Control Problems for Energy Harvester Model and Interpolation in Hardy Space

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Abstract

Three control problems for the system of two coupled differential equations, which govern the dynamics of an energy harvesting model are studied. The system consists of the equation of an Euler–Bernoulli beam model and the equation representing the Kirchhoff's law for an electric circuit. Both equations contain coupling terms representing the inverse and direct piezoelectric effects respectively. The system describes the deflection of the beam and the voltage generated between the top and bottom faces of a piezoelectric patch attached to the vibrating beam surface. The system is reformulated as a single evolution equation in the state space of 3–component vector functions. The control is introduced as a separable forcing term $\mathbf{g}(x)f(t)$ on the right–hand side of the operator equation with $\mathbf{g}(x)$ being a 3–component force profile function and $f(t)$ being the control function. The first control problem is concerned with an explicit construction of $f(t)$ that brings a given initial state to zero on a specific time interval $[0,T]$. The solution has been obtained via the spectral decomposition method. The second control problem is concerned with the derivation of $f(t)$ such that the voltage output is equal to some given function $\mathbf{v}(t)$ with $\mathbf{g}(x)$ being given as well. The solution is obtained in the form of an expansion with respect to non–harmonic exponentials built on the eigenvalues of the dynamics generator of the model. The third problem is concerned with an explicit construction of both the force profile, $\mathbf{g}(x)$, and the control, $f(t)$, which generate the desired voltage output $\mathbf{v}(t)$. A sufficient condition on $\mathbf{v}(t)$ that allows to reconstruct $\mathbf{g}(x)$ and $f(t)$ is discussed and explicit formulae for these functions are derived. Interpolation theory in the Hardy space of analytic functions is used in the solution of the second and third problems.

1 Introduction and statement of the problem

The present paper is the third in a row of three papers devoted to the asymptotic, spectral, and controllability results for a certain class of energy harvesting models. The harvester considered in the paper is transforming the mechanical energy of ambient vibrations into electric energy by using the direct piezoelectric effect. The contemporary literature on energy harvesting involves numerous papers of engineers and numerical analysts across several engineering disciplines, such as mechanical and electrical engineers, and material science researchers (see [2] and references therein). However, rigorous mathematical analysis is mostly an open research area. In our sequence of papers, we consider the model of a piezoelectric energy harvester, which is well known in engineering literature (see [2,3]) and is briefly described below. In the present paper, we discuss several control problems for this model.

Description of the model. The following sketch Fig.1(a) shows the construction of the harvester. The electrical circuit of the harvester is presented on Fig.1(b).

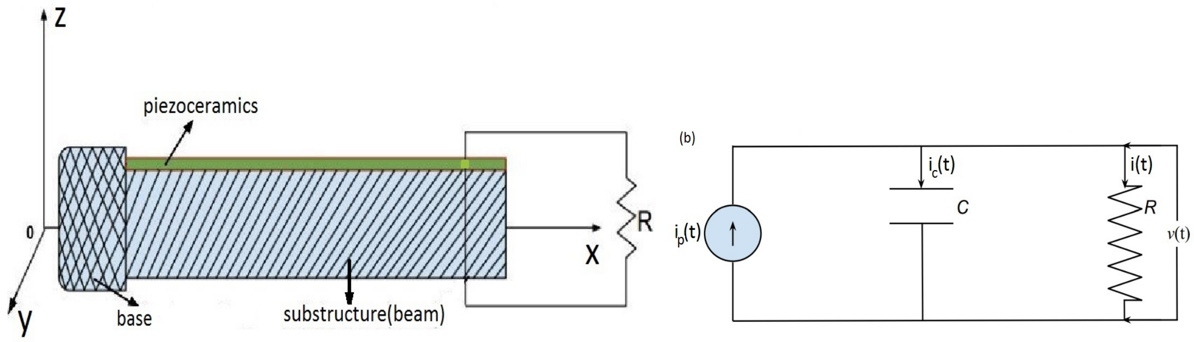

Figure 1. Unimorph cantilever with a piezoceramic layer connected to a resistive load.

We consider a piezoelectric energy harvester in the form of a thin cantilever (clamped–free) beam with a piezoceramic layer. The piezoceramic layer is attached to the top face of the beam (the unimorf configuration). A pair of perfectly conductive electrodes is covering the top and the bottom faces of the piezoceramic layer. The electrodes are assumed to be thin enough so that their thickness can be neglected. The electrodes are connected to a resistive load. The load is considered in an electrical circuit along with the internal capacitance of the piezoceramic layer, which is assumed to be a perfect insulator. If the beam vibrates due to some external load, then a dynamic strain appears in the piezoceramic layer. Due to the piezoelectric effect this strain results in an alternating voltage output across the electrodes. This output can be harvested for charging batteries or running low–powered electronic devices.

On the other hand, the electric potential difference between the electrodes generates an electric field in the piezoceramic layer. This electric field produces a stress on the piezoceramic due to the converse piezoelectric effect. So, the dynamics of the beam is affected by the electric circuit. As a result, the energy harvester is modeled by a coupled system of two differential equations. The first of them is the Euler–Bernoulli beam equation that contains an additional term depending on the voltage on the electrodes. This term represents the converse piezoelectric effect. In other words, the backward piezoelectric coupling effect modifies the vibration response of the cantilever. The second equation is just the Kirchhoff's law for the electric circuit. It is a linear first order equation with respect to the voltage across the electrodes. The equation contains an additional integral term depending on the transverse displacement of the beam. This term represents the direct piezoelectric effect. A detailed derivation of the model equations can be found in [2, 3, 11, 13].

Statement of the initial boundary–value problem. We consider a system of two coupled partial differential equations for two unknown scalar functions $w(x,t)$ and $\mathbf{v}(t)$ with $0 \leq x \leq L < \infty$, $t \geq 0$ (see [2, 3].) In what follows, we use the subindex notations for partial derivatives:

$$m\, w_{tt} + c_s I\, w_{xxxxt}(x,t) + c_a\, w_t + YI\, w_{xxxx}(x,t) - \theta \mathbf{v}(t) \left[\delta'(x) - \delta'(x-L)\right] = 0, \tag{1.1}$$

$$C\mathbf{v}_t(t) + \frac{1}{R}\mathbf{v}(t) = -\kappa \int_0^L w_{xxt}(x,t)dx = -\kappa[w_{tx}(L,t) - w_{tx}(0,t)]. \tag{1.2}$$

This system is equipped with the following set of boundary conditions:

$$w(0,t) = w_x(0,t) = 0, \quad YI\, w_{xx}(L,t) + c_s I\, w_{txx}(L,t) = 0, \quad YI\, w_{xxx}(L,t) + c_s I\, w_{txxx}(L,t) = 0, \tag{1.3}$$

where $w(x,t)$–the transverse displacement of the beam; m–mass per unit length; c_s–Kelvin–Voigt (strain–rate) damping coefficient; c_a–viscous air damping coefficient; Y–the Young modulus; I–cross section moment of inertia with respect to the neutral axis; (YI -the bending stiffness); θ–converse piezoelectric effect backward coupling coefficient. $\mathbf{v}(t)$–the output voltage across the electrodes of the piezoceramic layer; C–internal capacitance of the piezoceramic layer; R–resistance of the external load; κ–direct piezoelectric effect coupling coefficient; δ–Dirac delta–function. The first and fourth terms in (1.1) represent an undamped beam. Eq.(1.1) contains two damping mechanisms, i.e., Kelvin–Voigt and viscous air damping.

Using the variational approach, it has been shown in [11] that system (1.1) and (1.2) with boundary conditions (1.3) is equivalent to the following initial boundary-value problem:

$$mw_{tt}(x,t) + c_s I\, w_{txxxx}(x,t) + c_a w_t(x,t) + YI\, w_{xxxx}(x,t) = 0, \tag{1.4}$$

$$C\ \mathbf{v}_t(t) + \frac{1}{R}\ \mathbf{v}(t) + \kappa[w_{tx}(L,t) - w_{tx}(0,t)] = 0. \tag{1.5}$$

System (1.4) and (1.5) is equipped with a set of boundary conditions:

$$w(0,t) = w_x(0,t) = 0, \tag{1.6}$$

$$YI\, w_{xx}(L,t) + c_s I\, w_{txx}(L,t) = \theta\ \mathbf{v}(t), \qquad YI\, w_{xxx}(L,t) + c_s I\, w_{txxx}(L,t) = 0, \tag{1.7}$$

and a standard set of initial conditions

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad \mathbf{v}(0) = \mathbf{v}_0. \tag{1.8}$$

We point out that the only difference between the problems (1.1) – (1.3), (1.8) and (1.4) – (1.8) consists in the following. The equation of the beam (1.1) contains an additional distributional forcing term that is absent in (1.4). Instead the first (right end) boundary condition (1.7) contains a right hand side term which is absent in the corresponding second boundary condition (1.3).

Now we are in a position to describe the content of the present paper. As is shown in Section 2, problem (1.4)–(1.8) can be written as an evolution equation $U_t(x,t) = i(\mathscr{L}U)(x,t)$ in the state space $\mathscr{H}$, which is a Hilbert space of three–component vector valued functions equipped with the energy metric (i.e., in the above evolution equation $U(x,t) = (w(x,t), w_t(x,t), \mathbf{v}(t))^T$ where "T" means the transposition). The dynamics generator, a matrix differential operator $\mathscr{L}$, being the main object of interest, has been studied in the previous two papers of the author. (For precise definition of $\mathscr{L}$ see formulae (2.8)–(2.10) below.) For reader's convenience, we briefly describe some results from papers [11, 13], which will be used in the present paper. In particular, an asymptotic distribution of complex eigenvalues of the operator $\mathscr{L}$ as the number of an eigenvalue tends to infinity has been derived in [11], i.e., we performed the modal analysis for an electrically loaded (not short–circuit) system. (see Theorem 2.3 below). Since $\mathscr{L}$ is a non–selfadjoint operator with a compact resolvent, there may be multiple eigenvalues of a finite geometric and algebraic multiplicity each. In general, it is a difficult problem to control multiplicities of multiple eigenvalues. However, for this particular problem it is shown in [11] that the geometric multiplicity of a multiple eigenvalue is always 1, while an algebraic multiplicity is greater than 1, which means that every root space corresponding to a multiple eigenvalue has one eigenvector and a finite chain of the associate vectors. Based on the spectral asymptotics (see formula (2.13) below), one can see that the distant eigenvalues are simple, i.e., the operator $\mathscr{L}$ has at most a finite number of the associate vectors. The entire collection of the eigenvectors and the associate vectors is called the set of *the generalized eigenvectors*. To prove the asymptotic formula in [11], we have introduced a non–selfadjoint operator polynomial pencil $\mathscr{P}(\cdot)$ (for its definition see formulae (2.18) and (2.19) below), whose coefficients are differential operations. The pencil $\mathscr{P}(\cdot)$ and the operator $\mathscr{L}$ have the same eigenvalues and having a formula for an eigenfunction of $\mathscr{P}(\cdot)$, one can readily obtain a formula for an eigenvector of $\mathscr{L}$ (see (2.29) and (2.32) below). In paper [13] we have proven that the set of the generalized eigenvectors of $\mathscr{L}$ is a complete and minimal set in the state space of the system. Moreover, this set forms a Riesz basis, which is quadratically close to an orthonormal basis (a Bari basis). Hence, the problem discussed in the aforementioned series of papers, delivers an example from engineering sciences where Bari bases appear naturally.

In Section 3 we formulate and provide solutions of exact and approximate controllability problems for the evolution equation with a forcing term given in a separable form. Let $\mathbf{g}(x) = (g_1(x), g_2(x), g_3(x))^T$, then we consider the problem: $U_t = (i\mathscr{L}U)(x,t) + \mathbf{g}(x)f(t)$, $U(x,0) = U_0(x)$, $x \in [0,L]$, $t \geq 0$. The main question considered in Section 3 is the following: *Given U_0 and $\mathbf{g}(x)$, can we design a control function, $f(t)$, in such a way, that the solution of the above problem is equal to zero at a specific time moment $T > 0$?* This means that if $f(t) = 0$ for $t \geq T$, then the solution remains zero. The answer to this question is affirmative when certain condition is satisfied. The solution is given by an explicit formula for $f(t)$. (See Theorem 3.2 for the exact controllability case and Theorem 3.4 for the approximate controllability case.) To solve the problem we use the Riesz basis property of the generalized eigenvectors of the operator $\mathscr{L}$ and reduce the control problem to the moment problem via the spectral decomposition method. To write the formula for the desired control function we have to use the properties of non – harmonic exponentials $\left\{e^{i\lambda_n t}\right\}_{n\in\mathbb{Z}'}$ ($\mathbb{Z}' = \mathbb{Z}\setminus\{0\}$) with λ_n being exactly the eigenvalues of $\mathscr{L}$. In Section 3 we have collected those properties of non–harmonic exponentials that are important for all control problems considered in the paper. The main result of the section shows that for *any* $T > 0$, the system is controllable on time interval $[0,T]$ and the solution is given in terms of the parameters of the model and in the form of an expansion with respect to the set of the non–harmonic exponentials.

Section 4 is concerned with a *different type of control problems* that might be of a particular interest for the engineering community. As follows from system (1.4) and (1.5), the solution consists of two functions: the vertical displacement of a beam, $w(x,t)$, and the voltage across the electrodes, $\mathbf{v}(t)$. If we assume that the voltage is the output (*as follows from engineering literature*), then the control problem can be formulated as follows. Let the force profile function, $\mathbf{g}(x)$, and the voltage function, $\mathbf{v}(t)$, be given. *Can one design the control function, $f(t)$, in such a way that the voltage component of the solution is exactly equal to the given function* $\mathbf{v}(t)$? The solution to this control problem is presented in Theorem 4.6 below. It turns out that the main ingredient in the solution of the second control problem, which is a specific version of an *output tracking* problem, is the Interpolation Theorem in a subspace of Hardy space H^2_+. All information necessary for the construction of the above subspace and on convenient Riesz basis in this subspace is collected in Section 4. Finally, we discuss the third control problem, which is a modification of the second one. Namely, it is shown that for a given voltage output function, $\mathbf{v}(t)$, one can construct both the force profile function, $\mathbf{g}(x)$, and the time control function, $f(t)$, in such a way that the third component of the solution of problem (3.1) coincides with $\mathbf{v}(t)$. In the second control problem the class of admissible voltage outputs, $\mathbf{v}(t)$, for which the problem has a solution, depends on a given force profile function $\mathbf{g}(x)$. In contrast, the third control problem is solvable for a wider class of given voltage outputs. The explicit formulae for $\mathbf{g}(x)$ and $f(t)$ that provide the desired output are derived (see formulae (4.39) and (4.40) below).

2 Operator reformulation of the problem and main spectral results

In this paper, we assume that the Kelvin–Voigt damping is small enough to be neglected, i.e., $c_s = 0$. It is convenient to use scaled physical quantities with the new set of notations. Let

$$G = \frac{c_a}{m}, \qquad E = \frac{YI}{m}, \qquad H = \frac{1}{CR}, \qquad h = \frac{\kappa}{C}, \qquad \Theta = \frac{\theta}{m}. \tag{2.1}$$

Thus, equations (1.4), (1.5) in new notation have the form

$$\begin{aligned} &w_{tt}(x,t) + G\, w_t(x,t) + E\, w_{xxxx}(x,t) = 0,\\ &\mathbf{v}_t(t) + H\, \mathbf{v}(t) + h\,[w_{tx}(L,t) - w_{tx}(0,t)] = 0. \end{aligned} \tag{2.2}$$

The boundary and initial conditions are

$$w(0,t) = w_x(0,t) = 0, \qquad E\, w_{xx}(L,t) = \Theta \mathbf{v}(t), \qquad w_{xxx}(L,t) = 0. \tag{2.3}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad \mathbf{v}(0) = \mathbf{v}_0. \tag{2.4}$$

The operator reformulation. Let us rewrite problem (2.2)–(2.4) as an operator evolution equation in the state space of the system, which is the Hilbert space of the Cauchy data equipped with the energy norm.

