

Nonuniversal entanglement level statistics in projection-driven quantum circuits

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We study the level-spacing statistics in the entanglement spectrum of output states of random universal quantum circuits where qubits are subject to a finite probability of projection to the computational basis at each time step. We encounter two phase transitions with increasing projection rate. The first is the volume-to-area law transition observed in quantum circuits with projective measurements. We identify a second transition within the area law phase by repartitioning the system randomly into two subsystems and probing the entanglement level statistics. This second transition separates a pure Poisson level statistics phase at large projective measurement rates from a regime of residual level repulsion in the entanglement spectrum, characterized by nonuniversal level spacing statistics that interpolates between the Wigner-Dyson and Poisson distributions. By applying a tensor network contraction algorithm introduced in [Z.-C. Yang *et al.*, *Phys. Rev. E* **97**, 033303 (2018)] to the circuit spacetime, we identify this second projective-measurement-driven transition as a percolation transition of entangled bonds. The same behavior is observed in both circuits of random two-qubit unitaries and circuits of universal gate sets, including the set implemented by Google in its Sycamore circuits.

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I. INTRODUCTION

Closed quantum many-body systems undergoing unitary evolution generically reach a thermalized regime, exhibiting volume-law entanglement [1–5]. Exceptions to this scenario have drawn considerable attention, due to their relevance to experimentally controllable quantum systems. For example, many-body localization, which precludes thermalization and leads instead to area-law entanglement, has been the subject of extensive theoretical and experimental work [6–15]. Recently, failure to thermalize has also been reported in simulations of quantum circuits subjected to random measurement events that model coupling to a classical environment [16–23]. Interest in these models is fueled by the ongoing efforts to exploit noisy intermediate-scale quantum devices for tasks beyond the reach of classical computers.

The studies of Refs. [16–23] follow the time evolution of an initial product state of qubits arranged in a one-dimensional (1D) chain induced by local unitary gates randomly chosen from a volume-law entangling set (such as, e.g., the Clifford set), and subsequently measurement of each qubit with probability p . Intuitively, the nonunitary projective measurement operation effectively disentangle the state and, at sufficiently large measurement rate, results in a localization of the system in Hilbert space characterized by the volume-to-area law transition described in Refs. [16–23].

In this work, we study volume-law entangling unitary quantum circuits subjected to a different kind of disentangling perturbation, namely, projection operations that forcibly “reset” qubits to the computational basis, randomly inserted at a finite rate throughout the time evolution of the circuit. Furthermore, we employ the level spacing statistics of the

entanglement spectrum [24], referred to hereafter as “the entanglement spectrum statistics” (ESS), as a finer measure of thermalization and entanglement [25–27]. Our computations are carried out by adopting the iterative tensor network contraction method introduced in Ref. [28] to the 1+1D spacetime of the circuits.

The main result of this paper is that the ESS features three qualitatively different regimes separated by two transitions. The first transition displays the same phenomenology as discussed in the case of random projective measurements, namely, a volume-to-area law transition at a finite projection rate, $p = p_S$ [29], for contiguous partitions of the system into subsystems A and B . Within the area law regime, we analyze the ESS by changing the way the system is partitioned, assigning randomly the sites to either subsystem A or B . Probing this repartitioned system, we identify a second transition inside the area law regime for contiguous partitions. For $p \rightarrow p_{S+}$ the ESS obeys Wigner-Dyson statistics; for $p_S < p < p_c$, the ESS assumes a nonuniversal form that interpolates between the Wigner-Dyson and Poisson statistics (see Fig. 1); and for $p > p_c$ the ESS becomes Poisson-distributed. By resolving the entanglement bond dimensions of the 1+1D network spatially, we conclude that the transition from nonuniversal to Poisson statistics at p_c is associated with the percolation of entangled bonds in the spacetime geometry of the circuit. We observe the same behavior, with two transitions, in 1+1D circuits comprised of either two-qubit random-Haar unitaries or of gates drawn from universal sets, including that implemented by Google in its Sycamore circuits.

This paper is organized as follows. In Sec. II, we detail the construction of our quantum circuit and of the random

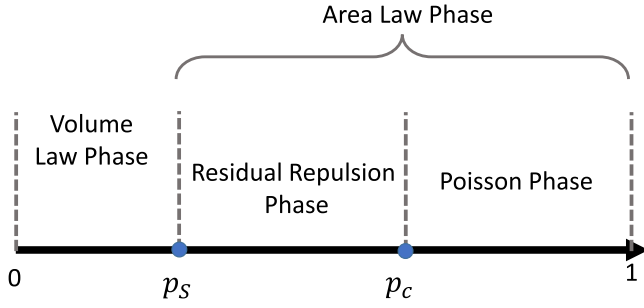


FIG. 1. Phase diagram as a function of projection probability p . We observe three phases separated by two phase transitions at p_s and p_c . The first transition, at p_s , is the volume-to-area transition identified in Refs. [16–23]. The second transition separates the pure Poisson level statistics phase ($p > p_c$) from a regime of residual level repulsion in the entanglement spectrum ($p < p_c$), characterized by nonuniversal level spacing statistics that interpolates between the Wigner-Dyson and Poisson distributions. We identify this second projective-measurement-driven transition as a percolation transition of entangled bonds in 1+1D spacetime.

