Communications in Partial Differential Equations

ISSN: 0360-5302 (Print) 1532-4133 (Online) Journal homepage: https://www.tandfonline.com/loi/lpde20

On stability of weak Navier–Stokes solutions with large $L^{3,\infty}$ initial data

T. Barker, G. Seregin & V. Šverák

To cite this article: T. Barker, G. Seregin & V. Šverák (2018) On stability of weak Navier–Stokes solutions with large $L^{3,\infty}$ initial data, Communications in Partial Differential Equations, 43:4, 628-651, DOI: 10.1080/03605302.2018.1449219

To link to this article: https://doi.org/10.1080/03605302.2018.1449219

Published online: 07 May 2018.

COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
2018, VOL. 43, NO. 4, 628–651
https://doi.org/10.1080/03605302.2018.1449219

On stability of weak Navier–Stokes solutions with large $L^{3,\infty}$ initial data

T. Barker[a], G. Seregin[a,b], and V. Šverák[c]

[a]Mathematical Institute, University of Oxford, Oxford, United Kingdom; [b]Laboratory of Mathematical Physics, St. Petersburg Department of V.A. Stekler Mathematical Institute, St. Petersburg, Russia; [c]School of Mathematics, University of Minnesota, Minneapolis, Minnesota, USA

ABSTRACT

We consider the Cauchy problem for the Navier–Stokes equation in $\mathbb{R}^3\times]0,\infty[$ with the initial datum $u_0 \in L^3_{\text{weak}}$, a critical space containing nontrivial $(-1)-$homogeneous fields. For small $||u_0||_{L^3_{\text{weak}}}$ one can get global well-posedness by perturbation theory. When $||u_0||_{L^3_{\text{weak}}}$ is not small, the perturbation theory no longer applies and, very likely, the local-in-time well-posedness and uniqueness fails. One can still develop a good theory of weak solutions with the following stability property: If $u^{(n)}$ are weak solutions corresponding the the initial datum $u_0^{(n)}$, and $u_0^{(n)}$ converge weakly* in L^3_{weak} to u_0, then a suitable subsequence of $u^{(n)}$ converges to a weak solution u corresponding to the initial condition u_0. This is of interest even in the special case $u_0 \equiv 0$.

ARTICLE HISTORY
Received 5 December 2016
Accepted 21 October 2017

KEYWORDS
Large initial data in critical spaces; Navier–Stokes solutions; stability of weak solutions

2010 MATHEMATICS SUBJECT CLASSIFICATION
35K55; 35Q30; 76D05

1. Introduction

We consider the Cauchy problem for the Navier–Stokes equation in $\mathbb{R}^3\times]0,\infty[$,

$$\left.\begin{aligned} \partial_t v + v.\Delta v + \nabla p - \Delta v &= 0 \\ \operatorname{div} v &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3\times]0,\infty[\,, \tag{1.1}$$

$$v|_{t=0} = u_0 \qquad \text{in } \mathbb{R}^3\,. \tag{1.2}$$

Our main assumption about u_0, in addition to $\operatorname{div} u_0 = 0$, is

$$u_0 \in L^{3,\infty}(\mathbb{R}^3)\,, \tag{1.3}$$

where the space $L^{3,\infty}$, sometimes also denoted by L^3_{weak}, is the Lorentz space consisting of measurable functions f with

$$||f||_{L^{3,\infty}} = \sup_{\kappa>0} \kappa|\{|f| > \kappa\}|^{\frac{1}{3}} < \infty\,. \tag{1.4}$$

The quantity $||f||_{L^{3,\infty}}$ is not really a norm, but there exists a norm equivalent to it, see for example [15] which will be denoted by $||f||^*_{L^{3,\infty}}$.

CONTACT T. Barker tobiasbarker5@gmail.com Andrew Wiles Building, University of Oxford, Radcliffe Observatory Quarter, Woodstock Rd, Oxford OX2 6GG, UK.

We note that $L^{3,\infty}(\mathbb{R}^3)$ is continuously imbedded into $\mathrm{BMO}^{-1}(\mathbb{R}^3)$, and hence, by well-known results of Koch and Tataru [23], the Cauchy problem (1.1), (1.2) is globally well-posed for $u_0 \in L^{3,\infty}$ when $||u||_{L^{3,\infty}}$ is sufficiently small. This can also be proved in many other ways, see for example [24].

What happens when $||u_0||_{L^{3,\infty}}$ is allowed to be large? For large initial conditions u_0 there is a significant difference between the spaces L_3 and $L^{3,\infty}$. They are both invariant under the Navier–Stokes scaling symmetry of the initial datum

$$u_0(x) \to \lambda u_0(\lambda x)\,, \tag{1.5}$$

but unlike L_3, the space $L^{3,\infty}$ contains nontrivial (-1)–homogeneous functions, which are of course invariant under the scaling (1.5). There are no nontrivial scale-invariant functions is L_3, and one in fact has

$$\lim_{\lambda\to 0}\int_{B_{x_0,R}} |\lambda f(\lambda x)|^3\,dx \to 0 \text{ uniformly in } x_0 \tag{1.6}$$

whenever $f \in L_3$ and $R > 0$ are fixed. The latter condition enables one to show local-in-time well-posedness for the Cauchy problem (1.1), (1.2) for any $u_0 \in L_3$ (in appropriate spaces of functions on $\mathbb{R}^3\times]0,T[$ for suitable $T = T(u_0) > 0$, such as $L_5(\mathbb{R}^3 \times [0,T[)$ and many others), see for example [22].

However, the Cauchy problem is conjecturally not well-posed locally-in-time in $L^{3,\infty}$ for large data, and in general it might presumably have many different solutions in $\mathbb{R}^3\times]0,T[$ for any small $T > 0$, see [21], with additional evidence in [16]. The regularity of these potential solutions is the same as the regularity of the caloric extension of u_0 to $t > 0$. (Here and below we use the term caloric extension of u_0 for the solution of the heat equation with the initial condition u_0.)

In addition to perturbation theory, there is a different approach through the theory of weak solutions. Pioneered by Leray [27], this approach relies mostly on the energy inequality and imbeddings. In the original version by Leray one needs $u_0 \in L_2$, but Lemarié-Rieusset [26] made an important observation that one can establish a local version of the energy estimates and prove the existence of the weak solutions (which we will sometimes refer to as Lemarié-Rieusset local energy solutions) only with $u_0 \in L^2_{\mathrm{loc}}$ and $\lim_{x_0\to\infty}\int_{B_{x_0,R}}|u_0(x)|^2\,dx = 0$ (for fixed $R > 0$). This approach covers $u_0 \in L^{3,\infty}$. The main shortcoming of the theory of the weak solutions is the possible lack of uniqueness. The best uniqueness results are the so-called weak-strong uniqueness results, which say that, modulo technical assumptions, if there is a sufficiently regular solution to the Cauchy problem, then any weak solution satisfying local energy estimates has to coincide with it. (Results of this form go back to Leray [27], see also [26] for more recent versions.) The possible examples of nonuniqueness for $u_0 \in L^{3,\infty}$ mentioned above are just outside of the regularity classes required by the uniqueness results, but they still satisfy all the requirements imposed on weak solutions.

Our goal in this paper is twofold. First, we develop a simple alternative approach to Lemarié-Rieusset's theory of local energy solutions in the case $u_0 \in L^{3,\infty}$. The main observation is that when $u_0 \in L^{3,\infty}$ and $V(x,t)$ is the caloric extension of u_0 to $\mathbb{R}^3\times]0,\infty[$, then the Navier–Stokes solution v can be sought in the form

$$v(x,t) = V(x,t) + u(x,t)\,, \tag{1.7}$$

where u is *globally* in the energy class in $\mathbb{R}^3\times]0,T[$ for any finite $T>0$. This motivates our definition of *weak* $L^{3,\infty}-$*solutions*, see Definition 1.4 below. When $u_0\in L^2\cap L^{3,\infty}$, it is not hard to verify that our definition gives the same class as Leray-Hopf solutions satisfying the local energy inequality (with the initial condition $u_0\in L^2\cap L^{3,\infty}$).

The following result is subsumed in results [26], but our approach gives an easier proof, through energy estimates for u in the above decomposition, see (3.31).

Proposition 1.1 (Existence of weak $L^{3,\infty}-$solutions). *For each $u_0\in L^{3,\infty}$ there exists at least one weak $L^{3,\infty}-$solution of the Cauchy problem (1.1), (1.2).*

We expect that our method can be fairly easily adapted to unbounded domains with boundaries, which seems to be an open problem for Lemarié-Rieusset's local energy solutions.

Decomposition (1.7) is of course not new. Its analogues have been used in the theory of dispersive equation, and in the context of the Navier–Stokes equations it has been used for example in [33].

Let us mention that in [25], prior to the development of Lemarié-Rieusset local energy solutions, Lemarié-Rieusset conceived a different notion of solution (v,q) (which we will refer to as L_2+L_p solutions) to the Navier–Stokes equations, with solenoidal initial data u_0 in L_2+L_p ($3<p<\infty$). The approach in [6, 25] is that one can split $u_0=u_0^1+u_0^2$, such that u_0^1 is solenoidal with sufficiently small L_p norm and u_0^2 is solenoidal in L_2. Then the paper [25] proceeds by constructing a mild solution of the Navier–Stokes equations $w\in C([0,2];L_p)$, with initial data u_0^1. Finally, the equation for $v-w$, with initial data u_0^2, is solved in the global energy class using methods related to those in Leray's paper [27].

Our second goal, and in fact the main goal of this paper, is to study the stability of the weak $L^{3,\infty}-$solutions under the weak convergence of the initial condition u_0.

When dealing with weak or distributional solutions, we always have to keep in mind that there might potentially be "anomalous weak solutions" which satisfy the equations in the sense of distributions but violate the energy inequality and have other counter-intuitive features. In the recent work [4] such solutions have been constructed for certain viscous SQG equations. In the inviscid case such examples go back to Scheffer [29], with later developments by De Lellis and Szekelyhidi [8], and Isett [18]. The purpose of the various technical requirements in the definition of the weak $L^{3,\infty}-$solutions is to rule out the anomalous solutions. (The situation is similar with the Leray-Hopf solutions.)

