SESHADRI CONSTANTS FOR VECTOR BUNDLES

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Abstract. We introduce Seshadri constants for line bundles in a relative setting. They generalize the classical Seshadri constants of line bundles on projective varieties and their extension to vector bundles studied by Beltrametti–Schneider–Sommese and Hacon. There are similarities to the classical theory. In particular, we give a Seshadri-type ampleness criterion, and we relate Seshadri constants to jet separation and to asymptotic base loci. Smoothness is generally not part of our assumptions. Thus we improve on some of the known results already for line bundles.

We give two applications of our new version of Seshadri constants. First, a celebrated result of Mori can be restated as saying that any Fano manifold whose tangent bundle has positive Seshadri constant at a point is isomorphic to a projective space. We conjecture that the Fano condition can be removed. Among other results in this direction, we prove the conjecture for surfaces. Second, we prove that our Seshadri constants can be used to control separation of jets for direct images of pluricanonical bundles, in the spirit of a relative Fujita-type conjecture of Popa and Schnell.

Contents

1. Introduction

Let X be a projective scheme over an algebraically closed field, and let $\mathcal{L}$ be an ample line bundle on X. In [Dem92, Section 6], Demailly defined the *Seshadri constant* $\varepsilon(\mathcal{L};x)$ of $\mathcal{L}$ at a closed point $x \in X$ by

$$\varepsilon(\mathcal{L};x) \coloneqq \sup\bigl\{t \in \mathbb{R}_{\geq 0} \bigm| \pi^* c_1(\mathcal{L}) - tE \text{ is nef}\bigr\},$$

where π is the blow-up of X at x with exceptional divisor E. Seshadri constants have attracted much attention as interesting invariants that capture subtle geometric properties of both X and $\mathcal{L}$; see [Laz04a, Chapter 5] and [BDRH$^+$09]. In higher rank, a version of Seshadri constants for ample vector bundles (of arbitrary rank) appears implicitly in work of Beltrametti–Schneider–Sommese [BSS93, BSS96], and has been further studied by Hacon [Hac00].

In this paper, we define a new version of Seshadri constants for line bundles in a relative setting, generalizing both Demailly's and Hacon's definitions. One advantage of this version is that it does

2010 *Mathematics Subject Classification.* Primary 14C17, 14C20, 14F06, 14J60.

The first author was partially supported by the Simons Foundation Collaboration Grant 579353.

The second author was partially supported by the National Science Foundation under Grant No. DMS-1501461, DMS-1701622, and DMS-1902616.

not impose any global positivity conditions on the line bundle or vector bundle in question. We refer to §3 for the precise definition. In the case of vector bundles $\mathcal{V}$ on X, loosely speaking

$$\varepsilon(\mathcal{V};x) \coloneqq \sup\left\{t \in \mathbb{R} \;\middle|\; \begin{array}{c}\pi^*\mathcal{V}\langle -tE\rangle \text{ is nef on curves that}\\ \text{meet } E \text{ properly in at least one point}\end{array}\right\}.$$

Many of the classical properties of Seshadri constants generalize to our new version.

(1) A Seshadri ampleness criterion holds (Theorem 3.11), generalizing [Laz04a, Theorem 1.4.13].
(2) We have homogeneity for vector bundles in the sense that $\varepsilon(\mathrm{S}^m\mathcal{V};x) = m\cdot\varepsilon(\mathcal{V};x)$ (Lemma 3.24) and $\varepsilon(\bigotimes^m\mathcal{V};x) = m\cdot\varepsilon(\mathcal{V};x)$ (Proposition 3.28). The case of line bundles is trivial.
(3) For ample vector bundles, the Seshadri constant measures asymptotic jet separation (Theorem 5.3). This generalizes Demailly's result [Dem92, Theorem 6.4], and is new even for Seshadri constants of line bundles at singular points.
(4) The Seshadri constants satisfy semicontinuity in both a convex geometric sense and in a variational sense (see §3.5).
(5) For nef vector bundles $\mathcal{V}$, the locus $\{x \in X \mid \varepsilon(\mathcal{V};x) = 0\}$ coincides with the non-ample locus $\mathbf{B}_+(\mathcal{V})$ (Proposition 6.9). The line bundle case, due to Nakamaye, can be found in [Nak03, ELM+09].
(6) For big and nef vector bundles, lower bounds on Seshadri constants lead to lower bounds on the order of jet separation for adjoint bundles (Proposition 5.7). These generalize the rank 1 case in [Dem92, Proposition 6.8].

1.1. Examples.

We describe our version of the Seshadri constant in some examples.

Example 1.1 (Vector bundles on curves)**.** In [Hac00, Theorem 3.1], Hacon proves that if $\mathcal{V}$ is a vector bundle on a smooth complex projective curve X, then

$$\varepsilon(\mathcal{V};x) = \mu_{\min}(\mathcal{V})$$

for all $x \in X$. Here, $\mu_{\min}(\mathcal{V})$ is the smallest slope in the Harder–Narasimhan filtration of $\mathcal{V}$. We prove a similar description in positive characteristic by replacing $\mathcal{V}$ with iterated Frobenius pullbacks of $\mathcal{V}$; see Example 3.20.

This example is fundamental to the development of the theory. It helps reduce many results to the case where X is a smooth projective curve, where they are significantly easier.

Example 1.2 (Toric bundles)**.** In [HMP10, Proposition 3.2], Hering, Mustaţă, and Payne compute Seshadri constants for *nef* toric bundles $\mathcal{V}$ on smooth toric varieties at the torus invariant points x_σ. They show that $\varepsilon(\mathcal{V};x_\sigma)$ is the smallest degree of any summand of the restrictions of $\mathcal{V}$ to the invariant $\mathbb{P}^1$'s through x_σ.

Example 1.3 (Tangent bundle of homogeneous spaces; see Examples 4.2 and 4.3)**.** Let X be a homogeneous space (e.g., a rational homogeneous space, or an abelian variety). Then,

$$\varepsilon(TX;x) = \begin{cases} 2 & \text{if } X \simeq \mathbb{P}^1;\\ 1 & \text{if } X \simeq \mathbb{P}^n, \text{ where } n \geq 2;\\ 0 & \text{otherwise.}\end{cases}$$

Example 1.4 (Picard bundles; see Example 3.22)**.** Let C be a smooth complex projective curve of genus $g \geq 1$, and let $d \geq 2g-1$. On the g-dimensional Picard variety $\mathrm{Pic}^d(C)$ of isomorphism classes of line bundles of degree d on C, there exists a vector bundle P_d of rank $d+1-g$ such that the fiber over $\lambda \in \mathrm{Pic}^d(C)$ is $H^0(C,\lambda)$. The dual $P_d^\vee$ is ample. Results of [EL92] imply that

$$\varepsilon(P_d^\vee;\lambda) \leq \frac{g}{d+1-g}$$

for all $\lambda \in \mathrm{Pic}^d(C)$. Furthermore, if $g = 2$ and $d = 3$ so that $P_3^\vee$ is ample of rank 2, then we show that

$$\varepsilon(P_3^\vee; \lambda) = \frac{1}{3}$$

for all $\lambda \in \mathrm{Pic}^3(C)$ that are not basepoint free. When $|\lambda|$ is basepoint free with only simple ramification, we bound the Seshadri constant in the interval $\left[\frac{3}{7}, \frac{1}{2}\right]$. Conjecturally the Seshadri constant is $\frac{1}{2}$. The proof makes use of the geometry of $\mathbb{P}(P_d^\vee)$ seen as the symmetric product C_d.

1.2. **Applications.** Our first application gives new characterizations of projective space. A celebrated result of Mori [Mor79] states that if X is an n-fold with ample tangent bundle, then $X \simeq \mathbb{P}^n$. Thus, $\mathbb{P}^n$ is the only projective manifold with "very positive" tangent bundle. It is natural to ask if any weaker positivity conditions on TX still ensure that $X \simeq \mathbb{P}^n$.

Theorem A (see Proposition 4.8 and Corollary 4.12)**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field k. Suppose $\varepsilon(TX; x_0) > 0$ for some closed point $x_0 \in X$, and suppose that one of the following conditions holds:*

1. *X is Fano;*
2. $\mathrm{char}\, k = 0$ *and x_0 is general in the sense of* [Keb02, Notation 2.2]*; or*
3. $\dim X = 2$.

Then, X is isomorphic to $\mathbb{P}^n$.

The theorem is also inspired by similar results for Seshadri constants of divisors due to Bauer–Szemberg [BS09], Liu–Zhuang [LZ18], the second author [Mur18], and Zhuang [Zhu18, Zhu17]. They find characterizations of projective spaces in terms of lower bounds of the form $\varepsilon(-K_X; x_0) > n$. We conjecture that Theorem A holds without any of the additional assumptions (1)–(3).

For (3), we show that the condition $\varepsilon(TX; x_0) > 0$ is preserved by smooth blow-downs away from x_0 (in arbitrary dimension). We then use the Enriques classification of minimal surfaces. Parts (1) and (2) follow from Mori's work and from [CMSB02], respectively.

Our second application shows that our version of Seshadri constants can be used to control jet separation of direct images of pluricanonical sheaves, in the spirit of a relative Fujita-type conjecture of Popa and Schnell [PS14, Conjecture 1.3]. This statement extends a result of Dutta and the second author [DM19, Theorem A] to vector bundles of higher rank, and to higher-order jets. See Theorem 7.1 and Corollary 7.2 for effective statements that do not mention $\varepsilon(\mathcal{V}; x)$.

Theorem B (see Theorem 7.1)**.** *Let $f\colon Y \to X$ be a surjective morphism of complex projective varieties, where X is of dimension n. Let (Y, Δ) be a log canonical $\mathbb{R}$-pair and let $\mathcal{V}$ be a locally free sheaf of finite rank $r \geq 1$ on X such that $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is big and nef. Consider a Cartier divisor P on Y such that $P \sim_{\mathbb{R}} k(K_Y + \Delta)$ for some integer $k \geq 1$, and consider a general smooth closed point $x \in X \setminus \mathbf{B}_+(\mathcal{V})$. If $\varepsilon(\mathcal{V}; x) > k \cdot \frac{n+s}{m+k(r-1)+1}$, then the sheaf*

$$f_*\mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathrm{S}^m \mathcal{V} \otimes_{\mathcal{O}_X} (\det \mathcal{V})^{\otimes k}$$

separates s-jets at x.

1.3. **Moving Seshadri constants.** For nef vector bundles $\mathcal{V}$, we can interpret $\mathbf{B}_+(\mathcal{V})$ as the locus where Seshadri constants vanish. When $\mathcal{V}$ is ample, the asymptotic order of jet separation at x is in fact equal to $\varepsilon(\mathcal{V}; x)$. For ample bundles $\mathcal{V}$ on complex projective manifolds, lower bounds on $\varepsilon(\mathcal{V}; x)$ give information about the jet separation of "adjoint-type" sheaves. These are all powerful applications of Seshadri constants, with the only drawback that they require strong global positivity conditions on $\mathcal{V}$ like nefness, or even ampleness.

In the line bundle case, on complex projective manifolds, [Nak03] introduced the moving Seshadri constant $\varepsilon(\|\mathcal{L}\|; x)$ of $\mathcal{L}$ at x. It is a refinement of $\varepsilon(\mathcal{L}; x)$, defined in terms of usual

Seshadri constants of certain ample Fujita approximations of $\mathcal{L}$. If $\mathcal{L}$ is a big and nef line bundle, then $\varepsilon(\|\mathcal{L}\|; x) = \varepsilon(\mathcal{L}; x)$. While the definition is less intuitive, the applications are compelling. [ELM+09] proves that the same properties mentioned in the previous paragraph are true of $\varepsilon(\|\mathcal{L}\|; x)$ for big line bundles $\mathcal{L}$ on complex projective manifolds.

In the forthcoming paper [FM19] we will extend these to arbitrary rank. We will also prove a version of Theorem B for moving Seshadri constants that does not assume the nefness of $\mathcal{V}$.

Acknowledgments. We thank Harold Blum, Yajnaseni Dutta, Lawrence Ein, Sándor J. Kovács, Yuchen Liu, Nicholas McCleerey, Mihnea Popa, Valentino Tosatti, and Yifei Zhao for useful discussions. We are especially grateful to Krishna Hanumanthu and the anonymous referee for helpful comments on a previous draft of this paper. The second author would also like to thank his advisor Mircea Mustaţă for his constant support and encouragement.

2. Background and notation

Let X be a projective scheme over an algebraically closed field. We denote by $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ the space of $\mathbb{R}$-Cartier $\mathbb{R}$-divisors, where $\mathrm{Div}(X)$ is the group of Cartier divisors on X.

2.1. **Formal twists of coherent sheaves.** We define formal twists of coherent sheaves. See [Laz04b, Section 6.2] for the case of bundles.

Definition 2.1. Let $\mathcal{V}$ be a coherent sheaf on X, and let $\lambda \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The *formal twist* of $\mathcal{V}$ by λ is the pair $(\mathcal{V}, \lambda)$, denoted by $\mathcal{V}\langle\lambda\rangle$.

When $D \in \mathrm{Div}(X)$, the formal twist $\mathcal{V}\langle D\rangle$ is set to be the usual twist $\mathcal{V} \otimes \mathcal{O}_X(D)$. The theory of twisted sheaves has natural pullbacks. In particular, when D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and $f\colon X' \to X$ is a finite morphism such that f^*D is actually Cartier, then $f^*\mathcal{V}\langle f^*D\rangle$ is $f^*\mathcal{V} \otimes \mathcal{O}_{X'}(f^*D)$. The Chern classes of twisted sheaves are natural for pullbacks.

For tensor powers and symmetric powers, we put $\mathcal{V}\langle\lambda\rangle \otimes \mathcal{V}'\langle\lambda'\rangle \coloneqq (\mathcal{V} \otimes \mathcal{V}')\langle\lambda + \lambda'\rangle$ and $\mathrm{S}^n(\mathcal{V}\langle\lambda\rangle) \coloneqq (\mathrm{S}^n \mathcal{V})\langle n\lambda\rangle$, respectively. Generally, when we talk about extensions, subsheaves, quotients of twisted sheaves, or morphisms between twisted sheaves, we understand that the twist is fixed. The exception is $\mathrm{S}^*(\mathcal{V}\langle\lambda\rangle) \coloneqq \bigoplus_{n \geq 0} \mathrm{S}^n \mathcal{V}\langle n\lambda\rangle$.

2.2. **Positivity for twisted coherent sheaves.** Let $\mathbb{P}_X(\mathcal{V}) = \mathrm{Proj}_{\mathcal{O}_X} \mathrm{S}^* \mathcal{V}$ denote the space of 1-dimensional quotients of (fibers of) $\mathcal{V}$. Usually, we suppress X from the notation. Let $\rho\colon \mathbb{P}(\mathcal{V}) \to X$ denote the natural projection map, and let ξ denote the first Chern class of the relative Serre $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ line bundle.

Definition 2.2. Let $\mathcal{V}$ be a coherent sheaf and let λ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Define $\mathbb{P}(\mathcal{V}\langle\lambda\rangle)$ as $\rho\colon \mathbb{P}(\mathcal{V}) \to X$, polarized with the ρ-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $\mathcal{O}_{\mathbb{P}(\mathcal{V}\langle\lambda\rangle)}(1) \coloneqq \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)\langle\rho^*\lambda\rangle$ whose first Chern class is $\xi + \rho^*\lambda$. As above, $\xi \coloneqq c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$.

This is in line with the classical formula $\mathcal{O}_{\mathbb{P}(\mathcal{V} \otimes \mathcal{O}_X(D))}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \rho^*\mathcal{O}_X(D)$.

Definition 2.3. The sheaf $\mathcal{V}$ is said to be *ample* (resp. *nef, effective*) if the Cartier divisor class ξ has the same property. This extends formally to twists.

Remark 2.4. For locally free sheaves $\mathcal{V}$ on the projective scheme X, the following three conditions are equivalent (see [Laz04b, Theorem 6.1.10]):

(i) $\mathcal{V}$ is ample.
(ii) (Global generation) For every coherent sheaf $\mathcal{F}$, the twist $\mathrm{S}^m \mathcal{V} \otimes \mathcal{F}$ is globally generated for m sufficiently large.
(iii) (Cohomological vanishing) For every coherent sheaf $\mathcal{F}$, the groups $H^i\big(X, \mathrm{S}^m \mathcal{V} \otimes \mathcal{F}\big)$ vanish for all $i > 0$ and all m sufficiently large.

The equivalence (i) $\Leftrightarrow$ (ii) holds for arbitrary coherent sheaves by [Kub70, Theorem 1]. The implication (iii) $\Rightarrow$ (ii) holds for arbitrary coherent sheaves (see the proof of (ii) $\Rightarrow$ (iii) in [Laz04b, Theorem 6.1.10]).

(i) also implies (iii) for *locally free* sheaves $\mathcal{F}$ (use the Leray spectral sequence, the relative ampleness of $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$, the projection formula, and cohomology vanishing for $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(m) \otimes \rho^*\mathcal{F}$ as in the proof for (i) $\Rightarrow$ (ii) in [Laz04b, Theorem 6.1.10]).

Lemma 2.5. *Let $\rho : Y \to X$ be a morphism of projective schemes, and let $\mathcal{L}$ be an ample invertible sheaf on Y. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F} \otimes \rho_*\big(\mathcal{L}^{\otimes n}\big)$ is ample and globally generated for all n sufficiently large.*

Proof. Let A be a very ample divisor on X such that there exists a surjection $\bigoplus \mathcal{O}_X(-A) \twoheadrightarrow \mathcal{F}$. Since ampleness and global generation descend to quotients, it is enough to prove the lemma for $\mathcal{F} = \mathcal{O}_X(-A)$. With the usual arguments of Castelnuovo–Mumford regularity [Laz04a, Theorem 1.8.5], it is enough to prove that if A is a very ample divisor on X, then $\rho_*\big(\mathcal{L}^{\otimes n}\big)$ is -2-regular with respect to A, i.e., $H^i\big(X, \rho_*\big(\mathcal{L}^{\otimes n}\big)(-(2+i)A)\big) = 0$ for all $i > 0$ for all n sufficiently large. This is because in this case $\rho_*\big(\mathcal{L}^{\otimes n}\big)(-2A)$ is globally generated, hence $\rho_*\big(\mathcal{L}^{\otimes n}\big)(-A)$ is ample and globally generated.

Since $\mathcal{L}$ is ample, it is in particular also ρ-ample. Hence for n large, we have $R^i\rho_*\big(\mathcal{L}^{\otimes n}\big) = 0$ for all $i > 0$. The Leray spectral sequence and the projection formula show that $H^i\big(X, \big(\rho_*\mathcal{L}^{\otimes n}\big)(-(2+i)A)\big) = H^i\big(Y, \mathcal{L}^{\otimes n} \otimes \rho^*(-(2+i)A)\big)$. The ampleness of $\mathcal{L}$ and Serre vanishing show that these cohomology groups are 0. $\square$

We can also define *big* or *pseudo-effective* coherent sheaves, cf. [BKK$^+$15, Definitions 5.1 and 6.1], but the definitions are more refined. See also Definition 6.8.