one-qubit projection operators, and outline the method used to compute the ESS. We then map each of these circuits into a tensor network and describe the algorithm for contracting these networks in Sec. III. In Sec. IV, we show numerical results for the ESS in the thermalizing, nonuniversal, and Poisson phases and locate the transition point, $p = p_c$, between the latter two, which we interpret as a two-dimensional percolation transition in the spacetime of the circuit in Sec. V. Section VI summarizes our conclusions.

II. QUANTUM CIRCUITS AND RANDOM MATRIX THEORY

We consider n qubits evolving in time t from an initial product state of the form

$$|\Psi(t=0)\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle, \quad (1)$$

where the single-qubit state for the j -th qubit is defined as $|\psi_j\rangle = \cos(\theta_j/2)|0\rangle + \sin(\theta_j/2)e^{i\phi_j}|1\rangle$ with arbitrary angles θ_j and ϕ_j . In what follows, the initial state evolves under the action of (i) random unitary gates, and (ii) single-qubit projection operators, randomly inserted after each gate with a finite probability p . The state at time t is

$$|\Psi(t)\rangle = M |\Psi(t=0)\rangle = \sum_x \Psi_x(t) |x\rangle, \quad (2)$$

where $|x\rangle = |x_1 x_2 \dots x_n\rangle$ is a configuration in the computational basis with $x_j = 0, 1$ for $j = 1, \dots, n$, and M is a $2^n \times 2^n$ nonunitary matrix describing both the unitary evolution and the projection operations. The resulting circuit is illustrated in Fig. 2, where two-qubit gates are represented as blocks and projection operators as circles.

We choose the projection operator acting on the j -th qubit to take the form $M_0 = I_1 \otimes I_2 \otimes \cdots \otimes |0_j\rangle\langle 0_j| \otimes \cdots \otimes I_n$, where I_j is the identity operator on a single qubit. Although we have chosen to project to the $|0\rangle$ instead of the $|1\rangle$ state, this choice is immaterial in what follows. Projection operators are not normpreserving, and hence the final state $|\Psi(t)\rangle$ is not nor-

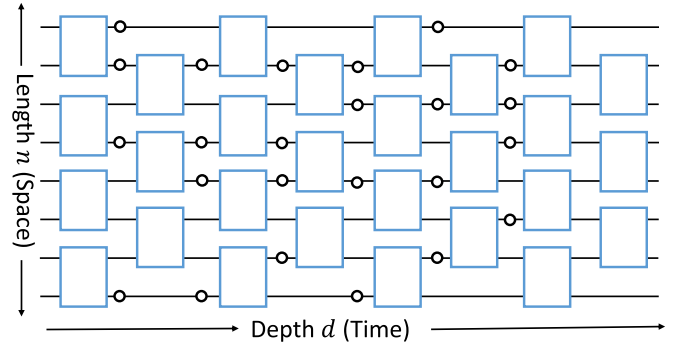


FIG. 2. Illustration of initial product state evolved in time in a quantum circuit. The circuit consists of local two-qubit unitary gates (blocks) and projection operators (circles). The latter are introduced randomly with probability p .

malized by default. We normalize final states for consistency. Projection operators can be physically interpreted as randomly picking a qubit and resetting it to the computational basis. As will become evident below, the projector operators have a disentangling effect, similar to that of the addition of measurement operators in random Clifford or Haar-random circuits [16–23]. In particular, projectors also lead to a volume-to-area law transition as a function of nonunitary operator density.

Now, consider the pure state $|\Psi\rangle = \sum_x \Psi_x |x\rangle$ after evolution with a quantum circuit as specified above, where x is the configuration of the qubits in the computational basis and Ψ_x is the coefficient for each x . Ψ_x can be reshaped into a matrix $\Psi_{A,B}$ by splitting the state into subsystems A and B . In this way, $|\Psi\rangle$ is expressed as

$$|\Psi\rangle = \sum_{x_A, x_B} \Psi_{A,B} |x_A\rangle \otimes |x_B\rangle, \quad (3)$$

where x_A and x_B are the local configurations for subsystems A and B , respectively. Notice that the partition into subsystems A and B can be chosen arbitrarily; for example, the subsystems can be each contiguous or not. The choice of contiguous subsystems is often made to study if the entanglement entropy satisfies volume or area law. Here we shall also consider random partitions, in which sites are randomly assigned to subsystem A or B . This partitioning allows us to probe the ESS even when the entanglement entropy is small for contiguous partition, e.g., in the area law regime.

