet functor as completion.** In this subsection N can be any unipotent group.

2.2.1. In this subsection we will consider the particular case of $\mathcal{C} = \mathfrak{n}$-mod, equipped with the natural N-action. Note that in this case $\mathcal{C}^N \simeq \mathrm{Rep}(N)$.

We will see that the endofunctor $(\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N$ of $\mathfrak{n}$-mod, can be described explicitly as the functor of $\mathfrak{n}$-adic completion.

2.2.2. Consider the inverse family of $\mathfrak{n}$-bimodules indexed by natural numbers

$$i \mapsto U(\mathfrak{n})^i := U(\mathfrak{n})/\mathfrak{n}^i \cdot U(\mathfrak{n}),$$

and the corresponding family of endofunctors of $\mathfrak{n}$-mod:

$$\mathcal{M} \mapsto U(\mathfrak{n})^i \underset{U(\mathfrak{n})}{\otimes} \mathcal{M} =: \mathcal{M}/\mathfrak{n}^i \cdot \mathcal{M}.$$

We claim:

Proposition 2.2.3. *The functor* $(\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N$ *identifies canonically with*

$$\mathcal{M} \mapsto \lim_i \mathcal{M}/\mathfrak{n}^i \cdot \mathcal{M}.$$

2.2.4. *Example.* Let $N = \mathbb{G}_a$. We identify the category $\mathfrak{n}$-mod with $k[t]$-mod, and

$$\mathrm{Rep}(N) \subset \mathfrak{n}\text{-mod}$$

with

$$k[t]\text{-mod}_{\{0\}} \subset k[t]\text{-mod},$$

the full subcategory consisting of modules, on whose cohomologies t acts locally nilpotently.

In this case, the assertion of the proposition is well-known: the functor in question is that of t-adic completion.

2.2.5. The rest of this subsection is devoted to the proof of Proposition 2.2.3.

Let $\mathfrak{n}\text{-mod}^{\text{non-nilp}} \subset \mathfrak{n}\text{-mod}$ be the full subcategory equal to the *right orthogonal* of $\operatorname{Rep}(N)$. The assertion of the proposition amounts to the following two statements:

(a) For $\mathcal{M} \in \mathfrak{n}\text{-mod}^{\text{non-nilp}}$, the limit $\lim\limits_{i} \mathcal{M}/\mathfrak{n}^i \cdot \mathcal{M}$ is zero.

(b) For any $\mathcal{M} \in \operatorname{Rep}(N)$, the cofiber of the canonical map

$$\mathcal{M} \to \lim_{i} \mathcal{M}/\mathfrak{n}^i \cdot \mathcal{M}$$

belongs to $\mathfrak{n}\text{-mod}^{\text{non-nilp}}$.

Let $\mathbf{triv} \in \operatorname{Rep}(N) \subset \mathfrak{n}\text{-mod}$ denote the trivial representation. First we notice:

Lemma 2.2.6. $\mathcal{M} \in \mathfrak{n}\text{-mod}^{\text{non-nilp}} \Leftrightarrow \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} \mathcal{M} = 0.$

Proof. The category $\operatorname{Rep}(N)$ is compactly generated by $\mathbf{triv}$. Hence,

$$\mathcal{M} \in \mathfrak{n}\text{-mod}^{\text{non-nilp}} \Leftrightarrow \mathcal{H}om_{\mathfrak{n}\text{-mod}}(\mathbf{triv}, \mathcal{M}) = 0 \Leftrightarrow \mathrm{C}^\bullet(\mathfrak{n}, \mathcal{M}) = 0.$$

Since, $\mathrm{C}^\bullet(\mathfrak{n}, -)$ is isomorphic to, up to a cohomological shift, to $\mathrm{C}_\bullet(\mathfrak{n}, -)$, we obtain that

$$\mathcal{M} \in \mathfrak{n}\text{-mod}^{\text{non-nilp}} \Leftrightarrow \mathrm{C}_\bullet(\mathfrak{n}, \mathcal{M}) = 0 \Leftrightarrow \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} \mathcal{M} = 0.$$

□

Proof of (a). Since each $U(\mathfrak{n})^i$ admits a filtration with subquotients isomorphic to $\mathbf{triv}$, from the Lemma, we obtain that if $\mathcal{M} \in \mathfrak{n}\text{-mod}^{\text{non-nilp}}$, then all the terms in $\lim\limits_{i} \mathcal{M}/\mathfrak{n}^i \cdot \mathcal{M}$ are zero. □

Proof of (b). We need to show that for any $\mathcal{M} \in \operatorname{Rep}(N)$, the map

$$\mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} \mathcal{M} \to \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} \left(\lim_{i} U(\mathfrak{n})^i \underset{U(\mathfrak{n})}{\otimes} \mathcal{M} \right)$$

is an isomorphism. We know that the functor

$$\mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} - \simeq \mathrm{C}_\bullet(\mathfrak{n}, -)$$

commutes with limits (again, because it is isomorphic up to a shift to $\mathrm{C}^\bullet(\mathfrak{n}, -)$), so we need to show that the map

$$\mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} \mathcal{M} \to \lim_{i} \left(\mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n})^i \underset{U(\mathfrak{n})}{\otimes} \mathcal{M} \right)$$

is an isomorphism. In other words, we need to show that

$$\lim_{i} \left(\operatorname{coFib}(\mathbf{triv} \to \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n})^i) \underset{U(\mathfrak{n})}{\otimes} \mathcal{M} \right)$$

is zero.

Since $\operatorname{Rep}(N)$ is *co-generated* by R_N, it suffices to show that

$$\lim_{i} \left(\operatorname{coFib}(\mathbf{triv} \to \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n})^i) \underset{U(\mathfrak{n})}{\otimes} R_N \right)$$

is zero.

Now, since R_N admits a *finite* resolution with terms of the form $U(\mathfrak{n})\otimes V$ (where $V \in \text{Vect}$), it suffices to show that

$$\lim_i \left(\text{coFib}(\mathbf{triv} \to \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n})^i) \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n}) \otimes V\right) \simeq \lim_i \left(\text{coFib}(k \to \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n})^i) \otimes V\right)$$

is zero.

We will show that the inverse system (in Vect)

$$i \mapsto \text{coFib}(k \to \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n})^i)$$

is *null*, i.e., for every i there exists $i' > i$, such that the map

$$\text{coFib}(k \to \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n})^{i'}) \to \text{coFib}(k \to \mathbf{triv} \underset{U(\mathfrak{n})}{\otimes} U(\mathfrak{n})^i)$$

is zero.

The objects of Vect involved are compact (i.e., have finitely many cohomologies and each cohomology is finite-dimensional). Hence, we can dualize, and the assertion becomes equivalent to the fact that the direct system

$$i \mapsto \text{Fib}\left(\text{C}^\bullet(\mathfrak{n}, (U(\mathfrak{n})^i)^*) \to k\right)$$

is null. Again, by compactness, the latter is equivalent to the fact that

$$\underset{i}{\text{colim}}\ \text{Fib}\left(\text{C}^\bullet(\mathfrak{n}, (U(\mathfrak{n})^i)^*) \to k\right) = 0,$$

i.e., that the map

$$\underset{i}{\text{colim}}\ \text{C}^\bullet(\mathfrak{n}, (U(\mathfrak{n})^i)^*) \to k$$

is an isomorphism. Since $\text{C}^\bullet(\mathfrak{n}, -)$ commutes with colimits (being isomorphic up to a shift to $\text{C}_\bullet(\mathfrak{n}, -)$), this is equivalent to the map

$$\text{C}^\bullet\left(\mathfrak{n}, (\underset{i}{\text{colim}}\,(U(\mathfrak{n})^i)^*\right) \to k \tag{2.1}$$

being an isomorphism.

We now notice that the pairing

$$f, u \mapsto u(f)(1), \quad f \in R_N,\ u \in U(\mathfrak{n})$$

defines an isomorphism

$$R_N \simeq \underset{i}{\text{colim}}\,(U(\mathfrak{n})^i)^*$$

as $\mathfrak{n}$-modules.

Under this isomorphism, the above map (2.1) identifies with

$$\text{C}^\bullet(\mathfrak{n}, R_N) \to R_N \overset{f\mapsto f(1)}{\longrightarrow} k,$$

which is evidently an isomorphism.

□

□[Proposition 2.2.3]

2.3. **The Casselman-Jacquet functor for A-modules.**

2.3.1. Let A be an associative algebra equipped with a Harish-Chandra structure with respect to N (see Sect. 1.1.2 for what this means). In particular, A-mod is acted on by N, and we have the restriction functor

$$A\text{-mod} \to \mathfrak{n}\text{-mod},$$

and its left adjoint $\mathfrak{n}\text{-mod} \to A\text{-mod}$, both compatible with N-actions.

