

## Learning interacting particle systems: Diffusion parameter estimation for aggregation equations

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In this paper, we study the parameter estimation of interacting particle systems subject to the Newtonian aggregation and Brownian diffusion. Specifically, we construct an estimator  $\hat{\nu}$  with partial observed data to approximate the diffusion parameter  $\nu$ , and the estimation error is achieved. Furthermore, we extend this result to general aggregation equations with a bounded Lipschitz interaction field.

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## 1. Introduction

Parameter estimation of (stochastic) dynamical systems is an exciting area of research with ubiquitous applications in many areas in science and technology, where it usually requires incorporating data into a model. This is often known as data assimilation (see Ref. 41 for a mathematical introduction) in particular in the context of numerical weather forecast. It is also known as system identification in the control literature (see for example Chaps. 11 and 12 of Ref. 40 for applications for modeling robots). In such problems, a physical model of the form of a dynamical system is derived from (partial) empirical observations and is usually calibrated with and improved by experimental data. The problem is also related to uncertainty quantification, which is important as it enables the building of more realistic models and making better predictions of their behavior in the future. In the modeling of self-organized systems, different ways to qualify uncertainties have been studied (see for example Refs. 2, 8, 15, 19, 37 and 49).

In this work, we are interested in the parameter estimation problems arising from a particular class of physical systems that can be modeled by interacting particle systems. This means that the dynamics of the system is determined by interactions between agents (particles) together with some intrinsic or extrinsic random effects. Such systems are widely used to establish different mathematical models describing collective behaviors of organisms and social aggregations, for instance flocks of birds,<sup>28</sup> aggregation of bacteria,<sup>4</sup> schools of fish,<sup>27</sup> swarms formed by insects,<sup>5</sup> opinion dynamics<sup>43</sup> and robotics and space missions.<sup>36</sup> Various types of diffusion are considered in these models: While linear diffusion is more commonly used,<sup>18</sup> the diffusion can be slow in areas with few particles, known as the degenerate (slow) diffusion model<sup>48</sup>; and similarly, the diffusion can also be fast.<sup>47</sup> One may also consider the nonlocal diffusion, where organisms adopt Lévy process search strategies which have continuous paths interspersed with random jumps.<sup>29</sup> Thus qualifying the type of the diffusion can significantly reduce the uncertainty in model predictions and is hence a very important step in many applications. This paper focuses on the case of Brownian diffusion with unknown diffusion parameter. We study the diffusion parameter estimation of such interacting particle systems with partial observed data.

More precisely, the microscopic agent-based model investigated here describes the evolution of positions of  $N$  agents, denoted by  $\{X_i^t\} \subset \mathbb{R}^d$ ,  $i = 1, \dots, N$ , whose evolution is governed by a system of stochastic differential equations (SDEs) of the type

$$dX_i^t = \frac{1}{N-1} \sum_{j \neq i}^N F(X_i^t - X_j^t) dt + \sqrt{2\nu} dB_i^t, \quad i = 1, \dots, N, \quad (1.1)$$

where  $F$  models some pairwise interaction between the agents and  $B_i^t$  are independent realizations of Brownian motions which count for extrinsic random perturbation of the agent positions. In such systems, the agents are assumed to be identical,

so that the noise level  $\nu$  is the same for each agent. In this work, we assume that the interaction kernel  $F$  between agents is known, while the noise level  $\nu$  is to be determined. More specifically, we will focus on the case when the interaction between agents is given by Newtonian type interaction for dimension  $d \geq 2$ , or more precisely, a regularized Newtonian interaction, to be specified below. Suppose we observe or track the trajectories of  $K$  agents on the time interval  $[0, T]$ , where  $1 \ll K \ll N$ , the question we address in this work is *how to estimate  $\nu$  and to quantify the error of the estimator*.

A more general situation one may consider is the problem for which the interaction kernel is also to be determined, this will be left for future works. In Ref. 21, authors solved the following inverse problem for aggregation equations: given an equilibrium state, they constructed a corresponding force  $F$  to ensure that equilibrium. We also note the recent work<sup>8</sup> which considers learning the interaction kernel for a deterministic interacting particle systems through a variational approach. While admittedly that we have taken a simple scenario and a somewhat simplistic model for interacting agents, already many interesting issues arise from both mathematical and application point of view. For instance, how accurate one can make the estimation by only observing/tracking a few agents. How the potential singularities of the interacting potential (such as Coulomb or Newtonian type) impact the estimation accuracy.

Observe that the scaling of (1.1) is chosen such that we are in the mean-field regime, as the interaction strength decreases as  $1/N$  as the number of agents  $N \rightarrow \infty$ . It is thus expected that in the limit  $N \rightarrow \infty$ , the system can be well described by a mean-field dynamics, which can be described as the following non-linear partial differential equation (PDE):

$$\partial_t \rho = \nu \Delta \rho - \nabla \cdot (\rho F * \rho), \quad x \in \mathbb{R}^d, \quad t > 0, \quad (1.2a)$$

$$\rho(x, 0) = \rho_0(x), \quad (1.2b)$$

where the noise level  $\nu > 0$  enters the PDE system as a diffusion parameter. In particular, here the interaction kernel is chosen as Newtonian:

$$F(x) = \mp \frac{C_* x}{|x|^d}, \quad \forall x \in \mathbb{R}^d \setminus \{0\}, \quad d \geq 2, \quad (1.3)$$

with  $C_* = \frac{\Gamma(d/2)}{2\pi^{d/2}}$ . Here the sign  $\mp$  indicates that the interaction between individuals can either be attraction or repulsion. Specifically, when the mechanism of interaction is attraction, the mean field equation (1.2) becomes the parabolic–elliptic Keller–Segel equation,<sup>38,44</sup> which is a prototypical model for chemotaxis and has been used in many related modeling scenarios. The analysis of the scaling limit of interacting particle system (1.1) is usually called the *mean-field limit*, which passes limits from microscopic discrete particle systems to macroscopic continuum models.

While it would be intriguing to study the parameter identification problem for the particle system (1.1) with the Newtonian interaction (1.3), such microscopic system is however ill-posed, as shown by the recent deep result by Fournier and

Jourdain (see Proposition 4 of Ref. 23): For any  $N \geq 2$  and  $T > 0$ , denote  $\{X_i(t); t \in [0, T], i = 1, \dots, N\}$  the solution to (1.1) with  $F$  given in (1.3), then

$$\mathbb{P}(\exists s \in [0, T], \exists 1 \leq i < j \leq N : X_i(s) = X_j(s)) > 0,$$

i.e. the singularity cannot be avoided in any finite time with a positive probability and thus the particle system is not well-defined.

Classical results of the mean-field limit require the kernel  $F \in W^{1,\infty}(\mathbb{R}^d)$ . One possible way to overcome singularity is to regularize the kernel  $F$ . In particular, in this work we consider the regularized kernel  $F^N$ :

$$F^N = F * \psi_N, \quad \psi_N(x) = N^{d\delta} \psi(N^\delta x), \quad (1.4)$$

where  $\delta$  the cut-off index and  $0 \leq \psi(x) \in C_0^\infty(\mathbb{R}^d)$  is a cut-off function, which satisfies  $\psi(x) = \psi(|x|)$  and  $\int_{\mathbb{R}^d} \psi(x) dx = 1$ . Then we have the regularized stochastic particle system  $\{X_i^t\}_{i=1}^N$  satisfying

$$dX_i^t = \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^t - X_j^t) dt + \sqrt{2\nu} dB_i^t, \quad i = 1, \dots, N, \quad (1.5)$$

where the initial data  $\{X_i^0\}_{i=1}^N$  are independently identically distributed (i.i.d.) with the common density function  $\rho_0$ . Since the regularized kernel is Lipschitz for any fixed  $N$ , the system above has a unique global strong solution. The corresponding aggregation equation has the form

$$\partial_t \rho = \nu \Delta \rho - \nabla \cdot (\rho F^N * \rho), \quad x \in \mathbb{R}^d, \quad t > 0, \quad (1.6a)$$

$$\rho(x, 0) = \rho_0(x). \quad (1.6b)$$

Classical results for mean-field limit with globally Lipschitz forces were obtained by Braun and Hepp<sup>10</sup> and Dobrushin.<sup>16</sup> Then Bolley, Cañizo and Carrillo<sup>7</sup> presented an extension of the classical theory to the particle system with only locally Lipschitz interacting force. The last few years have seen great progress in mean-field limits for singular forces by treating them with an  $N$ -dependent cut-off. In particular, the mean-field limit for the Keller–Segel model has been rigorously proved in Refs. 22, 23, 24, 30 and 31. And the deterministic particle method for aggregation equations can be found in Refs. 11 and 13. For a general overview of this topic we refer readers to Refs. 12, 25, 34, 35 and 46.

Considering the parameter estimation problem for diffusion processes, there is a huge literature in statistics and econometrics, often related to the estimation of volatility in financial models. A complete literature review is beyond our scope and we refer the readers to Ref. 45 for an overview. To make the scenario more realistic, instead of assuming the availability of some trajectories  $\{X_i^t\}_{i=1}^N$  for all time  $t \in [0, T]$ , we consider the case that trajectories are only observed at discrete time snapshots during the time interval. Diffusion parameter estimation problems based on discrete observations have been discussed by many authors.<sup>1,3,6,14,17,20,33,39,50</sup>

However, to our knowledge, no previous work has been done for diffusion estimation in the context of interacting particle systems. Specifically, there are a few differences between our work and these works: (1) We consider parameter estimation of an interacting particle system, however authors mentioned above studied a single diffusion process. (2) Our estimator (1.8) concerns the information of interacting particles, but they only investigated one trajectory and take the expectation value of this stochastic process. (3) In our setting, the interacting force  $F$  is singular, while the drift function is assumed to be regular enough in usual statistics literature as mentioned earlier. Our main result, given below in Theorem 1.1 after we make precise the estimator  $\hat{\nu}$ , quantifies the estimation error of the proposed estimator.