The entanglement spectrum can be obtained by a Schmidt decomposition [24]

$$|\Psi\rangle = \sum_k \lambda_k |x_A^k\rangle \otimes |x_B^k\rangle, \quad (4)$$

which is equivalent to singular value decomposition (SVD) of matrix $\Psi_{A,B}$ with singular values λ_k .

The set of entanglement levels λ_k defines the entanglement spectrum (ES). The entanglement entropy is given by

$$S = - \sum_k \lambda_k^2 \ln \lambda_k^2. \quad (5)$$

With the ES in descending order, $\lambda_k > \lambda_{k+1}$, the ratio of adjacent gaps in the spectrum can be defined as

$$r_k = \frac{\lambda_{k-1} - \lambda_k}{\lambda_k - \lambda_{k+1}}. \quad (6)$$

We remark that ratios of other functions of the λ_k can be defined, but these choices do not affect the ESS. For example, let us define

$$\begin{aligned} r_k^{(f)} &= \frac{f(\lambda_{k-1}) - f(\lambda_k)}{f(\lambda_k) - f(\lambda_{k+1})} \\ &= r_k \times \frac{f(\lambda_{k-1}) - f(\lambda_k)}{\lambda_{k-1} - \lambda_k} \bigg/ \frac{f(\lambda_k) - f(\lambda_{k+1})}{\lambda_k - \lambda_{k+1}}, \end{aligned} \quad (7)$$

which in the limit when the level spacings go to zero can be replaced by

$$r_k^{(f)} \rightarrow r_k \times \frac{f'(\lambda_{k-1})}{f'(\lambda_k)} \rightarrow r_k. \quad (8)$$

In other words, for any smooth function f the statistics for the ratios $r_k^{(f)}$ are the same as those for the ratios r_k , in the limit of dense spectra or small $\lambda_{k-1} - \lambda_k$ spacings.

For Haar-random states, the probability distribution for adjacent entanglement level ratios, which defines the ESS, follows Wigner-Dyson statistics from random matrix theory [30,31] and fits well the surmise [32]

$$P_{\text{WD}}(r) = \frac{1}{Z} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1+3\beta/2}} \quad (9)$$

with $Z = 4\pi/81\sqrt{3}$ and $\beta = 2$ for the Gaussian unitary ensemble (GUE) distribution. In contrast, the ESS for integrable systems takes the Poisson form

$$P_{\text{Poisson}}(r) = \frac{1}{(1 + r)^2}. \quad (10)$$

The most marked difference between the GUE and Poisson distributions is the level repulsion ($P_{\text{WD}} \rightarrow 0$ for $r \rightarrow 0$) in the former and its absence ($P_{\text{Poisson}} > 0$ at $r = 0$) in the latter.

III. QUANTUM CIRCUITS AS TENSOR NETWORKS

A. Tensor network mapping

In this section, we map the random quantum circuits introduced above and illustrated in Fig. 1 into a square-grid tensor network and detail the contraction algorithm we use to compute entanglement properties. The tensorial representation of the elements of the circuits is illustrated in Fig. 2. We use open boundary conditions in the space and time directions.

Each two-qubit gate g is expressed as a 4×4 unitary matrix $T_{(i_1 i_2)(o_1 o_2)}^g$, where $(i_1 i_2)$ and $(o_1 o_2)$ are combined indices corresponding to gate input and output qubit states, respectively. Each 4×4 matrix can be reshaped into a $2 \times 2 \times 2 \times 2$ tensor $T_{i_1 i_2 o_1 o_2}^g$. Since we want to transform the circuit into a square lattice geometry, we regroup the indices of each tensor to reshape it to a matrix $T_{(i_1 o_1)(i_2 o_2)}^g$ and use a SVD to

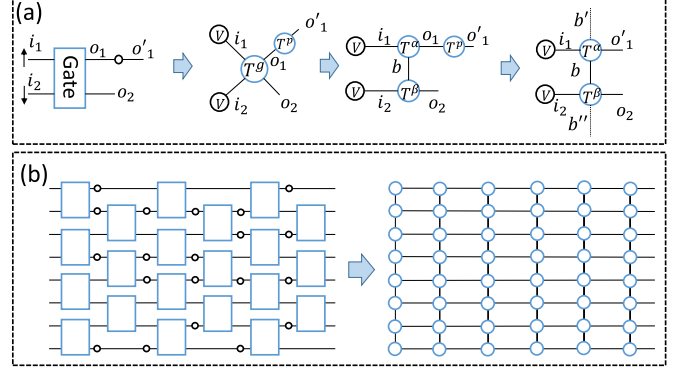


FIG. 3. Panel (a) shows how to rewrite a local two-qubit Haar-random gate followed by a projection operator into two tensors where each one has four active (i_1, i_2, o_1, o_2) and two dummy (b, b') indices. The active indices can be lumped together as $i_1 i_2$ and $o_1 o_2$. Panel (b) shows how to use the correspondence between two- and one-bit gates and a rank-4 tensor to map a quantum circuit into a rectangular tensor network.