We have commutative diagrams

$$\begin{array}{ccccc} A\text{-mod}^N & \xrightarrow{\mathbf{oblv}_N} & A\text{-mod} & \xrightarrow{\mathrm{Av}_*^N} & A\text{-mod}^N \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Rep}(N) & \xrightarrow{\mathbf{oblv}_N} & \mathfrak{n}\text{-mod} & \xrightarrow{\mathrm{Av}_*^N} & \mathrm{Rep}(N) \end{array}$$

and

$$\begin{array}{ccccc} A\text{-mod}^N & \xrightarrow{\mathbf{oblv}_N} & A\text{-mod} & \xrightarrow{\mathrm{Av}_*^N} & A\text{-mod}^N \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Rep}(N) & \xrightarrow{\mathbf{oblv}_N} & \mathfrak{n}\text{-mod} & \xrightarrow{\mathrm{Av}_*^N} & \mathrm{Rep}(N). \end{array}$$

By passing to right adjoints in the second diagram, we obtain that the following diagram commutes as well:

$$\begin{array}{ccccc} A\text{-mod} & \xrightarrow{\mathrm{Av}_*^N} & A\text{-mod}^N & \xrightarrow{(\mathrm{Av}_*^N)^R} & A\text{-mod} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{n}\text{-mod} & \xrightarrow{\mathrm{Av}_*^N} & \mathrm{Rep}(N) & \xrightarrow{(\mathrm{Av}_*^N)^R} & \mathfrak{n}\text{-mod}. \end{array} \tag{2.2}$$

2.3.2. From the commutation of (2.2) and Proposition 2.2.3 we obtain:

Corollary 2.3.3. *The endofunctor* $(\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N$ *on* A-mod *is right t-exact.*

Remark 2.3.4. By unwinding the constructions one can show that for $\mathcal{M} \in A\text{-mod}^\heartsuit$, the action of A on

$$H^0((\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N(\mathcal{M})) \simeq \lim_k H^0(\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M})$$

is given by the following formula:

For an element $a \in A$, let k be an integer so that

$$\mathrm{ad}_{\xi_1} \circ ... \circ \mathrm{ad}_{\xi_k}(a) = 0 \quad \forall \xi_1, ..., \xi_k \in \mathfrak{n}.$$

Then the action of a is well-defined as a map

$$H^0(\mathcal{M}/\mathfrak{n}^{k'+k} \cdot \mathcal{M}) \to H^0(\mathcal{M}/\mathfrak{n}^{k'} \cdot \mathcal{M}),$$

thereby giving a map on the inverse limit.

2.3.5. Let A be left-Noetherian. Let

$$A\text{-mod}^{\mathfrak{n}\text{-f.g.}} \subset A\text{-mod} \tag{2.3}$$

be the full subcategory consisting of modules that map to compact (i.e., finitely generated) objects under the forgetful functor $A\text{-mod} \to \mathfrak{n}\text{-mod}$. Note that $A\text{-mod}^{\mathfrak{n}\text{-f.g.}}$ is *not necessarily* contained in $A\text{-mod}^c$ (unless A has a finite cohomological dimension).

Let

$$\mathrm{Ind}(A\text{-mod}^{\mathfrak{n}\text{-f.g.}})$$

be the ind-completion of $A\text{-mod}^{\mathfrak{n}\text{-f.g.}}$. Ind-extending the tautological embedding

$$A\text{-mod}^{\mathfrak{n}\text{-f.g.}} \hookrightarrow A\text{-mod},$$

we obtain a functor

$$\mathrm{Ind}(A\text{-mod}^{\mathfrak{n}\text{-f.g.}}) \to A\text{-mod}. \tag{2.4}$$

Ind-extending the t-structure on $A\text{-mod}^{\mathfrak{n}\text{-f.g.}}$, we obtain one on $\mathrm{Ind}(A\text{-mod}^{\mathfrak{n}\text{-f.g.}})$. The functor (2.4) is t-exact, but not in general fully faithful. However, as in [Ga4, Proposition 2.3.3], one shows that the functors

$$\mathrm{Ind}(A\text{-mod}^{\mathfrak{n}\text{-f.g.}})^{\geq -n} \to A\text{-mod}^{\geq -n} \tag{2.5}$$

are fully faithful for every n.

Let $\mathrm{Ind}^\wedge(A\text{-mod}^{\mathfrak{n}\text{-f.g.}})$ denote the left-completion of $\mathrm{Ind}(A\text{-mod}^{\mathfrak{n}\text{-f.g.}})$ in its t-structure. Since A-mod is left-complete in its t-structure, the functor (2.4) extends to a functor

$$\mathrm{Ind}^\wedge(A\text{-mod}^{\mathfrak{n}\text{-f.g.}}) \to A\text{-mod}. \tag{2.6}$$

The functor (2.6) is fully faithful, since the functors (2.5) have this property. Its essential image consists of objects of A-mod, whose cohomologies are filtered colimits of objects from

$$A\text{-mod}^{\heartsuit,\mathfrak{n}\text{-f.g.}} = A\text{-mod}^{\heartsuit} \cap A\text{-mod}^{\mathfrak{n}\text{-f.g.}}.$$

Remark 2.3.6. Suppose that A has a finite cohomological dimension, in which case the functor (2.4) is fully faithful, and hence so is the functor

$$\mathrm{Ind}(A\text{-mod}^{\mathfrak{n}\text{-f.g.}}) \to \mathrm{Ind}^\wedge(A\text{-mod}^{\mathfrak{n}\text{-f.g.}}).$$

However, it is not clear to the authors whether the latter is an equivalence.

2.3.7. Assume now that A has a Harish-Chandra structure with respect to G. We claim:

Proposition 2.3.8.

(a) *The functor* $J : A\text{-mod} \to A\text{-mod}^{N^-}$ *is left t-exact when restricted to* $A\text{-mod}^{\mathfrak{n}\text{-f.g.}}$.

(b) *Assume that the corresponding functor* Υ *is an equivalence. Then the functor* J *is left t-exact when restricted to* $\mathrm{Ind}^\wedge(A\text{-mod}^{\mathfrak{n}\text{-f.g.}})$.

Proof. First off, point (b) reduces to point (a) as follows: if Υ is an equivalence, then by Proposition 2.1.4, the functor J is commutes with colimits. Then we use Lemma 1.1.9.

We now prove point (a).

The functor $\mathrm{Av}_*^{N^-}$, being the right adjoint of the t-exact functor $\mathbf{oblv}_{N^-}$, is left t-exact (on all of A-mod). Hence, it suffices to show that the functor $(\mathrm{Av}_*^N)^R \circ \mathrm{Av}_*^N$ is left t-exact when restricted to $A\text{-mod}^{\mathfrak{n}\text{-f.g.}}$. We will show that it is in fact t-exact.

Using the commutation of (2.2), it suffices to show that the functor $(\mathrm{Av}^N_*)^R \circ \mathrm{Av}^N_*$ is t-exact when restricted to $\mathfrak{n}\text{-mod}^{\text{f.g.}}$. Using Proposition 2.2.3, it suffices to show that the functor

$$\mathcal{M} \mapsto \lim_k \mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$$

sends $\mathcal{M} \in \mathfrak{n}\text{-mod}^{\text{f.g.},\heartsuit}$ to an object in $\mathrm{Vect}^\heartsuit$. However, the latter is known: this is Casselman's generalization of the Artin-Rees lemma.

□

2.3.9. Note that the functor

$$\mathcal{M} \mapsto H^0(\mathrm{Av}^{N^-}_*(\mathcal{M})), \quad A\text{-mod}^\heartsuit \to (A\text{-mod}^{N^-})^\heartsuit$$

is that of sending $\mathcal{M}$ to its submodule consisting of elements that are locally nilpotent with respect to the action of $\mathfrak{n}^-$.

Hence, from Proposition 2.3.8 we obtain:

Corollary 2.3.10.
(a) *Under the assumptions of Proposition 2.3.8(a), the functor*

$$\mathcal{M} \mapsto H^0(J(\mathcal{M})), \quad (A\text{-mod}^{\mathfrak{n}\text{-f.g.}})^\heartsuit \to (A\text{-mod}^{N^-})^\heartsuit$$

sends $\mathcal{M}$ to the submodule of $\lim\limits_k \mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$ consisting of elements that are locally nilpotent with respect to the action of $\mathfrak{n}^-$.
(b) *Under the assumptions of Proposition 2.3.8(b), ditto for the category* $(\mathrm{Ind}^\wedge(A\text{-mod}^{\mathfrak{n}\text{-f.g.}}))^\heartsuit$.

