On the Need for Switched-Gain Observers for Non-Monotonic Nonlinear Systems[○]

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Abstract

This paper focuses on the challenges in observer design for nonlinear systems which are non-monotonic. A class of nonlinear systems is considered in which the process dynamics and output equations are both composed of nonlinear vector functions of scalar combinations of the states. The nonlinear functions are assumed to be differentiable with bounded derivatives. An observer design algorithm that requires solving just a single linear matrix inequality for exponentially convergent state estimation is developed. The developed algorithm works effectively when the involved nonlinear functions are monotonic. However, it fails when all or even some of the system functions are non-monotonic. Both numerical computation and analytical results show that the observer design LMI has no feasible solutions when either all output functions or all process dynamics functions are non-monotonic. Further, other constant gain LMI-based observer design methods from literature also fail when the involved nonlinear functions are all non-monotonic, no matter how small the Lipschitz constant or the Jacobian bounds of the nonlinearities. This limitation has not previously been recognized in observer design literature. To overcome this limitation, a hybrid observer that switches between multiple constant observer gains is developed that can provide global asymptotic stability for systems with non-monotonic nonlinear functions. Hybrid observers with switched gains enable existing observer design methods to be utilized for non-monotonic nonlinear functions with finite local extrema. The application of the developed hybrid observer to two motion estimation applications, one a vehicle position tracking problem on roads and another a piston position estimation problem for an industrial actuator, are demonstrated.

Key words: Observers, nonlinear systems, hybrid systems, linear matrix inequalities, monotonic systems.

1. Introduction

Observer design for nonlinear systems continues to be a topic of significant research interest. Two major approaches to nonlinear observer design include the high gain observer approach (Khalil H. K., 2015) (Khalil & Praly, 2014) and the linear matrix inequality (LMI) based design approach (Zemouche, et al., Nov 2017). The high-gain observer approach is used for systems in triangular form or any system that can be transformed into a triangular structure (Khalil H. K., 2015). The advantage of the high gain methodology is that it always guarantees the existence of an exponentially convergent observer, thanks to the tuning of only one parameter that should be chosen large enough (Boizot, Busvelle, & Gauthier, 2010). Although the practicability of high-gain observer in output feedback control has been nicely analyzed by Khalil's work (Khalil H. K., 2015) (Khalil & Praly, 2014), the use of a large gain and the consequent sensitivity to noise as well as high frequency model uncertainty remains a drawback. To overcome this obstacle, many research papers have addressed high-gain observers with time-varying parameter adaptation, and a number of different switched-gain schemes (Khalil & Ahrens, 2009), (Boizot, Busvelle, & Gauthier, 2010), (Andrieu, Praly, & Astolfi, 2009).

The LMI-based observer design approaches have been developed in the literature by a number of different authors for different classes of nonlinear systems. For example, LMI-based observer design methods have been developed for systems with Lipschitz nonlinearities (Zemouche, et al., Nov 2017), (Rajamani R. , 1998), differentiable nonlinear systems with locally bounded Jacobians (Wang, Rajamani, & Bevly, April 2017), systems with nonlinearities satisfying an incremental quadratic inequality (Acikmese & M.Corless, 2005), and for monotonic nonlinear systems (Arcak & P. Kokotovic, 2001). Each new LMI technique aims to provide a better way to get less conservative LMI conditions compared to previous results for the class of systems under consideration. Despite recent theoretical advances in this field (Zemouche, et al., Nov 2017), (Oueder, Farza, Abdennour, & M'Saad, 2012), the search

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for a single widely-applicable powerful observer design method still remains open. A recent result of importance develops an observer design method which is a bridge between the LMI-based and the high-gain design methods (Zemouche, Zhang, Mazenc, & Rajamani, August 2019). Varying gain observers (in which the observer gain varies with parameters of the plant) for linear parameter varying systems (Bara, Daafouz, Kraatz, & Ragot, 2001) and for nonlinear parameter varying systems (Wang, Rajamani, & Bevly, April 2017) have also been explored under the LMI observer design framework.

While the above results from literature represent significant progress in developing viable observer design techniques, this paper demonstrates that the above LMI design techniques fall short when it comes to actual application to practical nonlinear systems. In particular, this paper shows that when the nonlinear functions are non-monotonic (whether in the process dynamics or in the output equations), none of the existing LMI design methods may yield feasible solutions. This constitutes a major shortcoming of all existing LMI-based methods that has not been explicitly recognized in literature. Further, previous approaches for converting a differentiable non-monotonic function to a monotonic function by subtraction of a linear function of states, are also shown in this paper to not succeed in enabling feasible observer solutions. Here, the monotonicity of a function refers to its being either a non-decreasing or a non-increasing function of its scalar argument.

Where do non-monotonic nonlinear functions arise? This is a valid question, since the casual reader might wonder if practical nonlinear functions encountered in the real world are usually monotonic. It turns out that many nonlinear functions encountered in modern applications are non-monotonic. For instance, robotic multi-link systems involve complex combinations of trigonometric functions which are non-monotonic, especially if the range of involved joint rotations are sufficiently large (Rajamani R. , 1998). State-of-charge estimation in batteries often involves non-monotonic output functions, when the outputs are either measured load cell force (Polóni, Figueroa-Santos, Siegel, & Stefanopoulou, 2018) or sometimes even measured terminal voltage (Tian, Fang, & Chen, 2019). Tracking of other vehicles on urban roads in autonomous vehicles often involves nonlinear dynamic models (Jeon, Zemouche, & Rajamani, 2019). Estimation of piston position in industrial actuators using magnetic sensors also involves nonlinear non-monotonic output functions (Madson & Rajamani, 2017).

The contributions of this paper are: the presentation of observer design LMIs for nonlinear systems with representation in the form of functions of scalar state combinations, the demonstration of infeasibility to solutions for these observer design LMIs when the nonlinear functions are all non-monotonic, the demonstration of continued infeasibility with standard linear subtraction conversion methods, the development of hybrid observer design methods that provide global stability for non-monotonic systems, and the application of the developed hybrid observer techniques for two practical applications, namely vehicle tracking on roads and piston position estimation in industrial actuators.

2. Observer Design for Systems with Nonlinear Functions of Scalar Variables

In this section, we develop an observer design method for a class of nonlinear systems in which the process dynamics and outputs both have vector nonlinear functions, with their components being functions of scalar variables. The class of systems is given by the following plant equations

$$\dot{x} = Ff(x) + g(y,u) \tag{1}$$

$$y(x) = h(x) \tag{2}$$

with

$$f(x) = \begin{Bmatrix} f_1(E_1x) \\ \vdots \\ f_r(E_rx) \end{Bmatrix} \text{ and } h(x) = \begin{Bmatrix} h_1(C_1x) \\ \vdots \\ h_m(C_mx) \end{Bmatrix} \tag{3}$$

where $x \in R^n$, $y \in R^m$, $F \in R^{n\,x\,r}$, $E_i^T \in R^n$, $f_i: R \to R$, $i = 1,2,\dots,r$ and $g(y,u): R^{m\,x\,q} \to R^n$. Thus, each of the f_i functions is a function of different scalar variables E_ix. Also, $C_j^T \in R^n$, $h_j: R \to R$ and $j = 1,2,\dots,m$. There are m outputs, but all of them are functions of different scalar variables C_jx. Note that the control input u is decoupled from $f(x)$, although the control input could be coupled to the state by replacing $f(x)$ with $f(x,u)$ if the control input and the Jacobian $\partial f/\partial x$ are both bounded, in spite of the presence of the control input in $f(x,u)$, as done for example in (Acikmese & M.Corless, 2005).