We recall that $L^{3,\infty}$ is the dual space of $L^{\frac{3}{2},1}$, and hence it is equipped with weak* topology. The weak*-convergence will be denoted by $\stackrel{*}{\rightharpoonup}$. It is easy to see that $f^{(k)}\stackrel{*}{\rightharpoonup}f$ in $L^{3,\infty}$ is equivalent the requirement that the norms $||f^{(k)}||_{L^{3,\infty}}$ are uniformly bounded and $f^{(k)}$ converge to f in distributions. By Banach-Alaoglu theorem, bounded sets in $L^{3,\infty}$ are weakly* pre-compact.

The strongest stability result would be that when $u_0^{(k)}\stackrel{*}{\rightharpoonup}u_0$ in $L^{3,\infty}$ (and are div-free)[1] the corresponding solutions $v^{(k)},v$ satisfy $v^{(k)}\to v$ (in a suitable sense). In view of the conjectured nonuniqueness discussed above, this statement probably fails, even under the additional assumptions that $u_0^{(k)}$ belong to $L^2\cap L^{3,\infty}$, are smooth, compactly supported,

[1]The div-free condition will always be assumed in this context, and will not be explicitly mentioned each time.

and converge to u_0 strongly in L^2. (Of course, u_0 cannot be smooth in such an example.) Nevertheless, one has the following statement.

Theorem 1.2 (Stability under weak convergence). *Let $u_0^{(k)} \stackrel{*}{\rightharpoonup} u_0$ in $L^{3,\infty}$ and let $v^{(k)}$ be a sequence of a global weak $L^{3,\infty}$-solutions to the Cauchy problem for the Navier–Stokes system with initial data $u_0^{(k)}$. Then there exists a subsequence, still denoted $v^{(k)}$, that converges to a global weak $L^{3,\infty}$-solution v to the Cauchy problem for the Navier–Stokes system with initial data u_0, in the sense of distributions.*

Remarks 1.3.

1. The theorem is not in contradiction with the possible nonuniqueness discussed above. The main tool in the proof of the theorem, inequality (3.31) gives among other things a quantitative upper bound on how fast the difference between two different solutions with the same initial datum $u_0 \in L^{3,\infty}$ can grow. Note that V in decomposition (1.7) is given uniquely by u_0, and hence the possible nonuniqueness can only be displayed by u, which is estimated by (3.31).
2. When $||u_0||_{L^{3,\infty}}$ is sufficiently small, so that one has global existence of regular solutions through perturbation arguments along the lines of [23], then the weak $L^{3,\infty}-$solution v coincides with the regular solution, and the whole sequence $v^{(k)}$ has to approach v.
3. It is not clear whether the stability remains true for Leray-Hopf weak solutions (even in a bounded domain) if we only assume only that $u_0^{(k)}$ are bounded in L_2. In particular, the following question seems to be open:
   (Q1) Assume that $u_0^{(k)} \in L_2$ are compactly supported in a fixed compact set and converge to $u_0 \equiv 0$ weakly in L_2. Let $v^{(k)}$ be the Leray-Hopf solution with the initial value $u_0^{(k)}$. Can we conclude that $v^{(k)}$ converge to $v \equiv 0$ in distributions?

   The question is related to the open problem whether the Leray-Hopf solutions satisfy the energy *identity*

$$\int_{\mathbb{R}^3} |v(x,t)|^2\, dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla v(x,s)|^2\, dx\, ds = \int_{\mathbb{R}^3} |u_0(x)|^2\, dx \tag{1.8}$$

   for every $t > 0$ (and not just the inequality $\leq$). An additional discussion of related issues can be found in [30].

   A negative answer to (Q1) would mean that in certain situations the energy of $u_0^{(k)}$ can be transported by the evolution extremely quickly from very high (spatial) Fourier modes into (relatively) low modes. One consequence of Theorem 1.2 is that such a fast transfer is not possible with the additional assumptions that the sequence $u_0^{(k)}$ is bounded in $L^{3,\infty}$ and v^k satisfies the local energy inequality for each k.
4. We do not know whether weak $L^{3,\infty}-$solutions can have singular points (x, t) with $t > 0$. Perturbation theory and weak-strong uniqueness results imply that there exists $\rho > 0$ such that no such singularities exist when $||u_0||^*_{L^{3,\infty}} < \rho$. Increasing ρ, we cannot rule out the existence of several solutions, which are smooth for $t > 0$ and have the same initial data u_0. Now, let $\rho_{\max}$ be the supremum of $\rho > 0$ for which all solutions starting with $||u_0||^*_{L^{3,\infty}} < \rho$ are smooth for all $t > 0$. Theorem 1.2, together with the results about

stability of singularities proved in [28] imply the following statement: If $\rho_{\max}$ is finite, then there exists $u_0 \in L^{3,\infty}$ with $||u_0||^*_{L^{3,\infty}} = \rho_{\max}$ such that that a weak $L^{3,\infty}-$solution v with the initial condition u_0 has a singularity for $t > 0$.

We proceed with more formal definition and additional results.
To define our weak solution, we need to introduce additional notation:

$$S(t)u_0(x) = \int_{\mathbb{R}^3} \Gamma(x-y,t)u_0(y)dy,$$

where Γ is a known heat kernel, $V(x,t) := S(t)u_0(x)$;
$L_s(\Omega)$ is a Lebesgue space in $\Omega \subseteq \mathbb{R}^3$ so that $L_s(\Omega) = L^{s,s}(\Omega)$ and abbreviations $L_s := L_s(\mathbb{R}^3)$ and $L^{s,l} := L^{s,l}(\mathbb{R}^3)$ are used;
J, $[C_{0,0}(\mathbb{R}^3)]^{L_s(\mathbb{R}^3)}$ and $\overset{\circ}{J}{}^{1}_{2}$ are the completion of the space

$$C^\infty_{0,0}(\mathbb{R}^3) := \{v \in C^\infty_0(\mathbb{R}^3) : \ \operatorname{div} v = 0\}$$

with respect to L_2-norm, L_s-norm and the Dirichlet integral

$$\Big(\int_{\mathbb{R}^3} |\nabla v|^2 dx\Big)^{\frac{1}{2}},$$

correspondingly. Additionally, we define the space-time domains $Q_T := \mathbb{R}^3 \times]0,T[$ and $Q_\infty := \mathbb{R}^3 \times]0,\infty[$.

Definition 1.4. Let $T > 0$ be finite. We say that v is a weak $L^{3,\infty}$-solution to Navier–Stokes IVP in Q_T if

$$v = V + u, \tag{1.9}$$

with $u \in L_\infty(0,T;J) \cap L_2(0,T;\overset{\circ}{J}{}^{1}_{2})$ and there exists $q \in L_{\frac{3}{2},\mathrm{loc}}(Q_T)$ such that u and q satisfy the perturbed Navier–Stokes system in the sense of distributions:

$$\partial_t u + v\cdot\nabla v - \Delta u = -\nabla q, \qquad \operatorname{div} u = 0 \tag{1.10}$$

in Q_T. Additionally, it is required that for any $w \in L_2$:

$$t \to \int_{\mathbb{R}^3} w(x)\cdot u(x,t)dx \tag{1.11}$$

is a continuous function on $[0,T]$. Moreover, u satisfies the energy inequality:

$$\|u(\cdot,t)\|^2_{L_2} + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u(x,t')|^2 dxdt' \leqslant$$

$$\leqslant 2\int_0^t \int_{\mathbb{R}^3} (V\otimes u + V\otimes V) : \nabla u dxdt' \tag{1.12}$$

for all $t \in [0,T]$.

Finally, it is required that v and q satisfy the local energy inequality. Namely, for a.a. $t \in]0,T[$,

$$\int_{\mathbb{R}^3} \phi(x,t)|v(x,t)|^2dx + 2\int_0^t\int_{\mathbb{R}^3} \phi|\nabla v|^2dxdt' \leqslant \int_0^t\int_{\mathbb{R}^3}[|v|^2(\partial_t\phi+\Delta\phi)+v\cdot\nabla\phi(|v|^2+2q)]dxdt' \tag{1.13}$$

for all non negative functions $\phi \in C_0^\infty(Q_T)$.

v is called a global weak $L^{3,\infty}$-weak solution if it is a weak solution in Q_T for any finite $T>0$.

Remark 1.5. One can see that the right-hand side in the energy inequality (1.12) is finite and thus the function u satisfies the initial condition in the strong L_2-sense, i.e., $u(\cdot,t)\to 0$ in L_2.

With regards to V, we can show that $\|V(\cdot,t)-u_0\|_{L_s,\mathrm{unif}} \to 0$ as $t\to 0$ for any $s<3$. In general, $V(\cdot,t)$ does not tends to u_0 in $L^{3,\infty}$ which can be easily seen for minus one homogeneous initial data, see [7].

As well as the main result stated in Theorem 1.2, we can prove additional facts, regarding uniqueness and regularity of global weak $L^{3,\infty}$-solutions on a finite time interval. These statements can be viewed as additional justification for Definition 1.4. In most cases, their proofs are based on comparison of energy and mild (perturbation theory) solutions and the corresponding results can be interpreted as weak-strong uniqueness statements. For other related weak-strong uniqueness statements, see for example [2, 9, 11, 12, 26].

Theorem 1.6. *Let v be a global weak $L^{3,\infty}$-solution to the Cauchy problem for the Navier–Stokes equations with the initial data $u_0 \in L^{3,\infty}$. There is a universal constant $\varepsilon_0>0$ with the following property. If*

$$\limsup_{R\to 0}\|u_0\|_{L^{3,\infty}(B(x_0,R))} < \varepsilon_0 \tag{1.14}$$

for any $x_0\in\mathbb{R}^3$ and

$$\|v(\cdot,t)-u_0(\cdot)\|_{L^{3,\infty}(\mathbb{R}^3)} < \varepsilon_0 \tag{1.15}$$

holds for all $t\in]0,T[$, then v is of class C^∞ in Q_T.