Let $w(x,t)$ and $\mathbf{v}(t)$ be a solution of the above system. Introduce a vector

$$U(x,t) = (w(x,t), w_t(x,t), \mathbf{v}(t))^T \quad \text{and} \quad \text{fix} \quad t > 0.$$

Consider a three–component vector–function of x. (Without misunderstanding we use the same notation U.)

$$U(x) = (u_0(x), u_1(x), u_2)^T. \tag{2.5}$$

Definition 2.1. The energy space $\mathscr{H} = \widetilde{H}_0^2(0,L) \times L^2(0,L) \times \mathbb{C}$ is the closure of smooth functions (2.5) satisfying the conditions $u_0(0) = u_0'(0) = 0$ in the norm

$$\|U\|_{\mathscr{H}}^2 = \frac{1}{2}\left[\int_0^L \left(E\left|u_0''(x)\right|^2 + |u_1(x)|^2\right) dx + |u_2|^2\right]. \tag{2.6}$$

We use the notation $\widetilde{H}^2(0,L)$ for the subspace of the Sobolev space $H^2(0,L)$ consisting of functions u satisfying $u(0) = u'(0) = 0$. The "tilde" is used to distinguish this space from the subspace $H_0^2(0,L)$ consisting of functions satisfying similar conditions at both ends of the interval $[0,L]$.

System (2.2)–(2.4) can be given in the form of an evolution equation in $\mathscr{H}$

$$U_t(x,t) = i\,(\mathscr{L}U)\,(x,t), \qquad U(0) = U_0(x) = (w_0(x), w_1(x), \mathbf{v}_0)^T. \tag{2.7}$$

The dynamics generator, the main object of interest, is a matrix differential operator given by the differential expression

$$\mathscr{L} = -i \begin{bmatrix} 0 & 1 & 0 \\ -E\frac{d^4}{dx^4} & -G & 0 \\ 0 & -h\,[A_L - A_0] & -H \end{bmatrix}, \tag{2.8}$$

with A_a being the differential operation defined on a smooth function $\varphi(x)$ by

$$A_a[\varphi] = \varphi'(a), \tag{2.9}$$

and $\mathscr{L}$ being defined on the domain

$$\begin{aligned}\mathscr{D}(\mathscr{L}) = \Big\{ U = (u_0, u_1, u_2)^T \in \mathscr{H} : \ \ u_0 \in H^4(0,L) \cap \widetilde{H}_0^2(0,L), \\ u_1 \in \widetilde{H}_0^2(0,L), \ \ u_2 \in \mathbb{C}; \ \ u_0'''(L) = 0, \quad E u_0''(L) = \Theta u_2 \} \, .\end{aligned} \tag{2.10}$$

We mention here that the reason for introducing the factor i into Eq.(2.7) and, correspondingly, the factor $(-i)$ into definition (2.8) of $\mathscr{L}$ consists in the following. The operator $\mathscr{L}$ considered in the energy space on the domain defined in (2.10) is a 'small perturbation' of a certain selfadjoint operator. This fact turns out to be convenient for the analysis of the Riesz basis property of the generalized eigenvectors of this operator [13].

The adjoint operator. Recall that the adjoint operator $\mathscr{L}^*$ satisfies

$$(\mathscr{L}U, V)_{\mathscr{H}} = (U, \mathscr{L}^* V)_{\mathscr{H}} \text{ for any } \quad U \in \mathscr{D}(\mathscr{L}), \quad V \in \mathscr{D}(\mathscr{L}^*).$$

Lemma 2.2, [11]. *The operator $\mathscr{L}^*$ adjoint to $\mathscr{L}$ is given by the differential expression*

$$\mathscr{L}^* = -i \begin{bmatrix} 0 & 1 & 0 \\ -E\frac{d^4}{dx^4} & G & 0 \\ 0 & -\Theta A_L & H \end{bmatrix}, \tag{2.11}$$

defined on the domain

$$\begin{aligned}\mathscr{D}(\mathscr{L}^*) = \Big\{ V = (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)^T \in \mathscr{H} : \ \ \mathbf{v}_0 \in H^4(0,L) \cap \widetilde{H}_0^2(0,L), \ \mathbf{v}_1 \in \widetilde{H}_0^2(0,L), \\ \mathbf{v}_1(0) = \mathbf{v}_1'(0) = \mathbf{v}_0'''(L) = 0; \ \ E\mathbf{v}_0''(L) = h\mathbf{v}_2 . \Big\}\end{aligned} \tag{2.12}$$

As one can see the reason for $\mathscr{L}$ to be non–selfadjoint is the fact that both the differential expression (2.12) and the domain (2.13) are different from the differential expression (2.8) and the domain (2.10).

The evolution semigroup.

Proposition 2.3. *The evolution problem (2.2)–(2.4) or, equivalently, (2.7) defines a C_0–semigroup in the state space $\mathscr{H}$. The operator $\mathbb{L} = i\mathscr{L}$, with $\mathscr{L}$ defined in (2.8), (2.10) is the generator of this semigroup.*

This proposition is an immediate corollary of the result proven in [13] (see Theorem 2.8 below). Namely, it follows from the fact that $\mathscr{L}$ is a non-selfadjoint operator with a discrete spectrum, whose generalized eigenvectors form a Riesz basis in $\mathscr{H}$. By Proposition 2.3 we point out that the evolution problem (2.2)–(2.4) is well–posed and that $i\mathscr{L}$ can be called the dynamics generator. Without misunderstanding, we use the same term for the operator $\mathscr{L}$ as well.

The spectral asymptotics. Below we reproduce the main result of our paper [11] that we need in the sequel.

Theorem 2.4. *1) Operator $\mathscr{L}$ is a nonselfadjoint matrix differential operator in $\mathscr{H}$, whose resolvent is a compact operator. The spectrum of $\mathscr{L}$ is discrete with the only accumulation point at infinity; each eigenvalue is normal [5], i.e. it is an isolated point of the spectrum with a finite multiplicity.*

2) The entire spectrum, with a possible exception of a finite number of the eigenvalues, is located in the upper half – plane of the complex plane in a strip parallel to the real axis. There may be a finite number of real eigenvalues and a finite number of the eigenvalues in the open lower half–plane. The spectrum is symmetric with respect to the imaginary axis, i.e., $\lambda_{-n} = -\overline{\lambda}_n$, $n = 1,2,3,\ldots$. (For this reason it is convenient to use the index $n \in \mathbb{Z}' = \mathbb{Z}\setminus\{0\}$ to number the eigenvalues.)

3) The following asymptotic approximation is valid for the eigenvalues as the number of an eigenvalue tends to infinity:

$$\lambda_n = \sqrt{E}\left(\frac{\pi}{L}\right)^2 (n^2+n) + \left[\left(\frac{\pi}{2L}\right)^2 \sqrt{E} + \frac{2\Theta h}{L}\right] + i\ \frac{G}{2} + O\left(\frac{1}{n}\right), \quad n \to \infty, \quad \lambda_{-n} = -\overline{\lambda}_n. \tag{2.13}$$

4) In general, the operator $\mathscr{L}$ is not a dissipative operator unless $\Theta = h$. (We accept the following definition of a dissipative operator (see [5,7]): a linear operator T in a Hilbert space $\mathscr{H}$ is dissipative if for any $F \in \mathscr{D}(T)$, one has $\Im\,(TF,F)_{\mathscr{H}} \geq 0$.)

Remark 2.5. As stated in the above theorem, the only case when $\mathscr{L}$ is a dissipative operator is when $\Theta = h$. As we know, the parameter $h = \kappa/C$ with κ being a direct piezoelectric effect coefficient and C being internal piezoelectric layer capacitance, and the parameter $\Theta = \theta/m$ with θ being an inverse piezoelectric effect coefficient and m being the density of structure. However, in practical ranges of the physical parameters Θ and h are quite different numerically, which means that we are dealing with nonselfadjoint and non–dissipative operators. The fact that $\mathscr{L}$ is not dissipative affects significantly the proof of the Riesz basis property of its generalized eigenvectors given in [13]. Namely, numerous fundamental results on the completness and the Riesz basis property for dissipative operators (see [5,7]) cannot be directly applied to $\mathscr{L}$.

The spectral equation. In paper [11], to derive the spectral asymptotics, we have reduced the spectral problem for the operator $\mathscr{L}$ to the spectral problem for the corresponding operator polynomial pencil. The eigenvalue–eigenvector equation for $\mathscr{L}$, i.e.,

$$\mathscr{L}U = \lambda U, \quad U = (u_0, u_1, u_2)^T \in \mathscr{D}(\mathscr{L}), \quad \lambda \in \mathbb{C}, \tag{2.14}$$

generates the following system for the components of U:

$$u_1(x) = i\lambda u_0(x), \qquad E u_0''''(x) + G u_1(x) = -i\lambda u_1(x), \qquad h u_1'(L) + H u_2 = -i\lambda u_2. \tag{2.15}$$

Eliminating u_1 from this system and taking into account that $u_2 = \Theta^{-1} E u_0''(L)$ yield the following boundary–value problem for u_0:

$$Eu_0''''(\lambda,x)+i\lambda Gu_0(\lambda,x)-\lambda^2u_0(\lambda,x)=0, \tag{2.16}$$

$$u_0(\lambda,0)=u_0'(\lambda,0)=0, \qquad u_0'''(L)=0, \qquad \Theta^{-1}E(H+i\lambda)u_0''(\lambda,L)+i\lambda hu_0'(L)=0. \tag{2.17}$$

It is clear that (2.16) and (2.17) represent the spectral problem for a quadratic operator pencil $\mathscr{P}(\cdot)$ (see [7]) defined by the formula

$$\mathscr{P}(\lambda)\varphi=E\varphi''''+i\lambda G\varphi-\lambda^2\varphi \tag{2.18}$$

on the domain

$$\begin{aligned}\mathscr{D}(\mathscr{P})=\{\varphi\in H^4(0,L):\varphi(0)=\varphi'(0)=\varphi''''(0)=0,\\ \Theta^{-1}E(H+i\lambda)\varphi''(L)+i\lambda h\varphi'(L)=0.\}\end{aligned} \tag{2.19}$$

Notice, $\mathscr{P}(\cdot)$ is a non–standard pencil since the spectral parameter λ enters the domain explicitly. This type of pencils have not been considered in monograph [7]. However, it is convenient to keep the terminology because there exists an extensive literature, in which the pencils with the spectral parameter–dependent boundary conditions appear naturally. A non–trivial solution $\varphi\in\mathscr{D}(\mathscr{P})$ of the pencil equation $\mathscr{P}(\lambda)\varphi=0$ will be called an *eigenfunction* of the pencil and the corresponding value of λ will be called an *eigenvalue*. The spectra of the operator $\mathscr{L}$ and the pencil $\mathscr{P}(\cdot)$ coincide. It is clear that having an eigenfunction of the pencil and using (2.15), we can find all components of the eigenvector of $\mathscr{L}$.

Now we return to Eq.(2.16) and construct the solution $\varphi(\lambda,x)$ of the equation $\mathscr{P}(\lambda)\varphi=0$, i.e.,

$$E\varphi''''(\lambda,x)+i\lambda G\varphi(\lambda,x)-\lambda^2\varphi(\lambda,x)=0, \tag{2.20}$$

satisfying all boundary conditions. It is convenient to introduce a new set of scaled parameters

$$\widetilde{\lambda}=\frac{\lambda}{\sqrt{E}}, \quad \widetilde{G}=\frac{G}{\sqrt{E}}, \quad \widetilde{H}=\frac{H}{\sqrt{E}}, \quad \widetilde{\Theta}=\frac{\Theta h}{\sqrt{E}}. \tag{2.21}$$

Rewriting problem (2.16)–(2.17) in the scaled parameters and omitting the “tilde” we obtain the following rescaled problem:

$$\varphi''''(\lambda,x)-\lambda^2\varphi(\lambda,x)+i\lambda G\varphi(\lambda,x)=0, \tag{2.22}$$

$$\varphi(\lambda,0)=\varphi'(\lambda,0)=0, \quad \varphi(\lambda,L)=0, \quad (H+i\lambda)\,\varphi''(\lambda,L)+i\lambda\Theta\varphi'(\lambda,L)=0. \tag{2.23}$$

As is shown in [11] the solution satisfying three homogeneous boundary conditions $\varphi(\lambda,0) = \varphi'(\lambda,0) = \varphi(\lambda,L) = 0$ can be given in the form

$$\begin{aligned}\varphi(\lambda,x) = &[\cosh(\mu(\lambda)L) + \cos(\mu(\lambda)L)]\,[\cosh(\mu(\lambda)x) - \cos(\mu(\lambda)x)] - \\ &[\sinh(\mu(\lambda)L) - \sin(\mu(\lambda)L)]\,[\sinh(\mu(\lambda)x) - \sin(\mu(\lambda)x)],\end{aligned} \tag{2.24}$$

where $\mu(\lambda)$ is defined as follows:

$$\mu(\lambda) = \sqrt[4]{\lambda^2 - i\lambda G} = \sqrt{\lambda} - \frac{iG}{4a\sqrt{\lambda}} + \frac{3G^2}{2^5\lambda^{3/2}} + O\left(\frac{1}{\lambda^{5/2}}\right), \qquad |\lambda| \to \infty, \tag{2.25}$$

and the fourth order root is fixed by the condition that $\Im\sqrt[4]{\alpha} \geq 0$ and $\Re\sqrt[4]{\alpha} \geq 0$ for $\Im\alpha \geq 0$. The fourth boundary condition (2.23) generates the following *spectral equation*:

$$\begin{aligned}&\mu(\lambda)(H + i\lambda)\,[1 + \cosh(\mu(\lambda)L)\cos(\mu(\lambda)L)] + \\ &i\lambda\Theta\,[\cosh(\mu(\lambda)L)\sin(\mu(\lambda)L) + \sinh(\mu(\lambda)L)\cos(\mu(\lambda)L)] = 0.\end{aligned} \tag{2.26}$$

The roots of Eq.(2.26) coincide with the eigenvalues of the pencil $\mathscr{P}(\cdot)$ and of the operator $\mathscr{L}$ if both the pencil and the operator are represented in terms of rescaled parameters (2.21). For this reason the asymptotics of the roots of Eq.(2.26) is given by formula (2.13) in which one sets $E = h = 1$.

The generalized eigenvectors. To proceed, we need several definitions.

