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Dynamic Leveraging–Deleveraging Games

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Abstract. We introduce a new mechanism for leverage dynamics, based on a multiperiod game of lenders with differentiated beliefs about the firm's fundamental returns. The game features strategic substitutability for low existing leverage and strategic complementarity for high existing leverage. The resulting leverage process exhibits a mean-reverting regime around a long-run level, as long as it stays below an instability level. Above the instability level, leverage becomes explosive. We validate our model empirically using aggregate returns of financial firms over the 10-year period 2001–2010. Our model is consistent with the leveraging/deleveraging of this period and with the 2008 collapse in short-term debt.

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1. Introduction

The last financial crisis provides prominent examples of excessive bank and fund leveraging using short-term debt, which ended with collapse (see, e.g., Brunnermeier 2009, Duffie 2010b, Acharya et al. 2011, Financial Crisis Inquiry Commission 2011, and Gorton and Metrick 2012) and led to systemic crises. A less recent example is the Long-Term Capital Management (LTCM) demise (see, e.g., Rubin et al. 1999). We provide a new mechanism of leverage dynamics. It is the lenders' aggregate decision that determines how much debt, and consequently leverage, the firm can attain based on its assets. Our goal is to endogenize the dynamics of the leverage. The same game of the lenders—the leveraging/deleveraging game—played period after period generates the entire leverage dynamics. We seek to characterize the level where lenders push the firm into a deleveraging spiral.

As model input, we have the fundamental trajectory of log-returns of the firm's investment strategy, which is fixed. Every potential lender has a belief about the mean log-return in the next period. The cross-distribution of beliefs is common knowledge, but the real-world drift is not known. Potential lenders take synchronous decisions whether to finance or not the borrower's asset (refinance in the case of existing lenders). The value of the option to finance varies with lenders' beliefs. A marginal lender (as a function of the existing leverage) is one for whom the option to lend equates the outside option (which is riskless cash with zero return).

The firm expands (contracts) its asset using the net inflow (outflow) of debt, and there may be transaction costs. We can think of a large leveraged hedge fund (or dealer bank) that pursues a core strategy. The heterogeneous lenders are the only lenders in the model, and they decide the inflows and outflows of debt. Given these flows, the management expands or reduces the asset position. Therefore, our model applies when the constraints on leverage imposed by the lenders are stricter than the constraints imposed by the management. The example of LTCM, where the management pursued their core strategy until the fall of the fund, naturally comes to mind. The game continues until either maturity, when the firm is liquidated, or until default, which occurs if the entire asset cannot cover a net debt outflow. In case of default, the asset is liquidated and proportionally distributed to outstanding lenders.

Our game-theoretic framework originates in Krishenik et al. (2015), who study a rollover game for the debt provision to a sovereign borrower. They provide an alternative to the global games setting of Carlsson and Van Damme (1993) and Morris and Shin (2001). It is more amenable to data and more suitable for a dynamic model: Heterogenous beliefs about the future evolution of the fundamentals replace the noisy observations of the fundamentals. Here, we keep the advantages of this framework, while we extend significantly the game.

Our leveraging–deleveraging game features complex dependencies that are novel in the literature. The value of lenders' options to finance (or refinance) the firm's asset changes with existing leverage. When the existing leverage is low, the payoffs of pessimistic lenders decrease when others lend as well. There are two sources for this strategic substitutability, totally absent in previous literature. First, under pessimists' belief, more debt increases default probability because the firm would become more leveraged for the next period and they expect low asset returns. Second, in the case of default, the firm's assets would be shared among more lenders, so the pessimists' expected recovery rates decrease. (The optimists are indifferent to the other lenders: Under their belief, they hold a default-free contract.)

In turn, when the existing leverage is high, then the firm cannot survive unless a large fraction of the current lenders roll over. Only in this regime the game behaves like a rollover game and features strategic complementarity similarly to Goldstein and Pauzner (2005) and Krishenik et al. (2015): Payoffs increase sharply with raised debt (up to a point where debt is sufficient for the firm to remain liquid). The alternation of leveraging and deleveraging phases and the roles played by pessimists and optimists are reminiscent of a leverage cycle in the collateral equilibrium model of Geanakoplos (2010); see also Fostel and Geanakoplos (2014) and the references therein.

1.1. Model Contributions and Financial Insights

Our contribution is to endogenize the dynamics of the leverage of a large borrower and to fully determine the regimes of this leverage process: We show that leverage is mean reverting around a long-run level and explosive above an instability level. The intuition comes from the changing nature of the lenders' game from strategic substitutability to one-sided strategic complementarity: When leverage is below the instability level, the firm is not in danger of default, and the strategic substitutability in the payoff structure acts as a counterbalance on leverage, which is pushed down to the long-run level. If leverage reaches above the instability level, then it becomes explosive: The strategic complementarity leads to spiraling effects that end in default. Default technically happens when leverage hits a ceiling. Determining the regimes of the leverage yields early warning indicators of default: When the leverage deviates from the long-run level and reaches above the instability level, then, in expectation, it will reach the debt ceiling because of the regime change from mean-reverting to explosive. We can moreover quantify sustainable debt levels: A wide mean-reverting regime around the long-run level is tantamount to stable short-term debt.

We expand the dynamic debt-run literature (He and Xiong 2012, Liang et al. 2014, Krishenik et al. 2015, Liang et al. 2015, Carmona et al. 2017, He et al. 2017). Our game has three main features: It is dynamic; lenders' decisions are synchronous; and the lenders' game drives both leveraging and deleveraging. Equally important, we have endogenous recovery rates in default, and these drive the strategic complementarity/substitutability profile of the game. With the exception of Krishenik et al. (2015) and Liang et al. (2014, 2015), who can allow for synchronous decisions in the global games setting, most other works use a staggered debt structure and thereby insulate their models from the multiplicity of equilibria. However, it is valuable to allow for synchronous decisions because short-term debt is the outcome of a maturity race; see Brunnermeier and Oehmke (2013). Likely, at the end of this race, many of the lenders will have a similar contract. It is anecdotal that in the recent crisis, large banks saw tens of billions of liquidity withdrawn in a matter of days, so there were large-scale synchronous decisions.

Our analysis exhibits two levels that are critical to the understanding of leverage stability: the long-run level and the instability level. Starting from zero debt, leverage reaches its long-run level, and there is a leveraging phase as the firm expands its asset position according to the provided debt. After this initial leveraging phase, leverage is stable for a while as it mean-reverts: Lenders adjust leverage (as outcome of their game) in response to asset returns. However, above the instability level, it is no longer possible to mean-revert: Existing leverage is too high, and a deleveraging spiral ensues with the expectation to end in default (leverage is explosive and expected to reach the debt ceiling).

By comparison, prior works find a "debt-run barrier" (Liang et al. 2015) or the "run threshold" (He and Xiong 2012): If the fundamental process touches these barriers or thresholds, then a debt run ensues. Touching the barrier is nonanticipative in these previous works. The "ceiling" in our model corresponds to these debt-run barriers.

Our analysis thus goes far beyond the default characterization in terms of the barrier (or ceiling, in our case), as we fully characterize the regimes of leverage and we quantify the stability of the leverage. These do not have a correspondent in the literature. Default becomes anticipative in our model: Leverage is expected to reach the ceiling as soon as it deviates from the long-run level and crosses the instability level: If leverage ratio tomorrow is an endogenous and, as we prove, convex function of the debt-to-asset ratio today, then its largest fixed point is the start of the spiral of lender withdrawals in which the leverage ratio switches from being mean-reverting to explosive. Reaching the instability level is thus an early indicator of default, and the collapse in debt capacity occurs over the time lag in which the debt-to-asset ratio process increases from the long-run level to the instability level (and further to the ceiling).

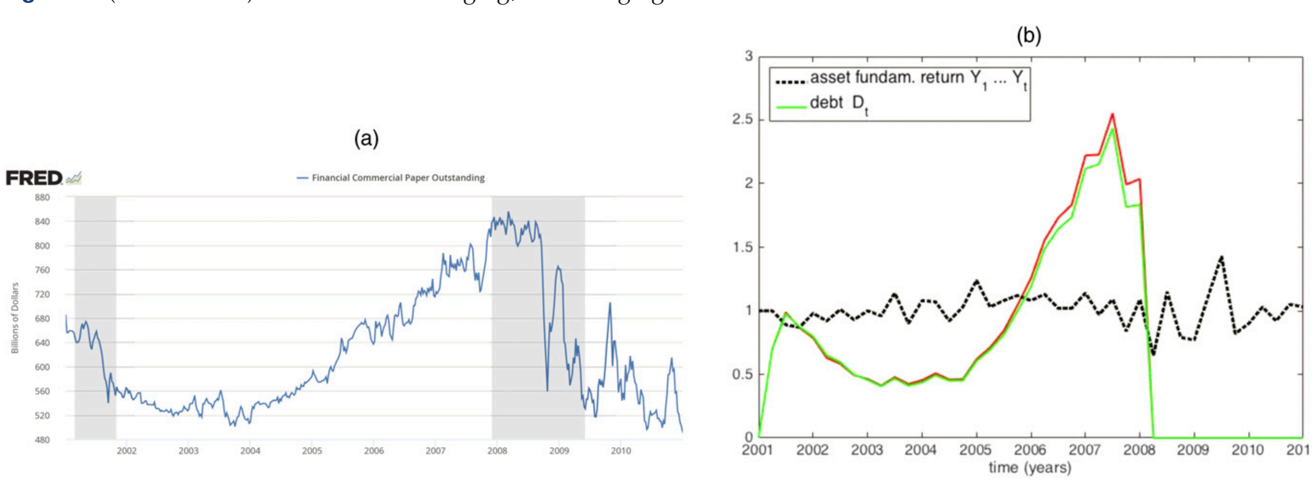
Our model provides a mechanism by which *pessimistic lenders transform a sequence of mild negative returns into large liquidity shocks*. When asset returns are high for a while and the firm builds equity, pessimistic lenders are drawn in, and the firm attains high leverage. Convexity effects are significant when leverage is high. Because of these effects, a sequence of mild negative returns is followed by a rapid increase of the marginal belief. Pessimists are the fastest to withdraw because the value of their put option (and expected default costs) increases most under negative returns. This leads to spiraling effects and a predictable collapse in debt capacity. The spiraling effects also depend on our belief distribution assumption: After a sequence of withdrawals, only the highly optimistic lenders stay, and they do not concentrate a large fraction of the capital mass.

Spiraling effects happen when the dependence of the marginal belief on leverage is stronger (higher convexity). When this dependence is weaker (lower convexity), then the firm will be able to recover from a sequence of negative returns. Deleveraging in not pronounced in this case: The marginal belief does not change too much, and the outflows are sufficiently small. Lower convexity comes along with lower leverage and a higher equity ratio. Combined, these two effects allow the firm to absorb the deleveraging. These weaker and stronger convexity effects concur to the existence of a mean-revering and an explosive leverage, which we will demonstrate in Section 4.

We provide a powerful model that can be calibrated to real data. We provide a "proof of concept" on how to use the model on real data as we use the financial commercial paper as a case study of short-term debt. The input is the aggregate fundamental returns of the securities dealers in the United States over a 10-year period. Over this period, the number of dealers is stationary, and we think of the aggregate as a "representative" dealer. We also validate the model using the single-firm example of Morgan Stanley. Our model is consistent with the leveraging/deleveraging witnessed over this period, and in particular with the financial commercial paper collapse of 2008; see Figure 1. We obtain similar results when using a constant spread equal to the average spread over the period 2001–2005. The difference in the model-predicted debt using the actual spreads and the constant spread is small during the 2007–2008 period, in which the actual spreads strongly increased. This implies that the (short) put option value for the marginal lender increases very fast after a sequence of negative returns (due to convexity effects). The implication is that increasing spreads during the deleveraging spiral has little effect on preventing the debt collapse.

Technical Contribution. Our main technical contribution is to provide a refinement that leads to a unique equilibrium. We are building on the game-theoretic foundations established in Krishenik et al. (2015). The proof technique there relies on eliminating weakly dominated strategies. This is not enough to obtain uniqueness here, but we can use those results to eliminate the multiplicity of equilibria in which the firm defaults. There remain two cutoff equilibria in which the firm survives: The presence of default recovery rates leads to nonmonotonous payoffs for a marginal lender with respect to the total debt. When the debt is high, the marginal lender is pessimistic, and her payoff decreases with debt. When the debt is low, the marginal lender is optimistic, and her payoff increases with debt. There are two possible marginal lenders that have the cutoff expected return zero. The technical innovation in this paper is the concept of strongly ϵ-coalition proof equilibria, and the cutoff

Figure 1. (Color online) Real-World Leveraging/Deleveraging vs. Model Predictions



Source. (a) Board of Governors of the Federal Reserve System (fred.stlouisfed.org).
Notes. (a) Financial outstanding commercial paper (data not used). (b) (Model) Debt dynamics with real-world spreads vs. constant spread equal to the average spread over 2001–2005. Real-world returns of FINRA members used as model input are plotted in the dashed line.

equilibrium with the lowest debt does not survive this refinement: If a small coalition of lenders just below the cutoff could coordinate to invest, then debt would increase, and the entire coalition (along with the marginal lender) would see increasing payoffs.

One assumption that leads to the elimination of multiple equilibria is that interest rates are constant. Most literature features constant interest rates, with the exception of Jarrow et al. (2018), where the firm optimizes the interest rate under uncertainty about the lenders' equilibrium, and multiplicity of equilibria cannot be eliminated. Spreads for short-term debt such as commercial paper did not fluctuate a lot before the collapse of the commercial paper outstanding notional (see, e.g., Kacperczyk and Schnabl 2010). The approximation by constant interest rates is thus valid in the mean-reverting regime of our model, when short-term debt provision is stable. More importantly, in light of Figure 1, we do not expect that endogenizing interest rates would have any effect on the leverage regimes. Once a deleveraging spiral starts, payoffs of pessimistic lenders have little sensitivity to interest rates, and, indeed, during the crisis, spread increases did not prevent the debt collapse.