We also assume that the functions $f(x)$ and $h(x)$ satisfy the following conditions:

$$-\infty < M_j \le \frac{\partial h_j}{\partial(C_jx)} \le N_j < +\infty, \qquad j = 1, \dots., m \tag{4}$$

$$-\infty < U_i \le \frac{\partial f_i}{\partial(E_ix)} \le V_i < +\infty, \qquad i = 1,2, \dots., r \tag{5}$$

Define the diagonal matrices of the bounds as: $M = diag(M_1, M_2, \dots, M_m)$, $N = diag(N_1, N_2, \dots, N_m)$, $U = diag(U_1, U_2, \dots, U_r)$ and $V = diag(V_1, V_2, \dots, V_r)$.

Note that equation (1) can certainly represent nonlinear systems in which each function $f_i(E_ix)$ is a nonlinear function of a scalar linear combination of the states. Further, $Ff(x)$ can represent linear combinations of nonlinear functions $f_i(E_ix)$. The reason it is necessary to consider functions of only scalar variables (and combinations of such functions) is because a monotonic function is properly defined in this manuscript as being either a non-decreasing or non-increasing function of its scalar argument.

Let the state observer be given by

$$\dot{\hat{x}} = Ff(\hat{x}) + g(y,u) + L[y - h(C\hat{x})] \tag{6}$$

where $C \in R^{m\,x\,n}$, and $C^T = [C_1^T \quad \dots \quad C_m^T]$. Note that there is a minor abuse of notation in using $h(Cx)$ instead of $h(C_1x, C_2x, \dots, C_mx)$, but provides more compact writing.

$$
\begin{bmatrix}
-\frac{1}{2}C^T(M^TN+N^TM)C-\frac{1}{2}E^T(V^TU+U^TV)E+\sigma P & PF+\frac{1}{2}(E^TU^T+E^TV^T) & -PL+\frac{1}{2}(C^TM^T+C^TN^T) \\
F^TP+\frac{1}{2}(VE+UE) & -I & 0 \\
-L^TP+\frac{1}{2}(NC+MC) & 0 & -I
\end{bmatrix} \leq 0 \quad (8)
$$

Box I.

Let the estimation error be $\tilde{x} = x - \hat{x}$. Then the estimation error dynamics obtained by subtracting equation (6) from equation (1) are given by:

$$\dot{\tilde{x}} = F\tilde{f}(x,\hat{x}) - L\tilde{h}(x,\hat{x}) \quad (7)$$

for $\tilde{f}(x,\hat{x}) = f(x) - f(\hat{x})$ and $\tilde{h}(x,\hat{x}) = h(Cx) - h(C\hat{x})$.

Theorem 1. If the LMI (8) in Box I has a feasible solution that yields an observer gain L and a symmetric positive definite matrix $P > 0$, then the observer of equation (6) using this observer gain is globally exponentially stable with a convergence rate of at least $\sigma/2$.

Proof. Consider the Lyapunov function candidate $V = \tilde{x}^T P\tilde{x}$, with $P > 0$. Substituting from equation (7),

$$\dot{V} = \tilde{f}^T F^T P\tilde{x} - \tilde{h}^T L^T P\tilde{x} + \tilde{x}^T PF\tilde{f} - \tilde{x}^T PL\tilde{h} \text{ , or}$$

$$\dot{V} = [\tilde{x}^T \quad \tilde{f}^T \quad \tilde{h}^T]\begin{bmatrix} 0 & PF & -PL \\ F^TP & 0 & 0 \\ -L^TP & 0 & 0 \end{bmatrix}\begin{bmatrix}\tilde{x} \\ \tilde{f} \\ \tilde{h}\end{bmatrix} \quad (9)$$

Using the differential mean value theorem, the output function difference is

$$\tilde{h}(x,\hat{x}) = \begin{Bmatrix} \vdots \\ h_i(C_ix) - h_i(C_i\hat{x}) \\ \vdots \end{Bmatrix} =$$

$$\begin{bmatrix} \left.\frac{\partial h_1}{\partial(C_1x)}\right|_{z_1=\bar{z}_1} & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & \left.\frac{\partial h_m}{\partial(C_mx)}\right|_{z_m=\bar{z}_m} \end{bmatrix}(Cx - C\hat{x}) \quad (10)$$

where $z_i = C_ix$. Then, using the lower and upper Jacobian bounds of $\frac{\partial h_j}{\partial(C_jx)}$ in equation (10),

$$\tilde{h}(x,\hat{x}) - MC\tilde{x} = diag\left\{\left.\frac{\partial h_1}{\partial(C_1x)}\right|_{z=z_1} - M_1, \ldots, \left.\frac{\partial h_m}{\partial(C_mx)}\right|_{z=z_m} - M_m\right\}C\tilde{x} \quad (11)$$

and

$$\tilde{h}(x,\hat{x}) - NC\tilde{x} = diag\left\{\left.\frac{\partial h_1}{\partial(C_1x)}\right|_{z=z_1} - N_1, \ldots, \left.\frac{\partial h_m}{\partial(C_mx)}\right|_{z=z_m} - N_m\right\}C\tilde{x} \quad (12)$$

From (11) and (12), due to M_i being a lower bound and N_i being an upper bound in each of the diagonal terms in the diagonal matrices, it follows that

$$\left[\tilde{h}(x,\hat{x}) - MC\tilde{x}\right]^T\left[\tilde{h}(x,\hat{x}) - NC\tilde{x}\right] \leq 0 \quad (13)$$

Equation (13) can be rewritten in matrix form as

$$[\tilde{x}^T \quad \tilde{h}(x,\hat{x})^T]\begin{bmatrix} C^TM^TNC & -C^TM^T \\ -NC & I \end{bmatrix}^T\begin{bmatrix}\tilde{x} \\ \tilde{h}(x,\hat{x})\end{bmatrix} < 0 \quad (14)$$

Since M and N can also be switched in (14), a symmetric form of the constant matrix in (14) is

$$\begin{bmatrix} 0.5(C^TM^TNC + C^TN^TMC) & -0.5(C^TM^T + C^TN^T) \\ -0.5(MC + NC) & I \end{bmatrix} \quad (15)$$

Similarly, for the difference $\tilde{f}(x,\hat{x}) = f(x) - f(\hat{x})$, it can be shown that the corresponding symmetric matrix is

$$\begin{bmatrix} 0.5(E^TU^TVE + E^TV^TUE) & -0.5(E^TU^T + E^TV^T) \\ -0.5(UV + VE) & I \end{bmatrix} \quad (16)$$

Combining matrices (15) and (16) into a larger matrix form, the constraint (20) in Box II on the nonlinear functions $\tilde{f}(x,\hat{x})$ and $\tilde{h}(x,\hat{x})$ and their Jacobian bounds is obtained.

Replacing the condition $\dot{V} \leq 0$ with the condition $\dot{V} + \sigma P < 0$ ensures that the estimation error has an exponential convergence rate of at least $\sigma/2$, as described in Chapter 4 of (Khalil H. , 2001). Using the S-procedure Lemma (Boyd, Ghaoui, Feron, & Balakrishnan, 1994), $\dot{V} + \sigma P < 0$ if and only if there exists $\epsilon > 0$ such that $\dot{V} + \sigma P \leq \epsilon V_1$ where V_1 is defined in equation (20) in Box II. Hence, equation (21) in Box III is obtained. Absorbing ϵ into the P matrix on the left-hand side of equation (21), the LMI of equation (8) then follows. ■

It should be noted that Theorem 1 is an observer design method for global exponential stability and is only a *sufficient* condition.