Take a time step  $\Delta t > 0$  and let  $t_n := n\Delta t$  and  $M := \frac{T}{\Delta t}$  (we assume that  $\frac{T}{\Delta t}$  is an integer). Denote  $X_i^{(n)} := X_i^{t_n}$  as the solution to (1.5) at time  $t_n$ . Namely, one has

$$\begin{aligned} X_i^{(n+1)} - X_i^{(n)} &= \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds + \sqrt{2\nu}(B_i^{t_{n+1}} - B_i^{t_n}) \\ &= \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds + \sqrt{2\nu\Delta t}\mathcal{N}_i^{(n)}, \end{aligned} \quad (1.7)$$

where  $\mathcal{N}_i^{(n)} \sim \mathcal{N}(0, 1)^d$ , i.e. the standard Gaussian distribution in dimension  $d$ .

Then we are ready to define our estimator for the diffusion parameter

$$\hat{\nu} := \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |X_i^{(n+1)} - X_i^{(n)}|^2, \quad (1.8)$$

where  $1 \ll K \ll N$ , which means we only have partial observations.

Our main result quantifies the estimation error of the proposed estimator (1.8), which is summarized as below.

**Theorem 1.1.** *Suppose the initial data  $0 \leq \rho_0(x) \in L^1 \cap L^\infty(\mathbb{R}^d)$  and let  $\rho(x, t)$  be the regular solution of the aggregation equation (1.2) up to the time  $T$  such that  $\rho \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^d))$ . Take a time step  $\Delta t > 0$  and let  $t_n := n\Delta t$  and  $M := \frac{T}{\Delta t}$ . Assume  $\{X_i^{(n)}\}_{i=1, n=0}^{K, M}$  be the  $K$  ( $1 \ll K \ll N$ ) sample trajectories satisfying (1.7) with the cut-off index  $0 < \delta < \frac{1}{d}$  at time  $t_n$ . For any  $\alpha > 0$ , there exists some constant  $N_0 > 0$  depending only on  $\nu, \alpha, T$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ , such that for  $N \geq N_0$ , the estimator  $\hat{\nu}$  defined in (1.8) is an approximation of  $\nu$ , and the following estimate holds:*

$$\mathbb{P}(|\hat{\nu} - \nu| \leq C_\alpha \nu^{\frac{1}{2}} \Delta t^{\frac{1}{2}} (1 + \nu^{\frac{1}{2}} N^{-\delta} \log(N)) + \gamma \nu) \geq 1 - N^{-\alpha} - 2e^{-\frac{dKM\gamma^2}{8}}, \quad (1.9)$$

for any  $\gamma \in (0, 1)$ , where  $C_\alpha > 0$  depends only on  $\alpha, T$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ .

Let us remark on two simple consequences from (1.9). If we consider  $N \rightarrow \infty$ , (1.9) simplifies to

$$\mathbb{P}(|\widehat{\nu} - \nu| \leq C_\alpha \nu^{\frac{1}{2}} \Delta t^{\frac{1}{2}} + \gamma \nu) \geq 1 - 2e^{-\frac{dK M \gamma^2}{8}} \quad (1.10)$$

for any  $\gamma \in (0, 1)$ . Thus despite that we are dealing with an interacting system, when  $N$  is large the accuracy of the estimator is similar to that based on using  $K$  (independent) trajectory observations of a *non-interacting* particle system. Moreover, to get from (1.9) a simpler looking error bound, we can choose  $\Delta t^{\frac{1}{2}} = \gamma$  (assuming we are able to adjust the frequency of observations) and get

$$\mathbb{P}(|\widehat{\nu} - \nu| \leq (C_\alpha \nu^{\frac{1}{2}} + \nu) \Delta t^{\frac{1}{2}}) \geq 1 - 2e^{-\frac{dKT}{8}}, \quad (1.11)$$

where we used the fact that  $M\gamma^2 = \frac{T}{\Delta t}\gamma^2 = T$ . Estimate (1.11) indicates that increasing the number  $K$  of the observed data improves the accuracy of our estimator  $\widehat{\nu}$  (the probability increase as  $K$  becomes larger).

To prove the theorem on the error of the estimator, we defined an intermediate estimator

$$\nu_{K,N} := \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| X_i^{(n+1)} - X_i^{(n)} - \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds \right|^2,$$

then we split the error into two parts

$$|\widehat{\nu} - \nu| \leq |\widehat{\nu} - \nu_{K,N}| + |\nu_{K,N} - \nu|. \quad (1.12)$$

Let us denote

$$a_i^n := X_i^{(n+1)} - X_i^{(n)}, \quad b_i^n := \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds,$$

then

$$\begin{aligned} & |\nu_{K,N} - \widehat{\nu}| \\ &= \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| |a_i^n - b_i^n|^2 - |a_i^n|^2 \right| \\ &\leq \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |b_i^n|^2 + \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |a_i^n b_i^n| \\ &\leq \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |b_i^n|^2 + \left( \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |a_i^n|^2 \right)^{\frac{1}{2}} \left( \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |b_i^n|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |b_i^n|^2 + (2\widehat{\nu})^{\frac{1}{2}} \left( \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |b_i^n|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Notice that

$$\begin{aligned}
\frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |b_i^n|^2 &= \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds \right|^2 \\
&\leq \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| \int_{t_n}^{t_{n+1}} \left( \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) \right. \right. \\
&\quad \left. \left. - \int_{\mathbb{R}^d} F^N(X_i^s - y) \rho(y, s) dy \right) ds \right|^2 \\
&\quad + \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} F^N(X_i^s - y) \rho(y, s) dy ds \right|^2 \\
&:= |\mathcal{I}_2| + |\mathcal{I}_3|,
\end{aligned}$$

which implies

$$|\nu_{K,N} - \hat{\nu}| \leq |\mathcal{I}_2| + |\mathcal{I}_3| + (2\hat{\nu})^{\frac{1}{2}} (2|\mathcal{I}_2| + 2|\mathcal{I}_3|)^{\frac{1}{2}}.$$

Collecting above inequality and (1.12), we conclude

$$|\hat{\nu} - \nu| \leq (1 + 2\hat{\nu}^{\frac{1}{2}})(|\mathcal{I}_2|^{\frac{1}{2}} + |\mathcal{I}_3|^{\frac{1}{2}}) + |\nu_{K,N} - \nu|, \quad (1.13)$$

when  $|\mathcal{I}_2|, |\mathcal{I}_3| < 1$ .

Moreover, notice that

$$\begin{aligned}
\hat{\nu} &= \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |a_i^n|^2 \leq \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |a_i^n - b_i^n|^2 + \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |b_i^n|^2 \\
&\leq 2\nu_{K,N} + 2|\mathcal{I}_2| + 2|\mathcal{I}_3| \\
&\leq 2|\nu_{K,N} - \nu| + 2|\mathcal{I}_2| + 2|\mathcal{I}_3| + 2\nu \leq 6 + 2\nu, \quad (1.14)
\end{aligned}$$

when  $|\nu_{K,N} - \nu|, |\mathcal{I}_2|, |\mathcal{I}_3| < 1$ . Combining (1.14) with (1.13), it yields that

$$|\hat{\nu} - \nu| \leq C\nu^{\frac{1}{2}} (|\mathcal{I}_2|^{\frac{1}{2}} + |\mathcal{I}_3|^{\frac{1}{2}}) + |\nu_{K,N} - \nu|, \quad (1.15)$$

where  $C$  is a positive number.

In the sequel, we will see that the estimate of  $|\mathcal{I}_3|$  is a direct result from the property of regularized kernel  $F^N$  (see estimate (3.2)). And the estimate of  $|\mathcal{I}_2|$  is an estimate of interaction (see Theorem 2.2), which follows from the mean-field limit result (see Theorem 2.1). As for the estimate of  $|\nu_{K,N} - \nu|$ , it can be deduced from a concentration inequality of Chi-squared distribution (see Theorem 3.1).

The work is organized as follows: In Sec. 2, we will give a rigorous proof of the mean-field limit for aggregation equations with Newtonian potential. Based on this, we also obtain an error estimate on interaction. Section 3 is devoted to prove that our estimator  $\hat{\nu}$  is a good approximation of  $\nu$  and the convergence rate between

them is achieved. Then in Sec. 4 we further extend our result to the case where the aggregation equation has a bounded Lipschitz interacting force.

## 2. Mean-Field Limit and Estimate on Interaction

In this section, we will prove the mean-field limit for particle system (1.5). Namely, given the solution  $\rho := \rho(x, t)$  to the mean-field equation (1.6), we construct a mean-field trajectories  $\{Y_i^t\}_{i=1}^N$  from (1.6), then we prove the closeness between  $X_t = (X_1^t, \dots, X_N^t)$  and  $Y_t = (Y_1^t, \dots, Y_N^t)$ . To do this, we shall consider again a Newtonian system with noise. However, this time not subject to the pair interaction but under the influence of the external mean field  $F^N * \rho(x, t)$

$$dY_i^t = \int_{\mathbb{R}^d} F^N(Y_i^t - y) \rho(y, t) dy dt + \sqrt{2\nu} dB_i^t, \quad i = 1, \dots, N, \quad (2.1)$$

here we let  $\{Y_i^t\}_{i=1}^N$  have the same initial condition as  $\{X_i^t\}_{i=1}^N$  (i.i.d. with the common density  $\rho_0$ ). Since the particles are subject to an external field, the independence is conserved. Therefore, the  $\{Y_i^t\}_{i=1}^N$  are distributed i.i.d. according to the common probability density  $\rho$ . We remark that the aggregation equation (1.6) is Kolmogorov's forward equation for any solution of (2.1), and in particular their probability distribution  $\rho$  solves (1.6).

### 2.1. Preliminaries

**Notations.** The generic constant will be denoted generically by  $C$ , even if it is different from line to line. The notation  $\|\cdot\|_p$  represents the usual  $L^p$ -norm of a function for any  $1 \leq p \leq \infty$ . For a vector  $X_t = (X_1^t, \dots, X_N^t)$ , we denote

$$\|X_t\|_\infty := \sup_{i=1, \dots, N} |X_i^t|.$$

Since error estimates obtained later are valid when the solution of PDE (1.6) is regular enough, we assume that

$$0 \leq \rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d), \quad (2.2)$$

then Eq. (1.6) has a unique local solution with the following regularity:

$$\|\rho\|_{L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^d))} \leq C(\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}) =: C_{\rho_0},$$

where  $C_{\rho_0}$  is independent of  $N$  and  $T > 0$  depends only on  $\nu$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ . The proof of this result is a standard process (see for example Proposition 4.1 of Ref. 22).