Our paper is also related to the literature on optimal capital structure and endogenous bankruptcy (see, e.g., Leland and Toft 1996). There, it is the firm that can choose its amount of debt. Here, it is lenders, and the firm's dynamic optimization problem in the optimal capital structure literature is replaced here by the lenders' dynamic game. Our game can be thought of as a "game of timing," as investigated in Carmona et al. (2017). They are focused on convergence results for games with finite numbers of players in a game of timing with strategic complementarities. Here, our setup is directly with a continuum of players, and we are focused on refinements that lead to interpretable "barriers" when the game has both strategic complementarities and substitutabilities.

The paper is organized as follows. Section 2 presents the model for the firm and introduces the lenders' game. Section 3 analyzes the games' equilibria and gives the uniqueness result. Section 4 contains the results on the regimes of the endogenous leverage process. Section 5 validates the model empirically by using as input the real-world fundamental returns, and, finally, Section 6 illustrates the dynamic behavior of our model under a variety of parameters.

2. The Model

We consider a firm that funds itself through short-term debt provided by a continuum of lenders. Our model is applicable to a variety of debt maturities; for example, 1 day, 1 month, 1 quarter. What makes the debt short-term is that the maturity of the debt contract matches the frequency with which the lenders can observe the firm's performance, and lenders can thus react to the firm's shocks. Time is discrete, and there is a finite horizon T. The firm invests all available funding in a portfolio of risky assets with given risk. Lenders' beliefs about this risk differ.

2.1. Fundamentals

Fundamental Trajectory. We start from a fundamental trajectory of the asset return, observable on a discrete time grid, $t = 1, 2, \dots T$, when the lenders can observe the firm's asset performance *and* can make decisions to invest, withdraw, or rollover. We denote by $\log Y_1, \dots, \log Y_T$ the sequence of fundamental log-returns of the firm's asset, and we assume that Y_t are independent log-normal random variables with $\mathrm{Var}[\log Y_t] = \sigma^2$. Under the real-world probability measure $\mathbb{P}$, the expected fundamental log-return is μ: $\mathbb{E}[Y_t] = e^{\mu}$.

We assume that there is a proportional cost both to liquidate and to purchase the asset, not necessarily the same. Liquidating the asset produces a loss equal to a fraction $\alpha \in [0, 1)$ of the traded volume if forced liquidation takes place before $t < T$. There are no liquidation costs at time T. The assumption of fixed and proportional, and known liquidation cost is common in the literature. We denote by $\alpha_1 \geq 0$ the corresponding cost when there are asset purchases (the asset is liquidated at time T, so the cost for asset purchases is relevant only for $t < T$). The firm's cash proceeds from liquidating an amount u of the asset value at time t is given by

$$f_t(u) := \begin{cases} u(1 - \alpha \mathbb{1}_{t<T}) & \text{if } u \geq 0 \\ u(1 + \alpha_1) & \text{if } u < 0. \end{cases}$$

Note that

$$f_t^{-1}(u) = \frac{u}{\left(1 - \mathbb{1}_{\{u \geq 0,\, t<T\}}\alpha + \mathbb{1}_{\{u<0\}}\alpha_1\right)},$$

and we have positive homogeneity: $f_t(\gamma u) = \gamma f_t(u)$ and $f_t^{-1}(\gamma u) = \gamma f_t^{-1}(u)$ for all $u \in \mathbb{R}$ and $\gamma \geq 0$.

Lenders' Maximum Exposure. We assume that the lenders' maximum investment scales linearly with the firm's size, with the *same* scaling factor for all lenders. Commercial paper resembles most this setting: It is short-term, and the size of the commercial-paper issue is proportional to the size of the firm's asset. Lenders will decide how much up to this maximum they will actually invest.

The linear-scaling assumption will allow us to preserve the homogeneity of the model and describe the evolution of the firm in terms of a single state variable. It can be relaxed to more sophisticated dependencies at the expense of an increase in dimension of the state space. We prefer the linear-dependence assumption because it implies homogeneity in firms' trajectories, which is reasonable given our setup with proportional transaction costs and no price impact. The resulting balance sheets scale linearly with the initial capital at any time: Two firms with the same characteristics—that is, asset returns and belief distribution among their lenders—will have the same trajectories up to a scaling factor.

Lenders' Beliefs. We start with the set of beliefs $\mathcal{B} := \mathbb{R}$ and a *decumulative* belief distribution function Φ, known by all lenders. Letting V_t be the size of the firm's asset at time t, the scalability assumption (with a scaling constant set to 1 for simplicity) states that the maximum investment by all lenders is V_t and the maximum investment by lenders with belief higher than b is

$$V_t\Phi(b). \tag{1}$$

A lender with *belief* b at time t measures risk using a probability measure $\mathbb{P}^b$ under which the fundamental log-return $\log Y_t$ is independent of everything else, with $\mathrm{Var}^b[\log Y_t] = \sigma^2$ and $\mathbb{E}^b[Y_t] = e^b$.

There are two interpretations of the notion of beliefs. Under the first interpretation, lenders are risk-neutral, and their differentiated beliefs stem from differences of opinion, as in Hong and Stein (2003). The second interpretation of belief is in the sense of the "pricing measure" of risk. A lender with belief b will evaluate at time t any payoff ψ at $t+1$ using $\mathbb{E}[S^b\psi(Y_{t+1})] = \mathbb{E}^b[\psi(Y_{t+1})]$, where S^b is a stochastic discount factor and encodes the lenders' subjective risk aversion. Using the stochastic discount factor is equivalent to discounting payoffs under the risk-neutral expectation but with the modified drift b of the return (and not the real-world return μ). Under the measure $\mathbb{P}^b$, the lender is risk-averse and puts additional weight on negative outcomes. The larger the difference $\mu - b$, the larger the subjective risk aversion of the lender.

Interest Rate. The lenders have as outside option a risk-free rate set to zero without loss of generality. The interest rate is constant r and is interpreted as a spread. The firm takes all credit that is provided to it and invests in the asset. The firm is assumed to act under a highly optimistic belief about the asset, so that under its belief, it is optimal to invest as many funds in this strategy as possible (recall that there is no price impact in our model, just transaction costs). The equity-maximization problem of the firm has been solved in the one-period case in Jarrow et al. (2018). They find that in the case of nonatomistic lenders, a sufficiently optimistic firm (e.g., a fund expecting high returns from its strategy) will place all available funds in the asset. This setting bears some resemblance to the manager in Hart and Moore (1995), whose "empire-building tendencies are sufficiently strong that it will always undertake the new investment if it can." There, the manager's financing is constraint by the maturity structure of the debt. Here, the belief distribution will play a critical role. When sufficiently many lenders are less optimistic than the firm, the ensemble of lenders will impose a stricter constraint on leverage than the firm would set. It is this endogenous leverage imposed by lenders with heterogenous beliefs that we seek to determine.

The following assumptions essentially mean that the belief distribution is not heavy-tailed (not more heavy than the exponential distribution). This excludes the possibility that highly optimistic lenders would concentrate a large fraction of the capital and could keep the firm liquid independently of its performance.

Assumption 1 (Log-Concavity of the Decumulative Belief Distribution). *We assume that the decumulative belief distribution admits a density ϕ: $\Phi' = -\phi$. Moreover, we shall assume that $b \mapsto \frac{\phi(b)}{\Phi(b)}$ is increasing on the interval $(-\infty, \Phi^{-1}(0))$. Here, $\Phi^{-1}(0) \in \mathbb{R} \cup \{\infty\}$, the supremum of the support of $\phi(\cdot)$, is the maximal belief.*

Remark 1. Assumption 1 is satisfied for a wide variety of distributions $\phi(\cdot)$, including exponential right tails, normal, or a uniform distribution.

2.2. Dynamics Under Given Lenders' Strategy

The lenders choose an investment strategy $\pi = (\pi_t)_{t=0,\ldots,T}$ given by the functions

$$\pi_t : \mathbb{R} \times [0,\infty)^t \to [0,1], \tag{2}$$

for $t < T$ and $\pi_T \equiv 0$—that is, there is no investment at time T. In words, a lender with belief b lends a fraction $\pi_t(b, Y_1, \ldots, Y_t)$ of her capital to the firm when the sequence of returns is $(Y_1, \ldots, Y_t)$.

Recall that the maximum investment for the lenders with belief b, whose density in the market is $\phi(b)$, scales linearly with V_t and the scaling factor is the same for all beliefs. Then, by (1), the debt capacity at time t under investment strategy π is given by

$$D_t^\pi = \int_{\mathcal{B}} \pi_t(b) V_t^\pi \phi(b)\,db, \tag{3}$$

with V_t^π the asset value at time t under given strategy of the lenders. Another interpretation is that in each period, debt is provided by a population whose size increases with the size of the firm: A firm that grows would have more and more investors. In a hypothetical model with V_t^π (or a scaling of that), lenders investing a fraction $\pi_t(b)$ of their one unit of capital, the equation is equivalent to summing up this investment.

Asset Value and Dynamics Under Given Strategy of the Lenders. Fix an investment strategy π of the lenders. At time 0, the firm starts with the initial asset value

$V_0^\pi = V_0 > 0$ and initial debt $D_{-1}^\pi = 0$. At each time $t = 0, 1, \ldots, T-1$, the firm starts with the asset value V_t^π. It accepts all debt D_t^π that is offered to it, due at time $t+1$ and bearing interest rate r. The firm then pays back its maturing debt plus interest $D_{t-1}^\pi(1+r)$.

If $(1+r)D_{t-1}^\pi - D_t^\pi < 0$, then the firm has a debt inflow, which it will use to expand the asset. If $(1+r)D_{t-1}^\pi - D_t^\pi \geq 0$, then the firm has a debt outflow and liquidates the minimum fraction of the asset in order to cover this outflow. If liquidating the entire asset cannot cover this outflow, then the firm defaults. In summary, at time t, the firm needs to liquidate an amount $f_t^{-1}\big((1+r)D_{t-1}^\pi - D_t^\pi\big)$ of its asset, in order to cover a debt outflow of $(1+r)D_{t-1}^\pi - D_t^\pi$. The asset value *after the debt flow* at time t is given by

$$V_t^\pi - f_t^{-1}\big((1+r)D_{t-1}^\pi - D_t^\pi\big). \tag{4}$$

This leads to an asset value *before the debt flow* at time $t+1$, which includes the fundamental log-return $\log Y_{t+1}$

$$V_{t+1}^\pi = \big(V_t^\pi - f_t^{-1}\big((1+r)D_{t-1}^\pi - D_t^\pi\big)\big)Y_{t+1}. \tag{5}$$

We define the **default time** τ^π of the firm as the first time t when the value of its asset after the debt flow becomes negative (i.e., liquidating the entire asset cannot cover the debt outflow),

$$\begin{aligned}\tau^\pi &= \min\{t = 1, \ldots, T \mid V_t^\pi - f_t^{-1}((1+r)D_{t-1}^\pi - D_t^\pi) < 0\} \\ &= \min\{t = 1, \ldots, T \mid f_t(V_t^\pi) < (1+r)D_{t-1}^\pi - D_t^\pi\}.\end{aligned} \tag{6}$$

Alternatively, we can write the default event in terms of the fundamental trajectory $\{\tau^\pi > t\} = \{(Y_1, \ldots, Y_t) \in \Gamma_t(\pi)\}$ with

$$\begin{aligned}\Gamma_t(\pi) := \{(y_1, \ldots, y_t) \mid &(1+r)V_{k-1}^\pi(y_1, \ldots, y_{k-1}) \\ &- V_k^\pi(y_1, \ldots, y_k) \leq f_t\big(V_k^\pi(y_1, \ldots, y_k)\big) \forall k \leq t\}.\end{aligned}$$

We call $\Gamma_t(\pi)$ the *survival set* of the investment strategy π.

Default relates to insolvency in a complex way, where insolvency is defined as the value of the asset being smaller than the value of the debt. At the horizon, all lenders must be paid back, so default is equivalent to insolvency. Before maturity however, the default event depends on the lenders' decisions, which in turn are based on their subjective valuation of lending to the firm (and the future default risk and recovery rate risk).

Remark 2. Note that liquidation costs (different for asset purchases and asset liquidations) are not the primary source of asymmetry between inflows and outflows. Outflows will play a much more important role in the firm dynamics than inflows, even in the absence of any transaction costs, because default can only occur under a debt-outflow scenario. We will show that asset purchase costs will not affect the long-run leverage level. This is because debt capacity is driven by default risk, and in the default scenario, only liquidation costs matter.

We will refer to

$$f_t(V_t^\pi) + D_t^\pi - (1+r)D_{t-1}^\pi, \tag{7}$$

as the **liquidity capacity**, and the default time can be expressed as the first time the liquidity capacity becomes negative.

On the set $\{\tau^\pi = t+1\}$, the asset is completely liquidated at time $t+1$, and the firm stops its operations. Note that $\tau^\pi = t+1$ implies $D_t^\pi > 0$ (there cannot be default in absence of debt).

Lenders' Recovery Rates. We assume that if the firm defaults at time $t+1$, all debt provided by *new lenders* (that is, debt provided by lenders that had not invested at time t) is immediately paid back in full by the firm, and all debt provided by *old lenders* is paid back partially as determined by the recovery rate. Note that old versus new lenders does not refer to the order in which they make decisions (which are simultaneous), and rather to the funds of new lenders (if any) being in a separate account. As such, at default time, any funds from new lenders can be fully returned because they are not yet invested in the asset. The debt provided by old lenders, on the other hand, has been invested in the asset. Upon default, the asset position is liquidated and distributed to the old lenders. The assumption that new lenders are not diluted by using the fresh cash to increase the recovery rates of the old lenders simplifies the analysis a lot. It is also innocuous in our setting: It cannot be that the firm is in a situation of default and that there is an inflow of lenders at the same time. That would mean that there was negative performance of the asset (otherwise, there would be no default), and at the same time, there were more lenders than before. This is impossible under any reasonable beliefs.