The following corollaries of Theorem 1 are presented below for the special cases where either only the process dynamics or only the output equations are nonlinear. In these cases, lower dimensional LMIs can be obtained in place of the LMI (8).

Corollary 1.1. Consider the case where the process dynamics are linear ($Ff(x) = Ax$) and the ouputs are nonlinear. In this case, if an observer gain L and a symmetric positive definite matrix $P > 0$ that satisfy equation (22) in Box IV can be obtained, then the observer with this gain is globally exponentially stable.

Proof. The estimation error dynamics in this case are

$$\dot{\tilde{x}} = A\tilde{x} - L\tilde{h}(x,\hat{x}) \quad (17)$$

$$V_1 = [\tilde{x}^T \quad \tilde{f}^T \quad \tilde{h}^T]\begin{bmatrix} C^T\left(\frac{M^TN+N^TM}{2}\right)C+E^T\left(\frac{V^TU+U^TV}{2}\right)E & -\left(\frac{E^TU^T+E^TV^T}{2}\right) & -\left(\frac{C^TM^T+C^TN^T}{2}\right) \\ -\left(\frac{VE+UE}{2}\right) & I & 0 \\ -\left(\frac{NC+MC}{2}\right) & 0 & I \end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{f}\\ \tilde{h}\end{bmatrix} \le 0 \tag{20}$$

Box II.

$$[\tilde{x}^T \quad \tilde{f}^T \quad \tilde{h}^T]\begin{bmatrix} \sigma P & PF & -PL \\ F^TP & 0 & 0 \\ -L^TP & 0 & 0 \end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{f}\\ \tilde{h}\end{bmatrix} \le \epsilon\begin{bmatrix} \frac{1}{2}C^T(M^TN+N^TM)C+\frac{1}{2}E^T(V^TU+U^TV)E & -\frac{1}{2}(E^TU^T+E^TV^T) & -\frac{1}{2}(C^TM^T+C^TN^T) \\ -\frac{1}{2}(VE+UE) & I & 0 \\ -\frac{1}{2}(NC+MC) & 0 & I \end{bmatrix} \tag{21}$$

Box III.

$$\begin{bmatrix} A^TP+PA-\frac{C^TM^TNC+C^TN^TMC}{2}+\sigma P & -PL+\frac{C^TM^T+C^TN^T}{2} \\ -L^TP+\frac{MC+NC}{2} & -I \end{bmatrix} \le 0 \tag{22}$$

Box IV.

$$\begin{bmatrix} -C^TL^TP-PLC-\frac{E^TU^TVE+E^TV^TUE}{2}+\sigma P & PF+\frac{E^TU^T+E^TV^T}{2} \\ F^TP+\frac{UE+VE}{2} & -I \end{bmatrix} \le 0 \tag{23}$$

Box V.

Using the same Lyapunov function as in Theorem 1,

$$\begin{aligned}\dot{V} &= \dot{\tilde{x}}^TP\tilde{x}+\tilde{x}^TP\dot{\tilde{x}} \\ &= \tilde{x}^TA^TP\tilde{x}-\tilde{h}^TL^TP\tilde{x}+\tilde{x}^TPA\tilde{x}-\tilde{x}^TPL\tilde{h}, \text{ or}\end{aligned}$$

$$\dot{V} = [\tilde{x}^T \quad \tilde{h}^T]\begin{bmatrix} A^TP+PA & -PL \\ -L^TP & 0 \end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{h}\end{bmatrix} \tag{18}$$

The output difference function in matrix form is

$$V_1 = [\tilde{x}^T \quad \tilde{h}(x,\hat{x})^T]\begin{bmatrix} \frac{C^TM^TNC+C^TN^TMC}{2} & -\frac{C^TM^T+C^TN^T}{2} \\ -\frac{MC+NC}{2} & I \end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{h}(x,\hat{x})\end{bmatrix} < 0 \tag{19}$$

Using the S-Procedure Lemma (Boyd, Ghaoui, Feron, & Balakrishnan, 1994) again, with $\dot{V} < \epsilon V_1$ yields the LMI

$$\begin{bmatrix} A^TP+PA & -PL \\ -L^TP & 0 \end{bmatrix} < \epsilon\begin{bmatrix} \frac{C^TM^TNC+C^TN^TMC}{2} & -\frac{C^TM^T+C^TN^T}{2} \\ -\frac{MC+NC}{2} & I \end{bmatrix}$$

Absorbing $1/\epsilon$ into the matrix P to define a new positive definite matrix and adding the term $+\sigma P$ to the (1,1) term above for a minimum convergence rate of $\sigma/2$, the final observer design LMI is obtained as equation (22) specified in the Corollary. ■

Corollary 1.2. Consider the case where the process dynamics are nonlinear and the outputs are linear ($y = Cx$). In this case, if an observer gain L that satisfies equation (23) in Box V can be obtained, then the observer with this gain is globally exponentially stable.

Proof. The estimation error dynamics in this case are

$$\dot{\tilde{x}} = F\tilde{f}(x,\hat{x}) - LC\tilde{x} \tag{24}$$

Using the same Lyapunov function as in Theorem 1,
$\dot{V} = \dot{\tilde{x}}^T P\tilde{x} + \tilde{x}^T P\dot{\tilde{x}} = \tilde{f}^T F^T P\tilde{x} - \tilde{x}^T C^T L^T P\tilde{x} + \tilde{x}^T PF\tilde{f} - \tilde{x}^T PLC\tilde{x}$, or in matrix form

$$\dot{V} = [\tilde{x}^T \quad \tilde{f}^T]\begin{bmatrix} -C^T L^T P - PLC & PF \\ F^T P & 0 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{f} \end{bmatrix} \tag{25}$$

The output difference function in matrix form is

$$V_1 = [\tilde{x}^T \quad \tilde{f}(x,\hat{x})^T]$$
$$\begin{bmatrix} \frac{E^T U^T V E + E^T V^T U E}{2} & -\frac{E^T U^T + E^T V^T}{2} \\ -\frac{UE + VE}{2} & I \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{f}(x,\hat{x}) \end{bmatrix} < 0 \tag{26}$$

Using the S-Procedure Lemma (Boyd, Ghaoui, Feron, & Balakrishnan, 1994) again, with $\dot{V} < \epsilon V_1$ yields the LMI

$$\begin{bmatrix} -C^T L^T P - PLC & PF \\ F^T P & 0 \end{bmatrix} < \epsilon \begin{bmatrix} \frac{E^T U^T V E + E^T V^T U E}{2} & -\frac{E^T U^T + E^T V^T}{2} \\ -\frac{UE + VE}{2} & I \end{bmatrix}$$

Absorbing $1/\epsilon$ into the matrix P to define a new positive definite matrix and adding the term $+\sigma P$ to the (1,1) term above for a minimum convergence rate of $\sigma/2$, the final observer design LMI is obtained as equation (23) specified in the Corollary. ■

3. Non-Existence of a Constant Observer Gain Solution for Non-Monotonic Systems

3.1 Non-existence for all non-monotonic functions

Theorem 2. If ALL of the output functions $h_j(C_j x)$, $j = 1,2,\dots,m$ as well as the process dynamic nonlinear functions $f_i(E_i x)$ $i = 1,2,\dots,r$ are non-monotonic, then a constant gain observer that satisfies the observer design LMI (8) does not exist.

Proof. A necessary condition for (8) to be satisfied is that

$$-\frac{1}{2}C^T(M^T N + N^T M)C - \frac{1}{2}E^T(V^T U + U^T V)E + \sigma P < 0 \tag{27}$$

If all nonlinear functions are non-monotonic, then $M < 0$, $N > 0$, $U < 0$ and $V > 0$. Hence, a solution to equation (27), and therefore to equation (8), can never exist.

