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# Higher rank hyperbolicity

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**Abstract** The large-scale geometry of hyperbolic metric spaces exhibits many distinctive features, such as the stability of quasi-geodesics (the Morse Lemma), the visibility property, and the homeomorphism between visual boundaries induced by a quasi-isometry. We prove a number of closely analogous results for spaces of rank $n \geq 2$ in an asymptotic sense, under some weak assumptions reminiscent of nonpositive curvature. For this purpose we replace quasi-geodesic lines with quasi-minimizing (locally finite) $n$-cycles of $r^n$ volume growth; prime examples include $n$-cycles associated with $n$-quasiflats. Solving an asymptotic Plateau problem and producing unique tangent cones at infinity for such cycles, we show in particular that every quasi-isometry between two proper CAT(0) spaces of asymptotic rank $n$ extends to a class of $(n-1)$-cycles in the Tits boundaries.

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## 1 Introduction

### 1.1 Overview

Since the appearance of Gromov's seminal paper [40] more than thirty years ago, hyperbolicity has played a central role in geometric group theory, and inspired a number of generalizations and variations. These include, among others, relative hyperbolicity [17,31,35,40,71], various notions of "directional" hyperbolicity inherent in stability/contraction properties of (quasi-)geodesics [12,22,24,49,52,76,77] (this in fact goes back to the notion of rank one geodesics [5,6] which predates hyperbolicity), acylindrical hyperbolicity [11,16,26,72], and hierarchical hyperbolicity [9,10,44,67]. (The literature is far richer than indicated here—we apologize for omissions.) These approaches provide unified descriptions of certain hyperbolicity phenomena in a variety of non-hyperbolic settings such as non-uniform lattices in rank one symmetric spaces, mapping class groups, Teichmüller space, and some CAT(0) cube complexes and three-manifold groups.

In this paper we develop a notion of higher rank hyperbolicity that complements, and partly overlaps with, the concepts mentioned above. We show that for metric spaces of asymptotic rank $n \geq 2$ satisfying certain weak convexity assumptions (see Sect. 1.2 below), characteristics of hyperbolicity such as slimness of (quasi-)geodesic triangles, stability of quasi-geodesics, and visibility remain valid when properly reformulated in terms of $n$-dimensional (relative) cycles. In particular, our results hold for proper and cocompact CAT(0) spaces of Euclidean rank $n$ and in that case they confirm several aspects of Gromov's discussion in Section 6 of [42], and also the well-known principle that in nonpositively curved spaces hyperbolic behavior should manifest itself in dimensions above the maximal dimension of a flat. Our approach also encompasses the stability properties of maximal quasiflats that were used

in the proofs of the quasi-isometric rigidity of higher rank symmetric spaces in [34,54,70].

We show further that a quasi-isometry between two proper CAT(0) spaces of asymptotic rank $n \geq 2$ naturally induces an isomorphism between the groups of compactly supported integral $(n-1)$-cycles — metric integral currents in the sense of Ambrosio–Kirchheim [1]—in their Tits boundaries. We remind the reader that in the (hyperbolic) rank one case, the usual visual boundaries are homeomorphic, whereas for $n \geq 2$ this can fail, even if the quasi-isometry is equivariant with respect to geometric actions of some finitely generated group [25]. The construction of the above isomorphism involves, on the one hand, an existence result for area-minimizing $n$-dimensional varieties with prescribed asymptotics. To our knowledge, this is the first general such result in a setting of nonpositive (rather than strictly negative) curvature (compare Section 1 in [41]). On the other hand, we show that $n$-dimensional (quasi-) minimizers with $r^n$ volume growth possess unique tangent cones at infinity, a phenomenon that occurs rather rarely (compare, for example, the discussion in [23]).

### 1.2 Setup

For simplicity, we assume throughout the paper that the underlying metric space $X = (X, d)$ is proper (that is, bounded closed subsets are compact). For a first set of results, described in Sect. 1.4 below, we assume that $X$ satisfies the following two conditions for some $n \geq 1$:

($\mathrm{CI}_n$) (Coning inequalities) There is a constant $c$ such that any two points $x, x'$ in $X$ can be joined by a curve of length $\leq c\, d(x, x')$, and for $k = 1, \ldots, n$, every $k$-cycle $R$ in some $r$-ball bounds a $(k+1)$-chain $S$ with mass

$$\mathbf{M}(S) \leq c\, r\, \mathbf{M}(R).$$

Here, for a general proper metric space $X$, we use metric integral currents (see Sect. 2). However, if $X$ is bi-Lipschitz homeomorphic to a finite-dimensional simplicial complex with standard metrics on the simplices, then (by a variant of the Federer–Fleming deformation theorem [37]) one may equivalently take simplicial chains or singular Lipschitz chains (with integer coefficients).

($\mathrm{AR}_n$) (Asymptotic rank $\leq n$) No asymptotic cone of $X$ contains an isometric copy of an $(n+1)$-dimensional normed space. Equivalently, $\operatorname{asrk}(X) \leq n$, where $\operatorname{asrk}(X)$ is defined as the supremal $k$ for which there exist a sequence $r_i \to \infty$ and subsets $Y_i \subset X$ such that the rescaled sets

$(Y_i, r_i^{-1}d)$ converge in the Gromov–Hausdorff topology to the unit ball in some $k$-dimensional normed space (see Sect. 4).

Condition $(\mathrm{CI}_n)$ is reminiscent of nonpositive curvature: if $X$ is a CAT(0) or Busemann space [73], the required inequality holds for the geodesic cone $S$ from the center of the $r$-ball over $R$ (see Sect. 2.7). Furthermore, any $n$-connected simplicial complex as above with a properly discontinuous and cocompact simplicial action of a combable group satisfies $(\mathrm{CI}_n)$; see Section 10.2 in [33]. Every combable group, in particular every automatic group, admits such an action.

When $X$ is a cocompact CAT(0) or Busemann space, the asymptotic rank $\mathrm{asrk}(X)$ equals the maximal dimension of an isometrically embedded Euclidean or normed space, respectively [53]. More generally, for spaces satisfying $(\mathrm{CI}_n)$, condition $(\mathrm{AR}_n)$ is equivalent to a *sub-Euclidean isoperimetric inequality* for $n$-cycles [82]; this result, restated in Theorem 4.4, plays a key role in this paper. If $X$ is a geodesic Gromov hyperbolic space, then every asymptotic cone of $X$ is an $\mathbb{R}$-tree, thus $\mathrm{asrk}(X) \leq 1$. Conversely, a space satisfying $(\mathrm{CI}_1)$ and $(\mathrm{AR}_1)$ is Gromov hyperbolic (compare Corollary 1.3 in [82] and the special case $n = 1$ of Theorem 1.1 below).

We remark that the asymptotic rank is a quasi-isometry invariant for metric spaces [82], whereas condition $(\mathrm{CI}_n)$ is preserved, for instance, by quasi-isometries between proper and cocompact, $n$-connected simplicial complexes with standard metrics on the simplices.

The main results discussed in the second half of the paper, starting from Sect. 7, involve actual convexity properties or the ideal boundary of $X$ (rather than condition $(\mathrm{CI}_n)$). For the outline of these results in Sects. 1.5 and 1.6, we will therefore assume that $X$ is CAT(0). In the body of the paper, we will work with the weaker sufficient condition that $X$ admits a *convex bicombing* — this disposes with geodesic uniqueness but retains Busemann convexity for a distinguished family of geodesics; see Definition 7.1 and the comments thereafter.

### 1.3 Quasi-minimizers with controlled density

We now discuss the objects we use to exhibit higher rank hyperbolic behavior, that is, $n$-dimensional replacements for quasi-geodesics.

One approach would be to study $n$-quasiflats, or more generally, images of quasi-isometric embeddings $W \to X$ for suitable subsets $W \subset \mathbb{R}^n$. (See Sect. 2.1 for the standard definitions of quasi-isometric maps.) However, since geodesics may be viewed either as isometric embeddings of intervals or as length minimizing curves, an alternative approach is to consider (relative) $n$-cycles which "quasi-minimize" area (compare [7,41], for example). We follow the latter approach in this paper: it turns out that it is not only more general,

but it also leads to cleaner and sharper results. The quasi-minimality condition will be used in conjunction with a polynomial growth bound of order $n$. We now provide more details.

We will work with the chain complexes $\mathbf{I}_{*,\mathrm{c}}(X)$ and $\mathbf{I}_{*,\mathrm{loc}}(X)$ of metric *integral currents* with compact support and *locally integral currents* introduced in [1,59]. This enables us in particular to pass to limits and to produce area-minimizers with sharp density and monotonicity properties. All relevant concepts and results will be reviewed in detail in Sect. 2. Every singular Lipschitz $n$-chain in $X$ with integer coefficients may be viewed as an element of $\mathbf{I}_{n,\mathrm{c}}(X)$ (and, conversely, every integral current in $\mathbb{R}^N$ admits an approximation by Lipschitz chains; see Theorem 5.8 in [37]). Similarly, $\mathbf{I}_{n,\mathrm{loc}}(X)$ comprises all locally finite Lipschitz $n$-chains. Associated with every $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ is a locally finite Borel measure $\|S\|$ on $X$ whose total *mass* is denoted $\mathbf{M}(S) := \|S\|(X)$, and the *support* $\mathrm{spt}(S) \subset X$ is the smallest closed set supporting $\|S\|$. We let $\mathbf{Z}_{n,\mathrm{c}}(X)$ and $\mathbf{Z}_{n,\mathrm{loc}}(X)$ denote the respective cycle groups for $n \geq 1$.

A local cycle $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ will be called (large-scale) *quasi-minimizing* if there exist constants $Q \geq 1$ and $a \geq 0$ such that, for every $x \in \mathrm{spt}(S)$ and almost every $r > a$, the restriction $S \llcorner B_x(r) \in \mathbf{I}_{n,\mathrm{c}}(X)$ of $S$ to the closed $r$-ball centered at $x$ satisfies

$$\mathbf{M}(S \llcorner B_x(r)) \leq Q\, \mathbf{M}(T)$$

for all $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ with $\partial T = \partial(S \llcorner B_x(r))$; then $S$ is $(Q, a)$-*quasi-minimizing*. A $(1, 0)$-quasi-minimizing local cycle is *(area-)minimizing*. Every quasiflat in $X$ may be viewed as a quasi-minimizer (see Propositions 3.6 and 3.7 for two precise statements).

We say that $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ has (large-scale) *controlled density* if there exist constants $C > 0$ and $a \geq 0$ such that

$$\Theta_{p,r}(S) := \frac{\|S\|(B_p(r))}{r^n} \leq C$$

for all $p \in X$ and $r > a$; then $S$ has $(C, a)$-*controlled density*. A generally weaker condition is that the *asymptotic density*

$$\Theta_\infty(S) := \limsup_{r\to\infty} \Theta_{p,r}(S)$$

of $S$ be finite; here $p$ is fixed, however the upper limit is independent of $p$. Similarly, for $Z \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ and any $p \in X$, we define the *asymptotic filling density*

$$F_\infty(Z) := \limsup_{r\to\infty} F_{p,r}(Z),$$

where $F_{p,r}(Z)$ denotes the infimum of $\mathbf{M}(V)/r^{n+1}$ over all $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ with $\mathrm{spt}(Z - \partial V) \cap B_p(r) = \emptyset$ (that is, $V$ "fills $Z$ in $B_p(r)$"). For $S, S' \in \mathbf{Z}_{n,\mathrm{loc}}(X)$, the relation $F_\infty(S - S') = 0$ will serve as an appropriate notion of asymptoticity.

We now discuss the main results in the paper.

### 1.4 Slim simplices, Morse Lemma, and asymptote classes

We first recall that a geodesic metric space $X$ is *(Gromov) hyperbolic* [40] if there exists a constant $\delta \geq 0$ such that every geodesic triangle in $X$ is $\delta$-*slim*, that is, each of its sides lies in the closed $\delta$-neighborhood of the union of the other two. According to the Morse Lemma (which for the real hyperbolic plane goes back to [68]), every $(L, a)$-quasi-geodesic segment in $X$ is then at Hausdorff distance at most $b$ from a geodesic segment connecting its endpoints, where the constant $b$ depends only on $L$, $a$ and $\delta$. Thus any triangle composed of three $(L, a)$-quasi-geodesic segments is still $(\delta + 2b)$-slim.

We prove the following higher rank analog of this property.

**Theorem 1.1** (slim simplices) *Let $X$ be a proper metric space satisfying conditions* $(\mathrm{CI}_n)$ *and* $(\mathrm{AR}_n)$ *for some $n \geq 1$. Let $\Delta$ be a Euclidean $(n+1)$-simplex, and let $f : \partial\Delta \to X$ be a map such that for every facet $W$ of $\Delta$, the restriction $f|_W$ is an $(L, a)$-quasi-isometric embedding. Then, for every facet $W$, the image $f(W)$ is contained in the closed $D$-neighborhood of $f\left(\overline{\partial\Delta \setminus W}\right)$ for some constant $D = D(X, n, L, a)$.*

Here $\Delta$ is the convex hull of a set of $n+2$ points in $\mathbb{R}^{n+1}$ such that $\Delta$ has non-empty interior, and a *facet* of $\Delta$ is the convex hull of $n+1$ of them.

The proof of this result depends, on the one hand, on an iterated application of the aforementioned sub-Euclidean isoperimetric inequality. For a cycle $Z \in \mathbf{Z}_{n,\mathrm{c}}(X)$ with controlled density, this provides an arbitrarily small upper bound $F_{p,r}(Z) < \epsilon$ on the filling density in any ball $B_p(r)$ of sufficiently large radius, depending on $\epsilon$ (Proposition 4.5). On the other hand, if $Z$ is "piecewise $(Q, a)$-quasi-minimizing", then $F_{x,r}(Z) \geq c = c(X, n, Q) > 0$ for any ball $B_x(r)$ with $r > 4a$ centered on one of the pieces and disjoint from the union of the remaining ones (Lemma 3.4); thus $x$ cannot be too far away from this union. For an appropriately chosen cycle $Z$ approximating the image of $f : \partial\Delta \to X$, this yields Theorem 1.1 (see Theorem 5.2).

In combination with the existence of area-minimizing integral currents with prescribed boundary, a similar argument yields a higher rank analog of the Morse Lemma stated above; see Theorem 5.4. We further establish the following asymptotic version of this result (see Theorem 5.7 for a generalization including boundaries).

**Theorem 1.2** (asymptotic Morse Lemma) *Let $X$ be a proper metric space satisfying conditions* ($\mathrm{CI}_n$) *and* ($\mathrm{AR}_n$) *for some $n \geq 1$. Suppose that $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is $(Q, a)$-quasi-minimizing and has $(C, a)$-controlled density. Then there exists an area-minimizing local cycle $\tilde{S} \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ such that $F_\infty(S - \tilde{S}) = 0$, and every such $\tilde{S}$ satisfies $\Theta_\infty(\tilde{S}) \leq \Theta_\infty(S)$ and $d_{\mathrm{H}}(\mathrm{spt}(S), \mathrm{spt}(\tilde{S})) \leq b$ for some constant $b = b(X, n, Q, C, a)$.*

This implies in particular the following analog of Morse's Theorem 1 [68] on the stability of geodesics in the hyperbolic plane. We remark that for a Riemannian manifold $X$, metric locally integral currents in $X$ can be identified with the classical ones from [36].

**Corollary 1.3** (persistence of minimizers) *Let $X = (X, g)$ be a Hadamard manifold of asymptotic rank $n \geq 1$, and suppose that $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is area-minimizing and has controlled density. Then for every Riemannian metric $\tilde{g}$ on $X$ bi-Lipschitz equivalent to $g$ there is an $\tilde{S} \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ that is area-minimizing with respect to $\tilde{g}$ and whose support is at finite Hausdorff distance from* $\mathrm{spt}(S)$.

Note that if $\tilde{d}$ is the distance function on $X$ induced by $\tilde{g}$, then $X = (X, \tilde{d})$ satisfies the assumptions of Theorem 1.2, and $S$ is quasi-minimizing and has controlled density with respect to $\tilde{d}$. Hence, the result follows. By regularity theory, $\mathrm{spt}(\tilde{S})$ is a smooth $n$-dimensional submanifold except for a closed singular set of Hausdorff dimension at most $n - 2$ (see [27] for a guide to the literature). For example, $S$ could be the current associated to an oriented $n$-flat in $(X, g)$ (but see also Theorem 1.6 below). The primary instance of Corollary 1.3 is when $(X, g)$ is the universal covering of a compact manifold of nonpositive sectional curvature such that $(X, g)$ contains no $(n + 1)$-flat, and $\tilde{g}$ is the lift of an arbitrary metric on the quotient.

Morse's result was generalized in various directions to surfaces of arbitrary dimension and codimension in spaces of negative curvature [7,41,55,56,58] and to totally geodesic hyperplanes in some product spaces [57]. There is a parallel development based on periodicity (rather than hyperbolicity) and limited to codimension one, starting with the work of Hedlund [45] on the two-dimensional torus and including the investigation of laminations of compact Riemannian manifolds by minimal hypersurfaces; see [4,20,69] and the references therein. Corollary 1.3 is now the first result in this area for higher rank and arbitrary codimension.

The tools developed so far enable us further to introduce visual metrics on sets of asymptote classes of local $n$-cycles, in analogy with the usual metrization of the visual boundary of a geodesic Gromov hyperbolic space. Let $X$ be a proper metric space satisfying condition ($\mathrm{CI}_n$) for $n = \mathrm{asrk}(X) \geq 1$. We consider the group

$$\mathbf{Z}^\infty_{n,\mathrm{loc}}(X) := \{S \in \mathbf{Z}_{n,\mathrm{loc}}(X) : \Theta_\infty(S) < \infty\}$$

and the quotient space $\mathscr{Z}X := \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)/\!\sim_F$ of *F-asymptote classes*, where $S \sim_F S'$ if and only if $F_\infty(S - S') = 0$. Making use of the existence of area-minimizers in each class $[S]$, we define an analog of the Gromov product of two points at infinity and show that for any constants $C > 0$ and $a \geq 0$, the set $\mathscr{Z}_{C,a}X$ of all classes represented by some element with $(C, a)$-controlled density admits an analog of Gromov's $\delta$-inequality (Proposition 6.2) and carries a family of visual metrics, with respect to which $\mathscr{Z}_{C,a}X$ is compact; see Theorem 6.3.