It follows directly from (5) that on the set $\{\tau^\pi = t+1\}$, the **recovery rate** for the debt provided at time t (by the old lenders) is given by

$$f_t\left(\frac{V_{t+1}^\pi}{D_t^\pi}\right) = f_t\left(\frac{V_t^\pi}{D_t^\pi} - f_t^{-1}\left((1+r)\frac{D_{t-1}^\pi}{D_t^\pi} - 1\right)\right)Y_{t+1}. \tag{8}$$

Finally, we assume that at time T, all debt must be paid back—that is, $\pi_T(\cdot,\cdot) \equiv 0$ and $D_T^\pi = 0$.

Remark 3. It follows recursively from $V_0^\pi = V_0$ and plugging (3) into (5) that the dependence of V_t^π on π is only through $\pi_0, \ldots, \pi_{t-1}$.

A key quantity in the subsequent analysis will be played by the firm's **leverage (or debt-to-asset) ratio** before the debt flow

$$X_t^\pi = \frac{D_{t-1}^\pi}{V_t^\pi},\ t = 1, \ldots, T \wedge \tau^\pi,$$

with the convention $X_t^\pi = \infty$ for $t > \tau^\pi$.

2.3. Lenders' Game

The lenders choose the strategy $(\pi_t)_{t=0,\ldots,T}$ in (2). Note in particular that all lenders with the same belief play the same strategy, so we classify lenders according to their beliefs and we refer to lender b to any lender with belief b.

Payoff Functions. Let π be a strategy and $t \le \tau^\pi$. We denote by $G_t^{\pi,b}$ the payoff of lender b under strategy π. The payoff has two branches.

• On $\{\tau^\pi > t\}$ (survival), lender b's expected return per dollar from lending at time t is

$$\begin{aligned} R_t^{\pi,b} &:= \mathbb{E}^b\Big[\mathbb{1}_{\{\tau^\pi > t+1\}}(1+r) \\ &\quad + \mathbb{1}_{\{\tau^\pi = t+1\}} f_t\left(\frac{V_{t+1}^\pi}{D_t^\pi}\right) \Big| (Y_1, \ldots, Y_t)\Big] - 1 \\ &= r - \mathbb{E}^b\left[\mathbb{1}_{\{\tau^\pi = t+1\}}\left(1 + r - f_t\left(\frac{V_{t+1}^\pi}{D_t^\pi}\right)\right) \Big| (Y_1, \ldots, Y_t)\right], \end{aligned} \tag{9}$$

and the payoff at time t is $G_t^{\pi,b} := \pi_t(b, Y_1, \ldots, Y_t) R_t^{\pi,b}$.

• On the set $\{\tau^\pi = t\}$ (default), the payoff of any old lender at time t is given by

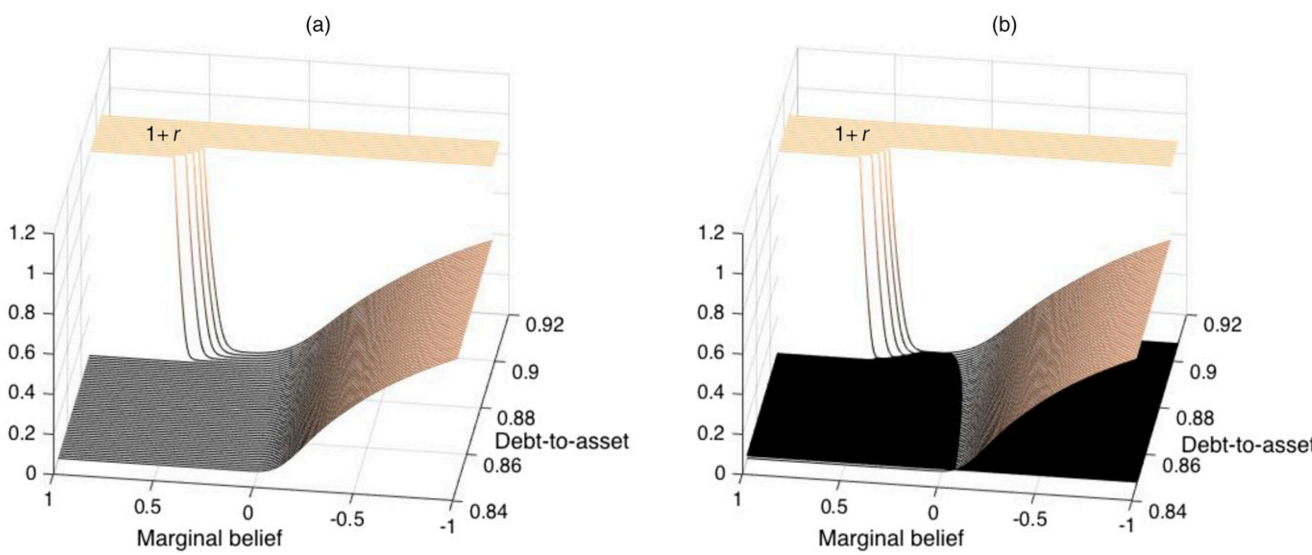
$G_t^{\pi,b} := \left(f_t\left(\frac{V_t^\pi}{D_{t-1}^\pi}\right) - 1\right)\pi_{t-1}(b, Y_1, \ldots, Y_{t-1}).$

Trade-off and the Lenders' Short Put Option. The expected return per dollar invested highlights the trade-off of a lender with belief b. On one side, she earns the interest r. On the other side, she bears the expected default cost. Her default cost depends on the other lenders' decisions both in the current period (the asset would be divided by the total amount of debt in case of default) and in the next period (because they determine the default time). The dependence on the belief is highly nonlinear, and any lender sees a strictly positive expected default cost. In fact, lenders' investment is akin them being short a put option on the (risky) recovery rate $f_t\left(\frac{V_{t+1}^\pi}{D_t^\pi}\right)$. The higher their belief, the lower their expected default cost.

At the core of our paper is the valuation of the put option at a marginal belief. The higher the valuation, the worse the payoff because lenders are short this option. Figure 2 plots the put-option surface as a function of the marginal belief and of the firms' leverage. Here, we are interested in a preview of the monotonicity properties; the specifics of the option valuation are detailed below in Proposition 2, and the analysis is in Online Appendix EC.0.2.

The parameters we used for the plots are typical to those we use in Section 6. The valuation is reminiscent of Merton's model: The debt-holders are short a put option on the firm's assets. Here, the lenders have differentiated values of this put option. Note that when the marginal belief is too high and the firm too leveraged, then the firm cannot survive (because the capital above that marginal belief is insufficient). In this case, the put option is valued at the maximum

Figure 2. (Color online) (Preview) Value of the Put Option by a Marginal Lender as a Function of the Marginal Belief and the Firm's Current Leverage



Notes. For high leverage, payoffs are nonmonotonous as a function of the marginal belief. This monotonicity are shown rigorously using option valuation techniques; see Proposition 2 and the analysis in Online Appendix EC.0.2. (a) Value of the put option. (b) The marginal lender is such that her value of the put option is equal to r (the intersecting plane) ($r = 1\%$ in this example).

$1+r$, and the lender's payoff is $r-(1+r)=-1$; that is, she expects full loss of her capital.

Figure 2(b) shows how the marginal lender is determined in equilibrium by equating the put option value with the outside option r. The curve where the option surface intersects the plane gives the marginal lender's belief as a function of the firm's leverage.

Equilibrium Definition. In a Nash equilibrium, no lender can increase her payoff by changing her strategies, if the other lenders keep theirs.

Definition 1 (Nash Equilibrium). By convention, any strategy is a Nash equilibrium at time T. A strategy π^* is called a *Nash equilibrium* at time $t<T$ if for each belief $b\in\mathcal{B}$ and each strategy π that satisfies $\pi_s=\pi_s^*$ for all $s<t$ and $\pi_t(b')=\pi_t^*(b')$ for $b'\neq b$ we have $G_t^{\pi^*,b}\geq G_t^{\pi,b}$. A strategy is said to be *Nash equilibrium* (*equilibrium* for short) if it is a Nash equilibrium at all times.

We now consider the two branches (survival and default) of the payoff G_t. Because the lenders are infinitesimally small, a unilateral deviation of lender b at time t cannot change if the firm defaults or not at that time and nor can it change b's payoff in the case of default (determined by the recovery rate). Therefore, any strategy that leads to default is automatically a Nash equilibrium. For strategies that do not lead to default, the Nash equilibrium condition translates into a cutoff property for the expected return R_t: Lenders invest if and only if their expected return is positive. We therefore have the following (trivial) proposition that characterizes Nash equilibria.

Proposition 1 (Nash Equilibrium Characterization). *A strategy π^* is called a Nash equilibrium at time $t<T$ if for every fundamental trajectory $(Y_1,\ldots,Y_t)$ we have that either*

- $(Y_1,\ldots,Y_t)\notin\Gamma_t(\pi^*)$ (*the firm defaults at time t*), *or*
- $(Y_1,\ldots,Y_t)\in\Gamma_t(\pi^*)$ (*the firm survives at time t*) *and for each belief $b\in\mathcal{B}$*

$$\pi_t^*(b,Y_1,\ldots,Y_t)=\begin{cases}1 & \text{if} \quad R_t^{\pi^*,b}\geq 0,\\ 0 & \text{if} \quad R_t^{\pi^*,b}<0.\end{cases} \tag{10}$$

Remark 4. If π^* is a Nash equilibrium and $(Y_1,\ldots,Y_t)\in\Gamma_t(\pi^*)$, then $D_t^{\pi^*}>0$. Suppose $D_t^{\pi^*}=0$. Then, there cannot be default at time $t+1$, so $R_t^{\pi,b}=r>0$, for all b. Therefore, all lenders would invest, in contradiction to $D_t^{\pi^*}=0$.

As soon as the firm attains high enough leverage, there are infinitely many Nash equilibria in which the firm defaults. For example, $\pi_t\equiv 0$ pushes the firm to default at time t as soon as $D_{t-1}(1+r)-f_t(V_t)>0$. This condition means that the early liquidation of the entire asset cannot cover existing debt plus interest, which is the typical case of a leveraged firm. In this case, there is an infinity of strategies π_t with $\int_{\mathcal{B}}\pi_t(b)V_t^\pi\Phi(db)<D_{t-1}(1+r)-f_t(V_t)$, and all these are Nash equilibria, which lead to default. Note that there may be infinitely many strategies π with survival—that is, $\int_{\mathcal{B}}\pi_t(b)V_t^\pi\Phi(db)\geq D_{t-1}(1+r)-f_t(V_t)$—but these are not Nash equilibria in general. Only if one finds a marginal lender whose value of the put option is equal to r (see Figure 2(b)), then we have a Nash equilibrium.

If the Nash equilibria in which the firm defaults coexist with Nash equilibria in which the firm survives, then these equilibria with default can be eliminated by removing weakly dominated strategies, as in Krishenik et al. (2015); see Online Appendix EC.0.5 for the definition and intuition behind weakly dominated strategies. We assume that lenders will not play weakly dominated strategies at time t. Because of the presence of recovery rates, which induce the nonmonotonicity in the expected return as a function of the belief of the marginal lender, there may be two Nash equilibria in which the firm survives.

We extend the solution concept "strongly coalition proof," introduced by Milgrom and Roberts (1996). Such an equilibrium is proof to any deviations by coalitions that are stable in the sense that they are Nash equilibria themselves, all else fixed outside the coalition. We will show in Theorem 1 that the first Nash equilibrium (with higher debt) is strongly coalition proof. The strongly coalition proof is a stronger requirement than coalition proof in the sense of Bernheim et al. (1987); see the discussion in Milgrom and Roberts (1996). Therefore, the first Nash equilibrium is automatically coalition proof.

We introduce the weaker condition "strongly ϵ-coalition proofness," in which we require stability with respect to deviations of coalitions that are Nash equilibria and in addition are arbitrarily small. If an equilibrium is not strongly ϵ-coalition proof, it automatically implies that it is not strongly coalition proof. We will show in Theorem 2 that the Nash equilibrium with lower debt is not strongly ϵ-coalition proof for any $\epsilon>0$ (and hence it is not strongly coalition proof). Note the importance of "for all $\epsilon>0$": It would be easy to show that a large group of lenders can be better off by jointly deviating from the second equilibrium. But our results are much stronger: *No matter how small* (*but positive*) *the size*, we can always find a stable lender coalition of that small size that can be better off by jointly deviating from the second equilibrium. Therefore, it is reasonable to exclude this second equilibrium. For a game with a continuum of players, it is very natural to consider coalitions of an arbitrarily small but positive fraction $\epsilon>0$ of all lenders: A player has zero mass in the continuum limit, whereas in any approximating game with a finite but large number of players, she would have a small but positive mass. Convergence results for finite games in the spirit of Carmona et al. (2017) are left for future research.