Moreover, if $\tilde v$ is another global weak $L^{3,\infty}$-solution to the Cauchy problem for the Navier–Stokes equations with the the same initial data u_0, then $\tilde v = v$ in Q_T.

Corollary 1.7. *Let v and $\tilde v$ be two global weak $L^{3,\infty}$-solution to the Cauchy problem for the Navier–Stokes equations with the same initial data u_0. Suppose that $v\in C([0,T];L^{3,\infty})$. Then $\tilde v = v$ in Q_T.*

As for regularity, we can state the following.

Theorem 1.8. *Suppose that $u_0\in L^{3,\infty}$. There exists a universal constant $\varepsilon>0$ such that if*

$$\langle V\rangle_{Q_T} := \sup_{0<t<T} t^{\frac{1}{5}}\|V(\cdot,t)\|_{L_5} \le \varepsilon, \tag{1.16}$$

where $V(\cdot,t) = S(t)u_0(\cdot)$, *then there exists a* v *that is a weak* $L^{3,\infty}$*-solution to the Cauchy problem for the Navier–Stokes system in* Q_T *and satisfies the property*

$$\langle v\rangle_{Q_T} < 2\langle V\rangle_{Q_T}. \tag{1.17}$$

Moreover, the following estimate is valid

$$\|v - V\|_{L_\infty(0,T;L_3)} < \langle V\rangle_{Q_T} \tag{1.18}$$

It is easy to verify that a solution of Theorem 1.8 is infinitely smooth in Q_T.

Although the main condition (1.16) holds for a wide class initial data, it does not work for large minus one homogeneous initial data, see details in [7].

Finally, we will show that under the additional condition (1.19), see [13] and [24], any global weak $L^{3,\infty}$-solution is unique and smooth on a short time interval.

Proposition 1.9. *Let* $u_0 \in L^{3,\infty}$. *There exists an* $\varepsilon_3 > 0$ *such that if*

$$\limsup_{\alpha\to\infty}\left(\alpha\big|\{|u_0| > \alpha\}\big|^{\frac{1}{3}}\right) < \varepsilon_3 \tag{1.19}$$

then there exists a $T = T(u_0) > 0$ *such that all global weak* $L^{3,\infty}$ *solutions, with initial data* $u_0 \in L^{3,\infty}$, *coincide on* Q_T.

2. Preliminaries

Now we state a fact concerning decompositions of Lorentz spaces. The proof can be found in [3]. This will be formulated as a Lemma. An analogous statement is Lemma II.I proven by Calderón in [6].

Lemma 2.1. *Take* $1 < t < r < s \leqslant \infty$, *and suppose that* $g \in L^{r,\infty}(\Omega)$. *For any* $N > 0$, *we let* $g_-^N := g\chi_{|g|\leqslant N}$ *and* $g_+^N := g - g_-^N$. *Then*

$$\|g_-^N\|_{L_s(\Omega)}^s \leqslant \frac{s}{s-r}N^{s-r}\|g\|_{L^{r,\infty}(\Omega)}^r - N^s d_g(N) \tag{2.1}$$

if $s < \infty$, *and*

$$\|g_+^N\|_{L_t(\Omega)}^t \leqslant \frac{r}{r-t}N^{t-r}\|g\|_{L^{r,\infty}(\Omega)}^r. \tag{2.2}$$

Moreover, for $\Omega = \mathbb{R}^3$, *if* $g \in L^{r,l}$ *with* $1 \leqslant l \leqslant \infty$ *and* $\mathrm{div}\ g = 0$, *then* $g = \bar{g}^N + \tilde{g}^N$ *where* $\bar{g}^N \in [C_{0,0}^\infty(\mathbb{R}^3)]^{L_s(\mathbb{R}^3)}$ $s < \infty$ *with*

$$\|\bar{g}^N\|_{L_s}^s \leqslant \frac{Cs}{s-r}N^{s-r}\|g\|_{L^{r,\infty}}^r \tag{2.3}$$

and $\tilde{g}^N \in [C_{0,0}^\infty(\mathbb{R}^3)]^{L_t(\mathbb{R}^3)}$ *with*

$$\|\tilde{g}^N\|_{L_t}^t \leqslant \frac{Cr}{r-t}N^{t-r}\|g\|_{L^{r,\infty}}^r. \tag{2.4}$$

Remark 2.2. Looking at the proof of the second part of of Lemma 2, we can easily see that

$$\|\bar{g}^N\|_{L^{r,\infty}} + \|\tilde{g}^N\|_{L^{r,\infty}} \leq c(r)\|g\|_{L^{r,\infty}}. \tag{2.5}$$

Let us recall the well known properties of $L^{s,1}$, for $1 < s < \infty$, such as separability and density of smooth compactly supported functions. Also, recall that

$$(L^{s,1})' = L^{s',\infty}, \qquad s' = \frac{s}{s-1}.$$

The identification is as follows, if $f \in L^{s',\infty}$ and $g \in L^{s,1}$:

$$T_f(g) = \int_{\mathbb{R}^3} fg dx.$$

The following proposition concerns weak-star approximation of $L^{3,\infty}$ functions.

Proposition 2.3. *Let $u_0 \in L^{3,\infty}$ be divergence free, in the sense of distributions. Then there exists a sequence $u_0^{(k)} \in C^\infty_{0,0}(\mathbb{R}^3)$ such that*

$$u_0^{(k)} \overset{*}{\rightharpoonup} u_0$$

in $L^{3,\infty}$.

The proof is based on the estimates of solutions to the Neumann boundary problem in the terms of the Lorentz space $L^{\frac{3}{2},1}$.

Now, consider the following Cauchy problem for the heat equation

$$\partial_t u - \Delta u = 0 \tag{2.6}$$

in Q_∞,

$$u(\cdot, 0) = u_0(\cdot) \in L^{3,\infty} \tag{2.7}$$

in $\mathbb{R}^3$.

Let us recall some known facts about solution operators of $S(t)$ for the corresponding semigroup. Indeed, $u(\cdot,t) = V(\cdot,t) = S(t)u_0(\cdot)$.

Proposition 2.4. *We have*

$$\|S(t)u_0\|_{L^{3,\infty}} \leqslant C\|u_0\|_{L^{3,\infty}}. \tag{2.8}$$

Moreover for $3 < r < \infty$, $m, k \in \mathbb{N}$:

$$\|\partial_t^m \nabla^k S(t)u_0\|_{L_r} \leqslant \frac{C\|u_0\|_{L^{3,\infty}}}{t^{m+\frac{k}{2}+\frac{3}{2}(\frac{1}{3}-\frac{1}{r})}}. \tag{2.9}$$

Furthermore for $1 \leqslant q < 3$ the following limits exist as $t \to 0$:

$$\|S(t)u_0 - u_0\|_{L_{q,unif}} \to 0, \tag{2.10}$$

$$S(t)u_0 \overset{*}{\rightharpoonup} u_0 \tag{2.11}$$

in $L^{3,\infty}$. Under the additional constraint that $u_0 \in \mathbb{L}^{3,\infty} := [C^\infty_{0,0}]^{L^{3,\infty}}$ Then we have that $S(t)u_0 \in \mathbb{L}^{3,\infty}$ and

$$\lim_{t\to 0} \|S(t)u_0 - u_0\|_{L^{3,\infty}} = 0. \tag{2.12}$$

Proof. The first two estimates are follows from convolution structure of the heat potential and the corresponding inequalities.

Recall the definition

$$\|f\|_{L_{p,unif}} := \sup_{x_0\in\mathbb{R}^3} \|f\|_{L_p(B(x_0,1))}.$$

Now, let us focus only on proving (2.10), as all other statements follow from this and (2.8). From Lemma 2.1 we can write

$$u_0 := \bar{u}_0^1 + \tilde{u}_0^1, \tag{2.13}$$

so that

$$\bar{u}_0^1 \in [C_{0,0}^\infty]^{L_s} \cap L^{3,\infty}, \qquad \tilde{u}_0^1 \in [C_{0,0}^\infty]^{L_q} \cap L^{3,\infty}$$

with $1 < q < 3 < s < \infty$. It is clear that

$$\lim_{t\to 0} \|S(t)\bar{u}_0^1 - \bar{u}_0^1\|_{L_s} = 0,$$

$$\lim_{t\to 0} \|S(t)\tilde{u}_0^1 - \tilde{u}_0^1\|_{L_q} = 0.$$

From here, (2.10) is obtained without difficulty. □

Proposition 2.5. *Let*

$$u_0^{(k)} \overset{*}{\rightharpoonup} u_0$$

in $L^{3,\infty}$. Then, for any $\phi \in C_0^\infty(Q_\infty)$:

$$\int_0^\infty \int_{\mathbb{R}^3} S(t)u_0^{(k)}(x)\phi(x,t)dxdt \to \int_0^\infty \int_{\mathbb{R}^3} S(t)u_0(x)\phi(x,t)dxdt. \tag{2.14}$$

Proof. By Lemma 2.1, we have

$$u_0^{(k)} := \bar{u}_0^{(k)1} + \tilde{u}_0^{(k)1}$$

and

$$\sup_k \|\bar{u}_0^{(k)1}\|_{L_s} + \sup_k \|\tilde{u}_0^{(k)1}\|_{L_q} \leqslant C(s,q) \sup_k \|u_0^{(k)}\|_{L^{3,\infty}}.$$

It is clear that $\bar{u}_0^{(k)1} \rightharpoonup \bar{u}_0$, $S(t)\bar{u}_0^{(k)1} \rightharpoonup S(t)\bar{u}_0$ in L_s and $\tilde{u}_0^{(k)1} \rightharpoonup \tilde{u}_0$, $S(t)\tilde{u}_0^{(k)1} \rightharpoonup S(t)\tilde{u}_0$ in L_q. Obviously, $u_0 = \bar{u}_0 + \tilde{u}_0$. From here the conclusion is easily reached. □