### 1.5 Asymptotic geometry of local cycles

For the remainder of the introduction, we will be mainly concerned with asymptotic properties of local $n$-cycles in spaces of asymptotic rank $n \geq 2$, and relations with the ideal boundary of $X$. For this reason we assume in Sects. 1.5 and 1.6 that $X$ is a CAT(0) space, so that we may make use of the boundary at infinity $\partial_\infty X$ and the compactification $\overline{X} := X \cup \partial_\infty X$—both equipped with the cone topology—as well as the Tits boundary $\partial_\mathrm{T} X$ and the Tits cone $\mathscr{C}_\mathrm{T} X$. As mentioned earlier, all of the results discussed here hold more generally if $X$ is a proper metric space equipped with a convex bicombing, and the respective statements will be given in the body of the paper.

A point in $\partial_\infty X$ is an asymptote class of unit speed rays in $X$. The Tits cone $\mathscr{C}_\mathrm{T} X$ may be defined as the set of asymptote classes of rays $\varrho\colon \mathbb{R}_+ \to X$ of arbitrary (constant) speed $s \geq 0$, endowed with the metric $d_\mathrm{T}$, where

$$d_\mathrm{T}([\varrho], [\varrho']) = \lim_{t\to\infty} \frac{1}{t}\, d(\varrho(t), \varrho'(t))$$

is the asymptotic slope of the convex function $t \mapsto d(\varrho(t), \varrho'(t))$. For every $p \in X$ there is a canonical 1-Lipschitz map

$$\mathrm{can}_p\colon \mathscr{C}_\mathrm{T} X \to X$$

such that $\mathrm{can}_p([\varrho]) = \varrho(1)$ for every ray $\varrho$ with $\varrho(0) = p$. The Tits boundary $\partial_\mathrm{T} X$ is the unit sphere in $\mathscr{C}_\mathrm{T} X$ and agrees with $\partial_\infty X$ as a set, but is endowed with the finer topology induced by $d_\mathrm{T}$. With respect to the (equivalent) angle metric $0 \leq \angle_\mathrm{T} \leq \pi$ characterized by the relation $2\sin(\angle_\mathrm{T}(u, v)/2) = d_\mathrm{T}(u, v)$, $\partial_\mathrm{T} X$ is a CAT(1) space, and $\mathscr{C}_\mathrm{T} X$ agrees with the Euclidean cone over $(\partial_\mathrm{T} X, \angle_\mathrm{T})$ and is thus a CAT(0) space. If $X$ is a symmetric space of non-compact type or a thick Euclidean building of rank $n \geq 2$, then $(\partial_\mathrm{T} X, \angle_\mathrm{T})$ has the structure of a thick $(n-1)$-dimensional spherical building.

For a local cycle $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$, we let

$$\Lambda(S) \subset \partial_\infty X$$

denote the *limit set* of $\mathrm{spt}(S)$, that is, the set of all points in $\partial_\infty X$ belonging to the closure of $\mathrm{spt}(S)$ in $\overline{X}$. We say that $S$ is *conical* with respect to some point $p \in X$ if $S$ is invariant, for every $\lambda \in (0, 1)$, under the $\lambda$-Lipschitz map $h_{p,\lambda} \colon X \to X$ that takes $x$ to $\sigma_{px}(\lambda)$, where $\sigma_{px} \colon [0, 1] \to X$ denotes the geodesic from $p$ to $x$.

The following result summarizes Theorem 7.3, Proposition 8.2, and Theorem 9.4 for the case when $X$ is CAT(0). It shows in particular that the group $\mathscr{Z}X = \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)/\!\sim_F$ of $F$-asymptote classes is canonically isomorphic to the group of integral $(n-1)$-cycles in $\partial_\mathrm{T} X$.

**Theorem 1.4** (Tits boundary) *Let $X$ be a proper* CAT(0) *space with* $\mathrm{asrk}(X) = n \geq 2$. *If $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$, then for every $p \in X$ there is a unique representative $S_{p,0} \in [S] \in \mathscr{Z}X$ that is conical with respect to $p$, and there is a unique local cycle $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_\mathrm{T} X)$ such that $\mathrm{can}_{p\#}\,\Sigma = S_{p,0}$ for all $p \in X$; furthermore, $\Sigma$ is conical with respect to the cone vertex $o$, and the spherical slice $\partial(\Sigma \llcorner B_o(1))$ defines an element $\partial_\mathrm{T} S = \partial_\mathrm{T}[S] \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X)$. This yields an isomorphism*

$$\partial_\mathrm{T} \colon \mathscr{Z}X \to \mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X).$$

*For every $p \in X$, $\mathrm{spt}(\partial_\mathrm{T} S) = \Lambda(S_{p,0}) \subset \Lambda(S)$, and if $S$ is quasi-minimizing, then $\Lambda(S_{p,0}) = \Lambda(S)$.*

We call $\partial_\mathrm{T} S = \partial_\mathrm{T}[S]$ the *Tits boundary* of $S$ or $[S]$, respectively. Due to the rank assumption, $\mathbf{I}_{m,\mathrm{c}}(\partial_\mathrm{T} X) = \{0\}$ for $m > n-1$, thus $\mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X)$ agrees with the homology group $\mathbf{H}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X)$ of integral currents, which is in turn isomorphic to the usual singular homology group $H_{n-1}(\partial_\mathrm{T} X)$ (see [74]). Hence, $\mathscr{Z}X$ is isomorphic to $H_{n-1}(\partial_\mathrm{T} X)$.

Regarding the last assertion of Theorem 1.4, we will in fact show that every quasi-minimizer $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is asymptotically conical in that $\mathrm{spt}(S)$ and $\mathrm{spt}(S_{p,0})$ lie within "sublinear" distance from each other, in terms of the distance to $p$; see (the proof of) Theorems 8.1 and 8.6. The following key result, which is part of the first of these two theorems, may be viewed as an analog of the *visibility axiom* for a Hadamard manifold $X$. This postulates that for all $p \in X$ and $\epsilon > 0$ there is an $r = r(p, \epsilon)$ such that every geodesic segment $[x, y] \subset X$ at distance at least $r$ from $p$ subtends an angle $\angle_p(x, y) \leq \epsilon$ at $p$; see Definition 4.2 and Remark 4.3 in [32] (compare pp. 294ff and 400 in [18] for a discussion in the context of CAT(0) spaces).

**Theorem 1.5** (visibility property) *Let $X$ be a proper* CAT(0) *space with* $\operatorname{asrk}(X) = n \geq 2$. *Suppose that $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is $(Q, a)$-quasi-minimizing and satisfies $\Theta_{p,r}(S) \leq C$ for some $p \in X$ and for all $r > a$. Then for every $\epsilon > 0$ there exists a constant $r_\epsilon = r_\epsilon(X, Q, C, a)$ such that for every $x \in \operatorname{spt}(S)$ with $d(p, x) \geq r_\epsilon$ there exists a unit speed ray $\varrho$ emanating from $p$ and representing a point in $\Lambda(S)$ such that $\inf_{t\geq 0} d(x, \varrho(t)) < \epsilon\, d(p, x)$.*

The next result solves an asymptotic Plateau problem (see also Theorems 8.3 and 9.5). This may be viewed as a higher rank analog of the property that any pair of distinct points in the visual boundary $\partial_\infty X$ can be joined by a geodesic line in $X$.

**Theorem 1.6** (minimizer with prescribed Tits data) *Let $X$ be a proper* CAT(0) *space with* $\operatorname{asrk}(X) = n \geq 2$. *Then for every cycle $R \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X)$ there exists an area-minimizing local cycle $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ with $\partial_\mathrm{T} S = R$. Every such $S$ satisfies $\Lambda(S) = \operatorname{spt}(R)$ and $\Theta_{p,r}(S) \leq \Theta_\infty(S) = \mathbf{M}(R)/n$ for all $p \in X$ and $r > 0$, in particular $S$ has controlled density, and $\mathbf{M}(R)/n = \Theta_\infty(S) \geq \omega_n$ whenever $R \neq 0$.*

Here $\omega_n$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The equality $\Theta_\infty(S) = \omega_n$ clearly holds if $S$ is the current associated with an oriented $n$-flat in $X$.

For ambient spaces of strictly negative curvature, minimal varieties of arbitrary dimension and codimension with prescribed limit sets were first constructed in [2,3]. We refer to [21,38,41,58] and the references therein for some generalizations and variations of these results. In Section 8.3 of [39], Gromov raised the question about the asymptotic behavior of minimal varieties in spaces of nonpositive curvature and symmetric spaces in particular. Theorem 1.6 addresses this for $n$-currents in spaces of rank $n$.

Theorems 1.4 and 1.6 show in particular that the three classes of conical, minimizing, or quasi-minimizing elements of $\mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ give rise to the same collection of limit sets, which also agrees with $\{\operatorname{spt}(R) : R \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X)\}$. We denote this canonical class of subsets of $\partial_\infty X$ by $\mathscr{L}X$. From Theorems 1.5 and 1.6 we deduce the following result (see Theorems 8.5, and [43] for a closely related discussion).

**Theorem 1.7** (dense orbit) *Let $X$ be a proper* CAT(0) *space of asymptotic rank $n \geq 2$, and suppose that $\Gamma$ is a cocompact group of isometries of $X$. Then, for every non-empty set $\Lambda \in \mathscr{L}X$, the orbit of $\Lambda$ under the action of $\Gamma$, extended to $\overline{X} = X \cup \partial_\infty X$, is dense in $\partial_\infty X$ (with respect to the cone topology).*

### 1.6 Applications to quasi-isometries

We recall that every quasi-isometric embedding $f\colon X \to \bar{X}$ between two geodesic Gromov hyperbolic spaces naturally induces a topological embedding $\partial_\infty f\colon \partial_\infty X \to \partial_\infty \bar{X}$ of their visual boundaries. In fact, $\partial_\infty f$ is a power quasi-symmetric (and hence bi-Hölder) embedding with respect to any pair of visual metrics on $\partial_\infty X$ and $\partial_\infty \bar{X}$ [15,19]. The proof is based on the Morse Lemma.

We now consider a quasi-isometric embedding $f\colon X \to \bar{X}$ between two proper CAT(0) spaces of asymptotic rank $n \geq 2$. Theorems 1.4 and 1.6 show that every $(n-1)$-cycle in $\partial_{\mathrm{T}} X$ corresponds to an $F$-asymptote class in $X$ which is represented by a minimizing local $n$-cycle with controlled density. Furthermore, for any quasi-minimizer $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ with controlled density, there exists a Lipschitz map $g\colon X \to \bar{X}$ such that $\sup_{x\in \mathrm{spt}(S)} d(f(x), g(x)) < \infty$, and this map takes $S$ to a local cycle $g_\# S \in \mathbf{Z}_{n,\mathrm{loc}}(\bar{X})$ that is again quasi-minimizing and has controlled density (see Proposition 10.3). The ambiguity in the choice of $g$ disappears on the level of $F$-asymptote classes. In fact, there is a unique monomorphism

$$\mathscr{Z} f\colon \mathscr{Z} X \to \mathscr{Z} \bar{X}$$

such that $\mathscr{Z} f\,[S] = [g_\# S]$ whenever $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ and $g$ is a Lipschitz map as above; see Theorem 10.6. Since classes in $\mathscr{Z}\bar{X}$ are, in turn, in bijective correspondence with $(n-1)$-cycles in $\partial_{\mathrm{T}}\bar{X}$, this provides a canonical map from $\mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}} X)$ into $\mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}}\bar{X})$ induced by $f$.

**Theorem 1.8** (mapping Tits cycles) *Let $X$, $\bar{X}$ be two proper* CAT(0) *spaces of asymptotic rank $n \geq 2$, and suppose that $f\colon X \to \bar{X}$ is a quasi-isometric embedding. Then there exists a unique monomorphism*

$$f_{\mathrm{T}}\colon \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}} X) \to \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}}\bar{X})$$

*such that $f_{\mathrm{T}}(\partial_{\mathrm{T}} S) = \partial_{\mathrm{T}}(g_\# S)$ whenever $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ and $g\colon X \to \bar{X}$ is a Lipschitz map with $\sup_{x\in\mathrm{spt}(S)} d(f(x), g(x)) < \infty$. If $f$ is a quasi-isometry, then $f_{\mathrm{T}}$ is an isomorphism.*

In particular, by the remark after Theorem 1.4, if $X$ and $\bar{X}$ are quasi-isometric, then $H_{n-1}(\partial_{\mathrm{T}} X)$ are $H_{n-1}(\partial_{\mathrm{T}}\bar{X})$ are isomorphic.

The next result describes the effect of a quasi-isometry on intersection patterns of limit sets. We let $\mathscr{P}(\mathscr{L}X)$ denote the set, partially ordered by inclusion, of all intersections $\bigcap_{i=1}^k \Lambda_i$ such that $1 \leq k < \infty$ and $\Lambda_i \in \mathscr{L}X$. Recall that $\mathscr{L}X = \{\mathrm{spt}(R) : R \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}} X)\}$.

**Theorem 1.9** (mapping limit sets) *Let $f\colon X \to \bar{X}$ be a quasi-isometry between two proper* CAT(0) *spaces of asymptotic rank $n \geq 2$. Then there exists an isomorphism (order preserving bijection)*

$$\mathscr{L} f\colon \mathscr{P}(\mathscr{L} X) \to \mathscr{P}(\mathscr{L} \bar{X})$$

*such that $\mathscr{L} f(\mathrm{spt}(R)) = \mathrm{spt}(f_{\mathrm{T}}(R))$ for all $R \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}} X)$. Furthermore, for every $P \in \mathscr{P}(\mathscr{L} X)$ and $\bar{P} := \mathscr{L} f(P)$ there is a pointed $L$-bi-Lipschitz homeomorphism between the cones $\mathbb{R}_+ P \subset \mathscr{C}_{\mathrm{T}} X$ and $\mathbb{R}_+ \bar{P} \subset \mathscr{C}_{\mathrm{T}} \bar{X}$, where $L$ is the multiplicative quasi-isometry constant of $f$.*

This follows from Theorem 11.2. For a higher rank symmetric space $X$ of non-compact type, the partially ordered set $\mathscr{P}(\mathscr{L} X)$ contains the simplicial building structure of $\partial_{\mathrm{T}} X$. This structure is pivotal in the proofs of both Mostow's rigidity Theorem [69] and the general non-equivariant rigidity Theorem [34,54] for such spaces. Indeed, the latter may be derived relatively quickly from Theorem 1.9 in conjunction with Tits' work [78] and the case $k = 1$ of the following result.

**Theorem 1.10** (structure of quasiflats) *Let $X$ be a proper* CAT(0) *space of asymptotic rank $n \geq 2$, and let $f\colon \mathbb{R}^n \to X$ be an $(L, a)$-quasi-isometric embedding with limit set $\Lambda := \partial_\infty(f(\mathbb{R}^n))$. Then the cone $\mathbb{R}_+\Lambda \subset \mathscr{C}_{\mathrm{T}} X$ is $L$-bi-Lipschitz homeomorphic to $\mathbb{R}^n$. Suppose that $\Lambda$ is contained in the union of the limit sets of $k$ $n$-flats in $X$ with a common point $p \in X$, and let $\mathrm{C}_p(\Lambda) \subset X$ denote the geodesic cone from $p$ over $\Lambda$. Then $f(\mathbb{R}^n)$ is within distance at most $b$ from $\mathrm{C}_p(\Lambda)$ for some constant $b$ depending only on $X, L, a, k$. In the case $k = 1$, $f(\mathbb{R}^n)$ is at Hausdorff distance at most $b$ from the flat $\mathrm{C}_p(\Lambda)$.*

We refer to Theorem 11.3 and the comments thereafter for a more general statement and some implications.

## 2 Preliminaries

### 2.1 Metric notions

Let $X = (X, d)$ be a metric space. We write

$$B_p(r) := \{x \in X : d(p, x) \leq r\}, \quad S_p(r) := \{x \in X : d(p, x) = r\}$$

for the closed ball and sphere with radius $r \geq 0$ and center $p \in X$.

A set $N \subset X$ is called *$\delta$-separated*, for a constant $\delta \geq 0$, if $d(x, y) > \delta$ for every pair of distinct points $x, y \in N$. For $A \subset X$, we call a subset $N \subset A$ a *$\delta$-net in $A$* if the family of all balls $B_x(\delta)$ with $x \in N$ covers $A$. Every maximal (with respect to inclusion) $\delta$-separated subset of $A$ is a $\delta$-net in $A$.

A map $f\colon X \to Y$ into another metric space $Y = (Y, d)$ is *$L$-Lipschitz*, for a constant $L \geq 0$, if $d(f(x), f(x')) \leq L\, d(x, x')$ for all $x, x' \in X$. The smallest such $L$ is the *Lipschitz constant* $\mathrm{Lip}(f)$ of $f$. The map $f\colon X \to Y$ is an *$L$-bi-Lipschitz embedding* if $L^{-1}d(x, x') \leq d(f(x), f(x')) \leq L\, d(x, x')$ for all $x, x' \in X$. For an $L$-Lipschitz function $f\colon A \to \mathbb{R}$ defined on a set $A \subset X$,

$$\bar{f}(x) := \sup\{f(a) - L\, d(a, x) : a \in A\} \quad (x \in X)$$

defines an $L$-Lipschitz extension $\bar{f}\colon X \to \mathbb{R}$ of $f$. Every $L$-Lipschitz map $f\colon A \to \mathbb{R}^n$, $A \subset X$, admits a $\sqrt{n}L$-Lipschitz extension $\bar{f}\colon X \to \mathbb{R}^n$.

A map $f\colon X \to Y$ between two metric spaces is called an *$(L, a)$-quasi-isometric embedding*, for constants $L \geq 1$ and $a \geq 0$, if

$$L^{-1}d(x, x') - a \leq d(f(x), f(x')) \leq L\, d(x, x') + a$$

for all $x, x' \in X$. A *quasi-isometry* $f\colon X \to Y$ has the additional property that $Y$ is within finite distance of the image of $f$. An *$(L, a)$-quasi-geodesic segment* in $X$ is the image of an $(L, a)$-quasi-isometric embedding of some compact interval. An $n$-dimensional *quasiflat* in $X$ is the image of a quasi-isometric embedding of $\mathbb{R}^n$.

### 2.2 Currents in metric spaces

Currents of finite mass in complete metric spaces were introduced by Ambrosio and Kirchheim in [1]. Here we will mainly work with the localized variant of this theory for locally compact metric spaces, as described in [59]. However, to avoid certain technicalities, we will assume throughout that the underlying metric space $X$ is proper, hence complete and separable.