Definition 2 (Strongly ϵ-Coalition Proofness). Let $\epsilon > 0$. A Nash equilibrium π^* is said to be **strongly ϵ-coalition proof** at time T by convention, and at time $t < T$ if:

For each (Borel) set of lenders $B \subseteq \mathcal{B}$ of Φ measure $\in (0, \epsilon)$ and each strategy π, which satisfies

1. $\pi_s(\cdot,\cdot) = \pi^*_s(\cdot,\cdot)$ for all $s \neq t$
2. $\pi_t(b',\cdot) = \pi^*_t(b',\cdot)$ for all $b' \in \mathcal{B}\backslash B$
3. (Stability of the coalition: All else fixed, the coalition plays a Nash equilibrium)

$$\pi_t(b, Y_1, \dots, Y_t) = \begin{cases} 1 & \text{if } R_t^{\pi,b} \geq 0, \\ 0 & \text{if } R_t^{\pi,b} < 0, \end{cases} \text{ for each } b \in B \text{ and } (Y_1, \dots, Y_t) \in \Gamma_t(\pi), \tag{11}$$

we have that the coalition is "worse off":

(a) $\Gamma_t(\pi) \subseteq \Gamma_t(\pi^*)$ (the survival set decreases), and

(b) $R_t^{\pi^*,b} \geq R_t^{\pi,b}$ for all $(Y_1, \dots, Y_t) \in \Gamma_t(\pi)$ and $b \in B$ (returns decrease on trajectories with survival).

A strategy is called *strongly ϵ-coalition proof* if it is strongly ϵ-coalition proof at all times t.

The solution concept strongly coalition proof is the same as strongly ∞-coalition proof.

3. Nash Equilibria for the Lenders' Game

The goal of this section is to determine the set of equilibria of the lenders' game. The Nash equilibria in which the firm survives are of cut-off type is straightforward (see Lemma EC.1 in the online appendix).

The subtlety is to show that the marginal belief is uniquely determined as a function of the firm's debt-to-asset ratio X_t^π. We will also need to show that the survival set $\Gamma_t(\pi)$ can be also expressed in terms of the debt-to-asset. As we will see in Section 3.1, there are two possible marginal belief functions. Note that in this case, the number of corresponding equilibria grows exponentially with the number of periods: One marginal belief function tomorrow yields two possible marginal belief functions today, and so on. We will prove uniqueness of an ϵ-coalition proof equilibrium and only then the leverage X_t^π will be a proper state variable.

Definition 3 (Strategy with Marginal Belief Function). A strategy π is said to have the marginal belief function $\beta_t(\cdot) : [0, \infty] \to \mathcal{B}$ at time t if

$$\pi_t(b) = \mathbb{1}_{\{b \geq \beta_t(X_t^\pi)\}} \text{ on } \Gamma_t(\pi). \tag{12}$$

If a Nash equilibrium has a marginal belief function $\beta_t(\cdot)$ at time t, then from Proposition 1 it follows that its corresponding marginal belief function $\beta_t(\cdot)$ satisfies

$$R_t^{\pi,\beta_t(X_t^\pi)} = 0. \tag{13}$$

Let now π be a strategy with marginal belief function $\beta_t(\cdot)$ at time t as in (12). By (3), the debt of the firm at time t on the set $\{\tau^\pi > t\}$ is given by

$$D_t^\pi = V_t^\pi \Phi(\beta_t(X_t^\pi)), \tag{14}$$

which is strictly positive. Recall that the liquidity capacity at time t of the firm is given by (7). On the set $\{\tau^\pi > t\}$, the ratio of the liquidity capacity of the firm and the asset can be written as $\lambda_t(X_t^\pi)$, as defined below.

Definition 4 (Liquidity Capacity Function). Let $\lambda_t : [0, \infty] \to [0, \infty]$ be defined as

$$\lambda_t(x) := 1 - \alpha \mathbb{1}_{\{t<T\}} + \Phi(\beta_t(x)) - (1 + r)x. \tag{15}$$

It now follows from (6) that

$$\tau^\pi > t \Rightarrow \lambda_t(X_t^\pi) \geq 0. \tag{16}$$

The converse is not a priori true because on the default set $\{\tau^\pi = t\}$, the new debt is not given by (14). The firm's equity typically cannot cover the liquidation costs for its entire asset. In such a case, the strategy in which no one lends $\pi \equiv 0$ is a trivial Nash equilibrium, which will not survive the elimination of weakly dominated strategies if the leverage is sufficiently low. We will show below that we do have equivalence, and the function λ_t is uniquely determined if π is a *ϵ-coalition proof Nash equilibrium* in which lenders do not use weakly dominated strategies. The proof is by backward induction and will use the following results as a building block.

3.1. One-Period Building Block for the Equilibrium

In this section, we give explicit formulas for the lenders' expected return R_t^π, akin to those of option prices in the Black and Scholes model. These formulas allow us to study analytically the monotonicity (at time t) of the expected return of the marginal lender as a function of her belief, which we illustrated in Figure 2. This analysis will be part of the induction step in our proof of the uniqueness Theorem 2.

Assume that we have equivalence in (16) for time $t+1$, that is,

$$\tau^\pi = t + 1 \quad \Leftrightarrow \quad \lambda_{t+1}(X_{t+1}^\pi) < 0, \tag{17}$$

and, moreover, that the marginal belief function $\beta_t(\cdot)$ is increasing and consequently $\lambda_t(\cdot)$ is decreasing (these assumptions will be part of the induction hypothesis in the proof of the uniqueness theorem). Then, on the set $\{\tau^\pi > t\}$, by (5), we have that the default event can be expressed as the discounted asset return being below a *discounted strike* k_t, where the discount rate is equal to the belief variable b

$$\begin{aligned}
\tau^\pi = t+1 &\Leftrightarrow \lambda_{t+1}\left(\frac{D_t^\pi}{V_{t+1}^\pi}\right) < 0 \Leftrightarrow \frac{D_t^\pi}{V_{t+1}^\pi} > \lambda_{t+1}^{-1}(0)\\
&\Leftrightarrow \frac{D_t^\pi}{V_t^\pi\left(1-f_t^{-1}\left((1+r)\frac{D_{t-1}^\pi}{V_t^\pi}-\frac{D_t^\pi}{V_t^\pi}\right)\right)Y_{t+1}} > \lambda_{t+1}^{-1}(0),\\
&\Leftrightarrow Y_{t+1}e^{-b} < \frac{e^{-b\frac{D_t^\pi}{V_t^\pi}}}{\lambda_{t+1}^{-1}(0)\left(1-f_t^{-1}\left((1+r)\frac{D_{t-1}^\pi}{V_t^\pi}-\frac{D_t^\pi}{V_t^\pi}\right)\right)}\\
&=: k_t\left(b,\frac{D_t^\pi}{V_t^\pi},\frac{D_{t-1}^\pi}{V_t^\pi}\right). \qquad (18)
\end{aligned}$$

Because we have log-normal returns, we can use the Black and Scholes machinery to express the expected return (similar to a short put position) as a function of X_t^π.

Proposition 2. *Let π be a strategy with a marginal belief function $\beta_{t+1}(\cdot)$ at time $t+1$, for some $t<T$, which satisfies (17). Then, on the set $\{\tau^\pi > t\}$, the expected return under belief b satisfies*

$$R_t^{\pi,b} = h_t\left(k_t\left(b,\frac{D_t^\pi}{V_t^\pi},X_t^\pi\right)\right), \qquad (19)$$

with ($\mathcal{N}$ denotes the cumulative distribution function of the Standard Normal)

$$h_t(K) = r-(1+r)\mathcal{N}(-d_-(K)) + \frac{1-\alpha}{\lambda_{t+1}^{-1}(0)K}\mathcal{N}(-d_+(K)),$$

$$d_\pm(K) = \frac{-\log K \pm \frac{1}{2}\sigma^2}{\sigma}.$$

Suppose moreover that $\beta_{t+1}(\cdot)$ is strictly increasing. Then, the function $h_t(K)$ is strictly decreasing in $K>0$ and satisfies $\lim_{K\to 0} h_t(K) = r$ and $\lim_{K\to\infty} h_t(K) = -1$.

By (19) and (14), the expected return at time t on the set $\{\tau^\pi > t\}$ under belief b for a strategy π as in Proposition 2 is given by

$$R_t^{\pi,b} = h_t\big(k_t\big(b,\Phi(\beta_t(X_t^\pi)),X_t^\pi\big)\big),$$

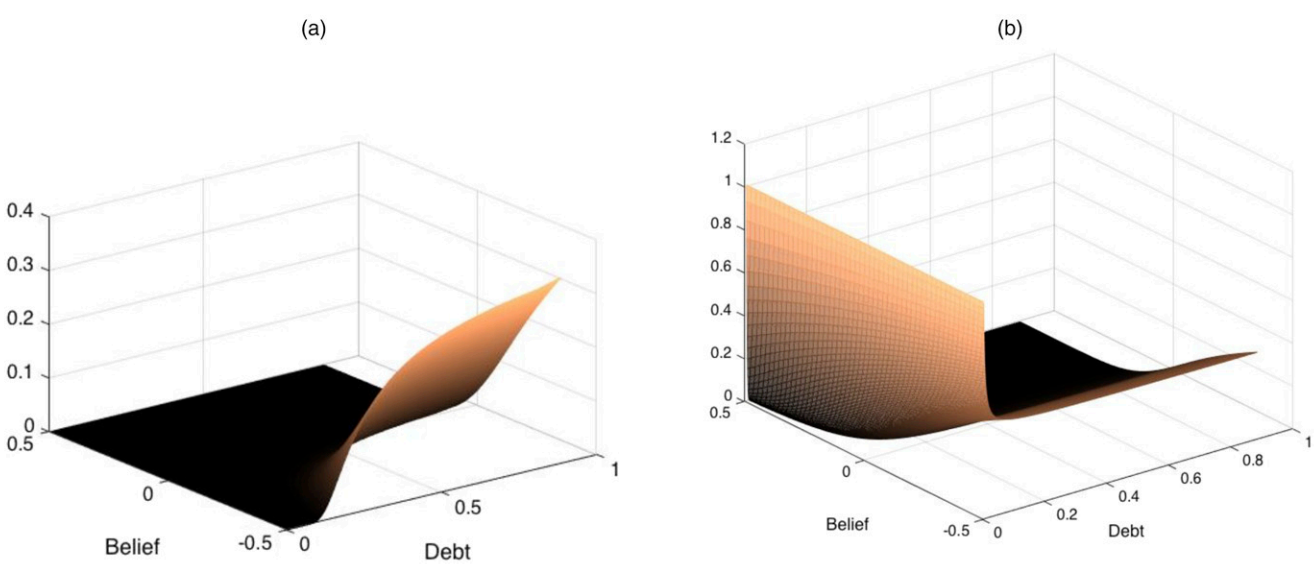
and $b \to R_t^{\pi,b}$ is increasing.

Strategic Substitutability and Complementarity. Having established the expected return, we are ready to plot the payoffs that arise in our leverage/deleveraging game—namely, we plot in Figure 3 the surface $(b,\Phi) \to h_t(k_t(b,\Phi,X_t))$ for two different values of the current leverage X_t. These plots illustrate the complex dependencies in our payoff structure. There is a change from global strategic substitutability to strategic complementarity as leverage increases. The sources of these properties are novel in the literature, as discussed in the introduction.

3.2. Existence and Uniqueness Results

In this section, we show that there exists a strongly coalition proof equilibrium—namely, the strategy $\hat{\pi}$ given in Definition 5. We also show that any other Nash equilibrium in which the firm would survive with lower debt is not proof to arbitrarily small deviations, in the sense that *no matter how small* (*but positive*) *the size*, we can always find a stable lender

Figure 3. (Color online) Lenders' Put Option Value as a Function of Belief and Concurrent Debt (Expressed as Percentage of Lenders' Capital)



Notes. Payoffs are given by r minus the value of the put option. When current leverage is lower (a), pessimists' put option value increases (and their payoff decreases) with the amount of concurrent debt. When current leverage is higher (b), pessimists' put option value decreases (and their payoff increases) with the amount of concurrent debt up to a certain point. (a) Put option surface (debt-to-asset = 0.8) shows strategic substitutability. (b) Put option surface (debt-to-asset = 0.9) shows strategic complementarity.

coalition of that size that can be better off by jointly deviating from that equilibrium.

From Proposition 2, Equation (13) takes the form

$$h_t\big(k_t\big(\beta_t(X_t^\pi), \Phi(\beta_t(X_t^\pi)), X_t^\pi\big)\big) = 0. \tag{20}$$

In Online Appendix EC.0.2, we show that this equation will have two solutions in the marginal belief function β_t. In the proof of Theorem 2, we will show that the larger solution cannot correspond to the marginal belief function in an ϵ-coalition proof equilibrium. Indeed, at the larger solution, the marginal lender is optimistic about the firm's asset, and higher debt will increase her expected payoff, because the firm will expand the asset. Therefore, slightly less optimistic potential lenders than the marginal one will join a coalition and invest, leading to positive returns for themselves and the former marginal one.

On the other hand, the equilibrium corresponding to the smaller solution of the equation, denoted $\hat{\beta}_t$, is not only ϵ-coalition proof, but coalition proof for any coalition size. This equilibrium is defined by the marginal belief functions $\hat{\beta}_t$, which satisfy the assumptions of Proposition 2 and (20).

Definition 5. Let $\hat{\pi}$ be the equilibrium with marginal belief functions $\hat{\beta}_t(\cdot)$, defined backward recursively for $t = T, \dots, 0$ as follows. Set $\hat{\beta}_T(\cdot) \equiv \infty$. Given the function $\hat{\beta}_{t+1}(\cdot)$ for $t < T$, let

$$\hat{\lambda}_{t+1}(x) = 1 - \alpha \mathbb{1}_{\{t+1<T\}} + \Phi(\hat{\beta}_{t+1}(x)) - (1+r)x,$$

$$\hat{h}_t(K) = r - (1+r)\mathcal{N}(-d_-(K)) + \frac{1-\alpha}{\hat{\lambda}_{t+1}^{-1}(0)K}\mathcal{N}(-d_+(K)),$$

$$\hat{k}_t(b,q,x) = \frac{e^{-b}q}{\lambda_{t+1}^{-1}(0)\big(1 - f_t^{-1}\big((1+r)x - q\big)\big)},$$

and then let $\hat{\beta}_t(x)$ denote the **smallest** solution of the equation

$$\hat{h}_t\Big(\hat{k}_t\big(\hat{\beta}_t(x), \Phi(\hat{\beta}_t(x)), x\big)\Big) = 0, \tag{21}$$

for all $x \ge 0$ for which there exists a solution, and $\hat{\beta}_t(x) = \infty$ otherwise.