2.4. **The Casselman-Jacquet functor for $\mathfrak{g}$-modules.**

2.4.1. The goal of this subsection is to prove the following:

Theorem 2.4.2. *Consider G acting on the category $\mathfrak{g}\text{-mod}_\chi$.*
(a) *There is a canonically defined natural transformation of functors*

$$\mathrm{Av}^{N^-}_* \circ \mathbf{oblv}_N \circ \mathrm{Av}^N_! \to \mathrm{Av}^{N^-}_! \circ \mathbf{oblv}_N \circ \mathrm{Av}^N_* \simeq J, \tag{2.7}$$

where the LHS is a partially defined functor.
(b) *The functor $\mathrm{Av}^N_!$ is defined and the above natural transformation is an isomorphism, when evaluated on* $\mathrm{Ind}^\wedge(\mathfrak{g}\text{-mod}^{\mathfrak{n}\text{-f.g.}}_\chi)$.

We will deduce Theorem 2.4.2 from the following result that will be proved in Sect. 3.4.5.

Let $\mathrm{D\text{-}mod}_\lambda(X)^{\mathrm{ULA}} \subset \mathrm{D\text{-}mod}_\lambda(X)$ be the full subcategory of objects that are ULA with respect to the projection $X \to N\backslash X$, see Sect. 3.2.1 for what this means. Consider the corresponding subcategory

$$\mathrm{Ind}^\wedge(\mathrm{D\text{-}mod}_\lambda(X)^{\mathrm{ULA}}) \subset \mathrm{D\text{-}mod}_\lambda(X),$$

see Sect. 3.2.3.

We claim:

Theorem 2.4.3. *Consider G acting on the category* $\mathrm{D\text{-}mod}_\lambda(X)$.
(a) *There is a canonically defined natural transformation of functors*

$$\mathrm{Av}^{N^-}_* \circ \mathbf{oblv}_N \circ \mathrm{Av}^N_! \to \mathrm{Av}^{N^-}_! \circ \mathbf{oblv}_N \circ \mathrm{Av}^N_* \simeq J,$$

where the LHS is a partially defined functor.
(b) *The functor $\mathrm{Av}^N_!$ is defined and the above natural transformation is an isomorphism, on objects from* $\mathrm{Ind}^\wedge(\mathrm{D\text{-}mod}_\lambda(X)^{\mathrm{ULA}})$.

2.4.4. *Proof of Theorem 2.4.2.* Consider the adjoint functors

$$\mathrm{Loc} : \mathfrak{g}\text{-mod}_\chi \rightleftarrows \text{D-mod}_\lambda(X) : \Gamma. \tag{2.8}$$

By Proposition 1.2.6, the functor $\mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N$ on $\mathfrak{g}\text{-mod}_\chi$ is defined and identifies with

$$\Gamma \circ \mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N \circ \mathrm{Loc}\,.$$

Similarly, the functor $\mathrm{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_!^N$, viewed as taking values in $\mathrm{Pro}(\mathfrak{g}\text{-mod}_\chi^{N^-})$, identifies with

$$\Gamma \circ \mathrm{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_!^N \circ \mathrm{Loc}\,.$$

Hence, point (a) of Theorem 2.4.2 follows from point (a) of Theorem 2.4.3.

For point (b), we will use Proposition 1.2.6 and the following assertion proved in Sect. 2.5:

Proposition 2.4.5.
(a) *The functor* Loc *sends objects in* $\mathrm{Ind}^\wedge(\mathfrak{g}\text{-mod}_\chi^{\mathfrak{n}\text{-f.g.}})$ *to objects in* $\mathrm{Ind}^\wedge(\text{D-mod}_\lambda(X)^{\mathrm{ULA}})$*.*
(b) *The functor* Γ *sends*

$$\text{D-mod}_\lambda(X)^{\mathrm{ULA}} \to \mathfrak{g}\text{-mod}_\chi^{\mathfrak{n}\text{-f.g.}}$$

and

$$\mathrm{Ind}^\wedge(\text{D-mod}_\lambda(X)^{\mathrm{ULA}}) \to \mathrm{Ind}^\wedge(\mathfrak{g}\text{-mod}_\chi^{\mathfrak{n}\text{-f.g.}}).$$

Now, the assertion of Theorem 2.4.2(b) follows from that of Theorem 2.4.3(b) and Proposition 1.2.6.

2.5. **ULA vs finite-generation.** In this subsection we prove Proposition 2.4.5.

2.5.1. Let $\mathfrak{n} \otimes \mathcal{O}_X$ be the Lie algebroid on X corresponding to the $\mathfrak{n}$-action on X. Let $'\mathrm{D}$ be its universal enveloping D-algebra. We have a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_\lambda(X) & \longrightarrow & '\text{D-mod}(X) \\ {\scriptstyle \Gamma(X,-)}\downarrow & & \downarrow{\scriptstyle \Gamma(X,-)} \\ \mathfrak{g}\text{-mod}_\chi & \longrightarrow & \mathfrak{n}\text{-mod}, \end{array} \tag{2.9}$$

where the vertical arrows are taken by taking global sections, and the horizontal arrows are given by restriction.

An object $\mathcal{M} \in \text{D-mod}_\lambda(X)$ is ULA with respect to $X \to N\backslash X$ if and only if $X|_{'\mathrm{D}}$ is finitely generated.

2.5.2. To prove Proposition 2.4.5(b) we have to show that the right vertical arrow in (2.9) sends finitely generated objects to finitely generated objects.

The latter is is enough to do at the associated graded level, and the assertion follows from the fact that X is proper.

2.5.3. Let us prove Proposition 2.4.5(a). It suffices to show that for $\mathcal{M} \in \mathfrak{g}\text{-mod}_\chi^\heartsuit \cap \mathfrak{g}\text{-mod}_\chi^{\mathfrak{n}\text{-f.g.}}$, the object $\text{Loc}(\mathcal{M})|_{'\text{D}}$ has finitely generated cohomologies.

We first consider the case when λ is dominant and *regular*, in which case the functor Loc is t-exact.

Let

$$\text{Loc}_\mathfrak{n} : \mathfrak{n}\text{-mod} \to' \text{D-mod}(X)$$

be the functor left adjoint to

$$\Gamma : '\text{D-mod}(X) \to \mathfrak{n}\text{-mod}.$$

From (2.9), we obtain a natural transformation

$$\text{Loc}_\mathfrak{n}(\mathcal{M}|_\mathfrak{n}) \to \text{Loc}(\mathcal{M})|_{'\text{D}}.$$

Moreover, for $\mathcal{M} \in \mathfrak{g}\text{-mod}_\chi^\heartsuit$, the above map is *surjective* at the level of H^0. Since $\text{Loc}(\mathcal{M}) \in \text{D-mod}_\lambda(X)^\heartsuit$, this proves the required assertion.

2.5.4. We now consider the case of a general dominant λ. Let μ be a dominant integral weight such that $\lambda' = \lambda + \mu$ is regular. Let χ' be the corresponding character of $Z(\mathfrak{g})$. Consider the translation functor

$$T_{\chi \to \chi'} : \mathfrak{g}\text{-mod}_\chi \to \mathfrak{g}\text{-mod}_{\chi'},$$

and the commutative diagram

$$\begin{array}{ccc} \text{D-mod}_\lambda(X) & \xrightarrow{-\otimes \mathcal{O}(\mu)} & \text{D-mod}_{\lambda'}(X) \\ {\scriptstyle \text{Loc}}\uparrow & & {\scriptstyle \text{Loc}}\uparrow \\ \mathfrak{g}\text{-mod}_\chi & \xrightarrow{T_{\chi \to \chi'}} & \mathfrak{g}\text{-mod}_{\chi'}, \end{array}$$

see Sect. 1.3.6.

It is clear that an object $\mathcal{F} \in \text{D-mod}_\lambda(X)$ is ULA with respect to $X \to N\backslash X$ if and only if $\mathcal{F} \otimes \mathcal{O}(\mu) \in \text{D-mod}_{\lambda'}(X)$ has the same property.

Furthermore, from the description of the functor $T_{\chi \to \chi'}$ in Corollary 1.3.9 it is clear that it sends objects in $\mathfrak{g}\text{-mod}_\chi^{\mathfrak{n}\text{-f.g.}}$ to objects in $\mathfrak{g}\text{-mod}_{\chi'}$ whose cohomologies are in $\mathfrak{g}\text{-mod}_{\chi'}^{\mathfrak{n}\text{-f.g.}}$. Hence, the assertion of Proposition 2.4.5(a) for λ follows from that for λ'.

Remark 2.5.5. The above prove shows not only that $\text{Loc}(\mathcal{M}) \in \text{Ind}^\wedge(\text{D-mod}_\lambda(X)^{\text{ULA}})$, but that it is actually in $\text{Ind}(\text{D-mod}_\lambda(X)^{\text{ULA}})$.

Indeed, Corollary 1.3.9 implies that $T_{\chi \to \chi'}$ sends objects in $\mathfrak{g}\text{-mod}_\chi^{\mathfrak{n}\text{-f.g.}}$ to objects that can be expressed as filtered colimits of objects in $\mathfrak{g}\text{-mod}_{\chi'}^{\mathfrak{n}\text{-f.g.}}$.