Let us recall some estimates of the regularized kernel  $F^N$  defined in (1.4).

**Lemma 2.1.** ([30, Lemma 2.1]) (1)  $F^N(0) = 0$ ,  $F^N(x) = F(x)$  for any  $|x| \geq N^{-\delta}$  and  $|F^N(x)| \leq |F(x)|$  for any  $x \in \mathbb{R}^d$ ;  
 (2)  $|\partial^\beta F^N(x)| \leq CN^{(d+|\beta|-1)\delta}$  for any  $x \in \mathbb{R}^d$ ;  
 (3)  $\|F^N\|_2 \leq CN^{(\frac{d}{2}-1)\delta}$ .

Next we define a cut-off function  $L^N$ , which will provide the local Lipschitz bound for  $F^N$ .

**Definition 2.1.** Let

$$L^N(x) = \begin{cases} \frac{6^d}{|x|^d} & \text{if } |x| \geq 6N^{-\delta}, \\ N^{d\delta} & \text{else,} \end{cases} \quad (2.3)$$

and  $\mathcal{L}^N : \mathbb{R}^{dN} \rightarrow \mathbb{R}^N$  be defined by  $(\mathcal{L}^N(X_t))_i := \frac{1}{N-1} \sum_{i \neq j}^N L^N(X_i^t - X_j^t)$ . Furthermore, we define  $\bar{\mathcal{L}}^N(Y_t)$  by  $(\bar{\mathcal{L}}^N(Y_t))_i := \int_{\mathbb{R}^d} L^N(Y_i^t - x) \rho(x, t) dx$ .

Denote

$$(\mathcal{F}^N(Y^t))_i := \frac{1}{N-1} \sum_{j \neq i}^N F^N(Y_i^t - Y_j^t), \quad (2.4)$$

then we have the local Lipschitz continuity of  $\mathcal{F}^N$ .

**Lemma 2.2.** (Lemma 2.3 of Ref. 32) *If  $\|X_t - Y_t\|_\infty \leq 2N^{-\delta}$ , then it holds that*

$$\|\mathcal{F}^N(X_t) - \mathcal{F}^N(Y_t)\|_\infty \leq C \|\mathcal{L}^N(Y_t)\|_\infty \|X_t - Y_t\|_\infty, \quad (2.5)$$

for some  $C > 0$  independent of  $N$ .

The following observations of  $F^N$  and  $L^N$  turn out to be very helpful in the sequel.

**Lemma 2.3.** (Lemma 2.4 of Ref. 32) *Let  $L^N(x)$  be defined in Definition 2.1 and  $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$ . Then there exists a constant  $C > 0$  independent of  $N$  such that*

$$\begin{aligned} \|L^N * \rho\|_\infty &\leq C \log(N) (\|\rho\|_1 + \|\rho\|_\infty), \\ \|(L^N)^2 * \rho\|_\infty &\leq C N^{d\delta} (\|\rho\|_1 + \|\rho\|_\infty); \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \|F^N * \rho\|_\infty &\leq C (\|\rho\|_1 + \|\rho\|_\infty), \\ \|\nabla F^N * \rho\|_\infty &\leq C \log(N) (\|\rho\|_1 + \|\rho\|_\infty). \end{aligned} \quad (2.7)$$

Also, we need the following concentration inequality to provide us the probability bounds of random variables.

**Lemma 2.4.** *Let  $Z_1, \dots, Z_N$  be i.i.d. random variables with  $\mathbb{E}[Z_i] = 0$ ,  $\mathbb{E}[Z_i^2] \leq g(N)$  and  $|Z_i| \leq C \sqrt{Ng(N)}$ . Then for any  $\alpha > 0$ , the sample mean  $\bar{Z} = \frac{1}{N} \sum_{i=1}^N Z_i$  satisfies*

$$\mathbb{P}\left(|\bar{Z}| \geq \frac{C_\alpha \sqrt{g(N)} \log(N)}{\sqrt{N}}\right) \leq N^{-\alpha},$$

where  $C_\alpha$  depends only on  $C$  and  $\alpha$ .

The proof can be seen in Lemma 1 of Ref. 26, which is a direct result of the Taylor expansion and the Markov's inequality.

## 2.2. Mean-field limit for the aggregation equation with Newtonian potential

In this section, we obtain the maximal distance between the exact microscopic dynamics (1.5) and the approximate mean-field dynamics (2.1). Denote

$$(\bar{\mathcal{F}}^N(Y_t))_i := \int_{\mathbb{R}^d} F^N(Y_i^t - x) \rho(x, t) dx, \quad (2.8)$$

then we can introduce the following lemma of law of large numbers.

**Lemma 2.5.** *At any fixed time  $t \in [0, T]$ , suppose that  $Y_t = (Y_i^t)_{i=1,\dots,N}$  satisfies the mean-field dynamics (2.1) with i.i.d initial data sharing the common density  $\rho_0$  satisfying (2.2). Assume that  $\mathcal{F}^N$  and  $\bar{\mathcal{F}}^N$  are defined in (2.4) and (2.8) respectively,  $\mathcal{L}^N$  and  $\bar{\mathcal{L}}^N$  are showed in Definition 2.1. For any  $\alpha > 0$  and  $0 < \delta \leq \frac{1}{d}$ , there exists a constant  $C_{1,\alpha} > 0$  depending only on  $\alpha$ ,  $T$  and  $C_{\rho_0}$  such that*

$$\mathbb{P}(\|\mathcal{F}^N(Y_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \geq C_{1,\alpha} N^{\frac{\delta(d-2)-1}{2}} \log(N)) \leq N^{-\alpha}, \quad (2.9)$$

and

$$\mathbb{P}(\|\mathcal{L}^N(Y_t) - \bar{\mathcal{L}}^N(Y_t)\|_\infty \geq C_{1,\alpha} N^{\frac{\delta\delta-1}{2}} \log(N)) \leq N^{-\alpha}. \quad (2.10)$$

**Proof.** We can prove this lemma by using Lemma 2.4. Due to the exchangeability of the particles, we are ready to bound

$$\begin{aligned} (\mathcal{F}^N(Y_t))_1 - (\bar{\mathcal{F}}^N(Y_t))_1 &= \frac{1}{N-1} \sum_{j=2}^N F^N(Y_1^t - Y_j^t) - \int_{\mathbb{R}^3} F^N(Y_1^t - x) \rho(x, t) dx \\ &= \frac{1}{N-1} \sum_{j=2}^N Z_j, \end{aligned}$$

where

$$Z_j := F^N(Y_1^t - Y_j^t) - \int_{\mathbb{R}^d} F^N(Y_1^t - x) \rho(x, t) dx.$$

Since  $Y_1^t$  and  $Y_j^t$  are independent when  $j \neq 1$  and  $F^N(0) = 0$ , let us consider  $Y_1^t$  as given and denote  $\mathbb{E}'[\cdot] = \mathbb{E}[\cdot | Y_1^t]$ . It is easy to show that  $\mathbb{E}'[Z_j] = 0$  since

$$\mathbb{E}'[F^N(Y_1^t - Y_j^t)] = \int_{\mathbb{R}^d} F^N(Y_1^t - x) \rho(x, t) dx.$$

To use Lemma 2.4, we need a bound for the variance

$$\mathbb{E}'[|Z_j|^2] = \mathbb{E}' \left[ \left| F^N(Y_1^t - Y_j^t) - \int_{\mathbb{R}^3} F^N(Y_1^t - x) \rho(x, t) dx \right|^2 \right].$$

Since it follows from Lemma 2.3 that

$$\int_{\mathbb{R}^3} F^N(Y_1^t - x) \rho(x, t) dx \leq C(\|\rho\|_1 + \|\rho\|_\infty),$$

it suffices to bound

$$\mathbb{E}'[F^N(Y_1^t - Y_j^t)] = \int_{\mathbb{R}^d} F^N(Y_1^t - x)\rho(x, t)dx \leq C(\|\rho\|_1 + \|\rho\|_\infty) \leq C(T, C_{\rho_0}),$$

and

$$\begin{aligned} \mathbb{E}'[F^N(Y_1^t - Y_j^t)^2] &= \int_{\mathbb{R}^d} F^N(Y_1^t - x)^2\rho(x, t)dx \\ &\leq \|\rho\|_\infty \|F^N\|_2^2 \leq C(T, C_{\rho_0})N^{\delta(d-2)}, \end{aligned}$$

where we have used  $\|F^N\|_2 \leq CN^{\delta(\frac{d}{2}-1)}$  in Lemma 2.1(iii). Hence one has

$$\mathbb{E}'[|Z_j|^2] \leq CN^{\delta(d-2)}.$$

So the hypotheses of Lemma 2.4 are satisfied with  $g(N) = CN^{\delta(d-2)}$ . In addition, it follows from (ii) in Lemma 2.1 that  $|Z_j| \leq CN^{\delta(d-1)} \leq C\sqrt{Ng(N)}$ . Hence, using Lemma 2.4, we have the probability bound

$$\mathbb{P}(|(\mathcal{F}^N(Y_t))_1 - (\bar{\mathcal{F}}^N(Y_t))_1| \geq C(\alpha, T, C_{\rho_0})N^{\frac{\delta(d-2)-1}{2}}\log(N)) \leq N^{-\alpha}.$$

Similarly, the same bound must also apply to the cases where  $i = 2, \dots, N$ ,

$$\mathbb{P}(\|\mathcal{F}^N(Y_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \geq C(\alpha, T, C_{\rho_0})N^{\frac{\delta(d-2)-1}{2}}\log(N)) \leq N^{1-\alpha}. \quad (2.11)$$

Let  $C_{1,\alpha}$  be the constant in (2.11), we conclude (2.9).