decompose it as

$$\begin{aligned} T_{(i_1 o_1)(i_2 o_2)}^g &= \sum_{m, m'} U_{(i_1 o_1)m} \Sigma_{mm'} V_{(i_2 o_2)m'}^* \\ &= \sum_{m, b, m'} U_{(i_1 o_1)m} \sqrt{\Lambda_{mb}} \sqrt{\Lambda_{bm'}} V_{(i_2 o_2)m'}^* \\ &= \sum_b T_{(i_1 o_1)b}^\alpha T_{(i_2 o_2)b}^\beta, \end{aligned} \quad (11)$$

where $U_{(i_1 o_1)m}$ and $V_{(i_2 o_2)m'}$ are unitary matrices and $\Lambda_{mm'}$ is a semipositive diagonal matrix containing the singular values. The two new matrices $T_{(i_1 o_1)b}^\alpha = \sum_m U_{(i_1 o_1)m} \sqrt{\Lambda_{mb}}$ and $T_{(i_2 o_2)b}^\beta = \sum_{m'} \sqrt{\Lambda_{bm'}} V_{(i_2 o_2)m'}^*$, with b an index running over singular values, are then reshaped to tensors $T_{i_1 o_1 b}^\alpha$ and $T_{i_2 o_2 b}^\beta$. To end up with a rectangular geometry, we add a fourth “dummy” index with dimension 1 to each tensor, connecting it with a neighboring tensor in the space dimension, as indicated by the faint vertical lines in Fig. 3(a).

Each single-qubit projector can also be expressed as a tensor $T_{o_1 o_1'}^P$, where o_1' has dimension 1. These can be contracted into gate tensors as

$$T_{i_1 o_1' b b'}^\alpha = \sum_{o_1} T_{i_1 o_1 b b'}^\alpha T_{o_1 o_1'}^P. \quad (12)$$

Since initial states are taken to be product states, they can be written simply as a tensor product of vectors, each vector corresponding to a single-qubit state. The state for the first qubit, for example, is $|\psi_{i_1}\rangle = [\cos(\theta_{i_1}/2), \sin(\theta_{i_1}/2)e^{i\phi_{i_1}}] = V_{i_1}$. This can be contracted into the first gate tensor as

$$T_{o_1' b b'}^\alpha = \sum_{i_1} V_{i_1} T_{i_1 o_1' b b'}^\alpha. \quad (13)$$

Finally, wherever no gates are applied to qubits at the top and bottom boundaries, a rank-3 identity tensor $\delta_{i_1 o_1 b}$ is added to complete the square lattice. With the above transformations, we map the evolution described by the quantum circuit into

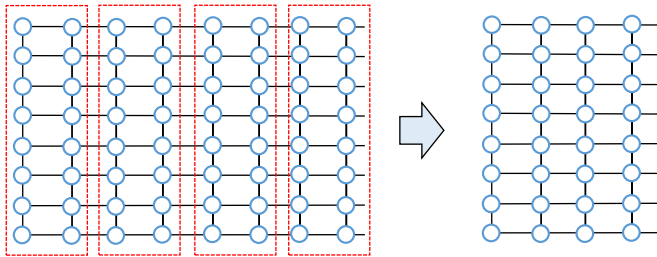


FIG. 4. Illustration of the coarse-graining process used to contract tensor networks in the time direction.

a tensor network, as shown in Fig. 3(b). Note that the final (right) column of n indices is left free.

Next, we adopt a common indexing scheme for all tensors in the network, where we denote every single tensor index as s . The set of all indices in the tensor network is thus $\{\mathbf{S}\} = \{s_1, s_2, \dots, s_N\}$, where $N = 2d(2n - 1)$ is the total number of indices and d is the circuit depth in time steps with two columns of two-qubit gates per time step. Each tensor can be uniquely determined by its subset of indices $\{s\} \subset \{\mathbf{S}\}$ as $T_{\{s\}}$. In this language, the final state $|\Psi_f\rangle$ is

$$|\Psi_f\rangle = \text{Tr} \prod_{\{s\}} T_{\{s\}}, \quad (14)$$

where Tr indicates a trace over all nonfree indices connecting tensors. Obtaining the final state is thus equivalent to partially contracting a tensor network.

B. Contraction algorithm

To contract tensor networks, we employ a variant of the iterative compression-decimation algorithm introduced in Ref. [28]. We perform iterations of alternating *compression* and *decimation* steps until the width of the lattice is fully contracted.