3.2 Non-existence for non-monotonic outputs

Consider the case where the process dynamics are linear and the output equations have non-monotonic nonlinear functions, as in Corollary 1.1.

Theorem 3. If ALL of the output functions $h_i(C_i x)$, $i = 1,2,\dots,m$ are non-monotonic, and if the open-loop system is not already asymptotically stable, then a constant gain observer that satisfies the design LMI (22) does not exist.

Proof. For this system where the process dynamics are linear and the outputs are described by nonlinear equations, Corollary 1.1 for observer design applies. Hence, if the LMI (22) were feasible, a globally exponentially stable observer would result. For (22) to be feasible, a necessary condition is that its (1,1) element be negative definite, or

$$A^T P + PA - \frac{C^T M^T N C + C^T N^T M C}{2} + \sigma P < 0 \tag{28}$$

If all the output functions are non-monotonic, then $M_i < 0$ and $N_i > 0$. This implies $M^T N < 0$ and $N^T M < 0$.

Since $C^T M^T N C + C^T N^T M C \le 0$, and $P > 0$, this implies $A^T P + PA < 0$. For $A^T P + PA$ to be negative definite, A must exponentially stable, which contradicts the assumptions of the Theorem. Hence, a constant gain observer that satisfies (22) cannot exist.

□

3.3 Non-existence for non-monotonic process dynamics

Consider the case where the process dynamics are nonlinear and the output measurement equations are linear.

Theorem 4. If ALL of the process dynamics functions $f_i(E_i x)$, $i = 1,2,\dots,n$ are non-monotonic, and if the output C matrix is not full rank, then a constant gain observer that satisfies the observer design LMI (8) does not exist.

Proof. For this system where the process dynamics are nonlinear and the output equations are linear, $h(x) = Cx$ and $M = N = I$. Then the LMI (8) can be rewritten as (30) in Box VI.

If all the process dynamics functions $f_i(E_i x)$ are non-monotonic, then $\frac{\partial f_i}{\partial(E_i x)}$ takes both positive and negative values. Hence $U_i < 0$ and $V_i > 0$. This implies that the diagonal matrices $U^T V < 0$ and $V^T U < 0$.

For equation (30) to be satisfied, a necessary condition is

$$-\frac{1}{2}C^T C - \frac{E^T U^T V E + E^T V^T U E}{2} + \sigma P \le 0 \tag{29}$$

Since $E^T U^T V E + E^T V^T U E < 0$ and $P > 0$, this implies $-\frac{1}{2}C^T C < 0$.

This implies that C is full rank, which contradicts the assumptions of the Theorem. Hence, a constant gain observer that satisfies the observer design LMI (8) cannot exist. ■

Corollary 3.1: If the system under consideration has nonlinear process dynamics and a linear output equation, and is a single output system, then a constant gain observer does not exist if the output function is non-monotonic.

Interpretation and Relation to Unobservability: The proof of this Corollary follows directly from Theorem 3. Further, the non-existence result is easy to interpret in the case of the single output system. If the output nonlinear function is non-monotonic, then it has both positive and negative values of the derivative with respect to its

$$\begin{bmatrix} -\frac{1}{2}C^TC - \frac{1}{2}E^T(V^TU + U^TV)E + \sigma P & PF + \frac{1}{2}(E^TU^T + E^TV^T) & -PL + \frac{1}{2}(C^T + C^T) \\ F^TP + \frac{1}{2}(VE + UE) & -I & 0 \\ -L^TP + \frac{1}{2}(C + C) & 0 & -I \end{bmatrix} \leq 0 \tag{30}$$

Box VI.

$$\begin{bmatrix} A^TP + PA - \frac{C^TM^TNC + C^TN^TMC}{2} + \sigma P & -PL + \frac{C^TM^T + C^TN^T}{2} \\ -L^TP + \frac{MC + NC}{2} & -I \end{bmatrix} \leq 0 \tag{31}$$

Box VII.

argument. This implies that for $A - L\frac{\partial h}{\partial(Cx)}C$ to be asymptotically stable, L has to change signs with the sign of $\frac{\partial h}{\partial(Cx)}$, or else the open-loop matrix A must already be stable. Hence, the non-existence result can be easily understood for this single output system. Further, note that when $\frac{\partial h}{\partial(Cx)}$ changes sign, it also goes through a value of zero, implying local loss of observability at one point in the operating domain of the system. Thus, the lack of a stabilizing observer gain for this non-monotonic system agrees with the *loss of observability* that occurs at the zero-slope point of the output nonlinear function.

3.4 Non-existence for partially non-monotonic outputs

The previous two sub-sections showed that a stabilizing constant observer gain does not exist for the cases where either all output functions, or all process dynamic functions are non-monotonic. This sub-section presents examples to show that a stabilizing observer gain may not exist even if only SOME of the outputs have nonmonotonic functions.

Example 1. Consider the special case where the plant equations are

$$\dot{x} = Ax + Bu \tag{32}$$

$$y = h(C_0x) \tag{33}$$

with $C_0^T \in R^n$ and $h: R \to R^m$. Thus, there are m outputs, but all of them are functions of the same single scalar variable C_0x.

In this case, the observer design condition is (31) in Box VII. Now, since C^TC is a rank one matrix (due to all rows of C being the same C_0),

$$\frac{C^TM^TNC + C^TN^TMC}{2} = \left(\sum_i M_iN_i\right)C_0^TC_0 \tag{34}$$

Now $\sum_i M_iN_i$ is a scalar and could be positive or negative, depending on how many output functions are non-monotonic and have negative values of M_i and positive values of N_i. If $\sum_i M_iN_i$ turns out to be negative, even if only some of the output functions are non-monotonic, then

$$\frac{C^TM^TNC + C^TN^TMC}{2} = \left(\sum_i M_iN_i\right)C_0^TC_0 \leq 0 \tag{35}$$

Hence

$$A^TP + PA - \frac{C^TM^TNC + C^TN^TMC}{2} + \sigma P < 0$$
$$\Rightarrow A^TP + PA < 0$$

which again would require A itself to be exponentially stable, which would contradict the assumptions of Theorem 3. Hence, a constant observer gain may not exist, even if only some of the output functions are non- monotonic.

Example 2. Consider the 3rd order system with

$C_1 = [a \quad b \quad c]$ and $C_2 = [d \quad e \quad f]$.

Without loss of generality, let the first output function be monotonic with $m_1 = 0$ and $n_1 > 0$. Let the second output function be non-monotonic with $m_2 < 0$ and $n_2 > 0$, with $m_2n_2 = -1$. Then

$$\frac{C^TM^TNC + C^TN^TMC}{2} = \begin{bmatrix} -d^2 & -de & -df \\ -de & -e^2 & -ef \\ -df & -ef & -f^2 \end{bmatrix}$$

It is easy to see that there are many values of d, e and f such that $\frac{C^TM^TNC+C^TN^TMC}{2} \leq 0$. This would again make the assumptions of Theorem 3 invalid, even though only one of the two output functions is non monotonic.

3.5 Non-Existence of a Constant Gain with Other LMI-Based Methods of Nonlinear Observer Design

It can be shown that the following popular methods of observer design for nonlinear systems from literature all fail to yield a solution with a constant observer gain for systems with all non-monotonic nonlinear functions:

a) Observer design method of Arcak and Kokotovic using the Circle Criterion (Arcak & P. Kokotovic, 2001)
b) Observer design method of Phanomcheong, et al for bounded Jacobian nonlinear systems (Phanomchoeng, Rajamani, & Piyabongkarn, May 2011)
c) High gain observer design method, when the output function is non-monotonic, as demonstrated in (Boizot, Busvelle, & Gauthier, 2010).