To prove (2.10), we follow the same procedure above

$$\begin{aligned} (\mathcal{L}^N(Y_t))_1 - (\bar{\mathcal{L}}^N(Y_t))_1 &= \frac{1}{N-1} \sum_{j=2}^N L^N(Y_1^t - Y_j^t) - \int_{\mathbb{R}^d} L^N(Y_1^t - x)\rho(x, t)dx \\ &= \frac{1}{N-1} \sum_{j=2}^N Z_j, \end{aligned}$$

where

$$Z_j = L^N(Y_1^t - Y_j^t) - \int_{\mathbb{R}^3} L^N(Y_1^t - x)\rho(x, t)dx.$$

It is easy to show that  $\mathbb{E}'[Z_j] = 0$ . To use Lemma 2.4, we need a bound for the variance. One computes that

$$\begin{aligned} \mathbb{E}'[L^N(Y_1^t - Y_j^t)] &= \int_{\mathbb{R}^d} L^N(Y_1^t - x)\rho(x, t)dx \leq C\log(N)(\|\rho\|_1 + \|\rho\|_\infty) \\ &\leq C(T, C_{\rho_0})\log(N), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}'[L^N(Y_1^t - Y_j^t)^2] &= \int_{\mathbb{R}^d} L^N(Y_1^t - x)^2\rho(x, t)dx \leq CN^{d\delta}(\|\rho\|_1 + \|\rho\|_\infty) \\ &\leq C(T, C_{\rho_0})N^{d\delta}, \end{aligned}$$

where we have used the estimates of  $L^N$  in Lemma 2.3. Hence one has

$$\mathbb{E}'[|Z_j|^2] \leq CN^{d\delta}.$$

So the hypotheses of Lemma 2.4 are satisfied with  $g(N) = CN^{d\delta}$ . In addition, it follows from Definition 2.1 that  $|Z_j| \leq CN^{d\delta} \leq C\sqrt{Ng(N)}$ . Hence, we have the probability bound

$$\mathbb{P}(|(\mathcal{L}^N(Y_t))_1 - (\bar{\mathcal{L}}^N(Y_t))_1| \geq C(\alpha, T, C_{\rho_0})N^{\frac{d\delta-1}{2}} \log(N)) \leq N^{-\alpha},$$

by Lemma 2.4, which leads to

$$\mathbb{P}(\|\mathcal{L}^N(Y_t) - \bar{\mathcal{L}}^N(Y_t)\|_\infty \geq C(\alpha, T, C_{\rho_0})N^{\frac{d\delta-1}{2}} \log(N)) \leq N^{1-\alpha}. \quad (2.12)$$

Thus, (2.10) follows from (2.12).  $\square$

Next we improve the consistency error to all time. To do this, we need the following lemma, where we temporarily set the time step size  $\Delta t = t_{n+1} - t_n = N^{-\frac{\beta}{d}}$  with  $\beta > 2$ , which is only for the purpose of proving Propositions 2.1 and 2.2. Here  $N^{-\frac{\beta}{d}}$  will not influence the choice of the  $\Delta t$  in Theorem 1.1.

**Lemma 2.6.** *Assume that the time step size  $\Delta t = t_{n+1} - t_n = N^{-\frac{\beta}{d}}$  for  $\beta > 2$  and  $Y_t$  satisfies the mean-field dynamics (2.1). There exists some constant  $C_B > 0$  depending only on  $T$  and  $C_{\rho_0}$ , such that it holds*

$$\mathbb{P}\left(\sup_n \sup_{t \in [t_n, t_{n+1}]} \|Y_t - Y_{t_n}\|_\infty \geq C_B N^{\frac{1}{2}} N^{-\frac{\beta+2}{2d}}\right) \leq C_B N^{\frac{2+\beta}{2d}} \exp(-C_B N^{\frac{\beta-2}{d}}).$$

**Proof.** Notice that for  $t \in [t_n, t_{n+1}]$

$$\begin{aligned} Y_t - Y_{t_n} &= \int_{t_n}^t \int_{\mathbb{R}^d} F^N(Y_i^s - y) \rho(y, s) dy ds + \sqrt{2\nu \Delta t} (B^t - B^{t_n}) \\ &=: I_1(t) + I_2(t). \end{aligned} \quad (2.13)$$

It follows from Lemma 2.3 that

$$\sup_{t_n \leq t \leq t_{n+1}} \|I_1(t)\|_\infty \leq C \Delta t \leq C N^{-\frac{\beta}{d}}, \quad (2.14)$$

where  $C$  depending only on  $T$  and  $C_{\rho_0}$ . To estimate  $I_2(t)$ , recall a basic property of the Brownian motion (Lemma 2.7 of Ref. 31):

$$\mathbb{P}\left(\sup_{t \leq s \leq t + \Delta t} \|B^s - B^t\|_\infty \geq b\right) \leq C_1 (\sqrt{\Delta t}/b) \exp(-C_2 b^2/\Delta t), \quad (2.15)$$

where  $C_1$  and  $C_2$  depend only on  $d$ . Choosing  $b = N^{-\frac{1}{d}}$  in (2.15), it leads to

$$\mathbb{P}\left(\sup_{t_n \leq t \leq t_{n+1}} \|I_2(t)\|_\infty \geq \sqrt{2\nu} N^{-\frac{\beta+2}{2d}}\right) \leq C_1 N^{\frac{2-\beta}{2d}} \exp(-C_2 N^{\frac{\beta-2}{d}}). \quad (2.16)$$

Collecting (2.14) and (2.16), it yields that

$$\mathbb{P} \left( \sup_n \sup_{t \in [t_n, t_{n+1}]} \|Y_t - Y_{t_n}\|_\infty \geq C \nu^{\frac{1}{2}} N^{-\frac{\beta+2}{2d}} \right) \leq C_1 N^{\frac{2+\beta}{2d}} \exp(-C_2 N^{\frac{\beta-2}{d}}),$$

for  $\beta > 2$ , which concludes the proof.  $\square$

Now we can prove the consistency error in all time.

**Proposition 2.1.** (Consistency) *Let  $Y_t = (Y_i^t)_{i=1,\dots,N}$  satisfies the mean-field dynamics (2.1) with i.i.d initial data sharing the common density  $\rho_0$  satisfying (2.2). Assume that  $\mathcal{F}^N$  and  $\bar{\mathcal{F}}^N$  be defined in (2.4) and (2.8), respectively. For any  $\alpha > 0$  and  $0 < \delta \leq \frac{1}{d}$ , there exists a constant  $C_{2,\alpha} > 0$  depending only on  $\alpha$ ,  $T$  and  $C_{\rho_0}$  such that*

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|\mathcal{F}^N(Y_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \geq C_{2,\alpha} \nu^{\frac{1}{2}} N^{\frac{\delta(d-2)-1}{2}} \log(N) \right) \leq N^{-\alpha}, \quad (2.17)$$

and

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|\mathcal{L}^N(Y_t) - \bar{\mathcal{L}}^N(Y_t)\|_\infty \geq C_{2,\alpha} \nu^{\frac{1}{2}} N^{\frac{d\delta-1}{2}} \log(N) \right) \leq N^{-\alpha}. \quad (2.18)$$

**Proof.** Denote events

$$\mathcal{H} := \left\{ \sup_n \sup_{t \in [t_n, t_{n+1}]} \|Y_t - Y_{t_n}\|_\infty \leq C_B \nu^{\frac{1}{2}} N^{-\frac{\beta+2}{2d}} \right\}, \quad (2.19)$$

and

$$\mathcal{C}_{t_n} := \{ \|\mathcal{F}^N(Y_{t_n}) - \bar{\mathcal{F}}^N(Y_{t_n})\|_\infty \geq C_{1,\alpha} N^{\frac{\delta(d-2)-1}{2}} \log(N) \},$$

where  $C_B$  and  $C_{1,\alpha}$  are used in Lemmas 2.5 and 2.6, respectively. According to Lemmas 2.5 and 2.6, one has

$$\mathbb{P}(\mathcal{C}_{t_n}^c) \leq N^{-\alpha}, \quad \mathbb{P}(\mathcal{H}^c) \leq C_B N^{\frac{2+\beta}{2d}} \exp(-C_B N^{\frac{\beta-2}{d}})$$

for any  $\alpha > 0$  and  $\beta > 2$ .

Furthermore, we denote

$$\mathcal{B}_{t_n} := \{ \|\mathcal{L}^N(Y_{t_n}) - \bar{\mathcal{L}}^N(Y_{t_n})\|_\infty \leq C_{1,\alpha} N^{\frac{d\delta-1}{2}} \log(N) \}, \quad (2.20)$$

then under the event  $\mathcal{B}_{t_n}$ , it holds that

$$\|\mathcal{L}^N(Y_{t_n})\|_\infty \leq \|\bar{\mathcal{L}}^N(Y_{t_n})\|_\infty + C_{1,\alpha} N^{\frac{d\delta-1}{2}} \log(N) \leq C(\alpha, T, C_{\rho_0}) \log(N) \quad (2.21)$$

and  $\mathbb{P}(\mathcal{B}_{t_n}^c) \leq N^{-\alpha}$  by Lemma 2.5.