The compression step is a sweep over lattice bonds where we first contract the tensors at the ends of each visited bond and then perform a SVD to restore the structure of the lattice, in a way reminiscent of the density-matrix renormalization group algorithm [33]. This step becomes significant when the projector density in quantum circuits is increased, as we will discuss below. We use the tools developed in Ref. [28] to implement compression efficiently.

The decimation step coarsens the lattice at the expense of increasing bond dimensions. As illustrated in Fig. 4 the tensor network coarse graining is performed in the time direction, so that every two columns of tensors are contracted into one. At the end of decimation, we get a tensor chain representing the final state.

IV. ENTANGLEMENT SPECTRUM STATISTICS AND PHASE TRANSITIONS

We now use the formulations and tools previously discussed to perform numerical investigations of the ESS as a function of projector density, p , in random quantum circuits satisfying the geometry given in Fig. 2. For each circuit realization, the initial state is of the form of Eq. (1), where all

θ_j and ϕ_j are selected uniformly at random. For comparative purposes, two separate gate sets were used to construct the random circuits. The first case is composed of two-qubit gates selected from the Haar-random measure, while the second consists of gates uniformly selected from the universal gate set $\mathbf{U}_{\text{sync}} = \{\sqrt{X}, \sqrt{Y}, \sqrt{W}, \text{fSim}\}$ [34]. A single-qubit projector is applied with probability p to each qubit after every gate and before the final time step. We calculate the ES of many random realizations and bin the spectra for the same p to obtain the ESS. Our results are summarized in Fig. 5.

We begin by characterizing the quantum chaotic and integrable regimes for these circuits with Wigner-Dyson and Poisson distribution ESS, respectively, and then proceed to describe a previously unforeseen intermediate regime. For $p < p_S$, the ESS follows the GUE distribution over the entire range of r , as seen in Figs. 5(a) and 5(d). This corresponds to a highly entangled final state (i.e., volume-law state), indicating that the system has settled into the quantum chaotic regime. On the other hand, for $p > p_c$, with $p_c \simeq 0.41$, the ESS follows the Poisson distribution, which indicates that the system is integrable — see Figs. 5(c) and 5(f). The intuition for this change in the ESS as a function of p is that as the frequency of projectors is increased, the system becomes frozen in local states, therefore failing to entangle [16–23]. The small deviations from the expected exact Poisson distribution are due to our choice to avoid placing projectors in the final time step of simulations. This choice prevents us from potentially reducing the number of qubits in the system at the final time step. This effect disappears in the thermodynamic limit, which is investigated below by analysis using finite size scaling.

The primary result in our work is the discovery of an intermediate regime between $p_S < p < p_c$, where the ESS smoothly transitions from the GUE distribution to the Poisson. We call this phase the *residual repulsion phase*. As shown in Figs. 5(b) and 5(e), the strict level repulsion emblematic of the chaotic regime disappears, i.e., $P(r)$ becomes nonzero for $r \rightarrow 0$, and is replaced by a distribution that is between the two regimes, having a maxima at a nonzero value of r . For the quantum circuits taken from the set \mathbf{U}_{sync} , this transitional phase exists within a shifted window of p values, specifically $0.15 < p < 0.3$. We infer that this quicker transition occurs as a result of the particular universal gate set that we have selected. The argument is as follows: from \mathbf{U}_{sync} , the gate responsible for introducing entanglement into the system is the fSim gate, which is an *i*SWAP gate concatenated with a controlled Z , having an internal block structure as $1 \times 1, 2 \times 2, 1 \times 1$. The entanglement created by this gate is not as robust as that introduced by a random-Haar unitary gate, having no predefined symmetries or structure. The entanglement arising from this structured gate is therefore more susceptible to the presence of projective measurements. Similar evidence for this argument is also found when considering a separate universal set $\mathbf{U} = \text{CNOT}, \text{T}, \text{Hadamard}$. For this gate set, the entangling gate (CNOT) has an internal structure of $2 \times 2, 2 \times 2$, and only encodes a bit flip operation. In this case, we found that the residual repulsion phase exists only within the narrow window $0.01 < p < 0.03$. In light of these narrower transition windows for the gate sets \mathbf{U}_{sync} and \mathbf{U} , we choose to focus our attentions on the Haar-random circuits when investigating the thermodynamic limit of the ES.