3. Existence of global weak $L^{3,\infty}(\mathbb{R}^3)$-solutions

3.1. Apriori estimates

Let $L_{s,l}(Q_T)$, $W_{s,l}^{1,0}(Q_T)$, $W_{s,l}^{2,1}(Q_T)$ be anisotropic (or parabolic) Lebesgues and Sobolev spaces with norms

$$\|u\|_{L_{s,l}(Q_T)} = \Big(\int_0^T \|u(\cdot,t)\|_{L_s}^l dt\Big)^{\frac{1}{l}}, \quad \|u\|_{W_{s,l}^{1,0}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},$$

$$\|u\|_{W_{s,l}^{2,1}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)}.$$

Lemma 3.1. *Assume that* $u \in L_\infty(0,T;J) \cap L_2(0,T;\overset{\circ}{J}{}^1_2)$ *and let* $u_0 \in L^{3,\infty}$ *be divergence free. Then*

$$V \cdot \nabla V \in L_{\frac{11}{7}}(Q_T), \tag{3.1}$$

$$V \cdot \nabla u + u \cdot \nabla V \in L_{\frac{5}{4},\frac{3}{2}}(Q_T), \tag{3.2}$$

$$V \otimes u : \nabla u \in L_1(Q_T). \tag{3.3}$$

Proof. By Hölder's inequality and Proposition 2.4:

$$\int_{\mathbb{R}^3} |V \cdot \nabla V|^{\frac{11}{7}} dx \leqslant \|V\|^{\frac{11}{7}}_{L_{\frac{22}{7}}} \|\nabla V\|^{\frac{11}{7}}_{L_{\frac{22}{7}}}$$

$$\leqslant c \frac{\|u_0\|^{\frac{22}{7}}_{L^{3,\infty}}}{t^{\frac{6}{7}}}.$$

From here, (3.1) is easily established. Again, by Hölder's inequality and Proposition 2.4:

$$\|u \cdot \nabla V\|_{L_{\frac{5}{4}}} \leqslant \|\nabla V\|_{L_{\frac{10}{3}}} \|u\|_{L_2} \leqslant c \frac{\|u_0\|_{L^{3,\infty}} \|u\|_{L_{2,\infty}(Q_T)}}{t^{\frac{11}{20}}}.$$

From this it is immediate that $u \cdot \nabla V \in L_{\frac{5}{4},\frac{3}{2}}(Q_T)$. Again by Hölder's inequality, it is not difficult to verify

$$\int_0^T \|V \cdot \nabla u\|^{\frac{3}{2}}_{L_{\frac{5}{4}}} dt \leqslant \left(\int_0^T \|\nabla u\|^2_{L_2} dt\right)^{\frac{3}{4}} \left(\int_0^T \|V\|^6_{L_{\frac{10}{3}}} dt\right)^{\frac{1}{4}}.$$

The conclusion is easily reached by noting that Proposition 2.4 gives:

$$\|V\|^6_{L_{\frac{10}{3}}} \leqslant c \frac{\|u_0\|_{L^{3,\infty}}}{t^{\frac{6}{20}}}.$$

The last estimate is known and shows why there are difficulties to prove energy estimate for u. By Hunt's inequality (Theorem 4.5, p. 271 of [17]) and Proposition 2.4:

$$\int_{\mathbb{R}^3} |V \otimes u : \nabla u| dx \leqslant \|V\|_{L^{3,\infty}} \|u\|_{L^{6,2}} \|\nabla u\|_{L_2}$$

$$\leqslant c \|u_0\|_{L^{3,\infty}} \|\nabla u\|^2_{L_2}.$$

We have used the fact that $L^{6,2}(\Omega) \hookrightarrow W^1_2(\Omega)$. See [1] for example. □

The next statement is a direct consequence of Lemma 3.1 and coercive estimates of solutions to the Stokes problem, which were developed by Solonnikov in [34][2] for equal space and time exponents and subsequently by Giga and Sohr in [14] for unequal space-time exponents.

[2]Specifically, Theorem 3.I p. 169 of [34].

Lemma 3.2. *Let v be a global weak $L^{3,\infty}$-solution with functions u and q as in Definition 1.4. Then*

$$(u,q)=\sum_{i=1}^{3}(u^i,p_i) \tag{3.4}$$

such that for any finite T:

$$(u^i,\nabla p_i)\in W^{2,1}_{s_i,l_i}(Q_T)\times L_{s_i,l_i}(Q_T) \tag{3.5}$$

and

$$(s_1,l_1)=(9/8,3/2), s_2=l_2=11/7, (s_3,l_3)=(5/4,3/2). \tag{3.6}$$

In addition (u^i,p_i) satisfy the following:

$$\partial_t u^1-\Delta u^1+\nabla p_1=-u\cdot\nabla u, \tag{3.7}$$

$$\partial_t u^2-\Delta u^2+\nabla p_2=-V\cdot\nabla V, \tag{3.8}$$

$$\partial_t u^3-\Delta u^3+\nabla p_3=-V\cdot\nabla u-u\cdot\nabla V \tag{3.9}$$

in Q_∞, and

$$\operatorname{div} u^i=0 \tag{3.10}$$

in Q_∞ for $i=1,2,3$,

$$u^i(\cdot,0)=0 \tag{3.11}$$

for all $x\in\mathbb{R}^3$ and $i=1,2,3$.

Let us introduce some notation. Let u, v and u_0 be as in Definition 1.4. Let $u_0=\bar{u}_0^N+\tilde{u}_0^N$ denote the splitting from Lemma 2.1. Let us define the following:

$$\bar{V}^N(\cdot,t):=S(t)\bar{u}_0^N(\cdot,t), \tag{3.12}$$

$$\tilde{V}^N(\cdot,t):=S(t)\tilde{u}_0^N(\cdot,t) \tag{3.13}$$

and

$$w^N(x,t):=u(x,t)+\tilde{V}^N(x,t). \tag{3.14}$$

We are going to prove

$$\|u(\cdot,t)\|_{L_2}^2+\int_0^t\int_{\mathbb{R}^3}|\nabla u|dxdt'\leq t^{\frac{1}{2}}F(\|u_0\|_{L^{3,\infty}})$$

by analyzing the global energy norm of w^N and a careful choice of the parameter N. Related splitting arguments have been used in [19], in the context of Lemarié-Rieusset local energy solutions with solenoidal initial data in L_3. First, let us state a relevant lemma.

Lemma 3.3. *In the above notation, we have the following global energy inequality*

$$\|w^N(\cdot,t)\|_{L_2}^2 + 2\int_0^t\int_{\mathbb{R}^3} |\nabla w^N(x,t')|^2 dxdt'$$
$$\leqslant \|\tilde{u}_0^N\|_{L_2}^2 + 2\int_0^t\int_{\mathbb{R}^3} (\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N dxdt' \tag{3.15}$$

that is valid for positive N and t.

Proof. The first stage is showing that w^N satisfies the local energy inequality. Let us briefly sketch how this can be done. Let $\phi \in C_0^\infty(Q_\infty)$ be a positive function. Observe that the assumptions in Definition 1.4 imply that the following function

$$t \to \int_\Omega w^N(x,t)\cdot \bar{V}^N(x,t)\phi(x,t)dx \tag{3.16}$$

is continuous for all $t \geqslant 0$. It is not so difficult to show that this term has the following expression:

$$\int_{\mathbb{R}^3} w^N(x,t)\cdot \bar{V}^N(x,t)\phi(x,t)dx$$
$$= \int_0^t\int_{\mathbb{R}^3} (w^N\cdot \bar{V}^N)(\Delta\phi + \partial_t\phi)dxdt'$$
$$-2\int_0^t\int_{\mathbb{R}^3} \nabla w^N : \nabla \bar{V}^N \phi dxdt' + \int_0^t\int_{\mathbb{R}^3} \bar{V}^N\cdot\nabla\phi q dxdt'$$
$$+\frac{1}{2}\int_0^t\int_{\mathbb{R}^3} (|v|^2 - |w^N|^2)v\cdot\nabla\phi dxdt'$$
$$-\int_0^t\int_{\mathbb{R}^3} (\bar{V}^N\otimes w^N + \bar{V}^N\otimes\bar{V}^N) : \nabla w^N\phi dxdt'$$
$$-\int_0^t\int_{\mathbb{R}^3} (\bar{V}^N\otimes\bar{V}^N + \bar{V}^N\otimes w^N) : (w^N\otimes\nabla\phi)dxdt'. \tag{3.17}$$

It is also readily shown that

$$\int_{\mathbb{R}^3} |\bar{V}^N(x,t)|^2\phi(x,t)dx = \int_0^t\int_{\mathbb{R}^3} |\bar{V}^N(x,t')|^2(\Delta\phi(x,t') + \partial_t\phi(x,t'))dxdt'$$
$$-2\int_0^t\int_{\mathbb{R}^3} |\nabla\bar{V}^N|^2\phi dxdt'. \tag{3.18}$$

Using (1.13), together with (3.17) and (3.18), we obtain that for all $t \in]0,\infty[$ and for all non negative functions $\phi \in C_0^\infty(Q_\infty)$:

$$\int_{\mathbb{R}^3} \phi(x,t)|w^N(x,t)|^2 dx + 2\int_0^t\int_{\mathbb{R}^3} \phi|\nabla w^N|^2 dxdt'$$
$$\leqslant \int_0^t\int_{\mathbb{R}^3} |w^N|^2(\partial_t\phi + \Delta\phi) + 2qw^N\cdot\nabla\phi dxdt'$$

$$+\int_0^t\int_{\mathbb{R}^3}|w^N|^2v\cdot\nabla\phi dxdt'$$

$$+2\int_0^t\int_{\mathbb{R}^3}(\bar{V}^N\otimes\bar{V}^N+\bar{V}^N\otimes w^N):(\nabla w^N\phi+w^N\otimes\nabla\phi)dxdt'. \quad (3.19)$$