3. The pseudo-identity functor

The goal of this section is to prove Theorem 2.4.3. In the process of doing so we will introduce the *pseudo-identity functor*, which will also be the main character of the sequel to this paper.

3.1. **The pseudo-identity functor: recollections.**

3.1.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack with an affine diagonal, and let us be given a twisting λ on $\mathcal{Y}$. Let $-\lambda$ denote the opposite twisting.

We identify

$$\text{D-mod}_\lambda(\mathcal{Y})^\vee \simeq \text{D-mod}_{-\lambda}(\mathcal{Y})$$

via Verdier duality. This allows to identify the category

$$\text{D-mod}_{-\lambda,\lambda}(\mathcal{Y} \times \mathcal{Y})$$

with that of continuous endofunctors on $\text{D-mod}_\lambda(\mathcal{Y})$. Explicitly,

$$\mathcal{Q} \in \text{D-mod}_{-\lambda,\lambda}(\mathcal{Y} \times \mathcal{Y}) \mapsto F_\mathcal{Q}, \quad F_\mathcal{Q}(\mathcal{F}) := (p_2)_*(\mathcal{Q} \overset{!}{\otimes} (p_1)^!(\mathcal{F})),$$

where p_1, p_2 are the two projections $\mathcal{Y} \times \mathcal{Y} \rightrightarrows \mathcal{Y}$, and where for a morphism f we denote by f_* the *renormalized* direct image functor (see [DrGa1, Sect. 9.3]).

Under this identification, the identity functor corresponds to

$$(\Delta_\mathcal{Y})_*(\omega_\mathcal{Y}) \in \text{D-mod}_{-\lambda,\lambda}(\mathcal{Y} \times \mathcal{Y}),$$

where we note that the functor

$$\Delta_* : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}_{-\lambda,\lambda}(\mathcal{Y} \times \mathcal{Y})$$

is well-defined because the pullback of the $(-\lambda, \lambda)$-twisting along the diagonal map is canonically trivialized.

3.1.2. The pseudo-identity functor

$$\text{Ps-Id}_\mathcal{Y} : \text{D-mod}_\lambda(\mathcal{Y}) \to \text{D-mod}_\lambda(\mathcal{Y})$$

is the functor corresponding to the object

$$(\Delta_\mathcal{Y})_!(k_\mathcal{Y}) \in \text{D-mod}_{-\lambda,\lambda}(\mathcal{Y} \times \mathcal{Y}),$$

where $k_\mathcal{Y}$ is the "constant sheaf" on $\mathcal{Y}$, i.e., the D-module Verdier dual to $\omega_\mathcal{Y}$.

3.1.3. A stack $\mathcal{Y}$ equipped with a twisting λ is said to be *miraculous* if the endofunctor $\text{Ps-Id}_\mathcal{Y}$ is an equivalence.

The following will be proved in the sequel to this paper (however, we do not use this result here):

Theorem 3.1.4. *Suppose that $\mathcal{Y}$ has a finite number of isomorphism classes of k-points. Then $\mathcal{Y}$ is miraculous.*

3.2. **Pseudo-identity, averaging and the ULA property.**

3.2.1. Let $f : \mathcal{Z} \to \mathcal{Y}$ be a smooth morphism between smooth algebraic stacks. Let us recall what it means for an object $\mathcal{F} \in \text{D-mod}_\lambda(\mathcal{Z})$ to be ULA with respect to f, see, e.g. [Ga1, Sect. 3.4].

The property of being ULA is local in the smooth topology on the source and the target, so we can assume that $\mathcal{Z} = Z$ and $\mathcal{Y} = Y$ are schemes.

In this case, the sheaf of rings D of differential operators on Z contains a subsheaf of rings, denoted $'\text{D}$, consisting of differential operators *vertical* with respect to f (i.e., these are those differential operators that commute with functions pulled-back from Y). Locally, $'\text{D}$ is generated by functions and vertical fields that are parallel to the fibers of f.

We shall say that $\mathcal{M} \in \text{D-mod}(Z)$ is ULA with respect to f if it is finitely generated when considered as a $'\text{D}$-module (i.e. it has finitely many non-zero cohomologies, and its cohomologies are locally finitely generated over $'\text{D}$).

Remark 3.2.2. The above definition of ULA can be thought of as the D-module version of the one given in [BG, Sect. 5] in the setting of étale sheaves.

3.2.3. Let $\text{D-mod}_\lambda(\mathcal{Z})^{\text{ULA}} \subset \text{D-mod}_\lambda(\mathcal{Z})$ be the full subcategory that consists of objects that are ULA with respect to f.

If $\mathcal{Z}$ is a scheme, then $\text{D-mod}_\lambda(\mathcal{Z})$ has finite cohomological dimension, and we have

$$\text{D-mod}_\lambda(\mathcal{Z})^{\text{ULA}} \subset \text{D-mod}_\lambda(\mathcal{Z})^c.$$

As in Sect. 2.3.5 we define the corresponding functor

$$\text{Ind}(\text{D-mod}_\lambda(\mathcal{Z})^{\text{ULA}}) \to \text{D-mod}_\lambda(\mathcal{Z})$$

and a fully faithful embedding

$$\text{Ind}^\wedge(\text{D-mod}_\lambda(\mathcal{Z})^{\text{ULA}}) \hookrightarrow \text{D-mod}_\lambda(\mathcal{Z}),$$

whose essential image consists of objects whose cohomologies are filtered colimits of objects from

$$\text{D-mod}_\lambda(\mathcal{Z})^{\heartsuit,\text{ULA}} := \text{D-mod}_\lambda(\mathcal{Z})^\heartsuit \cap \text{D-mod}_\lambda(\mathcal{Z})^{\text{ULA}}.$$

Remark 3.2.4. We note that as in Remark 2.3.6 it is not clear to the authors whether, when $\mathcal{Z}$ is a scheme, the inclusion $\text{Ind}(\text{D-mod}_\lambda(\mathcal{Z})^{\text{ULA}}) \subset \text{Ind}^\wedge(\text{D-mod}_\lambda(\mathcal{Z})^{\text{ULA}})$ is actually an equivalence.

3.2.5. We take X to be a smooth proper scheme acted on by a group H, and we take $\mathcal{Z} = X$ and $\mathcal{Y} = H\backslash X$.

We claim:

Theorem 3.2.6. *Let λ be a H-equivariant twisting on X.*

(a) *There exists a canonically defined natural transformation*

$$\text{Av}_!^H \to \text{Ps-Id}_{H\backslash X} \circ \text{Av}_*^H[2\dim(X)], \tag{3.1}$$

(where the left-hand side is a partially defined functor).

(b) *The map* (3.1) *is an isomorphism when evaluated on objects from* $\text{Ind}^\wedge(\text{D-mod}_\lambda(\mathcal{Z})^{\text{ULA}})$.

3.2.7. *Proof of Theorem 3.2.6, Step 0.* Consider the Cartesian diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\widetilde{\Delta}_{H\backslash X}} & X \times H\backslash X & \xrightarrow{\widetilde{p}_1} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times\text{id}} & & \downarrow{\scriptstyle f} \\
H\backslash X & \xrightarrow{\Delta_{H\backslash X}} & H\backslash X \times H\backslash X & \xrightarrow{p_1} & H\backslash X \\
& & \downarrow{\scriptstyle p_2} & & \\
& & H\backslash X. & &
\end{array}$$

For $\mathcal{F} \in \text{D-mod}_\lambda(X)$, the object $\text{Ps-Id}_{H\backslash X} \circ \text{Av}_*^H(\mathcal{F})$ identifies with

$$(p_2)_*\left((\Delta_{H\backslash X})_!(k_{H\backslash X}) \overset{!}{\otimes} (p_1^! \circ f_*(\mathcal{F}))\right) \simeq (p_2)_*\left((\Delta_{H\backslash X})_!(k_{H\backslash X}) \overset{!}{\otimes} ((f\times \text{id})_* \circ \widetilde{p}_1^!(\mathcal{F}))\right) \simeq$$
$$\simeq (p_2 \circ (f\times\text{id}))_*\left((f\times\text{id})^! \circ (\Delta_{H\backslash X})_!(k_{H\backslash X}) \overset{!}{\otimes} \widetilde{p}_1^!(\mathcal{F})\right) \overset{\text{smooth base change}}{\simeq}$$
$$\simeq (p_2 \circ (f\times\text{id}))_*\left((\widetilde{\Delta}_{H\backslash X})_! \circ f^!(k_{H\backslash X}) \overset{!}{\otimes} \widetilde{p}_1^!(\mathcal{F})\right) \simeq$$
$$\simeq (\widetilde{p}_2)_*\left((\widetilde{\Delta}_{H\backslash X})_!(k_X) \overset{!}{\otimes} \widetilde{p}_1^!(\mathcal{F})\right)[2\dim(H)],$$

where $\widetilde{p}_2 = p_2 \circ (f\times\text{id})$, and we have used the fact that

$$f^!(k_{H\backslash X}) \simeq k_X[2\dim(H)].$$

Note also that

$$\text{Av}_!^H(\mathcal{F}) \simeq f_!(\mathcal{F})[2\dim(H)] = (\widetilde{p}_2)_! \circ (\widetilde{\Delta}_{H\backslash X})_!(\mathcal{F})[2\dim(H)] =$$
$$= (\widetilde{p}_2)_! \circ (\widetilde{\Delta}_{H\backslash X})_! \circ (\widetilde{\Delta}_{H\backslash X})^* \circ \widetilde{p}_1^*(\mathcal{F})[2\dim(H)] \overset{\text{projection formula}}{\simeq}$$
$$\simeq (\widetilde{p}_2)_!\left((\widetilde{\Delta}_{H\backslash X})_!(k_X) \overset{*}{\otimes} \widetilde{p}_1^*(\mathcal{F})\right)[2\dim(H)].$$