For every integer $n \geq 0$, let $\mathscr{D}^n(X)$ denote the set of all $(n+1)$-tuples $(\pi_0, \ldots, \pi_n)$ of real valued functions on $X$ such that $\pi_0$ is Lipschitz with compact support $\mathrm{spt}(\pi_0)$ and $\pi_1, \ldots, \pi_n$ are locally Lipschitz. (In the case that $X = \mathbb{R}^N$ and the entries of $(\pi_0, \ldots, \pi_n)$ are smooth, this tuple should be thought of as representing the compactly supported differential $n$-form $\pi_0\, d\pi_1 \wedge \ldots \wedge d\pi_n$.) An *$n$-dimensional current* $S$ in $X$ is a function $S\colon \mathscr{D}^n(X) \to \mathbb{R}$ satisfying the following three conditions:

(1) $S$ is $(n+1)$-linear;
(2) $S(\pi_{0,k}, \ldots, \pi_{n,k}) \to S(\pi_0, \ldots, \pi_n)$ whenever $\pi_{i,k} \to \pi_i$ pointwise on $X$ with $\sup_k \mathrm{Lip}(\pi_{i,k}|_K) < \infty$ for every compact set $K \subset X$ $(i = 0, \ldots, n)$ and with $\bigcup_k \mathrm{spt}(\pi_{0,k}) \subset K$ for some such set;
(3) $S(\pi_0, \ldots, \pi_n) = 0$ whenever one of the functions $\pi_1, \ldots, \pi_n$ is constant on a neighborhood of $\mathrm{spt}(\pi_0)$.

We write $\mathscr{D}_n(X)$ for the vector space of all $n$-dimensional currents in $X$. The defining conditions already imply that every $S \in \mathscr{D}_n(X)$ is alternating in the last $n$ arguments and satisfies a product derivation rule in each of these. The definition is further motivated by the fact that every function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ induces a current $[\![w]\!] \in \mathscr{D}_n(\mathbb{R}^n)$ defined by

$$[\![w]\!](\pi_0, \ldots, \pi_n) := \int w\pi_0 \det\left[\partial_j \pi_i\right]_{i,j=1}^n dx$$

for all $(\pi_0, \ldots, \pi_n) \in \mathscr{D}^n(\mathbb{R}^n)$, where the partial derivatives $\partial_j \pi_i$ exist almost everywhere according to Rademacher's theorem. Note that this just corresponds to the integration of the differential form $\pi_0\, d\pi_1 \wedge \ldots \wedge d\pi_n$ over $\mathbb{R}^n$, weighted by $w$. For the characteristic function $\chi_W$ of a Borel set $W \subset \mathbb{R}^n$, we put $[\![W]\!] := [\![\chi_W]\!]$. (See Section 2 in [59] for details.)

### 2.3 Support, push-forward, and boundary

For every $S \in \mathscr{D}_n(X)$ there exists a smallest closed subset of $X$, the *support* $\mathrm{spt}(S)$ of $S$, such that the value $S(\pi_0, \ldots, \pi_n)$ depends only on the restrictions of $\pi_0, \ldots, \pi_n$ to this set. For a proper Lipschitz map $f\colon X \to Y$ into another proper metric space $Y$, the *push-forward* $f_\# S \in \mathscr{D}_n(Y)$ is defined simply by

$$(f_\# S)(\pi_0, \ldots, \pi_n) := S(\pi_0 \circ f, \ldots, \pi_n \circ f)$$

for all $(\pi_0, \ldots, \pi_n) \in \mathscr{D}^n(Y)$. This definition can be extended to proper Lipschitz maps $f\colon \mathrm{spt}(S) \to Y$ via appropriate extensions of the functions $\pi_i \circ f$ to $X$. In either case, $\mathrm{spt}(f_\# S) \subset f(\mathrm{spt}(S))$. For $n \geq 1$, the *boundary* $\partial S \in \mathscr{D}_{n-1}(X)$ of $S \in \mathscr{D}_n(X)$ is defined by

$$(\partial S)(\pi_0, \ldots, \pi_{n-1}) := S(\tau, \pi_0, \ldots, \pi_{n-1})$$

for all $(\pi_0, \ldots, \pi_{n-1}) \in \mathscr{D}^{n-1}(X)$ and for any compactly supported Lipschitz function $\tau$ that is identically 1 on some neighborhood of $\mathrm{spt}(\pi_0)$. If $\tilde{\tau}$ is another such function, then $\pi_0$ vanishes on a neighborhood of $\mathrm{spt}(\tau - \tilde{\tau})$ and $\partial S$ is thus well-defined by (1) and (3). Similarly one can check that $\partial \circ \partial = 0$. The inclusion $\mathrm{spt}(\partial S) \subset \mathrm{spt}(S)$ holds, and $f_\#(\partial S) = \partial(f_\# S)$ for $f\colon \mathrm{spt}(S) \to Y$ as above. (See Section 3 in [59].)

### 2.4 Mass

Let $S \in \mathscr{D}_n(X)$. A tuple $(\pi_0, \ldots, \pi_n) \in \mathscr{D}^n(X)$ will be called *normalized* if the restrictions of $\pi_1, \ldots, \pi_n$ to the compact set $\mathrm{spt}(\pi_0)$ are 1-Lipschitz.

For an open set $U \subset X$, the *mass* $\|S\|(U) \in [0,\infty]$ of $S$ in $U$ is then defined as the supremum of $\sum_j S(\pi_{0,j},\dots,\pi_{n,j})$ over all finite families of normalized tuples $(\pi_{0,j},\dots,\pi_{n,j}) \in \mathscr{D}^n(X)$ such that $\bigcup_j \operatorname{spt}(\pi_{0,j}) \subset U$ and $\sum_j |\pi_{0,j}| \le 1$. Note that $\|S\|(U) > 0$ if and only if $U \cap \operatorname{spt}(S) \neq \emptyset$. This induces a regular Borel measure $\|S\|$ on $X$, whose *total mass* $\|S\|(X)$ is denoted by $\mathbf{M}(S)$. For Borel sets $W, A \subset \mathbb{R}^n$, $\|[\![W]\!]\|(A)$ equals the Lebesgue measure of $W \cap A$. If $T \in \mathscr{D}_n(X)$ is another $n$-current in $X$, then clearly

$$\|S+T\| \le \|S\| + \|T\|.$$

We will now assume that the measure $\|S\|$ is locally finite (and hence finite on bounded sets, as $X$ is proper). Then it can be shown that

$$|S(\pi_0,\dots,\pi_n)| \le \int_X |\pi_0|\, d\|S\|$$

for every normalized tuple $(\pi_0,\dots,\pi_n) \in \mathscr{D}^n(X)$. This inequality allows to define the *restriction* $S \llcorner A \in \mathscr{D}_n(X)$ of $S$ to a Borel set $A \subset X$ by

$$(S \llcorner A)(\pi_0,\dots,\pi_n) := \lim_{k\to\infty} S(\tau_k,\pi_1,\dots,\pi_n)$$

for any sequence of compactly supported Lipschitz functions $\tau_k$ converging in $L^1(\|S\|)$ to $\chi_A \pi_0$. The measure $\|S \llcorner A\|$ equals the restriction $\|S\| \llcorner A$ of $\|S\|$. If $f\colon \operatorname{spt}(S) \to Y$ is a proper $L$-Lipschitz map into a proper metric space $Y$ and $B \subset Y$ is a Borel set, then $(f_\# S) \llcorner B = f_\#(S \llcorner f^{-1}(B))$ and

$$\|f_\# S\|(B) \le L^n \|S\|(f^{-1}(B)).$$

(See Section 4 in [59].)

### 2.5 Integral currents

A current $S \in \mathscr{D}_n(X)$ is called *locally integer rectifiable* if the measure $\|S\|$ is locally finite and concentrated on the union of countably many Lipschitz images of compact subsets of $\mathbb{R}^n$, and the following integer multiplicity condition holds: for every Borel set $A \subset X$ with compact closure and every Lipschitz map $\phi\colon X \to \mathbb{R}^n$, the current $\phi_\#(S \llcorner A) \in \mathscr{D}_n(\mathbb{R}^n)$ is of the form $[\![w]\!]$ for some *integer valued* $w = w_{A,\phi} \in L^1(\mathbb{R}^n)$. Then $\|S\|$ turns out to be absolutely continuous with respect to $n$-dimensional Hausdorff measure. Furthermore, push-forwards and restrictions to Borel sets of locally integer rectifiable currents are again locally integer rectifiable.

A current $S \in \mathscr{D}_n(X)$ is called a *locally integral current* if $S$ is locally integer rectifiable and, for $n \geq 1$, $\partial S$ satisfies the same condition. (Remarkably, this is the case already when $\|\partial S\|$ is locally finite, provided $S$ is locally integer rectifiable; see Theorem 8.7 in [59].) This yields a chain complex of abelian groups $\mathbf{I}_{n,\mathrm{loc}}(X)$. We write $\mathbf{I}_{n,\mathrm{c}}(X)$ for the respective subgroups of *integral currents* with compact support. For example, if $\Delta \subset \mathbb{R}^n$ is an $n$-simplex and $f\colon \Delta \to X$ is a Lipschitz map, then $f_{\#}[\![\Delta]\!] \in \mathbf{I}_{n,\mathrm{c}}(X)$. Thus every singular Lipschitz chain in $X$ with integer coefficients defines an element of $\mathbf{I}_{n,\mathrm{c}}(X)$. For $X = \mathbb{R}^N$, there is a canonical chain isomorphism from $\mathbf{I}_{*,\mathrm{c}}(\mathbb{R}^N)$ to the chain complex of "classical" integral currents in $\mathbb{R}^N$ originating from [37].

For $n \geq 1$, we let $\mathbf{Z}_{n,\mathrm{loc}}(X) \subset \mathbf{I}_{n,\mathrm{loc}}(X)$ and $\mathbf{Z}_{n,\mathrm{c}}(X) \subset \mathbf{I}_{n,\mathrm{c}}(X)$ denote the subgroups of currents with boundary zero. An element of $\mathbf{I}_{0,\mathrm{c}}(X)$ is an integral linear combination of currents of the form $[\![x]\!]$, where $[\![x]\!](\pi_0) = \pi_0(x)$ for all $\pi_0 \in \mathscr{D}^0(X)$. We let $\mathbf{Z}_{0,\mathrm{c}}(X) \subset \mathbf{I}_{0,\mathrm{c}}(X)$ denote the subgroup of linear combinations whose coefficients sum up to zero. The boundary of a current in $\mathbf{I}_{1,\mathrm{c}}(X)$ belongs to $\mathbf{Z}_{0,\mathrm{c}}(X)$. Given $Z \in \mathbf{Z}_{n,\mathrm{c}}(X)$, for $n \geq 0$, we will call $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ a *filling* of $Z$ if $\partial V = Z$.

### 2.6 Slicing

Let $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ be a locally integral current of dimension $n \geq 1$. Note that both $\|S\|$ and $\|\partial S\|$ are locally finite (that is, $S$ is *locally normal*, see Section 5 in [59]). Let $\varrho\colon X \to \mathbb{R}$ be a Lipschitz function, and let $B_s := \{\varrho \leq s\}$ denote the closed sublevel set for $s \in \mathbb{R}$. The corresponding *slice* of $S$ is the $(n-1)$-dimensional current

$$\langle S, \varrho, s\rangle := \partial(S \llcorner B_s) - (\partial S) \llcorner B_s$$

with support in $\{\varrho = s\} \cap \mathrm{spt}(S)$. We will use this construction exclusively in the case that $B_s \cap \mathrm{spt}(S)$ is compact for all $s$ (typically $\varrho$ will be the distance function to a point in $X$). Then, for almost every $s$, $\langle S, \varrho, s\rangle \in \mathbf{I}_{n-1,\mathrm{c}}(X)$ and hence $S \llcorner B_s \in \mathbf{I}_{n,\mathrm{c}}(X)$. Furthermore, for $a < b$, the coarea inequality

$$\int_a^b \mathbf{M}(\langle S, \varrho, s\rangle)\, ds \leq \mathrm{Lip}(\varrho)\, \|S\|(\{a < \varrho < b\})$$

holds. In particular, for every $c \in (0, b-a]$, the set of all $s \in (a, b)$ such that

$$\mathbf{M}(\langle S, \varrho, s\rangle) \leq c^{-1}\, \mathrm{Lip}(\varrho)\, \|S\|(\{a < \varrho < b\})$$

has measure $> b - a - c$. (See Section 6 and Theorem 8.5 in [59].)

### 2.7 Homotopies, cones, and isoperimetric inequality

Let $[\![0,1]\!] \in \mathbf{I}_{1,\mathrm{c}}([0,1])$ denote the current defined by

$$[\![0,1]\!](\pi_0,\pi_1) := \int_0^1 \pi_0(t)\pi_1'(t)\,dt.$$

Note that $\partial[\![0,1]\!] = [\![1]\!] - [\![0]\!]$. We endow $[0,1]\times X$ with the usual $l_2$ product metric. There exists a canonical product construction

$$S \in \mathbf{I}_{n,\mathrm{c}}(X) \rightsquigarrow [\![0,1]\!]\times S \in \mathbf{I}_{n+1,\mathrm{c}}([0,1]\times X)$$

for all $n \geq 0$. Suppose now that $Y$ is another proper metric space, $h\colon [0,1]\times X \to Y$ is a Lipschitz homotopy from $f = h(0,\cdot)$ to $g = h(1,\cdot)$, and $S \in \mathbf{I}_{n,\mathrm{c}}(X)$. Then $h_\#([\![0,1]\!]\times S)$ is an element of $\mathbf{I}_{n+1,\mathrm{c}}(Y)$ with boundary

$$\partial\, h_\#([\![0,1]\!]\times S) = g_\# S - f_\# S - h_\#([\![0,1]\!]\times \partial S)$$

(for $n=0$ the last term is zero). If $h(t,\cdot)$ is $L$-Lipschitz for every $t$, and $h(\cdot,x)$ is a geodesic of length at most $D$ for every $x \in \mathrm{spt}(S)$, then

$$\mathbf{M}(h_\#([\![0,1]\!]\times S)) \leq (n+1)L^n D\,\mathbf{M}(S).$$

(See Section 2.3 in [79].) An important special case of this is when $R \in \mathbf{Z}_{n,\mathrm{c}}(X)$ and $h(\cdot,x) = \sigma_{px}$ is a geodesic from some fixed point $p \in X$ to $x$ for every $x \in \mathrm{spt}(R)$. Then $h_\#([\![0,1]\!]\times R) \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ is the *cone from $p$ over $R$* determined by this family of geodesics, whose boundary is $R$. If the family of geodesics satisfies the convexity condition

$$d(h(t,x),h(t,x')) = d(\sigma_{px}(t),\sigma_{px'}(t)) \leq t\,d(x,x')$$

for all $x, x' \in \mathrm{spt}(R)$ and $t \in [0,1]$, and if $\mathrm{spt}(R) \subset B_p(r)$, then

$$\mathbf{M}(h_\#([\![0,1]\!]\times R)) \leq r\,\mathbf{M}(R).$$

Finally, if $X$ is a CAT(0) space, then this inequality holds with $r/(n+1)$ in place of $r$ (see Theorem 4.1 in [81]).

**Definition 2.1** *(coning inequalities)* For $n \geq 0$, we say that $X$ satisfies *condition* $(\mathrm{CI}_n)$ if for every $k \in \{0,\ldots,n\}$ there is a constant $c_k$ such that every $R \in \mathbf{Z}_{k,\mathrm{c}}(X)$ with support in some $r$-ball possesses a filling $S \in \mathbf{I}_{k+1,\mathrm{c}}(X)$ with mass

$$\mathbf{M}(S) \leq c_k r\,\mathbf{M}(R).$$

Condition ($\mathrm{CI}_0$) is satisfied if and only if $X$ is *quasi-convex*, that is, there is a constant $c_0'$ such that every pair of points $x, x'$ in $X$ can be joined by a curve of length less than or equal to $c_0' \, d(x, x')$.

Coning inequalities are instrumental for isoperimetric filling inequalities.

**Theorem 2.2** (isoperimetric inequality) *Let $n \geq 2$, and let $X$ be a proper metric space satisfying condition* ($\mathrm{CI}_{n-1}$). *Then every cycle $R \in \mathbf{Z}_{n-1,\mathrm{c}}(X)$ possesses a filling $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ with mass*

$$\mathbf{M}(T) \leq \gamma \, \mathbf{M}(R)^{n/(n-1)}$$

*for some constant $\gamma > 0$ depending only on the constants $c_1, \ldots, c_{n-1}$ from Definition 2.1.*

(Here the condition ($\mathrm{CI}_0$) is actually not needed.) This was shown in more general form for Ambrosio–Kirchheim currents in complete metric spaces in [79]; see Theorem 1.2 and the remark thereafter regarding compact supports. For earlier results of this type, see Remark 6.2 in [37] and the comments after Corollary 3.4.C in [39].

### 2.8 Convergence, compactness, and Plateau problem

A sequence $(S_i)$ in $\mathbf{I}_{n,\mathrm{loc}}(X)$ is said to converge *weakly* to a current $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ if $S_i \to S$ pointwise as functionals on $\mathscr{D}^n(X)$. Then

$$\|S\|(U) \leq \liminf_{i \to \infty} \|S_i\|(U)$$

for every open set $U \subset X$. Furthermore, weak convergence commutes with the boundary operator and with push-forwards.

A more geometric notion of convergence, with analogous properties, is given as follows. A sequence $(S_i)$ in $\mathbf{I}_{n,\mathrm{loc}}(X)$ converges in the *local flat topology* to a current $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ if for every compact set $K \subset X$ there exists a sequence $(V_i)$ in $\mathbf{I}_{n+1,\mathrm{loc}}(X)$ such that

$$(\|S - S_i - \partial V_i\| + \|V_i\|)(K) \to 0.$$

This implies that $S_i \to S$ weakly. The *flat distance* between two elements $S, S' \in \mathbf{I}_{n,\mathrm{c}}(X)$ is defined by

$$\mathscr{F}(S - S') := \inf\{\mathbf{M}(S - S' - \partial V) + \mathbf{M}(V) : V \in \mathbf{I}_{n+1,\mathrm{c}}(X)\};$$

this yields a metric on $\mathbf{I}_{n,\mathrm{c}}(X)$.