In the next part of the proof, let $\phi(x,t)=\phi_1(t)\phi_R(x)$. Here, $\phi_1\in C_0^\infty(0,\infty)$ and $\phi_R\in C_0^\infty(B(2R))$ are positive functions. Moreover, $\phi_R=1$ on $B(R)$, $0\leqslant\phi_R\leqslant 1$,

$$|\nabla\phi_R|\leqslant c/R,$$

$$|\nabla^2\phi_R|\leqslant c/R^2.$$

Since $\tilde{u}_0^N\in[C_{0,0}^\infty(\mathbb{R}^3)]^{L_2(\mathbb{R}^3)}$, it is obvious that for $\tilde{V}^N(\cdot,t):=S(t)\tilde{u}_0^N(\cdot,t)$ we have the energy equality:

$$\|\tilde{V}^N(\cdot,t)\|_{L_2}^2+\int_0^t\int_{\mathbb{R}^3}|\nabla\tilde{V}^N|^2dxdt'=\|\tilde{u}_0^N\|_{L_2}^2. \quad (3.20)$$

By semigroup estimates, we have for $2\leqslant p\leqslant\infty$, $10/3\leqslant q\leqslant\infty$:

$$\|\tilde{V}^N(\cdot,t)\|_{L_p}\leqslant\frac{C(p)}{t^{\frac{3}{2}(\frac{1}{2}-\frac{1}{p})}}\|\tilde{u}_0^N\|_{L_2}, \quad (3.21)$$

$$\|\bar{V}^N(\cdot,t)\|_{L_q}\leqslant\frac{C(q)}{t^{\frac{3}{2}(\frac{3}{10}-\frac{1}{q})}}\|\bar{u}_0^N\|_{L_{\frac{10}{3}}}. \quad (3.22)$$

Hence, we have $w^N\in C_w([0,T];J)\cap L_2(0,T;\overset{\circ}{J}{}^1_2)$. Here, T is finite and $C_w([0,T];J)$ denotes continuity with respect to the weak topology. Using these facts, and usual multiplicative inequalities, it is obvious that the following limits hold:

$$\lim_{R\to\infty}\int_{\mathbb{R}^3}\phi_R(x)\phi_1(t)|w^N(x,t)|^2dx+2\int_0^t\int_{\mathbb{R}^3}\phi_R\phi_1|\nabla w^N|^2dxdt'$$

$$=\int_{\mathbb{R}^3}\phi_1(t)|w^N(x,t)|^2dx+2\int_0^t\int_{\mathbb{R}^3}\phi_1|\nabla w^N|^2dxdt',$$

$$\lim_{R\to\infty}\int_0^t\int_{\mathbb{R}^3}(|w^N|^2\partial_t\phi_1\phi_R+2(\bar{V}^N\otimes w^N+\bar{V}^N\otimes\bar{V}^N):\nabla w^N\phi_1\phi_R)dxdt'$$

$$=\int_0^t\int_{\mathbb{R}^3}(|w^N|^2\partial_t\phi_1+2(\bar{V}^N\otimes w^N+\bar{V}^N\otimes\bar{V}^N):\nabla w^N\phi_1)dxdt',$$

$$\lim_{R\to\infty}\int_0^t\int_{\mathbb{R}^3}(|w^N|^2\phi_1\Delta\phi_R+\phi_1|w^N|^2v\cdot\nabla\phi_R$$

$$+2\phi_1(\bar{V}^N\otimes w^N+\bar{V}^N\otimes\bar{V}^N):(w^N\otimes\nabla\phi_R))dxdt'=0.$$

Let us focus on the term containing the pressure, namely

$$\int_0^t\int_{\mathbb{R}^3}qw^N\cdot\nabla\phi_R\phi_1dxdt'.$$

Define $T(R) := B(2R) \setminus B(R)$. We can instead treat

$$\int_0^t \int_{T_+(R)} (q - [q]_{B(2R)}) w^N \cdot \nabla \phi_R \phi_1 dxdt^{'}.$$

Using the Poincaré inequality, it is not so difficult to show:

$$\Big| \int_0^t \int_{T(R)} (p_1 - [p_1]_{B(2R)}) w^N \cdot \nabla \phi_R \phi_1 dxdt^{'} \Big|$$

$$\leqslant \frac{C\|\phi_1\|_{L_\infty(0,t)}}{R^{\frac{2}{3}}} \|w^N\|_{L_3(T(R)\times]0,t[)} \|\nabla p_1\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_t)}, \tag{3.23}$$

$$\Big| \int_0^t \int_{T(R)} (p_2 - [p_2]_{B(2R)}) w^N \cdot \nabla \phi_R \phi_1 dxdt^{'} \Big|$$

$$\leqslant C\|\phi_1\|_{L_\infty(0,t)} \|w^N\|_{L_{\frac{11}{4}}(T(R)\times]0,t[)} \|\nabla p_2\|_{L_{\frac{11}{7}}(Q_t)}, \tag{3.24}$$

$$\Big| \int_0^t \int_{T_+(R)} (p_3 - [p_3]_{B(2R)}) (w^N \cdot \nabla \phi_R) \phi_1 dxdt^{'} \Big|$$

$$\leqslant \frac{C\|\phi_1\|_{L_\infty(0,t)}}{R^{\frac{2}{5}}} \|w^N\|_{L_3(T(R)\times]0,t[)} \|\nabla p_3\|_{L_{\frac{5}{4},\frac{3}{2}}(Q_t)}. \tag{3.25}$$

Using (3.23)–(3.25), multiplicative inequalities and properties of the pressure decomposition in Definition 1.4 we infer that

$$\lim_{R\to\infty} \int_0^t \int_{T(R)} q w^N \cdot \nabla \phi_R \phi_1 dxdt^{'} = 0.$$

Thus, putting everything together, we get for an arbitrary positive function $\phi_1 \in C_0^\infty(0,\infty)$:

$$\int_{\mathbb{R}^3} \phi_1(t) |w^N(x,t)|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} \phi_1(t) |\nabla w^N|^2 dxdt^{'}$$

$$\leqslant \int_0^t \int_{\mathbb{R}^3} |w^N|^2 \partial_t \phi_1 + 2(\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N \phi_1 dxdt'. \tag{3.26}$$

From Remark 1.5, we see that

$$\lim_{t\to 0} \|w^N(\cdot,t) - \tilde{u}_0^N(\cdot)\|_{L_2} = 0. \tag{3.27}$$

Using known arguments from [3], we have the following estimates:

$$\int_0^t \int_{\mathbb{R}^3} |\bar{V}^N \otimes w^N : \nabla w^N| dxdt' \leqslant$$

$$\leqslant CN^{\frac{1}{10}} \|u_0\|_{L^{3,\infty}}^{\frac{9}{10}} \Big(\int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 dxdt' \Big)^{\frac{4}{5}} \Big(\int_0^t \frac{\|w^N(\cdot,\tau)\|_{L_2}^2}{\tau^{\frac{3}{4}}} d\tau \Big)^{\frac{1}{5}}, \tag{3.28}$$

$$\int_0^t \int_{\mathbb{R}^3} |\bar{V}^N \otimes \bar{V}^N : \nabla w^N| dxdt' \leqslant Ct^{\frac{7}{20}} N^{\frac{1}{5}} \|u_0\|_{L^{3,\infty}}^{\frac{9}{5}} \|\nabla w^N\|_{L_2(Q_t)}. \tag{3.29}$$

Using (3.26)–(3.29), we infer (3.15) by standard arguments involving an appropriate choices of $\phi_1(t) = \phi_\epsilon(t)$ and letting ϵ tend to zero. □

Lemma 3.4. *Let u, v and u_0 be as in Definition 1.4. Then the following estimate is valid for all $N, t > 0$:*

$$\|u(\cdot,t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxdt'$$

$$\leqslant C(N^{-1}\|u_0\|^3_{L^{3,\infty}} + t^{\frac{7}{10}} N^{\frac{2}{5}} \|u_0\|^{\frac{18}{5}}_{L^{3,\infty}})$$

$$+ C\exp(Ct^{\frac{1}{4}} N^{\frac{1}{2}} \|u_0\|^{\frac{9}{2}}_{L^{3,\infty}})(N^{-\frac{1}{2}} t^{\frac{1}{4}} \|u_0\|^{\frac{33}{8}}_{L^{3,\infty}} + t^{\frac{19}{20}} N^{\frac{9}{10}} \|u_0\|^{\frac{199}{40}}_{L^{3,\infty}}). \tag{3.30}$$

Hence, taking $N = t^{-\frac{1}{2}}$ gives the following scale invariant estimate:

$$\|u(\cdot,t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxdt'$$

$$\leqslant Ct^{\frac{1}{2}} \exp(C\|u_0\|^{\frac{9}{2}}_{L^{3,\infty}})(\|u_0\|^{\frac{9}{8}}_{L^{3,\infty}} + 1)(\|u_0\|^3_{L^{3,\infty}} + \|u_0\|^{\frac{18}{5}}_{L^{3,\infty}}). \tag{3.31}$$

Proof. First observe that $u = w^N - \tilde{V}^N$. Thus, using (3.20) we see that

$$\|u(\cdot,t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxdt'$$

$$\leqslant 2\|\tilde{u}_0^N\|_{L_2}^2 + 2\|w^N(\cdot,t)\|_{L_2}^2 + 2\int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 dxdt'.$$

By (2.4):

$$\|\tilde{u}_0^N\|_{L_2}^2 \leqslant CN^{-1}\|u_0\|^3_{L^{3,\infty}}. \tag{3.32}$$

Thus, it is sufficient to prove (3.30) for w^N in place of u. From now on, denote

$$y_N(t) := \|w^N(\cdot,t)\|_{L_2}^2.$$

Using (3.15), estimates (3.28) and (3.29), (3.32) and the Young's inequality, we obtain that

$$y_N(t) + \int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 dxdt' \leqslant CN^{\frac{1}{2}} \|u_0\|^{\frac{9}{2}}_{L^{3,\infty}} \int_0^t \frac{y_N(\tau)}{\tau^{\frac{3}{4}}} d\tau$$

$$+ C(N^{-1}\|u_0\|^3_{L^{3,\infty}} + t^{\frac{7}{10}} N^{\frac{2}{5}} \|u_0\|^{\frac{18}{5}}_{L^{3,\infty}}).$$