Definition 2.6. *i*) A set of vectors $\{\varphi_n\}_{n\in\mathbb{Z}'}$ in a Hilbert space H is *almost normalized* if there exist two positive constants C_1 and C_2 such that

$$0 < C_1 \leq \|\varphi_n\|_H \leq C_2 < \infty, \qquad n \in \mathbb{Z}', \qquad \mathbb{Z}' = \mathbb{Z}\setminus\{0\}. \tag{2.27}$$

ii) Two sets of vectors $\{\varphi_n\}_{n\in\mathbb{Z}'}$ and $\{\psi_n\}_{n\in\mathbb{Z}'}$ are *biorthogonal* if the following relation holds:

$$(\varphi_n, \psi_m)_H = \delta_{nm}, \tag{2.28}$$

iii) A set of vectors $\{\varphi_n\}_{n\in\mathbb{Z}'}$ in a Hilbert space H forms a *Riesz basis* if this set is a linear isomorphic image of an orthonormal basis $\left\{\varphi_n^0\right\}_{n\in\mathbb{Z}'}$ (i.e., a Riesz basis is an almost normalized unconditional basis.)

iv) A Riesz basis $\{\varphi_n\}_{n\in\mathbb{Z}'}$ which is quadratically close to an orthonormal basis, i.e., $\sum_{n\in\mathbb{Z}'}\left\|\varphi_n - \varphi_n^0\right\|_H^2 < \infty$, is called a *Bari basis*.

It is clear that if we evaluate the function $\varphi(\lambda,x)$ from (2.24) at those values of λ that satisfy Eq.(2.26), then we obtain the set of the eigenfunctions of the pencil $\mathscr{P}(\cdot)$. (written in terms of the rescaled parameters (2.21)). However, in the sequel we need the Riesz basis property of the eigenvectors of the operator $\mathscr{L}$ and, therefore, we will consider the set of the eigenfunctions of $\mathscr{P}(\cdot)$, which gives rise to an almost normalized set of the generalized eigenvectors of the operator $\mathscr{L}$. Namely, if $\mu_n = \mu(\lambda_n)$, with $\mu(\lambda)$ being defined in (2.25), then the corresponding eigenfunction of $\mathscr{P}(\cdot)$ can be given by the following formula:

$$\psi_n(x) = \varphi(\lambda_n, x)\ [\cosh(\mu_n L) + \cos(\mu_n L)]^{-1} = \cosh(\mu_n x) - \cos(\mu_n x) - \Gamma_n [\sinh(\mu_n x) - \sin(\mu_n x)], \tag{2.29}$$

with Γ_n being defined as

$$\Gamma_n = \frac{\sinh(\mu_n L) - \sin(\mu_n L)}{\cosh(\mu_n L) + \cos(\mu_n L)} = 1 + 2(-1)^{n+1} e^{-\mu_n L} + O\left(e^{-\mu_n L}\mu_n^{-1}\right). \tag{2.30}$$

In the sequel we need the asymptotic approximation for both ψ_n and ψ_n' as $|n| \to \infty$

$$\begin{aligned} \psi_n(x) &= e^{-\mu_n x} + [\sin(\mu_n x) - \cos(\mu_n x)] + 2(-1)^{n+1} e^{-\mu_n (L-x)} + O\left(\mu_n^{-1}\right), \\ \psi_n'(x) &= \mu_n \left\{ -e^{-\mu_n x} + [\sin(\mu_n x) + \cos(\mu_n x)] - 2(-1)^n e^{-\mu_n (L-x)} + O\left(\mu_n^{-1}\right) \right\}. \end{aligned} \tag{2.31}$$

It can be directly verified that the set of almost normalized generalized eigenvectors of the operator $\mathscr{L}$ (written in terms of the rescaled parameters (2.21)) can be given by the following formula:

$$\Psi_n(x) = \begin{pmatrix} \frac{1}{i\lambda_n} \psi_n(x) \\ \psi_n(x) \\ \frac{-h}{H + i\lambda_n} \psi_n'(L) \end{pmatrix}, \quad n \in \mathbb{Z}', \quad x \in [0, L], \tag{2.32}$$

One of the main results of paper [13] is the following statement.

Theorem 2.7. *If the operator $\mathscr{L}$ has a simple spectrum, then the set of the eigenvectors of $\mathscr{L}$ forms a Bari basis in $\mathscr{H}$. The set of the eigenvectors of the adjoint operator $\mathscr{L}^*$ forms a biorthogonal Bari basis in $\mathscr{H}$. If the operator $\mathscr{L}$ has a finite number of multiple eigenvalues, then the set of the generalized eigenvectors of $\mathscr{L}$ forms a Bari basis in $\mathscr{H}$. The corresponding vectors of the biorthogonal basis are constructed as linear combinations of the generalized eigenvectors of $\mathscr{L}^*$ (see, e.g., [13] for details).*

3 Exact and approximate controllability

The spectral decomposition. Let us consider the following nonhomogeneous problem in the state space $\mathscr{H}$

$$U_t(x,t) = i(\mathscr{L}U)(x,t) + \mathbf{g}(x) f(t), \qquad U(x,0) = U_0(x), \qquad 0 \le x \le L, \qquad t \ge 0. \tag{3.1}$$

The vector–valued function $\mathbf{g}(x) = (g_1(x), g_2(x), g_3(x))^T$ is called the force profile function and $f(t)$ is viewed as a control. Notice that if $g_1(x) = g_3(x) = 0$, then the only control term $g_2(x)f(t)$ appears in the right–hand side of the beam equation (1.4) and can be interpreted as a force acting on the beam in the transverse direction. However, the assumption $g_1 = g_3 = 0$ does not affect in any way the solution of the control problem below. For this reason we assume that $\mathbf{g}(x)$ may have all three nonzero components.

We assume that the force profile function $\mathbf{g} \in \mathscr{H}$. The control function, f, is called an *admissible control on the time interval* $[0,T]$, $0 < T < \infty$, if $f \in L^2(0,T)$. In what follows, we consider the case when the spectrum of $\mathscr{L}$ is simple, i.e., there are no associate vectors and the operator $\mathscr{L}$ has only the eigenvectors whose entire collection forms a Riesz basis in $\mathscr{H}$. More general situation is concerned with the case when there may be a finite number of multiple eigenvalues of $\mathscr{L}$ of a finite multiplicity each. As follows from Lemma 4.4 [11], in our problem the geometric multiplicity of each multiple eigenvalue is always 1 while an algebraic multiplicity can be greater than 1. In this case in addition to the eigenvector there exists a chain of associate vectors. We refer to [IMA J. on Cont. & Inform.] for a detailed analysis of an exact and approximate controllability problems in that case.

We look for the solution of problem (3.1) in the form of an expansion with respect to the Riesz basis of the eigenvectors of the operator $\mathscr{L}$:

$$U(x,t) = \sum_{n\in\mathbb{Z}'} u_n(t)\Psi_n(x). \tag{3.2}$$

In a similar fashion we obtain expansions for the force profile function, $\mathbf{g}$, and the initial state U_0:

$$\mathbf{g}(x) = \sum_{n\in\mathbb{Z}'} g_n\Psi_n(x) = \sum_{n\in\mathbb{Z}'} (\mathbf{g},\Psi_n^*)_{\mathscr{H}}\Psi_n(x), \tag{3.3}$$

$$U_0(x) = \sum_{n\in\mathbb{Z}'} u_n^0\ \Psi_n(x) = \sum_{n\in\mathbb{Z}'} (U_0,\Psi_n^*)_{\mathscr{H}}\Psi_n(x), \tag{3.4}$$

where $\{\Psi_n^*\}_{n\in\mathbb{Z}'}$ is the biorthogonal Riesz basis, which consists of the eigenvectors of the adjoint operator $\mathscr{L}^*$, described in Lemma 2.2 (see Theorem 2.7). Upon substitution of (3.2)–(3.4) into problem (3.1), one gets the following equation:

$$\sum_{n\in\mathbb{Z}'} u_n'(t)\Psi_n(x) = i\sum_{n\in\mathbb{Z}'} \lambda_n\Psi_n(x) + f(t)\sum_{n\in\mathbb{Z}'} u_n^0\ \Psi_n(x). \tag{3.5}$$

Using the Riesz basis property of $\{\Psi_n\}_{n\in\mathbb{Z}'}$ we obtain that each component, $u_n(t)$, has to satisfy the following initial–value problem:

$$u_n'(t) = i\lambda_n u_n(t) + g_n f(t), \qquad u_n(0) = u_n^0. \tag{3.6}$$

An explicit solution of problem (3.6) can be given in the form

$$u_n(t) = u_n^0 \; e^{i\lambda_n t} + \int_0^t g_n f(\tau) e^{i\lambda_n (t-\tau)} d\tau, \tag{3.7}$$

which yields the solution of problem (3.1)

$$U(x,t) = \sum_{n\in\mathbb{Z}'} u_n^0 \; e^{i\lambda_n t}\Psi_n(x) + \sum_{n\in\mathbb{Z}'} g_n \int_0^t f(\tau) e^{i\lambda_n (t-\tau)} d\tau \;\; \Psi_n(x). \tag{3.8}$$

Using an explicit representation (2.32) for $\Psi_n(x)$, we obtain the following formula for the voltage output:

$$\mathbf{v}(t) = -h \left\{ \sum_{n\in\mathbb{Z}'} \frac{u_n^0 \;\; \psi_n'(L)}{H + i\lambda_n} \; e^{i\lambda_n t} + \sum_{n\in\mathbb{Z}'} \frac{g_n \psi_n'(L)}{H + i\lambda_n} \int_0^t f(\tau) \; e^{i\lambda_n (t-\tau)} d\tau \right\}. \tag{3.9}$$

Now we formulate the first control problem considered in this paper.

Statement of the problem. The main exact controllability question can be formulated as follows. *Can one find a control function $f(t)$, $0 \le t \le T < \infty$, such that the solution $U(x,t)$ from (3.8) is equal to zero for all $t > T$?* As we show, the answer to this question is affirmative for any $T > 0$ if certain conditions on the initial state and the force distribution function $\mathbf{g}(x)$ are satisfied. (See Theorem 3.2 below for the exact controllability case.) If the aforementioned conditions are not satisfied, one has an approximate controllability case (see Theorem 3.4 below). To derive explicit formulae for the control laws, one needs the following information: (a) asymptotic distribution of the eigenvalues of the dynamics generator, $\mathscr{L}$, governing the harvester dynamics; (b) the Riesz basis property of the set of the generalized eigenvectors of the operator $\mathscr{L}$; (c) results on solvability of the moment problem generated by the evolution problem (3.1), which in turn requires (d) some information on the completeness, the minimality, and the basis property of the set of non–harmonic exponentials in $L^2(0,T)$.

The moment problem. Let initial condition, $U_0 \in \mathscr{H}$, and $T > 0$ be given. We are looking for an admissible control function, $f(t)$, on the interval $[0,T]$ such that the solution of problem (3.1) also satisfies an additional constraint

$$U(x,T) = 0, \qquad x \in [0,L]. \tag{3.10}$$

Due to the uniqueness theorem, (3.10) means that if $f(t) = 0$ for $t \ge T$, then $U(x,t) = 0$ for all $t \ge T$. If $U(x,T) = 0$, then from (3.8) it follows:

$$\sum_{n\in\mathbb{Z}'} \left[\left(u_n^0 \; + g_n \int_0^T f(\tau) e^{-i\lambda_n \tau} d\tau \right) e^{i\lambda_n T} \right] \Psi_n(x) = 0. \tag{3.11}$$

Since $\{\Psi_n\}_{n\in\mathbb{Z}'}$ forms a Riesz basis in $\mathscr{H}$, Eq.(3.11) yields the following fact: a control function,

f, steers the initial state to zero on the time interval $[0,\ T]$ if and only if it is a solution of the following system:

$$g_n \int_0^T \widetilde{f}(\tau) e^{i\lambda_n \tau} d\tau = -u_n^0\, e^{i\lambda_n T}, \qquad f(\tau) = \widetilde{f}(T-\tau), \qquad n \in \mathbb{Z}'. \tag{3.12}$$

Hence, the admissible control function, $f(t)$, exists if and only if system (3.12) has a non–trivial solution. The problem of reconstruction of the function f from system (3.12) is called *the moment problem*. The solution of this moment problem (under certain conditions on the parameters of the problem) can be given in terms of "non–harmonic" exponentials. At this moment, we would like to note that without loss of generality, we can assume that $\Im\lambda_n > 0$. Indeed, there could be only a finite number of of the eigenvalues with non–positive imaginary parts. Let $d_0 > 0$ be such that $\Im\lambda_n > -d_0,\ \ n \in \mathbb{Z}'$. Consider the following moment problem:

$$g_n \int_0^T \left[\widetilde{f}(\tau) e^{-id_0\tau}\right] e^{i(\lambda_n + d_0)\tau} d\tau \equiv g_n \int_0^T \widehat{f}(\tau) e^{i\widetilde{\lambda}_n \tau} d\tau = -u_n^0\, e^{i\lambda_n T}, \qquad \Im\widetilde{\lambda}_n > 0, \qquad n \in \mathbb{Z}'.$$

Obviously, if this moment problem is solvable, then the solution of the moment problem (3.12) can be given in the form $\widetilde{f}(\tau) = \widehat{f}(\tau)\ e^{id_0\tau}$. Hence in what follows we assume that $\Im\lambda_n > 0,\ n \in \mathbb{Z}'$.

For reader's convenience, we have reproduced below some results on the set of non–harmonic exponentials $\left\{e^{i\lambda_n t}\right\}_{n\in\mathbb{Z}'}$.

Some properties of the set of non–harmonic exponentials [1, 6]. Let $\mathscr{E}$ and $\mathscr{E}_n$ be two sets of functions defined as

$$\mathscr{E} = \left\{\exp\left(-i\overline{\lambda}_n t\right)\right\}_{m\in\mathbb{Z}'}, \qquad \mathscr{E}_n = \left\{\exp\left(-i\overline{\lambda}_n t\right)\right\}_{\substack{m\in\mathbb{Z}' \\ m\neq n}}. \tag{3.13}$$

1) For any $T > 0$ the set $\mathscr{E}$ is not complete in $L^2(0,T)$. Indeed, let $\mathfrak{E}(\mathscr{E},T)$ be the smallest closed subspace in $L^2(0,T)$ containing $\mathscr{E}$. As is well known [4, 6, 10, 11, 14], $\mathfrak{E}(\mathscr{E},T)$ is a proper subspace of $L^2(0,T)$ if and only if $\sum_{n\in\mathbb{Z}'}|\lambda_n|^{-1} < \infty$, which is the case in our problem due to asymptotics (2.13).

2) For any $T > 0$, the set $\mathscr{E}$ is almost normalized. Indeed, evaluating the norm of $\exp\left\{-i\overline{\lambda}_n t\right\}$ in $L^2(0,T)$, and using asymptotics (2.13) one gets for $\lambda_n \in \mathbb{C}^+$ (open upper half–plane)

$$\left\|e^{-i\overline{\lambda}_n t}\right\|^2_{L^2(0,T)} = \frac{e^{-2\ \Im\lambda_n T} - 1}{2\ \Im\lambda_n} \asymp \left(\frac{e^{-GT} - 1}{G}\right) \asymp 1.$$

(We have used the following notation: if $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers then $a_n \asymp b_n$ means that $c_1 a_n \le b_n \le c_2 a_n$ for some $c_1, c_2 > 0$.)

3) Since the set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}'}$ is located in a strip parallel to the real axis and is separated. i.e.,

$\inf_{n,\, m,\, n\neq m} |\lambda_n - \lambda_m| > 0$, the set $\mathscr{E}$ forms a Riesz basis in its closed linear span in $L^2(0,T)$ [6, 10, 18].