The definition of the previous functions uses tacitly that $\hat{\lambda}_{t+1}^{-1}(0)$ is well defined. This is indeed the case, by virtue of the following result.

Proposition 3. *The function $\hat{\beta}_t(\cdot)$ is increasing. Moreover, under the condition*

$$\phi'(\cdot) \le \Phi(\cdot) + 2\phi(\cdot), \tag{22}$$

the function $\hat{\beta}_t(\cdot)$ is strictly convex.

Together with Assumption 1, we have the condition $-\frac{\phi(\cdot)^2}{\Phi(\cdot)} \le \phi'(\cdot) \le \Phi(\cdot) + 2\phi(\cdot)$. Condition (22) is technical and allows us to control the left tail of the belief distribution. It is, for example, satisfied for the normal left tails and also for any distribution in which $\phi' \le 0$, such as the case when the density increases with the pessimism level. Although Assumption 1 controls the right tails (the capital of the optimists should decrease fast enough), the control of the left tails ensures, on the contrary, that the pessimists hold most of the capital.

Notation. In the sequel, we denote by $\pi^{(s)}$ any strategy with marginal belief functions $\hat{\beta}_t(\cdot)$ from time s on, $t \ge s$ and its survival sets are given by

$$\Gamma_t(\pi^{(s)}) = \Big\{y_t \,|\, \hat{\lambda}_t\Big(X_t^{\pi^{(s)}}(y_t)\Big) \ge 0\Big\}. \tag{23}$$

In particular, we have $\pi^{(0)} = \hat{\pi}$. This strategy is uniquely determined from time s on, whereas it is a generic strategy before time s. Any strategy satisfying these conditions is said to be *of the form* $\pi^{(s)}$. It is immediate to see that this strategy is a Nash equilibrium, so the proof is mainly focused on strongly coalition proofness.

Theorem 1 (Existence). *Any strategy of form $\pi^{(t)}$ is a strongly coalition proof Nash equilibrium at time t. In particular, $\hat{\pi}$ is a strongly coalition proof Nash equilibrium.*

The following uniqueness result states that any other Nash equilibrium than $\hat{\pi}$ can be blocked by deviations of arbitrarily small groups of lenders—that is, it is not strongly ϵ-coalition proof for any $\epsilon > 0$ (and in particular this makes $\hat{\pi}$ the unique strongly coalition proof equilibrium). The uniqueness result is valid among all strategies, not only those possessing a marginal belief function.

Theorem 2 (Uniqueness). *Let π be a strongly ϵ-coalition proof Nash equilibrium for an $\epsilon > 0$ and assume that no lender uses weakly dominated strategies. Then, we have that $\pi = \hat{\pi}$.*

Our existence and uniqueness results have critical implications for leverage stability. Indeed, because the second Nash equilibrium (with lower debt) can be blocked by any small stable coalition, it can be reasonably excluded, and lenders will select the equilibrium that gives the firm the higher leverage. The convexity effects of the marginal belief as function of leverage are properties of the equilibrium $\hat{\pi}$ and underlie the existence of explosive regimes of leverage that we discuss in Section 4.

The proof of the uniqueness result is by induction: We show successively that $\pi = \pi^{(t+1)}$ leads to $\pi = \pi^{(t)}$, for all $t = T-1, \dots, 0$, and thus we uniquely identify π as the equilibrium with marginal belief $\hat{\beta}$ at all times—namely, we have that $\pi = \hat{\pi}$.

We rely on the one period building block in the previous section. At step $s < T$, we use the induction hypothesis that $\pi = \pi^{(t+1)}$. The one period building block gives analytical formulas for the expected return as a function of the belief of the marginal lender, conditional on survival at time t. Using the elimination of weakly dominated strategies, we can establish whether the fundamental trajectory y_t leads to default or not: The firm survives if and only if there exists a marginal lender with zero expected return. Technically, this means that there is a solution to Equation (21).

Conditional on survival, there may be two marginal lenders with zero expected return, and they correspond to the solutions of Equation (21). Note the necessity of using backward induction: Assuming that we have removed uncertainty about the future strategies, we can use Equation (21) to define a Nash equilibrium in the current period. Using the ϵ-coalition proof refinement, we eliminate the largest solution to the equation in the current period and complete the induction step because the smallest equation gives the belief function $\hat{\beta}_t$, which defines the strategy $\pi^{(t)}$.

The proof relies on many results that fit together in a highly complex structure. As a final detail, we note that the assumptions of Proposition 2 in the building block hold due to the induction hypothesis. When $\pi = \pi^{t+1}$, by virtue of Proposition 3, we have that $\hat{\beta}_{t+1}(\cdot)$ is increasing.

3.3. Ceiling

By eliminating the multiplicity of equilibria, we eliminate the uncertainty about the value of the ceiling, because each equilibrium has its own ceiling associated with it. This unique equilibrium is characterized by the highest value for the ceiling. Removing uncertainty about the ceiling means that default becomes measurable with respect to the observation of the debt-to-asset. Indeed, from (23), it now follows that the company defaults as soon as the debt-to-asset ratio $X_t = \frac{D_{t-1}}{V_t}$ exceeds the value $\hat{\lambda}_t^{-1}(0)$, which henceforth we shall refer to as the firm's *ceiling*. The next result shows that the ceiling can be written explicitly in terms of a suitable deterministic function $m_t(\cdot)$ given in Equation (EC.7) in the online appendix.

Proposition 4. *The ceiling for the debt-to-asset ratio is given by*

$$\hat{\lambda}_t^{-1}(0) = m_t^{-1}\left(\frac{1}{\hat{h}_t^{-1}(0)}\right). \tag{24}$$

The debt ceiling gives the default condition. In the next section, we determine the regimes of leverage. We also give a measure of stability when the debt is in the mean-reverting regime.

4. Debt Dynamics and Debt Stability

For the remainder of the paper, we assume that lenders select the unique strongly coalition proof equilibrium $\hat{\pi}$ at all times $t = 0, 1, \ldots, T-1$, and we drop the superscript $\hat{\pi}$ from our notation of the processes D, V and X. We also drop the hat from the functions $\hat{\lambda}_t(\cdot)$, $\hat{h}_t(\cdot)$ and $\hat{k}_t(\cdot)$ in Definition 5. The debt-to-asset is a state variable, whose dynamics and stability can be characterized.

We consider that the debt-to-asset is below the debt ceiling that we established in the previous period—that is, $X_t \le \lambda_t^{-1}(0)$—and we characterize its dynamics. At the debt ceiling, the firm defaults. We can write the debt-to-asset ratio in the next period as a function of the debt-to-asset ratio in the current period, starting from (5):

$$\begin{aligned} X_{t+1} &= \frac{D_t}{V_t\left(1 - f_t^{-1}\left((1+r)\frac{D_{t-1}}{V_t} - \frac{D_t}{V_t}\right)\right)Y_{t+1}} \\ &= \frac{\lambda_{t+1}^{-1}(0)k_t\left(\beta_t(X_t), \frac{D_t}{V_t}, \frac{D_{t-1}}{V_t}\right)e^{\beta_t(X_t)}}{Y_{t+1}} \\ &= \lambda_{t+1}^{-1}(0)h_t^{-1}(0)\, e^{\beta_t(X_t)}\frac{1}{Y_{t+1}}, \end{aligned} \tag{25}$$

where in the second line we used the definition of k_t in Definition 5 and (14), and in the last line, we used (EC.12) in the online appendix.

Our notion of stability is that there exists a level x_t strictly below the debt ceiling $\lambda_t^{-1}(0)$, such that the process X_t is mean reverting to the level x_t under *the real-world measure* $\mathbb{P}$ as long as the process stays below an instability level $\bar{x}_t$, with $x_t < \bar{x}_t \le \lambda_t^{-1}(0)$.

Of course, because our model is in discrete time, the process X_t can jump from the long-run level over the instability level in one period. The probability of such a jump is a necessary addendum to the characterization of the stability of the process in terms of the various regimes. We define the stability measure (for debt with a mean-reverting regime) as a (conditional) probability to stay below the instability level $\bar{x}_t$ in one period if one forces leverage to start the long-run level x_t:

$$1 - \mathbb{P}[X_{t+1} > \bar{x}_t \mid X_t = x_t].$$

We define this measure using a conditional probability because we think of the assessment of the entire mean-reverting regime. If the stability measure is too low, then even if the process is in the mean-reverting regime, it can very easily "escape" in the explosive regime. Clearly, the stability measure increases the difference between the instability level and the long-run level. The existence of the mean-reverting regime and the stability measure together determine the sustainability of debt.

Definition 6. Let $p \in [0,1]$. We say that debt-to-asset has a mean-reverting regime $[0, \bar{x}_t)$ (with stability measure $1-p$) if for each t there exist $0<x_t<\bar{x}_t\leq \lambda_t^{-1}(0)$ such that

$$\mathbb{E}[X_{t+1}\,|\,(Y_1,\ldots,Y_t)]>X_t,\ \forall\ X_t\in(0,x_t)\cup(\bar{x}_t,\lambda_t^{-1}(0)]$$
$$\mathbb{E}[X_{t+1}\,|\,(Y_1,\ldots,Y_t)]<X_t,\ \forall\ X_t\in(x_t,\bar{x}_t),$$

and $\mathbb{P}[X_{t+1}>\bar{x}_t \mid X_t=x_t]<p$. We call x_t the *long-run level*, $\bar{x}_t$ the *instability level*, and $(\bar{x}_t,\lambda_t^{-1}(0)]$ the *explosive regime*. If debt-to-asset does not have a mean-reverting regime, then it is said to be explosive.

In the mean-reverting regime (if any), X_t reverts to the long-run level x_t as long as it stays below the instability level $\bar{x}_t$. Clearly, the definition uses the expected return of the asset under the real-world ("oracle") measure. We now let the real-world expected return be $e^{\mu}=\mathbb{E}[Y_t]$. Using the dynamics of X_t above, we can compute the *drift function* of the process X_t as

$$a_t(X_t):=\mathbb{E}[X_{t+1}\,|\,(Y_1,\ldots,Y_t)]=\lambda_{t+1}^{-1}(0)h_t^{-1}(0)e^{\beta_t(X_t)}e^{-\mu+\sigma^2}, \tag{26}$$

Remark 5 (Drift Invariance Under the Belief of the Marginal Lender). Similarly to computing the drift of the process X_t under the real-world measure, we can compute the drift under the belief of the marginal lender. At any time before default, the marginal lender's expectation of the future debt-to-asset is constant and given by

$$\mathbb{E}^{\beta_t(X_t)}[X_{t+1}\,|\,(Y_1,\ldots,Y_t)]=\lambda_{t+1}^{-1}(0)h_t^{-1}(0)e^{\sigma^2}.$$

The continuum of lenders adjust debt in each period such that the expectation of the future leverage from the perspective of the marginal lender is invariant. In turn, a marginal lender exists (and there is no default) as long as this adjustment of debt is possible.

This also explains why the instability level $\bar{x}_t$ can be different from the debt ceiling $\lambda_t^{-1}(0)$. Indeed, when debt-to-asset reaches the instability level, an oracle can detect that the process is explosive and expected to reach the debt ceiling. But the default does not happen yet, because a marginal lender can be found, and, under her belief, the debt-to-asset is not explosive yet. As we will later show, when lenders learn, then the instability level and the debt ceiling coincide: $\bar{x}_t=\lambda_t^{-1}(0)$. In this case, the expectation of any lender becomes close to the oracle's expectation. Therefore, all lenders can detect when the debt-to-asset becomes explosive, and default happens at this point.

We now assume that the belief distribution function satisfies (22). Then, by Proposition 3, we have that the function $\beta_t(\cdot)$ is increasing and convex, and so is the drift function $a_t(\cdot)$. We can then fully characterize the regimes of the debt-to-asset process using the fixed points of the drift function $a_t(x)=\lambda_{t+1}^{-1}(0)h_t^{-1}(0)\cdot e^{\beta_t(x)}e^{-\mu+\sigma^2}$, which incorporates all model parameters.