The conclusion is then easily reached using a Gronwall type lemma. □

3.2. Existence of global weak $L^{3,\infty}(\mathbb{R}^3)$-solutions

Proof of Theorem 1.2. We have

$$u_0^{(k)} \stackrel{*}{\rightharpoonup} u_0$$

in $L^{3,\infty}$ and we may assume that

$$M := \sup_k \|u_0^{(k)}\|_{L^{3,\infty}} < \infty.$$

Firstly, define

$$V^{(k)}(\cdot,t) := S(t)u_0^{(k)}(\cdot,t), \qquad V(\cdot,t) := S(t)u_0(\cdot,t).$$

By Proposition 2.5, we see that $V^{(k)}$ converges to V on Q_∞ in the sense of distributions. By Proposition 2.4, we see that

$$\|V^{(k)}(\cdot,t)\|_{L^{3,\infty}} \leqslant CM, \tag{3.33}$$

$$\|\partial_t^m \nabla^l V^{(k)}(\cdot,t)\|_{L_r} \leqslant \frac{CM}{t^{m+\frac{l}{2}+\frac{3}{2}(\frac{1}{3}-\frac{1}{r})}}. \tag{3.34}$$

Here $r \in]3,\infty]$. For $T<\infty$ and $l \in]1,\infty[$, we have the compact embedding

$$W_l^{2,1}(B(n)\times]0,T[) \hookrightarrow C([0,T];L_l(B(n))).$$

From this and (3.34) one immediately infers that for every $n \in \mathbb{N}$ and $l \in]1,\infty[$:

$$\partial_t^m \nabla^l V^{(k)} \to \partial_t^m \nabla^l V \ \text{ in } \ C([1/n,n];L_l(B(n))). \tag{3.35}$$

Fixing $N=1$ in Lemma 3.4 we have:

$$\|u^{(k)}(\cdot,t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u^{(k)}|^2 dxdt' \leqslant f_0(M,t). \tag{3.36}$$

By a Cantor diagonalisation argument, we can subtract a subsequence such that for any finite $T>0$:

$$u^{(k)} \overset{*}{\rightharpoonup} u \ \text{ in } \ L_{2,\infty}(Q_T), \tag{3.37}$$

$$\nabla u^{(k)} \rightharpoonup \nabla u \ \text{ in } \ L_2(Q_T). \tag{3.38}$$

Using (3.37), together with (3.31), we also get that:

$$\|u\|_{L_{2,\infty}(Q_t)} \leqslant C(M)t^{\frac{1}{2}}. \tag{3.39}$$

From (3.36) it is easily inferred that

$$\|u^{(k)}\cdot\nabla u^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_t)} \leqslant f_1(M,t). \tag{3.40}$$

By the same reasoning as in Lemma 3.1, we obtain:

$$\|V^{(k)}\cdot\nabla V^{(k)}\|_{L_{\frac{11}{7}}(Q_t)} \leqslant f_2(M,t), \tag{3.41}$$

$$\|V^{(k)}\cdot\nabla u^{(k)} + u^{(k)}\cdot\nabla V^{(k)}\|_{L_{\frac{5}{4},\frac{3}{2}}(Q_t)} \leqslant f_3(M,t). \tag{3.42}$$

Split $u^{(k)} = \sum_{i=1}^3 u^{i(k)}$ according to Definition 1.4, namely (3.4). By coercive estimates for the Stokes system, along with (3.40), we obtain:

$$\|u^{1(k)}\|_{W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q_t)} + \|\nabla p_1^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_t)} \leqslant Cf_1(M,t), \tag{3.43}$$

$$\|u^{2(k)}\|_{W^{2,1}_{\frac{11}{7}}(Q_t)} + \|\nabla p_2^{(k)}\|_{L_{\frac{11}{7}}(Q_t)} \leqslant Cf_2(M,t), \tag{3.44}$$

$$\|u^{3(k)}\|_{W^{2,1}_{\frac{5}{4},\frac{3}{2}}(Q_t)} + \|\nabla p_3^{(k)}\|_{L_{\frac{5}{4},\frac{3}{2}}(Q_t)} \leqslant Cf_3(M,t). \tag{3.45}$$

By the previously mentioned embeddings, we infer from (3.43)–(3.45) that for any $n \in \mathbb{N}$ we have the following convergence for a certain subsequence:

$$u^{(k)} \to u \ \text{ in } \ C([0,n]; L_{\frac{9}{8}}(B(n)). \tag{3.46}$$

Hence, using (3.36), it is standard to infer that for any $s \in]1, 10/3[$

$$u^{(k)} \to u \ \text{ in } \ L_s(B(n)\times]0,n[). \tag{3.47}$$

It is also not so difficult to show that for any $f \in L_2$ and for any $n \in \mathbb{N}$:

$$\int_{\mathbb{R}^3} u^{(k)}(x,t)\cdot f(x)dx \to \int_{\mathbb{R}^3} u(x,t).f(x)dx \ \text{ in } \ C([0,n]). \tag{3.48}$$

Using (3.39) with (3.48), we establish that

$$\lim_{t\to 0} \|u(\cdot,t)\|_{L_2} = 0. \tag{3.49}$$

All that remains to show is establishing the local energy inequality (1.13) for the limit and establishing the energy inequality (3.3) for u. Verifying the local energy inequality is not so difficult and hence omitted. Let us focus on verifying (3.3) for u. By identical reasoning to Lemma 3.3, we have that for an arbitrary positive function $\phi_1(t) \in C_0^\infty(0,\infty)$:

$$\int_{\mathbb{R}^3} \phi_1(t)|u(x,t)|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} \phi_1(t)|\nabla u|^2 dxdt'$$
$$\leqslant \int_0^t \int_{\mathbb{R}^3} |u|^2 \partial_t \phi_1 + 2(V\otimes u + V\otimes V) : \nabla u \phi_1 dxdt'. \tag{3.50}$$

From Lemma 3.1 and semigroup estimates, we have that

$$(V\otimes u + V\otimes V) : \nabla u \in L_1(Q_T)$$

for any positive finite T. Using these facts and (3.49), the conclusion is reached by choosing appropriate $\phi_\epsilon = \phi_1$ and taking a limit. □

Let us comment on Proposition 1.1. Recall that by Proposition 2.3, there exists a sequence $u_0^{(k)} \in C_{0,0}^\infty(\mathbb{R}^3)$ such that

$$u_0^{(k)} \overset{*}{\rightharpoonup} u_0$$

in $L^{3,\infty}$. It was shown in [33] that for any k there exists a global L_3-weak solution $v^{(k)}$. Now, Proposition 1.1 follows from Theorem 1.2.

4. Uniqueness

First we introduce the notation $Q(z_0,R) = B(x_0,R)\times]t-R^2,t[$. Here, $z_0 = (x_0,t) \in Q_\infty$.

Proof of Theorem 1.6.

Step I. Regularity

Our first remark is that, given $\varepsilon > 0$ and $R > 0$, there exists a number $R_*(T, R, \varepsilon) > 0$ such that if $B(x_0, R) \subset \mathbb{R}^3 \setminus B(R_*)$ and $t_0 - R^2 > 0$ then

$$\frac{1}{R^2}\int_{Q(z_0,R)} (|v|^3 + |q - [q]_{B(x_0,R)}|^{\frac{3}{2}})dxdt \leq \varepsilon.$$

For v it is certainly true. For q, we can use Lemma 3.3. Indeed, if $q = p_1 + p_2 + p_3$, then, for example, we have

$$\begin{aligned}
&\frac{1}{R^2}\int_{Q(z_0,R)} |p_1 - [p_1]_{B(x_0,R)}|^{\frac{3}{2}} dxds \\
&\quad\leq \frac{1}{R^2}\int_0^T \int_{B(x_0,R)} |p_1 - [p_1]_{B(x_0,R)}|^{\frac{3}{2}} dxds \\
&\quad\leq \frac{1}{R^{\frac{3}{2}}}\int_0^T \Big(\int_{B(x_0,R))} |\nabla p_1|^{\frac{9}{8}} dx\Big)^{\frac{4}{3}} dt \\
&\quad\leq \frac{1}{R^{\frac{3}{2}}}\int_0^T \Big(\int_{\mathbb{R}^3\setminus B(R_*))} |\nabla p_1|^{\frac{9}{8}} dx\Big)^{\frac{4}{3}} dt \to 0
\end{aligned}$$

as $R_* \to \infty$ for any fixed $R > 0$. Since the pair v and q satisfies the local energy inequality, by the ε-regularity theory developed in [5], we can claim that

$$|v(z_0)| \leq \frac{c}{R}$$

as long as z_0 and R satisfy the conditions above.

Now, our aim is to show that v is locally bounded. To this end, we can use condition (1.14) and state that there exists $R_0(x_0, \varepsilon_0) > 0$ such that

$$\|u_0\|_{L^{3,\infty}(B(x_0,R))} < \varepsilon_0$$

for all $0 < R < R_0(x_0, \varepsilon_0)$. Then

$$\|v(\cdot, t)\|_{L^{3,\infty}(B(x_0,R))} \leq \|u_0\|_{L^{3,\infty}(B(x_0,R))} + \varepsilon_0 < 2\varepsilon_0$$

for all $0 < R < R(x_0, \varepsilon_0)$ and for all $t \in]0, T[$. .

By Hunt's inequality (Theorem 4.5, p. 271 of [17]) for Lorentz spaces, we have

$$\begin{aligned}
&\frac{1}{r}\Big(\int_{t_0-r^2}^{t_0} \Big(\int_{B(x_0,r)} |v|^2 dx\Big)^2 dt\Big)^{\frac{1}{4}} \\
&\quad\leq c \sup_{t_0-r^2<t<t_0} \|v(\cdot, t)\|_{L^{3,\infty}(B(x_0,r))} \leq c\varepsilon_0
\end{aligned}$$

for all $t_0 \in]0, T]$, for all $0 < r < R_0(x_0, \varepsilon_0)$ satisfying $t_0 - r^2 > 0$, and c is a positive universal constant. Then the local boundedness follows from ε-regularity conditions derived in [35] with a suitable choice of the constant ε_0.