4) The Riesz basis biorthogonal to the set $\mathscr{E}$ can be described as follows. Let the set $\mathfrak{E}(\mathscr{E}_n, T)$, be the smallest subspace in $L^2(0,T)$ containing $\mathscr{E}_n$. This subspace does not include $\exp\left\{-i\overline{\lambda}_n t\right\}$. Using a standard argument from the Hilbert space theory, one can show that there exists a unique function $\tau_n \in \mathfrak{E}(\mathscr{E}_n, T)$, which is closest to the function $\exp\left(-i\overline{\lambda}_n t\right)$ in $L^2(0,T)$–norm. If $d_n(T)$ is the distance between $\exp\left(-i\overline{\lambda}_n t\right)$ and $\tau_n(t)$, then

$$(d_n(T))^2 = \int_0^T \left| e^{-i\overline{\lambda}_n t} - \tau_n(t) \right|^2 dt, \tag{3.14}$$

and the set of functions

$$w_n(t) = \frac{e^{-i\overline{\lambda}_n t} - \tau_n(t)}{(d_n(T))^2}, \qquad n \in \mathbb{Z}' \tag{3.15}$$

forms a biorthogonal set for $\mathscr{E}$ in $L^2(0,T)$. Obviously, since the set $\mathscr{E}$ is not complete in $L^2(0,T)$, the biorthogonal set is not unique. However, the set $\{w_n(t)\}_{n\in\mathbb{Z}'}$ is called the "optimal" biorthogonal set for $\mathscr{E}$. Indeed, assume that $\{\widetilde{w}_n(t)\}_{n\in\mathbb{Z}'}$ is another biorthogonal set for $\mathscr{E}$ in $L^2(0,T)$. It can be easily verified that

$$\widetilde{w}_n = w_n + \kappa_n, \qquad \kappa_n \in \mathfrak{E}(\mathscr{E}_n, T)^{\perp}, \qquad n \in \mathbb{Z}', \tag{3.16}$$

and hence, $\|w_n\|_{L^2(0,T)} \leq \|\widetilde{w}_n\|_{L^2(0,T)}$. From this inequality and (3.16), it follows that

$$\|w_n\|_{L^2(0,T)} = (d_n(T))^{-1}. \tag{3.17}$$

Since the system $\mathscr{E}$ is uniformly minimal in $L^2(0,T)$ the following bound holds:

$$\inf_{n\in\mathbb{Z}'} d_n(T) = C(T) > 0. \tag{3.18}$$

Solvability of the moment problem (3.12). We recall the basic result on the solvability of the general moment problem ([1, 6, 14]).

Theorem 3.1 *Let $\{\phi_n\}_{n\in\mathbb{Z}'}$ be a Riesz basis in a Hilbert space H and let $\{\phi_n^*\}_{n\in\mathbb{Z}'}$ be its biorthogonal Riesz basis. Consider the moment problem, i.e., the problem of restoration of an unknown vector f which satisfies an infinite system:*

$$a_n = (f, \phi_n)_H, \qquad n \in \mathbb{Z}'. \tag{3.19}$$

1. *If* $\{a_n\}_{n\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$ *and* $a_n \neq 0,\ n\in\mathbb{Z}'$, *then there exists a unique vector* $f\in H$ *that satisfies system (3.19). This vector is given by the formula*

$$f = \sum_{n\in\mathbb{Z}'} a_n \phi_n^*. \tag{3.20}$$

2. *If there exists a subset* $\mathscr{R}\subset\mathbb{Z}'$ *such that* $a_m = 0$ *for* $m\in\mathscr{R}$, *then the moment problem (3.19) has a solution under the assumption* $\{a_n\}_{n\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$. *This solution is not unique, it can be represented in the form*

$$f = f_0 + f_1, \quad \text{where} \quad f_0 = \sum_{n\in\mathbb{Z}'\setminus\mathscr{R}} a_n\phi_n^* \quad \text{and} \quad f_1 = \sum_{m\in\mathscr{R}} b_m\phi_m^*, \tag{3.21}$$

with $\{b_m\}_{m\in\mathscr{R}}$ *being an arbitrary sequence from* $l^2(\mathscr{R})$.

Theorem 3.2. *1. Assume that in decomposition (3.3)*

$$g_n \neq 0 \quad \text{for} \quad \text{all} \quad n\in\mathbb{Z}'. \tag{3.22}$$

The following statements are valid. System (3.1) is controllable on the time interval T *with* $T>0$ *if and only if*

$$\left\{\gamma_n \equiv \frac{u_n^0}{g_n}\right\}_{n\in\mathbb{Z}'} \in l^2(\mathbb{Z}'), \qquad i.e., \qquad \sum|\gamma_n|^2 < \infty. \tag{3.23}$$

The desired control function, $f(t)$, *which brings the system to zero state on the time interval* $[0,T]$, *can be defined by the formula*

$$f(t) = -\sum_{n\in\mathbb{Z}'} \gamma_n\, e^{i\lambda_n T}\, w_n(T-t), \tag{3.24}$$

where $\{w_n\}_{n\in\mathbb{Z}'}$ *is the Riesz basis in* $\mathfrak{E}(\mathscr{E},T)$, *which is biorthogonal to the Riesz basis of non–harmonic exponentials* $\mathscr{E}$ *of (3.13). There exist infinitely many control functions from* $L^2(0,T)$ *which bring the system to zero state. These functions have the form* $\widetilde{f} = f+\rho$, *where* $\rho\in\mathfrak{E}(\mathscr{E},T)^\perp$. *Therefore, the control function* f *defined by (3.24) has the minimal norm, i.e., if another function* $\widetilde{f}$, *brings the system to rest in the same time* T, *then*

$$\|f\|_{L^2(0,T)} < \left\|\widetilde{f}\right\|_{L^2(0,T)}. \tag{3.25}$$

2. *Assume that condition (3.22) is not satisfied and let* $\mathscr{R} = \{n\in\mathbb{Z}' : g_n = 0\}$ *and* $\mathscr{S} = \left\{n\in\mathbb{Z}' : u_n^0 = 0\right\}$; *let* γ_n *be defined by (3.23) only for* $n\in\mathbb{Z}'\setminus\mathscr{R}$. *Then the system is controllable in time* $T>0$ *if and only if* $\mathscr{R}\subseteq\mathscr{S}$ *and* $\{\gamma_n\}_{n\in\mathbb{Z}'} \in l^2(\mathbb{Z}'\setminus\mathscr{R})$. *The desired control function*

is not unique and can be given by the formula

$$f(t) = -\left[\sum_{n\in\mathbb{Z}'\setminus\mathscr{S}} \gamma_n\, e^{i\lambda_n T}\, w_n(T-t) + \sum_{m\in\mathscr{R}} b_m\, e^{i\lambda_m T}\, w_m(T-t)\right], \tag{3.26}$$

where $b_m \in \mathbb{C}$ *are arbitrary coefficients such that* $\sum_{m\in\mathscr{R}} |b_m|^2 < \infty$.

Remark 3.3. The biorthogonal basis function, $w_n(t)$, defined in (3.15) can be represented directly in terms of the eigenvalues $\{\lambda_n\}$. The formula is quite lengthy and can be given in terms of the truncated Blaschke product (see, e.g., [6]).

Proof of Theorem 3.2. If conditions (3.22) are satisfied, then the moment problem (3.12) can be written in the form

$$\left(\widetilde{f}, e^{-i\bar{\lambda}_n\cdot}\right)_{L^2(0,T)} = -\gamma_n\, e^{i\lambda_n T}, \qquad n\in\mathbb{Z}'. \tag{3.27}$$

Thus, $\left\{-\gamma_n\, e^{i\lambda_n T}\right\}_{n\in\mathbb{Z}'}$ is just the sequence of the generalized Fourier coefficients of the function $\widetilde{f}$ with respect to the Riesz basis $\mathscr{E}$. It follows from Theorem 3.1 that problem (3.27) has a unique solution $\widetilde{f}\in L^2(0,T)$ with the minimal norm if and only if $\gamma_n\neq 0$ and $\{\gamma_n e^{i\lambda_n T}\}_{n\in\mathbb{Z}'}\in l^2(\mathbb{Z}')$. This condition is equivalent to (3.23), i.e., $\{\gamma_n\}_{n\in\mathbb{Z}'}\in l^2(\mathbb{Z}')$, due to the fact that, according to asymptotic formula (2.13), $\sup_{n\in\mathbb{Z}'} |\Im\lambda_n| < \infty$. Hence the control problem solution is of the form

$$\widetilde{f}(t) = \sum_{n\in\mathbb{Z}'} \left(\widetilde{f}, e^{-i\bar{\lambda}_n\cdot}\right)_{L^2(0,T)} w_n(t), \tag{3.28}$$

where $\{w_n(t)\}_{n\in\mathbb{Z}'}$ is the Riesz basis biorthogonal to the Riesz basis $\mathscr{E}$. Substituting (3.27) into (3.28) and changing the time variable from t to $(T-t)$ yields precisely (3.24).

Recall, for any $T>0$, the system of exponentials $\mathscr{E}$ forms a Riesz basis in its closed linear span $\mathfrak{E}(\mathscr{E},T)$, but it is not complete in $L^2(0,T)$. Hence, the solution of the system (3.27) is defined up to an addition of a function $\widetilde{\varphi}\in L^2(0,T)$ and such that $\widetilde{\varphi}\in\mathfrak{E}(\mathscr{E},T)^\perp$.

Assume now that (3.22) is not valid, i.e., $g_n = 0$ for $n\in\mathscr{R}\subset\mathbb{Z}'$, but (3.23) holds. Then the moment problem in (3.12) is equivalent to the following one:

$$\int_0^T \widetilde{f}(\tau)\, e^{i\lambda_n\tau}\, d\tau = \begin{cases} -\gamma_n\, e^{i\lambda_n T}, & n\in\mathbb{Z}'\setminus\mathscr{R}, \\ b_n, & n\in\mathscr{R}, \end{cases} \tag{3.29}$$

with $\{b_n\}$ being an arbitrary sequence from $l^2(\mathscr{R})$. Since $\mathscr{E}$ is a Riesz basis, system has a solution if and only if $\sum_{n\in\mathbb{Z}'\setminus\mathscr{R}} |\gamma_n|^2 + \sum_{n\in\mathscr{R}} |b_n|^2 < \infty$ and this solution is of the form (3.24). Substituting (3.29) into (3.28) and taking into account that $\gamma_n\neq 0$ only if $n\in\mathbb{Z}'\setminus\mathscr{R}$, we obtain (3.26). ■

Approximate controllability. The following result is valid.

Theorem 3.4. *Assume that condition (3.23) is not satisfied, but let*

$$\{\gamma_n\}_{n\in\mathbb{Z}'} \in l^q(\mathbb{Z}'), \qquad (if \quad 2<q<\infty) \qquad or \quad \sup_{n\in\mathbb{Z}'} |\gamma_n| < \infty \qquad (if \;\; q=\infty). \tag{3.30}$$

Then for any $\varepsilon > 0$, there exists N such that for the control function

$$f_N(t) = -\sum_{|m|\le N} \gamma_m\, e^{i\lambda_m T}\, w_m(T-t), \tag{3.31}$$

the following result holds:

$$\|U(\cdot,T)|_{\mathscr{H}} \le \varepsilon \qquad for \qquad T>0. \tag{3.32}$$

However $\|f_N\|_{L^2(0,T)} \to \infty$ as $N\to\infty$.

Proof. If (3.30) holds, then the sequence $\{\gamma_n\}_{n\in\mathbb{Z}'}$ from (3.23) is bounded, i.e.,

$$|\gamma_n| \le \Gamma < \infty. \tag{3.33}$$

Let the control function be given in the form (3.31). Substituting the control function $f = f_N$ into the formula (3.8) for the solution and taking $t=T$ we obtain

$$U(x,T) = \left\{\sum_{|n|\le N} + \sum_{|n|\ge N+1}\right\}\left[u_n^0 e^{i\lambda_n T} + g_n \int_0^T e^{i\lambda_n(T-\tau)} f_N(\tau)d\tau\right]\Psi_n(x). \tag{3.34}$$

Consider the second sum of the integral terms from (3.34). Owing to the explicit formula (3.31) for each of those terms we have

$$\sum_{|m|\le N}\int_0^T e^{i\lambda_n(T-\tau)}\, w_m(T-\tau)\,\gamma_m\, e^{i\lambda_m T}\, d\tau = \sum_{|m|\le N}\gamma_m\, e^{i\lambda_m T}\int_0^T e^{i\lambda_n(\eta)}\, w_m(\eta)\, d\eta = 0, \tag{3.35}$$

since $n > m$ and $\{w_m\}$ is biorthogonal to the sequence of the exponentials. Thus, the second sum in (3.34) can be reduced to $\sum_{|n|\geq N+1} u_n^0 e^{i\lambda_n T}\Psi_n(x)$. For the first sum in (3.34) we have

$$\begin{gathered}\sum_{|n|\leq N}\left[u_n^0 e^{i\lambda_n T} - g_n \sum_{|m|\leq N}\int_0^T e^{i\lambda_n(T-\tau)}\, w_m(T-\tau)\,\gamma_m\, e^{i\lambda_m T}\, d\tau\right]\Psi_n(x) = \\ \sum_{|n|\leq N}\left[u_n^0 e^{i\lambda_n T} - g_n \sum_{|m|\leq N}\int_0^T e^{i\lambda_n(\eta)}\, w_m(\eta)\,\gamma_m\, e^{i\lambda_m T}\, d\eta\right]\Psi_n(x) = \\ \sum_{|n|\leq N}\left[u_n^0 e^{i\lambda_n T} - g_n \sum_{|m|\leq N}\gamma_m\, e^{i\lambda_m T}\,\delta_{m\,n}\right]\Psi_n(x) = 0.\end{gathered} \tag{3.36}$$

Hence, (3.34) can be reduced to the following form:

$$U(x,T) = \sum_{|n|\geq N+1} u_n^0 e^{i\lambda_n T}\Psi_n(x). \tag{3.37}$$

Now we estimate the $\mathscr{H}$–norm of the function in (3.37) using condition (3.33) and almost normalization of the basis functions $\{\Psi_n(x)\}_{n\in\mathbb{Z}'}$. We have

$$\|U(\cdot,T)\|^2_{\mathscr{H}} \asymp \sum_{|n|\geq N+1}\left|u_n^0\right|^2 = \sum_{|n|\geq N+1}|g_n|^2\,|\gamma_n|^2 \leq \Gamma^2 \sum_{|n|\geq N+1}|g_n|^2 \to 0 \qquad \text{as} \qquad N\to\infty. \tag{3.38}$$

Now we prove that $\|f_N\|_{L^2(0,T)} \to \infty$ as $N\to\infty$. We have the following estimates for f_N.

$$\|f_N\|^2_{L^2(0,L)} \asymp \sum_{|n|\leq N}|\gamma_n|^2 \to \infty \text{ as } N\to\infty \qquad \text{since} \qquad \{\gamma_n\}_{n\in\mathbb{Z}'} \notin l^2(\mathbb{Z}'). \qquad \blacksquare$$

4 Output tracking of a prescribed signal

Preliminary results. To formulate the second and third control problems and present their solutions, we need some information on the properties of simple fractions in Hardy class H^2_+. For reader's convenience, we reproduce the necessary facts.