Proposition 5. *The debt provision has a mean-reverting regime if and only if for each t, the drift function $a_t(\cdot)$ either has two fixed points $0<x_t<\bar{x}_t<\lambda_t^{-1}(0)$ or a unique fixed point $0<x_t<\lambda_t^{-1}(0)$, in which case we set by convention $\bar{x}_t:=\lambda_t^{-1}(0)$. The stability measure is given by $\mathcal{N}(\frac{\sigma}{2}+\frac{1}{\sigma}\log\frac{\bar{x}_t}{x_t})$. Moreover, if $\bar{x}_t<X_t<\lambda_t^{-1}(0)$, then we have $\mathbb{E}[X_{t+1}\,|\,(Y_1,\ldots,Y_t)]>X_t$.*

If the drift function does not have a fixed point, the debt-to-asset process is always explosive. If there exist $x_t,\bar{x}_t$ as in the proposition, then the debt-to-asset process is a mean-reverting process, as long as it does not exceed $\bar{x}_t$. We give its stability measure,

$$\mathbb{P}[X_{t+1}<\bar{x}_t \mid X_t=x_t]=\mathbb{P}\left[\frac{x_t}{Y_{t+1}e^{-\mu+\sigma^2}}<\bar{x}_t\right]$$
$$=\mathbb{P}\left[Y_{t+1}>\frac{x_te^{\mu-\sigma^2}}{\bar{x}_t}\right]$$
$$=\mathcal{N}\left(\frac{\sigma}{2}+\frac{1}{\sigma}\log\frac{\bar{x}_t}{x_t}\right). \tag{27}$$

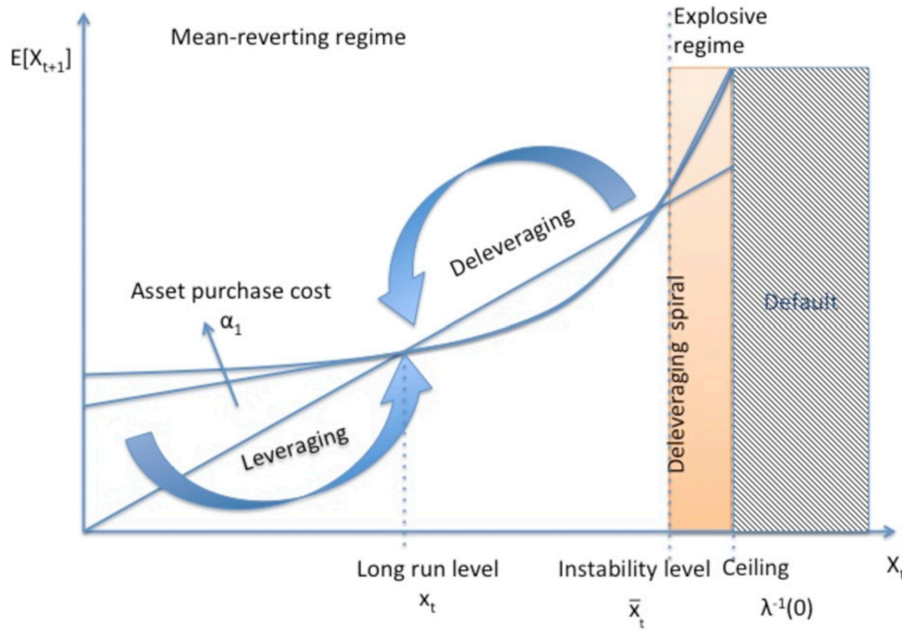
The larger the distance between the long-run level and the instability level (in relation to the asset volatility), the larger the stability measure of the debt when it is in the mean-reverting regime. When the stability measure is low, the debt-to-asset behaves for all practical purposes as an explosive process. If there is no fixed point of the drift function, then the debt only has an explosive regime.

Figure 4 illustrates the different regimes of the debt-to-asset. If the debt-to-asset process is in the mean-reverting regime, but below x_t, the firm will leverage, as it has high equity. A firm whose debt-to-asset is in the region $[x_t,\bar{x}_t]$ successfully deleverages by selling assets and repaying withdrawing lenders. In the explosive regime, it is not possible (under typical returns) to return to the mean-reverting level, and the debt-to-asset is instead moving away from $\bar{x}_t$ toward the debt ceiling, at which point the firm defaults.

Around the long-run leverage level, a positive fundamental return decreases the debt-to-asset ratio. Then, lenders increase leverage and push back the ratio to its long-run leverage. This is achieved by a decrease of the marginal belief—that is, more pessimists invest. After a sequence of positive fundamental returns, the marginal belief is low. However, pessimists are the fastest to exit because their payoffs decrease the fastest under negative fundamental returns. If the debt-to-asset goes into the explosive regime because of negative fundamental returns, pessimists exit and would require higher and higher returns to reenter, which will not occur on a typical sample path of the exogenous fundamental returns. The explosive regime happens due to a spiral of withdrawals by pessimists.

We end this section by showing that the mean-reverting level x_t is the level where the net inflow is

Figure 4. (Color online) The Expected Future Debt-to-Asset as a Function of the Current Debt-to-Asset



Note. Regimes of the debt-to-asset.

zero (in expectation). Such characterization fully justifies calling this level "long-run level" (even if the long-run level is not a constant and may change in time). Suppose that $(1+r)e^{-\mu+\sigma^2}=1$—that is, $\mathbb{E}(\frac{1}{Y_t})=\frac{1}{1+r}$: The expectation of the discount factor on the asset side is the same as the expectation of the discount factor on the debt side. Note that in this case, the asset grows faster than the debt ($e^{\mu}=(1+r)\,e^{\sigma^2}>(1+r)$). Intuitively, if the debt-to-asset process is mean reverting to the level x_t, then at this point the net debt inflow is zero—that is, $D_t=D_{t-1}(1+r)$ if X_t were deterministic and equal to x_t. This writes $\Phi(\beta(x_t))=x_t(1+r)$.

Proposition 6. *Suppose that* $(1+r)e^{-\mu+\sigma^2}=1$ *and that there exists a fixed point* $0<x_t<\lambda_t^{-1}(0)$ *such that* $a_t(x_t)=x_t$. *Then,* $\Phi(\beta(x_t))=x_t(1+r)$, *and* $D_t=D_{t-1}(1+r)$ *if* $X_t=x_t$. *Moreover, for* $x>x_t$, $\beta(x)$ *does not depend on the asset purchase cost* α_1.

The proof of Proposition 6 relies on the study of the dependence of the drift function on the asset purchase cost.

Remark 6. The proposition states that the long-run leverage x_t is the solution to the equation $\Phi(\beta(x_t))=x_t(1+r)$. This is a direct computation that does not rely on the fixed point of the drift function $a_t(\cdot)$. It sets instead the much simpler condition of zero net inflow. Because x_t is the solution to this equation and because $\beta(x)$ does not depend on the asset purchase costs for $x>x_t$, it follows that the long-run level x_t does not depend on the asset purchase costs.

The important implication is that asset purchase costs can only change the speed of leveraging: The higher the asset purchase costs, the higher the speed to reach the long-run leverage level (the proof shows that $\alpha_1\to a_t(\cdot)$ is increasing). But asset purchase costs do not change the long-run level, nor do they change the value of the drift function above the long-run level. Above that level, we are in a deleveraging phase, so only the liquidation costs are relevant, and not the asset purchase costs (see Figure 4).

5. Model Dynamics Under Real-World Returns

We illustrate the dynamics of debt capacity in our model when the input is the series of real-world asset returns. Notably, *we do not use any data on outstanding debt*. We use both a variable interest rate given by the actual spreads between short-term debt rates and t-bill rates and a constant interest rate, given by the average spread over the period 2001–2005. We use the constant spread to insulate the model predictions from any information included in the actual spreads. Even under constant spread, the model predicts the collapse in debt capacity observed in the real-world data.

We do not single out a firm, but we use aggregate data. We use the U.S. Securities Industry Financial Results, available at http://www.sifma.org/research/statistics.aspx, for 40 quarters, from 2001 to 2010. The data contain "Aggregated income statement, selected balance sheet, and employment data on the U.S. domestic broker-dealer operations of all FINRA member firms doing a public business derived from their Financial and Operational Combined Uniform Single (FOCUS) Report filings." The

Financial Industry Regulatory Authority (FINRA) is responsible for regulating every U.S. broker-dealer. The number of FINRA members in the time frame we consider averages 4,000. Specifically, we let the fundamental returns $Y(0) = 1$ and

$$Y(t) = \frac{\text{Aggregate Revenue FINRA Members}_t}{\text{Aggregate Revenue FINRA Members}_{t-1}}, \quad t = 1, \ldots, 40.$$

All results in this section will be showing single sample paths, based on the realization of $Y(t)$.

We divide the data into two parts and only use the first part, 2001–2005, to estimate the asset mean and the asset volatility. This is to ensure that model parameters are not calibrated by using data during the financial crisis. We use the 3-Month Treasury Bill: Secondary Market Rate as a proxy for the risk-free rate (the outside option). Monthly data are available at https://fred.stlouisfed.org/series/TB3MS. We aggregate monthly data to obtain quarterly data. The outside option has been set to zero in the model exposition, but here we use the actual interest-rate data to be consistent with using the returns of the asset as input. We use the "3-Month AA Financial Commercial Paper Rate," available at https://fred.stlouisfed.org/series/DCPF3M, to calibrate the interest rate offered to the lenders. The spread is plotted in Figure 5.

In summary, the calibrated model parameters are the following:

- $T = 40$ periods, and $\delta = 1/4$ for the time length of one period.
- Asset return (annualized) mean $\mu = 7\%$ and (annualized) volatility $\sigma = 20\%$.
- The outside interest rate is set to the 3-Month Treasury Bill (quarterly data series 2001–2010, average spread 2001–2005 is 18 bp).
- The interest rate is set to the 3-Month AA Financial Commercial Paper Rate.

The remaining parameters of the model could be calibrated by using data reported to regulators. The liquidation cost α and the asset purchase cost α_1 depend on the exact composition of assets in the firm's portfolio. Each type of asset can be assigned a liquidation cost (the parameter α is equivalent to Kyle's lambda introduced in Kyle 1985). Here, we set the liquidation cost $\alpha \in [0.08, 0.2]$ to illustrate the behavior of the model under varying values. The belief distribution is harder to infer from data currently collected by regulators, and we hope that our results would make a strong case for such data collection. Banks utilize a variety of statistical models to assess how "jittery" their short-term debt holders are. This would be analogous to the level of pessimism in our model.

We denote the variance of the belief distribution by σ_b. We let $\Phi(b) = (1 - \mathcal{N}((b - \mu)/\sigma_b))$. Results are robust to variations in the parameters of this distribution, as we let $\sigma_b \in [0.15, 0.4]$.

Figure 1(a) shows the real-world dynamics of the debt, specifically the financial outstanding commercial paper, available at https://fred.stlouisfed.org/series/FINCP. These data are not used, but plotted there to compare against the dynamics resulting from our model. Figure 1(b) shows the dynamics of debt predicted by our model. The corresponding dynamics of the debt-to-asset and the critical belief are shown in Figure 6. The parameters are: $\sigma_b = 0.2$, $\alpha = 0.1$ (liquidation cost).

In particular, as shown in Figure 1(b), our model provides a warning signal of the collapse of the financial outstanding commercial paper in Q1 2008, just before the actual event in Q3 2008 after the fall of Lehman. The early signal holds both when we use the real-world spread series and when we use the constant spreads. The constant spread is the average spread over the period 2001–2005 and thus does not include any information about spreads in the crisis period. The collapse is therefore not driven by spreads,

Figure 5. (Color online) Spread (Absolute Value) Between 3-Month AA Financial Commercial Paper and Effective Federal Funds Rate

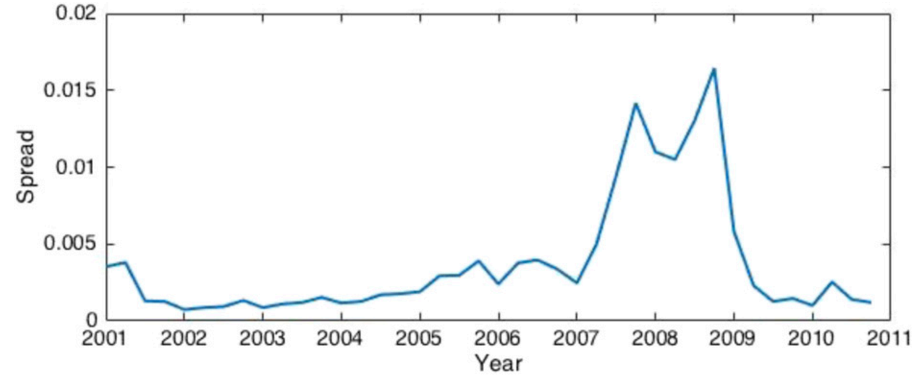
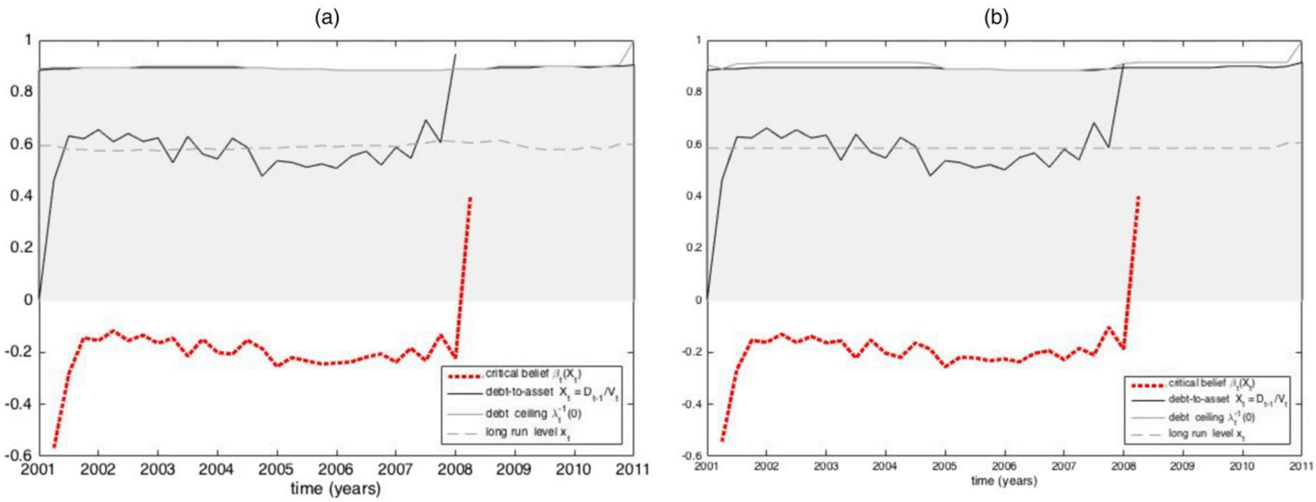


Figure 6. (Color online) Debt Collapses as Leverage Has a Big Deviation from the Long-Run Level Toward the Instability Level



Notes. (a) (Model) Debt-to-asset dynamics with real-world spreads. (b) (Model) Debt-to-asset dynamics with constant spread.

but by the debt-to-asset moving away from long-run level to the instability level.