3.2.8. *Proof of Theorem 3.2.6, Step 1.* Thus we are reduced to considering the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\widetilde{\Delta}_{H\backslash X}} & X \times H\backslash X & \xrightarrow{\widetilde{p}_1} & X \\ & & \Big\downarrow{\scriptstyle \widetilde{p}_2} & & \\ & & H\backslash X. & & \end{array}$$

Since X was assumed proper, the map $\widetilde{p}_2$ is proper, hence $(\widetilde{p}_2)_! \simeq (\widetilde{p}_2)_*$. The map $\widetilde{p}_1$ is smooth of relative dimension $\dim(X) - \dim(H)$, so

$$\widetilde{p}_1^*(\mathcal{F}) \simeq \widetilde{p}_1^!(\mathcal{F})[-2(\dim(X) - \dim(H))].$$

Hence, it suffices to construct a natural transformation

$$(\widetilde{\Delta}_{H\backslash X})_!(k_X) \overset{*}{\otimes} \widetilde{p}_1^!(\mathcal{F}) \to (\widetilde{\Delta}_{H\backslash X})_!(k_X) \overset{!}{\otimes} \widetilde{p}_1^!(\mathcal{F})[2(2\dim(X) - \dim(H))] \tag{3.2}$$

and show that it is an isomorphism if $\mathcal{F}$ is ULA with respect to f.

3.2.9. *Proof of Theorem 3.2.6, Step 2.* Consider the morphism

$$(f\times\text{id}) : X \times H\backslash X \to H\backslash X \times H\backslash X.$$

For any $\mathcal{F}_1 \in \text{D-mod}_{-\lambda,\lambda}(H\backslash X \times H\backslash X)$ and $\mathcal{F}_2 \in \text{D-mod}_{-\lambda,\lambda}(X \times H\backslash X)$ we have the canonical map (see [Ga1, Sect. 2.3]):

$$(f\times\text{id})^*(\mathcal{F}_1) \overset{*}{\otimes} \mathcal{F}_2 \to (f\times\text{id})^*(\mathcal{F}_1) \overset{!}{\otimes} \mathcal{F}_2[2(2\dim(X) - \dim(H))].$$

This morphism is an isomorphism for $\mathcal{F}_2$ that is ULA with respect to $f \times \text{id}$. Since all the functors involved are continuous, this remains true if $\mathcal{F}_2$ is a colimit of ULA objects. Further, since the functors involved have a bounded cohomological amplitude, the same is true if the cohomologies of $\mathcal{F}_2$ have this property.

We take $\mathcal{F}_2 = \widetilde{p}_1^!(\mathcal{F})$. The assumption that $\mathcal{F}$ has cohomologies that are ULA with respect to f implies that $\mathcal{F}_2$ has the same property with respect to $f \times \mathrm{id}$.

We take $\mathcal{F}_1 = (\Delta_{H\backslash X})_!(k_{H\backslash X})$. Then

$$(f \times \mathrm{id})^*(\mathcal{F}_1) \simeq (\widetilde{\Delta}_{H\backslash X})_!(k_X).$$

This yields the desired (iso)morphism (3.2).

□

Remark 3.2.10. In Theorem 3.2.6, we could have taken X to be a smooth and proper scheme, and $f : X \to \mathcal{Y}$ a smooth map (not necessarily a quotient map). Of course, Av_*^H would mean f_*, while $\mathrm{Av}_!^H$ would mean the (partially defined) left adjoint of f^*, i.e. $f_![2(\dim(X)-\dim(\mathcal{Y}))]$.

3.3. **A variant.**

3.3.1. Let $H' \subset H$ be a subgroup. Consider the forgetful functor

$$\mathbf{oblv}_{H/H'} : \text{D-mod}_\lambda(H\backslash X) \to \text{D-mod}_\lambda(H'\backslash X).$$

Its right adjoint $\mathrm{Av}_*^{H/H'}$ is given by *-direct image along

$$H'\backslash X \to H\backslash X,$$

and the partially defined left adjoint $\mathrm{Av}_!^{H/H'}$ is given by !-direct image along the above morphism, shifted by $2(\dim(H) - \dim(H'))$.

3.3.2. Let H'_{red} denote the reductive quotient of H'. We have:

Theorem 3.3.3.

(a) *There exists a canonically defined natural transformation*

$$\mathrm{Av}_!^{H/H'} \to \text{Ps-Id}_{H\backslash X} \circ \mathrm{Av}_*^{H/H'}[2\dim(X) - \dim(H'_{\mathrm{red}})], \tag{3.3}$$

(where the left-hand side is a partially defined functor).

(b) *The map* (3.3) *is an isomorphism when evaluated on objects from* $\mathrm{Ind}^\wedge(\text{D-mod}_\lambda(H'\backslash X)^{\mathrm{ULA}})$, *where the ULA condition is taken with respect to the projection* $f : H'\backslash X \to H\backslash X$.

3.3.4. The proof repeats verbatim that of Theorem 3.3.3 with the following modification:

Lemma 3.3.5. *Let X be a proper scheme acted on by a group H'. Let p denote the projection $H'\backslash X \to \mathrm{pt}$. Then we have a canonical isomorphism of functors*

$$p_* \simeq p_![-2\dim(H') + \dim(H'_{\mathrm{red}})].$$

Proof. Factor the map p as

$$H'\backslash X \to H'\backslash \mathrm{pt} \to \mathrm{pt}\,.$$

The first arrow is proper, and this reduces the assertion of the lemma to the case $X = \mathrm{pt}$. In the latter case, this is an easy verification.

□

Remark 3.3.6. In the above lemma, it is crucial that we understand p_* as the *renormalized* direct image functor of [DrGa1, Sect. 9.3] (i.e., the continuous extension of the restriction of the usual p_* to compact objects).

3.4. **First applications.** In this subsection we will take G to be a reductive group, X its flag variety, and $H = N$ the unipotent radical of a Borel.

3.4.1. We are going to show:

Theorem 3.4.2. *The functor* $\text{Ps-Id}_{N\backslash X}$ *induces a self-equivalence of* $\text{D-mod}_\lambda(N\backslash X)$*. Moreover,* $\text{Ps-Id}_{N\backslash X}[2\dim(X)]$ *identifies with the composite*

$$\text{D-mod}_\lambda(N\backslash X) \xrightarrow{\Upsilon^{-1}} \text{D-mod}_\lambda(N^-\backslash X) \xrightarrow{(\Upsilon^-)^{-1}} \text{D-mod}_\lambda(N\backslash X).$$

The proof is based on the following assertion, proved below:

Proposition 3.4.3. *Any object in the essential image of the forgetful functor*

$$\mathbf{oblv}_{N^-} : \text{D-mod}_\lambda(N^-\backslash X)^c \to \text{D-mod}_\lambda(X)$$

is ULA with respect to $X \to N\backslash X$*.*

Proof of Theorem 3.4.2. Since, by Proposition 1.4.2, the functors Υ and Υ^- are equivalences with

$$(\Upsilon^-)^{-1} \simeq \text{Av}_!^N \circ \mathbf{oblv}_{N^-},$$

it suffices to establish an isomorphism

$$\text{Ps-Id}_{N\backslash X} \circ \text{Av}_*^N \circ \mathbf{oblv}_{N^-}[2\dim(X)] \simeq \text{Av}_!^N \circ \mathbf{oblv}_{N^-}.$$

However, the latter follows from Proposition 3.4.3 and Theorem 3.2.6.

□

Remark 3.4.4. The fact that $\text{Ps-Id}_{N\backslash X}$ is an equivalence is also a special case of Theorem 3.1.4.

3.4.5. *Proof of Theorem 2.4.3.* We start with the (iso)morphism of Theorem 3.2.6 and compose it with $\text{Av}_*^{N^-} \circ \mathbf{oblv}_N$. We obtain a map

$$\text{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \text{Av}_!^N \to \text{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \text{Ps-Id}_{N\backslash X} \circ \text{Av}_*^N[2\dim(X)],$$

which is an isomorphism on objects whose cohomologies are ULA with respect to $X \to N\backslash X$.