We recall that the Hardy class H^2_+ is a Hilbert space that consists of functions $q(z)$, $z = x+iy$, analytic in the upper half–plane $\mathbb{C}^+$ and such that

$$\|q\|^2_{H^2_+} = \sup_{y>0}\frac{1}{2\pi}\int_{\mathbb{R}}|q(x+iy)|^2\,dx < \infty. \tag{4.1}$$

The supremum in (4.1) is attained when $y=0$, i.e., when the analytic function $q(z)$ is replaced by its limit value $q(x)$ on the real axis. The corresponding inner product coincides with the inner product in $L^2(\mathbb{R})$:

$$\langle q_1, q_2\rangle_{H^2_+} = \int_{-\infty}^{\infty} q_1(x)\overline{q_2(x)}dx.$$

Let $\mathscr{F}^*$ be the inverse Fourier transformation of a function $p(\gamma)$, $\gamma \in \mathbb{R}$, i.e.,

$$\mathscr{F}^* p(t) = \int_{\mathbb{R}} e^{it\gamma} p(\gamma) d\gamma, \qquad t \in \mathbb{R}. \tag{4.2}$$

Assume that $p(\gamma) = 0$ for $\gamma < 0$. Then for $p \in L^2(0,\infty)$ the inverse Fourier transform $\mathscr{F}^* p(t)$ can be extended analytically to $\mathbb{C}^+ :\ t \in \mathbb{C}^+$. By Paley–Wiener theorem [6], $\mathscr{F}^*$ is a one to one norm–preserving mapping of $L^2(0,\infty)$ onto H^2_+. By the inversion formula, we have

$$p(\gamma) = \mathscr{F} \cdot \mathscr{F}^* p(\gamma) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathscr{F}^* p(t) e^{-it\gamma} dt. \tag{4.3}$$

Let $\Lambda = \{\lambda_n,\ n \in \mathbb{Z}'\}$ be a fixed subset of $\mathbb{C}^+$ and let $a > 0$. Clearly, the family $\left\{e^{i\lambda_n x}\right\}_{n\in\mathbb{Z}'}$ forms an unconditional basis in $L^2(0,a)$ if and only if the family $\left\{e^{-i\overline{\lambda}_n x}\right\}_{n\in\mathbb{Z}'}$ does. Let $\Lambda^* = \left\{-\overline{\lambda}_n :\ n \in \mathbb{Z}'\right\}$. The inverse Fourier transformation $\mathscr{F}^*$ maps the closed linear span $\mathscr{E}_\Lambda$ of the family $\left\{e^{i\lambda x}\chi_{[0,\infty)}\right\}_{\lambda\in\Lambda^*}$ (we use the notation χ_A for the characteristic function of a set A) onto the subspace K defined by [6]

$$K = H^2_+ \ominus BH^2_+ \ . \tag{4.4}$$

Here $B(z)$ is the Blaschke product with the zeros $\{\lambda_n\}_{n\in\mathbb{Z}'}$ defined by

$$B(z) = \prod_{n\in\mathbb{Z}'} \varepsilon_n \ \frac{1-\frac{z}{\lambda_n}}{1-\frac{z}{\overline{\lambda}_n}} \ , \tag{4.5}$$

where ε_n is such that $|\varepsilon_n| = 1$ and ε_n makes each factor in the above product non–negative at the point $z = i$. By BH^2_+ we denote the subspace of functions of the form $B(z)g(z)$, where $g \in H^2_+$. The well–known Blaschke condition

$$\sum_{n\in\mathbb{Z}'} \frac{\Im\lambda_n}{|\lambda_n + i|^2} < \infty \tag{4.6}$$

is necessary and sufficient for the convergence of product (4.5).

Notice, $\mathscr{F}^*\left(e^{-i\overline{\lambda}\gamma}\chi_{[0,\infty)}\right)(z) = i\left(z-\overline{\lambda}\right)^{-1}$ and the closed linear span of simple fractions $\left\{\left(z-\overline{\lambda}\right)^{-1}\right\}_{\lambda\in\Lambda}$ coincides with K. In what follows, we need one more notion. We say that the set Λ satisfies the Carleson condition [1, 6] and write $\Lambda \in (\mathscr{C})$ if

$$\delta = \inf_n \prod_{k \neq n} \left| \frac{\lambda_k - \lambda_n}{\lambda_k - \overline{\lambda}_n} \right| > 0. \tag{4.7}$$

Obviously, condition (4.7) implies (4.6). Now we reproduce the following statement from [6].

Theorem 4.1 *Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}'\} \subset \mathbb{C}^+$. The following assertions are equivalent.*

1. *The family $\left\{ \left(z - \overline{\lambda}_n \right)^{-1} \right\}_{n \in \mathbb{Z}'}$ forms an unconditional basis in its closed linear span in H^2_+, i.e., in $K = H^2_+ \ominus BH^2_+$.*

2. *Each family $\left\{ e^{i\lambda x} \chi_{[0,\infty]} \right\}_{\lambda \in \Lambda}$* and $\left\{ e^{i\lambda x} \chi_{[0,\infty]} \right\}_{\lambda \in \Lambda^*}$ *forms an unconditional basis in its own closed linear span in $L^2(0, \infty)$.*

3. $\Lambda \in (\mathscr{C})$.

Corollary 4.2. Consider two sets of functions given by the formulas

$$\varphi_n(z) = \frac{\sqrt{2\ \Im \lambda_n}}{z - \overline{\lambda}_n}, \qquad \varphi_n^*(z) = \frac{\sqrt{2\ \Im \lambda_n}}{z - \overline{\lambda}_n} \ \frac{B_n(z)}{B_n(\lambda_n)}, \qquad n \in \mathbb{Z}', \tag{4.8}$$

with $B_n(z)$ being defined as

$$B_n(z) = \prod_{m \in \mathbb{Z}',\, m \neq n} \varepsilon_m \ \frac{1 - \frac{z}{\lambda_m}}{1 - \frac{z}{\overline{\lambda}_m}} \ . \tag{4.9}$$

Then $\{\varphi_n\}_{n \in \mathbb{Z}'}$ is a normalized Riesz basis in K and $\{\varphi_n^*\}_{n \in \mathbb{Z}'}$ is the corresponding biorthogonal basis.

Finally, we reproduce the interpolation theorem in the form convenient for applications [6].

Theorem 4.3. *The Carleson condition $\Lambda \in (\mathscr{C})$ is necessary and sufficient condition for solvability of the following interpolation problem in H^2_+. Let $\{a_n\}_{n \in \mathbb{Z}'}$ be any sequence from $l^2(\mathbb{Z}')$. Find an analytic function $q \in H^2_+$ such that*

$$q(\lambda_n) \sqrt{2\ \Im \lambda_n} = a_n, \qquad n \in \mathbb{Z}'. \tag{4.10}$$

The general solution can be given by an explicit formula

$$q(z) = i \sum_{n \in \mathbb{Z}'} \frac{\sqrt{2\ \Im \lambda_n}}{z - \overline{\lambda}_n} \ \frac{B_n(z)}{B_n(\lambda_n)} \ a_n + B(z) g(z), \qquad g \in H^2_+ \ . \tag{4.11}$$

The series converges unconditionally for every $\{a_n\}_{n \in \mathbb{Z}'} \in l^2(\mathbb{Z}')$. The solution corresponding to

$g = 0$ *has the minimal norm among all solutions and belongs to* $K = H^2_+ \ominus BH^2_+$.

Transformation of the formula for the voltage output. From this point on we assume that the set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}'}$ is the spectrum of the dynamics generator $\mathscr{L}$. We point out that Λ satisfies the Carleson condition (4.7), i.e., $\Lambda \in (\mathscr{C})$. Indeed, it is a well known fact [6] that for (4.7) to hold the following two conditions are sufficient: (a) the set Λ is contained in a strip parallel to the real axis and (b) the points of Λ are separated, i.e., $\inf_{n,\,m}|\lambda_m - \lambda_n| > 0$. It follows immediately from the spectral asymptotics (2.13) that both conditions are satisfied.

Without loss of generality we assume $\Lambda \subset \mathbb{C}^+$. Before formulating the control problems we modify the formula for the voltage output $\mathbf{v}(t)$ from (3.9). Let

$$\widetilde{u^0_n} = -h\ \frac{u^0_n\ \psi_n'(L)}{H + i\lambda_n}\,, \qquad \widetilde{g_n} = -h\ \frac{g_n \psi_n'(L)}{H + i\lambda_n}\,, \tag{4.12}$$

then formula (3.9) can be written in the form

$$\mathbf{v}(t) = \sum_{n\in\mathbb{Z}'} \widetilde{u^0_n}\ e^{i\lambda_n t} + \sum_{n\in\mathbb{Z}'} \widetilde{g_n}\ \int_0^t f(\tau) e^{i\lambda_n (t-\tau)} d\tau\,. \tag{4.13}$$

From this moment on we impose the following condition on the control function $f(t)$: $f(t) = 0$ for $t > 0$ and $f \in L^2(0,\infty)$. For any function $r(t)$ that satisfies this condition the Laplace transform $\widehat{r}(\mu) = \int_0^\infty e^{-\mu\, t}\, r(t) dt$ is well defined and is an analytic function in the right half–plane of the μ–complex plane. Moreover, if we replace the variable μ by $\nu = i\mu$, then the function

$$R(\nu) = \widehat{r}(-i\nu) = \mathscr{F}^* r(\nu) \tag{4.14}$$

is exactly the inverse Fourier transform of $r(t)$ analytically extended to the upper half–plane $\mathbb{C}^+$ (see (4.2) for the definition of $\mathscr{F}^*$). In the sequel, (4.14) will be called *the modified Laplace transform of* $r(t)$.

Let us apply the Laplace transformation to both sides of Eq.(4.13). Let $\widehat{\mathbf{v}}(\mu)$ and $\widehat{f}(\mu)$ be the Laplace transforms of the voltage and control functions respectively. Then from (4.13) we get

$$\begin{aligned}
\widehat{\mathbf{v}}(\mu) &= \sum_{n\in\mathbb{Z}'} \widetilde{u^0_n} \int_0^\infty e^{-\mu\tau} e^{i\lambda_n \tau} d\tau + \sum_{n\in\mathbb{Z}'} \widetilde{g_n} \int_0^\infty e^{-\mu t} dt \int_0^t f(\tau) e^{i\lambda_n (t-\tau)} d\tau \\
&= \sum_{n\in\mathbb{Z}'} \frac{\widetilde{u^0_n}}{\mu - i\lambda_n} + \sum_{n\in\mathbb{Z}'} \widetilde{g_n} \int_0^\infty \left(\frac{1}{-\mu + i\lambda_n} e^{-(\mu - i\lambda_n)t}\right)' dt \int_0^t f(\tau) e^{-i\lambda_n \tau} d\tau \\
&= \sum_{n\in\mathbb{Z}'} \frac{\widetilde{u^0_n}}{\mu - i\lambda_n} + \sum_{n\in\mathbb{Z}'} \frac{\widetilde{g_n}}{\mu - i\lambda_n} \int_0^\infty e^{-(\mu - i\lambda_n)t} f(t) e^{-i\lambda_n t} dt \\
&= -h \left\{ \sum_{n\in\mathbb{Z}'} \frac{u^0_n\ \psi_n'(L)}{H + i\lambda_n}\ \frac{1}{\mu - i\lambda_n} + \widehat{f}(\mu) \sum_{n\in\mathbb{Z}'} \frac{g_n\ \psi_n'(L)}{H + i\lambda_n}\ \frac{1}{\mu - i\lambda_n} \right\}, \qquad \Re\,\mu > 0.
\end{aligned} \tag{4.15}$$

On the third step of (4.15) we have used the integration by parts and have taken into account that the out of integral trems at $t=0$ and at $t=\infty$ vanish due to our assumptions on $f(t)$. Since $\{\lambda_n\}_{n\in\mathbb{Z}'}\subset\mathbb{C}^+$, the functions at the right hand–side of Eq.(4.15) are analytic in the right half–plane of the μ–complex plane. To deal with the problem in H^2_+, we replace μ with ν by the rule $\nu=i\mu$ and rewrite (4.15) as

$$\widehat{\mathbf{v}}(-i\nu)=h\sum_{n\in\mathbb{Z}'}\frac{u_n^0\,\psi_n'(L)}{H+i\lambda_n}\;\frac{1}{i(\nu+\lambda_n)}+h\left[\sum_{n\in\mathbb{Z}'}\frac{g_n\psi_n'(L)}{H+i\lambda_n}\;\frac{1}{i(\nu+\lambda_n)}\right]\widehat{f}(-i\nu). \tag{4.16}$$

Using the symmetry of the spectrum with respect to the imaginary axis, i.e., $\lambda_{-n}=-\overline{\lambda}_n$, introducing new notations

$$\mathbf{V}(\nu)=\widehat{\mathbf{v}}(-i\nu),\qquad F(\nu)=\widehat{f}(-i\nu), \tag{4.17}$$

and changing the index of summation to $m=-n$, we rewrite Eq.(4.16) in the form

$$\mathbf{V}(\nu)=-h\left\{\sum_{m\in\mathbb{Z}'}\frac{i\,u^0_{-m}\;\psi'_{-m}(L)}{H+i\lambda_{-m}}\;\frac{1}{\nu-\overline{\lambda}_m}+\left[\sum_{m\in\mathbb{Z}'}\frac{i\,g_{-m}\psi'_{-m}(L)}{H+i\lambda_{-m}}\;\frac{1}{\nu-\overline{\lambda}_m}\right]F(\nu)\right\}. \tag{4.18}$$

Introducing two sequences

$$\chi_m=\frac{-i\,h\,u^0_{-m}\;\psi'_{-m}(L)}{\sqrt{2\;\Im\lambda_m}\;\;(H+i\lambda_{-m})}\;,\qquad G_m=-\frac{i\,h\,g_{-m}\psi'_{-m}(L)}{\sqrt{2\;\Im\lambda_m}\;\;(H+i\lambda_{-m})}\;, \tag{4.19}$$

we rewrite Eq.(4.18) in the form

$$\mathbf{V}(\nu)=\sum_{n\in\mathbb{Z}'}\chi_n\;\frac{\sqrt{2\;\Im\lambda_n}}{\nu-\overline{\lambda}_n}+F(\nu)\sum_{n\in\mathbb{Z}'}G_n\;\frac{\sqrt{2\;\Im\lambda_n}}{\nu-\overline{\lambda}_n}\;. \tag{4.20}$$

Now we are in a position to formulate the *second and third control problems.*

The second control problem. *Let* $\mathbf{g},U_0\in\mathscr{H}$ *(the force profile function and the initial state from (3.1)) be given, i.e.,* $\mathbf{g}(x)=\sum_{n\in\mathbb{Z}'}g_n\;\Psi_n(x)$ *and* $U_0(x)=\sum_{n\in\mathbb{Z}'}u_n^0\;\Psi_n(x)$, $\{g_n\}_{n\in\mathbb{Z}'}\in l^2(\mathbb{Z}')$ *and* $\left\{u_n^0\right\}_{n\in\mathbb{Z}'}\in l^2(\mathbb{Z}')$. *Can we find such control function* $f(t)$, $0\le t<\infty$, *that the voltage output is equal to some given function* $\mathbf{v}(t)$*?*

Assumption 4.4. Let us agree that the given function $\mathbf{v}(t)$ (which is a signal to be tracked) is taken from a specific subspace $\mathfrak{M}$ of $L^2(0,\infty)$. Namely, let $\mathfrak{M}$ be a closed linear span of the set $\left\{e^{i\lambda_n t}\right\}_{n\in\mathbb{Z}'}$ in $L^2(0,\infty)$. Then $\mathbf{v}(t)$ can be represented in the form

$$\mathbf{v}(t)=\sum_{n\in\mathbb{Z}'}\mathbf{v}_n\,e^{i\lambda_n t},\qquad\{\mathbf{v}_n\}_{n\in\mathbb{Z}'}\in l^2(\mathbb{Z}'). \tag{4.21}$$

From this moment on we assume that on the one hand $\mathbf{v}(t)$ is the voltage output, i.e., the third component of the solution $U(x,t)$ of the problem (3.1). Therefore $\mathbf{v}(t)$ is given by the formula (3.9) or (4.13) and it's modified Laplace transform is given by (4.20). On the other hand we assume that $\mathbf{v}(t)$ is the prescribed signal to be tracked defined in (4.21).