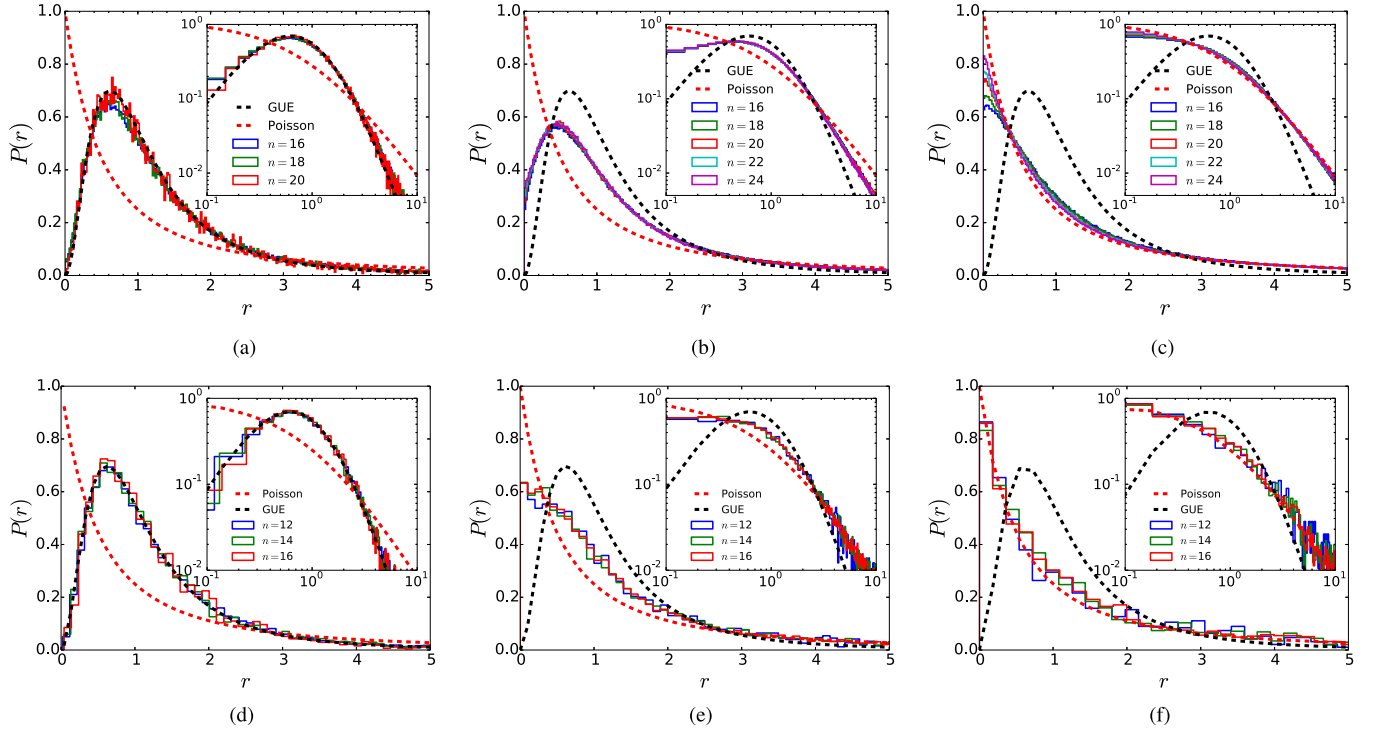


FIG. 5. Level spacing ratio distributions for the entanglement spectrum in the three phases of Fig. 1. From (a) to (c), the distributions are taken from circuits constructed from the random-Haar measure with N bits and with depth $d = N$, whereas from (d) to (f) the distributions are taken from circuits constructed from the universal gate set $\mathbf{U}_{\text{sys}} = \sqrt{X}, \sqrt{Y}, \sqrt{W}$, fSim with N bits and depth $d = 2N$. (a) Volume-law phase at $p = 0.2$, as indicated by the GUE distribution with level repulsion in the limits $r \rightarrow 0$ and a Gaussian tail at $r \rightarrow \infty$. (b) Residual repulsion phase at $p = 0.35$. Level repulsion disappears at $r \rightarrow 0$ while a majority of levels show level repulsion as indicated by the presence of a peak at finite r . (c) Poisson phase at $p = 0.5$. The spectrum displays an absence of level repulsion in similarity to the Poisson distribution. (d) Volume-law phase at $p = 0$, again indicated by the GUE statistics in the level spacing. (e) Residual repulsion phase at $p = 0.2$ exhibiting a shoulder instead of the shifted peak seen in (b). (f) Poisson phase at $p = 0.4$. The insets show the distributions in log-log scales in order to capture their behavior at the tails. In (a), results are obtained from 500 realizations for $n = 20$ and from 1000 realizations for $n < 20$. For (b) and (c), results are obtained from 1000 realizations for up to $n = 24$. In (d), (e), and (f) results are obtained from 500 realizations up to $n = 16$.