So, we can ensure that $v \in L_\infty(Q_{\delta,T})$ for any $\delta > 0$. Here, $Q_{\delta,T} = \mathbb{R}^3 \times]\delta, T[$. Then, we can easily deduce that, for any $\delta > 0$, $u \in W^{2,1}_2(Q_{\delta,T})$, $\nabla u \in L_{2,\infty}(Q_{\delta,T})$, and $\nabla q \in L_2(Q_{\delta,T})$. By iterative arguments, we complete the proof of the theorem.

Step II. Uniqueness
Regularity results proved above allow us to state that the energy identity

$$\frac{1}{2}\int_{\mathbb{R}^3}|u(x,t)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2dxds=\int_0^t\int_{\mathbb{R}^3}V\otimes v:\nabla udxds$$

holds for any $t>0$ and, moreover,

$$\int_{\mathbb{R}^3}\Big(\partial_t u(x,t)\cdot w(x)+(v(x,t)\cdot\nabla v(x,t))\cdot w(x)+\nabla u(x,t):\nabla w(x)\Big)dx=0$$

for any $w\in C^\infty_{0,0}(\mathbb{R}^3)$ and for all $t\in]0,T[$.

Letting $\tilde{u}=\tilde{v}-V$ and $w=\tilde{u}-u$, we can repeat the same arguments as in [33] to obtain

$$\frac{1}{2}\int_{\mathbb{R}^3}|w(x,t_0)|^2dx+\int_0^{t_0}\int_{\mathbb{R}^3}|\nabla w|^2dxdt$$

$$\leq\int_0^{t_0}\int_{\mathbb{R}^3}\Big(\tilde{v}\otimes\tilde{v}:\nabla w-v\otimes v:\nabla w\Big)dxdt$$

$$=\int_0^{t_0}\int_{\mathbb{R}^3}(w\otimes v+v\otimes w):\nabla wdxdt.$$

So, finally,

$$I:=\int_{\mathbb{R}^3}|w(x,t_0)|^2dx+\int_0^{t_0}\int_{\mathbb{R}^3}|\nabla w|^2dxdt$$

$$\leq c\int_0^{t_0}\int_{\mathbb{R}^3}|v|^2|w|^2dxdt.$$

Let us fix $s\in]0,T[$, then

$$I\leq cI_1+cI_2+cI_3,$$

where

$$I_1=\int_0^{t_0}\int_{\mathbb{R}^3}|v(x,t)-u_0(x)|^2|w(x,t)|^2dxdt,$$

$$I_2=\int_0^{t_0}\int_{\mathbb{R}^3}|v(x,s)-u_0(x)|^2|w(x,t))|^2dxdt,$$

$$I_3=\int_0^{t_0}\int_{\mathbb{R}^3}|v(x,s)|^2|w(x,t)|^2dxdt.$$

The first two integrals are evaluated in the same way with the help of Hunt's inequality (Theorem 4.5, p. 271 of [17]) for Lorentz spaces:

$$c(I_1+I_2)\leq c\int_0^{t_0}(\|v(\cdot,t)-u_0(\cdot)\|^2_{L^{3,\infty}}$$

$$+\|v(\cdot,s)-u_0(\cdot)\|^2_{L^{3,\infty}})\|w(\cdot,t)\|^2_{L^{6,2}}dt.$$

By the assumptions of the theorem,

$$c(I_1 + I_2) \leq c\varepsilon_0 \int_0^{t_0} \|w(\cdot, t)\|^2_{L^{6,2}} dt.$$

It remains to apply the Sobolev inequality $L^{6,2}(\Omega) \hookrightarrow W^1_2(\Omega)$ to conclude that

$$c(I_1 + I_2) \leq c\varepsilon_0 \int_0^{t_0} \|\nabla w(\cdot, t)\|^2_{L_2}.$$

To estimate I_3, we are going to use the fact that $v(\cdot, s)$ is bounded for positive $s \leq T$, i.e.,

$$\|v(\cdot, s)\|_{L_\infty} \leq g(s).$$

Here, it might happen that $g(s) \to \infty$ if $s \to 0$. So,

$$I_3 \leq g^2(s) \int_0^{t_0} \int_{\mathbb{R}^3} |w(x, t)|^2 dxdt.$$

Then reducing ε_0 if necessary, we find

$$\int_{\mathbb{R}^3} |w(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dxdt \leq cg^2(s) \int_0^{t_0} \int_{\mathbb{R}^3} |w(x, t)|^2 dxdt$$

for all $t_0 \in]0, T[$, which implies that $w(\cdot, t) = 0$ for the same t. □

To justify Corollary 1.7, we can argue as follows. First, it can be shown that

$$\|u_0\|_{L^{3,\infty}(B(x_0,R))} \to 0$$

as $R \to 0$. Indeed, if v is a weak $L^{3,\infty}$-solution in Q_T, then for a.a. $t \in]0, T[$ we have $v(\cdot, t) \in L^{3,\infty}$ along with the following property. Namely, for all $x_0 \in \mathbb{R}^3$:

$$\|v(\cdot, t)\|_{L^{3,\infty}(B(x_0,R))} \to 0$$

as $R \to 0$. Since it is assumed that $v \in C([0, T]; L^{3,\infty})$, the above property in fact holds for all $t \in [0, T]$.

Now, one should split the interval $[0, T]$ into sufficiently small pieces by points t_k with $k = 1, 2, \dots, N$ and $t_N = T$ so that

$$\|v(\cdot, t) - v(\cdot, t_{k-1})\|_{L^{3,\infty}(\mathbb{R}^3)} < \varepsilon_0$$

for any $t \in [t_{k-1}, t_k]$ and

$$\lim_{s \to t_k^+} \|u(\cdot, s) - u(\cdot, t_k)\|_{L_2} = 0$$

($k = 1, 2, \dots, N$). It is not difficult to see that for $k = 1, 2, \dots, N$, $v(\cdot, t_k + s)$ is a weak $L^{3,\infty}$ solution on Q_{T-t_k}, with initial value $v(\cdot, t_k)$. It remains to apply Theorem 1.6 successively for $k = 1, 2, \dots, N$.

5. Regularity

Proof of Theorem 1.8. We use the Kato iteration scheme. Let us define the following, for $k = 1, 2, \dots,$

$$v^{(1)} = V, \qquad V^{(k+1)} = V + u^{(k+1)},$$

where $u^{(k+1)}$ solves the following problem

$$\partial_t u^{(k+1)} - \Delta u^{(k+1)} + \nabla q^{(k+1)} = -\mathrm{div}\, v^{(k)} \otimes v^{(k)}, \quad \mathrm{div}\, u^{(\mathrm{k}+1)} = 0$$

in Q_T,

$$u^{(k+1)}(\cdot, 0) = 0$$

in $\mathbb{R}^3$. It is easy to check that for solutions to the above linear problem the following estimates are true

$$\langle u^{(k+1)} \rangle_{Q_T} \leq c \langle v^{(k)} \rangle^2_{Q_T},$$

$$\|u^{(k+1)}\|_{L_\infty(0,T;L_3)} \leq c \langle v^{(k)} \rangle^2_{Q_T}$$

and thus we have

$$\langle v^{(k+1)} \rangle_{Q_T} \leq \langle V \rangle_{Q_T} + c \langle v^{(k)} \rangle^2_{Q_T},$$

$$\|v^{(k+1)}\|_{L_\infty(0,T;L^{3,\infty})} \leq \|V\|_{L_\infty(0,T;L^{3,\infty})} + c \langle v^{(k)} \rangle^2_{Q_T},$$

and

$$\|v^{(k+1)} - V\|_{L_\infty(0,T;L_3)} \leq c \langle v^{(k)} \rangle^2_{Q_T}$$

for all $k = 1, 2, \dots$. Using Kato's arguments from [22], one easily show that for $\varepsilon < \frac{1}{4c}$ we shall have

$$\langle v^{(k)} \rangle_{Q_T} < 2 \langle V \rangle_{Q_T} \tag{5.1}$$

for all $k = 1, 2, \dots$. We get, in addition, that

$$\|v^{(k)}\|_{L_\infty(0,T;L^{3,\infty})} \leq \|V\|_{L_\infty(0,T;L^{3,\infty})} + \langle V \rangle_{Q_T}, \tag{5.2}$$

$$\|v^{(k+1)} - V\|_{L_\infty(0,T;L_3)} \leq \langle V \rangle_{Q_T} \tag{5.3}$$

for all $k = 1, 2, \dots$. Furthermore, Kato's arguments in [22] also give that there is a $v = V + u$ such that

$$\langle v^{(k)} - v \rangle_{Q_T}, \langle u^{(k)} - u \rangle_{Q_T} \to 0, \tag{5.4}$$

$$\|v^{(k)} - v\|_{L_\infty(0,T;L^{3,\infty})}, \|u^{(k)} - u\|_{L_\infty(0,T;L_3)} \to 0. \tag{5.5}$$

Next we note that by interpolation:

$$t^{\frac{1}{8}} \|g(\cdot, t)\|_{L_4} \leq C(\|g(\cdot, t)\|_{L^{3,\infty}})^{\frac{3}{8}} (t^{\frac{1}{5}} \|g(\cdot, t)\|_{L_5})^{\frac{5}{8}}. \tag{5.6}$$

Using this and (5.4) and (5.5), we immediately see that

$$\|v^{(k)} - v\|_{L_4(Q_T)}, \|u^{(k)} - u\|_{L_4(Q_T)} \to 0. \tag{5.7}$$

We also can exploit our equation, together with the pressure equation, to derive the following estimate for the energy and pressure:

$$\|u^{(k)} - u^{(m)}\|^2_{2,\infty,Q_T} + \|\nabla u^{(k)} - u^{(m)}\|^2_{2,Q_T} + \|q^{(k)} - q^{(m)}\|^2_{2,Q_T}$$
$$\leq c \int_0^T \int_{\mathbb{R}^3} |v^{(k)} \otimes v^{(k)} - v^{(m)} \otimes v^{(m)}|^2 dxdt. \tag{5.8}$$