We now state the compactness theorem for locally integral currents and minimizing locally integral currents. An element $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ is *(area-)minimizing* if

$$\mathbf{M}(S \llcorner B) \leq \mathbf{M}(T)$$

whenever $B \subset X$ is a Borel set such that $S \llcorner B \in \mathbf{I}_{n,\mathrm{c}}(X)$ and $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ satisfies $\partial T = \partial(S \llcorner B)$.

**Theorem 2.3** (compactness) *Let $X$ be a proper metric space, and let $n \geq 1$. Suppose that $(S_i)$ is a sequence in $\mathbf{I}_{n,\mathrm{loc}}(X)$ such that*

$$\sup_i (\|S_i\| + \|\partial S_i\|)(K) < \infty$$

*for every compact set $K \subset X$.*

*(1) There is a subsequence $(S_{i_j})$ that converges weakly to a current $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$.*
*(2) Suppose, in addition, that $X$ satisfies condition $(\mathrm{CI}_n)$. Then there is a subsequence $(S_{i_j})$ that converges in the local flat topology to a current $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$. If each $S_i$ is area-minimizing, then so is $S$.*

For (2), a uniformly local version of condition $(\mathrm{CI}_n)$ suffices; compare the assumptions in [80].

*Proof* For (1), see Theorem 8.10 in [59].

For the proof of (2), pick a base point $p \in X$. By passing to a further subsequence, denoted again by $(S_{i_j})$, one can arrange that there exists a sequence of radii $0 < r_k \uparrow \infty$ such that for every $B_k := B_p(r_k)$, the restrictions $S_{i_j} \llcorner B_k$ and $S \llcorner B_k$ are in $\mathbf{I}_{n,\mathrm{c}}(X)$,

$$\sup_j \big(\mathbf{M}(S_{i_j} \llcorner B_k) + \mathbf{M}(\partial(S_{i_j} \llcorner B_k))\big) < \infty,$$

and $S_{i_j} \llcorner B_k \to S \llcorner B_k$ weakly, as $j \to \infty$ (see the proof of Proposition 6.6 in [59]). Now, to show that $S_{i_j} \to S$ in the local flat topology, fix an index $k$. Since $X$ satisfies condition $(\mathrm{CI}_n)$, it follows from Theorem 1.4 in [80] that $\mathscr{F}((S - S_{i_j}) \llcorner B_k) \to 0$. Hence, there exists a sequence $(V_j)$ in $\mathbf{I}_{n+1,\mathrm{c}}(X)$ such that, for $T_j := (S - S_{i_j}) \llcorner B_k - \partial V_j$,

$$\mathbf{M}(T_j) + \mathbf{M}(V_j) \to 0.$$

Since $\|S - S_{i_j} - \partial V_j\|(B_k) \leq \|T_j\|(B_k) + \|(S - S_{i_j}) \llcorner (X \setminus B_k)\|(B_k) \leq \mathbf{M}(T_j)$ and $\|V_j\|(B_k) \leq \mathbf{M}(V_j)$, this gives the result.

Suppose now that each $S_i$ is minimizing. To prove that $S$ is minimizing, it suffices to show that for every fixed $k$,

$$\mathbf{M}(S \llcorner B_k) \le \mathbf{M}(T)$$

for all $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ with $\partial T = \partial(S \llcorner B_k)$. Let $V_j$ and $T_j$ be given as above, and note that then $\partial(T - T_j) = \partial(S_{i_j} \llcorner B_k)$. By the minimality of $S_{i_j}$,

$$\mathbf{M}(S_{i_j} \llcorner B_k) \le \mathbf{M}(T - T_j) \le \mathbf{M}(T) + \mathbf{M}(T_j).$$

Since $S_{i_j} \llcorner B_k \to S \llcorner B_k$ weakly and $\mathbf{M}(T_j) \to 0$, it follows that

$$\mathbf{M}(S \llcorner B_k) \le \liminf_{j\to\infty} \mathbf{M}(S_{i_j} \llcorner B_k) \le \mathbf{M}(T),$$

as desired. □

From Theorem 2.2 and the first part of Theorem 2.3 one obtains a solution of the Plateau problem in spaces with coning inequalities (compare also Theorem 10.6 in [1] and Theorem 1.6 in [79]).

**Theorem 2.4** (minimizing filling) *Let $n \ge 1$, and let $X$ be a proper metric space satisfying condition* ($\mathrm{CI}_{n-1}$). *Then for every $R \in \mathbf{Z}_{n-1,\mathrm{c}}(X)$ there exists a filling $S \in \mathbf{I}_{n,\mathrm{c}}(X)$ of $R$ with mass*

$$\mathbf{M}(S) = \inf\{\mathbf{M}(S') : S' \in \mathbf{I}_{n,\mathrm{loc}}(X),\ \partial S' = R\}.$$

*Furthermore,* $\mathrm{spt}(S)$ *is within distance at most* $(\mathbf{M}(S)/\delta)^{1/n}$ *from* $\mathrm{spt}(R)$ *for some constant $\delta > 0$ depending only on $n$ and the constants $c_1, \ldots, c_{n-1}$ from Definition 2.1.*

*Proof* Let $\mathscr{S}$ denote the set of all $S' \in \mathbf{I}_{n,\mathrm{loc}}(X)$ with $\partial S' = R$. By condition ($\mathrm{CI}_{n-1}$), $\mathscr{S}$ is non-empty. Choose a sequence $(S_i)$ in $\mathscr{S}$ such that

$$\mathbf{M}(S_i) \to \mu := \inf\{\mathbf{M}(S') : S' \in \mathscr{S}\} \quad \text{for } i \to \infty.$$

By Theorem 2.3, some subsequence $(S_{i_j})$ converges weakly to a current $S \in \mathscr{S}$, and $\mathbf{M}(S) \le \liminf_{j\to\infty} \mathbf{M}(S_{i_j})$, thus $\mathbf{M}(S) = \mu$. It is well-known that an isoperimetric inequality of Euclidean type as in Theorem 2.2 leads to a lower density bound for minimizing $n$-currents: if $x \in \mathrm{spt}(S)$ and $r > 0$ are such that $B_x(r) \cap \mathrm{spt}(\partial S) = \emptyset$, then $\|S\|(B_x(r)) \ge \delta r^n$, where $\delta := n^{-n}\gamma^{1-n}$ for $n \ge 2$ and $\delta := 2$ for $n = 1$ (see Theorem 9.13 in [37] and the special case $(Q, a) = (1, 0)$ of Lemma 3.3 below). This gives the desired distance bound and shows in particular that $\mathrm{spt}(S)$ is compact. □

## 3 Quasi-minimizers

We now introduce the main objects of study and discuss some basic properties and examples.

**Definition 3.1** *(quasi-minimizer)* Suppose that $X$ is a proper metric space, $n \geq 1$, and $Q \geq 1$, $a \geq 0$ are constants. For a closed set $Y \subset X$, a local cycle

$$S \in \mathbf{Z}_{n,\mathrm{loc}}(X, Y) := \{S \in \mathbf{I}_{n,\mathrm{loc}}(X) : \mathrm{spt}(\partial S) \subset Y\}$$

relative to $Y$ will be called $(Q, a)$-*quasi-minimizing mod* $Y$ if, for all $x \in \mathrm{spt}(S)$ and almost all $r > a$ such that $B_x(r) \cap Y = \emptyset$, the inequality

$$\mathbf{M}(S \llcorner B_x(r)) \leq Q\,\mathbf{M}(T)$$

holds whenever $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ and $\partial T = \partial(S \llcorner B_x(r))$ (recall that $S \llcorner B_x(r) \in \mathbf{I}_{n,\mathrm{c}}(X)$ for almost all $r > 0$, see Sect. 2.6). A current $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ is $(Q, a)$-*quasi-minimizing* or a $(Q, a)$-*quasi-minimizer* if $S$ is $(Q, a)$-quasi-minimizing mod $\mathrm{spt}(\partial S)$, and we say that $S$ is *quasi-minimizing* or a *quasi-minimizer* if this holds for some $Q \geq 1$ and $a \geq 0$.

Obviously every minimizing $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ is $(1, 0)$-quasi-minimizing.

**Definition 3.2** *(density/filling density)* Suppose that $X$ is a proper metric space, $n \geq 1$, and $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$. For $p \in X$ and $r > 0$, put

$$\Theta_{p,r}(S) := \frac{1}{r^n}\|S\|(B_p(r)),$$
$$F_{p,r}(S) := \frac{1}{r^{n+1}} \inf\{\mathbf{M}(V) : V \in \mathbf{I}_{n+1,\mathrm{c}}(X),\ \mathrm{spt}(S - \partial V) \cap B_p(r) = \emptyset\}$$

(where $\inf \emptyset := \infty$). Furthermore, for any $p \in X$, put

$$\Theta_\infty(S) := \limsup_{r\to\infty} \Theta_{p,r}(S),$$
$$F_\infty(S) := \limsup_{r\to\infty} F_{p,r}(S);$$

the upper limits are clearly independent of the choice of $p \in X$. For constants $C > 0$ and $a \geq 0$, we say that $S$ has $(C, a)$-*controlled density* if $\Theta_{p,r}(S) \leq C$ for all $p \in X$ and $r > a$, and $S$ has *controlled density* if this holds for some such constants.

Note that if $\mathrm{spt}(\partial S) \cap B_p(r) \neq \emptyset$, then there is no $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ with $\mathrm{spt}(S - \partial V) \cap B_p(r) = \emptyset$, thus $F_{p,r}(S) = \infty$. Note also that if $S, S' \in$

$\mathbf{I}_{n,\mathrm{loc}}(X)$, then

$$\Theta_{p,r}(S+S') \le \Theta_{p,r}(S) + \Theta_{p,r}(S')$$

for all $p \in X$ and $r > 0$, hence $\Theta_\infty(S+S') \le \Theta_\infty(S) + \Theta_\infty(S')$. Likewise, $F_{p,r}$ and $F_\infty$ satisfy the triangle inequality.

If $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ has $(C,a)$-controlled density, then obviously $\Theta_\infty(S) \le C$. However, an $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ with $\Theta_\infty(S) < \infty$ need not have controlled density. For example, it is not difficult to see that there exists a complete Riemannian metric on $\mathbb{R}^2$ with bounded curvature $|K| \le 1$ and with arbitrarily large disks of constant curvature $-1$ such that the associated current $S = [\![\mathbb{R}^2]\!] \in \mathbf{Z}_{2,\mathrm{loc}}(\mathbb{R}^2)$ is of this type.

**Lemma 3.3** (density) *Let $n \ge 1$, let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_{n-1})$*, and let $Y \subset X$ be a closed set. If $S \in \mathbf{Z}_{n,\mathrm{loc}}(X,Y)$ is $(Q,a)$-quasi-minimizing mod $Y$, and if $x \in \mathrm{spt}(S)$ and $r > 2a$ are such that $B_x(r) \cap Y = \emptyset$, then*

$$\Theta_{x,r}(S) \ge \delta$$

*for some constant $\delta > 0$ depending only on $n$, the constants $c_1, \ldots, c_{n-1}$ from Definition 2.1, and $Q$.*

*Proof* Let first $n \ge 2$. Define $\mu\colon (0,r] \to \mathbb{R}$ by $\mu(s) := \|S\|(B_x(s))$. Note that $\mu$ is non-decreasing, and $\mu > 0$ since $x \in \mathrm{spt}(S)$. For almost every $s \in (0,r)$, the derivative $\mu'(s)$ exists, and the slice $R_s := \partial(S \llcorner B_x(s))$ is in $\mathbf{Z}_{n-1,\mathrm{c}}(X)$ and satisfies $\mathbf{M}(R_s) \le \mu'(s)$. It follows from the quasi-minimality of $S$ and Theorem 2.2 (isoperimetric inequality) that for almost every $s \in (a,r)$, there is a filling $T_s \in \mathbf{I}_{n,\mathrm{c}}(X)$ of $R_s$ such that

$$\begin{aligned}\mu(s) = \mathbf{M}(S \llcorner B_x(s)) &\le Q\,\mathbf{M}(T_s) \le Q\gamma\,\mathbf{M}(R_s)^{n/(n-1)} \\ &\le Q\gamma\,\mu'(s)^{n/(n-1)}\end{aligned}$$

and hence $\mu'(s)\mu(s)^{(1-n)/n} \ge (Q\gamma)^{(1-n)/n}$. Now integration from $a$ to $r$ yields $\mu(r) \ge n^{-n}(Q\gamma)^{1-n}(r-a)^n$. Since $r - a > r/2$, this gives the result.

In the case $n = 1$, since $S$ is $(Q,a)$-quasi-minimizing mod $Y$ and $x \in \mathrm{spt}(S)$, the 0-dimensional slice $R_s = \partial(S \llcorner B_x(s))$ is a non-zero integral boundary for almost every $s \in (a,r)$, so in fact $\mathbf{M}(R_s) \ge 2$, and the coarea inequality gives $\|S\|(B_x(r)) \ge 2(r-a) > r$. □

We show two direct consequences of this lemma.

**Lemma 3.4** (filling density) *Let $n \ge 1$, let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_{n-1})$*, and let $Y \subset X$ be a closed set. If $S \in \mathbf{Z}_{n,\mathrm{loc}}(X,Y)$*

*is $(Q, a)$-quasi-minimizing mod $Y$, and if $x \in \operatorname{spt}(S)$ and $r > 4a$ are such that $B_x(r) \cap Y = \emptyset$, then*

$$F_{x,r}(S) \geq c$$

*for some constant $c > 0$ depending only on $n$, the constant $\delta$ from Lemma 3.3, and $Q$.*

*Proof* Let $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ be such that $\operatorname{spt}(S - \partial V) \cap B_x(r) = \emptyset$. For almost every $s \in (0, r)$, the slice $T_s := \partial(V \llcorner B_x(s)) - (\partial V) \llcorner B_x(s)$ is in $\mathbf{I}_{n,\mathrm{c}}(X)$, and $\partial T_s = -\partial(S \llcorner B_x(s))$ because $(\partial V) \llcorner B_x(s) = S \llcorner B_x(s)$. By the quasi-minimality of $S$ and Lemma 3.3, for almost every $s \in (2a, r)$,

$$Q\,\mathbf{M}(T_s) \geq \mathbf{M}(S \llcorner B_x(s)) = \|S\|(B_x(s)) \geq \delta s^n.$$

Since $\mathbf{M}(V) \geq \int_{2a}^{r} \mathbf{M}(T_s)\,ds$ and $2a < r/2$, the result follows. □

Recall that a subset $A$ of a metric space $X$ is *doubling* if there is a constant $M \geq 1$ such that every bounded subset $B \subset A$ can be covered by at most $M$ sets of diameter less than or equal to $\operatorname{diam}(B)/2$. The *Assouad dimension* of a set $A \subset X$ is the infimum of all $\alpha > 0$ for which there exists $L \geq 1$ such that for all $\lambda \in (0, 1)$, every bounded set $B \subset A$ can be covered by no more than $L\lambda^{-\alpha}$ sets of diameter $\leq \lambda \operatorname{diam}(B)$. The set $A$ has finite Assouad dimension if and only if it is doubling. (See [46].)

**Lemma 3.5** (doubling) *Let $n \geq 1$, and let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_{n-1})$. *Suppose that $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is a $(Q, a)$-quasi-minimizer with $(C, a)$-controlled density. Then every $s$-separated subset of* $\operatorname{spt}(S)$ *with $s \geq 4a$ has Assouad dimension at most $n$ and is thus doubling. The doubling constant depends only on $n$, the constant $\delta$ from Lemma 3.3, and $C$.*

Note that if $a = 0$, then $\operatorname{spt}(S)$ itself has Assouad dimension at most $n$.

*Proof* Let $A \subset \operatorname{spt}(S)$ be an $s$-separated set, where $s \geq 4a$. Suppose that $B \subset A$ is a bounded set with $D := \operatorname{diam}(B) > 0$. Let $\lambda \in (0, 1)$, and let $N \subset B$ be a $(\lambda D/2)$-separated $(\lambda D/2)$-net in $B$. Put $r := \max\{s/2, \lambda D/4\}$. The balls $B_x(r)$ in $X$ with $x \in N$ are pairwise disjoint, and the corresponding sets $B_x(2r) \cap B$ have diameter at most $\lambda D$ and cover $B$. Since $r \geq 2a$, Lemma 3.3 shows that $\|S\|(B_x(r)) \geq \delta r^n$ for all these balls, and their union $U$ is contained in $B_p(D + r)$ for any $p \in N$. Note that $D \leq 4r/\lambda$, thus $\|S\|(U) \leq C(5r/\lambda)^n$. It follows that the covering has cardinality $|N| \leq 5^n C\delta^{-1}\lambda^{-n}$. □

The doubling property will be used in Sect. 10 in order to approximate quasi-isometric embeddings by Lipschitz maps. We will then show that if $S$ is a quasi-minimizing local $n$-cycle with controlled density in a proper CAT(0)

space (or a space with a convex bicombing) of asymptotic rank $n \geq 2$, and if $g$ is a Lipschitz quasi-isometric embedding of $\mathrm{spt}(S)$ into another proper metric space, then $g_\# S$ is again quasi-minimizing and has controlled density (see Proposition 10.3).

Here we first prove a simpler result for Lipschitz quasiflats, which also allows for boundaries.

**Proposition 3.6** (Lipschitz quasiflats) *For all $n$, $L \geq 1$ and $a_0 \geq 0$ there exist $Q \geq 1$, $C > 0$, and $a \geq 0$ such that the following holds. Let $W \subset \mathbb{R}^n$ be any closed set such that the associated current $E := [\![W]\!]$ is in $\mathbf{Z}_{n,\mathrm{loc}}(\mathbb{R}^n, \partial W)$. Suppose that $g\colon W \to X$ is a map into a proper metric space $X$ such that for all $x, y \in W$,*

$$L^{-1} d(x, y) - a_0 \leq d(g(x), g(y)) \leq L\, d(x, y).$$

*Then $S := g_\# E \in \mathbf{Z}_{n,\mathrm{loc}}(X, g(\partial W))$ is $(Q, a)$-quasi-minimizing mod $g(\partial W)$ and has $(C, a)$-controlled density, furthermore $d(g(x), \mathrm{spt}(S)) \leq a$ for all $x \in W$ with $d(x, \partial W) \geq a$.*

The condition on $W$ is satisfied if and only if $W$ has locally finite perimeter (that is, $\chi_W$ has locally bounded variation; see Theorem 7.2 in [59]). We will use this result only for $W$ equal to $\mathbb{R}^n$ or an $n$-simplex in $\mathbb{R}^n$.