Applying the modified Laplace transformation (4.14) to such function $\mathbf{v}(t)$, we get

$$\mathbf{V}(\nu) = \sum_{n\in\mathbb{Z}'} \frac{i\mathbf{v}_n}{\nu+\lambda_n} = \sum_{n\in\mathbb{Z}'} \mathbf{V}_n \frac{\sqrt{2\ \Im\lambda_n}}{\nu-\overline{\lambda}_n}, \text{ where } \mathbf{V}_n = \frac{i\mathbf{v}_{-n}}{\sqrt{2\ \Im\lambda_n}} \ . \tag{4.22}$$

In terms of (4.22), Eq.(4.20) can be rewritten in the form

$$\sum_{n\in\mathbb{Z}'} \mathbf{V}_n \ \frac{\sqrt{2\ \Im\lambda_n}}{\nu-\overline{\lambda}_n} = \sum_{n\in\mathbb{Z}'} \chi_n \ \frac{\sqrt{2\ \Im\lambda_n}}{\nu-\overline{\lambda}_n} + F(\nu)\sum_{n\in\mathbb{Z}'} G_n \ \frac{\sqrt{2\ \Im\lambda_n}}{\nu-\overline{\lambda}_n}. \tag{4.23}$$

Proposition 4.5. *The following two functions are well defined and belong to the space* $K \subset H^2_+$*:*

$$\mathbb{V}(\nu) = \sum_{m\in\mathbb{Z}'} (\mathbf{V}_m - \chi_m) \ \frac{\sqrt{2\ \Im\lambda_m}}{\nu-\overline{\lambda}_m} \quad \text{and} \quad \mathbb{G}(\nu) = \sum_{m\in\mathbb{Z}'} G_m \ \frac{\sqrt{2\ \Im\lambda_m}}{\nu-\overline{\lambda}_m} \ . \tag{4.24}$$

Proof. Since the simple fractions $\varphi_m(\nu) = \sqrt{2\ \Im\lambda_m}\,(\nu-\overline{\lambda}_m)^{-1}$ form a Riesz basis in K, it is sufficient to show that the sequences $\{\chi_n\}_{n\in\mathbb{Z}'}$, $\{G_n\}_{n\in\mathbb{Z}'}$, $\{\mathbf{V}_n\}_{n\in\mathbb{Z}'}$ defined in (4.18) and (4.22) belong to the space $l^2(\mathbb{Z}')$. As shown in [11], the following estimate holds: $|\psi_n'(L)| \asymp |n|$. From the asymptotics (2.13) one gets $|\lambda_n| \asymp n^2$ and $|\Im\lambda_n| \asymp 1$. Using these facts and owing to formulae (4.19), (4.22), and Assumption 4.4, we immediately obtain the desired result. (In fact it is easy to see that all these sequences belong to $l^p(\mathbb{Z}')$ for any $p>1$. However, we do not use this fact at the present point.) ■

In terms of functions (4.24), Eq.(4.23) can be rewritten in the form

$$\mathbb{V}(\nu) = F(\nu)\mathbb{G}(\nu). \tag{4.25}$$

Now we can give an equivalent reformulation of the above control problem. *Given 3 sequences* $\{V_n\}_{n\in\mathbb{Z}'}$, $\{\chi_n\}_{n\in\mathbb{Z}'}$, $\{G_n\}_{n\in\mathbb{Z}'}$, *under what conditions on those sequences, there exists a function* $F \in H^2_+$ *such that Eq.(4.25) holds? Can the solution be given by an explicit formula?*

The answers to the above questions are given in Theorem 4.6 below. To formulate this theorem we need the following notations.

Let $\{\mathbb{V}_n\}_{n\in\mathbb{Z}'}$ and $\{\mathbb{G}_n\}_{n\in\mathbb{Z}'}$ be two sequences defined by

$$\{\mathbb{V}_n = \mathbb{V}(\lambda_n)\}_{n\in\mathbb{Z}'} \quad \text{and} \quad \{\mathbb{G}_n = \mathbb{G}(\lambda_n)\}_{n\in\mathbb{Z}'} \,. \tag{4.26}$$

Now we present the main result on the second control problem..

Theorem 4.6. *Let Assumption 4.4 on the output* $\mathbf{v}(t)$ *be satisfied. Let* $\{\mathbb{V}_m\}_{m\in\mathbb{Z}'}$ *and* $\{\mathbb{G}_m\}_{m\in\mathbb{Z}'}$ *be the sequences defined in (4.26) in terms of functions (4.24), which in turn are defined in terms of the sequences from (4.19) and (4.22). Let* $\{\beta_m\}_{m\in\mathbb{Z}'}$ *be a new sequence defined by*

$$\beta_m = \frac{\mathbb{V}_m}{\mathbb{G}_m}, \qquad m \in \mathbb{Z}' \,. \tag{4.27}$$

If $\{\beta_m\}_{m\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$*, then equation (4.25) for an unknown function* $F(\nu)$ *has a solution in* H^2_+*. This solution is not unique. It can be given in the form*

$$\mathbb{F}(\nu) = F(\nu) + B(\nu)J(\nu), \tag{4.28}$$

where $F(\nu)$ *is defined uniquely by*

$$F(\nu) = i \sum_{n\in\mathbb{Z}'} \beta_n \, \frac{2\ \Im\lambda_n}{\nu - \overline{\lambda}_n} \, \frac{B_n(\nu)}{B_n(\lambda_n)}, \qquad F \in K \tag{4.29}$$

and $J(\nu)$ *is any function from* H^2_+*. Solution* $F(\nu)$ *has a minimal norm. Having* $\mathbb{F}(\nu)$ *from (4.28) one can reconstruct the desired control function* $f(t)$ *by using the change of variable* $\mu = -i\nu$ *and applying the inverse Laplace transformation, which is equivalent to restricting* $\mathbb{F}(\nu)$ *to the real axis and applying the Fourier transformation (4.3)*

Proof. The proof follows as an immediate application of Theorem 4.3. Indeed, substitute $\nu = \lambda_n$ into Eq.(4.25) and rewrite the result in the form

$$F\,(\lambda_n) = \beta_n, \qquad n \in \mathbb{Z}'. \tag{4.30}$$

It remains to notice that (4.30) coincides with (4.10) from Theorem 4.3 if one replaces q by F and takes $a_n = \beta_n\sqrt{2\ \Im\lambda_n}$. (Recall that $|a_n| \asymp |\beta_n|$ since $|\Im\lambda_n| \asymp 1$ and, therefore, $\{a_n\}_{n\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$). Formulae (4.28) and (4.29) for the solution are just (4.11) written in terms of F and β_n. ■

In the conclusion of our discussion of the second control problem we point out a clear similarity between the moment problem (3.12) in Sec.3 and the factorization problem (4.25) considered in this section. In both cases the main condition for the solvability of the problem under consideration was the requirement that the ratio of two sequences belonged to the class $l^2(\mathbb{Z}')$ (see (3.23)) and (4.27)). In the case of (3.23) both sequences, $\{u_n^0\}$ and $\{g_n\}$, were themselves sequences from

$l^2(\mathbb{Z}')$. It turns out that the same is true for the sequences from (4.26). Also the computations that we carry out in the proof of Proposition 4.7 will be used below in the analysis of the third control problem.

Proposition 4.7. *Each sequence* $\{\mathbb{V}_n\}_{n\in\mathbb{Z}'}$ *and* $\{\mathbb{G}_n\}_{n\in\mathbb{Z}'}$ *belongs to the space* $l^2(\mathbb{Z}')$.

Proof. Let $\{a_n\}_{n\in\mathbb{Z}'}$ be any sequence from $l^2(\mathbb{Z}')$. Since the set of simple fractions forms a Riesz basis in $K = H^2_+ \ominus BH^2_+$, the series

$$\mathscr{A}(\nu) = \sum_{n\in\mathbb{Z}'} a_n \, \frac{\sqrt{2\,\Im\lambda_n}}{\nu - \overline{\lambda}_n} \tag{4.31}$$

defines an analytic function from K. Consider the numerical sequence $\{\mathscr{A}_m\}_{m\in\mathbb{Z}'}$ defined by

$$\mathscr{A}_m \equiv \mathscr{A}(\lambda_m) = \sum_{n\in\mathbb{Z}'} a_n \, \frac{\sqrt{2\,\Im\lambda_n}}{\lambda_m - \overline{\lambda}_n} \ . \tag{4.32}$$

Notice, all we need to show is that $\{\mathscr{A}_m\}_{m\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$. Indeed, the sequences $\{\mathbb{V}_n\}$ and $\{\mathbb{G}_n\}$ are particular cases of $\{\mathscr{A}_n\}$ with $a_n = \mathbf{V}_n - \chi_n$ or $a_n = G_n$ respectively. First we notice that $\left(\lambda_m - \overline{\lambda}_n\right)^{-1}$ can be obtained as a result of the contour integration over the closed contour C_R, which consists of the semi–circle $|\xi| = R$, $\Im\xi > 0$ and the interval $[-R,R]$, i.e.,

$$\begin{aligned} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\left(\xi-\overline{\lambda}_n\right)(\xi-\lambda_m)} = \frac{1}{2\pi i}\ \lim_{R\to\infty}\Bigg[\int_{-R}^{R} \frac{d\xi}{\left(\xi-\overline{\lambda}_n\right)(\xi-\lambda_m)} + \\ \int_0^{\pi} \frac{Rie^{i\varphi}d\varphi}{\left(Re^{i\varphi}-\overline{\lambda}_n\right)(Re^{i\varphi}-\lambda_m)}\Bigg] = \frac{1}{2\pi i}\ \lim_{R\to\infty}\int_{C_R} \frac{d\xi}{\left(\xi-\overline{\lambda}_n\right)(\xi-\lambda_m)} = \frac{1}{\lambda_m-\overline{\lambda}_n}\ . \end{aligned} \tag{4.33}$$

Here on the last step, we used the Residue Theorem. Using (4.33), we can represent the sequence $\{\mathscr{A}_m\}_{m\in\mathbb{Z}'}$, in the form

$$\mathscr{A}_m = \frac{1}{2\pi i\sqrt{2\,\Im\lambda_m}} \ \sum_{n\in\mathbb{Z}'} a_n \int_{-\infty}^{\infty} \frac{\sqrt{2\,\Im\lambda_n}\ \sqrt{2\,\Im\lambda_m}}{\left(\xi-\overline{\lambda}_n\right)(\xi-\lambda_m)}\ d\xi, \tag{4.34}$$

which means that the sequence $\left\{2\pi i\sqrt{2\,\Im\lambda_m}\ \mathscr{A}_m\right\}_{m\in\mathbb{Z}'}$ can be viewed as the result of application of the Gram matrix $[\langle\varphi_n,\varphi_m\rangle]_{m,n\in\mathbb{Z}'}$ (φ_n is defined in (4.8)) to the vector $\{a_n\}_{n\in\mathbb{Z}'}$ from $l^2(\mathbb{Z}')$. (By the angular brackets we denote the standard inner product in $L^2(\mathbb{R})$.) As is known [5, 6], the Gram matrix built on a Riesz basis vectors generates a bounded and boundedly invertible operator in $l^2(\mathbb{Z}')$. Since in our case, $|\Im\lambda_m| \asymp 1$, the desired result, $\{\mathscr{A}_m\}_{m\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$, follows immediately.

■

The third control problem. As shown in Theorem 4.6, the solution of the second control problem depends on the properties of the force profile function, $\mathbf{g}(x)$. Namely, the desired control function $f(t)$ exists only if the signal to be tracked, $\mathbf{v}(t)$, satisfies two conditions: a) Assumption 4.4; b) the sequence $\{\beta_m\}_{m\in\mathbb{Z}'}$ defined in (4.27) must belong to $l^2(\mathbb{Z}')$. It follows from the second condition b) that the class of admissible signals that can be tracked depends on the force profile function $\mathbf{g}(x)$. Now our goal is to relax conditions induced by external factors (such as the force profile function) and using the properties of the voltage output $\mathbf{v}(t)$ only to solve the next control problem. Namely, let us assume that we are given just $\mathbf{v}(t)$. The main question is the following. *Can one construct explicitly the control (both the force profile function,* $\mathbf{g}(x)$*, and the time factor,* $f(t)$*) in such a way that the output voltage is exactly equal to* $\mathbf{v}(t)$ with the class of admissible signals to be tracked, $\mathbf{v}(t)$, being independent of $\mathbf{g}(x)$. In the answer to this question, we give an explicit construction of $\mathbf{g}$ and f in terms of their modified Laplace transforms (see (4.14)).

Let again $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}'}$ be the spectrum of the dynamics generator $\mathscr{L}$ defined in (2.8), (2.10). Let Assumption 4.4 be satisfied, i.e., the voltage signal to be tracked, $\mathbf{v}(t)$ is given by (4.21).

Now we are in a position to reformulate the third control problem.

Given the coefficients $\{\mathbf{v}_n\}_{n\in\mathbb{Z}'}$ *from (4.21) find* $\mathbf{g}(x)$ and $f(t)$ *in such a way that the voltage output produced by problem* (3.1) *is exactly equal to* $\mathbf{v}(t)$ *of* (4.21).

Lemma 4.8. *Let* $\mathbf{V}(\nu)$ *be the modified Laplace transform (4.22) of the function* $\mathbf{v}(t)$ *from (4.21) and let* $\left\{\widetilde{\mathbf{V}}_n\right\}_{n\in\mathbb{Z}'}$ *be the sequence defined by* $\widetilde{\mathbf{V}}_n = \mathbf{V}(\lambda_n)$, $n\in\mathbb{Z}'$*. Assume that there exists a numerical sequence* $\{\alpha_n\}_{n\in\mathbb{Z}'}\in l^1(\mathbb{Z}')$ *such that*

$$\left\{\alpha_n^{-1}\widetilde{\mathbf{V}}_n\right\}_{n\in\mathbb{Z}'}\in l^2(\mathbb{Z}'). \tag{4.35}$$

Let the functions $\mathbf{G}$ *and* $\mathbf{F}$ *be defined by the formulae*

$$\mathbf{G}(\nu) = \sum_{m\in\mathbb{Z}'} j_m\,\frac{\sqrt{2\ \Im\lambda_m}}{\nu-\overline{\lambda}_m},\qquad j_m = -\frac{2\sqrt{2\ \Im\lambda_m}}{\overline{B_m(\lambda_m)}}\,\varepsilon_m\sum_{n\in\mathbb{Z}'}\frac{\sqrt{2\ \Im\lambda_n}\ \bar{\varepsilon}_n\left(\alpha_n^{-1}\widetilde{\mathbf{V}}_n\right)}{\left(\lambda_n-\overline{\lambda}_m\right)B_n(\lambda_n)}, \tag{4.36}$$

$$\mathbf{F}(\nu) = \sum_{m\in\mathbb{Z}'} h_m\,\frac{\sqrt{2\ \Im\lambda_m}}{\nu-\overline{\lambda}_m},\qquad h_m = -\frac{2\sqrt{2\ \Im\lambda_m}}{\overline{B_m(\lambda_m)}}\,\varepsilon_m\sum_{n\in\mathbb{Z}'}\frac{\sqrt{2\ \Im\lambda_n}\ \bar{\varepsilon}_n(2\ \Im\lambda_n\ \alpha_n)}{\left(\lambda_n-\overline{\lambda}_m\right)B_n(\lambda_n)}. \tag{4.37}$$

Then the following statements hold:

a) $\quad \{j_m\}_{m\in\mathbb{Z}'}\in l^2(\mathbb{Z}'),\qquad \{h_m\}_{m\in\mathbb{Z}'}\in l^2(\mathbb{Z}'),$

and hence the series for $\mathbf{G}$ *and* $\mathbf{F}$ *from (4.36) and (4.37) are convergent in the Hardy space* H^2_+ *and define functions from the subspace* $K\subset H^2_+$*, where* K *is defined in (4.4).*

b) The following factorization of $\mathbf{V}(\nu)$ *holds:*

$$\mathbf{V}(\nu) = \mathbf{G}(\nu)\mathbf{F}(\nu). \tag{4.38}$$

The proof of the lemma is given after the formulation of Theorem 4.9 below. We point out that Lemma 4.8 and Theorem 4.9 would still remain valid if we relax the assumption on $\{\alpha_n\}_{n\in\mathbb{Z}'}$. It is sufficient to assume that $\{\alpha_n\}_{n\in\mathbb{Z}'} \in l^p(\mathbb{Z}')$ with any $1 \le p < 2$.