3. Definition and properties of Seshadri constants

Notation 3.1. Let $\rho\colon Y \to X$ be a morphism of projective schemes over an algebraically closed field, and fix a closed point $x \in X$. Let $\pi\colon \operatorname{Bl}_x X \to X$ be the blow-up at x with Cartier exceptional divisor E. We then consider the commutative square

$$\begin{tikzcd} Y' \arrow[r, "\pi'"] \arrow[d, "\rho'"'] & Y \arrow[d, "\rho"] \\ \operatorname{Bl}_x X \arrow[r, "\pi"'] & X \end{tikzcd}$$

where $Y' \coloneqq \operatorname{Bl}_{Y_x} Y$ and $Y_x \coloneqq \rho^{-1}(x)$. The exceptional divisor of π' is ρ'^*E. Note that the square is cartesian when ρ is flat at x. In any event, the π'-ampleness of $-\rho'^*E$ implies that the induced map $Y' \to Y \times_X \operatorname{Bl}_x X$ is finite.

Let $\mathcal{C}_{\rho,x}$ denote the set of irreducible curves on Y that meet Y_x, but are not contained in the support of Y_x. Let $\mathcal{C}'_{\rho,x}$ denote their strict transforms via π'. Let $\xi \in N^1(Y)$. Usually ξ will be ρ-nef, meaning $\xi|_{Y_t}$ is nef for all $t \in X$, or even ρ-ample.

A case that we are particularly interested in is when $Y = \mathbb{P}_X(\mathcal{V})$ for some coherent sheaf $\mathcal{V}$ on X, often locally free. In this case, $\rho\colon \mathbb{P}(\mathcal{V}) \to X$ is the bundle map, and $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$. We denote $\mathcal{C}_{\mathcal{V},x} \coloneqq \mathcal{C}_{\rho,x}$ and $\mathcal{C}'_{\mathcal{V},x} \coloneqq \mathcal{C}'_{\rho,x}$.

3.1. Definition and basic properties.

We begin by defining the notion of local nefness.

Definition 3.2 (Local nefness)**.** Suppose ξ is ρ-nef. We say that ξ is *nef at x* if $\xi \cdot C \geq 0$ for all $C \in \mathcal{C}_{\rho,x}$. When $Y = \mathbb{P}(\mathcal{V})$, we also say that $\mathcal{V}$ is *nef at x* when the same condition holds for $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$.

Example 3.3. If a coherent sheaf $\mathcal{V}$ is globally generated at x, i.e., $H^0(X,\mathcal{V}) \otimes \mathcal{O}_X \to \mathcal{V}$ is surjective at x, then $\mathcal{V}$ is nef at x. (Since $\rho^*\mathcal{V} \to \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is surjective, we find that $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is globally generated along the fiber $\rho^{-1}x = \mathbb{P}(\mathcal{V}(x))$. If C is a curve that meets $\mathbb{P}(\mathcal{V}(x))$ without being contained in it, and if $y \in C \cap \mathbb{P}(\mathcal{V}(x))$, then we can find an effective representative of ξ that does not pass through y, hence it does not contain C. It follows that $\xi \cdot C \geq 0$.) $\square$

Remark 3.4. If ξ is ρ-nef, then ξ is nef on Y if and only if ξ is nef at all $x \in X$. (One direction is clear. The other is immediate from the ρ-nefness of ξ.)

We now define the following measure of local nefness at x. We believe these constants were first defined explicitly for ample locally free sheaves by Hacon [Hac00, p. 769], although they appear implicitly in the work of Beltrametti, Schneider, and Sommese [BSS93, BSS96].

Definition 3.5. The *Seshadri constant* of ξ at x is

$$\varepsilon\,(\xi; x) \coloneqq \inf_{C \in \mathcal{C}_{\rho,x}} \left\{ \frac{\xi \cdot C}{\operatorname{mult}_x \rho_* C} \right\}.$$

When $Y = \mathbb{P}(\mathcal{V})$, put $\varepsilon\,(\mathcal{V}; x) \coloneqq \varepsilon\,(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1); x)$. When $\mathcal{C}_{\rho,x}$ is empty, set $\varepsilon\,(\xi; x) = \infty$.

Note that the curves in $\mathcal{C}_{\rho,x}$ are precisely the irreducible curves C on Y for which $\operatorname{mult}_x \rho_* C > 0$.

Remark 3.6. The Seshadri constant descends to a well-defined function $\varepsilon\,(-; x) \colon N^1(Y) \to \mathbb{R}$ that is homogeneous and concave, i.e., $\varepsilon\,\big((1-t)\xi + t\xi'; x\big) \geq (1-t) \cdot \varepsilon\,(\xi; x) + t \cdot \varepsilon\,(\xi'; x)$ for all $t \in [0,1]$.

Proposition 3.7. *If ξ is ρ-nef, then*

$$\varepsilon\,(\xi; x) = \sup\big\{ t \bigm| (\pi'^*\xi - t\rho'^*E) \cdot C' \geq 0 \textit{ for all } C' \in \mathcal{C}'_{\rho,x} \big\}.$$

See also [Hac00] for the case of bundles.

Proof. Let C' be the strict transform of C on Y' via π'. We then have

$$\operatorname{mult}_x \rho_* C = E \cdot \rho'_* C' = \rho'^* E \cdot C',$$

hence $(\pi'^*\xi - t\rho'^*E) \cdot C' \geq 0$ if and only if $\frac{\xi \cdot C}{\operatorname{mult}_x \rho_* C} \geq t$. $\square$

Example 3.8. When ρ is the identity morphism $X \to X$ and ξ is nef, then $\varepsilon\,(\xi; x)$ is the classical Seshadri constant of the divisor class ξ at x; see [Laz04a, Proposition 5.1.5].

Example 3.9. When $\rho = \pi$ is the blow-up of x and $\xi = -E$, then $\varepsilon\,(\xi; x) = -1$. In fact for all curves C on $\operatorname{Bl}_x X$ that meet E, without being contained in it, we have $\frac{-E \cdot C}{\operatorname{mult}_x \pi_* C} = -1$.

Remark 3.10. Assume that ξ is ρ-nef. We have the following:

(a) $\varepsilon\,(\xi; x) \geq 0$ if and only if ξ is nef at x.

(b) If C' is an irreducible curve on Y' that is contained in the exceptional locus $\rho'^{-1}E$ of π', then

$$(\pi'^*\xi - t\rho'^*E) \cdot C' = \xi \cdot \pi'_* C' - tE \cdot \rho'_* C' \geq 0$$

for all $t \geq 0$. The inequality is strict if $t > 0$ and C' is not contracted by ρ', or if C' is not contracted by π' and ξ is ρ-ample. (Use that ξ is nef on Y_x, and that $-E$ is ample on E.)

(c) If ξ is nef, then $\varepsilon\,(\xi; x) = \sup\big\{ t \in \mathbb{R}_{\geq 0} \bigm| \pi'^*\xi - t\rho'^*E \in \operatorname{Nef}^1(Y') \big\}$. In particular, if $\mathcal{V}$ is nef, then

$$\varepsilon\,(\mathcal{V}; x) = \sup\big\{ t \in \mathbb{R}_{\geq 0} \bigm| \pi^*\mathcal{V}\langle -tE \rangle \text{ is nef} \big\}.$$

(The twisted bundle $\pi^*\mathcal{V}\langle -tE \rangle$ is nef if and only if $\pi'^*\xi - t\rho'^*E$ is nef on $\mathbb{P}(\pi^*\mathcal{V})$. The irreducible curves on $\mathbb{P}(\pi^*\mathcal{V})$ are either in $\mathcal{C}'_{\mathcal{V},x}$, are in the exceptional locus of π', or do not intersect the support of ρ'^*E. From part (b), and using the nefness of $\pi'^*\xi$, we find that the nefness of $\pi'^*\xi - t\rho'^*E$ can be verified on the curves in $\mathcal{C}'_{\mathcal{V},x}$. The case for general Y and ξ is analogous.)

As is the case for divisors [Laz04a, Theorem 1.4.13], Seshadri constants can detect whether ξ is ample.

Theorem 3.11 (Seshadri ampleness criterion)**.** *If ξ is ρ-ample, then ξ is ample if and only if*

$$\inf_{x\in X} \varepsilon\,(\xi; x) > 0. \tag{3.11.1}$$

In particular, if $\mathcal{V}$ is a locally free sheaf on X, then $\mathcal{V}$ is ample if and only if $\inf_{x\in X} \varepsilon\,(\mathcal{V}; x) > 0$.

See also [Laz04a, Example 6.1.20].

Proof. Assume that the infimum in (3.11.1) is positive, but that ξ is not ample. In any case, ξ is nef by Remarks 3.4 and 3.10.(a). By the Seshadri ampleness criterion for divisors [Laz04a, Theorem 1.4.13], $\inf_{y\in Y} \varepsilon\,(\xi; y) = 0$. Hence there exist closed points $y_m \in Y$ and irreducible curves C_m through y_m with

$$\xi \cdot C_m < \frac{1}{m} \operatorname{mult}_{y_m} C_m.$$

We claim that C_m is not contracted by ρ for infinitely many m. Indeed, suppose that the curves C_m are contracted by ρ for all m, in which case

$$\inf_{x\in X} \inf_{y\in Y_x} \varepsilon\left(\xi|_{Y_x}; y\right) = 0. \tag{3.11.2}$$

Let h be a sufficiently ample divisor class on X such that $\xi + \rho^* h$ is ample. Then, $\inf_{y\in Y} \varepsilon\,(\xi + \rho^* h; y) > 0$, and in particular, $\inf\limits_{x\in X} \inf\limits_{y\in Y_x} \varepsilon\left((\xi+\rho^* h)|_{Y_x}; y\right) > 0$. But $(\xi+\rho^* h)|_{Y_x} = \xi|_{Y_x}$, contradicting (3.11.2). This shows the claim.

From the claim, $\rho|_{C_m}$ is finite for all sufficiently large m. Writing $x_m \coloneqq \rho(y_m)$, the inequality $\operatorname{mult}_{x_m} \rho_* C_m \geq \operatorname{mult}_{y_m} C_m$ (see [Ful17, Lemma 2.3]) leads to a contradiction.

Conversely, assume that ξ is ample. Let h be ample on X. Then, $\xi - \epsilon\rho^* h$ is ample for sufficiently small $\epsilon > 0$, and for all $C \in \mathcal{C}_{\rho,x}$, since $\rho|_C$ is finite,

$$\frac{\xi \cdot C}{\operatorname{mult}_x \rho_* C} = \frac{(\xi - \epsilon\rho^* h) \cdot C}{\operatorname{mult}_x \rho_* C} + \frac{\epsilon\rho^* h \cdot C}{\operatorname{mult}_x \rho_* C} \geq \epsilon \frac{h \cdot \rho_* C}{\operatorname{mult}_x \rho_* C} \geq \epsilon \cdot \varepsilon\,(h; x).$$

Taking the infimum over all $x \in X$, we see that ξ is ample by the classical Seshadri ampleness criterion for divisors [Laz04a, Theorem 1.4.13]. $\square$

Remark 3.12. In the case of sheaves, the first part of the previous proof can be adapted to show the following: *If there exists $y \in \mathbb{P}(\mathcal{V}(x))$ such that $0 \leq \varepsilon\,(\xi; y) < 1$, then $\varepsilon\,(\mathcal{V}; x) \leq \varepsilon\,(\xi; y)$.* For arbitrary ρ and ρ-ample ξ, a similar statement holds with 1 replaced by $\inf_{y\in Y_x} \varepsilon\,(\xi|_{Y_x}; y)$, which is in any case strictly positive. (The inequality $\varepsilon\,(\xi; y) < 1$ proves that the Seshadri constant of ξ at y is not approximated by intersecting with curves in $\mathbb{P}(\mathcal{V}(x))$, since $\varepsilon\,(\xi|_{\mathbb{P}(\mathcal{V}(x))}; y) = 1$. For curves in $\mathcal{C}_{\mathcal{V},x}$ that pass through y, use the inequality $\operatorname{mult}_y C \leq \operatorname{mult}_x \rho_* C$ from [Ful17, Lemma 2.3].)

Furthermore, for arbitrary ρ and ρ-ample ξ, we have the following: *If there exists $y \in Y_x$ such that $\varepsilon\,(\xi; y) < 0$, then $\varepsilon\,(\xi; x) < 0$.* (If $\varepsilon\,(\xi; y) < 0$, then there exists $C \in \mathcal{C}_{\rho,x}$ through y with $\xi \cdot C < 0$.) $\square$

One can also characterize Seshadri constants in terms of all varieties intersecting Y_x, instead of just curves.

Proposition 3.13. *If ξ is nef, then*

$$\varepsilon\,(\xi; x) \leq \left(\frac{\xi^{\dim W} \cdot [W]}{\binom{\dim W}{\dim \rho(W)} \cdot \operatorname{mult}_x \rho(W) \cdot (\xi^{\dim W_{x'}} [W_{x'}])}\right)^{1/\dim \rho(W)}, \tag{3.13.1}$$

as W ranges through the subvarieties of Y that meet Y_x without being contained in it. In the above, $W_{x'}$ is a fiber over the flat locus of $W \to \rho(W)$.

If X is a variety and $Y=\mathbb{P}(\mathcal{V})$ for a locally free sheaf $\mathcal{V}$ of rank r, then in particular by considering $W=Y$, we obtain

$$\varepsilon\left(\mathcal{V};x\right)\leq\sqrt[n]{\frac{s_n(\mathcal{V}^\vee)}{\binom{n+r-1}{n}\cdot\operatorname{mult}_x X}},\tag{3.13.2}$$

where $s_n(\mathcal{V}^\vee)=(\xi^{n+r-1})$ is the n-th Segre class of $\mathcal{V}^\vee$ (see [Ful98, §3.1][1]). This is a generalization of the rank one case $\varepsilon\left(\mathcal{L};x\right)\leq\sqrt[n]{\frac{(\mathcal{L}^n)}{\operatorname{mult}_x X}}$ in [Laz04a, Proposition 5.1.9]. A transcendental generalization is [Tos18, Theorem 4.6].

Example 3.14. Put $n\coloneqq\dim X$ and assume that $\mathcal{V}$ is locally free of rank r and nef. When considering $W=\rho^{-1}Z\subseteq\mathbb{P}(\mathcal{V})$ for some subvariety $Z\subseteq X$ of codimension i, we obtain

$$\varepsilon\left(\mathcal{V};x\right)\leq\left(\frac{\xi^{n-i+r-1}\cdot[\rho^*Z]}{\binom{n-i+r-1}{n-i}\cdot\operatorname{mult}_x Z}\right)^{\frac{1}{n-i}}=\left(\frac{s_{n-i}(\mathcal{V}^\vee)\cap[Z]}{\binom{n-i+r-1}{n-i}\cdot\operatorname{mult}_x Z}\right)^{\frac{1}{n-i}},$$

where $s_{n-i}(\mathcal{V}^\vee)\cap[Z]=\xi^{n-i+r-1}\cdot[\rho^*Z]$ is the evaluation of the Segre class of degree $n-i$ of $\mathcal{V}^\vee$ on the fundamental class of Z (see [Ful98, §3.1]). These bounds are similar to the ones appearing in [Hac00, Theorem 1.5.a].

We thank Valentino Tosatti for suggesting this example.

Remark 3.15 (Relation with other Seshadri constants)**.** With hypotheses as in the previous example, taking the infimum over all Z of fixed codimension i, we obtain

$$\varepsilon\left(\mathcal{V};x\right)\leq\left(\frac{1}{\binom{n-i+r-1}{n-i}}\cdot\varepsilon\left(s_{n-i}(\mathcal{V}^\vee);x\right)\right)^{\frac{1}{n-i}},$$

where the Seshadri constant of the nef dual class $s_{n-i}(\mathcal{V}^\vee)$ on the right is defined as in [Ful17, §8].

We thank Nicholas M$^{\text{c}}$Cleerey for suggesting this example.

Proof of Proposition 3.13. Let W be as above, and let W' be its strict transform in Y'. By Remark 3.10.(c) we have $(\pi'^*\xi-\varepsilon\left(\xi;x\right)\rho'^*E)^{\dim W'}\cdot[W']\geq 0$. By restricting to W' we can assume without loss of generality that $W'=Y'$, that ρ is surjective, and that X is a variety. Let $n\coloneqq\dim X$ and $e\coloneqq\dim Y-n$, with $e\geq 0$. We have

$$\begin{aligned}0\leq\left(\pi'^*\xi-\varepsilon\left(\xi;x\right)\rho'^*E\right)^{n+e}&=\sum_{k=0}^{n}\binom{n+e}{k}\left(-\varepsilon\left(\xi;x\right)\rho'^*E\right)^k\pi'^*\xi^{n+e-k}\\&\leq\xi^{n+e}+\binom{n+e}{n}\left(-\varepsilon\left(\xi;x\right)\rho'^*E\right)^n\cdot\pi'^*\xi^e\\&=\xi^{n+e}-\binom{n+e}{n}\cdot\operatorname{mult}_x X\cdot\varepsilon^{\,n}(\xi;x)\cdot(\xi^e\cdot[Y_{x'}]).\end{aligned}$$

The first equality holds since $(E^k)=0$ for $k>n$. The second inequality is a consequence of the projection formula for π'. Pushing forward $-(-\rho'^*E)^k$ produces a pseudo-effective class, since $-E|_E$ is ample. In the last equality, we used that $(-E)^n=-\operatorname{mult}_x X$. This implies $\pi'_*(\rho'^*(-E)^n)=-\operatorname{mult}_x X\cdot F$, where F is a fiber over the flat locus of $\rho|_W:W\to\rho(W)$. $\square$

Remark 3.16. With hypotheses as in the proposition, assume that ξ is ample. We show that there exists a subvariety $W'\subseteq Y'$, which is the strict transform of some $W\subseteq Y$ that meets Y_x without being contained in it, such that

$$(\pi'^*\xi-\varepsilon\left(\xi;x\right)\rho'^*E)^{\dim W'}\cdot[W']=0.$$

[1] Duality is present because [Ful98] uses projective bundles of lines instead of quotients.

For this, let $W' \subset Y'$ be a subvariety that observes the failure of ampleness of $\pi'^*\xi - \varepsilon(\xi; x)\rho'^*E$, i.e., $(\pi'^*\xi - \varepsilon(\xi; x)\rho'^*E)^{\dim W'} \cdot [W'] = 0$. These exist by [CP90, Bir17] over arbitrary fields for nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisors, extending the Nakai–Moishezon criterion for nef Cartier divisors. That $W \coloneqq \pi'(W')$ meets Y_x without being contained in it follows from the finiteness of $Y' \to Y \times_X \mathrm{Bl}_x X$. □

Remark 3.17. When $\mathcal{V}$ is ample on X, equality in the proposition is not necessarily achieved by some $W = \rho^{-1}Z$ for Z a subvariety of X containing x. (As in [Hac00, p. 771], consider $X = \mathbb{P}^1$ and $\mathcal{V} = \mathcal{O}_X(1) \oplus \mathcal{O}_X(2)$. In this case $E = Y_x$. From Remark 3.10.(c), we deduce $\varepsilon(\mathcal{V}; x) = 1$. The only subvariety of $\mathbb{P}(\mathcal{V})$ that achieves equality in (3.13.1) is $W = \mathbb{P}(\mathcal{O}_X(1))$, embedded via the quotient $\mathcal{O}_X(1) \oplus \mathcal{O}_X(2) \twoheadrightarrow \mathcal{O}_X(1)$.)