The difference in debt under the actual spreads and under the average spread is minimal. Moreover, as shown in Figure 6, (a) and (b), the long-run level and the instability level given by the model change little when using the actual spreads versus the average spread. As implied by our theory, debt collapses, as leverage has a big deviation from the long-run level toward the instability level. The only difference from using the actual spreads is the more accurate calculation of the debt-to-asset X_t. Because spreads vary when the debt-to-asset was well above the mean-reverting level, it is not surprising that they have very little influence on the outcome of the deleveraging game: The pessimists' payoffs are too low at this level of leverage, and they are not very sensitive to the interest rate. Debt does not recover in our model after the collapse. Because of the convexity of their short put option, the pessimists would have required much higher positive returns to compensate for the negative returns in order to stay with the firm.

In Online Appendix EC.0.7, we demonstrate robustness of the model behavior under the real-world return data for a variety of parameters.

5.1. Single-Firm Case

We end this section with the application of our model to the case of Morgan Stanley (a large broker dealer, prone to the kind of deleveraging crisis that our paper models; see, e.g., Duffie 2010a). We consider two periods, 2001–2010 and 2009–2018 (including the crisis and after the crisis). As proxy for short-term debt, we use current liabilities, available at Compustat-Capital IQ. As before, the asset fundamental return is the quarterly percentage change in revenue. We check that our early warning signal does not produce false positives in the postcrisis period. The single-firm case is more suited as a control case for fictitious defaults (in the aggregate case, the number of firms before and after the crisis would be nonstationary). The results are given in Figure 7, (a)–(d). Before the crisis, the short-term debt collapsed in reality and as predicted by the model. After the crisis, the model produces a similar debt pattern in the period 2009–2012 as in the real world. Remark that the only input of the model is the revenue of the bank. Given the complexity of the current liabilities of an investment bank, it is remarkable that the model implied short-term debt is a close match to reality in the 12 quarters after the crisis.

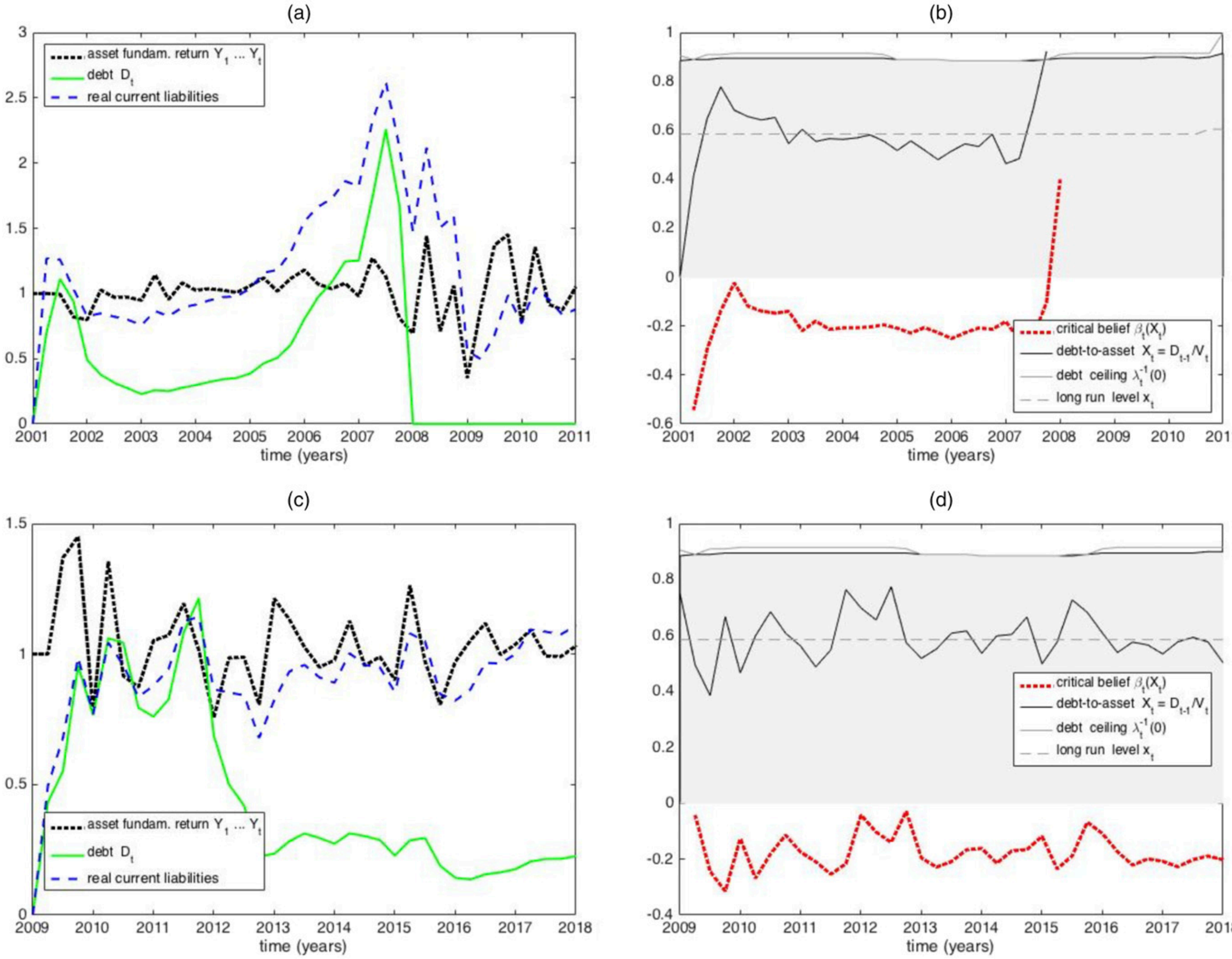
After the year 2012, the model implies lower debt than in reality, but there is no trigger of default. One explanation is that Morgan Stanley has expanded to other sources of short-term debt—for example, deposits by less sophisticated investors. These would be captured in the "real-world" current liabilities but not necessarily by our model of strategic lenders. The model does not produce false positives for any of the 38 postcrisis quarters, despite several quarters of significant drop in revenue.

We also note a drop in debt in 2002 in Figure 7(a) and in 2012 in Figure 7(c). These drops are not deleveraging spirals in the sense of our model; they are not genuine "collapses." The leverage ratio, although it increases above the long-run level, stays in both cases below the instability level. It is then possible for the leverage ratio to revert to the long-run level and for debt to recover.

6. Comparative Dynamics and Financial Insights

In the following sections, we illustrate the dynamic behavior of our model under a variety of parameters

Figure 7. (Color online) Real-World Current Liabilities vs. Model Implied Short-Term Debt in the Precrisis and Postcrisis Period for Morgan Stanley



Notes. (a) Real current liabilities vs. model-implied debt (2001–2010). (b) Debt-to-asset dynamics (2001–2010). (c) Real current liabilities vs. model-implied debt (2009–2018). Fundam., fundamental. (d) Debt-to-asset dynamics (2009–2018).

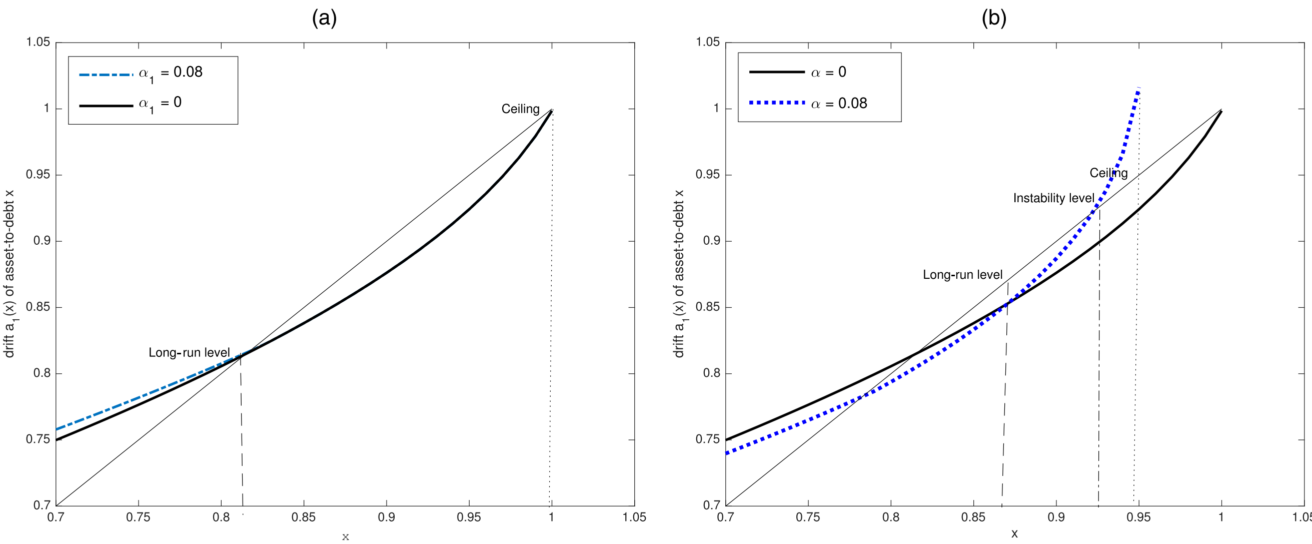
and simulated paths of the fundamental trajectory. Financial insights are highlighted. We assume that the beliefs are normally distributed around the true mean μ, with variance σ_b. In unreported results, we also considered the case of exponential distribution and uniform distribution with finite upper support, both of which satisfy the log-concavity assumption on the decumulative distribution function. The results are robust.

We fix a time horizon of 10 years and set $\delta = 1/4$ for the length of one period—that is, $T = 40$. This means that lenders observe the firm's performance quarterly, and the short-term debt is rolled over on a quarterly basis. We set $\sigma_b = 0.2$. The asset and money-market parameters are given by an annualized volatility $\sigma = 6\%$, an annualized expected asset return $\mu = 3\%$, and an annualized short-term interest rate of $r = 1\%$, both in excess of the risk-free rate. The parameter values μ and σ for the log return distribution are typical values for a diversified bond asset. If the firm is a large bank, the TED spread can be seen as a proxy for r. Its long-term mean has been around 0.3%, but the spread has varied considerably over time, averaging between 1% and 2% during periods of financial distress, such as in 2008 and 2009.

To make the analysis easy to follow, there is an overlapping case in each pair of consecutive figures, which show the drift function $a_t(x)$ of the debt-to-asset process for different parameters.

We start by analyzing the impact of the liquidation cost α and the asset purchase cost α_1. We illustrate numerically Proposition 6 in Figure 8(a). We set the liquidation cost $\alpha = 0$. We verify the points in Remark 6: (1) The drift function increases with the asset purchase cost (because the marginal belief function

Figure 8. (Color online) The Effect of the Asset Purchase Cost (a) and the Liquidation Cost (b) on Long-Run Level, Instability Level, and Ceiling



Notes. Levels are read where vertical lines intersect the diagonal. (a) Liquidation cost α is zero. (b) Asset purchase cost α_1 is zero.

increases with the asset purchase cost); (2) the long-run level (the first fixed point of the drift function) is the same for different asset purchase costs, and the drift functions coincide above this level. *Asset purchase costs have no impact on long-run levels of the leverage.* Of course, because asset purchase costs do lower the asset value, it means that debt levels are lower, so that the long-run leverage is invariant with the purchase costs. (3) Below the first fixed point of the drift function—that is, in the leveraging phase—the drift function increases with purchase costs. *Leveraging toward the long-run level is faster when purchase costs are higher.*

In the sequel, we set the asset purchase cost to zero. Figure 8(b) shows the effect of liquidation costs on the drift function. Unlike asset purchase costs, liquidation costs affect the long-run level for the debt-to-asset ratio. One can see that: (1) The first fixed point of the drift function is lower if liquidation costs are higher. This means that *under higher liquidation costs, lenders impose a lower long-run leverage level*. (2) With zero liquidation costs, the ceiling is one. This means that *insolvency and default coincide under zero liquidation costs*: The firm defaults when the debt-to-asset reaches one (its equity is zero). In contrast, *when liquidation costs are higher, the ceiling is below one and the firm defaults while it is solvent* (its equity is nonzero).

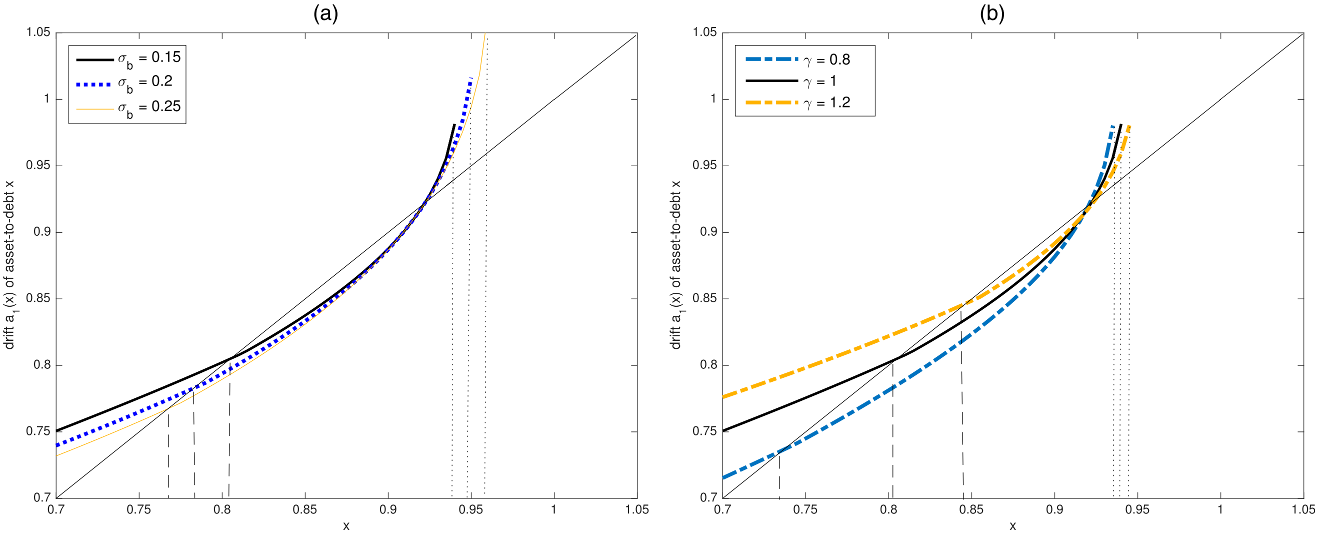
In Figures 9(a) and 9(b), we analyze the role of the belief distribution and the exposure constraint. We set the liquidation cost $\alpha = 0.08$. In Figure 9(a), we investigate the effect of the variance of the belief distribution σ_b on the debt regimes. Larger heterogeneity of the beliefs translates in both a lower long-run leverage and in a higher ceiling. The instability level is the same for all levels of belief heterogeneity. This means that the stability measure (given in (27)) increases with the belief heterogeneity as the ratio $\frac{\bar{x}_t}{x_t}$ between the instability level and the long-run level is higher. When lenders are less certain about the true mean of the asset returns, they impose a precautionary lower long-run leverage on the firm. At the same time, the ceiling is higher, so they allow the firm's debt-to-asset process to spend a longer time in the explosive regime. This makes the early warning indicator more effective. *More belief heterogeneity makes debt more stable.* Under more heterogeneity, capital decreases more slowly as we go toward more optimistic beliefs. As we approach a point mass, the debt provision is most unstable: Assume that the marginal lender is slightly below the real-world mean. Then, a very small increase in the marginal lender is accompanied by a large loss of lender capital.