*Proof* If $p \in X$ and $r > a_0$, and if $B := B_p(r)$ and $x, y \in g^{-1}(B)$, then $d(x, y) \leq L(d(g(x), g(y)) + a_0) \leq 3Lr$, thus

$$\|S\|(B) = \|g_\# E\|(B) \leq L^n \, \|E\|(g^{-1}(B)) \leq C r^n$$

for some constant $C$ depending only on $n$ and $L$. Hence $S$ has $(C, a_0)$-controlled density.

Let $N \subset W$ be a $2La_0$-separated $2La_0$-net in $W$. If $x, y \in N$ are distinct, then $d(g(x), g(y)) \geq L^{-1} d(x, y) - a_0 \geq (2L)^{-1} d(x, y)$, thus $g|_N$ is injective, and $(g|_N)^{-1}\colon g(N) \to N$ admits an $\bar{L}$-Lipschitz extension $\bar{g}\colon X \to \mathbb{R}^n$, where $\bar{L} := 2\sqrt{n}L$. Put $h := \bar{g} \circ g$. For every $x \in W$ there is a $y \in N$ such that $d(x, y) \leq 2La_0$. Then $h(y) = y$, thus

$$d(h(x), x) \leq d(h(x), h(y)) + d(y, x) \leq (\bar{L}L + 1)\, d(x, y) \leq b$$

for $b := 2(\bar{L}L + 1)La_0$.

Suppose now that $x \in W$ and $r > 2Lb$ are such that $B_{\bar{x}}(r) \cap g(\partial W) = \emptyset$, where $\bar{x} := g(x)$. For almost every such $r$, both $S' := S \llcorner B_{\bar{x}}(r)$ and $E' := E \llcorner g^{-1}(B_{\bar{x}}(r))$ are integral currents, and $g_\# E' = S'$. Since $g^{-1}(B_{\bar{x}}(r)) \cap \mathrm{spt}(\partial E) = \emptyset$, the support of $\partial E'$ is in $g^{-1}(S_{\bar{x}}(r))$ and thus at distance at least $r/L$ from $x$. Note that $\partial(\bar{g}_\# S') = h_\#(\partial E')$. Using the geodesic homotopy from

$\mathrm{id}_W$ to $h$, we get a current $R \in \mathbf{I}_{n,\mathrm{c}}(\mathbb{R}^n)$ with boundary $\partial R = \partial(\bar{g}_\# S') - \partial E'$ and support within distance $b$ from $\mathrm{spt}(\partial E')$; in fact $R = \bar{g}_\# S' - E'$, because $\mathbf{Z}_{n,\mathrm{c}}(\mathbb{R}^n) = \{0\}$. Since $r/L - b > r/(2L)$, the support of $R$ lies outside $B_x(r/(2L))$. It follows that

$$\mathbf{M}(\bar{g}_\# S') = \mathbf{M}(E' + R) \geq \|E\|(B_x(r/(2L))) \geq \epsilon r^n$$

for some constant $\epsilon > 0$ depending only on $n$ and $L$.

Now if $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ is such that $\partial T = \partial S'$, then $\bar{g}_\# T = \bar{g}_\# S'$, and

$$\mathbf{M}(S') \leq C r^n \leq C\epsilon^{-1}\mathbf{M}(\bar{g}_\# T) \leq Q\,\mathbf{M}(T)$$

for $Q := C\epsilon^{-1}\bar{L}^n$. Since $\mathrm{spt}(S) \subset g(W)$, this shows that $S$ is $(Q, 2Lb)$-quasi-minimizing mod $g(\partial W)$.

To prove the last assertion, choose any $a > L(2Lb + a_0)$ and let $x \in W$ be a point with $d(x, \partial W) \geq a$. Then $d(g(x), g(\partial W)) \geq L^{-1}a - a_0 > 2Lb$. For a suitable $r \in (2Lb, a]$, the above argument then shows that $\mathbf{M}(\bar{g}_\# S') > 0$, thus $S' = S \llcorner B_{g(x)}(r) \neq 0$, and this implies that $d(g(x), \mathrm{spt}(S)) \leq r \leq a$. □

The following variant of the above result applies to situations where quasiflats can possibly not be approximated by Lipschitz ones (see the proof of Theorem 5.2). Here we call a compact set $W \subset \mathbb{R}^n$ *triangulated* if $W$ has the structure of a finite simplicial complex all of whose maximal cells are Euclidean $n$-simplices (thus $W$ is polyhedral). We denote by $W^0$ and $(\partial W)^0$ the set of vertices and boundary vertices of the triangulation, respectively.

**Proposition 3.7** (triangulated quasiflats) *Let $n \geq 1$, and let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_{n-1})$. *Then for all $C_0$, $D_0 > 0$ and $L$, $a_0$ there exist $Q$, $C$, $a$ such that the following holds. Suppose that $W \subset \mathbb{R}^n$ is a compact triangulated set with simplices of diameter at most $D_0$, and such that every $r$-ball in $\mathbb{R}^n$ with $r \geq D_0$ meets at most $C_0 r^n$ $n$-simplices. Let $\mathscr{P}_*(W)$ denote the corresponding chain complex of simplicial integral currents. If $f\colon W \to X$ is an $(L, a_0)$-quasi-isometric embedding, then there exists a chain map $\iota\colon \mathscr{P}_*(W) \to \mathbf{I}_{*,\mathrm{c}}(X)$ such that*

*(1) $\iota$ maps every vertex $[\![x_0]\!] \in \mathscr{P}_0(W)$ to $[\![f(x_0)]\!]$ and, for $1 \leq k \leq n$, every basic oriented simplex $[\![x_0, \ldots, x_k]\!] \in \mathscr{P}_k(W)$ to a current with support in $N_a(f(\{x_0, \ldots, x_k\}))$;*
*(2) $S := \iota[\![W]\!] \in \mathbf{I}_{n,\mathrm{c}}(X)$ is $(Q, a)$-quasi-minimizing mod $N_a(f((\partial W)^0))$ and has $(C, a)$-controlled density;*
*(3) $d(f(x), \mathrm{spt}(S)) \leq a$ for all $x \in W$ with $d(x, \partial W) \geq a$.*

Here $N_a(\cdot)$ stands for the closed $a$-neighborhood of a set. Note that by (1), $\mathrm{spt}(S) \subset N_a(f(W^0))$ and $\mathrm{spt}(\partial S) \subset N_a(f((\partial W)^0))$. An analogous result for

closed sets with locally finite triangulations and locally integral currents also holds.

*Proof* Put $\mathscr{S}_* := \bigcup_{k=0}^n \mathscr{S}_k$, where $\mathscr{S}_k$ denotes the set of all basic simplices $s = [\![x_0, \dots, x_k]\!] \in \mathscr{P}_k(W)$ (compare p. 365 in [36] for the notation). Using Theorem 2.4 (minimizing filling), we can inductively build a map $\iota\colon \mathscr{S}_* \to \mathbf{I}_{*,\mathrm{c}}(X)$ as follows. For every $[\![x_0]\!] \in \mathscr{S}_0$, $\iota[\![x_0]\!] := f_\#[\![x_0]\!] = [\![f(x_0)]\!]$. Now let $k \geq 1$, and suppose that $\iota$ is defined on $\mathscr{S}_{k-1}$. For every $k$-cell of $W$, we pick an orientation $s = [\![x_0, \dots, x_k]\!] \in \mathscr{S}_k$, then we let $\iota(s) \in \mathbf{I}_{k,\mathrm{c}}(X)$ be a minimizing filling of

$$\sum_{i=0}^{k} (-1)^i \, \iota[\![x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k]\!] \in \mathbf{Z}_{k-1,\mathrm{c}}(X),$$

and we put $\iota(-s) := -\iota(s)$. The resulting map on $\mathscr{S}_*$ readily extends to a chain map $\iota\colon \mathscr{P}_*(W) \to \mathbf{I}_{*,\mathrm{c}}(X)$. It follows inductively from condition $(\mathrm{CI}_{k-1})$ and the distance bound in Theorem 2.4 for $k = 1, \dots, n$ that for all $s = [\![x_0, \dots, x_k]\!] \in \mathscr{S}_k$,

$$\mathbf{M}(\iota(s)) \leq M$$

and $\operatorname{spt}(\iota(s)) \subset N_{a'}(f(\{x_0, \dots, x_k\}))$ for some constants $M, a'$ depending on $D_0, L, a_0$ and the constants $c_0, \dots, c_{n-1}$ implicit in condition $(\mathrm{CI}_{n-1})$. In the following we assume that $a' \geq LD_0 + a_0$.

Let now $\mathscr{S}_n^+ \subset \mathscr{S}_n$ be the set of all positively oriented $n$-simplices, whose sum is $[\![W]\!]$. Put $S := \iota[\![W]\!]$. To show that $S$ has controlled density, let $p \in X$ and $r > a'$, and consider the set of all $s \in \mathscr{S}_n^+$ for which $\operatorname{spt}(\iota(s)) \cap B_p(r) \neq \emptyset$. Every such $s$ has a vertex $x^s$ with $f(x^s) \in B_p(r + a')$, thus the set of all $x^s$ has diameter at most $L(2(r + a') + a_0) \leq 5Lr$. It follows that there are at most $C_0(5Lr)^n$ such simplices and that $\Theta_{p,r}(S) \leq C := C_0(5L)^n M$ for $p \in X$ and $r > a'$.

Similarly as in the proof of Proposition 3.6, there exists an $\bar{L}$-Lipschitz map $\bar{f}\colon X \to \mathbb{R}^n$ such that $h := \bar{f} \circ f$ satisfies $d(h(x), x) \leq b'$ for all $x \in W$, where $\bar{L} := 2\sqrt{n}L$ and $b'$ depends on $n, L, a_0$. Then

$$\bar{\iota} := \bar{f}_\# \circ \iota\colon \mathscr{P}_*(W) \to \mathbf{I}_{*,\mathrm{c}}(\mathbb{R}^n)$$

is a chain map that sends every $[\![x_0]\!] \in \mathscr{S}_0$ to $[\![h(x_0)]\!]$ and every $[\![x_0, \dots, x_k]\!] \in \mathscr{S}_k$ to a current with support in $N_{\bar{L}a'+b'}(\{x_0, \dots, x_k\})$. Note that $\bar{\iota}(\partial[\![W]\!]) = \partial(\bar{f}_\# S)$. Similarly as above, using geodesic cone fillings of cycles in $\mathbb{R}^n$, we can inductively construct a chain homotopy between $\mathrm{id}_\#, \bar{\iota}\colon \mathscr{P}_*(\partial W) \to \mathbf{I}_{*,\mathrm{c}}(\mathbb{R}^n)$. This yields an $R \in \mathbf{I}_{n,\mathrm{c}}(\mathbb{R}^n)$ such that $\partial R = \partial(\bar{f}_\# S) - \partial[\![W]\!]$ and

$\mathrm{spt}(R) \subset N_b((\partial W)^0)$ for some constant $b \geq D_0$ depending on $n$, $D_0$, $L$, $a'$. Since $\mathbf{Z}_{n,c}(\mathbb{R}^n) = \{0\}$, in fact $R = \bar{f}_\# S - [\![W]\!]$.

Note that $\mathrm{spt}(S) \cup f(W) \subset N_{a'}(f(W^0))$. Let now $\bar{x} \in N_{a'}(f(W^0))$ and $r > 0$ be such that $B_{\bar{x}}(r) \cap f((\partial W)^0) = \emptyset$ and $S' := S \llcorner B_{\bar{x}}(r) \in \mathbf{I}_{n,c}(X)$. Pick an $x \in W^0$ with $d(f(x), \bar{x}) \leq a'$. For all $y \in (\partial W)^0$,

$$r < d(\bar{x}, f(y)) \leq d(f(x), f(y)) + a' \leq L\, d(x, y) + 2a'.$$

Assuming that $r > 2Lb + 4a'$, we get that $d(x, y) > r/(2L) + b$; hence

$$(\mathrm{spt}(R) \cup \partial W) \cap B_x(r/(2L)) = \emptyset.$$

Moreover, for every $\bar{y} \in \mathrm{spt}(S - S') \subset \mathrm{spt}(S)$ there exists a $y \in W^0$ such that $d(f(y), \bar{y}) \leq a'$ and $r \leq d(\bar{x}, \bar{y}) \leq d(f(x), f(y)) + 2a' \leq L\, d(x, y) + 3a'$, thus

$$\begin{aligned} d(x, \bar{f}(\bar{y})) &\geq d(x, y) - d(y, h(y)) - d(\bar{f}(f(y)), \bar{f}(\bar{y})) \\ &\geq L^{-1}(r - 3a') - b' - \bar{L}a'. \end{aligned}$$

By increasing $b$ if necessary, so that $r$ is large enough, we arrange that this last expression is bigger than $r/(2L)$. This then shows that

$$\mathrm{spt}(\bar{f}_\#(S - S')) \cap B_x(r/(2L)) = \emptyset.$$

Since $\bar{f}_\# S' = [\![W]\!] + R - \bar{f}_\#(S - S')$, it follows that

$$\mathbf{M}(\bar{f}_\# S') \geq \|[\![W]\!]\|(B_x(r/(2L))) \geq \epsilon r^n$$

for some $\epsilon > 0$ depending on $n$ and $L$. The proof may now be completed as for Proposition 3.6. For assertion (3), choose $a > L(2Lb + 4a' + a_0)$. □

## 4 Asymptotic rank

In this section we will first discuss the notion of asymptotic rank and the sub-Euclidean isoperimetric inequality from [82]. Then we will derive a localized version of this result as well as various characterizations of quasi-minimizing local $n$-cycles in spaces of asymptotic rank at most $n$.

In [42], Section $6.B_2$, Gromov defined a number of different large-scale notions of rank for spaces of nonpositive curvature. Many of the ensuing questions were then answered in [53] (see the discussion in Section 9 therein). Theorem D in that paper shows in particular the following.

**Theorem 4.1** (rank conditions) *Let $X$ be a proper Busemann space with cocompact isometry group. Then for every $n \geq 1$ the following are equivalent:*

*(1) $X$ contains an isometric copy of some $n$-dimensional normed space;*
*(2) there exists a quasi-isometric embedding of $\mathbb{R}^n$ into $X$;*
*(3) there exist a sequence of subsets $Y_i \subset X$ and a sequence $0 < r_i \to \infty$ such that the rescaled sets $(Y_i, r_i^{-1}d)$ converge in the Gromov–Hausdorff topology to the closed unit ball in some $n$-dimensional normed space.*

Stronger conclusions hold if $X$ is a proper and cocompact CAT(0) space. Then any normed space isometrically embedded in $X$ is necessarily Euclidean; furthermore, the *Euclidean rank* of $X$, the maximal $n$ for which $X$ contains an $n$-flat, is equal to the *geometric dimension* or the *compact topological dimension* (that is, the supremum of the topological dimensions of compact subsets) of the Tits cone $\mathscr{C}_{\mathrm{T}}X$ or of any asymptotic cone $X_\omega$ and also agrees with the maximal $n$ for which $H_{n-1}(\partial_{\mathrm{T}}X) \neq \{0\}$, where $\partial_{\mathrm{T}}X$ denotes the Tits boundary. See Theorems A and C in [53].

Property (3) above suggests the following notion of asymptotic rank that was investigated in [82].

**Definition 4.2** *(asymptotic subset, asymptotic rank)* Let $X = (X, d)$ be a metric space. Any compact metric space $(Y, d_Y)$ that can be obtained as the Gromov–Hausdorff limit of a sequence $(Y_i, r_i^{-1}d)$ as in (3) above will be called an *asymptotic subset* of $X$. The *asymptotic rank* $\operatorname{asrk}(X)$ of $X$ is the supremum of all integers $n \geq 0$ such that there exists an asymptotic subset of $X$ isometric to the unit ball in some $n$-dimensional normed space.

*Remark 4.3* Alternatively, $\operatorname{asrk}(X)$ may be defined as the supremum of all $n$ such that $X$ admits an asymptotic subset bi-Lipschitz homeomorphic to a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure. The equivalence of the two definitions is shown by means of a metric differentiation argument (see [82]).

We remark that every asymptotic subset of $X$ embeds isometrically into some asymptotic cone of $X$. Conversely, every compact subset $Y \subset X_\omega$ of an asymptotic cone is an asymptotic subset of $X$, and the respective sets $Y_i \subset X$ may be chosen to be finite. If $f\colon X \to \bar{X}$ is a quasi-isometric embedding into another metric space $\bar{X}$, then

$$\operatorname{asrk}(X) \leq \operatorname{asrk}(\bar{X});$$

thus asrk is a quasi-isometry invariant for metric spaces (see Corollary 3.3 in [82]).

In a nonpositively curved symmetric space $X$, every $n$-cycle $Z$ with $n$ greater than or equal to the rank of $X$ admits a filling $V$ with mass

$$\mathbf{M}(V) \leq \text{const} \cdot \mathbf{M}(Z)$$

(see p. 105 in [42], and [65]), whereas in smaller dimensions, the optimal isoperimetric inequalities in $X$ are of Euclidean type, as in Theorem 2.2. It is not known whether the linear inequalities for $n$-cycles remain valid, for example, in cocompact Hadamard manifolds containing no $(n+1)$-flat. However, the following key result due to Stefan Wenger provides a substitute for spaces of asymptotic rank at most $n$.

**Theorem 4.4** (sub-Euclidean isoperimetric inequality) *Let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_n)$ *for some $n \geq 1$, and suppose that* $\operatorname{asrk}(X) \leq n$. *Then for all $C, \epsilon > 0$ there is a constant $a_\epsilon \geq 0$ (depending on $X, n, C, \epsilon$) such that if $r > a_\epsilon$, then every cycle $Z \in \mathbf{Z}_{n,\mathrm{c}}(X)$ with $\mathbf{M}(Z) \leq Cr^n$ and $\operatorname{spt}(Z) \subset B_p(r)$ for some $p \in X$ possesses a filling $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ with mass*

$$\mathbf{M}(V) < \epsilon r^{n+1}.$$

This is shown in a more general form, for complete metric spaces, and without restrictions on $\operatorname{spt}(Z)$, in Theorem 1.2 in [82]. The stated version suffices for our purposes, and the proof could be slightly simplified under these assumptions.