We describe the behavior of Seshadri constants under pullback.

Lemma 3.18 (Quotients)**.** *Assume that ξ is ρ-nef. Let $\imath\colon Z \to Y$ be a morphism of projective schemes. Then,*

$$\varepsilon(\imath^*\xi; x) \geq \varepsilon(\xi; x), \tag{3.18.1}$$

and equality holds if $\imath$ is surjective. In particular, if $\mathcal{V} \to \mathcal{Q}$ is a surjective morphism of coherent sheaves on X, then $\varepsilon(\mathcal{Q}; x) \geq \varepsilon(\mathcal{V}; x)$.

Proof. Let $C \in \mathcal{C}_{\rho\circ\imath,x}$, and write $C' \coloneqq \imath(C) \in \mathcal{C}_{\rho,x}$. We have $\imath_*C = dC'$ for some $d \geq 1$. By the projection formula, $\frac{\imath^*\xi\cdot C}{\mathrm{mult}_x(\rho\circ\imath)_*C} = \frac{\xi\cdot dC'}{\mathrm{mult}_x\,\rho_*(dC')} = \frac{\xi\cdot C'}{\mathrm{mult}_x\,\rho_*C'}$. Taking the infimum over all $C \in C_{\rho\circ\imath,x}$, since $\mathcal{C}_{\rho,x}$ may contain curves that are not of form C' as above, we deduce $\varepsilon(\imath^*\xi; x) \geq \varepsilon(\xi; x)$. When $\imath$ is surjective, every curve in $\mathcal{C}_{\rho,x}$ is of form C' as above, hence equality holds in (3.18.1).

For the last statement, note that there is a closed immersion $\mathbb{P}_X(\mathcal{Q}) \hookrightarrow \mathbb{P}_X(\mathcal{V})$ such that the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is $\mathcal{O}_{\mathbb{P}(\mathcal{Q})}(1)$. □

3.2. **Restrictions to curves.** Every curve in $\mathcal{C}_{\rho,x}$ is mapped by ρ to a unique curve through x in X. It is fundamental then for the development of the theory to understand the case where X itself is a curve.

Remark 3.19. Assume that X is a projective curve and that ξ is ρ-nef. Then,

$$\varepsilon(\xi; x) = \frac{1}{\mathrm{mult}_x X} \cdot \sup\{t \mid \xi - tf \in \mathrm{Nef}^1(Y)\},$$

where f is the class of a general fiber of ρ. In particular, $\mathrm{mult}_x X \cdot \varepsilon(\xi; x)$ is independent of x in this case. If $\mathcal{V}$ is a coherent sheaf on X, then

$$\varepsilon(\mathcal{V}; x) = \frac{1}{\mathrm{mult}_x X} \cdot \sup\{t \mid \mathcal{V}\langle -tx_0\rangle \text{ is nef}\},$$

where x_0 denotes a $\mathbb{Q}$-Cartier $\mathbb{Q}$-class of degree 1. (Since X is a curve, $\deg E = \mathrm{mult}_x X$. The set $\mathcal{C}_{\rho,x}$ is the set of curves in Y that dominate X, hence it is independent of x. Using the ρ-nefness of ξ, it follows that $(\pi'^*\xi - t\rho'^*E) \cdot C' \geq 0$ for all $C' \in \mathcal{C}'_{\rho,x}$ if and only if $\pi'^*\xi - t\rho'^*E$ is nef. However, $\pi'^*\xi - t\rho'^*E = \pi'^*(\xi - (\mathrm{mult}_x X)tf)$ is nef on Y' if and only if $\xi - (\mathrm{mult}_x X)tf$ is nef on Y.)

We can now give the following generalization of [Hac00, Theorem 3.1].

Example 3.20 (Curves)**.** If X is a (possibly singular) integral projective curve over an algebraically closed field k and $\nu\colon X' \to X$ denotes the normalization, and if $\mathcal{V}$ is a coherent sheaf on X, then

$$\varepsilon(\mathcal{V}; x) = \frac{\overline{\mu}_{\min}(\nu^*\mathcal{V})}{\mathrm{mult}_x X}. \tag{3.20.1}$$

For the purpose of explaining notation, assume that X is a smooth projective curve. The slope of a bundle $\mathcal{V}$ on X is

$$\mu(\mathcal{V}) \coloneqq \frac{\deg \mathcal{V}}{\mathrm{rank}\,\mathcal{V}}.$$

By convention, the slope of torsion sheaves is infinite. The smallest slope of any quotient (of positive rank) of $\mathcal{V}$ is denoted by $\mu_{\min}(\mathcal{V})$. A quotient of $\mathcal{V}$ with minimal slope exists, and is determined by the Harder–Narasimhan filtration of $\mathcal{V}$. In characteristic 0, set $\overline{\mu}_{\min}(\mathcal{V}) \coloneqq \mu_{\min}(\mathcal{V})$. In characteristic $p > 0$, let $F\colon X \to X$ be the absolute Frobenius morphism, and consider

$$\overline{\mu}_{\min}(\mathcal{V}) \coloneqq \lim_{n\to\infty} \frac{\mu_{\min}\big((F^n)^*\mathcal{V}\big)}{p^n}.$$

The sequence is weakly decreasing and eventually stationary. In fact, [Lan04, Theorem 2.7] proves that there exists $\delta = \delta_{\mathcal{V}} \geq 0$ such that the Harder–Narasimhan filtration of $(F^{\delta+n})^*\mathcal{V}$ is the pullback of the Harder–Narasimhan filtration of $(F^{\delta})^*\mathcal{V}$. In particular, $\overline{\mu}_{\min}(\mathcal{V}) = \frac{\mu_{\min}((F^{\delta})^*\mathcal{V})}{p^{\delta}}$ is the smallest normalized slope of any quotient of any iterated Frobenius pullback $(F^n)^*\mathcal{V}$.

Note that torsion is irrelevant when computing $\mu_{\min}$ or $\overline{\mu}_{\min}$.

(For the proof of (3.20.1), assume first that X is smooth. From Remark 3.19, the Seshadri constant is independent of $x \in X$, and verifies the linearity $\varepsilon(\mathcal{V}\langle\lambda\rangle; x) = \varepsilon(\mathcal{V}; x) + \deg\lambda$. Furthermore, slopes respect the same formula $\mu(\mathcal{V}\langle\lambda\rangle) = \mu(\mathcal{V}) + \deg\lambda$, and similarly for $\mu_{\min}$ and $\overline{\mu}_{\min}$.

In characteristic zero, we are then free to assume that $\mu_{\min}(\mathcal{V}) = 0$. Hartshorne's Theorem [Laz04a, Theorem 6.4.15] (which is only valid in characteristic zero; see [Har71, Example 3.2]) shows that $\mathcal{V}$ is nef. In particular, $\varepsilon(\mathcal{V}; x) \geq 0$ for all $x \in X$. By the assumption $\mu_{\min}(\mathcal{V}) = 0$, there exists a quotient map $\mathcal{V} \twoheadrightarrow \mathcal{Q}$ with $\mathcal{Q}$ nonzero, nef, semistable, and $\mu(\mathcal{Q}) = 0$. Since $\varepsilon(\mathcal{V}; x) \leq \varepsilon(\mathcal{Q}; x)$ by Lemma 3.18, it is then enough to treat the case when $\mathcal{V} = \mathcal{Q}$ is nef of degree 0. In this case, one can use Remark 3.10(c), where the blow-up π of $x \in X$ is the identity, and the "exceptional" divisor E is $\mathcal{O}_X(x)$.

In positive characteristic, the proof is analogous after replacing $\mu_{\min}$ with $\overline{\mu}_{\min}$, in view of [BP14, Theorem 1.1], which proves that $\mathcal{V}$ is nef iff $\overline{\mu}_{\min}(\mathcal{V}) \geq 0$. The result was seemingly first proved by Barton [Bar71, Theorem 2.1], and stated explicitly by Brenner in [Bre04, Theorem 2.3] and [Bre06, p. 534], Biswas in [Bis05, Theorem 1.1], and Zhao in [Zha17, Theorem 4.3].

When X is singular, then from the projection formula, one finds $\varepsilon(\mathcal{V}; x) = \frac{\varepsilon(\nu^*\mathcal{V})}{\operatorname{mult}_x X}$, where $\varepsilon(\nu^*\mathcal{V})$ is the Seshadri constant of $\nu^*\mathcal{V}$ at any point of X'.) ☐

Corollary 3.21 (Seshadri constants for sheaves via restrictions to curves)**.** *Let X be a projective scheme of arbitrary dimension over an algebraically closed field. Fix $x \in X$ a closed point and $\mathcal{V}$ a coherent (twisted) sheaf on X. Then*

$$\varepsilon(\mathcal{V}; x) = \inf_{x\in C\subset X} \frac{\overline{\mu}_{\min}(\nu^*\mathcal{V})}{\operatorname{mult}_x C},$$

where C ranges through the set of irreducible curves through x on X, where $\nu\colon C' \to C$ is the normalization, and $\overline{\mu}_{\min}$ is defined as above.

Note that torsion subsheaves whose supports have positive dimension may influence the result.

Proof. Use $\mathcal{C}_{\mathcal{V},x} = \bigcup_{x\in C\subset X} \mathcal{C}_{\mathcal{V}|_C,x}$ to deduce that $\varepsilon(\mathcal{V}; x) = \inf_{x\in C\subset X} \varepsilon(\mathcal{V}|_C; x)$. The result then follows from Example 3.20. ☐

Example 3.22 (Dual Picard bundles)**.** Let C be a smooth complex projective curve of genus g. For $d \geq 1$, denote by C_d the d-th symmetric product of C and set $J_d \coloneqq \operatorname{Pic}^d(C)$. The Abel–Jacobi map $\alpha_d\colon C_d \to J_d$ is given by $D \mapsto \mathcal{O}_C(D)$ for all effective divisors D of degree d on C. For $d \geq 2g-1$ we have $C_d = \mathbb{P}_{J_d}(P_d^\vee)$ and α_d is the bundle map, where P_d is the Picard bundle on J_d of rank $d+1-g$. It is uniquely determined by the choice of a fixed point $x_0 \in C$. Furthermore, $P_d^\vee$ is ample ([Laz04b, Theorem 6.3.48]), and the relative $\mathcal{O}(1)$ sheaf has a section vanishing along $C_{d-1} + x_0$.

Assume $d \geq 2g-1$. Denote by ξ_d the ample class of the relative $\mathcal{O}(1)$ on $C_d = \mathbb{P}(P_d^\vee)$. We have $\xi_d^d = 1$ and $\xi_d \cdot C = 1$, where C is the class of any embedding $C = C_1 \subset C_d$ by $x \mapsto x + D'$, where

$D' \in C_{d-1}$. Since such curves pass through any point of C_d and map isomorphically onto their image by α_d, we deduce $\varepsilon(P_d^\vee, \lambda) \le 1$ for all $\lambda \in J_d$. In fact, since $P_d|_C$ is semistable (cf. [EL92]) for any embedding $C \hookrightarrow J_d : x \mapsto x + \lambda'$ with $\lambda' \in J_{d-1}$ fixed, and since $c_1(P_d^\vee) = \theta \in N^1(J_d)$ is the class of a theta divisor on the Jacobian, we obtain by Example 3.20 that

$$\varepsilon(P_d^\vee; \lambda) \le \mu(P_d^\vee|_C) = \frac{\theta \cdot C}{\operatorname{rk} P_d} = \frac{g}{d+1-g} \le 1$$

for all $\lambda \in J_d$.

For a concrete situation, assume $g = 2$ and $d = 2g-1 = 3$. We prove that

$$\varepsilon(P_3^\vee, \lambda) \text{ is } \begin{cases} = \frac{1}{3}, & \text{if } \operatorname{Bs}|\lambda| \neq \emptyset \\ \le \frac{1}{2}, & \text{if } \operatorname{Bs}|\lambda| = \emptyset \\ \in \left[\frac{3}{7}, \frac{1}{2}\right], & \text{if } \operatorname{Bs}|\lambda| = \emptyset \text{ and } |\lambda| \text{ has only simple ramification} \end{cases}$$

In this case simple ramification means that $|\lambda|$ contains no divisor of form $3x$ for some $x \in C$. Note that $\lambda \in J_3$ is not basepoint free if and only if $\lambda = \mathcal{O}_C(K_C + x)$ for some $x \in C$. Note also that $|K_C|$ is the hyperelliptic canonical pencil.

Let $\mathbb{P}^1_\lambda := \alpha_3^{-1}\{\lambda\}$ and put $\widetilde{C_3} := \operatorname{Bl}_{\mathbb{P}^1_\lambda} C_3$ with blow-down map π_3' and exceptional divisor E_3'.

Case $\lambda \in J_3$ is not basepoint free. Let x be its unique basepoint. The fiber $\mathbb{P}^1_\lambda$ is contained in $C_2 + x$ as a divisor, the hyperelliptic $\mathbb{P}^1 \subset C_2$. The strict transform $\overline{C_2 + x}$ in $\widetilde{C_3}$ is then isomorphic to C_2 and has class ${\pi_3'}^*\xi_3 - E_3'$.

The restriction $E_3'|_{\overline{C_2+x}}$ is identified with $\mathbb{P}^1_\lambda$ in $C_2 + x$, and ${\pi_3'}^*\xi_3$ restricts to ξ_2.

We have ${\pi_3'}^*\xi_3 - \frac{1}{3}E_3' = [\overline{C_2 + x}] + \frac{2}{3}E_3'$.

As in Remark 3.10.(b), we have that $\left({\pi_3'}^*\xi_3 - \frac{1}{3}E_3'\right)|_{E_3'}$ is nef. By Remark 3.10.(c), to show $\varepsilon(P_3^\vee, \lambda) = \frac{1}{3}$, it is enough to prove that $\left(3{\pi_3'}^*\xi_3 - E_3'\right)|_{\overline{C_2+x}}$ is nef, but not ample.

This restriction has class $3\xi_2 - [\mathbb{P}^1_\lambda]$. Its lift via the quotient map $C \times C \to C_2$ is the pullback of the theta class by the difference map $C \times C \to J_0$ given by $(x, y) \mapsto \mathcal{O}_C(x - y)$. In particular it is nef, and not ample because the lift vanishes on the diagonal of $C \times C$.

Case $|\lambda|$ is basepoint free. Since $|\lambda|$ is basepoint free it determines a g^1_3, a $3:1$ map to $\mathbb{P}^1$. Let $S_\lambda \subset C_3$ be the image of the map $F\colon C \times C \to C_3$ given by $(x, y) \mapsto x_2 + x_3 + y$, where $x + x_2 + x_3$ is the unique member of $|\lambda|$ determined by x. Observe that F is $1:1$ away from the diagonal, and $3:1$ on the diagonal. Then S_λ has multiplicity 3 along $\mathbb{P}^1_\lambda$, the image of the diagonal of $C \times C$ via F.

There exists a Cartier divisor $\frac{\Delta_3}{2}$ on C_3 with class $\frac{\delta_3}{2}$ such that $2 \cdot \frac{\delta_3}{2}$ is the class of the big diagonal Δ_3, the image of the map $G\colon C \times C \to C_3$ given by $(x, y) \mapsto 2x + y$.

One computes that $[S_\lambda] = 6\xi_3 - \frac{\delta_3}{2}$, hence the strict transform of S_λ in $\widetilde{C_3}$ satisfies $[\overline{S}_\lambda] = 6\pi_3'^*\xi_3 - \pi_3'^*\frac{\delta_3}{2} - 3E_3'$. For this consider $T \subset C \times_{\mathbb{P}^1} C \subset C \times C$ the closure of the complement of the diagonal inside the self fiber product over the g^1_3. It has class $3f_1 + 3f_2 - \delta$, where f_1, f_2 are the classes of the fibers of the two projections of $C \times C$, and δ is the class of the diagonal. Then S_λ is the image of $T \times C \subset (C \times C) \times C$ under the quotient map $C^3 \to C_3$.

The map F factors through $\widetilde{C_3}$. The pullback of $\pi_3'^*\xi_3 - \frac{1}{2}E_3'$ under F is $2f_1 + f_2 - \frac{1}{2}\delta$. This class is $f_1 + \frac{1}{2}[\Gamma]$, where Γ is the graph of the hyperelliptic involution and $[\Gamma] = 2f_1 + 2f_2 - \delta$. The intersection with Γ is 0, showing that it is nef and not ample. This shows $\varepsilon(P_3^\vee, \lambda) \le \frac{1}{2}$.

Case $|\lambda|$ is basepoint free with only simple ramification. Δ_3 meets $\mathbb{P}^1_\lambda$ in 8 points $2x_i + y_i$, the nonreduced fibers of the g^1_3. Then, the strict transform $\overline{\Delta}_3$ of Δ_3 in $\widetilde{C_3}$ is the blow-up of Δ_3 at the 8 points, and has class $\pi_3'^*\delta_3$. Let $\widetilde{C \times C}$ be the blow-up of $C \times C$ at the 8 points (x_i, y_i). Denote by E the sum of the 8 exceptional divisors, and by abuse denote by f_1, f_2, δ the pullbacks of the respective classes on $C \times C$. The pullback of $\pi_3'^*\xi_3 - \frac{1}{2}E_3'$ induced by G is $2f_1 + f_2 - \frac{1}{2}E$. It is unclear to us if this is nef.

The assumption that $|\lambda|$ has only simple ramification implies that the curve T constructed above is irreducible (see [Laz04a, Proof of Theorem 1.5.8]). Note also that $S_\lambda \cap \Delta_3 = G(T)$ set theoretically. Since $\overline{S}_\lambda$ contains no π_3'-exceptional curves, we have $\overline{S}_\lambda \cap \overline{\Delta}_3 = \overline{G(T)}$, and we deduce that the pullback of $\overline{S}_\lambda$ to $\widetilde{C \times C}$ is supported on an irreducible curve, the strict transform of T. The pullback has class $14f_1 + 6f_2 - 2\delta - 3E$. It is nef since it has self-intersection $8 > 0$.

We have $\pi_3'^*\xi_3 - \frac{3}{7}E_3' = \frac{1}{7}[\overline{S}_\lambda] + \frac{1}{14}\pi_3'^*(2\xi_3 + \delta_3)$. To prove its nefness, it is enough to verify the nefness of its restriction to $\overline{\Delta}_3$. From the previous paragraph we know that $\overline{S}_\lambda|_{\overline{\Delta}_3}$ is nef. It suffices to prove that $G^*(2\xi_3 + \delta_3)$ is nef. This class is $(4f_1 + 2f_2) + (-2f_1 + 2\delta) = 2(f_1 + f_2 + \delta)$, which is nef because it has intersection 0 with δ, and f_1, f_2 are nef. □

Remark 3.23. We expect that $\varepsilon\,(P_3^\vee, \lambda) = \frac{1}{2}$ when $|\lambda|$ is basepoint free (at least in the simple ramification case). This comes down to showing that $4f_1 + 2f_2 - E$ is nef on the blow-up of $C \times C$ at the 8 points (x_i, y_i) such that $2x_i + y_i$ are the nonreduced fibers of the g_3^1 determined by λ.