For all  $t \in [t_n, t_{n+1}]$ , under the event  $\mathcal{B}_{t_n} \cap \mathcal{C}_{t_n} \cap \mathcal{H}$ , we obtain

$$\begin{aligned}
& \|\mathcal{F}^N(Y_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \\
& \leq \|\mathcal{F}^N(Y_t) - \mathcal{F}^N(Y_{t_n})\|_\infty + \|\mathcal{F}^N(Y_{t_n}) - \bar{\mathcal{F}}^N(Y_{t_n})\|_\infty \\
& \quad + \|\bar{\mathcal{F}}^N(Y_{t_n}) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \\
& \leq C\|\mathcal{L}^N(Y_{t_n})\|_\infty \|Y_t - Y_{t_n}\|_\infty + C_{1,\alpha} N^{\frac{\delta(d-2)-1}{2}} \log(N) \\
& \quad + C \log(N) \|Y_t - Y_{t_n}\|_\infty \\
& \leq C(\alpha, T, C_{\rho_0}) \nu^{\frac{1}{2}} \log(N) N^{-\frac{\beta+2}{2d}} + C_{1,\alpha} N^{\frac{\delta(d-2)-1}{2}} \log(N) \\
& \leq C(\alpha, T, C_{\rho_0}) \nu^{\frac{1}{2}} N^{\frac{\delta(d-2)-1}{2}} \log(N), \quad (\beta > (d-2)(1-d\delta)),
\end{aligned}$$

where in the second inequality we have used the local Lipschitz bound of  $\mathcal{F}^N$

$$\|\mathcal{F}^N(Y_t) - \mathcal{F}^N(Y_{t_n})\|_\infty \leq C\|\mathcal{L}^N(Y_{t_n})\|_\infty \|Y_t - Y_{t_n}\|_\infty,$$

under the event  $\mathcal{H}$  (see in Lemma 2.2). It yields that

$$\sup_{t \in [0, T]} \|\mathcal{F}^N(Y_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \leq C(\alpha, T, C_{\rho_0}) \nu^{\frac{1}{2}} N^{\frac{\delta(d-2)-1}{2}} \log(N),$$

holds under the event  $\bigcap_{n=0}^{M-1} (\mathcal{B}_{t_n} \cap \mathcal{C}_{t_n}) \cap \mathcal{H}$ . Therefore,

$$\begin{aligned}
& \mathbb{P} \left( \sup_{t \in [0, T]} \|\mathcal{F}^N(Y_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \geq C(\alpha, T, C_{\rho_0}) \nu^{\frac{1}{2}} N^{\frac{\delta(d-2)-1}{2}} \log(N) \right) \\
& \leq \sum_{n=0}^{M-1} P(\mathcal{B}_{t_n}^c) + \sum_{n=0}^{M-1} P(\mathcal{C}_{t_n}^c) + P(\mathcal{H}^c) \\
& \leq TN^{-\frac{d\alpha-\beta}{d}} + TN^{-\frac{d\alpha-\beta}{d}} + C_B N^{\frac{2+\beta}{2d}} \exp(-C_B N^{\frac{\beta-2}{d}}) \leq N^{-\alpha'}. \quad (2.22)
\end{aligned}$$

Denote  $C_{2,\alpha'}$  to be the constant  $C(\alpha, T, C_{\rho_0})$  in (2.22). Since  $\alpha > 0$  is arbitrary and so is  $\alpha'$ , hence (2.17) holds true. The proof of (2.18) can be done similarly.  $\square$

In order to prove the convergence, we still need the stability result which states the following.

**Proposition 2.2.** (Stability) *Assume that trajectories  $X_t = (X_i^t)_{i=1,\dots,N}$ ,  $Y_t = (Y_i^t)_{i=1,\dots,N}$  satisfy (1.5) and (2.1) respectively with the initial data  $X_0 = Y_0$ , which are i.i.d. sharing the common density  $\rho_0$  satisfying (2.2). Let events  $\mathcal{B}_{t_n}$  and  $\mathcal{H}$  be defined in (2.20) and (2.19) respectively,  $\mathcal{F}^N$  be defined in (2.4). Denote events*

$$\mathcal{A} := \left\{ \sup_{t \in [0, T]} \|X_t - Y_t\|_\infty < N^{-\delta} \right\}, \quad (2.23)$$

and

$$\mathcal{S}(\Lambda) := \{\|\mathcal{F}^N(X_t) - \mathcal{F}^N(Y_t)\|_\infty \leq \Lambda \log(N) \|X_t - Y_t\|_\infty + \Lambda \nu^{\frac{1}{2}} \log(N) N^{-\frac{\beta+2}{2d}}, \forall t \in [0, T]\}.$$

For any  $\alpha > 0$ , there exists some  $C_{3,\alpha} > 0$  depending only on  $\alpha$ ,  $T$  and  $C_{\rho_0}$  such that

$$\bigcap_{n=0}^{M-1} \mathcal{B}_{t_n} \cap \mathcal{A} \cap \mathcal{H} \subset \mathcal{S}(C_{3,\alpha}).$$

Here the event  $\mathcal{S}(C_{3,\alpha})$  can be seen as the stability result and the events  $\mathcal{B}_{t_n}$ ,  $\mathcal{A}$  and  $\mathcal{H}$  can be treated as the stability conditions.

**Proof.** First, we split  $\mathcal{S}(\Lambda)$  into the union of non-overlapping sets  $\{\mathcal{S}_n(\Lambda)\}_{n=0}^{M-1}(\Lambda)$ , where

$$\begin{aligned} \mathcal{S}_n(\Lambda) := \{& \|\mathcal{F}^N(X_t) - \mathcal{F}^N(Y_t)\|_\infty \leq \Lambda \log(N) \|X_t - Y_t\|_\infty \\ & + \Lambda \log(N) N^{-\frac{\beta+2}{2d}}, \forall t \in [t_n, t_{n+1}]\}. \end{aligned}$$

Notice that for any  $t \in [t_n, t_{n+1}]$ , under the event  $\mathcal{A} \cap \mathcal{H}$ , one has

$$\begin{aligned} \sup_{t \in [t_n, t_{n+1}]} \|X_t - Y_{t_n}\|_\infty & \leq \sup_{t \in [t_n, t_{n+1}]} \|X_t - Y_t\|_\infty + \sup_{t \in [t_n, t_{n+1}]} \|Y_t - Y_{t_n}\|_\infty \\ & \leq N^{-\delta} + C_B \nu^{\frac{1}{2}} N^{-\frac{\beta+2}{2d}} < 2N^{-\delta} \quad (\beta > 2d\delta - 2) \end{aligned}$$

and

$$\sup_{t \in [t_n, t_{n+1}]} \|Y_t - Y_{t_n}\|_\infty < C_B \nu^{\frac{1}{2}} N^{-\frac{\beta+2}{2d}} < N^{-\delta} \quad (\beta > 2d\delta - 2).$$

Then applying the local Lipschitz bound of  $F^N$  (see in Lemma 2.2) leads to

$$\begin{aligned} & \|\mathcal{F}^N(X_t) - \mathcal{F}^N(Y_t)\|_\infty \\ & \leq \|\mathcal{F}^N(X_t) - \mathcal{F}^N(Y_{t_n})\|_\infty + \|\mathcal{F}^N(Y_{t_n}) - \mathcal{F}^N(Y_t)\|_\infty \\ & \leq C \|\mathcal{L}^N(Y_{t_n})\|_\infty (\|X_t - Y_{t_n}\|_\infty + \|Y_{t_n} - Y_t\|_\infty) \\ & \leq C \|\mathcal{L}^N(Y_{t_n})\|_\infty \|X_t - Y_t\|_\infty + 2C \|\mathcal{L}^N(Y_{t_n})\|_\infty \|Y_{t_n} - Y_t\|_\infty \end{aligned}$$

under the event  $\mathcal{A} \cap \mathcal{H}$ .

Furthermore, under the event  $\mathcal{B}_{t_n}$ , it follows from (2.21) that

$$\|\mathcal{L}^N(Y_{t_n})\|_\infty \leq C \log(N),$$

Hence, for all  $t \in [t_n, t_{n+1}]$  one has

$$\begin{aligned} \|\mathcal{F}^N(X_t) - \mathcal{F}^N(Y_t)\|_\infty &\leq C(\alpha, T, C_{\rho_0}) \log(N) \|X_t - Y_t\|_\infty \\ &\quad + C(\alpha, T, C_{\rho_0}) \nu^{\frac{1}{2}} \log(N) N^{-\frac{\beta+2}{2d}}, \end{aligned}$$

under event  $\mathcal{A} \cap \mathcal{H} \cap \mathcal{B}_{t_n}$ . Denote the  $C(\alpha, T, C_{\rho_0})$  in the above as  $C_{3,\alpha}$ . This implies  $\mathcal{A} \cap \mathcal{H} \cap \mathcal{B}_{t_n} \subset \mathcal{S}_n(C_{3,\alpha})$ , which yields

$$\bigcap_{n=0}^{M-1} \mathcal{B}_{t_n} \cap \mathcal{H} \cap \mathcal{A} \subset \mathcal{S}(C_{3,\alpha}).$$

Thus, the proposition has been proved.  $\square$

Before proving the result on mean-field limit, let us recall a Gronwall-type inequality in Ref. 32.

**Lemma 2.7.** *For any  $T > 0$ , let  $e(t)$  be a non-negative continuous function on  $[0, T]$  with the initial data  $e(0) = 0$  and  $\lambda, \delta$  be two universal constants satisfying the following differential inequality that holds*

$$\frac{de(t)}{dt} \leq C \log(N) e(t) + C \log(N) N^{-\lambda}, \quad 0 < t \leq T_1, \quad (2.24)$$

*provided that*

$$\sup_{t \in [0, T_1]} e(t) \leq N^{-\delta} \quad (2.25)$$

*holds. Then  $e(t)$  is uniformly bounded on  $[0, T]$ . Furthermore, there is a  $N_0 \in \mathbb{N}$  depending only on  $C$  and  $T$  such that for all  $N \geq N_0$*

$$\sup_{t \in [0, T]} e(t) \leq N^{-\delta}. \quad (2.26)$$

**Proof.** This lemma has been proved in Lemma 3.3 of Ref. 32. For completeness, we provide a proof here, which is done by contradiction. We assume that there is a  $t \in [0, T]$  with  $e(t) \geq N^{-\lambda_2}$  and show that for  $N \geq N_0$  with some  $N_0 \in \mathbb{N}$  specified below, we get a contradiction.