We claim that the RHS, i.e.,

$$\text{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \text{Ps-Id}_{N\backslash X} \circ \text{Av}_*^N[2\dim(X)],$$

is canonically isomorphic to

$$\text{Av}_!^{N^-} \circ \mathbf{oblv}_N \circ \text{Av}_*^N.$$

In fact, we claim that there is a canonical isomorphism

$$\text{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \text{Ps-Id}_{N\backslash X}[2\dim(X)] \simeq \text{Av}_!^{N^-} \circ \mathbf{oblv}_N \tag{3.4}$$

as functors $\text{D-mod}_\lambda(N\backslash X) \to \text{D-mod}_\lambda(N^-\backslash X)$.

Since the functors $\text{Av}_*^N \circ \mathbf{oblv}_{N^-} = \Upsilon$ and $\text{Av}_!^{N^-} \circ \mathbf{oblv}_N$ are mutually inverse, the latter isomorphism is equivalent to

$$\Upsilon^- \circ \text{Ps-Id}_{N\backslash X}[2\dim(X)] \simeq \Upsilon^{-1},$$

while the latter is the assertion of Theorem 3.4.2.

□

3.4.6. As another application of Theorem 3.2.6, we will now prove:

Theorem 3.4.7. *let $\mathcal{F} \in \text{D-mod}(X)$ be (N^-, ψ)-equivariant, where $\psi : N^- \to \mathbb{G}_a$ is a non-degenerate character. Then there exists a functorial isomorphism (depending on a certain choice)*

$$\text{Av}_!^N(\mathcal{F}) \simeq \text{Av}_*^N(\mathcal{F})[2\dim(X)].$$

Proof. By a variant of Proposition 3.4.3 (where we replace equivariance by twisted equivariance), we have a canonical isomorphism

$$\text{Ps-Id}_{N\backslash X} \circ \text{Av}_*^N(\mathcal{F})[2\dim(X)] \simeq \text{Av}_!^N(\mathcal{F}).$$

By Theorem 3.4.2, the left-hand side can be further rewritten as

$$\text{Av}_!^N \circ w_0 \cdot \text{Av}_!^N \circ w_0 \cdot \text{Av}_*^N(\mathcal{F}),$$

where $w_0 \cdot -$ is the functor of translation by (a representative of) the longest element of the Weyl group.

We claim that there is a functorial isomorphism

$$\text{Av}_!^N \circ w_0 \cdot \text{Av}_*^N(\mathcal{F}) \simeq \text{Av}_*^N(\mathcal{F})[\dim(X)],$$

depending on a certain choice.

Namely, it is known that objects of the form $\text{Av}_*^N(\mathcal{F})$ for $\mathcal{F} \in \text{D-mod}(X)^{N^-,\psi}$ are *canonically* of the form

$$\mathcal{M} \underset{\text{End}(\Xi)}{\otimes} \Xi, \quad \mathcal{M} \in \text{End}(\Xi)^{\text{op}}\text{-mod},$$

for a choice of the "big projective" $\Xi \in (\text{D-mod}(X)^N)^\heartsuit$. (Indeed, such objects are right-orthogonal to the other indecomposable projectives in $(\text{D-mod}(X)^N)^\heartsuit$).

Denote

$$\Xi' := \text{Av}_!^N \circ w_0(\Xi)[-\dim(X)].$$

We obtain

$$\text{Av}_!^N \circ w_0 \cdot \text{Av}_*^N(\mathcal{F}) \simeq \text{Av}_!^N \circ w_0(\mathcal{M} \underset{\text{End}(\Xi)}{\otimes} \Xi) \simeq \mathcal{M} \underset{\text{End}(\Xi)}{\otimes} \Xi'[\dim(X)].$$

Now, it is also known that

$$\Xi' := \text{Av}_!^N \circ w_0(\Xi)[-\dim(X)]$$

is *non-canonically* isomorphic again to Ξ, in a way compatible with the action of $\text{End}(\Xi)$.

A choice of such an isomorphism gives rise to an identification

$$\mathcal{M} \underset{\text{End}(\Xi)}{\otimes} \Xi'[\dim(X)] \simeq \mathcal{M} \underset{\text{End}(\Xi)}{\otimes} \Xi[\dim(X)] \simeq \text{Av}_*^N(\mathcal{F}),$$

as desired.

□

3.5. **Transversality and the proof of Proposition 3.4.3.**

3.5.1. Let H_1 and H_2 be two groups acting on a smooth variety X. We shall say that these two actions are *transversal* if for every point $x \in X$, the orbits

$$H_1 \cdot x \subset X \supset H_2 \cdot x$$

are transversal at x.

Lemma 3.5.2. *The actions of H_1 and H_2 on X are transversal if and only if the map*

$$H_1 \times H_2 \times X \to X \times X, \quad (h_1, h_2, x) \mapsto (h_1 \cdot x, h_2 \cdot x) \tag{3.5}$$

is smooth.

3.5.3. *Example.* It is easy to see that for X being the flag variety of G and $H_1 = N$, and $H_2 = N^-$, then the corresponding actions are transversal.

3.5.4. We have the following generalization of Proposition 3.4.3:

Proposition 3.5.5. *Let the actions of H_1 and H_2 on X be transversal. Then for any (twisted) D-module $\mathcal{F}$ on $H_1\backslash X$, if the (twisted) D-module $\mathbf{oblv}_{H_1}(\mathcal{F})$ on X is coherent, it is ULA with respect to the projection $X \to H_2\backslash X$.*

Proof. Consider the Cartesian diagram

$$\begin{array}{ccccc}
H_1 \times H_2 \times X & \longrightarrow & H_1 \times X & \xrightarrow{\mathrm{act}_1} & X \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle f_1} \\
H_2 \times X & \xrightarrow{\mathrm{pr}_2} & X & \xrightarrow{f_1} & H_1\backslash X \\
\downarrow{\scriptstyle \mathrm{act}_2} & & \downarrow{\scriptstyle f_2} & & \\
X & \xrightarrow{f_2} & H_2\backslash X. & &
\end{array}$$

The property of being ULA is smooth-local with respect to the base, so it is enough to show that the pullback of $\mathbf{oblv}_{H_1}(\mathcal{F})$ to $H_2 \times X$ is ULA with respect to the map act_2.

The ULA property is also smooth-local with respect to the source. Hence, it suffices to show that the further pullback of $\mathcal{F}$ to $H_1 \times H_2 \times X$ is ULA with respect to the composite left vertical arrow.

However, the latter arrow factors as

$$H_1 \times H_2 \times X \to X \times X \xrightarrow{p_2} X \to H_1\backslash X.$$

Since the map $H_1 \times H_2 \times X \to X \times X$ is smooth, it suffices to show that the pullback of $\mathcal{F}' := \mathbf{oblv}_{H_1}(\mathcal{F})$ along $X \times X \xrightarrow{p_2} X$ is ULA with respect to p_1. However, this is true for any coherent object $\mathcal{F}'$.

□

4. The case of a symmetric pair

4.1. **Adjusting the previous framework.**

4.1.1. In this section we will take G equipped with an involution θ; set $K := G^\theta$. Let P be a minimal parabolic compatible with θ (i.e., minimal among parabolics for which $\theta(P)$ is opposite to P); denote $P^- := \theta(P)$.

We change the notations, and in this section denote by N (resp., N^-) the unipotent radical of P (resp., P^-) and

$$M_K := P \cap P^- \cap K.$$

We have the following basic assertion:

Lemma 4.1.2.
(a) *The groups K, $M_K \cdot N$ and $M_K \cdot N^-$ act on X with finitely many orbits.*
(b) *The actions of $M_K \cdot N$ and K on X, as well as the actions of $M_K \cdot N$ and $M_K \cdot N^-$ on X, are transversal.*

4.1.3. The discussion in Sect. 2 needs to be modified as follows: instead of the functor

$$\mathbf{oblv}_N : \mathcal{C}^N \to \mathcal{C}$$

and its right and (partially defined) left adjoints Av_*^N and $\mathrm{Av}_!^N$, we consider the analogous functor

$$\mathbf{oblv}_{M_K \cdot N/M_K} : \mathcal{C}^{M_K \cdot N} \to \mathcal{C}^{M_K}$$

and its right and (partially defined) left adjoints $\mathrm{Av}_*^{M_K \cdot N/M_K}$ and $\mathrm{Av}_!^{M_K \cdot N/M_K}$. However, by abuse of notation, we will still write $\mathbf{oblv}_N$ instead of $\mathbf{oblv}_{M_K \cdot N/M_K}$, etc.

In Proposition 2.1.4 we take Υ to be the functor

$$\mathcal{C}^{M_K \cdot N^-} \to \mathcal{C}^{M_K \cdot N}$$

given by $\mathrm{Av}_*^N \circ \mathbf{oblv}_{N^-}$.