Since the Seshadri constant of the theta divisor on $\operatorname{Jac}(C)$ is $\frac{4}{3}$ ([Ste98]), the class $10f_1 + 4f_2 - 2\delta - \frac{4}{3}E$ is nef, and in fact in the boundary of the nef cone. It is the pullback of the corresponding class on the blow-up $\mathrm{Bl}_\lambda J_3$. At least when $|\lambda|$ has only simple ramification, we have that $14f_1 + 6f_2 - 2\delta - 3E$ is nef. It is the class of the pullback of $\overline{S}_\lambda$. We were not able to deduce the nefness of $4f_1 + 2f_2 - E$ from these two classes.

If L is an ample line bundle on a smooth projective variety X, it is conjectured ([EKL95]) that $\varepsilon\,(L; x) \geq 1$ for $x \in X$ very general. When $\mathcal{V}$ is an ample vector bundle of rank $r > 1$, the example of curves 3.20 shows that we cannot expect 1 as a universal lower bound at very general points. A conceivable generalization is a lower bound of $\frac{1}{r}$ at a very general point. When X is a smooth curve, this is proved in [Hac00, Corollary 3.1]. The lower bound $\frac{1}{r} = \frac{1}{2}$ is what we expect to see in the previous example.

3.3. **Functoriality.** We use restriction to curves to describe important properties of Seshadri constants.

Lemma 3.24 (Homogeneity)**.** *Assume that $\mathcal{V}$ is a coherent (twisted) sheaf on a projective scheme X. Fix $x \in X$. Then*
$$\varepsilon\,(\mathrm{S}^d\,\mathcal{V}; x) = d \cdot \varepsilon\,(\mathcal{V}; x).$$

Proof. By Corollary 3.21 and since symmetric powers are compatible with pullbacks, it is enough to consider the case of curves. By normalizing, we may assume that X is a smooth projective curve. After iterated Frobenius pullback, we may assume that $\overline{\mu}_{\min} = \mu_{\min}$ throughout. Note that slopes respect the formula $\mu(\mathrm{S}^d\,\mathcal{V}) = d \cdot \mu(\mathcal{V})$ for locally free sheaves $\mathcal{V}$. From any quotient $\mathcal{V} \twoheadrightarrow \mathcal{Q}$ of slope $\mu(\mathcal{Q})$ we obtain the quotient $\mathrm{S}^d\,\mathcal{V} \twoheadrightarrow \mathrm{S}^d\,\mathcal{Q}$ of slope $d \cdot \mu(\mathcal{Q})$. This proves the "$\leq$" inequality by Lemma 3.18.

For the inequality "$\geq$", we note that $\mathcal{V}\langle -\mu_{\min}(\mathcal{V})\rangle$ is nef and not ample (cf. [BP14, Theorem 1.1]). Thus, [Laz04b, Theorem 6.2.12(iii)] implies that so is $\mathrm{S}^d\big(\mathcal{V}\langle -\mu_{\min}(\mathcal{V})\rangle\big) = \big(\mathrm{S}^d\,\mathcal{V}\big)\langle -d \cdot \mu_{\min}(\mathcal{V})\rangle$. In particular, the latter can have no (twisted) quotients of negative slope, proving "$\geq$". □

Corollary 3.25. *Let $\mathcal{V}$ be a (twisted) locally free sheaf of finite rank on the projective variety X. Let $\nu_d : \mathbb{P}(\mathcal{V}) \to \mathbb{P}(\mathrm{S}^d\,\mathcal{V})$ denote the relative Veronese embedding. Then*
$$\nu_d^*\,\mathrm{Nef}^1\big(\mathbb{P}(\mathrm{S}^d\,\mathcal{V})\big) = \mathrm{Nef}^1\big(\mathbb{P}(\mathcal{V})\big) \qquad \textit{and} \qquad \nu_{d*}\,\overline{\mathrm{Eff}}_1\big(\mathbb{P}(\mathcal{V})\big) = \overline{\mathrm{Eff}}_1\big(\mathbb{P}(\mathrm{S}^d\,\mathcal{V})\big).$$

Proof. The second equality follows from the first by duality. Let $\rho_d : \mathbb{P}(\mathrm{S}^d\,\mathcal{V}) \to X$ be the bundle map with relative Serre bundle ξ_d such that $\nu_d^*\xi_d = d\xi$. If $\delta \in N^1(X)$, it is enough to prove that $d(\xi - \rho^*\delta)$ is nef if and only if $\xi_d - d\rho_d^*\delta$ is nef. In other words, that $\mathcal{V}\langle -\delta\rangle$ is nef if and only if $(\mathrm{S}^d\,\mathcal{V})\langle -d\delta\rangle = \mathrm{S}^d(\mathcal{V}\langle -\delta\rangle)$ is nef. This is immediate from Lemma 3.24 and from Remark 3.10.(a). □

Remark 3.26. With notation as in the corollary, when the characteristic of the base field is zero, then $\nu_d^* \overline{\mathrm{Eff}}^1(\mathbb{P}(\mathrm{S}^d \mathcal{V})) \supseteq \overline{\mathrm{Eff}}^1(\mathbb{P}(\mathcal{V}))$. By the duality of [BDPP13], we also deduce $\nu_{d*} \overline{\mathrm{Mov}}_1(\mathbb{P}(\mathcal{V})) \supseteq \overline{\mathrm{Mov}}_1(\mathbb{P}(\mathrm{S}^d \mathcal{V}))$.

For the first inclusion, we essentially want to show that if some $\mathrm{S}^p \mathcal{V}$ has sections, then some $\mathrm{S}^k \mathrm{S}^d \mathcal{V}$ has sections. In characteristic zero, $\mathrm{S}^{kd} \mathcal{V}$ is a direct summand of $\mathrm{S}^k \mathrm{S}^d \mathcal{V}$. Then one can take $k = p$.

Lemma 3.27 (Determinants)**.** *If $\mathcal{V}$ is locally free of rank r, then for all $x \in X$, we have*

$$\varepsilon(\mathcal{V}; x) \leq \frac{1}{r} \varepsilon(\det \mathcal{V}; x).$$

Proof. Immediate from Corollary 3.21 and from $\mu_{\min}(\nu^*\mathcal{V}) \leq \mu(\nu^*\mathcal{V}) = \frac{\deg_{C'}(\nu^*\mathcal{V})}{r} = \frac{\det \mathcal{V} \cdot C}{r}$. □

Lemma 3.28 (Tensor products)**.** *Let $\mathcal{V}$ and $\mathcal{V}'$ be (twisted) coherent sheaves on X. Then*

$$\varepsilon(\mathcal{V} \otimes \mathcal{V}'; x) \geq \varepsilon(\mathcal{V}; x) + \varepsilon(\mathcal{V}'; x)$$

for all $x \in X$. If X is a curve, or if $\mathcal{V}' = \mathcal{V}$, then equality holds.

Proof. Corollary 3.21 allows to reduce to the case of possibly singular curves. By normalizing we can assume that X is a smooth projective curve. Pulling back by a sufficiently large iteration of the Frobenius, we may assume that $\overline{\mu}_{\min} = \mu_{\min}$ for all the (finitely many) sheaves involed. Then in fact we claim

$$\mu_{\min}(\mathcal{V} \otimes \mathcal{V}') = \mu_{\min}(\mathcal{V}) + \mu_{\min}(\mathcal{V}').$$

Up to twisting, we may assume $\mu_{\min}(\mathcal{V}) = \mu_{\min}(\mathcal{V}') = 0$, so $\mathcal{V}$ and $\mathcal{V}'$ are nef. Then $\mathcal{V} \otimes \mathcal{V}'$ is also nef (cf. [Laz04a, Theorem 6.2.12]), hence $\mu_{\min}(\mathcal{V} \otimes \mathcal{V}') \geq 0$. If $\mathcal{Q}$ and $\mathcal{Q}'$ are quotients of slope 0 of $\mathcal{V}$ and $\mathcal{V}'$ respectively, then $\mathcal{Q} \otimes \mathcal{Q}'$ is a quotient of slope 0 of $\mathcal{V} \otimes \mathcal{V}'$, giving the remaining inequality $\mu_{\min}(\mathcal{V} \otimes \mathcal{V}') \leq 0$. □

Note that equality on curves does not lead to equality in arbitrary dimension in general, since the Seshadri constants $\varepsilon(\mathcal{V}; x)$ and $\varepsilon(\mathcal{V}'; x)$ could be approximated on different curves through x. We observe this below already for line bundles.

Example 3.29. On $X \coloneqq \mathbb{P}^1 \times \mathbb{P}^1$, we have $\varepsilon(\mathcal{O}(1,0); x) = \varepsilon(\mathcal{O}(0,1); x) = 0$ for all $x \in X$, since the line bundles in question are nef and have trivial restrictions on the fibers of the respective natural projection. On the other hand, as in [Laz04a, Example 5.1.7], we find $\varepsilon(\mathcal{O}(1,1); x) = 1$ for all $x \in X$. □

Corollary 3.30. *Let X be a projective scheme over an algebraically closed field, and let $\mathcal{V}$ and $\mathcal{V}'$ be (twisted) sheaves on X. Assume that $\mathcal{V}$ is ample (resp. nef), and that $\mathcal{V}'$ is nef. Then $\mathcal{V} \otimes \mathcal{V}'$ is ample (resp. nef). Furthermore, all Schur functors $S_\lambda \mathcal{V}$ are ample (resp. nef), where λ is a partition of some positive integer.*[2]

Compare with [Laz04b, Corollary 6.1.6] and [Bar71].

Proof. Immediate from Lemma 3.28 and Theorem 3.11 (resp. Remark 3.10.(a)). For the last part, use the construction of $S_\lambda \mathcal{V}$ as quotient of $\mathrm{S}^{\lambda_1} \mathcal{V} \otimes \ldots \otimes \mathrm{S}^{\lambda_r} \mathcal{V}$. This reduces the problem to showing that $\mathrm{S}^n \mathcal{V}$ is ample (resp. nef) if $\mathcal{V}$ is ample (resp. nef). This follows from Lemma 3.24 and from Theorem 3.11 (resp. Remark 3.10.(a)). □

Lemma 3.31. *Let $\mathcal{K} \to \mathcal{V} \to \mathcal{Q} \to 0$ be an exact sequence of (twisted) coherent sheaves on X. Then, we have*

$$\varepsilon(\mathcal{V}; x) \geq \min\{\varepsilon(\mathcal{K}; x), \varepsilon(\mathcal{Q}; x)\}$$

[2] Here, $S_\lambda \mathcal{V}$, where $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r \geq 0) \vdash n$, is understood as a quotient of $\mathrm{S}^{\lambda_1} \mathcal{V} \otimes \ldots \otimes \mathrm{S}^{\lambda_r} \mathcal{V}$ as in [Ful97, Chapter 8.3, Example 10].

for all $x \in X$. In particular, if $\varepsilon(\mathcal{K}; x) \geq \varepsilon(\mathcal{Q}; x)$, then $\varepsilon(\mathcal{V}; x) = \varepsilon(\mathcal{Q}; x)$. Furthermore, if $\mathcal{V} = \mathcal{K} \oplus \mathcal{Q}$, then $\varepsilon(\mathcal{V}; x) = \min\{\varepsilon(\mathcal{K}; x), \varepsilon(\mathcal{Q}; x)\}$.

Proof. By Corollary 3.21, as above, we can assume that X is a smooth curve, and that $\overline{\mu}_{\min} = \mu_{\min}$ for all the sheaves involved. Let $\mathcal{V} \twoheadrightarrow A$ be the quotient of minimal slope in the Harder–Narasimhan filtration of $\mathcal{V}$. In particular, A is semistable. If the induced map $\mathcal{K} \to A$ is nonzero, then its image has slope at most $\mu(A) = \mu_{\min}(\mathcal{V})$, and $\mu_{\min}(\mathcal{K}) \leq \mu_{\min}(\mathcal{V})$. If $\mathcal{K} \to A$ is zero, then we obtain an induced nonzero map $\mathcal{Q} \to A$ and argue as before. The last part follows from Lemma 3.18. $\square$

3.4. **Pseudo-effectivity.** Using results from [BDPP13] (which hold in arbitrary characteristic by [FL17, Section 2.2]), we show that Seshadri constants for non-pseudo-effective divisors are negative.

Lemma 3.32. *Let X be a projective variety of dimension n over an algebraically closed field, and let $L \in N^1(X)$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor class outside the pseudo-effective cone $\overline{\mathrm{Eff}}^1(X)$. Then, $\varepsilon(L; x) = -\infty$ for general $x \in X$. Furthermore, $\varepsilon(L; x) < 0$ for all x.*

Proof. By [BDPP13, Theorem 2.2], there exists a birational model $f\colon X' \to X$ and ample divisor classes $H_1, \ldots, H_{n-1}$ on X' such that $L \cdot f_*(H_1 \cdot \ldots \cdot H_{n-1}) < 0$. Since there exist complete intersection curves through every point of X', their images pass through every point of X. This implies $\varepsilon(L; x) < 0$ for all x.

Let $x \in X$ be a general smooth point where f is an isomorphism, and denote the inverse image of x in X' also by x. By Bertini, for all large m there exists a curve C_m smooth at $x \in X'$, of class $m^{n-1}(H_1 \cdot \ldots \cdot H_{n-1})$. Then $\varepsilon(L; x) \leq \frac{L \cdot f_* C_m}{\operatorname{mult}_x f_* C_m}$. The latter tends to $-\infty$ as $m \to \infty$. $\square$

Corollary 3.33. *Let X be a projective variety of dimension n over an algebraically closed field. If $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is not pseudo-effective, then $\varepsilon(\mathcal{V}; x) < 0$ for all $x \in X$.*

Proof. Immediate from Lemma 3.32 and the negative case of Remark 3.12. $\square$

3.5. **Semicontinuity.** We end this section with two semicontinuity results. The first concerns semicontinuity in the $\mathbb{R}$-twists λ for a twisted sheaf of the form $\mathcal{V}\langle\lambda\rangle$, which is a consequence of our functoriality results.

Corollary 3.34. *Let X be a projective scheme over an algebraically closed field, and fix a closed point $x \in X$. Let $\mathcal{V}$ be a coherent sheaf, with $\varepsilon(\mathcal{V}; x) > -\infty$. Let h be an ample divisor class on X. Consider the function $\epsilon(t) \coloneqq \varepsilon(\mathcal{V}\langle th\rangle; x)$. Then, ϵ is nondecreasing, continuous at all $t > 0$ and lower-semicontinuous at $t = 0$.*

Proof. Lemma 3.28 and homogeneity for divisors imply
$$\epsilon(t) \geq \epsilon(t') + (t - t')\,\varepsilon(h; x) > \epsilon(t')$$
for all $t > t' \geq 0$. Furthermore $\epsilon(t) + \epsilon(t') \leq \varepsilon\big(\mathcal{V} \otimes \mathcal{V}\big\langle(t + t')h\big\rangle; x\big) = 2\epsilon\big(\frac{t+t'}{2}\big)$. Finite concave functions are continuous on open intervals. Lower-semicontinuity follows because ϵ is nondecreasing. $\square$

Proposition 3.35 (Semicontinuity in families)**.** *Let T be a smooth connected variety over an uncountable algebraically closed field. Let $p\colon \mathscr{X} \to T$ be a smooth projective family of varieties with connected fibers and a section $T \to \mathscr{X}$ which maps $t \mapsto x_t \in X_t \coloneqq p^{-1}\{t\}$. Let $\mathscr{V}$ be a locally free sheaf on $\mathscr{X}$, and denote $\mathcal{V}_t$ the corresponding restriction to X_t.*

Let $\epsilon \geq 0$, and let $t_0 \in T$ such that $\mathcal{V}_{t_0}$ is nef and $\varepsilon(\mathcal{V}_{t_0}; x_{t_0}) \geq \epsilon$. Then, $\varepsilon(\mathcal{V}_t; x_t) \geq \epsilon$ for very general $t \in T$. In particular, under the positivity assumptions above, the Seshadri constants are constant outside an at most countable union of proper closed subsets, on which they may decrease.

Proof. Let $\rho\colon \mathbb{P}(\mathscr{V}) \to \mathscr{X}$ be the bundle map with fiberwise restrictions $\rho_t\colon \mathbb{P}(\mathcal{V}_t) \to X_t$. If the conclusion fails, then standard relative Hilbert scheme arguments produce a scheme of finite type H with a dominant morphism $f\colon H \to T$ (by restriction to a closed subset we may assume that f is generically finite) and a relative flat curve $\mathscr{C} \subset H \times_T \mathbb{P}(\mathscr{V})$ over H such that the fibers $C_h \subset \mathbb{P}(\mathcal{V}_{f(h)})$ are irreducible, and moreover in $\mathcal{C}_{\mathcal{V}_{f(h)}, x_{f(h)}}$ for all $h \in H$. Furthermore $\frac{\xi_{f(h)} \cdot C_h}{\mathrm{mult}_{x_{f(h)}} \rho_{t*} C_h} < \epsilon$.

Let $Y \subset \mathbb{P}(\mathscr{V})$ be the closure of $(f \times_T \mathbb{P}(\mathscr{V}))(\mathscr{C}) = \cup_{h \in H} C_h \subset \mathbb{P}(\mathscr{V})$. For any $t \in T$, denote by $[Y_t]$ the Chow class of the restriction $Y|_{\mathbb{P}(\mathcal{V}_t)}$ in the sense of [Ful98, Chapter 8]. This is an effective curve class (even if the scheme theoretic fiber Y_t may have dimension greater than 1). See [FL16, Lemma 4.10] for details. For very general $t \in T$, the class $[Y_t]$ is represented by the fundamental cycle of the scheme theoretic Y_t which is just the sum (with multiplicity) of the finitely many C_h with $h \in f^{-1}t$. By abuse, we write $[Y_t] = Y_t = \sum_{h \in f^{-1}t} C_h$ in this case.

For $t \in T$ very general, let $Z_0 + Z_0'$ be a flat degeneration over t_0 of the restriction of Y over some irreducible curve $T' \subset T$ connecting t and t_0. In fact, $Z_0 + Z_0'$ is the fundamental cycle of the fiber over t_0 of the irreducible component of $Y_{T'}$ that dominates T'. Here Z_0' is the part that does not come from $\mathcal{C}_{\mathcal{V}_{x_{t_0}}, x_{t_0}}$. Since multiplicity is upper semicontinuous in families, and $\rho_{t_0 *} Z_0'$ does not have x_{t_0} in its support, we have

$$\mathrm{mult}_{x_{t_0}} \rho_{t_0 *} Z_0 \geq \mathrm{mult}_{x_t} \rho_{t*}[Y_t].$$

Since ξ_{t_0} is nef, we have $\xi_{t_0} \cdot [Y_{t_0}] \geq \xi_{t_0} \cdot Z_0$. We reach the contradiction

$$\begin{aligned} \epsilon \leq \frac{\xi_{t_0} \cdot Z_0}{\mathrm{mult}_{x_{t_0}} \rho_{t_0 *} Z_0} \leq \frac{\xi_{t_0} \cdot [Y_{t_0}]}{\mathrm{mult}_{x_{t_0}} \rho_{t_0 *} Z_0} \leq \frac{\xi_t \cdot [Y_t]}{\mathrm{mult}_{x_t} \rho_{t*} Y_t} &= \frac{\xi_t \cdot \sum_{h \in f^{-1}t} C_h}{\sum_{h \in f^{-1}t} \mathrm{mult}_{x_t} \rho_{t*} C_h} \\ &\leq \max_{h \in f^{-1}t} \frac{\xi_t \cdot C_h}{\mathrm{mult}_{x_t} \rho_{t*} C_h} < \epsilon. \end{aligned}$$

□

Remark 3.36. The same results work in the more general setting of a smooth projective morphism $\rho\colon \mathscr{Y} \to \mathscr{X}$ of T-schemes with ρ-ample polarization ξ. The only step in the proof of Proposition 3.35 where the nefness of $\mathcal{V}_{t_0}$ is used is in the inequality $\xi_{t_0} \cdot Z_0 \leq \xi_{t_0} \cdot [Y_{t_0}]$. In the absence of the nefness condition (and of the positivity of ϵ), [Ful17, Example 3.15] observes that this form of lower semicontinuity fails already for line bundles on toric surfaces.