As for the extended Kalman filter (EKF), it is not related to the non-existence results as introduced in this paper because the EKF uses a time-varying observer gain, instead of a constant one.

Another popular method of nonlinear observer design is through transformation to a normal form under which the observer design can be done simply with eigenvalue assignment with a constant observer gain. However, it is also important to note that in the original coordinates, the gain may not be constant because it depends on the (left) inverse of the state transformation Jacobian, which is not constant if the coordinate transformation is not constant.

Further, finding a nonlinear transformation to put the nonlinear system under a normal form is not an easy task in general, and sometimes requires solvability of a set of partial differential equations. We recognize that normal-form-based methods are useful for observer design, but the use of the switched gain approach proposed in this manuscript does not require any transformation of the system nor any changes to the structure of the LMI conditions obtained with a constant observer gain. We need only to switch between regions of monotonicity.

4. No Benefits from Conversion to Monotonicity by Linear Subtraction

A non-monotonic function that has a bounded Jacobian can be converted to a monotonic function by subtracting a linear function of the states from it. This conversion aspect has been described in previous observer design results from literature (Arcak & P. Kokotovic, 2001). This section shows that such a conversion does not help in observer design. Consider a system with nonlinear output functions and linear process dynamics as follows:

$$\dot{x} = Ax + Bu \tag{36}$$

$$y = h(Cx) \tag{37}$$

with $C \in R^{m \times n}$, and $h: R \to R$. Let the original nonlinear output functions be *non-monotonic* so that the diagonal matrices satisfy $M < 0$ and $N > 0$. The original observer design LMI (as derived in Corollary 1.1) is (22) in box IV.

Conversion to a monotonic nonlinear function:

Let $\phi(Cx) = h(Cx) - MCx$. Then $\frac{\partial \phi}{\partial z} = \left.\frac{\partial h}{\partial z}\right|_{z=Cx} - M$

Then, it is easy to see that

$$0 \le \left.\frac{\partial \phi}{\partial z}\right|_{z=Cx} \le N - M \tag{38}$$

Therefore $\frac{\partial \phi}{\partial z} \ge 0$ and hence all the functions in $\phi(Cx)$ are monotonic. Can we use the new nonlinear function $\phi(Cx)$ to re-define the output? Rewrite the original output as

$$y = MCx + h(Cx) - MCx = MCx + \phi(Cx) \tag{39}$$

Rewrite the plant dynamics as

$$\dot{x} = (A - LMC)x + LMCx + Bu \tag{40}$$

and the observer as

$$\begin{aligned}\dot{\hat{x}} &= (A - LMC)\hat{x} + LMC\hat{x} + Bu \\ &\quad + L(h(Cx) - h(C\hat{x}))\end{aligned} \tag{41}$$

Then, the estimation error dynamics are

$$\dot{\tilde{x}} = (A - LMC)\tilde{x} + L(\phi(Cx) - \phi(C\hat{x})) \tag{42}$$

with $\phi(Cx)$ being a monotonic function.

The new observer design LMI uses $(A - LMC)$ instead of A. Also, the lower Jacobian bound of $\phi(Cx)$ is 0 and the upper Jacobian bound is $N - M$ because of the monotonicity of the new output. Then the new observer design LMI, by applying equation (22) is (45) in Box VIII.

This requires $(A - LMC)^T P + P(A - LMC) + \sigma P < 0$ as a necessary condition. Hence $(A - LMC)$ needs to be an asymptotically stable matrix, with $M < 0$.

Theorem 5. The observer design for the new system (39)-(40) in which the nonlinear function has been converted to a monotonic function continues to be infeasible, if it was infeasible for the original system before conversion.

Proof. By using the Schur complements Lemma, the observer LMI (45) for the system (39)-(40) is equivalent to

$$\begin{aligned}&(A - LMC)^T P + P(A - LMC) + \sigma P \\ &+ \left(-PL + \tfrac{C^T(N-M)^T}{2}\right)\left(-L^T P + \tfrac{(N-M)}{2}C\right) \le 0 \text{ , i.e.}\end{aligned}$$

$$\begin{aligned}&A^T P + PA + PLL^T P - C^T\left(\tfrac{N+M}{2}\right)^T L^T P - \\ &PL\left(\tfrac{N+M}{2}\right)C + \tfrac{1}{4}C^T(N-M)^T(N-M)C + \sigma P \le 0\end{aligned} \tag{43}$$

On the other hand, the original observer design LMI for the untransformed system (36)-(37), using equation (22), is equivalent to (46) in Box IX.

$$\begin{aligned}&A^T P + PA - \frac{C^T M^T NC + C^T N^T MC}{2} + \sigma P \\ &\quad + \left(-PL + \tfrac{C^T M^T + C^T N^T}{2}\right)\left(-L^T P + \tfrac{MC + NC}{2}\right) \le 0 \text{ , i.e.}\end{aligned}$$

$$\begin{aligned}&A^T P + PA + PLL^T P - C^T\left(\tfrac{N+M}{2}\right)^T L^T P - \\ &PL\left(\tfrac{N+M}{2}\right)C + \tfrac{1}{4}C^T(N-M)^T(N-M)C + \sigma P \le 0\end{aligned} \tag{44}$$

Thus, inequality (43) turns out to be completely equivalent to inequality (44). ■

Hence the conversion of the non-monotonic output function to a monotonic one by subtracting a linear term did

$$\begin{bmatrix} (A-LMC)^T P + P(A-LMC) + \sigma P & -PL + \frac{(N-M)C^T}{2} \\ -L^T P + \frac{(N-M)C}{2} & -I \end{bmatrix} \leq 0 \tag{45}$$

Box VIII.

$$\begin{bmatrix} A^T P + PA - \frac{C^T M^T NC + C^T N^T MC}{2} + \sigma P & -PL + \frac{C^T M^T + C^T N^T}{2} \\ -L^T P + \frac{MC+NC}{2} & -I \end{bmatrix} \leq 0 \tag{46}$$

Box IX.

not help. A constant gain observer continues to be infeasible, if the original function is non-monotonic.

5. Hybrid Observer Design Using Switched Gains and Switched Lyapunov Functions

For the plant with nonlinear process dynamics and nonlinear output equations, as given in (1)-(3), consider a hybrid observer with two constant-gain regions, as shown in Figure 1, with no loss of generality. Let the observer be designed with an observer gain L_1 in region 1 and L_2 in region 2. Let the two observers be designed to be exponentially stable in each of the two regions using the following two LMIs: (52) in Box X for all $y \leq y_{switch} + \epsilon$, and (53) in Box XI for all $y \geq y_{switch} - \epsilon$. Here y_{switch} is the nominal switching point between the two regions and the variable ϵ is the hysteresis added to the switching to ensure a minimum dwell time after each switch.

Note that a nonlinear function on a compact set has finite local extrema and can always therefore be represented using piecewise monotonic functions. Hence, the observer can be designed using a finite set of piecewise regions with the monotonicity being ensured in each region.

Theorem 6. Let P_1, L_1 and P_2, L_2 be the Lyapunov function matrices and observer gain matrices in regions 1 and 2 respectively, chosen so as to satisfy equations (52) and (53). Let σ_1 and σ_2 be the minimum exponential convergence rates in the two regions. Choose a value of τ such that the following equations are satisfied:

$$P_1 \geq P_2 e^{-\tau\sigma_2} \tag{47}$$

and

$$P_2 \geq P_1 e^{-\tau\sigma_1} \tag{48}$$

Then, if the switching between regions does not occur faster than τ, the hybrid observer system will be globally asymptotically stable.