It follows that the infimum over all times  $t$  where  $e(t)$  is larger than or equal to  $N^{-\lambda_2}$  exists and we define

$$T_* = \inf\{0 \leq t \leq T : e(t) \geq N^{-\lambda_2}\}.$$

We get by continuity of  $e(t)$  together with  $e(0) = 0$  that  $T^* > 0$ ,

$$e(T_*) = N^{-\lambda_2} \quad \text{and} \quad \max_{0 \leq t \leq T_*} e(t) = N^{-\lambda_2}. \quad (2.27)$$

Since (2.25) implies (2.26), we get for  $T_1 = T_*$  that

$$\frac{de(t)}{dt} \leq C \sqrt{\log(N)} e(t) + C \log^2(N) N^{-\lambda_3}, \quad 0 < t \leq T_*.$$

Gronwall's Lemma gives that

$$e(t) \leq e^{C\sqrt{\log(N)}t} \log^2(N) N^{-\lambda_3},$$

in particular

$$e(T_*) \leq e^{C\sqrt{\log(N)}T_*} \log^2(N) N^{-\lambda_3}.$$

Since  $e^{C\sqrt{\log(N)}T_*}$  and  $\log^2(N)$  are asymptotically bounded by any positive power of  $N$ , we can find a  $N_0 \in \mathbb{N}$  depending only on  $C$  and  $T_*$  such that for any  $N \geq N_0$

$$e^{C\sqrt{\log(N)}T_*} \log^2(N) < N^{\lambda_3 - \lambda_2} \quad \text{for } 0 < \lambda_2 < \lambda_3,$$

and hence

$$e(T_*) < N^{-\lambda_2} \quad \text{for any } N \geq N_0.$$

Thus we get a contradiction to (2.27) for all  $N \geq N_0$  and the lemma is proven.  $\square$

Our next theorem states that the  $N$ -particle trajectory  $X_t = (X_i^t)_{i=1,\dots,N}$  starting from  $X_0$  (i.i.d. with common density  $\rho_0$ ) remains close to the mean-field trajectory  $Y_t = (Y_i^t)_{i=1,\dots,N}$  with the same initial configuration  $Y_0 = X_0$ . More precisely, we prove that the measure of the set where the maximal distance  $\sup_{t \in [0,T]} \|X_t - Y_t\|_\infty$  on  $[0, T]$  exceeds  $N^{-\delta}$  decreases exponentially with the number of particles  $N$ , as  $N$  grows to infinity.

**Theorem 2.1.** (Convergence) *Assume that trajectories  $X_t = (X_i^t)_{i=1,\dots,N}$ ,  $Y_t = (Y_i^t)_{i=1,\dots,N}$  satisfy (1.5) and (2.1) respectively with the initial data  $X_0 = Y_0$ , which is i.i.d. sharing the common density  $\rho_0$  satisfying (2.2). Then for any  $\alpha > 0$ , there exists some constant  $N_0 > 0$  depending only on  $\nu$ ,  $\alpha$ ,  $T$  and  $C_{\rho_0}$ , such that for  $N \geq N_0$ , the following estimate holds with the cut-off index  $0 < \delta < \frac{1}{d}$*

$$\mathbb{P} \left( \sup_{t \in [0,T]} \|X_t - Y_t\|_\infty \leq N^{-\delta} \right) \geq 1 - N^{-\alpha}.$$

**Proof.** We can prove the convergence result by using the consistency from Proposition 2.1, the stability from Proposition 2.2 and Lemma 2.7. Denote the event

$$\mathcal{C} := \left\{ \sup_{t \in [0,T]} \|\mathcal{F}^N(Y_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \leq C_{2,\alpha} \nu^{\frac{1}{2}} N^{\frac{\delta(d-2)-1}{2}} \log(N) \right\}.$$

Consider the quantity  $e(t)$  defined as

$$e(t) := \|X_t - Y_t\|_\infty.$$

Computing under the event  $\mathcal{C} \cap \mathcal{S}(C_{3,\alpha})$  and using the fact  $\frac{d\|x\|_\infty}{dt} \leq \|\frac{dx}{dt}\|_\infty$ , one has

$$\begin{aligned}
\frac{de(t)}{dt} &\leq \|\mathcal{F}^N(X_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \\
&\leq \|\mathcal{F}^N(X_t) - \mathcal{F}^N(Y_t)\|_\infty + \|\mathcal{F}^N(Y_t) - \bar{\mathcal{F}}^N(Y_t)\|_\infty \\
&\leq C_{3,\alpha} \log(N) \|X_t - Y_t\|_\infty + C_{3,\alpha} \nu^{\frac{1}{2}} \log(N) N^{-\frac{\beta+2}{2d}} \\
&\quad + C_{2,\alpha} \nu^{\frac{1}{2}} N^{\frac{\delta(d-2)-1}{2}} \log(N) \\
&\leq C(\alpha, T, C_{\rho_0}) \log(N) e(t) + C(\alpha, T, C_{\rho_0}) \nu^{\frac{1}{2}} N^{\frac{\delta(d-2)-1}{2}} \log(N). \tag{2.28}
\end{aligned}$$

According to Proposition 2.2 one has

$$\mathcal{C} \cap \bigcap_{n=0}^{M-1} \mathcal{B}_{t_n} \cap \mathcal{H} \cap \mathcal{A} \subset \mathcal{C} \cap \mathcal{S}(C_{3,\alpha}). \tag{2.29}$$

Thus it follows from (2.28) that for any  $0 < T_1 \leq T$ , it holds

$$\frac{de(t)}{dt} \leq C \log(N) e(t) + C(\alpha, T, C_{\rho_0}) \nu^{\frac{1}{2}} N^{-\lambda} \log(N) \quad \text{for all } t \in (0, T_1],$$

under the event  $\mathcal{C} \cap \bigcap_{n=0}^{M-1} \mathcal{B}_{t_n} \cap \mathcal{H} \cap \mathcal{A}$ , where

$$-\lambda := \frac{\delta(d-2)-1}{2}.$$

And for  $0 < \delta < \frac{1}{3}$  we have  $-\lambda < -\delta$ .

Recall the event

$$\mathcal{A} := \left\{ \sup_{t \in [0, T]} e(t) \leq N^{-\delta} \right\} \subseteq \left\{ \sup_{t \in [0, T_1]} e(t) \leq N^{-\delta} \text{ for any } 0 < T_1 \leq T \right\}.$$

We deliberately take the event  $\mathcal{A}$  out as the condition (2.25) in Lemma 2.7. Hence it yields that

$$\sup_{t \in [0, T]} e(t) \leq N^{-\delta}$$

under the event  $\mathcal{C} \cap \bigcap_{n=0}^{M-1} \mathcal{B}_{t_n} \cap \mathcal{H}$ . Then we arrive at that

$$\begin{aligned}
&\mathbb{P} \left( \sup_{t \in [0, T]} \|X_t - Y_t\|_\infty \geq N^{-\delta} \right) \\
&\leq \sum_{n=0}^{M-1} \mathbb{P}(\mathcal{B}_{t_n}^c) + \mathbb{P}(\mathcal{H}^c) + \mathbb{P}(\mathcal{C}^c) \\
&\leq TN^{\frac{\beta}{d}-\alpha} + C_B N^{\frac{2+\beta}{2d}} \exp(-C_B N^{\frac{\beta-2}{d}}) + N^{-\alpha} \leq N^{-\alpha'},
\end{aligned}$$

by using Proposition 2.1, Lemmas 2.6 and 2.5. Since  $\alpha > 0$  is arbitrary and so is  $\alpha'$ , we have proved Theorem 2.1.  $\square$

### 2.3. The error estimate on interaction

Using Theorem 2.1, we obtain the error estimate on interaction.

**Theorem 2.2.** *Under the same assumption as Theorem 2.1, let  $\rho(x, t)$  be the regular solution to the aggregation equation (1.6) up to time  $T$  such that  $\rho \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^d))$ . Assume that  $\{X_i^t\}_{i=1}^N$  satisfy the particle system (1.5) and  $F^N$  satisfies (1.4). Then for any  $\alpha > 0$ , there exists some constant  $C_{4,\alpha} > 0$  depending only on  $\alpha$ ,  $T$  and  $C_{\rho_0}$  such that the following estimate holds with the cut-off index  $0 < \delta < \frac{1}{3}$ :*

$$\begin{aligned} & \mathbb{P} \left( \sup_{t \in [0, T]} \sup_{i=1, \dots, N} \left| \int_{\mathbb{R}^d} F^N(X_i^t - y) \rho(y, t) dy - \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^t - X_j^t) \right| \right. \\ & \quad \left. \leq C_{4,\alpha} \nu^{\frac{1}{2}} N^{-\delta} \log(N) \right) \geq 1 - N^{-\alpha}. \end{aligned}$$

**Proof.** For  $i = 1$ , let us denote

$$e_1^t := \left| \int_{\mathbb{R}^d} F^N(X_1^t - y) \rho(y, t) dy - \frac{1}{N-1} \sum_{j=2}^N F^N(X_1^t - X_j^t) \right|,$$

then one splits it into two parts

$$\begin{aligned} e_1^t & \leq \left| \int_{\mathbb{R}^d} F^N(X_1^t - y) \rho(y, t) dy - \frac{1}{N-1} \sum_{j=2}^N F^N(X_1^t - Y_j^t) \right| \\ & \quad + \left| \frac{1}{N-1} \sum_{j=2}^N F^N(X_1^t - Y_j^t) - \frac{1}{N-1} \sum_{j=2}^N F^N(X_1^t - X_j^t) \right| \\ & =: e_{11}^t + e_{12}^t, \end{aligned}$$

where  $Y_t = (Y_i^t)_{i=1, \dots, N}$  satisfies (2.1).

To estimate  $e_{11}^t$ , we use the law of large number estimates. In particular, similar to the estimate (2.9) in Lemma 2.5, we can prove that at any fix time  $t \in [0, T]$

$$\begin{aligned} & \mathbb{P} \left( \left| \int_{\mathbb{R}^d} F^N(X_1^t - y) \rho(y, t) dy - \frac{1}{N-1} \sum_{j=2}^N F^N(X_1^t - Y_j^t) \right| \geq C N^{\frac{\delta(d-2)-1}{2}} \log(N) \right) \\ & \leq N^{-\alpha}, \end{aligned} \tag{2.30}$$

where  $C$  depends only on  $\alpha$ ,  $T$  and  $C_{\rho_0}$ . Then following the procedure in Proposition 2.1, we can get the estimate of  $e_{11}^t$  for all the time  $t \in [0, T]$ . Hence one has

$$\mathbb{P} \left( \sup_{t \in [0, T]} e_{11}^t \geq C \nu^{\frac{1}{2}} N^{\frac{\delta(d-2)-1}{2}} \log(N) \right) \leq N^{-\alpha}, \quad (2.31)$$

where  $C$  depends only on  $\alpha$ ,  $T$  and  $C_{\rho_0}$ .