4. Tangent bundles

Let X be a smooth projective variety, and let TX be the tangent sheaf. We are interested in the Seshadri constants of this bundle and in how they recover some of the birational geometry of X. The motivation is given by the following easy consequence of the Seshadri ampleness criterion (Theorem 3.11) and Mori's characterization of projective space [Kol96, V.3.3 Corollary].

Corollary 4.1. *Let X be a smooth projective variety. If $\inf_{x \in X} \varepsilon(TX; x) > 0$, then $X \simeq \mathbb{P}^n$.*

4.1. Examples.
We start by computing some examples.

Example 4.2 (Seshadri constants for $T\mathbb{P}^n$)**.** We have

$$\varepsilon(T\mathbb{P}^n; x) = \begin{cases} 2 & \text{if } n = 1; \\ 1 & \text{if } n \geq 2. \end{cases}$$

(Since $\mathbb{P}^n$ is a toric variety, and $T\mathbb{P}^n$ is ample, by [HMP10, Proposition 3.2], the Seshadri constant is computed by restricting to invariant $\mathbb{P}^1$'s. For $n = 1$, the restriction is $\mathcal{O}_{\mathbb{P}^1}(2)$. For $n \geq 2$, the restriction is $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, and we see that the minimal slope is 1.) □

Example 4.3 (Homogeneous varieties)**.** If X is a homogeneous variety (e.g., abelian or rational homogeneous space like a Grassmann variety or smooth quadric), not isomorphic to a projective space, then TX is globally generated but not ample. Since X has a transitive algebraic group action, $\varepsilon(TX; x)$ is independent of $x \in X$. Then $\varepsilon(TX; x) = 0$ for all $x \in X$ by the Seshadri ampleness criterion (Theorem 3.11).

Example 4.4 (Varieties of general type)**.** Assume that X is smooth projective variety over an algebraically closed field, with K_X big (or even pseudo-effective, but not numerically trivial). Then

$$\varepsilon(TX; x) = -\infty \quad \forall\, x \in X.$$

(Let C_d be a smooth curve through x with $\lim_{d\to\infty} K_X \cdot C_d = \infty$. General complete intersections through x of large degree will do. Then $\varepsilon(-K_X; x) = -\infty$. Conclude by Lemma 3.27.) □

Example 4.5 (Calabi–Yau type manifolds)**.** Assume that X is a smooth projective variety of dimension n over an algebraically closed field, with K_X numerically trivial. Then

$$\varepsilon(TX; x) \leq 0 \quad \forall\, x \in X.$$

(Indeed $\varepsilon(TX; x) \leq \frac{1}{n}\varepsilon(\det TX; x) = 0$.) □

Corollary 4.6 (Uniruledness and Separably rationally connectedness (SRC) criterion)**.** *Let X be a smooth projective variety over an algebraically closed field. Assume there exists $x_0 \in X$ such that $\varepsilon(TX; x_0) > 0$. Then X is uniruled, even SRC, and $\mathcal{O}_{\mathbb{P}(TX)}(1)$ is pseudo-effective.*

Proof. The previous two examples show that K_X is not pseudo-effective. Then X is uniruled by [BDPP13] (whose results hold in arbitrary characteristic by [FL17, Section 2.2]).

We now show that X is SRC. Since $\frac{1}{\dim X}\varepsilon(-K_X; x_0) \geq \varepsilon(TX; x_0) > 0$ by Lemma 3.27, we see that $-K_X \cdot C > 0$ for every curve C through x_0. By bend and break [Kol96, II.5.14 Theorem], there therefore exists a rational curve D through x_0. Since $\varepsilon(TX; x_0) > 0$, we see that $TX|_D$ is very free by Example 3.20, and it follows that X is SRC by [Kol96, IV.3.7 Theorem].

For the pseudo-effectivity statement, see Corollary 3.33. □

Remark 4.7. The previous criterion is not a characterization of uniruled or SRC varieties. If $f\colon X \to Y$ is a smooth morphism of smooth projective varieties with positive dimensional fibers and $\dim Y > 0$, we claim that $\varepsilon(TX; x) \leq 0$ for all $x \in X$. (Let $y \coloneqq \pi(x) \in Y$. From the surjections

$$TX \twoheadrightarrow f^*TY \twoheadrightarrow f^*TY|_{X_y} = \mathcal{O}_{X_y}^{\oplus \dim Y},$$

by Lemma 3.18 we deduce $\varepsilon(TX; x) \leq \varepsilon(\mathcal{O}_{X_y}^{\oplus \dim Y}; x) = 0$.) □

4.2. **Characterizations of projective space.** In particular cases, we can say something stronger than Corollary 4.6 when $\varepsilon(TX; x_0) > 0$ for a point $x_0 \in X$.

Proposition 4.8 (Fano manifolds)**.** *Let X be a smooth projective variety over an algebraically closed field k. Suppose that one of the following conditions holds:*

(1) *X is Fano and some $x_0 \in X$ verifies $\varepsilon(TX; x_0) > 0$;*
(2) $\operatorname{char} k = 0$ *and a* general *point $x_0 \in X$ verifies $\varepsilon(TX; x_0) > 0$.*

Then, $X \simeq \mathbb{P}^n$.

We note that the notion of general point in (2) is that in [Keb02, Notation 2.2].

Proof. Let $f\colon \mathbb{P}^1 \to X$ be a rational curve passing through x_0. From Corollary 3.21 we immediately find that f^*TX is ample, hence

$$f^*TX \simeq \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \oplus \cdots \oplus \mathcal{O}(d_n) \tag{4.8.1}$$

and $d_i \geq 1$ for all i. In situation (1), we conclude that $X \simeq \mathbb{P}^n$ from [Kol96, V.3.2 Theorem].

In situation (2), we have that $d_i \geq 2$ for some i in (4.8.1) since there is a non-zero natural homomorphism $\mathcal{O}(2) \simeq T\mathbb{P}^1 \to f^*TX$. Thus, $\deg f^*TX = -\deg f^*\omega_X \geq n+1$ for every rational curve passing through x_0. Since X is uniruled by Corollary 4.6, we conclude that $X \simeq \mathbb{P}^n$ from [CMSB02, Corollary 0.4(11)]. □

See also Corollary 6.7. Inspired by Proposition 4.8, we pose the following:

Conjecture 4.9. *Let X be a smooth projective variety over an algebraically closed field. If there exists $x_0 \in X$ such that $\varepsilon(TX; x_0) > 0$, then $X \simeq \mathbb{P}^n$.*

We now show the case when $\dim X = 2$. We start with the following:

Lemma 4.10. *Let $Z \subset X$ be a smooth closed subvariety of a smooth variety. Consider the blow-up cartesian diagram*

$$\begin{tikzcd} E \arrow[r, hook, "\jmath"] \arrow[d, "\pi|_E"'] \arrow[dr, phantom, "\square"] & \widetilde{X} \arrow[d, "\pi"] \\ Z \arrow[r, hook, "\imath"'] & X \end{tikzcd}.$$

Identify all $x \in X \setminus Z$ with their preimages in $\widetilde{X} \setminus E$. Then for all $x \in X \setminus Z$,

$$\varepsilon\left(\pi^*TX(-E); x\right) \le \varepsilon\left(T\widetilde{X}; x\right) \le \varepsilon\left(TX; x\right) \le \varepsilon\left(T\widetilde{X}(E); x\right) \le \varepsilon\left(\pi^*TX(E); x\right).$$

Proof. We have a short exact sequence $0 \to \pi^*\Omega_X \to \Omega_{\widetilde{X}} \to \jmath_*\Omega_{E/Z} \to 0$. By duality, from the long $\mathcal{E}xt$ sequence we extract

$$0 \longrightarrow T\widetilde{X} \longrightarrow \pi^*TX \longrightarrow \jmath_*T_{E/Z}(E) \longrightarrow 0. \tag{4.10.1}$$

The second inequality now follows from Lemma 3.31. We use here that $\varepsilon\left(\jmath_*T_{E/Z}(E); x\right) = \infty$, because x is not in the support and $\varepsilon\left(\pi^*TX; x\right)$ (computed an $\tilde{X}$) is the same as $\varepsilon\left(TX; x\right)$ (computed on X). The fourth inequality is similar. Twist (4.10.1) by E.

For the first inequality, the main ingredient is a short exact sequence

$$0 \longrightarrow \pi^*TX(-E) \longrightarrow T\widetilde{X} \longrightarrow \jmath_*Q \longrightarrow 0 \tag{4.10.2}$$

Assuming it, we conclude again by Lemma 3.31. For the third inequality, twist (4.10.2) by E.

From the normal bundle sequence $0 \to TE \to T\widetilde{X}|_E \to \mathcal{O}_E(E) \to 0$ and the relative tangent bundle sequence $0 \to T_{E/Z} \to TE \to \pi|_E^*TZ \to 0$, we find a bundle Q defined by the sequence

$$0 \longrightarrow T_{E/Z} \longrightarrow T\widetilde{X}|_E \longrightarrow Q \longrightarrow 0, \tag{4.10.3}$$

sitting in $0 \to \pi|_E^*TZ \to Q \to \mathcal{O}_E(E) \to 0$. Restrict (4.10.1) over E, obtaining $T\widetilde{X}|_E \to \pi^*TX|_E \to T_{E/Z}(E) \to 0$. The first map is the restriction of the differential $d\pi$. Its kernel is clearly $T_{E/Z}$, included in $T\widetilde{X}|_E$ by (4.10.3). We obtain another short exact sequence

$$0 \longrightarrow Q \longrightarrow \pi^*TX|_E \longrightarrow T_{E/Z}(E) \longrightarrow 0. \tag{4.10.4}$$

From the snake lemma for (4.10.1) and (4.10.4), we obtain (4.10.2). ☐

Corollary 4.11. *With notation as in the lemma, if $\varepsilon(T\widetilde{X}; x_0) > 0$ for some $x \in X \setminus Z = \widetilde{X} \setminus E$, then $\varepsilon(TX; x_0) > 0$.*

Corollary 4.12. *Let X be a smooth projective* surface *over an algebraically closed field. If there exists $x_0 \in X$ such that $\varepsilon(TX; x_0) > 0$, then $X \simeq \mathbb{P}^2$.*

Proof. Let $E \subset X$ be a smooth curve with negative self-intersection. Then from the surjection $TX|_E \twoheadrightarrow \mathcal{O}_E(E)$ we deduce that $\varepsilon(TX; x) < 0$ for all $x \in E$.

Let $\pi\colon X \to X'$ be a minimal model of X constructed by blowing-down smooth -1 curves. By the previous observation, x_0 is not on any of the contracted curves, so it is in the isomorphism locus of π. By the previous corollary, $\varepsilon(TX'; \pi(x_0)) > 0$.

The examples at the beginning of the section show that X' is uniruled. In the Kodaira classification of minimal surfaces, X' is then either $\mathbb{P}^2$, or a ruled surface (possibly a Hirzebruch surface). Remark 4.7 excludes ruled surfaces. Therefore $X' \simeq \mathbb{P}^2$.

If π is not an isomorphism, then it factors through the blow-up of one point on $\mathbb{P}^2$. This is the Hirzebruch surface $\mathbb{F}_1$. Apply the previous corollary and Remark 4.7 again to find a contradiction. ☐

5. Separation of jets

In this section we give a characterization of Seshadri constants in terms of separation of jets following [Laz04a, Chapter 5]. First, recall the following:

Definition 5.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on a projective scheme X, and fix a closed point $x \in X$ defined by the ideal $\mathfrak{m}_x \subseteq \mathcal{O}_X$. We say that $\mathcal{F}$ *separates s-jets at x* if the restriction map

$$H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}/\mathfrak{m}_x^{s+1}\mathcal{F})$$

is surjective. With the convention $\mathfrak{m}_x^0 = \mathcal{O}_X$, all sheaves separate -1-jets. We denote by $s(\mathcal{F}; x)$ the largest integer $s \geq -1$ such that $\mathcal{F}$ separates s-jets at x.

Remark 5.2. If $\mathcal{F} \to \mathcal{G}$ is a morphism of quasi-coherent $\mathcal{O}_X$-modules, surjective at x, then $s(\mathcal{G}; x) \geq s(\mathcal{F}; x)$.

We show the following analogue of [Laz04a, Theorem 5.1.17] for higher ranks. The statement for x a singular point is new even for line bundles.

Theorem 5.3. *Let $\mathcal{V}$ be an ample coherent sheaf on a projective scheme X, and let $x \in X$ be a closed point. Then,*

$$\varepsilon(\mathcal{V}; x) \leq \lim_{k\to\infty} \frac{s(\mathrm{S}^k \mathcal{V}; x)}{k},$$

and equality holds if $\mathcal{V}$ is locally free at x.

When $\mathcal{V}$ is locally free, for any cartesian diagram

$$\begin{tikzcd} \mathbb{P}(f^*\mathcal{V}) \arrow[r, "f'"] \arrow[d, "\rho'"'] \arrow[dr, phantom, "\square"] & \mathbb{P}(\mathcal{V}) \arrow[d, "\rho"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

and any $k \geq 0$, the base change map

$$f^*\rho_*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(k) \longrightarrow \rho'_* f'^*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(k) \tag{5.3.1}$$

is an isomorphism. Both terms are isomorphic to $\mathrm{S}^k f^*\mathcal{V}$. When $\mathcal{V}$ is an arbitrary coherent sheaf, then the same conclusion holds for k sufficiently large. We will also need the following lemma.

Lemma 5.4 (cf. [Ito13, Proof of Lem. 3.7])**.** *Let X be a scheme, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X with $s(\mathcal{F}; x) \geq 0$ and $s(\mathcal{G}; x) \geq 0$. Then, for every closed point $x \in X$, we have*

$$s(\mathcal{F}; x) + s(\mathcal{G}; x) \leq s(\mathcal{F} \otimes \mathcal{G}; x).$$

Furthermore,

$$s(\mathrm{S}^m \mathcal{F}; x) + s(\mathrm{S}^n \mathcal{F}; x) \leq s(\mathrm{S}^{m+n} \mathcal{F}; x)$$

for all $m, n \geq 0$.

Proof. We first show that a coherent sheaf $\mathcal{F}$ separates s-jets if and only if

$$H^0(X, \mathfrak{m}_x^i\mathcal{F}) \longrightarrow H^0(X, \mathfrak{m}_x^i\mathcal{F}/\mathfrak{m}_x^{i+1}) \tag{5.4.1}$$

is surjective for every $i \in \{0, 1, \ldots, s\}$. We proceed by induction on s. If $s = 0$, then there is nothing to show. Now suppose $s > 0$. By induction and the fact that a coherent sheaf separating s-jets also separates all lower order jets, it suffices to show that if $\mathcal{F}$ separates $(s-1)$-jets, then $\mathcal{F}$ separates s-jets if and only if (5.4.1) is surjective for $i = s$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_x^s\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}/\mathfrak{m}_x^s\mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{m}_x^s\mathcal{F}/\mathfrak{m}_x^{s+1}\mathcal{F} \arrow[r] & \mathcal{F}/\mathfrak{m}_x^{s+1}\mathcal{F} \arrow[r] & \mathcal{F}/\mathfrak{m}_x^s\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Taking global sections, we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathfrak{m}_x^s\mathcal{F}) \arrow[r] \arrow[d] & H^0(X, \mathcal{F}) \arrow[r] \arrow[d] & H^0(X, \mathcal{F}/\mathfrak{m}_x^s\mathcal{F}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^0(X, \mathfrak{m}_x^s\mathcal{F}/\mathfrak{m}_x^{s+1}) \arrow[r] & H^0(X, \mathcal{F}/\mathfrak{m}_x^{s+1}\mathcal{F}) \arrow[r] & H^0(X, \mathcal{F}/\mathfrak{m}_x^s\mathcal{F})
\end{tikzcd}$$

where the top row remains exact by the assumption that $\mathcal{F}$ separates $(s-1)$-jets. By the snake lemma, we see that the left vertical arrow is surjective if and only if the middle vertical arrow is surjective, as desired.

We now prove the lemma. Suppose $\mathcal{F}$ separates i-jets and $\mathcal{G}$ separates j-jets. We then have the commutative diagram

$$\begin{tikzcd}
H^0(X, \mathfrak{m}_x^i\mathcal{F}) \otimes H^0(X, \mathfrak{m}_x^j\mathcal{G}) \arrow[r] \arrow[d] & H^0(X, \mathfrak{m}_x^i\mathcal{F}/\mathfrak{m}_x^{i+1}\mathcal{F} \otimes \mathfrak{m}_x^j\mathcal{G}/\mathfrak{m}_x^{j+1}\mathcal{G}) \arrow[d] \\
H^0\big(X, \mathfrak{m}_x^{i+j}(\mathcal{F} \otimes \mathcal{G})\big) \arrow[r] & H^0\big(X, \mathfrak{m}_x^{i+j}(\mathcal{F} \otimes \mathcal{G})/\mathfrak{m}_x^{i+j+1}(\mathcal{F} \otimes \mathcal{G})\big)
\end{tikzcd}$$

Since the top horizontal arrow is surjective by assumption, and the right vertical arrow is surjective, essentially by the the surjectivity of

$$\mathfrak{m}_x^i/\mathfrak{m}_x^{i+1} \otimes \mathfrak{m}_x^j/\mathfrak{m}_x^{j+1} \simeq (\mathfrak{m}_x^i \otimes \mathfrak{m}_x^j) \otimes \mathcal{O}_X/\mathfrak{m}_x \twoheadrightarrow \mathfrak{m}_x^{i+j}/\mathfrak{m}_x^{i+j+1},$$

we see that the composition from the top left corner to the bottom right corner is surjective, hence the bottom horizontal arrow is surjective. By running through all combinations of integers $i \leq s(\mathcal{F}; x)$ and $j \leq s(\mathcal{G}; x)$, we see that $s(\mathcal{F}; x) + s(\mathcal{G}; x) \leq s(\mathcal{F} \otimes \mathcal{G}; x)$ by the argument in the previous paragraph.