The key fact is that this functor is an equivalence for $\mathcal{C} = \text{D-mod}_\lambda(X)$ (with the same proof), and hence also for $\mathfrak{g}\text{-mod}_\chi$. It's inverse is again given by $\Upsilon^{-1} = \mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N$.

The assertions of Theorems 2.4.2 and 2.4.3 should be modified as follows. Let

$$(\mathfrak{g}\text{-mod}_\chi^{M_K})^{\mathfrak{n}\text{-f.g.}} \subset \mathfrak{g}\text{-mod}_\chi^{M_K}$$

be the full subcategory equal to the preimage of $\mathfrak{g}\text{-mod}_\chi^{\mathfrak{n}\text{-f.g.}} \subset \mathfrak{g}\text{-mod}_\chi$ under the forgetful functor $\mathbf{oblv}_{M_K} : \mathfrak{g}\text{-mod}_\chi^{M_K} \to \mathfrak{g}\text{-mod}_\chi$.

Theorem 4.1.4.
(a) *There is a canonically defined natural transformation of functors*

$$\mathrm{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_!^N \to \mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N \simeq J,$$

considered as functors $\mathfrak{g}\text{-mod}_\chi^{M_K} \to \mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-}$.
(b) *The above natural transformation is an isomorphism when evaluated on objects from $\mathrm{Ind}^\wedge((\mathfrak{g}\text{-mod}_\chi^{M_K})^{\mathfrak{n}\text{-f.g.}})$.*

Theorem 4.1.5.
(a) *There is a canonically defined natural transformation of functors*

$$\mathrm{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_!^N \to \mathrm{Av}_!^{N^-} \circ \mathbf{oblv}_N \circ \mathrm{Av}_*^N \simeq J,$$

considered as functors

$$\text{D-mod}_\lambda(M_K\backslash X) \to \text{D-mod}_\lambda(M_K \cdot N^-\backslash X).$$

(b) *The above natural transformation is an isomorphism on* $\mathrm{Ind}^\wedge(\text{D-mod}_\lambda(M_K\backslash X)^{\mathrm{ULA}})$*, where the ULA condition is taken with respect to the projection* $M_K\backslash X \to M_K N\backslash X$.

4.1.6. Theorem 4.1.5 is proved in the same manner as Theorem 2.4.3, using Proposition 3.3.3 (with $H = M_K \cdot N$ and $H' = M_K$) and the following analog of Theorem 3.4.2:

Theorem 4.1.7. *The functor* $\text{Ps-Id}_{M_K\cdot N\backslash X}$ *is a self-equivalence of* $\text{D-mod}_\lambda(M_K\cdot N\backslash X)$*. Moreover,* $\text{Ps-Id}_{M_K\cdot N\backslash X}[2\dim(X)-\dim(M_K)]$ *identifies with the composite*

$$\text{D-mod}_\lambda(M_K\cdot N\backslash X) \overset{\Upsilon^{-1}}{\longrightarrow} \text{D-mod}_\lambda(M_K\cdot N^-\backslash X) \overset{(\Upsilon^-)^{-1}}{\longrightarrow} \text{D-mod}_\lambda(M_K\cdot N\backslash X).$$

Theorem 4.1.4(a) is proved as Theorem 2.4.2(a). Theorem 4.1.4(b) is proved using the following version of Proposition 2.4.5(a):

Proposition 4.1.8.
(a) *The functor* Loc *sends*

$$\mathrm{Ind}^\wedge((\mathfrak{g}\text{-mod}_\chi^{M_K})^{\mathfrak{n}\text{-f.g.}}) \to \mathrm{Ind}^\wedge(\text{D-mod}_\lambda(M_K\backslash X)^{\mathrm{ULA}}).$$

(b) *The functor* Γ *sends*

$$\text{D-mod}_\lambda(M_K\backslash X)^{\mathrm{ULA}} \to (\mathfrak{g}\text{-mod}_\chi^{M_K})^{\mathfrak{n}\text{-f.g.}}$$

and

$$\mathrm{Ind}^\wedge(\text{D-mod}_\lambda(M_K\backslash X)^{\mathrm{ULA}}) \to \mathrm{Ind}^\wedge((\mathfrak{g}\text{-mod}_\chi^{M_K})^{\mathfrak{n}\text{-f.g.}}).$$

Proof. For point (a), it is enough to see that Loc sends an object $\mathcal{F}$ in $(\mathfrak{g}\text{-mod}_\chi^{M_K})^{\mathfrak{n}\text{-f.g.}}$ to an object whose cohomologies are ULA with respect to $M_K\backslash X \to M_K\cdot N\backslash X$.

Being ULA is smooth-local on the source, hence it is enough to see that the cohomologies of $\mathbf{oblv}_{M_K}\circ \mathrm{Loc}(\mathcal{F})$ are ULA with respect to $X \to M_K\cdot N\backslash X$.

For this, it is enough to see that the cohomologies of $\mathbf{oblv}_{M_K}\circ\mathrm{Loc}(\mathcal{F}) \simeq \mathrm{Loc}\circ\mathbf{oblv}_{M_K}(\mathcal{F})$ are ULA with respect to $X \to N\backslash X$, which is the case by Proposition 2.4.5.

Point (b) is proved similarly.

□

4.2. The Casselman-Jacquet functor for $(\mathfrak{g}, K)$-modules.

4.2.1. In this subsection we will prove:

Theorem 4.2.2.
(a) *The functor*

$$\mathfrak{g}\text{-mod}_\chi^K \overset{\mathbf{oblv}_{K/M_K}}{\longrightarrow} \mathfrak{g}\text{-mod}_\chi^{M_K} \overset{J}{\to} \mathfrak{g}\text{-mod}_\chi^{M_K\cdot N^-}$$

identifies canonically with

$$\mathrm{Av}_*^{N^-}\circ\mathbf{oblv}_N\circ\mathrm{Av}_!^N\circ\mathbf{oblv}_{K/M_K}$$

and is t-exact.
(b) *The functor*

$$\text{D-mod}_\lambda(X)^K \overset{\mathbf{oblv}_{K/M_K}}{\longrightarrow} \text{D-mod}_\lambda(X)^{M_K} \overset{J}{\to} \text{D-mod}_\lambda(X)^{M_K\cdot N^-}$$

identifies canonically with

$$\mathrm{Av}_*^{N^-}\circ\mathbf{oblv}_N\circ\mathrm{Av}_!^N\circ\mathbf{oblv}_{K/M_K}$$

and is t-exact.

4.2.3. We will first prove:

Proposition 4.2.4.
(a) *The functor* $\mathbf{oblv}_{K/M_K}$ *maps* $(\mathfrak{g}\text{-mod}_\chi^K)^c$ *to* $(\mathfrak{g}\text{-mod}_\chi^{M_K})^{\mathfrak{n}\text{-f.g.}}$.
(b) *The functor* $\mathbf{oblv}_{K/M_K}$ *maps* $\text{D-mod}_\lambda(K\backslash X)^c$ *to objects in* $\text{D-mod}_\lambda(M_K\backslash X)^{\text{ULA}}$*, where the ULA condition is taken with respect to* $M_K\backslash X \to M_K N\backslash X$.
(c) *The functor* $\mathbf{oblv}_K$ *maps* $\text{D-mod}_\lambda(K\backslash X)^c$ *to objects in* $\text{D-mod}_\lambda(X)$ *that are holonomic.*

Proof. Point (a) is well-known: it is enough to show that objects of the form

$$(U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho) \underset{Z(\mathfrak{g})}{\otimes} k, \quad \rho \in \text{Rep}(K)^{\text{f.d.}}$$

where $Z(\mathfrak{g}) \to k$ is given by χ, belong to $\mathfrak{g}\text{-mod}_\chi^{\mathfrak{n}\text{-f.g.}}$. For that it suffices to show that $U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho$ is finitely generated over $U(\mathfrak{n}) \otimes Z(\mathfrak{g})$, and this follows from the corresponding assertion at the associated graded level.

To show point (b), since the property of being ULA is smooth-local on the source, it is enough to show that $\mathbf{oblv}_K$ maps $\text{D-mod}_\lambda(K\backslash X)^c$ to objects in $\text{D-mod}_\lambda(X)$ that are ULA with respect to $X \to M_K \cdot N\backslash X$. This follows from Proposition 3.5.5 and Lemma 4.1.2(b).

Alternative proof: change the twisting λ by an integral amount to make Γ an equivalence. Then point (b) follows from point (a), combined with Proposition 4.1.8.

Finally, point (c) follows from the fact that the group K has finitely many orbits on X.