Our main result on the third control problem is the following statement.

Theorem 4.9. *Let a function* $\mathbf{v}(t)$ *be given as an expansion (4.21) and let its modified Laplace transform satisfy condition (4.35) of Lemma 4.8. Let* $\mathbf{g}$ *and* f *be defined as*

$$\mathbf{g}(x) = \sum_{n\in\mathbb{Z}'} r_n\, j_{-n}\, \Psi_n(x), \qquad r_n = \frac{-i(H+i\lambda_n)}{h\psi_n'(L)}\sqrt{2\ \Im\lambda_n}, \qquad x\in[0,L], \tag{4.39}$$

(notice that $\psi_n'(L) \neq 0$ since, as shown in [11], $\psi_n'(L)\psi_n(L) > 0$),

$$f(t) = \sum_{n\in\mathbb{Z}'} h_n\, \frac{1}{\sqrt{2\ \Im\lambda_n}}\, e^{i\lambda_n t}, \qquad t \ge 0, \tag{4.40}$$

where $\{j_n\}_{n\in\mathbb{Z}'}$ *and* $\{h_n\}_{n\in\mathbb{Z}'}$ *are given in Lemma 4.8, the constants* H *and* h *are defined in (2.1), and* $\Psi_n(x)$ *are the eigenvectors (2.32) of the dynamics generator* $\mathscr{L}$*. Then the following statements hold.*

a) *The series in (4.39) converges in the space* $\mathbb{H} = H^1(0,L)\times H^{-1}(0,L)\times\mathbb{C}$ *and, therefore, defines* $\mathbf{g}(x)$ *as an element of this space.*

b) *The series in (4.40) converges in* $L^2(0,\infty)$ *and defines* $f(t)$ *as an element of the subspace* $\mathfrak{M} \subset L^2(0,\infty)$ *defined in Assumption 4.4.*

c) *If one takes the function* $\mathbf{g}(x)f(t)$ *as the forcing term in Eq.(3.1), then the voltage component of the solution of (3.1) coincides with* $\mathbf{v}(t)$*.*

Proof of Lemma 4.8. 1) Using Theorem 4.3 and condition (4.35) one can check that there exists a unique function $\widetilde{\mathbf{G}} \in K$, which solves the following interpolation problem and has the minimal H^2_+– norm:

$$\widetilde{\mathbf{G}}(\lambda_n)\sqrt{2\ \Im\lambda_n} = \alpha_n^{-1}\widetilde{\mathbf{V}}_n, \qquad n\in\mathbb{Z}'. \tag{4.41}$$

This function, $\widetilde{\mathbf{G}}$, can be given by the formula, which follows from (4.11), i.e.,

$$\widetilde{\mathbf{G}}(\nu) = i\sum_{n\in\mathbb{Z}'} \frac{\sqrt{2\ \Im\lambda_n}}{\nu-\overline{\lambda}_n}\, \frac{B_n(\nu)}{B_n(\lambda_n)}\left(\alpha_n^{-1}\widetilde{\mathbf{V}}_n\right). \tag{4.42}$$

The calculation presented below shows that the function $\mathbf{G}(\nu)$ defined in (4.36) coincides with the function $\widetilde{\mathbf{G}}$ defined in (4.41) and (4.42).

It is clear that (4.42) represents an expansion of $\widetilde{\mathbf{G}}$ with respect to the Riesz basis $\{\varphi_n^*\}_{n\in\mathbb{Z}'}$ in the space K, which is biorthogonal to the Riesz basis $\{\varphi_n\}_{n\in\mathbb{Z}'}$ of simple fractions (see Corollary 4.2). Let us expand the function $\widetilde{\mathbf{G}}$ given in (4.42) with respect to the basis $\{\varphi_n\}_{n\in\mathbb{Z}'}$. We denote the coefficients of this expansion by $\widetilde{j}_m$ and demonstrate that they are given by the same formula as in (4.36), i.e., $\widetilde{j}_m = j_m$ and, therefore, $\widetilde{\mathbf{G}} = \mathbf{G}$. We have

$$\widetilde{\mathbf{G}}(\nu) = \sum_{m\in\mathbb{Z}'} \widetilde{j}_m \ \frac{\sqrt{2\ \Im\lambda_m}}{\nu - \overline{\lambda}_m},$$

where

$$\widetilde{j}_m = \langle \widetilde{\mathbf{G}}, \varphi_m^* \rangle_{H^2_+} = \frac{1}{\pi}\int_{-\infty}^{\infty} \widetilde{\mathbf{G}}(\xi) \ \frac{\sqrt{2\ \Im\lambda_m}}{\xi - \lambda_m} \ \frac{\overline{B_m(\xi)}}{\overline{B_m(\lambda_m)}} \ d\xi. \tag{4.43}$$

Here we use the formula for the inner product in H^2_+ (see [4, 6] or (4.1) and the subsequent remark). Using formula (4.42) for $\widetilde{\mathbf{G}}$ we represent (4.43) in the form

$$\widetilde{j}_m = \frac{i}{\pi}\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}'} \frac{\sqrt{2\ \Im\lambda_n}}{\xi - \overline{\lambda}_n} \ \frac{B_n(\xi)}{B_n(\lambda_n)} \left(\alpha_n^{-1}\widetilde{\mathbf{V}}_n\right) \frac{\sqrt{2\ \Im\lambda_m}}{\xi - \lambda_m} \ \frac{\overline{B_m(\xi)}}{\overline{B_m(\lambda_m)}} \ d\xi. \tag{4.44}$$

By definition (4.9) of $B_n(\nu)$, one can write

$$B_n(\nu) = \left(\prod_{\kappa\neq n,\ \kappa\neq m} \varepsilon_\kappa \frac{\nu - \lambda_\kappa}{\nu - \overline{\lambda}_\kappa}\right) \varepsilon_m \frac{\nu - \lambda_m}{\nu - \overline{\lambda}_m}, \qquad B_m(\nu) = \left(\prod_{\kappa\neq n,\ \kappa\neq m} \varepsilon_\kappa \frac{\nu - \lambda_\kappa}{\nu - \overline{\lambda}_\kappa}\right) \varepsilon_n \frac{\nu - \lambda_n}{\nu - \overline{\lambda}_n}, \tag{4.45}$$

which yields the following representation for the product of $B_n(\nu)$ and $\overline{B_m(\nu)}$ when $\nu \in \mathbb{R}$:

$$B_n(\nu)\overline{B_m(\nu)} = \bar{\varepsilon}_n \varepsilon_m \ \frac{\nu - \lambda_m}{\nu - \overline{\lambda}_m} \ \frac{\nu - \overline{\lambda}_n}{\nu - \lambda_n}. \tag{4.46}$$

Substituting (4.46) into (4.44), we evaluate the integral and show that $\widetilde{j}_m$ is given by the same formula (4.36) as j_m:

$$\begin{aligned}\widetilde{j}_m &= \frac{i}{\pi}\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}'} \frac{\sqrt{2\ \Im\lambda_n}\sqrt{2\ \Im\lambda_m}}{\left(\xi - \overline{\lambda}_n\right)(\xi - \lambda_m)} \ \varepsilon_m\bar{\varepsilon}_n \ \frac{(\xi - \lambda_m)\left(\xi - \overline{\lambda}_n\right)}{\left(\xi - \overline{\lambda}_m\right)(\xi - \lambda_n)} \ \frac{\left(\alpha_n^{-1}\widetilde{\mathbf{V}}_n\right)}{B_n(\lambda_n)\overline{B_m(\lambda_m)}} \ d\xi \\ &= \frac{i}{\pi}\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}'} \frac{\sqrt{2\ \Im\lambda_n}\sqrt{2\ \Im\lambda_m}\varepsilon_m\bar{\varepsilon}_n\left(\alpha_n^{-1}\widetilde{\mathbf{V}}_n\right)}{(\xi - \lambda_n)\left(\xi - \overline{\lambda}_m\right)B_n(\lambda_n)\overline{B_m(\lambda_m)}} = -2\sum_{n\in\mathbb{Z}'} \frac{\sqrt{2\ \Im\lambda_n}\sqrt{2\ \Im\lambda_m}\varepsilon_m\bar{\varepsilon}_n\left(\alpha_n^{-1}\widetilde{\mathbf{V}}_n\right)}{\left(\lambda_n - \overline{\lambda}_m\right)B_n(\lambda_n)\overline{B_m(\lambda_m)}}.\end{aligned} \tag{4.47}$$

On the last step, we have used the Residue Theorem in exactly same way it was done in the proof of Proposition 4.7 (see (4.33)).

Now it is obvious that $\{j_m\}_{m\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$. Indeed, $j_m = \widetilde{j}_m$ are the coefficients in the expansion of the function $\widetilde{\mathbf{G}} \in K \subset H^2_+$ with respect to the Riesz basis $\{j_n\}_{n\in\mathbb{Z}'}$ in K.

2) Using Theorem 4.3 one can see that there exists a unique function $\widetilde{\mathbf{F}} \in K$, which solves the following interpolation problem and has the minimal H^2_+ – norm:

$$\widetilde{\mathbf{F}}(\lambda_n)\sqrt{2\ \Im\lambda_n} = 2\ \Im\lambda_n\ \alpha_n, \qquad n \in \mathbb{Z}'. \tag{4.48}$$

Notice that problem (4.48) is solvable since $\{\alpha_n\}_{n\in\mathbb{Z}'} \in l^1(\mathbb{Z}') \subset l^2(\mathbb{Z}')$ and $|\Im\lambda_n| \asymp 1$. This function can be given by the formula

$$\widetilde{\mathbf{F}}(\nu) = i \sum_{n\in\mathbb{Z}'} \frac{\sqrt{2\ \Im\lambda_n}}{\nu - \overline{\lambda}_n}\ \frac{B_n(\nu)}{B_n(\lambda_n)}\ 2\ \Im\lambda_n\ \alpha_n. \tag{4.49}$$

If we repeat all steps (4.43)–(4.47), we immediately arrive at the fact that $\widetilde{\mathbf{F}} = \mathbf{F}$ with $\mathbf{F}$ given in (4.37) and to the fact that $\{h_m\}_{m\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$.

3) Now we prove statement $b)$ of the Lemma. Let us establish that the product of the functions $\mathbf{G}$ and $\mathbf{F}$ belongs to $K = H^2_+ \ominus BH^2_+$. Using the fact that $\{\alpha_n\}_{n\in\mathbb{Z}'} \in l^1(\mathbb{Z})$ and taking into account that the Blaschke product $B_m(\nu)$ can be treated as an element of the Hardy class H^∞, we obtain that there exists a constant $C > 0$ such that $|\widetilde{\mathbf{F}}(\nu)| \le C < \infty$, $\nu \in \mathbb{C}^+$. Indeed,

$$|\mathbf{F}(\nu)| \le \sum_{m\in\mathbb{Z}'} |\alpha_m|\ \frac{\sqrt{2\ \Im\lambda_m}}{\left|\nu - \overline{\lambda}_m\right|}\ \frac{|B_m(\nu)|}{|B_m(\lambda_m)|} \le C_0 \sum_{m\in\mathbb{Z}'} |\alpha_m|\ \frac{1}{d_0|B_m(\lambda_m)|}, \tag{4.50}$$

where $d_0 = \inf_{m\in\mathbb{Z}'} \operatorname{dist}\{\lambda_m, \mathbb{R}\} > 0$. Due to the Carleson condition (4.7), we have $\inf_{m\in\mathbb{Z}'} |B_m(\lambda_m)| > 0$. Hence

$$|\mathbf{F}(\nu)| \le C_1 \sum_{m\in\mathbb{Z}'} |\alpha_m| = C < \infty. \tag{4.51}$$

To show that $\mathbf{GF} \in K$, it suffices to show that the following sequence of functions:

$$S_N(\nu) = \mathbf{F}(\nu) \sum_{|n|\le N} j_n\ \frac{\sqrt{2\ \Im\lambda_n}}{\nu - \overline{\lambda}_n}, \qquad S_N \in K, \tag{4.52}$$

is converging to $\mathbf{G}(\nu)\mathbf{F}(\nu)$ in H^2_+. We have

$$\|\mathbf{G}(\nu)\mathbf{F}(\nu)-S_N(\nu)\|^2_{H^2_+}\le \sup_{y>0}\int_{-\infty}^{\infty}|\mathbf{F}(x+iy)|^2\left|\sum_{|n|\ge N+1} j_n\,\frac{\sqrt{2\ \Im\lambda_n}}{x+iy-\overline{\lambda}_n}\right|^2 dx$$

$$\le C\sup_{y>0}\int_{-\infty}^{\infty}\left|\sum_{|n|\ge N+1} j_n\frac{\sqrt{2\ \Im\lambda_n}}{x+iy-\overline{\lambda}_n}\right|dx = C\left\|\sum_{|n|\ge N+1} j_n\,\frac{\sqrt{2\ \Im\lambda_n}}{\nu-\overline{\lambda}_n}\right\|^2_{H^2_+}.$$

Since the set of the normalized simple fractions forms a Riesz basis in K, we obtain that $S_N(\nu)\mapsto \mathbf{G}(\nu)\mathbf{F}(\nu)$ in the sense of H^2_+–space, which yields the result.

Evaluating $\mathbf{G}(\nu)\mathbf{F}(\nu)$ at the points λ_n, we have (due to (4.41) and (4.48))

$$\mathbf{G}(\lambda_n)\mathbf{F}(\lambda_n)=\frac{\alpha_n^{-1}\mathbf{V}_n}{\sqrt{2\ \Im\lambda_n}}\ \sqrt{2\ \Im\lambda_n}\ \alpha_n=\mathbf{V}_n,$$

which means that two functions, **GF** and **V**, from the space K have the same interpolating values. Thus, they have to coincide. ■

Now we are in a position to prove our main result.