The statement on symmetric powers is similar. Use the commutative diagram

$$\begin{tikzcd}
H^0\big(X, \mathfrak{m}_x^i \operatorname{S}^m \mathcal{F}\big) \otimes H^0\big(X, \mathfrak{m}_x^j \operatorname{S}^n \mathcal{F}\big) \arrow[r] \arrow[d] & H^0\big(X, \mathfrak{m}_x^i \operatorname{S}^m \mathcal{F}/\mathfrak{m}_x^{i+1} \operatorname{S}^m \mathcal{F} \otimes \mathfrak{m}_x^j \operatorname{S}^n \mathcal{F}/\mathfrak{m}_x^{j+1} \operatorname{S}^n \mathcal{F}\big) \arrow[d] \\
H^0\big(X, \mathfrak{m}_x^{i+j} \operatorname{S}^{m+n} \mathcal{F}\big) \arrow[r] & H^0\big(X, \mathfrak{m}_x^{i+j} \operatorname{S}^{m+n} \mathcal{F}/\mathfrak{m}_x^{i+j+1} \operatorname{S}^{m+n} \mathcal{F}\big)
\end{tikzcd}$$

□

Proposition 5.5. *Let X be a projective scheme, and let $\mathcal{V}$ be a coherent sheaf on it. Assume that $\mathcal{V}$ is locally free around x and $\varepsilon(\mathcal{V}; x) \geq 0$. Then*

$$\varepsilon(\mathcal{V}; x) \geq \limsup_{k\to\infty} \frac{s(\operatorname{S}^k \mathcal{V}; x)}{k}.$$

In particular, $\varepsilon(\mathcal{V}; x) \geq s(\mathcal{V}; x)$.

Proof. Note that the second statement follows from the first by Lemma 5.4, since the limit supremum is a supremum by Fekete's lemma. We have natural maps $\mathfrak{m}_x^s \subseteq \pi_*\mathcal{O}_{\mathrm{Bl}_x X}(-sE)$ for all $s \geq 0$. They are equalities if x is smooth, or if s is sufficiently large. In either case, for all coherent $\mathcal{V}$ that are locally free around x, they induce isomorphisms

$$H^0\big(X, \mathfrak{m}_x^s\mathcal{V}/\mathfrak{m}_x^{s+1}\mathcal{V}\big) \simeq H^0\big(\mathbb{P}(\mathcal{V}(x)), \mathcal{O}(1)\big) \otimes H^0\big(E, \mathcal{O}_E(-sE)\big). \tag{5.5.1}$$

This is because $\mathfrak{m}_x^s\mathcal{V}/\mathfrak{m}_x^{s+1}\mathcal{V} \simeq \mathfrak{m}_x^s/\mathfrak{m}_x^{s+1} \otimes \mathcal{V}(x)$ by the fact that $\mathcal{V}$ is flat at x, and because $\mathfrak{m}_x^s/\mathfrak{m}_x^{s+1} = \pi_*\mathcal{O}_E(-sE)$, under our assumptions on x and s. When $s \geq 1$, these assumptions also imply that $\mathcal{O}_E(-sE)$ is very ample on E. When $s = 0$, it is globally generated. The same are true of $\mathcal{O}_{\mathbb{P}(\mathcal{V}(x))}(1) \boxtimes \mathcal{O}_E(-sE) = \mathcal{O}_{\rho'^{-1}E}((\pi'^*\xi - s\rho'^*E)|_{\rho'^{-1}E})$.

Let $s \coloneqq s(\mathcal{V}; x)$. Assume $s \geq 0$. As in the proof of Lemma 5.4, we have a surjection

$$H^0(X, \mathfrak{m}_x^s\mathcal{V}) \twoheadrightarrow H^0\big(X, \mathfrak{m}_x^s\mathcal{V}/\mathfrak{m}_x^{s+1}\mathcal{V}\big). \tag{5.5.2}$$

When x is smooth or s is large, then $\pi'^*\xi - s\rho'^*E$ is globally generated along $\rho'^{-1}E$. For this, in view of (5.5.1) and (5.5.2), it is enough to show that $H^0(X, \mathfrak{m}_x^s\mathcal{V})$ determine naturally a subspace of $H^0\big(Y', \pi'^*\mathcal{O}_Y(1) \otimes \rho'^*\mathcal{O}_{\mathrm{Bl}_x X}(-sE)\big)$. Consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{m}_x^s\mathcal{V} \arrow[r, two heads] \arrow[d, "\simeq"'] & \mathfrak{m}_x^s\mathcal{V}/\mathfrak{m}_x^{s+1}\mathcal{V} \arrow[d, "\simeq"] \\
\pi_*\big(\pi^*\mathcal{V} \otimes \mathcal{O}_{\mathrm{Bl}_x X}(-sE)\big) \arrow[r] \arrow[d] & \pi_*\big(\pi^*\mathcal{V} \otimes \mathcal{O}_E(-sE)\big) \arrow[d, "\simeq"] \\
\pi_*\big(\pi^*\rho_*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \mathcal{O}_{\mathrm{Bl}_x X}(-sE)\big) \arrow[r] \arrow[d] & \pi_*\big(\pi^*\rho_*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \mathcal{O}_E(-sE)\big) \arrow[d, "\simeq"] \\
\pi_*\big(\rho'_*\pi'^*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \mathcal{O}_{\mathrm{Bl}_x X}(-sE)\big) \arrow[r] \arrow[d, "\simeq"'] & \pi_*\big(\rho'_*\pi'^*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \mathcal{O}_E(-sE)\big) \arrow[d, "\simeq"] \\
(\pi \circ \rho')_*\big(\pi'^*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \rho'^*\mathcal{O}_{\mathrm{Bl}_x X}(-sE)\big) \arrow[r] & (\pi \circ \rho')_*\big(\pi'^*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \rho'^*\mathcal{O}_E(-sE)\big)
\end{tikzcd}$$

where the top vertical arrows are isomorphisms by the fact that $\mathcal{V}$ is locally free at x, and the vertical arrows in the second row are obtained from the natural map $\mathcal{V} \to \rho_*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$; the map on the right is an isomorphism since $\mathcal{V}$ is locally free at x. The arrows in the third row are obtained from base change for the cartesian diagram in Notation 3.1, where the right arrow is an isomorphism by cohomology and base change since $\mathcal{V}$ is locally free at x, hence π is flat around x. The bottom vertical arrows are isomorphisms by the projection formula. After taking global sections, the bottom horizontal arrow is still surjective by the commutativity of the diagram. Thus, since $\pi'^*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \rho'^*\mathcal{O}_E(-sE)$ is globally generated, we see that $\pi'^*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \rho'^*\mathcal{O}_{\mathrm{Bl}_x X}(-sE)$ is globally generated along $\rho'^{-1}(E)$.

Let $C' \in \mathcal{C}'_{\mathcal{V},x}$. By Proposition 3.7, when $s \geq 0$, to show $\varepsilon(\mathcal{V}; x) \geq s$, it is enough to prove that

$$(\pi'^*\xi - s\rho'^*E) \cdot C' \geq 0.$$

Use global generation along $\rho'^{-1}E$ to produce an effective divisor in the class $\pi'^*\xi - s\rho'^*E$ that does not pass through y, where y is any point of $C' \cap \rho'^{-1}E$.

If x is smooth, the argument above works when $s \geq 0$. When $s = -1$, there is nothing to prove.

If x is singular, and if $s(\mathrm{S}^k\,\mathcal{V}; x) > 0$ for some k, then by Lemma 5.4 we have that $s(\mathrm{S}^k\,\mathcal{V}; x)$ is arbitrarily large as k grows. Repeat the arguments above for all $\mathrm{S}^k\,\mathcal{V}$, and use the homogeneity of $\varepsilon(-; x)$ from Lemma 3.24. Assume $s(\mathrm{S}^k\,\mathcal{V}; x) \leq 0$ for all k. If $s(\mathrm{S}^k\,\mathcal{V}; x) = 0$, then $\mathrm{S}^k\,\mathcal{V}$ is globally generated at x, therefore $\varepsilon(\mathrm{S}^k\,\mathcal{V}; x) \geq 0$ by Example 3.3. By homogeneity, $\varepsilon(\mathcal{V}; x) \geq 0$. If $s(\mathrm{S}^k\,\mathcal{V}; x) = -1$ for all k, then there is nothing to prove. □

Proof of Theorem 5.3. Write $\varepsilon = \varepsilon(\mathcal{V}; x)$ and $s_k = s(\mathrm{S}^k\,\mathcal{V}; x)$. Let H be a very ample divisor on X that separates 1-jets. Since $\mathcal{V}$ is ample, $\mathrm{S}^k\,\mathcal{V} \otimes \mathcal{O}_X(-H)$ is eventually globally generated by Remark

2.4, hence $s_k \geq 1$ for k sufficiently large by Lemma 5.4. By Proposition 5.5, it is enough to prove

$$\varepsilon \leq \lim_{k\to\infty} \frac{s_k}{k}.$$

Note that the limit exists by Fekete's Lemma, since the sequence s_k is superadditive by Lemma 5.4.

Let $0 < \delta \ll 1$ be arbitrary, and fix positive integers p_0, q_0 such that

$$\varepsilon - \delta < \frac{p_0}{q_0} < \varepsilon.$$

Then, $q_0\pi'^*\xi - p_0\rho'^*E$ is ample. Indeed, the cone generated by $\pi'^*\xi$ and $\pi'^*\xi - \varepsilon\rho'^*E$ is contained in the nef cone, and meets the ample cone because $-\rho'^*E$ is π'-ample and ξ is ample. Consequently, all the classes in its interior are ample. By Serre vanishing, there exists a natural number m_0 such that $H^1\big(Y', \mathcal{O}_{Y'}\big(mq_0\pi'^*\xi - mp_0\rho'^*E\big)\big) = 0$ for all $m \geq m_0$, where $Y' = \mathrm{Bl}_{\mathbb{P}(\mathcal{V}(x))}\mathbb{P}(\mathcal{V})$ as in Notation 3.1. By the Leray spectral sequence [Laz04a, Lemma 5.4.24], this cohomology group is isomorphic to $H^1\big(\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(mq_0) \otimes \mathcal{I}^{mp_0}_{\mathbb{P}(\mathcal{V}_x)}\big)$ for $m \gg 0$, even if the point x is singular. Now for $m \gg 0$, the right vertical arrow in the commutative diagram

$$\begin{tikzcd}
H^0(X, \mathrm{S}^{mq_0}\mathcal{V}) \arrow[r] \arrow[d, equal] & H^0(X, \mathrm{S}^{mq_0}\mathcal{V} \otimes \mathcal{O}_X/\mathfrak{m}_x^{mp_0}) \arrow[d, "\simeq"] \\
H^0\big(\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(mq_0)\big) \arrow[r] & H^0\big(\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(mq_0) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V})}/\mathcal{I}^{mp_0}_{\mathbb{P}(\mathcal{V}_x)}\big)
\end{tikzcd}$$

is an isomorphism by the base change isomorphism (5.3.1) applied to $\mathrm{Spec}(\mathcal{O}_X/\mathfrak{m}_x^{mp_0}) \subseteq X$. The bottom arrow is therefore surjective for $m \gg 0$. Thus, $\mathrm{S}^{mq_0}\mathcal{V}$ separates $mp_0 - 1$ jets, and

$$\frac{s_{mq_0}}{mq_0} \geq \frac{mp_0 - 1}{mq_0} > \varepsilon - \delta - \frac{1}{mq_0}.$$

Then $\lim_{k\to\infty} \frac{s_k}{k} \geq \varepsilon - \delta$. The conclusion follows. □

It is known that lower bounds on Seshadri constants of big and nef invertible sheaves $\mathcal{L}$ lead to lower bounds on the jet separation of adjoint bundles $\omega_X \otimes \mathcal{L}$. See [Dem92, Proposition 6.8]. In this direction, Hacon proves

Theorem 5.6 ([Hac00, Theorem 1.7])**.** *Let $\mathcal{V}$ be an ample locally free sheaf of finite rank r on a complex projective manifold of dimension n. Let $\beta \in \mathbb{Q}_+$ such that $\pi^*\mathcal{V}^\vee\langle\beta\xi\rangle$ is ample. Set*

$$M := \min_{0\leq i\leq n-1}\left[\frac{1}{\binom{n+r-i}{r}^{\frac{1}{n-i}}} \cdot \frac{1}{n-i}\right]. \tag{5.6.1}$$

Then for any integer $\lambda > n\beta/M$, the locally free sheaf $\omega_X \otimes \mathrm{S}^\lambda\mathcal{V} \otimes \det\mathcal{V}$ is generated by global sections at all very general points $x \in X$.

[dC98b, Theorem 5.2.2.1′] is a result of similar flavor. Hacon's global generation result is a corollary of his lower bounds on Seshadri constants [Hac00, Theorem 1.5.a.i]. These generalize the line bundle case of [EKL95]. Theorem 5.6 is then an instance of the following jet separation bound:

Proposition 5.7. *Let X be a complex projective manifold of dimension n, and let $\mathcal{V}$ be an ample (or $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is only big and nef) locally free sheaf of finite rank $r \geq 1$ on X. If $p \geq 0$ is such that $\varepsilon(\mathcal{V}; x) > \frac{n+s}{p+r}$, then $\omega_X \otimes \mathrm{S}^p\mathcal{V} \otimes \det\mathcal{V}$ separates s-jets at x.*

In particular, if $\varepsilon(\mathcal{V}; x) > \frac{n}{r}$ for all $x \in X$, then $\omega_X \otimes \det\mathcal{V}$ is globally generated.

Compare with [Dem92, Proposition 6.8]. A relative version of this argument yields a higher-rank analogue of [dC98a, Theorem 2.2]; see Theorem 7.1.

Proof. We follow the proof of the Griffiths vanishing result in [Laz04b, Theorem 7.3.1]. We prove that $H^1\big(X, \omega_X \otimes \mathrm{S}^p\,\mathcal{V} \otimes \det \mathcal{V} \otimes \mathfrak{m}_x^{s+1}\big) = 0$. This is equivalent to

$$H^1\big(\mathbb{P}(\pi^*\mathcal{V}), \omega_{\mathbb{P}(\pi^*\mathcal{V})} \otimes \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(p+r) \otimes \rho'^*\mathcal{O}(-(n+s)E)\big) = 0.$$

By Remark 3.10, we know that $(p+r)\pi'^*\xi - (n+s)\rho'^*E$ is nef. It is also big as it is a positive combination between the big divisor $\pi'^*\xi$ and the nef (so pseudo-effective) $\pi'^*\xi - \varepsilon(\mathcal{V}; x)\rho'^*E$. The conclusion follows from the Kawamata–Viehweg vanishing theorem. □

5.1. **Separation of jets in positive characteristic.** In this subsection, we prove the following weaker version of Proposition 5.7 in positive characteristic. Our proof is based on the proof sketched in [PST17, Exercise 6.3], which applies to the case when $\mathcal{V}$ is invertible.

Proposition 5.8. *Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic $p > 0$, and let $\mathcal{V}$ be an ample locally free sheaf of finite rank $r \geq 1$ on X. If $q \geq 0$ is such that $\varepsilon(\mathcal{V}; x) > \frac{n}{q+r}$, then $\omega_X \otimes \mathrm{S}^q\,\mathcal{V} \otimes \det \mathcal{V}$ is globally generated at x.*

In particular, if $\varepsilon(\mathcal{V}; x) > \frac{n}{r}$ for all $x \in X$, then $\omega_X \otimes \det \mathcal{V}$ is globally generated.

Proof. Following the notation in Notation 3.1, we consider the cartesian diagram

$$\begin{tikzcd} \mathbb{P}(\pi^*\mathcal{V}) \arrow[r, "\pi'"] \arrow[d, "\rho'"'] \arrow[dr, phantom, "\square"] & \mathbb{P}(\mathcal{V}) \arrow[d, "\rho"] \\ \mathrm{Bl}_x X \arrow[r, "\pi"'] & X \end{tikzcd} \tag{5.8.1}$$

As in the proof of Proposition 5.7, it suffices to show that the restriction homomorphism

$$\begin{aligned} &H^0\big(\mathbb{P}(\pi^*\mathcal{V}), \omega_{\mathbb{P}(\pi^*\mathcal{V})} \otimes \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(q+r) \otimes \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(-(n-1)E')\big) \\ &\qquad \longrightarrow H^0\big(E', \omega_{\mathbb{P}(\pi^*\mathcal{V})} \otimes \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(q+r) \otimes \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(-(n-1)E')\big) \end{aligned}$$

is surjective, where E is the exceptional divisor of π, and $E' \coloneqq \rho'^{-1}E$. Since the morphisms ρ and ρ' are smooth, the Cartier divisor E' is smooth, and the target of this homomorphism can be written as

$$H^0\big(E', \omega_{E'} \otimes \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(q+r) \otimes \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(-nE')\big).$$

By Remark 3.10, we know that $(q+r)\pi'^*\xi - nE'$ is nef, and is moreover ample since ξ is ample and $-E'$ is π'-ample; see the proof of Theorem 5.3. Thus, the associated invertible sheaf $\mathcal{L} \coloneqq \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(q+r) \otimes \mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}(-nE')$ is ample.

We now consider the commutative diagram

$$\begin{tikzcd} F^e_*\big(\omega_{\mathbb{P}(\pi^*\mathcal{V})}(E')\big) \arrow[r] \arrow[d, "\mathrm{Tr}^e_{\mathbb{P}(\pi^*\mathcal{V}),E'}"'] & F^e_*\omega_{E'} \arrow[d, "\mathrm{Tr}^e_{E'}"] \\ \omega_{\mathbb{P}(\pi^*\mathcal{V})}(E') \arrow[r] & \omega_{E'} \end{tikzcd}$$

of $\mathcal{O}_{\mathbb{P}(\pi^*\mathcal{V})}$-modules. Here, the vertical morphisms are induced by the Grothendieck trace morphisms associated to the e-th iterates of the absolute Frobenius on $\mathbb{P}(\pi^*\mathcal{V})$ and E' [Tan15, Lemma 2.6(1)]. Tensoring by $\mathcal{L}$ and taking global sections, we obtain the commutative diagram

$$\begin{tikzcd} H^0\big(\mathbb{P}(\pi^*\mathcal{V}), \omega_{\mathbb{P}(\pi^*\mathcal{V})}(E') \otimes \mathcal{L}^{p^e}\big) \arrow[r] \arrow[d] & H^0\big(E', \omega_{E'} \otimes \mathcal{L}^{p^e}|_{E'}\big) \arrow[d] \\ H^0\big(\mathbb{P}(\pi^*\mathcal{V}), \omega_{\mathbb{P}(\pi^*\mathcal{V})}(E') \otimes \mathcal{L}\big) \arrow[r] & H^0\big(E', \omega_{E'} \otimes \mathcal{L}|_{E'}\big) \end{tikzcd} \tag{5.8.2}$$

We now fix $e \gg 0$ such that the top horizontal arrow is surjective, which exists since

$$H^1\big(\mathbb{P}(\pi^*\mathcal{V}), \omega_{\mathbb{P}(\pi^*\mathcal{V})} \otimes \mathcal{L}^{p^e}\big) = 0$$

for $e \gg 0$ by Serre vanishing. It therefore suffices to show that the right vertical arrow in (5.8.2) is surjective, since this would imply that the composition from the top left corner to the bottom right corner in the commutative diagram is surjective, in which case the bottom horizontal arrow is necessarily surjective.