The following result may be viewed as a localized version of Theorem 4.4 and will be used repeatedly throughout the paper. The main content is that if a cycle $Z \in \mathbf{Z}_{n,\mathrm{c}}(X)$ satisfies $\|Z\|(B_p(r)) \leq Cr^n$ for some $p \in X$ and for all $r > a \geq 0$, then for every $\epsilon > 0$ and every sufficiently large $r > 0$ there exists a "partial filling" $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ such that $\operatorname{spt}(Z - \partial V) \cap B_p(r) = \emptyset$ and $\mathbf{M}(V) < \epsilon r^{n+1}$; that is, $F_{p,r}(Z) < \epsilon$. We formulate this more generally for local cycles of the form $Z = S - S'$ with $F_\infty(Z) < \infty$, where only $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ is required to satisfy a density bound with respect to $p$ and $S' \in \mathbf{I}_{n,\mathrm{loc}}(X)$ (possibly zero) is area-minimizing.

**Proposition 4.5** (partial filling) *Let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_n)$ *for some $n \geq 1$, and suppose that* $\operatorname{asrk}(X) \leq n$. *Then for all $C, \epsilon > 0$ and $a \geq 0$ there is a constant $a'_\epsilon \geq 0$ such that the following holds. Suppose that $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ satisfies $\Theta_{p,r}(S) \leq C$ for some point $p \in X$ and for all $r > a$, and $S' \in \mathbf{I}_{n,\mathrm{loc}}(X)$ is minimizing with $\partial S' = \partial S$ and $F_\infty(S - S') < \infty$. Then*

$$\Theta_{p,r}(S') < C + \epsilon \quad \text{and} \quad F_{p,r}(S - S') < \epsilon$$

*for all $r > a'_\epsilon$, in particular $\Theta_\infty(S') \le C$ and $F_\infty(S - S') = 0$.*

This shows in particular the following dichotomy: if $Z \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ and $\Theta_\infty(Z) < \infty$, then $F_\infty(Z)$ is either 0 or $\infty$.

*Proof* We write $B_r := B_p(r)$ for $r > 0$. Choose a constant $D > F_\infty(S - S')$. Then, for every sufficiently large $r_0 > 0$, there is a $V_0 \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ such that

$$\mathbf{M}(V_0) < D r_0^{n+1}$$

and $\mathrm{spt}(S - S' - \partial V_0) \cap B_{r_0} = \emptyset$. We fix such $r_0$ and $V_0$ for the moment, and we put $r_i := \eta^i r_0$ for some fixed $\eta \in [1/2, 1)$ and every integer $i \ge 1$.

There exists an $s \in (r_1, r_0)$ such that both $S' \llcorner B_s$ and the slice

$$T_s := \partial(V_0 \llcorner B_s) - (\partial V_0) \llcorner B_s = \partial(V_0 \llcorner B_s) - (S - S') \llcorner B_s$$

belong to $\mathbf{I}_{n,\mathrm{c}}(X)$, and $\mathbf{M}(T_s) \le \mathbf{M}(V_0)/(r_0 - r_1) \le (1-\eta)^{-1} D r_0^n$. Note that $\partial(S' \llcorner B_s) = \partial(S \llcorner B_s + T_s)$. Using the minimality of $S'$ and assuming that $r_1 > a$, so that $\Theta_{p,s}(S) \le C$, we infer that

$$\mathbf{M}(S' \llcorner B_s) \le \mathbf{M}(S \llcorner B_s) + \mathbf{M}(T_s) \le C s^n + (1-\eta)^{-1} D r_0^n \le \bar{C} r_1^n$$

for $\bar{C} := \eta^{-n}(C + (1-\eta)^{-1} D)$. Thus $\Theta_{p,r_1}(S') \le \bar{C}$, and the cycle $Z_s := (S - S') \llcorner B_s + T_s$ satisfies $\mathbf{M}(Z_s) \le 2\bar{C} r_1^n$ and $\mathrm{spt}(Z_s) \subset B_{2r_1}$. Let $\delta > 0$. By Theorem 4.4 there exists a constant $\bar{a}_\delta \ge a$, depending only on $n$, $X$, $\bar{C}$, $a$, $\delta$, such that if $r_1 > \bar{a}_\delta$, then $Z_s$ possesses a filling $V_1 \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ with mass

$$\mathbf{M}(V_1) < \delta r_1^{n+1}.$$

Note that the support of $S - S' - \partial V_1 = S - S' - Z_s$ lies outside $B_{r_1}$, thus $F_{p,r_1}(S - S') < \delta$. Note further that for $\delta \le D$, $V_1$ replicates the properties of $V_0$ at the next smaller scale $r_1$.

Now, given any $\delta \in (0, D]$ and $r > \bar{a}_\delta$, we can choose $r_0$ initially such that $r = r_k = \eta^k r_0$ for some $k \ge 1$. In the case that $k \ge 2$, we repeat the slicing and filling procedure described in the preceding paragraph successively for $i = 2, \ldots, k$, with $(r_i, r_{i-1})$ and $V_{i-1}$ in place of $(r_1, r_0)$ and $V_0$. This produces a sequence of partial fillings $V_1, \ldots, V_k \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ of $S - S'$, with $\mathrm{spt}(S - S' - \partial V_i) \cap B_{r_i} = \emptyset$, such that $\Theta_{p,r_i}(S') \le \bar{C}$ and $\mathbf{M}(V_i) < \delta r_i^{n+1}$. For $i = k$, this shows that

$$\Theta_{p,r}(S') \le \bar{C} = \eta^{-n}(C + (1-\eta)^{-1} D) \quad \text{and} \quad F_{p,r}(S - S') < \delta$$

whenever $0 < \delta \le D$ and $r > \bar{a}_\delta$.

In particular, $F_\infty(S - S') = 0$, and we may thus repeat the above argument for arbitrarily small $D > 0$. Let $\epsilon > 0$. Choosing $\eta \in [1/2, 1)$ and $D$ such that $\bar{C} < C + \epsilon$, and putting $\delta := \min\{\epsilon, D\}$, we conclude that $\Theta_{p,r}(S') < C + \epsilon$ and $F_{p,r}(S - S') < \epsilon$ whenever $r > a'_\epsilon := \bar{a}_\delta$. Note that $a'_\epsilon$ depends only on $n, X, C, a, \epsilon$. □

The following result is included mainly for illustration. It shows that for local $n$-cycles with controlled density in spaces satisfying the assumptions of Theorem 4.4, quasi-minimality is equivalent to several other conditions, among them the lower bound on the filling density obtained in Lemma 3.4.

**Proposition 4.6** (characterizing quasi-minimizers) *Let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_n)$ *for some $n \geq 1$, and suppose that* $\mathrm{asrk}(X) \leq n$. *For an $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ with $(C, a)$-controlled density, the following are equivalent:*

*(1) There exist $Q \geq 1$ and $a_1 \geq 0$ such that $S$ is $(Q, a_1)$-quasi-minimizing.*
*(2) There exist $c_2 > 0$ and $a_2 \geq 0$ such that if $x \in \mathrm{spt}(S)$, then $\mathbf{M}(T) \geq c_2 r^n$ for almost every $r > a_2$ and every $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ with $\partial T = \partial(S \llcorner B_x(r))$.*
*(3) There exist $c_3 > 0$ and $a_3 \geq 0$ such that if $x \in \mathrm{spt}(S)$, then $\mathbf{M}(T) \geq c_3 r^n$ for almost every $r > a_3$ and every $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ with $\partial T = \partial(S \llcorner B_x(r))$ and $\mathrm{spt}(T) \subset S_x(r)$.*
*(4) There exist $c_4 > 0$ and $a_4 \geq 0$ such that $F_{x,r}(S) \geq c_4$ for all $x \in \mathrm{spt}(S)$ and $r > a_4$.*

Notice that (3) is a divergence condition for $S$; compare, for example, the definition of the divergence of a geodesic line in Section 3 of [51].

*Proof* The implications (1) ⇒ (2) and (2) ⇒ (1) follow easily from Lemma 3.3 (density) and the fact that $S$ has controlled density, respectively. The implication (2) ⇒ (3) holds trivially, and (3) ⇒ (4) is shown by a simple integration as in the proof of Lemma 3.4 (filling density).

To prove that (4) ⇒ (2), let $x \in \mathrm{spt}(S)$, $r > a$, and $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ be such that $S' := S \llcorner B_x(r) \in \mathbf{I}_{n,\mathrm{c}}(X)$ and $\partial T = \partial S'$. By Theorem 2.4 (minimizing filling), we can assume that $T$ is minimizing and $\mathrm{spt}(T)$ is within distance $(\mathbf{M}(T)/\delta)^{1/n}$ from $\mathrm{spt}(\partial S')$ for some constant $\delta = \delta(n, X) > 0$. Then the cycle $Z := S' - T \in \mathbf{Z}_{n,\mathrm{c}}(X)$ has mass at most $2Cr^n$. Now if $\mathbf{M}(T) < \delta(r/2)^n$, say, then $\mathrm{spt}(Z) \subset B_x(3r/2)$ and $\mathrm{spt}(S - Z) \cap B_x(r/2) = \emptyset$, and it follows from Theorem 4.4 that $F_{x,r/2}(S) < c_4$, provided $r$ is sufficiently large. Thus (4) implies that $\mathbf{M}(T) \geq \delta(r/2)^n$ for large enough $r$. □

## 5 Morse Lemmas

In this section we will prove some higher rank analogs of the Morse Lemma, replacing quasi-geodesics with $n$-dimensional quasi-minimizers with controlled density. Here we will also establish Theorem 1.1.

A first result follows very quickly from Lemma 3.4 (filling density) and Proposition 4.5 (partial filling).

**Theorem 5.1** (Morse Lemma I) *Let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_n)$ *for some $n \geq 1$, and suppose that* $\operatorname{asrk}(X) \leq n$. *Then for all $Q \geq 1$, $C > 0$, and $a \geq 0$ there is a constant $b \geq 0$ such that the following holds. Suppose that $Z \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ has $(C, a)$-controlled density and satisfies $F_\infty(Z) < \infty$. If $Y \subset X$ is a closed set such that $Z$ is $(Q, a)$-quasi-minimizing mod $Y$, then* $\operatorname{spt}(Z)$ *lies within distance at most $b$ from $Y$.*

Note that if $S, S' \in \mathbf{I}_{n,\mathrm{loc}}(X)$ are two $(Q, a)$-quasi-minimizers with $(C/2, a)$-controlled density and $\partial S = \partial S'$, then $Z := S - S' \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is $(Q, a)$-quasi-minimizing mod $\operatorname{spt}(S)$ as well as mod $\operatorname{spt}(S')$ and has $(C, a)$-controlled density. Theorem 5.1 then shows that the Hausdorff distance $d_{\mathrm{H}}(\operatorname{spt}(S), \operatorname{spt}(S'))$ is at most $b$, provided $F_\infty(Z) < \infty$ (which holds trivially if $S, S' \in \mathbf{I}_{n,\mathrm{c}}(X)$).

*Proof* Since $Z$ is $(Q, a)$-quasi-minimizing mod $Y$, Lemma 3.4 shows that

$$F_{x,r}(Z) \geq c = c(n, X, Q) > 0$$

whenever $x \in \operatorname{spt}(Z)$, $r > 4a$, and $B_x(r) \cap Y = \emptyset$. On the other hand, since $Z$ has $(C, a)$-controlled density and satisfies $F_\infty(Z) < \infty$, we may apply Proposition 4.5 with $p = x$ and $S = Z$, $S' = 0$. Taking $\epsilon = c$, we infer that there is a constant $b \geq 4a$, depending only on $n$, $X$, $Q$, $C$, $a$, such that

$$F_{x,r}(Z) < c$$

for $r > b$. This shows that $r \leq b$ (in particular $Y \neq \emptyset$). □

As a first application, we deduce Theorem 1.1, which we restate for convenience.

**Theorem 5.2** (slim simplices) *Let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_n)$ *for some $n \geq 1$, and suppose that* $\operatorname{asrk}(X) \leq n$. *Let $\Delta$ be a Euclidean $(n+1)$-simplex, and let $f : \partial\Delta \to X$ be a map such that for every facet $W$ of $\Delta$, the restriction $f|_W$ is an $(L, a_0)$-quasi-isometric embedding. Then, for every facet $W$, the image $f(W)$ is contained in the closed $D$-neighborhood of $f\left(\overline{\partial\Delta \setminus W}\right)$ for some constant $D \geq 0$ depending only on $X, n, L, a_0$.*

*Proof* Let $W_0, \dots, W_{n+1}$ be an enumeration of the facets of $\Delta$, and write the cycle $\partial[\![\Delta]\!] \in \mathbf{Z}_{n,\mathrm{c}}(\mathbb{R}^{n+1})$ as $\sum_{i=0}^{n+1} E_i$ for $E_i := (\partial[\![\Delta]\!]) \llcorner W_i \in \mathbf{I}_{n,\mathrm{c}}(\mathbb{R}^{n+1})$. Choose a triangulation of $\partial\Delta$ with simplices of diameter at most $D_0$ such that every $r$-ball in $\mathbb{R}^{n+1}$ with $r \geq D_0$ meets at most $C_0 r^n$ $n$-simplices in each $W_i$, for some constants $C_0, D_0 > 0$ depending only on $n$. Consider the corresponding chain complex $\mathscr{P}_*(\partial\Delta)$ of simplicial integral currents and proceed as in the proof of Proposition 3.7 (triangulated quasiflats) to get a chain map $\iota\colon \mathscr{P}_*(\partial\Delta) \to \mathbf{I}_{*,\mathrm{c}}(X)$ such that the following properties hold for every $S_i := \iota(E_i) \in \mathbf{I}_{n,\mathrm{c}}(X)$ and for some constants $Q, C, a$ depending only on $X, n, L, a_0$:

(1) $\mathrm{spt}(S_i) \subset N_a(f(W_i))$ and $\mathrm{spt}(\partial S_i) \subset N_a(f(\partial W_i))$;
(2) $S_i$ is $(Q, a)$-quasi-minimizing mod $N_a(f(\partial W_i))$ and has $(C, a)$-controlled density;
(3) $d(f(x), \mathrm{spt}(S_i)) \leq a$ for all $x \in W_i$ with $d(x, \partial W_i) \geq a$.

(Here $N_a$ stands again for the closed $a$-neighborhood, and $\partial W_i$ denotes the relative boundary of $W_i$.) Let $M_i$ denote the union of all $W_j$ with $j \neq i$. The cycle $Z := \iota(\partial[\![\Delta]\!]) = \sum_{i=0}^{n+1} S_i$ is $(Q, a)$-quasi-minimizing mod $N_a(f(M_i))$ for every $i$ and has $((n+2)C, a)$-controlled density. It then follows from Theorem 5.1 that the set $\mathrm{spt}(S_i) \setminus N_a(f(M_i)) = \mathrm{spt}(Z) \setminus N_a(f(M_i))$ lies within distance at most $b$ from $N_a(f(M_i))$ for some constant $b$ depending only on $X, n, L, a_0$. Hence, for $x \in W_i$, it follows from (3) that $d(f(x), f(M_i))$ is less than or equal to $2a + b$ if $d(x, \partial W_i) \geq a$ and less than $La + a_0$ otherwise. □

*Remark 5.3* If, for $f\colon \partial\Delta \to X$ as above, there exists a map $g\colon \partial\Delta \to X$ such that $g|_W$ is $L'$-Lipschitz for every facet $W$ and $d(f(x), g(x)) \leq b'$ for all $x \in \partial\Delta$, for some constants $L', b'$ depending on $n, L, a_0$, then one may use Proposition 3.6 (Lipschitz quasiflats) instead of Proposition 3.7 in the above argument. Such a map $g$ exists if $X$ is Lipschitz $(n-1)$-connected (compare Corollary 1.7 in [62]), in particular if $X$ is CAT(0) or a space with a convex bicombing.

We now prove an analog of the Morse Lemma for quasi-geodesic segments.

**Theorem 5.4** (Morse Lemma II) *Let $X$ be a proper metric space satisfying condition* $(\mathrm{CI}_n)$ *for some $n \geq 1$, and suppose that* $\mathrm{asrk}(X) \leq n$. *Then for all $Q \geq 1$, $C > 0$, and $a \geq 0$ there is a constant $b \geq 0$ such that the following holds. If $S \in \mathbf{I}_{n,\mathrm{c}}(X)$ is a $(Q, a)$-quasi-minimizer with $(C, a)$-controlled density, then there exists a minimizing $\tilde{S} \in \mathbf{I}_{n,\mathrm{c}}(X)$ such that $\partial S = \partial\tilde{S}$, and every such $\tilde{S}$ satisfies*

$$d_{\mathrm{H}}(\mathrm{spt}(S), \mathrm{spt}(\tilde{S})) \leq b.$$

*Proof* A minimizing $\tilde{S} \in \mathbf{I}_{n,\mathrm{c}}(X)$ with $\partial\tilde{S} = \partial S$ exists by Theorem 2.4 (minimizing filling). Since $S$ has $(C, a)$-controlled density, it follows from Proposition 4.5 (partial filling) that every such $\tilde{S}$ has $(\tilde{C}, \tilde{a})$-controlled density for some constants $\tilde{C} \geq C$ and $\tilde{a} \geq a$ depending only on $n, X, C, a$. Then the cycle $S - \tilde{S}$ has $(2\tilde{C}, \tilde{a})$-controlled density and is $(Q, a)$-quasi-minimizing mod $\mathrm{spt}(\tilde{S})$ as well as mod $\mathrm{spt}(S)$; the result thus follows from Theorem 5.1. □

Our next goal is to extend this last result to local chains. We state an auxiliary lemma.