The Kullback-Leibler (KL) divergence, defined as

$$D_{KL}(P(x)||Q(x)) = \sum_x P(x) \ln \left(\frac{P(x)}{Q(x)} \right), \quad (15)$$

provides a measure of distance between two distributions $P(x)$ and $Q(x)$. If we calculate the D_{KL} between numerically calculated ESS distributions in the residual repulsion phase and the Poisson distribution, there should exist a point p where D_{KL} goes to zero, indicating that the numerical distributions have definitively become Poisson. However, as previously mentioned, the numeric distributions will necessarily deviate from the Poisson distribution for small- n circuits, such that the D_{KL} takes on a nonzero even in the Poisson phase. We therefore instead calculate the quantity

$$\Delta D_{KL} = D_{KL}(P_{\text{final}}||P_{\text{Poisson}}) - D_{KL}(P_{\text{2layer}}||P_{\text{Poisson}}), \quad (16)$$

where P_{final} is the calculated final state ESS distribution and P_{2layer} is the ESS distribution obtained by evolving random initial n -qubit product states with a single time step of Haar-random two-qubit gates. This quantity is positive in the residual repulsion phase but vanishes in the Poisson phase, and can hence be used as an indicator of the transition between the two phases with varying p .

In Fig. 6 we locate the transition point p_c by use of two distinct figures of merit. The first one is the aforementioned ΔD_{KL} , shown in Fig. 6(a). In the inset of Fig. 6(b), we plot ΔD_{KL} as a function of $1/n$ and use the data for various n to extrapolate linearly to $n \rightarrow \infty$. For small p , ΔD_{KL} extrapolates to a positive value at infinite size. The slope of the finite-size scaling curve increases with increasing p and at $p_c \simeq 0.41$ the curves start intersecting the $\Delta D_{KL} = 0$ axis at finite n , indicating a transition to the Poisson phase. Additionally, in Fig. 6(c) we show how the position of the maximum of $P_{\text{final}}(r)$ changes with p . For the Poisson distribution the maximum is at $r = 0$, whereas the GUE distribution has its maximum $P(r)$ at $r = (\sqrt{5} - 1)/2 \approx 0.618$. Figure 6(c) shows that the maximum of $P_{\text{final}}(r)$ decreases from $r \approx 0.618$, reaching zero close to p_c .

V. ESS TRANSITION AT p_c AND BOND PERCOLATION IN SPACETIME

We propose that the transition at p_c can be explained as bond percolation in a square lattice [35]. Percolation theory, however, predicts a transition at $p = 0.5$ and not at the observed $p_c \approx 0.41$. The reason for this discrepancy is that a single projector may affect multiple bonds.

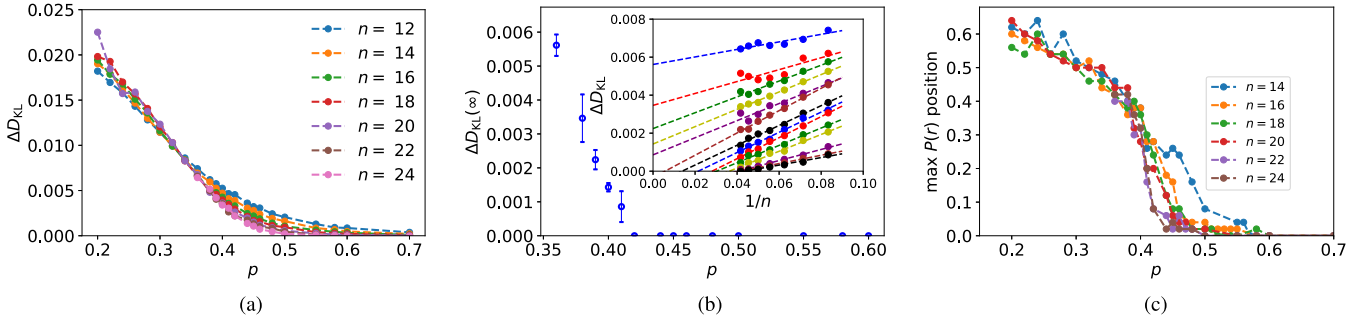


FIG. 6. (a) KL divergence ΔD_{KL} between $P(r)$ of evolved states and that of Poisson distribution as a function of projection rate p at different system size. (b) ΔD_{KL} at infinite size as a function of projection rate p . Inset: finite-size scaling for ΔD_{KL} . (c) Position of maximum $P(r)$ as a function of projection rate p . The position is obtained from a histogram plot (not shown).