Proof of Theorem 4.9. a) To proceed with the proof we have to recall some information about the eigenvalues λ_n and the eigenvectors, $\Psi_n(x)$, of the dynamics generator $\mathscr{L}$. The asymptotic formula (2.13) implies that $|\lambda_n|\asymp n^2$, $|\Im\lambda_n|\asymp 1$. The eigenvectors $\Psi_n(x)$ are given in (2.32) in terms of eigenfunctions, $\psi_n(x)$, of the operator pencil (2.18), (2.19). The eigenfunctions $\psi_n(x)$ and their asymptotic behavior are described in (2.29) – (2.31). We need the following facts about $\psi_n(x)$. The entire set $\{\psi_n\}_{n\in\mathbb{Z}'}$ is doubly complete in $L^2(0,L)$, which means that each set, $\{\psi_n\}_{n\ge1}$ and $\{\psi_n\}_{n\le-1}$, forms a Riesz basis in $L^2(0,L)$ (see [7]). As shown in [11], the following estimate holds: $|\psi'(L)|\asymp|n|$.

Now we can turn to the convergence of the series (4.39) for the force profile function $\mathbf{g}(x)$. Using formula (2.32) for $\Psi_n(x)$ we can represent this series in the form a

$$\mathbf{g}(x)=\begin{bmatrix}\displaystyle\sum_{n\in\mathbb{Z}'}\frac{1}{\lambda_n}r_n j_{-n}\psi_n(x)\\ \displaystyle\sum_{n\in\mathbb{Z}'}r_n j_{-n}\psi_n(x)\\ \displaystyle-\sum_{n\in\mathbb{Z}'}\frac{h}{H+i\lambda_n}r_n j_{-n}\psi_n'(L)\end{bmatrix}. \tag{4.53}$$

Consider the first component series from (4.53). To show that this series defines a function from

$H^1(0,L)$, it suffices to show that the function defined by

$$\mathfrak{S}(x) = \sum_{n\in\mathbb{Z}'} \frac{1}{\lambda_n} r_n j_{-n} \psi_n'(x) \tag{4.54}$$

belongs to $L^2(0,L)$. Based on the asymptotic approximations for $\psi_n'(x)$ from (2.31) it is convenient to represent $\mathfrak{S}(x)$ as $\mathfrak{S}(x) = \sum_{j=1}^4 \mathfrak{S}_j(x)$, where

$$\mathfrak{S}_1(x) = -\sum_{n\in\mathbb{Z}'} \mu_n r_n j_{-n} \lambda_n^{-1} e^{-\mu_n x}, \qquad \mathfrak{S}_2(x) = \sum_{n\in\mathbb{Z}'} \sqrt{2}\mu_n r_n j_{-n} \lambda_n^{-1} \sin\left(\mu_n x + \frac{\pi}{4}\right), \tag{4.55}$$

$$\mathfrak{S}_3(x) = \sum_{n\in\mathbb{Z}'} 2(-1)^{n+1} \mu_n r_n j_{-n} \lambda_n^{-1} e^{-\mu_n (L-x)}, \qquad \mathfrak{S}_4(x) = \sum_{n\in\mathbb{Z}'} \mu_n r_n j_{-n} \lambda_n^{-1} O\left(\frac{1}{\mu_n}\right). \tag{4.56}$$

Now we show that $\mathfrak{S}_i \in L^2(0,L),\ \ i = 1,...,4$. Using asymptotics (2.13), (2.25), and the explicit formula for r_n we obtain that $\left|\mu_n r_n \lambda_n^{-1}\right| \asymp 1$ and thus, $\left\{b_n \equiv \mu_n r_n \lambda_n^{-1} j_{-1}\right\}_{n\in\mathbb{Z}'} \in l^2(\mathbb{Z}')$. In what follows, it is convenient to write that two positive sequences $\{c_n\}$ and $\{d_n\}$ are related by $c_n \prec d_n$ if there exists an absolute constant C such that $c_n \le C\, d_n$. One can check that

$$\begin{aligned} \|\mathfrak{S}_1\|^2_{L^2(0,L)} &\prec \int_0^L \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n b_m |e^{-(\mu_n+\mu_m)x}| dx + \int_0^L \sum_{n=-\infty}^{-1}\sum_{m=-\infty}^{-1} b_n b_m |e^{-(\mu_n+\mu_m)x}| dx \\ &\prec \sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} b_m |\mu_n+\mu_m|^{-1} + \sum_{n=-\infty}^{-1} b_n \sum_{m=-\infty}^{-1} b_m |\mu_n+\mu_m|^{-1} \\ &\prec \sum_{n\in\mathbb{Z}'} b_n \sum_{m\in\mathbb{Z}'} b_m (|n|+|m|)^{-1} \prec \sum_{n\in\mathbb{Z}'} b_n^2 \prec \sum_{n\in\mathbb{Z}'} |j_n|^2. \end{aligned} \tag{4.57}$$

The estimate for $\|\mathfrak{S}_3\|^2_{L^2(0,L)}$ can be obtained in a similar fashion. For $\mathfrak{S}_4$ we proceed as

$$\|\mathfrak{S}_4\|^2_{L^2(0,L)} \prec \left(\sum_{n\in\mathbb{Z}'} |n|^{-1} |j_{-n}|\right)^2 \prec \sum_{n\in\mathbb{Z}'} |j_n|^2. \tag{4.58}$$

Finally, to prove that $\mathfrak{S}_2 \in L^2(0,L)$ one has to show $\left\|\sum_{n\in\mathbb{Z}'} \mu_n r_n j_{-n} \lambda_n^{-1} \sin\left(\mu_n x + \frac{\pi}{4}\right)\right\|_{L^2(0,L)} < \infty$. Taking into account that each set $\left\{e^{i\mu_n x}\right\}_{n\in\mathbb{Z}'}$ and $\left\{e^{-i\mu_n x}\right\}_{n\in\mathbb{Z}'}$ forms a Riesz basis in $L^2(-L,L)$, we obtain that the following estimates are valid:

$$\|\mathfrak{S}_2\|^2_{L^2(0,L)} \prec \left\|\sum_{n\in\mathbb{Z}'} b_n e^{i\mu_n x}\right\|^2_{L^2(0,L)} + \left\|\sum_{n\in\mathbb{Z}'} b_n e^{-i\mu_n x}\right\|^2_{L^2(0,L)} \prec \sum_{n\in\mathbb{Z}'} |b_n|^2 \prec \sum_{n\in\mathbb{Z}'} |j_n|^2. \tag{4.59}$$

Collecting (4.56) – (4.59) we obtain that the first component from (4.55) belongs to $H^1(0,L)$.

Now we have to show that the series in the second component of (4.53) converges in $H^{-1}(0,L)$. We have already shown that the series in the first component of (4.53) converges in $H^1(0,L)$. From this fact we conclude that the series $\sum_{n\in\mathbb{Z}'}\frac{1}{\lambda_n}r_n j_{-n}\psi_n''(x)$ converges in $H^{-1}(0,L)$. Using formula (2.29) for $\psi_n(x)$ we can evaluate $\psi_n''(x)$ and represent the above series in the form

$$\sum_{n\in\mathbb{Z}'}\frac{1}{\lambda_n}r_n j_{-n}\psi_n''(x) = \sum_{n\in\mathbb{Z}'} r_n j_{-n}\psi_n(x) + \sum_0, \tag{4.60}$$

where

$$\sum_0 = \sum_{n\in\mathbb{Z}'}\left(\frac{\mu_n^2}{\lambda_n}-1\right) r_n j_{-n}\psi_n(x) - \sqrt{2}\sum_{n\in\mathbb{Z}'}\frac{\mu_n^2}{\lambda_n}\Gamma_n r_n j_{-n}\sin\left(\mu_n x+\frac{\pi}{4}\right). \tag{4.61}$$

Our goal is to show that the first series on the right in (4.60) converges in $H^{-1}(0,L)$. To show this it is sufficient to demonstrate that both series (4.61) are convergent in $H^{-1}(0,L)$ since, as we stated above, the series on the left in (4.60) converges in $H^{-1}(0,L)$. Using asymptotic formulas (2.25) and (2.13) we obtain the estimate $\left|\frac{\mu_n^2}{\lambda_n}-1\right| \asymp \frac{1}{n^2}$. It follows from (2.30) that $|\Gamma_n| \asymp 1$. Recall also that $|r_n| \asymp n$. Collecting all these facts and substituting into (4.61) we conclude that the convergence of the series in (4.61) is equivalent to the convergence of the following two series: $\sum_1 = \sum_{n\in\mathbb{Z}'}\frac{1}{|n|}j_{-n}\psi_n(x)$ and $\sum_2 = \sum_{n\in\mathbb{Z}'}|n| j_{-n}\sin\left(\mu_n x+\frac{\pi}{4}\right)$.

We already know from the above that $\sum_1$ is convergent in $H^1(0,L)$. Since $\sum_2$ is an expansion with respect to simple harmonics and $\{j_n\}_{n\in\mathbb{Z}'}\in l^2(\mathbb{Z}')$ we calculate that $\sum_2$ is convergent in $H^{-1}(0,L)$. This completes the proof of the convergence of the series in the second component of (4.53).

Finally, we consider the numerical series in the third component of (4.53). Based on the above facts we conclude that the convergence of this series is equivalent to the convergence of $\sum_{n\in\mathbb{Z}'}|j_n|$, since $\left|\frac{h}{H+i\lambda_n}r_n\psi_n'(L)\right| \asymp 1$. So, we have to show that, in fact, $\{j_n\}_{n\in\mathbb{Z}'}\in l^1(\mathbb{Z}')$, which is stronger than the statement made in Lemma 4.8. The proof is given below. In these estimates we put the symbol $\asymp$ between the series and/or integrals whose convergence is equivalent to each other. Also, $p>1$ and $q>1$ are Hölder conjugate and we recall that $|B_n(\lambda_n)|\geq C>0$. We have

$$\begin{aligned}\sum_{m\in\mathbb{Z}'}|j_m| &= \sum_{m\in\mathbb{Z}'}\left|\frac{2\sqrt{2}\ \Im\lambda_m\,\varepsilon_m}{\overline{B}_m(\lambda_m)}\sum_{n\in\mathbb{Z}'}\frac{2\sqrt{2}\ \Im\lambda_n\,\bar{\varepsilon}_n\,2\lambda_m\,\alpha_m}{\left(\lambda_n-\overline{\lambda}_m\right)B_n(\lambda_n)}\right| \asymp \sum_{n\in\mathbb{Z}'}|\alpha_n|\sum_{m\in\mathbb{Z}'}\frac{1}{\left|\lambda_n-\overline{\lambda}_m\right|}\\ &\leq \left(\sum_{n\in\mathbb{Z}'}|\alpha_n|^q\right)^{\frac{1}{q}}\left[\sum_{n\in\mathbb{Z}'}\left(\sum_{m\in\mathbb{Z}'}\frac{1}{\left|\lambda_n-\overline{\lambda}_m\right|}\right)^p\right]^{\frac{1}{p}}.\end{aligned} \tag{4.62}$$

Now we estimate the second sum in the second factor of (4.62). We use the asymptotic formula

(2.13) and a constant c, whose value is immaterial to us

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}'}\frac{1}{\left|\lambda_n-\overline{\lambda}_m\right|}\asymp\sum_{m\in\mathbb{Z}'}\frac{1}{\left|\left(n+\frac{1}{2}\right)^2-\left(m+\frac{1}{2}\right)^2+2ic\right|}\\
&\asymp\sum_{|m|\geq|n|+1}\frac{1}{\left|\left(n+\frac{1}{2}\right)^2-\left(m+\frac{1}{2}\right)^2+2ic\right|}\asymp\int_{|n|+\frac{3}{2}}^{\infty}\frac{dx}{\left[\left(x^2-\left(n+\frac{1}{2}\right)^2\right)^2+4c^2\right]^{1/2}}\\
&=\int_1^{\infty}\frac{d\xi}{\left[\xi^2\left(\xi+2|n|+1\right)^2+4c^2\right]^{1/2}}\leq\int_1^{\infty}\frac{d\xi}{\xi(\xi+2|n|+1)}=\frac{\ln(2|n|+2)}{2|n|+1}\asymp\frac{\ln|n|}{|n|}.
\end{aligned}
\tag{4.63}
$$

Substituting this result into (4.62) we see that the second factor in (4.62) is dominated by a convergent series $\sum_{n\in\mathbb{Z}'}\left(\frac{\ln|n|}{|n|}\right)^p$, $p>1$. The convergence of the first factor is obvious since $\{\alpha_n\}_{n\in\mathbb{Z}'}\in l^1(\mathbb{Z}')$.

b) The convergence of the series (4.40) in $L^2(0,\infty)$ is obvious since by Lemma 4.8, $\{h_n\}_{n\in\mathbb{Z}'}\in l^2(\mathbb{Z}')$ and $|\Im\lambda_n|\asymp 1$.

c) Now we are in a position to complete the proof of the theorem. Let us take as the force profile $\mathbf{g}$ from Eq.(3.1), the function defined in (4.39). Without loss of generality, we assume the initial state from problem (3.1), $U_0=0$. We return to the initial–value problem

$$
U_t(x,t)=i\left(\mathscr{L}U\right)(x,t)+\mathbf{g}(x)\widetilde{f}(t),\qquad U(x,0)=0,
\tag{4.64}
$$

with $\mathbf{g}$ being defined in (4.39) and $\widetilde{f}(t)$ being some control function.

Assume that the voltage output $\mathbf{v}(t)$ of the problem (4.64) coincides with the signal to be tracked, which is given in (4.21). On the one hand $\mathbf{v}(t)$ is the third component of the solution $U(x,t)$ of problem (4.64) and its modified Laplace transform is given by (4.19), (4.20). On the other hand, $\mathbf{v}(t)$ is given by (4.21) and its modified Laplace transform is presented in (4.22). We are going to show that in this case the control function $\widetilde{f}(t)$ must coincide with the function $f(t)$ defined in (4.37).

Recall formulas (4.19), (4.20). Due to the fact that $\chi_m=0$, formula (4.20) for the modified Laplace transform of the voltage output takes the form:

$$
\mathbb{V}(\nu)=\mathbf{V}(\nu)=\widetilde{F}(\nu)\sum_{n\in\mathbb{Z}'}G_n\,\frac{\sqrt{2}\;\Im\,\lambda_n}{\nu-\overline{\lambda}_n}\,,
\tag{4.65}
$$

where $\widetilde{F}(\nu)$ is the modified Laplace transform of $\widetilde{f}(t)$. Since $\mathbf{g}(x)$ is given by (4.39) we should take $g_n=r_n j_{-n}$ in formula (4.19) for G_n. (Recall that r_n is defined in (4.39) and j_n is given in Lemma (4.8) – formula (4.36).) Substituting the above g_n with r_n defined in (4.39) into (4.19) we obtain after a straightforward calculation that $G_n=j_n$. (In this calculation we have taken into account that $\Im\lambda_n=\Im\lambda_{-n}$ due to the symmetry of the spectrum: $\lambda_{-n}=-\overline{\lambda}_n$.) Therefore, the series in (4.59) coincides with the function $\mathbf{G}(v)$ defined in (4.36). So, (4.36) takes from

$$\mathbf{V}(\nu) = \widetilde{F}(\nu)\mathbf{G}(\nu), \tag{4.66}$$

(4.66) is a factorization of $\mathbf{V}(v)$. However, by Lemma (4.8) we already have another factorization of $\mathbf{V}(v)$ given in (4.38). As we know, formula (4.66) viewed as an equation for $\widetilde{F}(v)$ has a unique solution in the space $K \subset H^2_+$. Since we already have such solution $\mathbf{F}(v)$ given in (4.37) we can conclude that $\widetilde{F}(v) = \mathbf{F}(v)$. It remains to notice that $\mathbf{F}(v)$ is the modified Laplace transform of $f(t)$ defined in (4.40) and thus $\widetilde{f}(t) = f(t)$.

The proof is complete. ■

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