To show that the right vertical arrow in (5.8.2) is surjective, we first show that E' is globally F-split, i.e., that the Frobenius morphism $F\colon \mathcal{O}_{E'} \to F_*\mathcal{O}_{E'}$ splits as a morphism of $\mathcal{O}_{E'}$-modules. By base changing the diagram (5.8.1) along the inclusion $\{x\} \hookrightarrow X$, we see that E' is a product of projective spaces. By [Smi00, Proposition 6.4], the scheme E' is therefore globally F-split (even globally F-regular).

We now show that the right vertical arrow in (5.8.2) is surjective. Since the Frobenius morphism $F\colon \mathcal{O}_{E'} \to F_*\mathcal{O}_{E'}$ splits, we see that $F^e\colon \mathcal{O}_{E'} \to F^e_*\mathcal{O}_{E'}$ splits. The Grothendieck dual $\mathrm{Tr}^e_{E'}\colon F^e_*\omega_{E'} \to \omega_{E'}$ therefore has a section. Finally, this implies that the right vertical arrow in (5.8.2) has a section, and in particular is surjective. $\square$

6. Base loci

Building on ideas of Nakamaye, [ELM+09, Remark 6.5] proves that if D is a big and nef divisor on a smooth projective variety, then the Seshadri constants of D determine the augmented base locus:

$$\mathbf{B}_+(D) = \{x \in X \mid \varepsilon(D; x) = 0\}.$$

We aim to prove a generalization to sheaves. Let $\mathcal{V}$ be a coherent sheaf on a protective scheme X. [BKK+15, Definition 2.1] defines the *base locus* of $\mathcal{V}$ as

$$\mathrm{Bs}(\mathcal{V}) \coloneqq \{x \in X \mid H^0(X, \mathcal{V}) \to \mathcal{V}(x) \text{ is not surjective}\}.$$

With notation as in Notation 3.1, when $\mathcal{V}$ is locally free, the relation with the base locus of $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is given by $\rho\big(\mathrm{Bs}\big(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)\big)\big) = \mathrm{Bs}(\mathcal{V})$.

Remark 6.1. More precisely, $\mathrm{Bs}(\mathcal{V}) = \mathrm{Supp}\ \mathcal{Q}$ and, $\mathrm{Bs}(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)) \subseteq \mathbb{P}(\mathcal{Q})$, with equality when $\mathcal{V}$ is locally free. Here $\mathcal{Q}$ determined by

$$H^0(X, \mathcal{V}) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} \mathcal{V} \longrightarrow \mathcal{Q} \longrightarrow 0. \qquad \square$$

The *stable base locus* of a coherent sheaf $\mathcal{V}$ is

$$\mathbf{B}(\mathcal{V}) \coloneqq \bigcap_{k\geq 1} \mathrm{Bs}(\mathrm{S}^k \mathcal{V}).$$

Let $gg(\mathcal{V}) \coloneqq X \setminus \mathrm{Bs}(\mathcal{V})$ be the globally generated locus of $\mathcal{V}$. From

$$gg(\mathcal{V}) \subseteq gg(\mathrm{S}^m \mathcal{V}) \subseteq gg(\mathrm{S}^k \mathrm{S}^m \mathcal{V}) \subseteq gg(\mathrm{S}^{km} \mathcal{V}), \tag{6.1.1}$$

we deduce that $\mathbf{B}(\mathcal{V}) \subseteq \mathbf{B}(\mathrm{S}^m \mathcal{V})$ for all $m \geq 1$. While the inclusion $\rho\big(\mathbf{B}\big(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)\big)\big) \subseteq \mathbf{B}(\mathcal{V})$ is easy to prove (see [BKK+15, p. 233]), equality may fail, even when $\mathcal{V}$ is locally free (see [MU19, Example 3.2]). However, equality does hold if one allows perturbations.

Definition 6.2 ([BKK+15, Definition 2.4])**.** The *augmented base locus* of a coherent sheaf $\mathcal{V}$ is

$$\mathbf{B}_+(\mathcal{V}) \coloneqq \bigcap_{k\geq 0} \mathbf{B}\big(\mathrm{S}^k \mathcal{V} \otimes \mathcal{O}_X(-H)\big),$$

where H is any ample divisor on X.

Remark 6.3. It is a standard argument that the definition is independent of the choice of H. Furthermore,

$$\mathbf{B}_+(\mathcal{V}) = \bigcap_{k\geq 0} \operatorname{Bs}\bigl(\mathrm{S}^k \mathcal{V} \otimes \mathcal{O}_X(-H)\bigr).$$

When H is ample and globally generated, then

$$\mathbf{B}_+(\mathcal{V}) = \operatorname{Bs}\bigl(\mathrm{S}^k \mathcal{V} \otimes \mathcal{O}_X(-H)\bigr)$$

for all sufficiently divisible k.

The relation between $\mathbf{B}_+(\mathcal{V})$ and $\mathbf{B}_+(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$ is given by the following:

Proposition 6.4. *Let $\mathcal{V}$ be a coherent sheaf on a projective scheme over an algebraically closed field. Then,*

$$\mathbf{B}_+(\mathcal{V}) \supseteq \rho\bigl(\mathbf{B}_+\bigl(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)\bigr)\bigr).$$

Equality holds when intersecting with the open locally free locus of $\mathcal{V}$.

Proof. [BKK+15, Proposition 3.2] proves that when $\mathcal{V}$ is locally free on complex projective manifolds, and the proof in general is essentially the same. Let H be a very ample divisor on X such that $\mathcal{V}(H)$ is globally generated. We obtain a surjection $H^0(X,\ \mathcal{V}(H)) \otimes \mathcal{O}_X(H) \twoheadrightarrow \mathcal{V}(2H)$, which shows that $A \coloneqq \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes \rho^*\mathcal{O}_X(2H)$ is very ample on $\mathbb{P}(\mathcal{V})$.

Assume $x \in gg\bigl(\mathrm{S}^k \mathcal{V} \otimes \mathcal{O}_X(-H)\bigr)$. Then $\rho^{-1}\{x\} = \mathbb{P}(\mathcal{V}(x)) \subseteq gg\bigl(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(2k) \otimes \rho^*\mathcal{O}_X(-2H)\bigr)$. We have $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(2k) \otimes \rho^*\mathcal{O}_X(-2H) = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(2k+1) \otimes A^\vee$. These show the "$\supseteq$" inclusion.

Assume now $\mathbb{P}(\mathcal{V}(x)) \subseteq \bigcup_{k\geq 0} gg\bigl(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(k) \otimes A^\vee\bigr)$. Since A is very ample, we have inclusions $gg\bigl(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(k) \otimes A^\vee\bigr) \subseteq gg\bigl(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(mk) \otimes A^\vee\bigr)$ for all $m \geq 1$ and all $k \geq 0$. We deduce that $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(k) \otimes A^\vee = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(k-1) \otimes \rho^*\mathcal{O}_X(-2H)$ is globally generated along $\mathbb{P}(\mathcal{V}(x))$ for sufficiently divisible k. Pushing forward to X, since $\rho_*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(k) = \mathrm{S}^k \mathcal{V}$ for k large enough, we find that the canonical map

$$H^0\bigl(X,\ \mathrm{S}^{k-1} \mathcal{V} \otimes \mathcal{O}_X(-2H)\bigr) \otimes \rho_*\mathcal{O}_{\mathbb{P}(\mathcal{V})} \to \mathrm{S}^{k-1} \mathcal{V} \otimes \mathcal{O}_X(-2H)$$

is surjective at x. If x is in the locally free locus of $\mathcal{V}$, then the natural map $\mathcal{O}_X \to \rho_*\mathcal{O}_{\mathbb{P}(\mathcal{V})}$ is an isomorphism around x, hence $x \in gg\bigl(\mathrm{S}^{k-1} \mathcal{V} \otimes \mathcal{O}_X(-2H)\bigr)$. ☐

Remark 6.5. [ELM+09] and [Bir17] define augmented base loci of $\mathbb{R}$-Cartier $\mathbb{R}$-divisors. If $\mathcal{V}$ is locally free, one can use the result above to define $\mathbf{B}_+(\mathcal{V}\langle\lambda\rangle) \coloneqq \rho(\mathbf{B}_+(\xi + \rho^*\lambda))$.

We start relating $\mathbf{B}_+(\mathcal{V})$ to Seshadri constants.

Lemma 6.6. *Let $\mathcal{V}$ be a coherent sheaf on a projective scheme X. If $x \notin \mathbf{B}_+(\mathcal{V})$, then $\varepsilon(\mathcal{V}; x) > 0$.*

Proof. The assumptions imply that for every ample Cartier divisor H on X there exists $k > 0$ such that $\mathrm{S}^{mk} \mathcal{V} \otimes \mathcal{O}_X(-mH)$ is globally generated at x for sufficiently large m. Then $\varepsilon(\mathrm{S}^{mk} \mathcal{V} \otimes \mathcal{O}_X(-mH); x) \geq 0$ by Example 3.3. Conclude by Lemmas 3.24 and 3.28. ☐

Corollary 6.7. *Let X be a smooth projective variety over an algebraically closed field of characteristic zero. If $\mathbf{B}_+(TX) \subsetneq X$, then $X \simeq \mathbb{P}^n$.*

Proof. Lemma 6.6 implies $\varepsilon(TX; x) > 0$ for x a general point on X. Now use Proposition 4.8(2). ☐

Definition 6.8 (see [BKK+15, Theorem 6.4])**.** A sheaf $\mathcal{V}$ is called *V-big*[3] if $\mathbf{B}_+(\mathcal{V}) \neq X$.

[Jab09, Examples 1.7 and 1.8] shows that this is usually stronger than asking for $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ to be big, even when $\mathcal{V}$ is locally free. See also [BKK+15, Remark 6.6].

The main result of this section is the following:

[3] "V" stands for Viehweg.

Proposition 6.9. *Let $\mathcal{V}$ be a locally free sheaf of finite rank on a projective scheme X over an algebraically closed field, and suppose that $\mathcal{V}$ is nef. Then,*

$$\mathbf{B}_+(\mathcal{V}) = \bigl\{x \in X \bigm| \varepsilon(\mathcal{V}; x) = 0\bigr\}.$$

If $\mathcal{V}$ is only a coherent sheaf (but still nef), and x is in the locally free locus of $\mathcal{V}$, then $x \in \mathbf{B}_+(\mathcal{V})$ if and only if $\varepsilon(\mathcal{V}; x) = 0$.

Proof. In view of Lemma 6.6, it is enough to justify the "$\subseteq$" inclusion. Let $x \in \mathbf{B}_+(\mathcal{V})$ such that $\mathcal{V}$ is locally free around x. By Proposition 6.4, there exists $y \in \mathbb{P}(\mathcal{V}(x))$ such that $y \in \mathbf{B}_+(\xi)$. Since ξ is nef, [Bir17] proves that there exists a subvariety $Z \subseteq \mathbb{P}(\mathcal{V})$ through y such that $\xi^{\dim Z} \cdot Z = 0$. By [Laz04a, Proposition 5.1.9], we deduce $\varepsilon(\xi; y) = 0$. Conclude by Remark 3.12. □

We obtain an immediate improvement of Theorem 3.11.

Corollary 6.10. *Let X be a projective scheme. Let $\mathcal{V}$ be a (nef) locally free sheaf of finite rank on X. Then, $\mathcal{V}$ is ample if and only if $\varepsilon(\mathcal{V}; x) > 0$ for all $x \in X$.*

The following lemma will be used in the proof of Theorem 7.1.

Lemma 6.11. *Let X be a projective scheme, and let $\mathcal{V}$ be a coherent sheaf on X. If $x \notin \mathbf{B}_+(\mathcal{V})$ is a closed point, then for every coherent sheaf $\mathcal{F}$ on X and every integer $s \geq 0$, the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathrm{S}^m \mathcal{V}$ separates s-jets at x for all m sufficiently large.*

Proof. Let H be a very ample divisor on X that separates s-jets at x. Since $x \notin \mathbf{B}_+(\mathcal{V})$, there exists $m \geq 1$ such that $x \in gg\bigl(\mathrm{S}^m \mathcal{V} \otimes \mathcal{O}_X(-H)\bigr)$. Let n_0 be sufficiently large so that $\mathcal{F} \otimes \mathrm{S}^r \mathcal{V} \otimes \mathcal{O}_X(nH)$ separates s-jets at x for all $0 \leq r < m$ and all $n \geq n_0$. Such n_0 exists by Lemma 5.4. For $M \geq mn_0$, write $M = mq + r$ with $0 \leq r < m$ and $q \geq n_0$. Then $\mathcal{F} \otimes \mathrm{S}^M \mathcal{V}$ is a quotient of $\mathcal{F} \otimes \mathrm{S}^r \mathcal{V} \otimes \mathrm{S}^q \mathrm{S}^m \mathcal{V} = \bigl(\mathcal{F} \otimes \mathrm{S}^r \mathcal{V} \otimes \mathcal{O}_X(qH)\bigr) \otimes \mathrm{S}^q(\mathrm{S}^m \mathcal{V} \otimes \mathcal{O}_X(-H))$. Conclude by Lemma 5.4. □

7. Direct images of pluricanonical sheaves

In this section, we prove the following analogue of [DM19, Theorem A] for higher-rank bundles and for higher-order jets, in the spirit of a relative Fujita-type conjecture of Popa and Schnell [PS14, Conjecture 1.3].

Theorem 7.1. *Let $f \colon Y \to X$ be a surjective morphism of complex projective varieties, where X is of dimension n. Let (Y, Δ) be a log canonical $\mathbb{R}$-pair and let $\mathcal{V}$ be a locally free sheaf of finite rank $r \geq 1$ on X such that $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is big and nef. Consider a Cartier divisor P on Y such that $P \sim_{\mathbb{R}} k(K_Y + \Delta)$ for some integer $k \geq 1$, and consider a general smooth closed point $x \in X \setminus \mathbf{B}_+(\mathcal{V})$. If we have*

$$\varepsilon(\mathcal{V}; x) > k \cdot \frac{n+s}{m + k(r-1) + 1}, \tag{7.1.1}$$

then the sheaf

$$f_*\mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathrm{S}^m \mathcal{V} \otimes_{\mathcal{O}_X} (\det \mathcal{V})^{\otimes k} \tag{7.1.2}$$

separates s-jets at x.

In particular, if X is smooth, $\mathcal{V}$ is ample, and $\beta > 0$ is such that $\pi^*\mathcal{V}^\vee\langle\beta\xi\rangle$ is ample, then with M as in (5.6.1), for every integer

$$\lambda > k \cdot \left(\frac{\beta}{M}(n+s) - (r-1)\right) - 1$$

the sheaf $f_*\mathcal{O}_Y(P) \otimes \mathrm{S}^\lambda \mathcal{V} \otimes (\det \mathcal{V})^{\otimes k}$ separates s-jets at all general points $x \in X$.

Note that by Proposition 6.9, the condition $x \notin \mathbf{B}_+(\mathcal{V})$ follows from the condition on $\varepsilon(\mathcal{V};x)$ in (7.1.1). This condition also implies $\mathcal{V}$ is V-big in the sense of Definition 6.8. We also note that the last statement follows in the same way as in [Hac00, Theorem 1.7], using the lower bound for Seshadri constants in [Hac00, Theorem 1.5.a.i], hence it suffices to show the first statement. Finally, our statement has "general" instead of "very general" since separating s-jets is an open condition.

Proof. By applying Lemma 6.11 to $\mathcal{F} = f_*\mathcal{O}_Y(P) \otimes (\det \mathcal{V})^{\otimes k}$, there exists a smallest positive integer m_0 such that the sheaf (7.1.2) separates s-jets at x for $m \geq m_0$. We will prove that the sheaf (7.1.2) separates s-jets at x for a suitable choice of a general point x, if

$$\varepsilon(\mathcal{V};x) > \frac{n+s}{m+r-\frac{k-1}{k}m_0}. \tag{7.1.3}$$

The choice of the general point x will be detailed momentarily, but first we explain how the conclusion of the theorem follows from (7.1.3). This inequality is equivalent to

$$m > \frac{n+s}{\varepsilon(\mathcal{V};x)} + \frac{k-1}{k}m_0 - r,$$

and by the minimality of m_0, we see that

$$m_0 \leq \left\lfloor \frac{n+s}{\varepsilon(\mathcal{V};x)} + \frac{k-1}{k}m_0 - r \right\rfloor + 1 \leq \frac{n+s}{\varepsilon(\mathcal{V};x)} + \frac{k-1}{k}m_0 - r + 1.$$

Rearranging this inequality yields

$$m_0 \leq k \cdot \left(\frac{n+s}{\varepsilon(\mathcal{V};x)} - r + 1 \right),$$

and substituting this upper bound for m_0 into the inequality for m above, we see that the sheaf (7.1.2) separates s-jets at x if

$$m > \frac{n+s}{\varepsilon(\mathcal{V};x)} + (k-1)\cdot\left(\frac{n+s}{\varepsilon(\mathcal{V};x)} - r + 1 \right) - r = k \cdot \frac{n+s}{\varepsilon(\mathcal{V};x)} - k(r-1) - 1$$

which is equivalent to the inequality (7.1.1). This idea was inspired by the proof of [PS14, Theorem 1.7].

We now explain the choice of the general point x. Following Steps 0 and 1 in the proof of [DM19, Theorem A], we may assume that Y is smooth, that Δ has simple normal crossings support and coefficients in $(0,1]$, and that the image of the adjunction morphism

$$f^*f_*\mathcal{O}_Y(P) \longrightarrow \mathcal{O}_Y(P)$$

is of the form $\mathcal{O}_Y(P-G)$ for a divisor G such that $\Delta + G$ has simple normal crossings support. We will show that under these assumptions, the sheaf (7.1.2) separates s-jets at all smooth closed points $x \in X \setminus \mathbf{B}_+(\mathcal{V})$ satisfying (7.1.1), such that f is smooth at x and such that the fiber $Y_x := f^{-1}(x)$ over x intersects each component of Δ transversely.