To illustrate this, we consider the more direct mapping from circuit to tensor network shown in Fig. 7(a), which implements only the first step of Fig. 3(a). Without projectors, this yields a rotated square lattice with uniform bond dimension 2. Adding a projection operator to a bond reduces the dimension of that bond to 1. Naively, one would expect

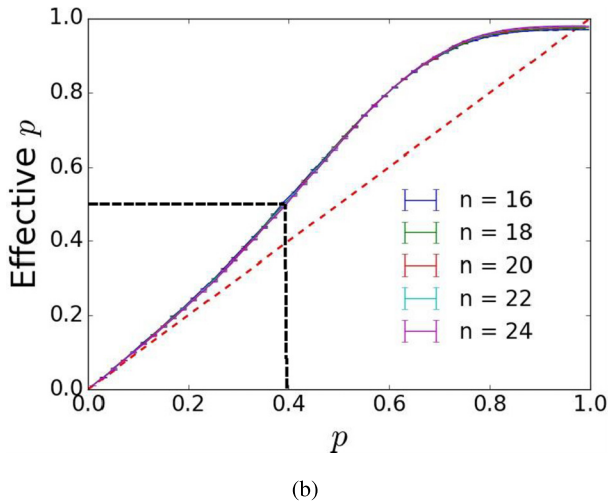
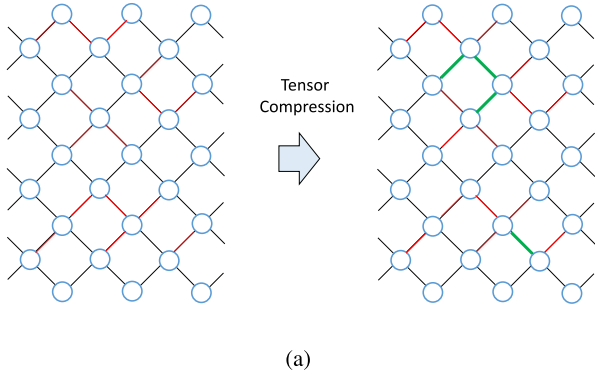


FIG. 7. (a) Bond dimension distribution before (left) and after (right) compression in a tensor network corresponding to circuit with projectors inserted in bonds marked red. Bonds whose dimension is reduced to 1 after compression are indicated by green lines. (b) Numerical results for effective projection rate as a function of projector insertion rate p .

that percolation occurs when a giant component of bonds with a projector forms, which for the square lattice would happen at density $p = 0.5$. However, the projection of a qubit to the computational basis has a disentangling effect also in its vicinity in spacetime and not only at the particular point of insertion of a projector. This effect is resolved by the compression step of our algorithm. In the example of Fig. 7(a), after the compression step, all bonds indicated by a green line are also reduced to dimension 1. This reasoning suggests a modified percolation threshold based on the density of dimension-1 bonds after a compression sweep, which we call the *effective* projection rate.

We verify this numerically. Figure 7(b) shows the effective projection rate as a function of the density of projectors p . Dashed black lines indicate that an effective projection ratio of 0.5 is achieved at $p \approx p_c$. This agrees well with our ESS-based estimate for the transition point. As the two-dimensional bond percolation transition is characterized by absence of long range correlations, this reflects the fact that the system becomes integrable at $p > p_c$. It should be noted that the compression algorithm is exact to machine precision and hence accuracy does not factor appreciably into this reasoning.

VI. CONCLUSION

The work presented here explores projection-driven quantum circuits from the perspective of the ESS of the output state. Our results uncover three distinct behaviors of the entanglement spectrum with increasing the rate, p , of projection of qubits to the computational basis. The first regime, $0 < p < p_s$, displays volume-law entanglement entropy and Wigner-Dyson statistics of the ESS. At $p = p_s$ the systems undergoes a volume-to-area-law transition similar to that studied in Refs. [16–23]. The principal result of this paper is that the ESS in the area law phase emerging at $p = p_s$ is nonuniversal and interpolates between Wigner-Dyson and Poisson statistics, with the region of residual level repulsion extending up to a second transition, $p = p_c > p_s$, beyond which the ESS of the system is Poisson.

This second transition within the area law phase is revealed by repartitioning the system randomly into two subsystems and probing the entanglement level statistics. In particular, our tensor network algorithm, which resolves entanglement by monitoring the distribution of bond dimensions across the

1+1D spacetime of the circuit, allows us to interpret this transition as a percolation transition of entangled bonds and to locate the corresponding critical value $p = p_c$ via finite size scaling. We note that, unlike previous results in Refs. [21,22] obtained in the limit of large local Hilbert space dimension that associate the volume-to-area law transition to percolation in a classical Potts model, here we find that the ESS transition from nonuniversal to Poisson statistics is due to percolation of entangled bonds in the circuit spacetime itself.

This work leaves open the question of the origin of the intermediate regime with nonuniversal statistics of the entanglement spectrum. The nature of level statistics is determined by the details of the interactions between eigenvalues which can induce complex nonuniversal level statistics, including a Griffiths-like phase observed in studies of many body localization [36,37]. More detailed work is needed to elucidate the nonuniversal regime in the ESS identified in this paper.

Nevertheless, the methodology and findings presented here enable a more fine-grained characterization of entanglement buildup in noisy quantum circuits compared to analyses based solely on entanglement entropy, and can be used to study near-term noisy quantum chips via quantum state tomography.

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