Note: It is always possible to find a $\tau > 0$ sufficiently large such that both equations (47) and (48) are satisfied.

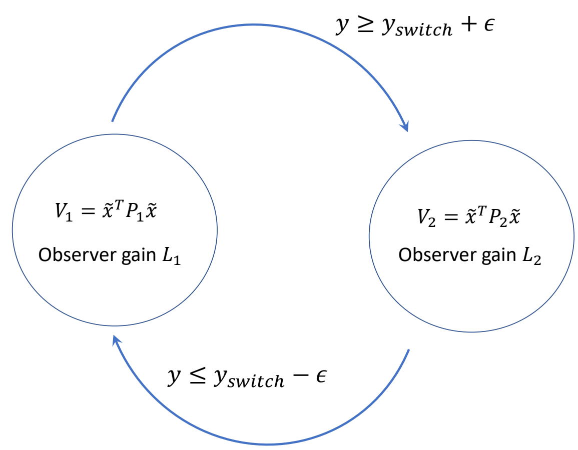

Fig. 1: Hybrid Observer

Proof. Without loss of generality, consider a switching from region 1 to region 2. Let the switching occur at time t_s. Then, according to the assumption in the theorem, the switch back cannot occur before $t_s + \tau$. At the time of switching, the value of the Lyapunov function in region 1 is

$$V_1(t_s) = \tilde{x}(t_s)^T P_1 \tilde{x}(t_s)$$

and in region 2 is

$$V_2(t_s) = \tilde{x}(t_s)^T P_2 \tilde{x}(t_s) \ .$$

Since the convergence rate in region 2 is at least σ_2,

$$\dot{V}_2(t) \leq -\sigma_2 V_2(t) \tag{49}$$

Integrating both sides to obtain a relationship between $V_2(t_s+\tau)$ and $V_2(t_s)$, it can be shown that

$$V_2(t_s+\tau) \leq V_2(t_s) e^{-\tau\sigma_2} \tag{50}$$

Equation (50) implies

$$\tilde{x}(t_s+\tau)^T P_2 \tilde{x}(t_s+\tau) \leq \tilde{x}(t_s)^T P_2 \tilde{x}(t_s) e^{-\tau\sigma_2}$$

But, according to equation (47), $P_1 \geq P_2 e^{-\tau\sigma_2}$. Hence

$$V_2(t_s+\tau) \leq V_1(t_s) \tag{51}$$

$$\begin{bmatrix} -\frac{1}{2}C^T(M^TN+N^TM)C-\frac{1}{2}E^T(V^TU+U^TV)E+\sigma_1P_1 & P_1F+\frac{1}{2}(E^TU^T+E^TV^T) & -P_1L_1+\frac{1}{2}(C^TM^T+C^TN^T) \\ F^TP_1+\frac{1}{2}(VE+UE) & -I & 0 \\ -L_1^TP_1+\frac{1}{2}(NC+MC) & 0 & -I \end{bmatrix} \leq 0 \tag{52}$$

Box X.

$$\begin{bmatrix} -\frac{1}{2}C^T(M^TN+N^TM)C-\frac{1}{2}E^T(V^TU+U^TV)E+\sigma_2P_2 & P_2F+\frac{1}{2}(E^TU^T+E^TV^T) & -P_2L_2+\frac{1}{2}(C^TM^T+C^TN^T) \\ F^TP_2+\frac{1}{2}(VE+UE) & -I & 0 \\ -L_2^TP_2+\frac{1}{2}(NC+MC) & 0 & -I \end{bmatrix} \leq 0 \tag{53}$$

Box XI.

Subsequently, for all $t \geq t_s + \tau$, $V_2(t)$ further keeps decreasing exponentially with an exponential time constant of at least σ_2, as long as the system remains in region 2.

After each switch, the Lyapunov function always decreases to a value below the value at the time of transition and subsequently continues decreasing exponentially. Hence, the values of the Lyapunov function candidate at consecutive switching points t_{s_1} and t_{s_2} can be related by

$$V_2(t_{s_2}) \leq \alpha V_1(t_{s_1}) \tag{54}$$

where $0 \leq \alpha < 1$. Hence, after k switches with a minimum dwell time τ after each switch, we have

$$V_j(t_{s_k}) \leq \alpha^{k-1} V_1(t_{s_1}) \tag{55}$$

where V_j can be V_1 or V_2, depending on k being even or odd. Equation (55) for the repeated decay of the Lyapunov function at consecutive switching points, together with the exponential decay that occurs in each region when there is no switching, ensures that the Lyapunov function converges to zero (Goebel, R.G.Sanfelice, & A.R.Teel, 2012). Hence the estimation error also converges asymptotically to zero.

While Theorem 6 considered only two regions, the proof holds for switching from any region to any other region, as long as the minimum dwell constraint is met. In order to meet the dwell time constraint, hysteresis can be introduced into the switching region, as shown in Figure 1 with the parameter ϵ which provides different points for switching into and out of an observer gain region. While the proposed switching algorithm works well for the applications studied in this paper, there are no switching results necessarily available for all cases of switching when the switching instant is not known apriori. Literature on the design of switching observers and on the stability analysis of switched hybrid systems with adequate dwell time is available in (Liberzon, 2003), (Alessandri & Coletta, 2001), (Alessandri, Baglietto, & Battistelli, 2005), and (Goebel, R.G.Sanfelice, & A.R.Teel, 2012).

6. Application to Vehicle Tracking on Highways and Local Roads

This section develops a nonlinear observer for an autonomous vehicle to estimate motion variables of other vehicles on a road, based on measurements from an on-board radar sensor (Rajamani R. , 2012). The estimation algorithm uses a vehicle tracking algorithm based on a single model to represent all possible vehicle motions involving both longitudinal and lateral maneuvers. By using a single vehicle model, stability of the state observer can be guaranteed and the real-time computational effort in estimating trajectories of multiple vehicles on the road is reduced in comparison with switched model approaches, such as the interacting multiple model (IMM) approach.

Since the proposed vehicle model is nonlinear, an effective nonlinear observer design technique is required to ensure a stable observer. The observer design LMIs developed in this paper, together with the hybrid observer technique are used for the nonlinear observer design.

6.1 Observer design

A bicycle model is used for each tracked vehicle with X being its relative longitudinal position, Y its relative lateral position, and ψ the yaw angle of vehicle, as shown in Fig. 2. The variables X and Y are measured by a radar tracking sensor. The vehicle motion model uses the states:

$$x = [X \quad Y \quad \psi \quad \delta_f\,]^T \tag{56}$$

Under the assumption that the derivative of the steering angle is zero, then the process dynamics are

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{\psi} \\ \dot{\delta}_f \end{bmatrix} = \begin{bmatrix} V cos(\psi) \\ V sin(\psi) \\ \frac{V}{l_f + l_r}\tan(\delta_f) \\ 0 \end{bmatrix} \tag{57}$$

where l_f and l_r are distances to front and rear tires from c.g. of vehicle, δ_f is steering angle of front wheels, V is total velocity at c.g. of vehicle. We assume V is a constant or slowly varying.