**Lemma 5.5** ($F$-convergence) *Let $X$ be a proper metric space satisfying condition* ($\mathrm{CI}_n$) *for some $n \geq 1$. Then a sequence $(Z_j)$ in $\mathbf{Z}_{n,\mathrm{loc}}(X)$ converges in the local flat topology to* 0 *if and only if* $\lim_{j\to\infty} F_{p,r}(Z_j) = 0$ *for all $p \in X$ and $r > 0$.*

*Proof* Suppose that $Z_j \to 0$ in the local flat topology, and let $p \in X$ and $r > 0$. There is a sequence $(V_j)$ in $\mathbf{I}_{n+1,\mathrm{loc}}(X)$ such that

$$(\|Z_j - \partial V_j\| + \|V_j\|)(B_p(2r)) \to 0.$$

Note that $Z_j - \partial V_j \in \mathbf{Z}_{n,\mathrm{loc}}(X)$. Pick $s \in (r, 2r)$ such that, for $K := B_p(s)$, the slice $\partial((Z_j - \partial V_j) \llcorner K)$ is in $\mathbf{Z}_{n-1,\mathrm{c}}(X)$ for all $j$, and furthermore $V_j \llcorner K \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ for all $j$. By Theorem 2.4 (minimizing filling), there exists a minimizing current $T_j \in \mathbf{I}_{n,\mathrm{c}}(X)$ with $\partial T_j = \partial((Z_j - \partial V_j) \llcorner K)$, and since $\mathrm{spt}(\partial T_j) \subset S_p(s)$ and $\mathbf{M}(T_j) \leq \mathbf{M}((Z_j - \partial V_j) \llcorner K) \to 0$, it follows that $\mathrm{spt}(T_j) \subset B_p(2r) \setminus B_p(r)$ for $j$ sufficiently large. The cycles $(Z_j - \partial V_j) \llcorner K - T_j$ converge to zero in mass, by condition ($\mathrm{CI}_n$) they thus possess fillings $W_j \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ such that also $\mathbf{M}(W_j) \to 0$. Now define $V_j' := V_j \llcorner K + W_j \in \mathbf{I}_{n+1,\mathrm{c}}(X)$. Then $\mathbf{M}(V_j') \leq \mathbf{M}(V_j \llcorner K) + \mathbf{M}(W_j) \to 0$, and the support of

$$\begin{aligned} Z_j - \partial V_j' &= Z_j - \partial(V_j \llcorner K) - (Z_j - \partial V_j) \llcorner K + T_j \\ &= Z_j \llcorner (X \setminus K) - \big(\partial(V_j \llcorner K) - (\partial V_j) \llcorner K\big) + T_j \end{aligned}$$

is disjoint from $B_p(r)$ for all sufficiently large $j$. This shows that $F_{p,r}(Z_j) \leq \mathbf{M}(V_j')/r^{n+1} \to 0$.

The reverse implication is clear. □

We now establish a basic existence theorem for minimizing local $n$-chains in spaces of asymptotic rank at most $n$.

**Theorem 5.6** (constructing minimizers) *Let $X$ be a proper metric space satisfying condition* ($\mathrm{CI}_n$) *for some $n \geq 1$, and suppose that* $\mathrm{asrk}(X) \leq n$.

*Then for every* $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ *with* $\Theta_\infty(S) < \infty$ *there exists a minimizing* $\tilde{S} \in \mathbf{I}_{n,\mathrm{loc}}(X)$ *such that* $\partial\tilde{S} = \partial S$ *and* $F_\infty(S - \tilde{S}) = 0$*, and every such* $\tilde{S}$ *satisfies* $\Theta_\infty(\tilde{S}) \le \Theta_\infty(S)$.

Note that $\partial S$ may well be zero; the assertion $F_\infty(S-\tilde{S}) = 0$ then guarantees that $\tilde{S}$ is non-zero, provided $F_\infty(S) \neq 0$. Conversely, if $F_\infty(S) = 0$, then $\partial S = 0$, and it follows from Lemma 3.4 (filling density) that there is no minimizer $\tilde{S} \neq 0$ with $\partial\tilde{S} = 0$ and $F_\infty(\tilde{S}) = 0$.

*Proof* Fix a base point $p \in X$, and choose a sequence $0 < r_i \uparrow \infty$ such that $S_i := S \llcorner B_p(r_i) \in \mathbf{I}_{n,\mathrm{c}}(X)$ for all $i$. Theorem 2.4 (minimizing filling) provides a corresponding sequence of minimizing currents $\tilde{S}_i \in \mathbf{I}_{n,\mathrm{c}}(X)$ with $\partial\tilde{S}_i = \partial S_i$. Since $\Theta_\infty(S) < \infty$, there exist $C > 0$ and $a \ge 0$ such that $\Theta_{p,r}(S_i) \le \Theta_{p,r}(S) \le C$ for all $r > a$. Proposition 4.5 (partial filling) shows that for every $\epsilon > 0$ there is a constant $\tilde{a}_\epsilon \ge 0$ such that, for all $i$ and $r > \tilde{a}_\epsilon$,

$$\|\tilde{S}_i\|(B_p(r)) < (C+\epsilon)r^n \quad \text{and} \quad F_{p,r}(S_i - \tilde{S}_i) < \epsilon.$$

Note also that if $K \subset X$ is a compact set, then $\|\partial\tilde{S}_i\|(K) = \|\partial S_i\|(K) = \|\partial S\|(K)$ for all but finitely many indices $i$. By Theorem 2.3 (compactness), some subsequence $(\tilde{S}_{i_j})$ converges in the local flat topology to a minimizing current $\tilde{S} \in \mathbf{I}_{n,\mathrm{loc}}(X)$ with $\partial\tilde{S} = \partial S$.

To show that $F_\infty(S - \tilde{S}) = 0$, put $Z_j := S - S_{i_j} - (\tilde{S} - \tilde{S}_{i_j}) \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ and note that $Z_j \to 0$ in the local flat topology. If $\epsilon > 0$ and $r > \tilde{a}_\epsilon$, then

$$F_{p,r}(S - \tilde{S}) \le F_{p,r}(Z_j) + F_{p,r}(S_{i_j} - \tilde{S}_{i_j}) < F_{p,r}(Z_j) + \epsilon.$$

Hence, $F_{p,r}(S - \tilde{S}) \le \epsilon$ by Lemma 5.5.

Finally, a simple slicing argument shows that $\Theta_\infty(\tilde{S}) \le \Theta_\infty(S)$ for every minimizing $\tilde{S} \in \mathbf{I}_{n,\mathrm{loc}}(X)$ with $\partial\tilde{S} = \partial S$ and $F_\infty(S - \tilde{S}) = 0$. (This also follows from Proposition 4.5.) □

The next result generalizes Theorem 5.4 to local currents. Theorem 1.2 in the introduction corresponds to the case $\partial S = 0$.

**Theorem 5.7** (Morse Lemma III) *Let* $X$ *be a proper metric space satisfying condition* $(\mathrm{CI}_n)$ *for some* $n \ge 1$*, and suppose that* $\mathrm{asrk}(X) \le n$*. Then for all* $Q \ge 1$*,* $C > 0$*, and* $a \ge 0$ *there is a constant* $b \ge 0$ *such that the following holds. If* $S \in \mathbf{I}_{n,\mathrm{loc}}(X)$ *is a* $(Q, a)$*-quasi-minimizer with* $(C, a)$*-controlled density, then there exists a minimizing* $\tilde{S} \in \mathbf{I}_{n,\mathrm{loc}}(X)$ *such that* $\partial S = \partial\tilde{S}$ *and* $F_\infty(S - \tilde{S}) = 0$*, and every such* $\tilde{S}$ *satisfies* $\Theta_\infty(\tilde{S}) \le \Theta_\infty(S)$ *and*

$$d_{\mathrm{H}}(\mathrm{spt}(S), \mathrm{spt}(\tilde{S})) \le b.$$

*Proof* Since $\Theta_\infty(S) < \infty$, Theorem 5.6 shows that there exists a minimizing $\tilde{S} \in \mathbf{I}_{n,\mathrm{loc}}(X)$ with $\partial S = \partial \tilde{S}$ and $F_\infty(S - \tilde{S}) = 0$, and every such $\tilde{S}$ satisfies $\Theta_\infty(\tilde{S}) \le \Theta_\infty(S)$. The rest of the proof is the same as for Theorem 5.4. □

## 6 Asymptote classes and visual metrics

We now consider asymptote classes of local $n$-cycles in spaces of asymptotic rank $n$.

**Definition 6.1** *(asymptote classes)* Let $X$ be a proper metric space that satisfies condition $(\mathrm{CI}_n)$ for $n = \mathrm{asrk}(X) \ge 1$. We put

$$\mathbf{Z}^\infty_{n,\mathrm{loc}}(X) := \{S \in \mathbf{Z}_{n,\mathrm{loc}}(X) : \Theta_\infty(S) < \infty\}$$

and call two elements $S, S'$ of this group *$F$-asymptotic* if $F_\infty(S - S') = 0$ (or, equivalently, $F_\infty(S - S') < \infty$; see Proposition 4.5 (partial filling)). This defines an equivalence relation $\sim_F$ on $\mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$. We denote the quotient space by

$$\mathscr{Z}X := \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)/\!\sim_F$$

(note that $n = \mathrm{asrk}(X)$ is implicit in $X$) and the equivalence class of $S$ by $[S] \in \mathscr{Z}X$. The addition $[S] + [S'] := [S + S']$ is clearly well-defined, thus $\mathscr{Z}X$ is an abelian group.

As stated in Theorem 1.4, when $X$ is a CAT(0) space, $\mathscr{Z}X$ turns out to be canonically isomorphic to the group $\mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X)$ of integral $(n-1)$-cycles in the Tits boundary of $X$. This will be discussed in Sect. 9.

Theorem 5.6 (constructing minimizers) shows that every class $[S] \in \mathscr{Z}X$ contains an area-minimizing $\tilde{S} \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$, and furthermore every such $\tilde{S}$ has minimal asymptotic density among all members of $[S]$. We will now show that for any $C > 0$ and $a \ge 0$, the set

$$\mathscr{Z}_{C,a}X := \{[S] \in \mathscr{Z}X : S \text{ has } (C,a)\text{-controlled density}\}$$

carries a family of metrics analogous to the visual metrics on $\partial_\infty X$ in the hyperbolic case. With the present hypotheses ($X$ satisfies condition $(\mathrm{CI}_n)$ for $n = \mathrm{asrk}(X) \ge 1$), a class in $\mathscr{Z}X$ need not contain a representative with controlled density; however, under the stronger assumptions of the subsequent sections, in particular when $X$ is CAT(0), every minimizer $\tilde{S} \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ has controlled density (see Proposition 7.4 and Remark 7.5). Note also that every quasiflat $f\colon \mathbb{R}^n \to X$ yields an $S$ with controlled density (compare Proposition 3.7).

First, for any reference point $p \in X$ and any $[S] \in \mathscr{Z}X$, we put

$$\langle [S] \rangle_p := \inf\{d(p, \mathrm{spt}(\tilde{S})) : \tilde{S} \in [S] \text{ is minimizing}\}.$$

Note that $\langle [S] \rangle_p = \infty$ if and only if $[S] = [0]$, that is, $F_\infty(S) = 0$ (see the remark after Theorem 5.6). Clearly $\langle [-S] \rangle_p = \langle [S] \rangle_p$ and

$$\left| \langle [S] \rangle_p - \langle [S] \rangle_q \right| \le d(p, q).$$

If $X$ is a geodesic $\delta$-hyperbolic space ($n = 1$) and $S$ corresponds to a geodesic $\varrho : \mathbb{R} \to X$ connecting two points $u, v \in \partial_\infty X$, then $\langle [S] \rangle_p$ agrees, up to a bounded additive error, with the Gromov product $(u \mid v)_p$. The following result mimics the $\delta$-inequality

$$(u \mid w)_p \ge \min\{(u \mid v)_p, (v \mid w)_p\} - \delta$$

for the Gromov product of points at infinity (see p. 89 in [40] and Section 2.2 in [19]).

**Proposition 6.2** ($D$-inequality) *Let $X$ be a proper metric space that satisfies condition* ($\mathrm{CI}_n$) *for $n = \mathrm{asrk}(X) \ge 1$. Then for all $C > 0$ and $a \ge 0$ there exists $D \ge 0$ such that*

$$\langle [S + S'] \rangle_p \ge \min\{\langle [S] \rangle_p, \langle [S'] \rangle_p\} - D$$

*for all $p \in X$ and $[S], [S'] \in \mathscr{Z}_{C,a}X$.*

*Proof* Let $\epsilon > 0$. Pick minimizers $\tilde{S} \in [S]$, $\tilde{S}' \in [S']$, and $\hat{S} \in [S + S']$ such that

$$d(p, \mathrm{spt}(\hat{S})) < \langle [S + S'] \rangle_p + \epsilon.$$

Note that $[S + S'] \in \mathscr{Z}_{2C,a}X$. Applying Proposition 4.5 (partial filling) to each of $\tilde{S}$, $\tilde{S}'$, $\hat{S}$, we infer that $Z := \hat{S} - (\tilde{S} + \tilde{S}')$ has $(\tilde{C}, \tilde{a})$-controlled density for some constants $\tilde{C}, \tilde{a}$ depending only on $X, C, a$. Note that $F_\infty(Z) = 0$. Since $Z$ is minimizing mod $Y := \mathrm{spt}(\tilde{S}) \cup \mathrm{spt}(\tilde{S}')$, Theorem 5.1 (Morse Lemma I) shows that $\mathrm{spt}(\hat{S})$ is within distance at most $D$ from $Y$ for some constant $D$ depending only on $X, C, a$. Hence,

$$d(p, Y) \le d(p, \mathrm{spt}(\hat{S})) + D < \langle [S + S'] \rangle_p + \epsilon + D.$$

Since $\min\{\langle [S] \rangle_p, \langle [S'] \rangle_p\} \le d(p, Y)$, this gives the result. □

We call a metric $\nu$ on $\mathscr{Z}_{C,a}X$ *visual* if there are $p \in X$, $b > 1$ and $c \geq 1$ such that

$$c^{-1} b^{-\langle [S-S'] \rangle_p} \leq \nu([S],[S']) \leq c\, b^{-\langle [S-S'] \rangle_p}$$

for all $[S], [S'] \in \mathscr{Z}_{C,a}X$. It is easily seen that any two metrics that are visual with respect to the same parameter $b$ but different base points are bi-Lipschitz equivalent, whereas any two visual metrics are snowflake equivalent (compare Theorem 3.2.4 in [66]). In particular, all visual metrics induce the same topology on $\mathscr{Z}_{C,a}X$.

**Theorem 6.3** (visual metrics) *Let $X$ be a proper metric space that satisfies condition* $(\mathrm{CI}_n)$ *for $n = \mathrm{asrk}(X) \geq 1$, and let $C > 0$ and $a \geq 0$. Then for every $p \in X$ and every sufficiently small $b > 1$ there exists a metric $\nu$ on $\mathscr{Z}_{C,a}X$ that is visual with respect to $p$ and $b$. Furthermore, $\mathscr{Z}_{C,a}X$ is compact with respect to any visual metric.*

*Proof* Let $p \in X$ and $b > 1$, and put $\tilde{\nu}([S],[S']) := b^{-\langle [S-S'] \rangle_p}$; then

$$\tilde{\nu}([S],[S'']) \leq \kappa \max\{\tilde{\nu}([S],[S']), \tilde{\nu}([S'],[S''])\}$$

for all $[S], [S'], [S''] \in \mathscr{Z}_{C,a}X$, where $\kappa = b^D$ and $D$ is the constant from Proposition 6.2 associated with the parameters $2C$ and $a$. Note that $\tilde{\nu}([S],[S']) = 0$ if and only if $[S] = [S']$. If $\kappa \leq 2$, then a standard chain construction yields a metric $\nu$ on $\mathscr{Z}_{C,a}X$ such that

$$\frac{1}{2\kappa}\, \tilde{\nu}([S],[S']) \leq \nu([S],[S']) \leq \tilde{\nu}([S],[S'])$$

(see Lemma 2.2.5 in [19]). Thus $\nu$ is visual with respect to $p$ and $b$.

To prove the compactness assertion, let $(S_i)$ be a sequence in $\mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ such that each $S_i$ has $(C,a)$-controlled density. By Theorem 2.3 (compactness), some subsequence $(S_{i_j})$ converges in the local flat topology, hence also weakly, to an $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$. For all $p \in X$ and $s > r > a$,

$$\|S\|(B_p(r)) \leq \liminf_{j\to\infty} \|S_{i_j}\|(B_p(s)) \leq C s^n$$

by the lower semicontinuity of mass on open sets; thus $S$ has $(C,a)$-controlled density. Suppose now that $\nu$ is a visual metric on $\mathscr{Z}_{C,a}X$ with respect to $p \in X$, and note that $\nu([S_{i_j}],[S]) \to 0$ if and only if $\langle [Z_j] \rangle_p \to \infty$, where $Z_j := S_{i_j} - S$. Consider a sequence of minimizers $\tilde{Z}_j \in [Z_j]$, and let $\epsilon > 0$. Since $\Theta_{p,r}(Z_j) \leq 2C$ for all $r > a$, Proposition 4.5 (partial filling) shows that if $r$ is sufficiently large, then $F_{p,r}(\tilde{Z}_j - Z_j) < \epsilon/2$ for all $j$. Furthermore,

it follows from Lemma 5.5 ($F$-convergence) that $F_{p,r}(Z_j) \to 0$ for every $r > 0$. Hence, for every sufficiently large $r > 0$, there is an index $j_0$ such that

$$F_{p,r}(\tilde{Z}_j) \le F_{p,r}(\tilde{Z}_j - Z_j) + F_{p,r}(Z_j) < \epsilon$$

for all $j \ge j_0$. Let $V_j \in \mathbf{I}_{n+1,c}(X)$ be such that $\operatorname{spt}(\tilde{Z}_j - \partial V_j) \cap B_p(r) = \emptyset$ and $\mathbf{M}(V_j) < \epsilon r^{n+1}$. For a point $x \in \operatorname{spt}(\tilde{Z}_j) \cap B_p(r/2)$, Lemma 3.4 (filling density) then shows that $F_{x,r/2}(\tilde{Z}_j) \ge c = c(X) > 0$, thus $\mathbf{M}(V_j) \ge c(r/2)^{n+1}$. Choosing $\epsilon = c/2^{n+1}$ we conclude that for every sufficiently large $r$ there is a $j_0$ such that $d(p, \operatorname{spt}(\tilde{Z}_j)) > r/2$ for all $j \ge j_0$. This shows that $\langle [Z_j] \rangle_p \to \infty$ as desired. □

Visual metrics will be discussed further in Remarks 9.6 and 10.7.

## 7 Conical representatives

Our next goal is to relate $F$-asymptote classes to geodesic cones and to cycles at infinity. For this purpose, we now impose a convexity condition on the metric space $X$.

A curve $\varrho\colon I \to X$ defined on some interval $I \subset \mathbb{R}$ is a *geodesic* if there is a constant $s \ge 0$, the *speed* of $\varrho$, such that $d(\varrho(t), \varrho(t')) = s|t - t'|$ for all $t, t' \in I$. A geodesic defined on $I = \mathbb{R}_+ := [0, \infty)$ is called a *ray*.