Step 1. *Reduction to the case $k = 1$ for a suitable pair.*

By assumption on m_0, we know that the sheaf (7.1.2) separates s-jets at x for $m = m_0$, and in particular, is globally generated at x. This implies that the sheaf

$$\mathcal{O}_Y(P-G) \otimes \mathrm{S}^{m_0} f^*\mathcal{V} \otimes (\det f^*\mathcal{V})^{\otimes k}$$

is globally generated along Y_x. By pulling back along the bundle map

$$\rho_Y \colon \mathbb{P}_Y(f^*\mathcal{V}) \longrightarrow Y,$$

and using the m_0th symmetric power of the tautological quotient map, the invertible sheaf

$$\mathcal{O}_{\mathbb{P}_Y(f^*\mathcal{V})}\big(\rho_Y^*(P-G)\big) \otimes \mathcal{O}_{\mathbb{P}_Y(f^*\mathcal{V})}(m_0) \otimes \big(\det(f\circ\rho_Y)^*\mathcal{V}\big)^{\otimes k}$$

on $\mathbb{P}_Y(f^*\mathcal{V})$ is globally generated along $\rho_Y^{-1}(Y_x)$. Now let $c_1(f^*\mathcal{V})$ denote the divisor class of the determinant of $f^*\mathcal{V}$ on Y, and let $\eta = c_1(\mathcal{O}_{\mathbb{P}_Y(f^*\mathcal{V})}(1))$. Switching to divisor notation,

$$\begin{aligned}\rho_Y^*(P-G) + k\,\rho_Y^* c_1(f^*\mathcal{V}) + m_0\eta &\sim_{\mathbb{R}} \rho_Y^*(k\Delta - G) + k\,\rho_Y^* K_Y + k\,\rho_Y^* c_1(f^*\mathcal{V}) + m_0\eta\\ &\sim_{\mathbb{R}} \rho_Y^*(k\Delta - G) + k\,K_{\mathbb{P}_Y(f^*\mathcal{V})} + (m_0+kr)\eta.\end{aligned}$$

By Bertini's theorem, we can therefore choose a general divisor

$$\mathfrak{D} \in \bigl|\rho_Y^*(P-G) + k\,\rho_Y^* c_1(f^*\mathcal{V}) + m_0\eta\bigr|$$

that is smooth along $\rho_Y^{-1}(Y_x)$, and intersects both $\rho_Y^{-1}(Y_x)$ and the supports of $\rho_Y^*\Delta$ and ρ_Y^*G transversely (see [Laz04a, Lemma 4.1.11]) in a neighborhood of $\rho_Y^{-1}(Y_x)$. We then have

$$k\bigl(K_{\mathbb{P}_Y(f^*\mathcal{V})} + \rho_Y^*\Delta\bigr) \sim_{\mathbb{R}} K_{\mathbb{P}_Y(f^*\mathcal{V})} + \rho_Y^*\Delta + \frac{k-1}{k}\mathfrak{D} + \frac{k-1}{k}\rho_Y^* G - \frac{k-1}{k}(m_0+kr)\eta.$$

We now want to rewrite the right-hand side as the sum of a log canonical divisor coming from a log canonical pair on $\mathbb{P}_Y(f^*\mathcal{V})$ and a multiple of η. Since $\Delta + \frac{k-1}{k}G$ may have some coefficients greater than one, we first adjust the coefficients of Δ and G. Applying [DM19, Lemma 2.18] to $c = \frac{k-1}{k}$, there exists an effective Cartier $\mathbb{Z}$-divisor $G' \preceq G$ such that

$$\Delta' \coloneqq \Delta + \frac{k-1}{k}G - G'$$

is effective with simple normal crossings support, with components intersecting Y_x transversely, and with coefficients in $(0,1]$. Since ρ_Y is a smooth morphism, the pullback $\rho_Y^*\Delta'$ also has these same properties on $\mathbb{P}_Y(f^*\mathcal{V})$. We then have

$$\begin{aligned}\rho_Y^*\bigl(P + k\,c_1(f^*\mathcal{V}) - G'\bigr) &\sim_{\mathbb{R}} k\bigl(K_{\mathbb{P}_Y(f^*\mathcal{V})} + r\eta + \rho_Y^*\Delta\bigr) - \rho_Y^* G'\\ &\sim_{\mathbb{R}} K_{\mathbb{P}_Y(f^*\mathcal{V})} + \rho_Y^*\Delta' + \frac{k-1}{k}\mathfrak{D} + \biggl(r - \frac{k-1}{k}m_0\biggr)\eta.\end{aligned} \tag{7.1.4}$$

This $\mathbb{R}$-linear equivalence will be used to reduce the case $k>1$ for the pair (Y,Δ) to the case $k=1$ for the pair $(\mathbb{P}_Y(f^*\mathcal{V}), \rho_Y^*\Delta' + \frac{k-1}{k}\mathfrak{D})$.

Step 2. *Replacing $\mathfrak{D}$ with a divisor with simple normal crossings support.*

Let $\mu\colon Z \to \mathbb{P}_Y(f^*\mathcal{V})$ be a common log resolution for $\mathfrak{D}$ and $(\mathbb{P}_Y(f^*\mathcal{V}), \rho_Y^*\Delta')$. Note that we can choose μ to be an isomorphism along $\rho_Y^{-1}(Y_x)$, since $\mathfrak{D}$ and $\rho_Y^*\Delta'$ intersect transversely and have simple normal crossings support in a neighborhood of $\rho_Y^{-1}(Y_x)$. We can then write

$$\mu^*\mathfrak{D} = \mathfrak{D}_1 + F, \qquad (\rho_Y\circ\mu)^*\Delta' = \mu_*^{-1}(\rho_Y^*\Delta') + F_1,$$

where $\mathfrak{D}_1$ is a smooth divisor intersecting $(\rho_Y\circ\mu)^{-1}(Y_x)$ transversely and F, F_1 are supported away from $(\rho_Y\circ\mu)^{-1}(Y_x)$. Define

$$\begin{gathered}F' \coloneqq \biggl\lfloor \frac{k-1}{k}F + F_1\biggr\rfloor, \qquad \widetilde{\Delta} \coloneqq (\rho_Y\circ\mu)^*\Delta' + \frac{k-1}{k}\mu^*\mathfrak{D} - F',\\ \widetilde{P} \coloneqq (\rho_Y\circ\mu)^*\bigl(P + k\,c_1(f^*\mathcal{V})\bigr) + K_{Z/\mathbb{P}_Y(f^*\mathcal{V})}.\end{gathered}$$

Note that $\widetilde{\Delta}$ has simple normal crossings support and coefficients in $(0,1]$ by assumption on the log resolution μ and by the definition of F', and also has components intersecting $(\rho_Y\circ\mu)^{-1}(Y_x)$ transversely. Pulling back the decomposition in (7.1.4) via μ and adding $K_{Z/\mathbb{P}_Y(f^*\mathcal{V})} - F'$ yields

$$\widetilde{P} - (\rho_Y\circ\mu)^*G' - F' \sim_{\mathbb{R}} K_Z + \widetilde{\Delta} + \biggl(r - \frac{k-1}{k}m_0\biggr)\mu^*\eta. \tag{7.1.5}$$

Step 3. *To show the sheaf* (7.1.2) *separates s-jets at x, it suffices to show that the sheaf*

$$(f \circ \rho_Y \circ \mu)_* \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr) \tag{7.1.6}$$

separates s-jets at x.

Consider the commutative diagram

$$\begin{tikzcd}
H^0\bigl(X, (f \circ \rho_Y \circ \mu)_* \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr)\bigr) \arrow[r] \arrow[d, hook] & H^0\bigl(X, (f \circ \rho_Y \circ \mu)_* \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr) \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x^{s+1}}\bigr) \arrow[d, "\simeq"] \\
H^0\bigl(X, (f \circ \rho_Y \circ \mu)_* \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' + m\mu^*\eta\bigr)\bigr) \arrow[r] \arrow[d, "\simeq"'] & H^0\bigl(X, (f \circ \rho_Y \circ \mu)_* \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' + m\mu^*\eta\bigr) \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x^{s+1}}\bigr) \arrow[d, "\simeq"] \\
H^0\bigl(X, f_*\mathcal{O}_Y(P - G') \otimes \mathrm{S}^m \mathcal{V} \otimes (\det \mathcal{V})^{\otimes k}\bigr) \arrow[r] \arrow[d, "\simeq"'] & H^0\bigl(X, f_*\mathcal{O}_Y(P - G') \otimes \mathrm{S}^m \mathcal{V} \otimes (\det \mathcal{V})^{\otimes k} \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x^{s+1}}\bigr) \arrow[d, "\simeq"] \\
H^0\bigl(X, f_*\mathcal{O}_Y(P) \otimes \mathrm{S}^m \mathcal{V} \otimes (\det \mathcal{V})^{\otimes k}\bigr) \arrow[r] & H^0\bigl(X, f_*\mathcal{O}_Y(P) \otimes \mathrm{S}^m \mathcal{V} \otimes (\det \mathcal{V})^{\otimes k} \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x^{s+1}}\bigr)
\end{tikzcd}$$

where the top right isomorphism holds since F' is supported away from $(\rho_Y \circ \mu)^{-1}(Y_x)$. The vertical isomorphisms in the middle row follow from the projection formula, the fact that $K_{Z/\mathbb{P}_Y(f^*\mathcal{V})}$ is μ-exceptional, and the fact that $\mathbb{R}\rho_{Y*}\mathcal{O}_{\mathbb{P}_Y(f^*\mathcal{V})}(m)$ is quasi-isomorphic to $\mathrm{S}^m f^*\mathcal{V}$ for $m \geq 0$. The vertical isomorphisms in the bottom row follow from [DM19, Lemma 2.17]. If the top horizontal arrow is surjective, then the commutativity of the diagram implies that the bottom horizontal arrow is also surjective, i.e., the sheaf in (7.1.2) separates s-jets at x.

Step 4. *The sheaf* (7.1.6) *separates s-jets at x if*

$$\varepsilon\,(\mathcal{V}; x) > \frac{n+s}{m + r - \frac{k-1}{k} m_0}.$$

Consider the commutative diagram

$$\begin{tikzcd}
Z' \arrow[r, "\pi_Z"] \arrow[d, "\mu'"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\mu"] \\
\mathbb{P}_{Y'}(\mathcal{W}) \arrow[r, "\pi'_Y"] \arrow[d, "\rho_{Y'}"'] \arrow[dr, phantom, "\square"] & \mathbb{P}_Y(f^*\mathcal{V}) \arrow[d, "\rho_Y"] \\
Y' \arrow[r, "\pi_Y"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\
X' \arrow[r, "\pi"] & X
\end{tikzcd}$$

with cartesian squares, where $X' = \mathrm{Bl}_x X$, where $Y' = \mathrm{Bl}_{Y_x} Y$, and $\mathcal{W} = (f \circ \pi_Y)^*\mathcal{V} = (\pi \circ f')^*\mathcal{V}$. The bottom square is cartesian since f is flat at x. Since ρ_Y is smooth and therefore flat, we also have $\mathbb{P}_{Y'}(\mathcal{W}) = \mathrm{Bl}_{\rho_Y^{-1} Y_x} \mathbb{P}_Y(f^*\mathcal{V})$. In the top square, π_Z is the blow-up of Z along $(\rho_Y \circ \mu)^{-1}(Y_x)$ since μ is an isomorphism over $\rho_Y^{-1}(Y_x)$. Consider the commutative diagram

$$\begin{tikzcd}
H^0\bigl(Z', \pi_Z^*\mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr)\bigr) \arrow[r] & H^0\bigl(Z', \pi_Z^*\mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr)\big|_{(t+1)\mu'^*E}\bigr) \\
H^0\bigl(Z, \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr)\bigr) \arrow[r] \arrow[u, "\simeq"] & H^0\bigl(Z, \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr)/\mathcal{I}^{t+1}_{(\rho_Y \circ \mu)^{-1}(Y_x)}\bigr) \arrow[u, "\simeq"'] \\
H^0\bigl(X, (f \circ \rho_Y \circ \mu)_* \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr)\bigr) \arrow[r] \arrow[u, "\simeq"] & H^0\bigl(X, (f \circ \rho_Y \circ \mu)_* \mathcal{O}_Z\bigl(\widetilde{P} - (\rho_Y \circ \mu)^* G' - F' + m\mu^*\eta\bigr) \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x^{t+1}}\bigr) \arrow[u, "\alpha_t(x)"']
\end{tikzcd} \tag{7.1.7}$$

where E denotes the exceptional divisor of the blow-up π'_Y, and where the vertical arrows in the top row are isomorphisms by the fact that π_Z is the blow-up along the smooth subscheme $(\rho_Y \circ$

$\mu)^{-1}(Y_x) \subseteq Z$; see [Laz04a, Lemma 4.3.16]. We will show that the top horizontal arrow is surjective for $t = 0$ and $t = s$. The $t = 0$ statement will show that $\alpha_t(x)$ is surjective by the commutativity of the diagram, hence an isomorphism for all t by cohomology and base change [Ill05, Corollary 8.3.11], using the flatness of f at x. The surjectivity of the top horizontal arrow for $t = s$ will then show that the sheaf (7.1.6) separates s-jets at x.

Choose a sufficiently small positive rational number δ such that

$$\varepsilon(\mathcal{V}; x) > \frac{n+s+\delta}{m+r-\frac{k-1}{k}m_0}.$$

Let D denote the exceptional divisor for the blow-up $\mathbb{P}_{X'}(\pi^*\mathcal{V}) \to \mathbb{P}_X(\mathcal{V})$ along $\mathbb{P}(\mathcal{V}(x))$ and let ξ denote the Serre class on $\mathbb{P}_{X'}(\pi^*\mathcal{V})$. The $\mathbb{Q}$-divisor

$$\begin{aligned}\mu'^*\biggl(\Bigl(m+r-\frac{k-1}{k}m_0\Bigr)\pi_Y'^*\eta-(n+t+\delta)E\biggr)\\ =(\rho_{Y'}\circ\mu')^*\biggl(\Bigl(m+r-\frac{k-1}{k}m_0\Bigr)\xi-(n+t+\delta)D\biggr)\end{aligned} \tag{7.1.8}$$

is big and nef for $t \in \{0, s\}$ by assumption on $\varepsilon(\mathcal{V}; x)$ and Remark 3.10(c). By the definition of $\varepsilon(\mathcal{V}; x)$ and [ELM+09, Remark 6.5], the stable base locus of the divisor (7.1.8) is disjoint from μ'^*E (cf. the proof of [DM19, Lemma 3.3]). By Bertini's theorem, for ℓ a sufficiently large and divisible integer, we can therefore choose a general divisor

$$\mathfrak{E} \in \left|\ell\biggl(\mu'^*\biggl(\Bigl(m+r-\frac{k-1}{k}m_0\Bigr)\pi_Y'^*\eta-(n+t+\delta)E\biggr)\biggr)\right|$$

that is smooth along μ'^*E, and intersects every component of the support of $\pi_Z^*\widetilde{\Delta}$ transversely in a neighborhood of μ'^*E.

Choose a common log resolution $\nu\colon \widetilde{Z}' \to Z'$ for $\mathfrak{E}$ and $(Z', \pi_Z^*\widetilde{\Delta})$ that is an isomorphism along μ'^*E. We then write

$$\nu^*\mathfrak{E} = \mathfrak{E}_1 + B, \qquad (\pi_Z \circ \nu)^*\widetilde{\Delta} = \nu_*^{-1}\pi_Z^*\widetilde{\Delta} + B_1$$

where $\mathfrak{E}_1$ is a smooth prime divisor intersecting $(\mu' \circ \nu)^*E$ transversely and B, B_1 are supported away from $(\mu' \circ \nu)^*E$. Define

$$B' := \left\lfloor \frac{1}{\ell}B + B_1 \right\rfloor, \qquad \Gamma := (\pi_Z \circ \nu)^*\widetilde{\Delta} + \frac{1}{\ell}\nu^*\mathfrak{E} - B' + \delta(\mu' \circ \nu)^*E,$$
$$Q := (\pi_Z \circ \nu)^*\widetilde{P} + K_{\widetilde{Z}'/Z'},$$

where we note that Γ has simple normal crossings support and coefficients in $(0, 1]$, since $\pi_Z^*\widetilde{\Delta}$ has simple normal crossings support and coefficients in $(0, 1]$ by the condition that $\widetilde{\Delta}$ has components intersecting $(\rho_Y \circ \mu)^{-1}(Y_x)$ transversely; see [Ful98, Corollary 6.7.2]. By the $\mathbb{R}$-linear equivalence (7.1.5), we have that

$$\begin{aligned}\pi_Z^*\widetilde{P} - (\rho_Y \circ \mu \circ \pi_Z)^*G' - \pi_Z^*F' + m(\mu \circ \pi_Z)^*\eta - (t+1)\mu'^*E\\ \sim_{\mathbb{R}} K_{Z'} + \pi_Z^*\widetilde{\Delta} + \delta\mu'^*E + \frac{1}{\ell}\mathfrak{E}\end{aligned}$$

where we use the fact that π_Z is the blow-up along the smooth subvariety $(\rho_Y \circ \mu)^{-1}(Y_x)$ of codimension n. Pulling back along ν and adding $K_{\widetilde{Z}'/Z} - B'$, we obtain

$$\begin{aligned}Q - (\rho_Y \circ \mu \circ \pi_Z \circ \nu)^*G' - (\pi_Z \circ \nu)^*F' - B' + m(\mu \circ \pi_Z \circ \nu)^*\eta - (t+1)(\mu' \circ \nu)^*E\\ \sim_{\mathbb{R}} K_{\widetilde{Z}'} + \Gamma.\end{aligned} \tag{7.1.9}$$

Since B' is supported away from $(\mu'\circ\nu)^*E$ and $K_{\widetilde{Z}'/Z}$ is ν-exceptional, an argument similar to Step 3 shows that to show the surjectivity of the top horizontal arrow in (7.1.7), it suffices to show that the morphism

$$\begin{aligned} &H^1\bigl(\widetilde{Z}',\mathcal{O}_{\widetilde{Z}'}\bigl(Q-(\rho_Y\circ\mu\circ\pi_Z\circ\nu)^*G'-(\pi_Z\circ\nu)^*F'-B'+m(\mu\circ\pi_Z\circ\nu)^*\eta-(t+1)\mu'^*E\bigr)\bigr)\\ &\qquad\longrightarrow H^1\bigl(\widetilde{Z}',\mathcal{O}_{\widetilde{Z}'}\bigl(Q-(\rho_Y\circ\mu\circ\pi_Z\circ\nu)^*G'-(\pi_Z\circ\nu)^*F'-B'+m(\mu\circ\pi_Z\circ\nu)^*\eta\bigr)\bigr)\end{aligned}$$

is injective. This injectivity follows from Fujino's Kollár-type injectivity theorem [Fuj17, Theorem 5.4.1] by using the $\mathbb{R}$-linear equivalence (7.1.9) and the fact that Γ contains $(\mu'\circ\nu)^*E$ in its support.

The argument above works for $t=0$ or $t=s$, hence the sheaf (7.1.6) separates s-jets at x. □

Specializing to the case when $\mathcal{V}$ is an invertible sheaf, we obtain the following version of [DM19, Theorem A] for higher-order jets using the lower bound on Seshadri constants in [EKL95]. This also gives a generic version of [SZ20, Corollary 1.9(2)] for big and nef line bundles that are not necessarily globally generated, albeit with weaker bounds.

Corollary 7.2. *Let $f\colon Y\to X$ be a surjective morphism of complex projective varieties, where X is of dimension n. Let (Y,Δ) be a log canonical $\mathbb{R}$-pair and let $\mathcal{L}$ be a big and nef invertible sheaf on X. Consider a Cartier divisor P on Y such that $P\sim_{\mathbb{R}} k(K_Y+\Delta)$ for some integer $k\geq 1$. Then, the sheaf*

$$f_*\mathcal{O}_Y(P)\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes\ell}$$

separates s-jets at all general points $x\in X$ for all $\ell\geq k(n(n+s)+1)$.

Just as in the case when $s=0$, one can replace the lower bound $\ell\geq k(n(n+s)+1)$ with the lower bound $\ell\geq k((n-1)(n+s)+1)$ when X is smooth of dimension at most three and $\mathcal{L}$ is ample; see [DM19, Remark 5.2].

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