□

4.3. **Proof of Theorem 4.2.2.**

4.3.1. First, we note that the fact that $J \circ \mathbf{oblv}_{K/M_K}$ is isomorphic to

$$\text{Av}_*^{N^-} \circ \mathbf{oblv}_N \circ \text{Av}_!^N \circ \mathbf{oblv}_{K/M_K}$$

follows from Theorem 4.1.4(b) and Proposition 4.2.4(a) (resp., Theorem 4.1.5(b) and Proposition 4.2.4(b)).

To prove the t-exactness, we proceed as follows:

4.3.2. *Step 1.* First, we claim that the functor $J \circ \mathbf{oblv}_{K/M_K}$ is left t-exact for $\text{D-mod}_\lambda(K\backslash X)$.

This statement is insensitive to changing λ by an integral twisting. Hence, we can assume that λ is such that Γ is an equivalence. Since Γ is t-exact, the assertion now follows from Propositions 4.2.4(a) and 2.3.8.

4.3.3. *Step 2.* We now claim that the functor $J \circ \mathbf{oblv}_{K/M_K}$ is right t-exact, still for $\text{D-mod}_\lambda(K\backslash X)$.

Indeed, this follows by Verdier duality from the previous step, using Proposition 4.2.4(c).

4.3.4. *Step 3.* It remains to show that $J \circ \mathbf{oblv}_{K/M_K}$ is right t-exact on $\mathfrak{g}\text{-mod}_\chi^K$. Since λ was chosen so that Γ is t-exact, the functor Loc is right t-exact.

We have:

$$J \circ \mathbf{oblv}_{K/M_K} \simeq \Gamma \circ J \circ \mathbf{oblv}_{K/M_K} \circ \text{Loc},$$

where the right-hand side is a composition of t-exact and right t-exact functors.

□[Theorem 2.4.2]

4.4. **The "2nd adjointness" conjecture.** In the previous subsection we studied the functors

$$J \circ \mathbf{oblv}_{K/M_K} : \mathfrak{g}\text{-mod}_\chi^K \to \mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-}$$

and

$$J \circ \mathbf{oblv}_{K/M_K} : \text{D-mod}_\lambda(K\backslash X) \to \text{D-mod}_\lambda(M_K \cdot N^- \backslash X).$$

In this subsection we will study functors in the opposite direction, namely,

$$\text{Av}_!^{K/M_K} \text{ and } \text{Av}_*^{K/M_K}$$

that go from $\mathfrak{g}\text{-mod}_\chi^{M_K \cdot N}$ (or $\mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-}$) to $\mathfrak{g}\text{-mod}_\chi^K$ and from $\text{D-mod}_\lambda(M_K \cdot N\backslash X)$ (or $\text{D-mod}_\lambda(M_K \cdot N^-\backslash X)$) to $\text{D-mod}_\lambda(K\backslash X)$, respectively.

4.4.1. First, we note the following consequence of Lemma 4.1.2, Proposition 3.5.5 and Theorem 3.3.3:

Corollary 4.4.2. *We have a canonical isomorphism*

$$\text{Av}_!^{K/M_K} \simeq \text{Ps-Id}_{K\backslash X} \circ \text{Av}_*^{K/M_K}[2\dim(X) - \dim(M_K)]$$

as functors $\text{D-mod}_\lambda(M_K \cdot N\backslash X) \to \text{D-mod}_\lambda(K\backslash X)$.

Combining with Proposition 1.2.6, we obtain:

Corollary 4.4.3. *The partially defined functor* $\text{Av}_!^{K/M_K}$ *is defined on the essential image of*

$$\mathbf{oblv}_N : \mathfrak{g}\text{-mod}_\chi^{M_K \cdot N} \to \mathfrak{g}\text{-mod}_\chi^K.$$

4.4.4. For a group H, let us write

$$\mathfrak{l}_H = \Lambda^{\dim(H)}(\mathfrak{h})[\dim H].$$

For a pair of groups $H_1 \subset H_2$, set $\mathfrak{l}_{H_2/H_1} = \mathfrak{l}_{H_2} \otimes \mathfrak{l}_{H_1}^{-1}$.

The same symbols will also stand for the functors of tensoring by those lines. Set $\mathcal{C} = \mathfrak{g}\text{-mod}_\chi$ or $\mathcal{C} = \text{D-mod}_\lambda(X)$. We propose the following conjecture:

Conjecture 4.4.5. *There exists a canonical isomorphism*

$$\text{Av}_*^{K/M_K} \simeq \mathfrak{l}_{K/M_K}^{-1} \circ \text{Av}_!^{K/M_K} \circ \Upsilon$$

as functors

$$\mathcal{C}^{M_K \cdot N^-} \to \mathcal{C}^K.$$

In light of Corollary 4.4.2, in the case of $\mathcal{C} = \text{D-mod}_\lambda(X)$, we can reformulate Conjecture 4.4.5 as follows:

Conjecture 4.4.6. *The following diagram of functors commutes:*

$$\begin{array}{ccc} \mathcal{C}^K & \xleftarrow{\text{Av}_*^{K/M_K}} & \mathcal{C}^{M_K \cdot N} \\ {\scriptstyle \mathfrak{l}_{K/M_K}^{-1} \circ \text{Ps-Id}_{K\backslash X}[2\dim(X) - \dim(M_K)]} \big\downarrow & & \big\downarrow {\scriptstyle \Upsilon^{-1}} \\ \mathcal{C}^K & \xleftarrow{\text{Av}_*^{K/M_K}} & \mathcal{C}^{M_K \cdot N^-}. \end{array}$$

Note that Conjecture 4.4.6 can be thought of as a sort of functional equation for the functor Av^{K/M_K}, cf. [Ga3].

4.4.7. Note that we have two adjoint pairs of functors

$$\mathrm{Av}_!^{N^-} : \mathcal{C}^K \rightleftarrows \mathcal{C}^{M_K \cdot N^-} : \mathrm{Av}_*^{K/M_K}$$

and

$$\mathrm{Av}_!^{K/M_K} : \mathcal{C}^{M_K \cdot N^-} \rightleftarrows \mathcal{C}^K : \mathrm{Av}_*^{N^-}.$$

We obtain that Conjecture 4.4.5 is equivalent to the following one:

Conjecture 4.4.8. *The* right *adjoint functor to*

$$\mathrm{Av}_*^{K/M_K} \circ \mathbf{oblv}_{N^-} : \mathcal{C}^{M_K \cdot N^-} \to \mathcal{C}^K$$

is given by

$$J \circ \mathbf{oblv}_{K/M_K} \circ \mathfrak{l}_{K/M_K}.$$

4.4.9. Combining with Theorem 4.1.4, we further obtain that Conjecture 4.4.8 is equivalent to:

Conjecture 4.4.10. *The* right *adjoint functor to*

$$\mathrm{Av}_*^{K/M_K} \circ \mathbf{oblv}_{N^-} : \mathcal{C}^{M_K \cdot N^-} \to \mathcal{C}^K$$

is given by

$$\Upsilon^- \circ \mathrm{Av}_!^N \circ \mathbf{oblv}_{K/M_K} \circ \mathfrak{l}_{K/M_K}.$$

4.4.11. We regard Conjecture 4.4.10 as an analog of Bernstein's *2nd adjointness theorem* for $\mathfrak{p}$-adic groups.

Recall that the latter says that in addition to the tautological adjunction (the 1st adjointness)

$$r : \mathbf{G}\text{-mod} \rightleftarrows \mathbf{M}\text{-mod} : i$$

(here we denote by G a $\mathfrak{p}$-adic group, by M its Levi subgroup, by i the normalized parabolic induction functor, and by r the Jacquet functor), we also have an adjunction

$$i : \mathbf{M}\text{-mod} \rightleftarrows \mathbf{G}\text{-mod} : \overline{r},$$

where $\overline{r}$ is the Jacquet functor with respect to the opposite parabolic.

Here is the table of analogies/points of difference between $\mathfrak{p}$-adic groups and symmetric pairs:

- The analog of $\mathbf{G}$-mod is the category $\mathfrak{g}\text{-mod}_\chi^K$;
- The analog of $\mathbf{M}$-mod is *not* $\mathfrak{m}\text{-mod}_\chi^{M_K}$, but rather $\mathfrak{g}\text{-mod}_\chi^{M_K \cdot N}$ (or $\mathfrak{g}\text{-mod}_\chi^{M_K \cdot N^-}$); note that this category explicitly depends on the choice of the parabolic or its opposite.
- The analog of the tautological identification $\mathbf{M}$-mod $=$ $\mathbf{M}$-mod is the intertwining functor Υ;
- The analog of the induction functor i is Av_*^{K/M_K};
- The analog of the Jacquet functor r (resp., $\overline{r}$) is $\mathrm{Av}_!^N$ (resp., $\mathrm{Av}_!^{N^-}$).

With these analogies, Conjecture 4.4.10 says that the right adjoint to the induction functor Av_*^{K/M_K} is isomorphic to the the Jacquet functor $\mathrm{Av}_!^N$, up to replacing N by N^-, inserting the intertwining functor, and a twist.

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