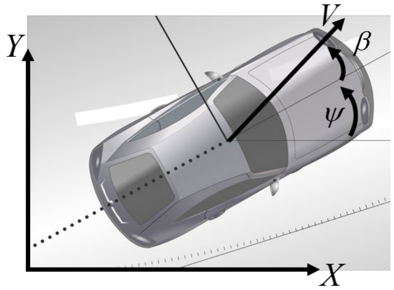

Fig. 2. Vehicle motion model

The output matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{58}$$

Thus, the process dynamics are nonlinear while the output equations are linear. It is also clear that $f_1(\,)$ and $f_2(\,)$ are functions of the state variable ψ while $f_3(\,)$ is a function of the state variable δ_f. The Jacobian is found to be

$$\frac{\partial f_i}{\partial(E_i x)} = \begin{bmatrix} -V\sin(\psi) \\ V\cos(\psi) \\ \dfrac{V}{l_f + l_r}\sec^2(\delta_f) \\ 0 \end{bmatrix} \tag{59}$$

where $E_1 = E_2 = [0 \quad 0 \quad 1 \quad 0]$, and $E_3 = [0 \quad 0 \quad 0 \quad 1]$. Using the observer

$$\dot{\hat{x}} = Ff(\hat{x}) + L(Cx - C\hat{x}) \tag{60}$$

it is found that for the limited operating range $0 < \psi < 90^o$, the functions $f_i(E_i x)$ are monotonic. Hence, it is possible to find a constant observer gain matrix L for the operating range $0 < \psi < 90^o$. However, it is impossible to find a constant gain matrix L that makes the observer stable for the entire operating range $0 < \psi < 360^o$. Hence a switched gain hybrid observer is developed for two different operating regimes as follows

Gain L_1 for the operating range $0^\circ \le \psi \le 80^\circ$, $-10^\circ \le \delta_f \le 10^\circ$:

$$L_1 = \begin{bmatrix} 42.1703 & -23.6081 \\ -51.8942 & 41.5463 \\ -121.0412 & 86.8002 \\ -1.7662 & 1.2666 \end{bmatrix} \tag{61}$$

Gain L_2 for the operating range $60^\circ \le \psi \le 140^\circ$, $-10^\circ \le \delta_f \le 10^\circ$:

$$L_2 = \begin{bmatrix} 46.4268 & 3.2988 \\ 8.4042 & 5.0011 \\ -75.2145 & -6.8793 \\ -1.1358 & -0.1039 \end{bmatrix} \tag{62}$$

V is assumed as 10m/s. It can be seen that the two gains overlap over 20 degrees of yaw angle regime.

6.2 Simulation results

The following simulation scenario is utilized. The vehicle is driving with 10m/s constant velocity and performs the following maneuvers:

i) Straight driving for 5 seconds
ii) Left turning until vehicle yaw angle becomes 120 degrees
iii) Straight driving with its yaw angle

The observer gain is switched at 60 degrees of the estimated yaw angle. Initially, the vehicle estimates are located at (0,0) and oriented with 0 degree yaw angle. We assume that vehicle itself is located at (-5,-5) and oriented with 30 degree. The region of the initial condition is determined to be either $0 \le \psi \le 80^o$ or $60 \le \psi \le 140^o$ by ad-hoc computation of the initial direction of the vehicle from the first few samples of measurements. Whether the target vehicle is traveling parallel or perpendicular to the ego vehicle determines its initial condition region.

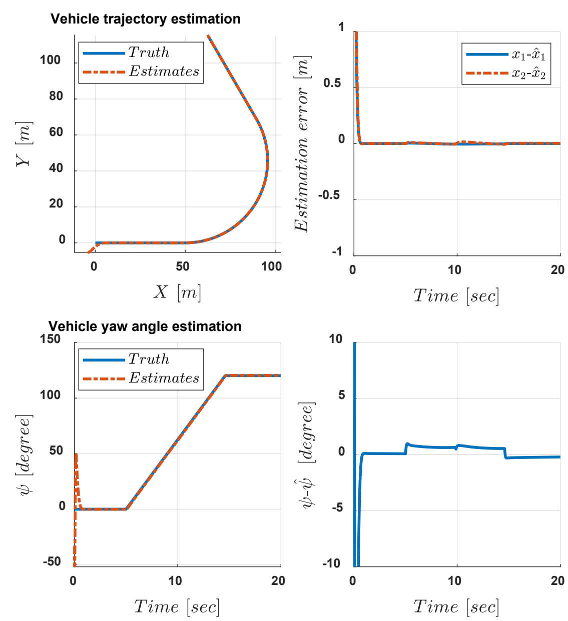

Fig. 3. Simulation results with a switched gain approach

The hybrid observer with the switched gain method provides very good estimation performance, as seen in Fig. 3. The estimated and actual values for both the vehicle trajectory and for vehicle orientation track each other very closely. The estimation error converges to zero from the initial condition error and is subsequently very small during straight driving, increasing slightly only during the turning motion as shown in Fig. 3.

Next, consider the case where a single gain observer is utilized instead of the hybrid observer. The single gain for the first regime from the above observer is utilized for the entire range of vehicle operation in Fig. 4. It can be seen that the yaw angle error increases as vehicle yaw angle increases to be away from the operating range for which the observer was originally designed, as shown in Fig. 4. The error grows significantly after the first 15 seconds when the orientation exceeds the observer's stable operating regime.

These simulation results clearly show both the stability of the switched gain observer and also its superiority compared to a constant gain observer.

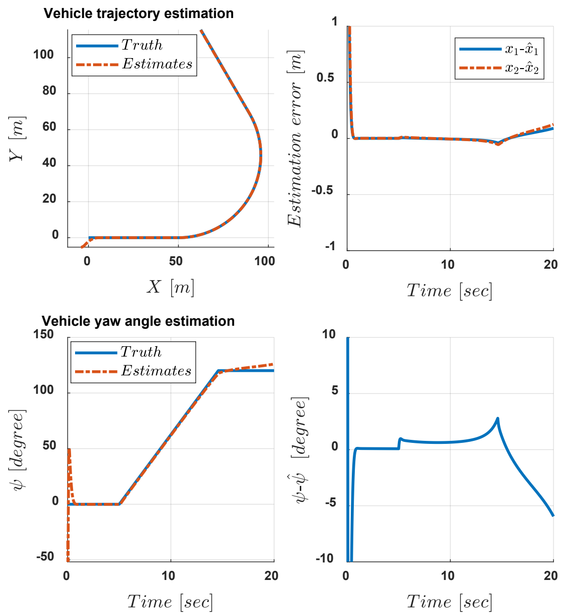

Fig. 4. Simulation results without a switched gain approach

7. Application to Magnetic Position Estimation in Industrial Actuators

This section focuses on a different motion estimation problem, one about estimating the position of a moving piston inside an industrial piston-cylinder actuator. Magnetic position estimation offers an excellent inexpensive and non-contacting method of obtaining piston position on such actuators, including on pneumatic cylinders, hydraulic actuators and IC engines. In magnetic position estimation, a magnet is placed on the moving object, such as the moving piston shown in Fig. 5 (Movahedi, Zemouche, & Rajamani, 2019). A sensor board containing one or more magnetic sensors is placed on the outside cylinder, again as shown in Fig. 5. Such magnetic sensors are inexpensive (as low as $1 each when purchased in large quantities). At the same time, they enable non-contact estimation of position of the piston. Traditional sensors such as potentiometers and LVDTs require the sensor to be connected co-axially to the moving piston. This requires significant installation effort, results in contacting motion and in shear loads on the sensor during operation, often resulting in sensor failure. Furthermore, potentiometers and LVDTs can be significantly more expensive than the low-cost magnetic sensors considered in this paper.