To estimate  $e_{12}^t$ , we shall use the result from Theorem 2.1. Let us recall the event

$$\mathcal{A} = \left\{ \sup_{t \in [0, T]} \|X_t - Y_t\|_{\infty} \leq N^{-\delta} \right\},$$

then it follows from Theorem 2.1 that

$$\mathbb{P}(\mathcal{A}^c) \leq N^{-\alpha}.$$

For any  $\xi \in \mathbb{R}^d$  with  $|\xi| < 4N^{-\delta}$ , it follows from [42, Lemma 6.3] that

$$|F^N(x + \xi) - F^N(x)| \leq CL^N(x)|\xi|,$$

where  $L^N$  is defined in (2.3). Therefore, it holds

$$\begin{aligned} & \frac{1}{N-1} \sum_{j=2}^N |F^N(X_1^t - Y_j^t) - F^N(X_1^t - X_j^t)| \\ & \leq \frac{1}{N-1} \sum_{j=2}^N CL^N(X_1^t - Y_j^t) |X_j^t - Y_j^t| \\ & \leq C \frac{1}{N-1} \sum_{j=2}^N L^N(X_1^t - Y_j^t) \|X_t - Y_t\|_{\infty} \\ & \leq CN^{-\delta} \frac{1}{N-1} \sum_{j=2}^N L^N(X_1^t - Y_j^t), \end{aligned} \quad (2.32)$$

under the event  $\mathcal{A}$ . Next we denote the event

$$\begin{aligned} \mathcal{B}_1 := & \left\{ \sup_{t \in [0, T]} \left| \frac{1}{N-1} \sum_{j=2}^N L^N(X_1^t - Y_j^t) - \int_{\mathbb{R}^d} L^N(X_1^t - y) \rho(y, t) dy \right| \right. \\ & \left. \leq C \nu^{\frac{1}{2}} N^{\frac{d\delta-1}{2}} \log(N) \right\}. \end{aligned}$$

Similar to the law of large numbers estimate (2.10) in Lemma 2.5, we can prove that

$$\mathbb{P}(\mathcal{B}_1^c) \leq N^{-\alpha}.$$

Hence it follows from (2.32) and Lemma 2.3 that

$$\begin{aligned} \sup_{t \in [0, T]} e_{12}^t &\leq C \left( \left| \int_{\mathbb{R}^d} L^N(X_1^t - y) \rho(y, t) dy \right| + C \nu^{\frac{1}{2}} N^{\frac{d\delta-1}{2}} \log(N) \right) N^{-\delta} \\ &\leq C \nu^{\frac{1}{2}} \log(N) N^{-\delta}, \end{aligned}$$

under the event  $\mathcal{A} \cap \mathcal{B}_1$ , which implies that

$$\mathbb{P} \left( \sup_{t \in [0, T]} e_{12}^t \leq C \nu^{\frac{1}{2}} \log(N) N^{-\delta} \right) \geq 1 - N^{-\alpha}, \quad (2.33)$$

where  $C$  depends only on  $\nu$ ,  $\alpha$ ,  $T$  and  $C_{\rho_0}$ .

Collecting estimates (2.31) and (2.33), it yields that

$$\mathbb{P} \left( \sup_{t \in [0, T]} e_1^t \leq C \nu^{\frac{1}{2}} N^{-\delta} \log(N) \right) \geq 1 - N^{-\alpha}, \quad (2.34)$$

where  $C$  depends only on  $\alpha$ ,  $T$  and  $C_{\rho_0}$ . Similarly, we can arrive at the same estimate for  $i = 2, \dots, N$ , which finishes the proof.  $\square$

### 3. Parameter Estimation and the Proof of Theorem 1.1

In this section, we obtain the diffusion parameter estimation and prove our main theorem (Theorem 1.1).

Let us recall (1.15) that

$$|\hat{\nu} - \nu| \leq C \nu^{\frac{1}{2}} (|\mathcal{I}_2|^{\frac{1}{2}} + |\mathcal{I}_3|^{\frac{1}{2}}) + |\nu_{K,N} - \nu|,$$

where

$$\nu_{K,N} := \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| X_i^{(n+1)} - X_i^{(n)} - \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds \right|^2, \quad (3.1)$$

and

$$\begin{aligned} |\mathcal{I}_2| &= \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| \int_{t_n}^{t_{n+1}} \left( \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) \right. \right. \\ &\quad \left. \left. - \int_{\mathbb{R}^d} F^N(X_i^s - y) \rho(y, s) dy \right) ds \right|^2, \end{aligned}$$

and

$$|\mathcal{I}_3| = \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} F^N(X_i^s - y) \rho(y, s) dy ds \right|^2.$$

According to Lemma 2.3, one has

$$|\mathcal{I}_3| \leq C\Delta t, \quad (3.2)$$

where  $C$  depends only on  $T$  and  $C_{\rho_0}$ . Then it follows from Theorem 2.2 that

$$\mathbb{P}(|\mathcal{I}_2| \leq C\nu\Delta t N^{-2\delta} \log^2(N)) \geq 1 - N^{-\alpha}, \quad (3.3)$$

where  $C$  depends only on  $\alpha$ ,  $T$  and  $C_{\rho_0}$ . It is left to estimate the error between  $\nu_{K,N}$  and  $\nu$ , which can be done by using the concentration property of  $\chi^2$  random variable.

**Theorem 3.1.** *Under the assumption as in Theorem 1.1. Suppose that  $\nu_{K,N}$  satisfies (3.1), then the following estimate holds:*

$$\mathbb{P}(|\nu_{K,N} - \nu| > \gamma\nu) \leq 2e^{-\frac{dKM\gamma^2}{8}} \quad \text{for any } \gamma \in (0, 1). \quad (3.4)$$

**Proof.** Recall that

$$X_i^{(n+1)} = X_i^{(n)} + \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds + \sqrt{2\nu\Delta t} \mathcal{N}_i^{(n)}, \quad i = 1, \dots, K,$$

then we know

$$\frac{X_i^{(n+1)} - X_i^{(n)} - \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds}{\sqrt{2\nu\Delta t}} \sim \mathcal{N}(0, 1)^d.$$

Notice that the random variable

$$S := \frac{1}{2\nu\Delta t} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| X_i^{(n+1)} - X_i^{(n)} - \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds \right|^2$$

is distributed according to the chi-squared distribution with  $dNM$  degrees of freedom. This is usually denoted as

$$S \sim \chi^2(dKM).$$

Recall a simple fact from probability theory, we know  $\mathbb{E}[S] = dKM$  and

$$\text{Var}[S] = \mathbb{E}[(S - dKM)^2] = 2dKM.$$

Recall that the estimate of  $\nu$  is given by

$$\nu_{K,N} = \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| X_i^{(n+1)} - X_i^{(n)} - \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F^N(X_i^s - X_j^s) ds \right|^2,$$

which leads to

$$\mathbb{E} \left[ \left( \frac{\nu_{K,N}}{\nu} - 1 \right)^2 \right] = \frac{2}{dKM}.$$

Hence we have

$$\mathbb{E}[(\nu_{K,N} - \nu)^2] = \frac{2\nu^2}{dKM}.$$

Also by the concentration of  $\chi^2$  variable, we have the following two sided tail bound

$$\mathbb{P}\left(\left|\frac{S}{dKM} - 1\right| > \gamma\right) \leq 2e^{-\frac{dKM\gamma^2}{8}} \quad \text{for any } \gamma \in (0, 1),$$

which is a direct result from the Bernstein's inequality as the form showed in Corollary 2.11 of Ref. 9. And it leads to

$$\mathbb{P}(|\nu_{K,N} - \nu| > \gamma\nu) \leq 2e^{-\frac{dKM\gamma^2}{8}} \quad \text{for any } \gamma \in (0, 1). \quad (3.5)$$

Hence it concludes the proof.  $\square$

Collecting estimates (3.5), (3.3) and (3.2), one has

$$\mathbb{P}(|\hat{\nu} - \nu| \leq C\nu^{\frac{1}{2}}\Delta t^{\frac{1}{2}}(1 + \nu^{\frac{1}{2}}N^{-\delta}\log(N)) + \gamma\nu) \geq 1 - N^{-\alpha} - 2e^{-\frac{dKM\gamma^2}{8}},$$

for any  $\gamma \in (0, 1)$ . Hence Theorem 1.1 has been proved.

#### 4. Extension to Regular Interacting Kernel $F \in W^{1,\infty}(\mathbb{R}^d)$

In this section, we will extend our result to the particle system with regular interacting force  $F$ , which satisfies

$$F \in W^{1,\infty}(\mathbb{R}^d). \quad (4.1)$$

Since  $F$  is non-singular, there is no need to mollify the force  $F$  anymore. To be more specific, we consider trajectories  $\{X_i^t\}_{i=1}^N$  satisfying SDEs:

$$dX_i^t = \frac{1}{N-1} \sum_{j \neq i}^N F(X_i^t - X_j^t) dt + \sqrt{2\nu} dB_i^t, \quad i = 1, \dots, N, \quad (4.2)$$

where the initial data  $\{X_i^0\}_{i=1}^N$  are i.i.d. sharing the common density  $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ . Then the solution  $\rho$  to the mean field equation

$$\partial_t \rho = \nu \Delta \rho - \nabla \cdot (\rho F * \rho), \quad x \in \mathbb{R}^d, \quad t > 0, \quad (4.3a)$$

$$\rho(x, 0) = \rho_0(x), \quad (4.3b)$$

has the following regularity for any  $T > 0$

$$\|\rho\|_{L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^d))} \leq C(T, \|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}, \|F\|_{W^{1,\infty}(\mathbb{R}^d)}) =: C_{F,\rho_0}.$$

Take a time step  $\Delta t > 0$  and let  $t_n := n\Delta t$  and  $M := \frac{T}{\Delta t}$  (we assume that  $\frac{T}{\Delta t}$  is an integer). Denote  $X_i^{(n)} := X_i^{t_n} = X_i^{n\Delta t}$  as the solution to (4.2) at time  $t_n$ . Namely, one has

$$X_i^{(n+1)} - X_i^{(n)} = \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F(X_i^s - X_j^s) ds + \sqrt{2\nu\Delta t} \mathcal{N}_i^{(n)},$$

where  $\mathcal{N}_i^{(n)} \sim \mathcal{N}(0, 1)^d$ , i.e. the standard Gaussian distribution in dimension  $d$ .