6.1. Debt Capacity of One Unit of the Asset

As the lenders select the marginal belief $\beta(X_t)$, the total debt provided is $V_t\Phi(\beta(X_t))$. Recall that, so far, we have chosen 1 as scaling constant of the lenders' maximum exposure. We now vary the scaling of the maximum exposure: Lenders scale their maximum exposure to the borrower by γV_t. The total debt provided is $\gamma V_t\Phi(\beta(X_t))$. The quantity $\gamma\Phi(\beta(X_t))$ is then **the debt capacity at time** t **of one unit of the firm's asset**, and we clearly have $\gamma\Phi(\beta(X_t)) \le \gamma$. Therefore, γ is an upper limit on the long-run leverage level.

Figure 9(b) shows the effect of this scaling. We set the variance of the belief distribution $\sigma_b = 0.15$. We verify that the long-run leverage levels are below the respective values for γ. As expected, the ceiling

Figure 9. (Color online) The Effect of the Belief Distribution and Exposure Constraint on Long-Run Level, Instability Level, and Ceiling



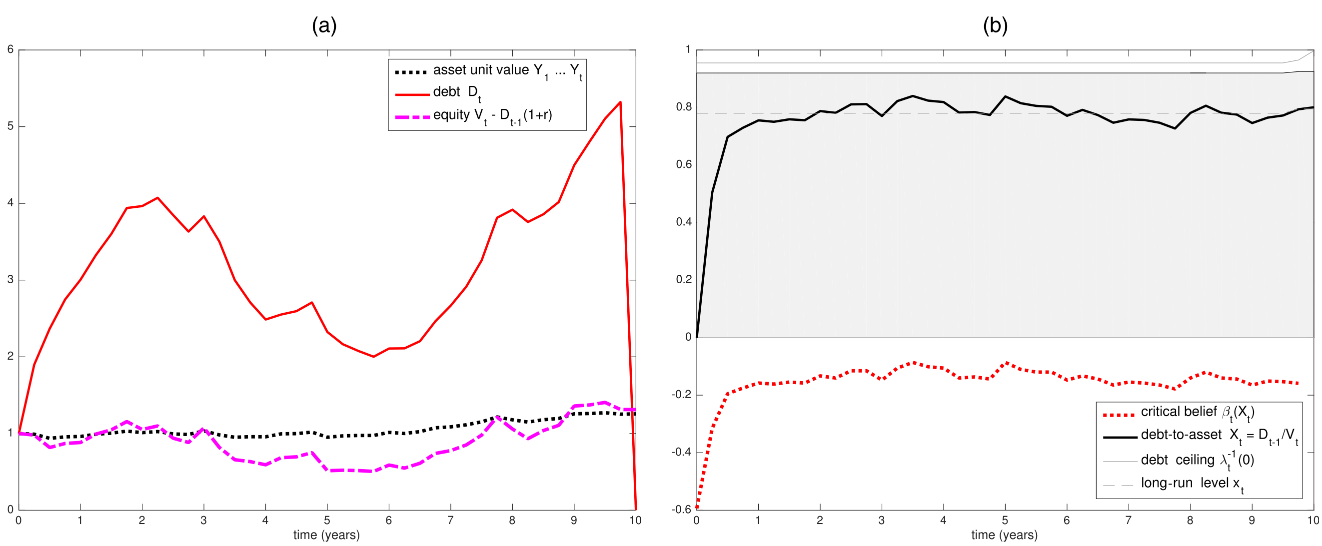
Notes. Dashed vertical lines mark the long-run average. Dotted vertical lines mark the ceiling. (a) The effect of the belief variance σ_b. (b) The effect of the exposure constraint γ. $\sigma_b = 0.15$.

increases with γ. However, it is not the ceiling that drives debt stability but the ratio between the instability level and the long-run level; see (27). The instability level is the same for all γ considered, whereas the long-run leverage level decreases as γ decreases. When lenders are more constraint, they impose a precautionary long-run leverage, which is lower. When they are less constraint, they allow for a higher long-run leverage of the firm. *Debt is more unstable when lenders' constraints relax.*

6.2. Debt Dynamics

We now investigate the dynamics of the balance sheet. For the following simulations, we set $\sigma_b = 0.2$, and $\alpha = 0.08$. For these parameters, the debt process has both a mean-reverting and an explosive regime. We assume that the company starts with an initial asset value of $V_0 = 1$. The company's expected equity at the time horizon T under the (real-world) probability measure is given by $\mathbb{E}[(V_T - D_{T-1}(1+r))\mathbb{1}_{\{\tau > T\}}]$. We find this value by Monte Carlo simulation to be

Figure 10. (Color online) Debt Dynamics with Mean-Reverting Leverage



Notes. The dynamics is in steady state (the long-run level, the instability level, and the ceiling are time-independent) for all but the last two periods ($t \in [9.5, 10]$). (a) A typical trajectory of the asset unit value and associated balance-sheet dynamics. (b) The debt-to-asset ratio and the marginal belief exhibit a stationary behavior. In gray, the mean-reverting regime $(0, \bar{x}_t)$, with $\bar{x}_t$ the instability level.

around 2.1. By raising short-term debt, the company therefore raises its annual expected rate of return on its initial equity from the fundamental asset's return of 3% to the rate of $\frac{1}{10}\log 2.1 = 7.4\%$. This increase in the expected return rate comes at the expense of introducing default risk; the Monte Carlo simulation yields a default probability under the real-world measure of around 0.01 over the 10-year period. The corresponding annualized default probability p satisfies $(1-p)^{10} = 1 - 0.01$—that is, approximately $p = 0.1\%$. The expected value of one unit of the fundamental asset at the time horizon is 1.35, whereas the expected equity is 2.1. The Monte Carlo results indicate that only a small fraction of the paths end with default. In the vast majority of paths, for the considered parameters, the debt-to-asset process stays in the mean-reverting regime for the entire period.

In Figures 10 and 11, we plot in (a) the sample paths of the cumulative asset return (or the value of one unit of fundamental asset over time) $F_t = Y_1, \ldots, Y_t$ along with the resulting debt process D_t, the equity process $V_t - D_{t-1}(1+r)$. In (b), we plot the debt-to-asset process (which reflects the marginal belief, also plotted).

Figure 10 shows a typical sample in which the debt-to-asset process stays in the mean-reverting regime for the entire period. Figure 11 selects one of the paths with default and shows that on that path the debt-to-asset switches from the mean-reverting regime to the explosive regime.

Steady-State Dynamics. For identification of the regimes of the debt-to-asset, we plot the long-run level x_t, the instability level $\overline{x}_t$, and the ceiling. These quantities are deterministic, and of course coincide in Figures 10(b) and 11(b). We require to be sufficiently far away from the time horizon to ensure that the system is in the "steady state" and that time to maturity is not a driver of the debt-provision stability (the debt capacity is trivially equal to zero at maturity, because all debt is repaid at the horizon).

For time-independent spreads, risk-free rate, and belief distribution, the long-run level x_t, the instability level $\overline{x}_t$, and the ceiling for the debt-to-asset ratio are also time-independent when the system is away from the time horizon. For the parameters we considered, the dynamics is in steady state for all but the last 2 (out of 40) periods.

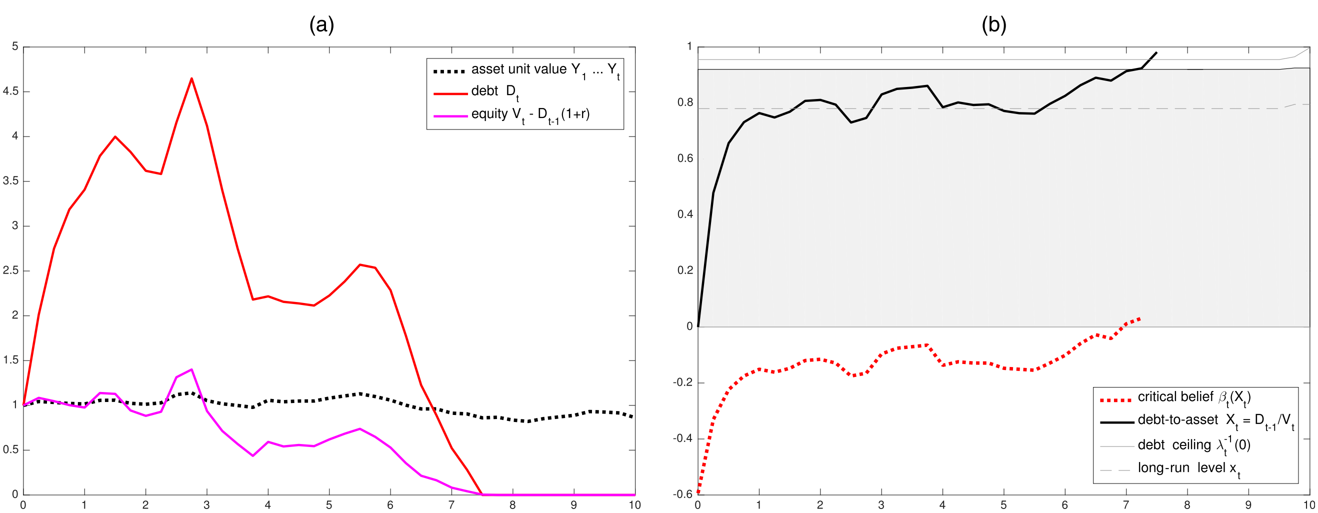
Studying the stability of the debt-to-asset process necessitates sufficiently many periods to be far away from the time horizon. When we refer to the collapse in debt capacity, we refer to the collapse in debt capacity that occurs in steady state and not at maturity, as illustrated in Figure 11(a).

7. Concluding Remarks and Implications

We have endogenized the short-term debt by showing uniqueness of a Nash equilibrium in leveraging/deleveraging games with heterogeneous lenders. We have shown that the debt-to-asset has a mean-reverting and an explosive regime that we fully determine.

We constructed a parsimonious dynamic model that quantifies short-term debt stability. The characteristics of the firm's asset can be estimated from data. The same is true for the frequency. Our work has

Figure 11. (Color online) Debt Dynamics with Default



Notes. The dynamics is in steady state (the long-run level, the instability level, and the ceiling are time independent) for all but the last two periods ($t \in [9.5, 10]$). Default happens while in steady state, as the debt-to-asset switches from the mean-reverting regime to the explosive regime as it crosses the instability level. (a) A sample trajectory of asset unit value and associated balance-sheet dynamics in which the firm defaults. (b) After a sequence of negative fundamental returns, the debt-to-asset ratio reaches the instability level. After this, the debt-capacity collapses, ultimately leading to the firm's default. In gray, the mean-reverting regime $(0, \overline{x}_t)$, with $\overline{x}_t$ the instability level.

implications for data collection. Much work remains to be done, in particular, to retrieve from data the belief distribution, which is the critical driver of our results. Our results make the case to collect additional data that would allow us to calibrate the lenders' belief distribution, which we have shown is a critical driver of debt stability. In an empirical study, we have used as an input to the model the actual fundamental returns of the aggregate assets of FINRA members (security dealers). Our model provides an early warning signal for the collapse of the short-term debt just before the actual event in Q3 2008. This signal is robust and can be used as well in the case of a single-broker dealer that finances itself via significant amounts of short-term debt.

Like other works on the lenders' debt-provision game, we restrict the firm from issuing equity. In contrast to debt financing, with equity financing, only optimists invest. Because there is no put option, a model with equity financing would not show spiraling effects unless there is a shift in the belief distribution.

In future work, we plan to extend the model to a setting where lenders learn the distribution of the asset return. Learning will reduce the heterogeneity of lenders' beliefs, and, thus, we expect that the long-run level will increase and that debt will be more unstable. We also expect that early indicator of default to disappear as lenders learn the true mean, as any marginal lender would have the same belief as an oracle who detects the early indicator.

Having shown the existence of an instability level for the debt-to-asset ratio, perhaps the most important practical implication emerging out of our study is the timing for intervention strategies. Such strategies would involve changing the firm's portfolio in conjunction with a signal to influence the lenders' belief distribution. These strategies would need to be implemented as soon as leverage approaches the instability level.

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