Using (5.7), we immediately see the following

$$u^{(k)} \to u \text{ in } W^{1,0}_2(Q_T) \cap C([0,T]; L_2(\mathbb{R}^3)) \cap L_4(Q_T), \tag{5.9}$$

$$u(\cdot, 0) = 0, \tag{5.10}$$

$$q^{(k)} \to q \text{ in } L_2(Q_T). \tag{5.11}$$

Clearly, the pair v and q satisfies the Navier–Stokes equations, in a distributional sense. It is easily verified that

$$S(t)u_0 \in L_4(Q_T) \cap L_{2,\infty}(B(R)\times]0,T[) \cap W^{1,0}_2(B(R)\times]\epsilon,T[) \tag{5.12}$$

for any $0 < R, 0 < \varepsilon < T$. By (5.9)–(5.11), v has the same property. It is known that this, along with $q \in L_2(Q_T)$, is sufficient to infer that the pair v and q satisfies the local energy equality. This can be shown by a mollification argument. Showing that u satisfies the energy inequality (on Q_T) present in our definition of global weak $L^{3,\infty}$ solution (in fact, in this case it is an equality), can now be performed in a similar way to Lemma 3.3. Here, certain decay properties of u, q from (5.9)–(5.11) are needed, as well as the fact that $\lim_{t\to 0^+} \|u(\cdot,t)\|_{L_2(\mathbb{R}^3)} = 0$. □

Proof of Theorem 1.9. Condition (1.19) ensures that there exists an $N > 0$ such that

$$\|(u_0)^N_+\|_{L^{3,\infty}} < \varepsilon_3.$$

Thus, by the convolution inequality,

$$< S(t)(u_0)^N_+ >_{Q_T}, \ \|S(t)(u_0)^N_+\|_{L_\infty(0,T;L^{3,\infty})} < C\varepsilon_3.$$

By Lemma 2.1, we have that

$$\|(u_0)^N_-\|_{L_5} \le CN^{\frac{2}{5}} \|u_0\|^{\frac{3}{5}}_{L^{3,\infty}}.$$

Thus,

$$\langle V\rangle_{Q_T} < C\varepsilon_3 + T^{\frac{1}{5}} CN^{\frac{2}{5}} \|u_0\|^{\frac{3}{5}}_{L^{3,\infty}}.$$

Taking $T := T(u_0)$ and ε_3 sufficiently small gives, by Theorem 1.8, the existence of weak $L^{3,\infty}$ solution on Q_T such that

$$\begin{aligned} &\|v - S(t)(u_0)^N_-\|_{L_\infty(0,T;L^{3,\infty})} \le \\ &\le \|v - V\|_{L_\infty(0,T;L^{3,\infty})} + \|S(t)(u_0)^N_+\|_{L_\infty(0,T;L^{3,\infty})} < \\ &< V >_{Q_T} + C\varepsilon_3 < \varepsilon_0. \end{aligned} \tag{5.13}$$

Next we notice that $S_1(t)(u_0)^N_-$ is bounded in Q_T and moreover

$$\|S(t)(u_0)^N_-\|_{L^{3,\infty}(B(x_0,R))} \le CRN. \tag{5.14}$$

These facts, along with (5.13), are enough to conclude using minor adaptations to the proof of Theorem 1.6. □

Remark 5.1. Furthermore, there is the lower bound for T:

$$T \geq \frac{\min(\varepsilon^5, \varepsilon_0^5)}{CN^2 \|u_0\|^3_{L^{3,\infty}}}. \tag{5.15}$$

Here, C is a universal constant. Moreover, ε and ε_0 are from Theorems 1.6 and 1.8 respectively.

Acknowledgement

The authors wish to warmly thank the anonymous reviewer for their helpful comments and discussion.

References

[1] Adams, R.A., Fournier, J.F. (2003). *Sobolev Spaces*. Pure and Applied Mathematics, Vol. 140. 2nd ed. Amsterdam: Academic Press.
[2] Barker, T. (2017). Uniqueness results for weak Leray-Hopf solutions of the Navier–Stokes system with initial values in critical spaces. *J. Math. Fluid Mech.* 20:133–160. doi: 10.1007/s00021-017-0315-8.
[3] Barker, T., Seregin, G. (2017). A necessary condition of potential blowup for the Navier–Stokes system in half-space. *Math. Ann.* 369:1327–1352.
[4] Buckmaster, T., Shkoller, S., Vicol, V. Nonuniqueness of weak solutions to the SQG equation, preprint arXiv:1610.00676.
[5] Caffarelli, L., Kohn, R.-V., Nirenberg, L. (1982). Partial regularity of suitable weak solutions of the Navier–Stokes equations. *Commun. Pure Appl. Math.* XXXV:771–831.
[6] Calderón, C.P. (1990). Existence of weak solutions for the Navier–Stokes equations with initial data in Lp. *Trans. Am. Math. Soc.* 318:179–200.
[7] Cannone, M. (1997). A generalization of a theorem by Kato on Navier–Stokes equations. *Rev. Mat. Iberoam.* 13:515–541.
[8] De Lellis, C., Szekelyhidi, L. (2012). The h-principle and the equations of fluid dynamics. *Bull. Am. Math. Soc. (N.S.)* 49:347–375.
[9] Dubois, S. (2003). Uniqueness for some Leray-Hopf solutions to the Navier–Stokes equations. *J. Differ. Equations* 189:99–147.
[10] Fujita, H., Kato, T. (1964). On the Navier–Stokes initial value problem. *I. Arch. Rat. Mech. Anal.* 16:269–315.
[11] Gallagher, I., Planchon, F. (2002). On global infinite energy solutions to the Navier–Stokes equations in two dimensions. *Arch. Rat. Mech. Anal.* 161:307–337.
[12] Germain, P. (2006). Multipliers, paramultipliers, and weak-strong uniqueness for the Navier–Stokes equations. *J. Differ. Equations* 226:373–428.
[13] Giga, Y., Miyakawa, T. (1989). Navier–Stokes flow in R3 with measures as initial vorticity and Morrey spaces. *Commun. Part. Differ. Equations* 14:577–618.
[14] Giga, Y., Sohr, H. (1991). Abstract L_p estimates for the Cauchy problem with applications to the Navier–Stokes equations in exterior domains. *J. Funct. Anal.* 102:72–94.
[15] Grafakos, L. (2008). *Classical Fourier Analysis*. Graduate Texts in Mathematics, Vol. 249, 2nd ed. Berlin, New York: Springer-Verlag.
[16] Guillod, J., Šverák, V. (2017). Numerical investigations of non-uniqueness for the Navier–Stokes initial value problem in borderline spaces. arXiv:1704.00560.
[17] Hunt, R.A. (1966). On $L(p, q)$ spaces. *Enseign. Math.* 12:249–276.
[18] Isett, P. A proof of Onsager's conjecture, preprint, arXiv:1608.08301.
[19] Jia, H., Sverak, V. (2013). Minimal L_3-initial data for potential Navier–Stokes singularities. *SIAM J. Math. Anal.* 45:1448–1459.
[20] Jia, H., Sverak, V. (2014.) Local-in-space estimates near initial time for weak solutions of Navier–Stokes equations and forward self-similar solutions. *Invent. Math.* 196:233–265.

[21] Jia, H., Sverak, V. (2015). Are the incompressible 3d Navier–Stokes equations locally ill-posed in the natural energy space? *J. Funct. Anal.* 268:3734–3766.
[22] Kato, T. (1984). Strong L_p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. *Math. Z.* 187:471–480.
[23] Koch, H., Tataru, D. (2001). Well-posedness for the Navier–Stokes equations. *Adv. Math.* 157: 22–35.
[24] Kozono, H., Yamazaki, M. (1995). Local and global unique solvability of the Navier–Stokes exterior problem with Cauchy data in the space $L^{n,\infty}$. *Houston J. Math.* 21:755–799.
[25] Lemarié-Rieusset, P. G. (1998.) Quelques remarques sur les équations de Navier–Stokes dans $\mathbb{R}^3$. *Sém. Équations Aux dér. Part. (Polytech.)* 1997:1–8.
[26] Lemarié-Rieusset, P. G. *Recent Developments in the Navier–Stokes Problem*. Chapman&Hall/CRC, Boca Roton, FL, 2002. XIV + 395 pp.
[27] Leray, J. (1934). Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* 63: 193–248.
[28] Rusin, W., Sverak, V. (2011). Minimal initial data for potential Navier–Stokes singularities. *J. Funct. Anal.* 260:879–891.
[29] Scheffer, V. (1993).An inviscid flow with compact support in space-time. *J. Geom. Anal.* 3:343–401.
[30] Seregin, G. (2012). *Leray-Hopf Solutions to Navier–Stokes Equations with Weakly Converging Initial Data*. Mathematical Aspects of Fluid Mechanics, London Math. Soc. Lecture Note Ser., Vol. 402, pp. 251–258. Cambridge: Cambridge University Press.
[31] Seregin, G. A. (2012). A certain necessary condition of potential blow up for Navier–Stokes equations. *Commun. Math. Phys.* 312:–845.
[32] Seregin, G. (2015). *Lecture Notes on Regularity Theory for the Navier–Stokes Equations*. Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd. x+258 pp. ISBN:78-981-4623-40-7.
[33] Seregin, G., Šverák, V. (2016). On global weak solutions to the Cauchy problem for the Navier–Stokes equations with large L_3-initial data. *Nonlinear Anal.* 154:269–296. doi: 10.1016/j.na.2016.01.018.
[34] Solonnikov, V. A. (1973). Estimates of solutions to the non-stationary Navier–Stokes system. *Zap. Nauchn. Semin. LOMI* 28:153–231.
[35] Zajaczkowski, W., Seregin, G.A. (2006). A sufficient condition of local regularity for the Navier–Stokes equations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 336. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 37, 46–54, 274; translation in J. Math. Sci. (N. Y.) 143 (2007), no. 2, 2869–2874.