Then we are ready to define our estimator for the diffusion parameter as before

$$\hat{\nu} := \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} |X_i^{(n+1)} - X_i^{(n)}|^2, \quad (4.4)$$

where  $1 \ll K \ll N$ , which means we only have partial observations.

The extended result can be described in the following theorem.

**Theorem 4.1.** *Suppose that  $F(x) \in W^{1,\infty}(\mathbb{R}^d)$  and  $0 \leq \rho_0(x) \in L^1 \cap L^\infty(\mathbb{R}^d)$ . For any  $T > 0$ , take a time step  $\Delta t > 0$  and define  $t_n := n\Delta t$  and  $M := \frac{T}{\Delta t}$ . Let  $\{X_i^{(n)}\}_{i=1,n=0}^{K,M}$  be the sample trajectories satisfying (4.2) at time  $t_n$ . Then there exists some constant  $N_0 > 0$  depending only on  $\nu$ ,  $\alpha$ ,  $T$ ,  $\|F\|_{W^{1,\infty}(\mathbb{R}^d)}$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ , such that for  $N \geq N_0$ , the estimator  $\hat{\nu}$  defined in (4.4) is an approximation of  $\nu$ , and the following estimate holds:*

$$\begin{aligned} \mathbb{P}(|\hat{\nu} - \nu| \leq C_\alpha \nu^{\frac{1}{2}} \Delta t^{\frac{1}{2}} (1 + \nu^{\frac{1}{2}} N^{-\frac{1}{2}} \log(N)) + \nu \gamma) \\ \geq 1 - N^{-\alpha} - 2e^{-\frac{dKM\gamma^2}{8}}, \end{aligned} \quad (4.5)$$

for any  $\gamma \in (0, 1)$ , where  $C_\alpha > 0$  depends only on  $\alpha$ ,  $T$ ,  $\|F\|_{W^{1,\infty}(\mathbb{R}^d)}$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ . In particular, let  $N$  goes to infinity and choose  $\Delta t^{\frac{1}{2}} = \gamma$ , it follows from (4.5) that

$$\mathbb{P}(|\hat{\nu} - \nu| \leq C_\alpha (\nu^{\frac{1}{2}} + \nu) \Delta t^{\frac{1}{2}}) \geq 1 - 2e^{-\frac{dKT}{8}}. \quad (4.6)$$

**Proof.** Again, we defined an intermediate estimator

$$\nu_{K,N} := \frac{1}{2dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| X_i^{(n+1)} - X_i^{(n)} - \int_{t_n}^{t_{n+1}} \frac{1}{N-1} \sum_{j \neq i}^N F(X_i^s - X_j^s) ds \right|^2$$

then we split the error into two parts

$$|\hat{\nu} - \nu| \leq |\hat{\nu} - \nu_{K,N}| + |\nu_{K,N} - \nu| \quad (4.7)$$

and we can prove that there exists a positive number  $C$  such that

$$|\hat{\nu} - \nu| \leq C\nu^{\frac{1}{2}} (|\mathcal{I}_2|^{\frac{1}{2}} + |\mathcal{I}_3|^{\frac{1}{2}}) + |\nu_{K,N} - \nu| \quad (4.8)$$

with

$$\begin{aligned} |\mathcal{I}_2| := \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} & \left| \int_{t_n}^{t_{n+1}} \left( \frac{1}{N-1} \sum_{j \neq i}^N F(X_i^s - X_j^s) \right. \right. \\ & \left. \left. - \int_{\mathbb{R}^d} F(X_i^s - y) \rho(y, s) dy \right) ds \right|^2, \end{aligned}$$

and

$$|\mathcal{I}_3| := \frac{1}{dKT} \sum_{i=1}^K \sum_{n=0}^{M-1} \left| \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} F(X_i^s - y) \rho(y, s) dy ds \right|^2.$$

According to Lemma 2.3, one has

$$|\mathcal{I}_3| \leq C\Delta t, \quad (4.9)$$

where  $C$  depends only on  $T$ ,  $\|F\|_{W^{1,\infty}(\mathbb{R}^d)}$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ . It follows from Theorem 3.1 that

$$\mathbb{P}(|\nu_{K,N} - \nu| > \gamma\nu) \leq 2e^{-\frac{dKM\gamma^2}{8}} \quad \text{for any } \gamma \in (0, 1). \quad (4.10)$$

Now it is left to get the estimate of  $\mathcal{I}_2$ . The main idea behind the proof is also to construct a mean-field dynamic system  $\{Y_i^t\}_{i=1}^N$  without interaction

$$dY_i^t = \int_{\mathbb{R}^d} F(Y_i^t - y) \rho(y, t) dy dt + \sqrt{2\nu} dB_i^t, \quad i = 1, \dots, N, \quad (4.11)$$

here again we let  $\{Y_i^t\}_{i=1}^N$  has the same initial condition as  $\{X_i^t\}_{i=1}^N$  (i.i.d. with common density  $\rho_0$ ). Consider the quantity  $e(t)$  defined as

$$e(t) := \|X_t - Y_t\|_\infty.$$

Following the same procedure as in Lemma 2.5 and Proposition 2.1, one can prove that there exists some  $C_{1,\alpha}$  depending only on  $\alpha$ ,  $T$ ,  $\|F\|_{W^{1,\infty}(\mathbb{R}^d)}$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$  such that

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|\mathcal{F}(Y_t) - \bar{\mathcal{F}}(Y_t)\|_\infty \geq C_{1,\alpha} \nu^{\frac{1}{2}} N^{-\frac{1}{2}} \log(N) \right) \leq N^{-\alpha},$$

where

$$(\mathcal{F}(Y_t))_i := \frac{1}{N-1} \sum_{j \neq i}^N F(Y_i^t - Y_j^t),$$

and

$$(\bar{\mathcal{F}}(Y_t))_i := \int_{\mathbb{R}^d} F(Y_i^t - y) \rho(y, t) dy.$$

We denote the event

$$\mathcal{C} := \left\{ \sup_{t \in [0, T]} \|\mathcal{F}(Y_t) - \bar{\mathcal{F}}(Y_t)\|_\infty \leq C_{1,\alpha} \nu^{\frac{1}{2}} N^{-\frac{1}{2}} \log(N) \right\}.$$

Then using the fact  $\frac{d\|x\|_\infty}{dt} \leq \|\frac{dx}{dt}\|_\infty$ , one concludes that under the event  $\mathcal{C}$

$$\begin{aligned} \frac{de(t)}{dt} &\leq \|\mathcal{F}(X_t) - \bar{\mathcal{F}}(Y_t)\|_\infty \\ &\leq \|\mathcal{F}(X_t) - \mathcal{F}(Y_t)\|_\infty + \|\mathcal{F}(Y_t) - \bar{\mathcal{F}}(Y_t)\|_\infty \\ &\leq C\|X_t - Y_t\|_\infty + C\nu^{\frac{1}{2}} N^{-\frac{1}{2}} \log(N), \end{aligned} \quad (4.12)$$

which leads to

$$\sup_{t \in [0, T]} \|X_t - Y_t\|_\infty \leq C\nu^{\frac{1}{2}} N^{-\frac{1}{2}} \log(N), \quad (4.13)$$

where  $C$  depends only on  $\alpha$ ,  $T$ ,  $\|F\|_{W^{1,\infty}(\mathbb{R}^d)}$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ . Based on this mean-field limit result, we can prove error estimate on interaction as in Theorem 2.2.

Let us split the error

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} F(X_1^t - y) \rho(y, t) dy - \frac{1}{N-1} \sum_{j=2}^N F(X_1^t - X_j^t) \right| \\
& \leq \left| \int_{\mathbb{R}^d} F(X_1^t - y) \rho(y, t) dy - \frac{1}{N-1} \sum_{j=2}^N F(X_1^t - Y_j^t) \right| \\
& \quad + \left| \frac{1}{N-1} \sum_{j=2}^N F(X_1^t - Y_j^t) - \frac{1}{N-1} \sum_{j=2}^N F(X_1^t - X_j^t) \right| \\
& =: e_{11}^t + e_{12}^t.
\end{aligned}$$

Similar to estimates (2.31) and (2.33), it is easy to compute that

$$\mathbb{P} \left( \sup_{t \in [0, T]} e_{11}^t \leq C \nu^{\frac{1}{2}} N^{-\frac{1}{2}} \log(N) \right) \geq 1 - N^{-\alpha}, \quad (4.14)$$

and

$$\mathbb{P} \left( \sup_{t \in [0, T]} e_{12}^t \leq C \nu^{\frac{1}{2}} N^{-\frac{1}{2}} \log(N) \right) \geq 1 - N^{-\alpha}. \quad (4.15)$$

where  $C$  depends only on  $\alpha$ ,  $T$ ,  $\|F\|_{W^{1,\infty}(\mathbb{R}^d)}$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ .

Combining (4.14) and (4.15), it leads to

$$\begin{aligned}
& \mathbb{P} \left( \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} F(X_1^t - y) \rho(y, t) dy - \frac{1}{N-1} \sum_{j=2}^N F(X_1^t - X_j^t) \right| \leq C N^{-\frac{1}{2}} \log(N) \right) \\
& \leq 1 - N^{-\alpha},
\end{aligned}$$

which yields

$$\mathbb{P}(|\mathcal{I}_2| \leq C \nu^{\frac{1}{2}} \Delta t N^{-1} \log^2(N)) \geq 1 - N^{-\alpha}, \quad (4.16)$$

where  $C$  depends only on  $\alpha$ ,  $T$ ,  $\|F\|_{W^{1,\infty}(\mathbb{R}^d)}$  and  $\|\rho_0\|_{L^1 \cap L^\infty(\mathbb{R}^d)}$ .

Collecting (4.10), (4.16) and (4.9), we obtain our result

$$\mathbb{P}(|\hat{\nu} - \nu| \leq C \nu^{\frac{1}{2}} \Delta t^{\frac{1}{2}} (1 + \nu^{\frac{1}{2}} N^{-\frac{1}{2}} \log(N)) + \nu \gamma) \geq 1 - N^{-\alpha} - 2e^{-\frac{d K M \gamma^2}{8}},$$

for any  $\gamma \in (0, 1)$ .  $\square$

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