The variation of the magnetic field with piston position is shown in Fig. 6 for an example electrohydraulic actuator with a magnet installed on its piston. The model for the position estimation dynamic system for the EHA when using two magnetic sensor outputs can be represented as:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} h_1(C_1 x) \\ h_2(C_2 x) \end{bmatrix} \end{aligned} \tag{63}$$

where $x = \begin{bmatrix} z \\ v \\ a \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $C_1 = C_2 = [1\ 0\ 0]$.

with z, v and a being the position, velocity and acceleration of the piston. Note that the output equations in (63) are highly nonlinear functions of the position z. The functions $h_1(C_1 x)$ and $h_2(C_2 x)$ were defined using polynomial curves to fit the experimentally measured data of Fig. 6. These functions are seen to be not only nonlinear but also non-monotonic with both positive and negative slopes.

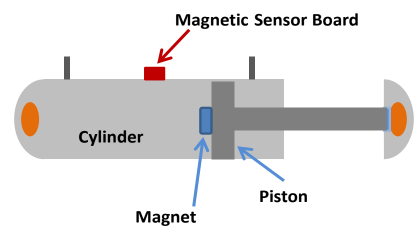

Fig. 5. Sensor Configuration for position estimation of EHA

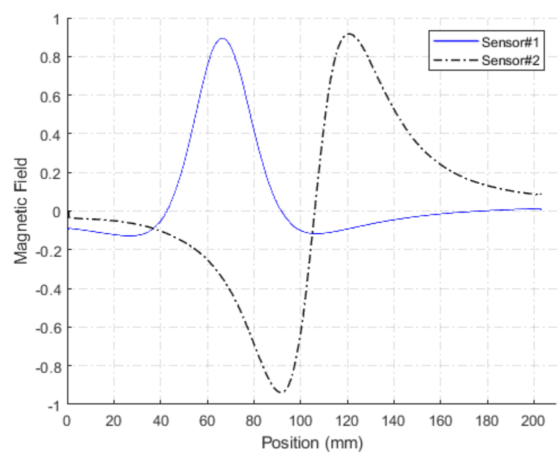

Fig. 6. Non-monotonic measurement functions of magnetic sensors, after removal of hysteresis

From the theoretical results in section II (Corollary 1.1), we have seen that if both output functions are non-monotonic, we cannot find a feasible solution to the observer design LMI (22). With the monotonicity requirement in mind, the position range of 0 – 203 mm can be divided piecewise into different regions in a manner that in each region at least one of the output functions is a monotonic function of position. Such a piecewise division of the position range into regions R_1 to R_{11} is shown in Fig. 7. Note that the boundaries of the regions lie near the slope change points (of one or the other output function). For example, R_4 is a narrow region in which the slope of the output y_1 is close to zero. In this region, only the output y_2 will be used by the observer, since

y_2 is monotonic in this region. Regions R_3 and R_5 lie on either side of R_4 and both of these regions can utilize both outputs y_1 and y_2, since they are monotonic in these regions.

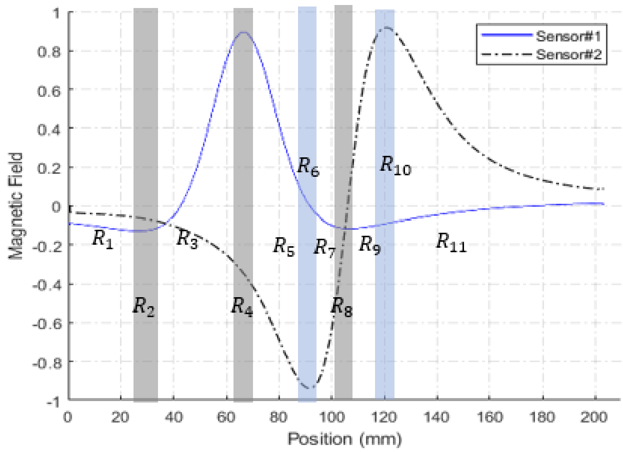

Fig. 7. Creating regions around slope-change points of output functions

It should be noted that we have the liberty of relying on only one of the output measurements in the narrow regions with zero slope, because even with one output the system is still observable, although the result of estimation might not be as accurate as the case when we use both outputs. Hence, the width of these regions was kept narrow so as to minimize regions with use of only 1 output by the observer. It is ideal to have these regions to be as narrow as possible, but in practice their width is determined by the accuracy of the measurement models. For example, if we anticipate a considerable horizontal uncertainty or shift in the output functions, we are forced to sacrifice the estimation accuracy for the sake of stability by widening the low observability regions.

A switched gain observer can be developed using the regions defined in Fig. 7 (Movahedi, Zemouche, & Rajamani, 2019). The switched gain observer uses different gains in each of the discrete piecewise regions. Since each region R_1 through R_{11} has monotonic output function properties, a constant stabilizing observer gain exists in each of these regions. As the operating region changes, the observer gains switch in value accordingly using a finite state machine of the type shown in Fig. 1.

One obstacle that could affect the performance of this piecewise nonlinear observer is the initial condition. If we pick the initial condition to be in the wrong region (with the wrong observer gain), it might result in a divergence of the observer estimates. However, thanks to the specific shape of output functions for this application, there is an easy solution that can remedy this shortcoming. From Fig. 6, since there is a one-to-one relationship between the position and the ordered pair that is constructed by the two output functions y_1 and y_2, we can identify the correct region for the initial condition accurately.

8. Conclusions

This paper considered the design of observers for nonlinear systems and the aspect of how observers can be designed in nonlinear systems which are non-monotonic. The plant considered is one in which the process dynamics and output equations are both composed of nonlinear vector functions of scalar combinations of the states. The nonlinear functions are assumed to be differentiable with bounded derivatives. An observer design algorithm that requires solving just a single linear matrix inequality for exponentially convergent state estimation was developed. The developed algorithm worked effectively when the involved nonlinear functions were monotonic. Since each component of the nonlinear functions was a function of a scalar variable, it could be analyzed as being either monotonic or non-monotonic.

The developed observer design method was seen to fail in yielding an observer solution when all or sometimes even some of the system functions were non-monotonic. Analytical results were presented to show that no solutions exist to the observer design LMIs when either all output functions or all process dynamics functions are non-monotonic. Further, other observer design methods from literature also fail when the involved nonlinear functions are non-monotonic. This relationship between the nonlinear functions being non-monotonic and the feasibility of solutions to the observer design LMIs has not previously been recognized in observer design literature. Previous observer design results in literature have focused on the size of the Lipschitz constant or on the size of the Jacobian bounds in influencing the existence of a stabilizing observer gain. The result in this paper shows that these LMI-based observer design methods will not succeed for a full non-monotonic system, no matter how small the Lipschitz constant or the Jacobian bounds of the nonlinearity.

Finally, a hybrid observer technique that switches between multiple constant observer gains was developed that can provide global asymptotic stability for systems with non-monotonic nonlinear functions. The need for hybrid observers with switched gains becomes important for such non-monotonic systems. The global stability of the hybrid observer was established when there is sufficient dwell time in each locally stable constant gain observer region.

The application of the developed hybrid observer to two different motion estimation problems was presented. One motion estimation problem involved tracking of vehicles on a road using radar sensors and handled a plant with nonlinear process dynamics. Another estimation problem involved a position estimation problem for an industrial actuator using a magnetic sensor and handled a plant with nonlinear output equations. Both applications demonstrated that while a constant gain observer could not be globally stable in either case, a hybrid observer can perform well and be globally stable.

Acknowledgements

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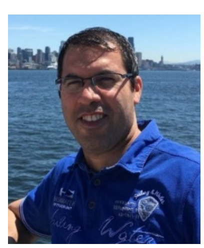

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