**Definition 7.1** *(convex bicombing)* By a *convex bicombing* $\sigma$ on a metric space $X$ we mean a map $\sigma\colon X \times X \times [0,1] \to X$ such that

(1) $\sigma_{xy} := \sigma(x, y, \cdot)\colon [0,1] \to X$ is a geodesic from $x$ to $y$ for all $x, y \in X$;
(2) $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex on $[0,1]$ for all $x, y, x', y' \in X$;
(3) $\operatorname{im}(\sigma_{pq}) \subset \operatorname{im}(\sigma_{xy})$ whenever $x, y \in X$ and $p, q \in \operatorname{im}(\sigma_{xy})$.

A geodesic $\varrho\colon I \to X$ is then called a *$\sigma$-geodesic* if $\operatorname{im}(\sigma_{xy}) \subset \operatorname{im}(\varrho)$ whenever $x, y \in \operatorname{im}(\varrho)$. A convex bicombing $\sigma$ on $X$ is *equivariant* if $\gamma \circ \sigma_{xy} = \sigma_{\gamma(x)\gamma(y)}$ for every isometry $\gamma$ of $X$ and for all $x, y \in X$.

Note that in (3), we do not specify the order of $p$ and $q$ with respect to the parameter of $\sigma_{xy}$, in particular $\sigma_{yx}(t) = \sigma_{xy}(1-t)$. In the terminology of [29], $\sigma$ is a *reversible* and *consistent convex geodesic bicombing* on $X$. In Section 10.1 of [53], metric spaces with such a structure $\sigma$ are called *often convex*. This class of spaces includes all CAT(0) and Busemann spaces as well as (linearly) convex subsets of normed spaces; at the same time, it is closed under various limit and product constructions such as ultralimits, (complete) Gromov–Hausdorff limits, and $l_p$ products for $p \in [1, \infty]$.

A large part of the theory of spaces of nonpositive curvature extends to this more general setting, see [8,29,30]. Furthermore, as was shown in [29,60],

every word hyperbolic group acts geometrically on a proper metric space of finite topological dimension with an equivariant convex bicombing $\sigma$. In the recent paper [28] it is shown that Theorem 4.1 (rank conditions) still holds for every proper and cocompact metric space $X$ with a convex bicombing. In fact, Theorem 1.1 in that paper shows that if the unit ball of some $n$-dimensional normed space $V$ is an asymptotic subset of $X$, then $V$ itself embeds isometrically into $X$.

Let now $X$ be a proper metric space with a convex bicombing $\sigma$. It follows from Sect. 2.7 that $X$ satisfies condition $(\mathrm{CI}_n)$ for every $n \geq 1$, thus all the preceding results are still at our disposal. The boundary at infinity of $(X, \sigma)$ is defined in the usual way, as for CAT(0) spaces, except that only $\sigma$-rays are taken into account. Specifically, we let $\mathrm{R}^\sigma X$ and $\mathrm{R}^\sigma_1 X$ denote the sets of all $\sigma$-rays and $\sigma$-rays of speed one, respectively, in $X$. For every pair of rays $\varrho, \varrho' \in \mathrm{R}^\sigma X$, the function $t \mapsto d(\varrho(t), \varrho'(t))$ is convex, and $\varrho$ and $\varrho'$ are called *asymptotic* if this function is bounded. This defines an equivalence relation $\sim$ on $\mathrm{R}^\sigma X$ as well as on $\mathrm{R}^\sigma_1 X$. The *boundary at infinity* or *visual boundary* of $(X, \sigma)$ is the set

$$\partial_\infty X := \mathrm{R}^\sigma_1 X/\!\sim$$

(whereas $\mathrm{R}^\sigma X/\!\sim$ is the set underlying the Tits cone of $X$, see the end of Sect. 8). Given $\varrho \in \mathrm{R}^\sigma_1 X$ and $p \in X$, there is a unique ray $\varrho_p \in \mathrm{R}^\sigma_1 X$ asymptotic to $\varrho$ with $\varrho_p(0) = p$. The set

$$\overline{X} := X \cup \partial_\infty X$$

carries a natural metrizable topology, analogous to the cone topology for CAT(0) spaces. With this topology, $\overline{X}$ is a compact absolute retract, and $\partial_\infty X$ is a $Z$-set in $\overline{X}$. See Section 5 in [29] for details, and [14] for more background. For a subset $A \subset X$, the limit set $\partial_\infty(A)$ is defined as the set of all points in $\partial_\infty X$ that belong to the closure of $A$ in $\overline{X}$. For a point $p \in X$ we define the geodesic homotopy

$$h_p \colon [0, 1] \times X \to X$$

by $h_p(\lambda, x) := h_{p,\lambda}(x) := \sigma_{px}(\lambda)$. Note that the map $h_{p,\lambda} \colon X \to X$ is $\lambda$-Lipschitz. For a set $A \subset X$,

$$\mathrm{C}_p(A) := h_p([0, 1] \times A)$$

denotes the geodesic cone from $p \in X$ over $A$, and $\overline{\mathrm{C}}_p(A)$ denotes its closure in $X$. Similarly, if $\Lambda \subset \partial_\infty X$, then $\mathrm{C}_p(\Lambda) \subset X$ denotes the union of the traces of the rays emanating from $p$ and representing points of $\Lambda$.

Let now $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$. We write

$$\Lambda(S) := \partial_\infty(\mathrm{spt}(S)) \subset \partial_\infty X$$

for the *limit set* of (the support of) $S$, and we call $S$ *conical* if there is a point $p \in X$ such that

$$h_{p,\lambda\#}S = S$$

for all $\lambda \in (0, 1)$. The following lemma collects a number of basic properties.

**Lemma 7.2** (conical) *Let $X$ be a proper metric space with a convex bicombing $\sigma$, and suppose that $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is conical with respect to some point $p \in X$. Then*

*(1) $S \llcorner B_p(r) \in \mathbf{I}_{n,\mathrm{c}}(X)$ and $h_{p,\lambda\#}(S \llcorner B_p(r)) = S \llcorner B_p(\lambda r)$ for all $r > 0$ and $\lambda \in (0, 1)$;*
*(2) the functions $r \mapsto \Theta_{p,r}(S)$ and $r \mapsto F_{p,r}(S)$ are non-decreasing on $(0, \infty)$;*
*(3) if $S \neq 0$, then $F_\infty(S) > 0$ (possibly $F_\infty(S) = \infty$);*
*(4) $\mathrm{spt}(S) \subset \mathrm{C}_p(\Lambda(S))$.*

*Proof* Put $B_r := B_p(r)$ for all $r > 0$. To see that $S \llcorner B_r \in \mathbf{I}_{n,\mathrm{c}}(X)$ for *every* $r > 0$, note that $S \llcorner B_s \in \mathbf{I}_{n,\mathrm{c}}(X)$ for almost every $s > 0$, pick such an $s > r$, and put $\lambda := r/s$. Now $h_{p,\lambda\#}(S \llcorner B_s) \in \mathbf{I}_{n,\mathrm{c}}(X)$, and since $B_s = h_{p,\lambda}^{-1}(B_r)$ and $h_{p,\lambda\#}S = S$, it follows that

$$h_{p,\lambda\#}(S \llcorner B_s) = (h_{p,\lambda\#}S) \llcorner B_r = S \llcorner B_r.$$

From this, the second assertion of (1) is also clear.

We show (2). For any $s > r > 0$,

$$\|S\|(B_r) = \|h_{p,r/s\#}S\|(B_r) \leq (r/s)^n \|S\|(B_s),$$

thus $\Theta_{p,r}(S)$ is non-decreasing in $r$. Similarly, if there exists $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ such that $\mathrm{spt}(S - \partial V) \cap B_s = \emptyset$, then $\mathbf{M}(h_{p,r/s\#}V) \leq (r/s)^{n+1}\mathbf{M}(V)$, and the support of $S - \partial(h_{p,r/s\#}V) = h_{p,r/s\#}(S - \partial V)$ is disjoint from $B_r$, thus $F_{p,r}(S) \leq \mathbf{M}(V)/s^{n+1}$. Taking the infimum over all such $V$, we get that $F_{p,r}(S) \leq F_{p,s}(S)$ (where $F_{p,s}(S) = \infty$ if no such $V$ exists).

As for (3), note that if $S \neq 0$, there is an $s > 0$ such that $\mathrm{spt}(S) \cap B_s \neq \emptyset$. Then any $V$ as above must be non-zero, thus $F_{p,s}(S) \in (0, \infty]$, and $F_\infty(S) \geq F_{p,s}(S)$ by monotonicity.

Finally, observe that $\mathrm{spt}(S) = \mathrm{spt}(h_{p,\lambda\#}S) \subset h_{p,\lambda}(\mathrm{spt}(S))$ for all $\lambda \in (0, 1]$. Hence, for every $x_1 \in \mathrm{spt}(S)$ there exist $x_2, x_3, \ldots \in \mathrm{spt}(S)$ such that $h_{p,1/k}(x_k) = x_1$, and (4) follows. □

We now prove a first part of Theorem 1.4 stated in the introduction.

**Theorem 7.3** (conical representative) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with* $\operatorname{asrk}(X) = n \geq 2$. *Suppose that $S \in \mathbf{Z}^{\infty}_{n,\mathrm{loc}}(X)$ and $p \in X$. Then there exists a unique local cycle $S_{p,0} \in \mathbf{Z}^{\infty}_{n,\mathrm{loc}}(X)$ that is conical with respect to $p$ and $F$-asymptotic to $S$. Furthermore, $\Theta_\infty(S_{p,0}) \leq \Theta_\infty(S)$, $\Lambda(S_{p,0}) \subset \Lambda(S)$, and* $\operatorname{spt}(S_{p,0}) \subset \mathrm{C}_p(\Lambda(S))$.

Note that by uniqueness, $S_{p,0} = 0$ if and only if $F_\infty(S) = 0$. For the proof of Theorem 7.3, we consider the family of all

$$S_{p,\lambda} := h_{p,\lambda\#} S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$$

for $\lambda \in (0, 1]$. We show that, as $\lambda \to 0$, this family converges in the local flat topology to the desired local cycle $S_{p,0}$.

*Proof* Pick any $C > \Theta_\infty(S)$. Then there exists an $a \geq 0$ such that $\Theta_{p,r}(S) \leq C$ for all $r > a$. We write again $B_r := B_p(r)$. Since $h_{p,\lambda}$ is $\lambda$-Lipschitz,

$$\|S_{p,\lambda}\|(B_r) \leq \lambda^n \|S\|(B_{\lambda^{-1}r}) \leq C r^n$$

for all $r > \lambda a$ (see Sect. 2.4), thus $\Theta_{p,r}(S_{p,\lambda}) \leq C$ for all $r > \lambda a$.

First we construct partial fillings of $S_{p,\lambda} - S$ for a fixed $\lambda \in (0, 1)$. Let $R' > a/2$. Then $\|S\|(B_{2R'}) \leq C(2R')^n$, hence there exists an $R \in (R', 2R')$ such that $S \llcorner B_R \in \mathbf{I}_{n,\mathrm{c}}(X)$,

$$\mathbf{M}(\partial(S \llcorner B_R)) \leq 2^n C (R')^{n-1} \leq 2^n C R^{n-1},$$

and $h_{p,\lambda\#}(S \llcorner B_R) = S_{p,\lambda} \llcorner B_{\lambda R} \in \mathbf{I}_{n,\mathrm{c}}(X)$. The truncated geodesic cone $T := h_{p\#}\big([\![\lambda, 1]\!] \times \partial(S \llcorner B_R)\big) \in \mathbf{I}_{n,\mathrm{c}}(X)$ with boundary

$$\partial T = \partial(S \llcorner B_R) - h_{p,\lambda\#} \partial(S \llcorner B_R)$$

satisfies $\mathbf{M}(T) \leq R\, \mathbf{M}(\partial(S \llcorner B_R)) \leq 2^n C R^n$ (see Sect. 2.7). It follows that $\|T\|(B_r) \leq 2^n C r^n$ for all $r > 0$. Hence,

$$Z := S_{p,\lambda} \llcorner B_{\lambda R} - S \llcorner B_R + T$$

is a cycle satisfying $\Theta_{p,r}(Z) \leq C'$ for all $r > a$ and for some constant $C' = C'(C, n)$. Proposition 4.5 (partial filling) shows that for every $\epsilon > 0$ there is an $a'_\epsilon = a'_\epsilon(X, C', a) \geq 0$ such that if $r > a'_\epsilon$, there exists $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ with $\operatorname{spt}(Z - \partial V) \cap B_r = \emptyset$ and $\mathbf{M}(V) < \epsilon r^{n+1}$. If we choose $R'$ sufficiently large, so that $\lambda R > r$, then $\operatorname{spt}(T) \cap B_r = \emptyset$ and $\operatorname{spt}(S_{p,\lambda} - S - \partial V) \cap B_r = \emptyset$. This shows that $F_{p,r}(S_{p,\lambda} - S) < \epsilon$ whenever $\epsilon > 0$ and $r > a'_\epsilon$.

Next, suppose that $0 < \lambda' < \lambda \leq 1$. Let $\epsilon > 0$ and $r > \lambda a'_\epsilon$. Since $\lambda'/\lambda < 1$ and $r/\lambda > a'_\epsilon$, the above result yields that $F_{p,r/\lambda}(S_{p,\lambda'/\lambda} - S) < \epsilon$. Since $h_{p,\lambda}$ is $\lambda$-Lipschitz, it follows that $F_{p,r}(S_{p,\lambda'} - S_{p,\lambda}) < \epsilon$.

We can now conclude the proof. Since $\Theta_{p,r}(S_{p,\lambda}) \leq C$ for $r > \lambda a$, Theorem 2.3 (compactness) shows that for some sequence $\lambda_i \downarrow 0$, the respective $S_{p,\lambda_i}$ converge in the local flat topology to a limit $S_{p,0} \in \mathbf{Z}_{n,\mathrm{loc}}(X)$. By Lemma 5.5 ($F$-convergence), $F_{p,r}(S_{p,0} - S_{p,\lambda_i}) \to 0$ for every fixed $r > 0$. Using the inequality

$$F_{p,r}(S_{p,0} - S_{p,\lambda}) \leq F_{p,r}(S_{p,0} - S_{p,\lambda_i}) + F_{p,r}(S_{p,\lambda_i} - S_{p,\lambda}),$$

we infer that $F_{p,r}(S_{p,0} - S_{p,\lambda}) \leq \epsilon$ whenever $\lambda \in (0, 1]$, $\epsilon > 0$, and $r > \lambda a'_\epsilon$. This shows at once that $F_\infty(S_{p,0} - S) = 0$ and that $S_{p,\lambda} \to S_{p,0}$ in the local flat topology, as $\lambda \to 0$. To see that $S_{p,0}$ is conical with respect to $p$, note that for any $\mu \in (0, 1)$, $h_{p,\mu\#}S_{p,0}$ is the weak limit of $h_{p,\mu\#}S_{p,\lambda} = S_{p,\mu\lambda}$ for $\lambda \to 0$, which is again $S_{p,0}$.

Next we show that $\Theta_\infty(S_{p,0}) \leq \Theta_\infty(S)$. For all pairs $s > r > 0$,

$$\|S_{p,0}\|(B_p(r)) \leq \liminf_{\lambda\to 0} \|S_{p,\lambda}\|(B_p(s)) \leq Cs^n$$

by the lower semicontinuity of mass with respect to weak convergence and since $\Theta_{p,s}(S_{p,\lambda}) \leq C$ for $s > \lambda a$. As $C > \Theta_\infty(S)$ was arbitrary, this gives the result. In particular $S_{p,0} \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$.

If $S' \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is another local cycle that is conical with respect to $p$ and $F$-asymptotic to $S$, then $S' \sim_F S_{p,0}$ and so $S' = S_{p,0}$ by Lemma 7.2.

By construction, $\mathrm{spt}(S_{p,\lambda}) \subset \mathrm{C}_p(\mathrm{spt}(S))$ for all $\lambda \in (0, 1)$. Therefore $\mathrm{spt}(S_{p,0}) \subset \overline{\mathrm{C}}_p(\mathrm{spt}(S))$ and thus

$$\Lambda(S_{p,0}) \subset \partial_\infty(\overline{\mathrm{C}}_p(\mathrm{spt}(S))) = \Lambda(S).$$

Hence, by Lemma 7.2, $\mathrm{spt}(S_{p,0}) \subset \mathrm{C}_p(\Lambda(S_{p,0})) \subset \mathrm{C}_p(\Lambda(S))$. □

A consequence of Theorem 7.3 (and Proposition 4.5) is the following uniform density bound.

**Proposition 7.4** (controlled density) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with* $\mathrm{asrk}(X) = n \geq 2$. *Then for all $C, \epsilon > 0$ there is a constant $a \geq 0$ such that every minimizing $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ with $\Theta_\infty(S) \leq C$ has $(C + \epsilon, a)$-controlled density.*

*Proof* Let $p \in X$. Since $F_\infty(S - S_{p,0}) = 0$ and $\Theta_{p,r}(S_{p,0}) \leq \Theta_\infty(S_{p,0}) \leq \Theta_\infty(S) \leq C$ for all $r > 0$, Proposition 4.5 (partial filling) shows that for every $\epsilon > 0$ there is an $a = a(X, C, \epsilon) \geq 0$ such that $\Theta_{p,r}(S) \leq C + \epsilon$ for all $r > a$. As $p$ was arbitrary, this yields the result. □

*Remark 7.5* When $X$ is a proper CAT(0) space, it follows more directly that every minimizing $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ has $(C, 0)$-controlled density for $C := \Theta_\infty(S)$, regardless of the asymptotic rank of $X$. In fact, for every fixed $p \in X$, the function $r \mapsto \Theta_{p,r}(S)$ is non-decreasing on $(0, \infty)$. This monotonicity property is shown by an argument very similar to the proof of Lemma 3.3 (density), using the sharp cone inequality $\mathbf{M}(T_s) \le (s/n)\,\mathbf{M}(R_s)$ instead of the Euclidean isoperimetric inequality $\mathbf{M}(T_s) \le \gamma\,\mathbf{M}(R_s)^{n/(n-1)}$ (compare Corollary 4.4 in [81]).

## 8 Visibility and applications

Theorem 7.3 (conical representative) shows in particular that for every $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ and $p \in X$ there is an $S_{p,0} \in [S]$ with support in the geodesic cone $\mathrm{C}_p(\Lambda(S))$. We now assume in addition that $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is quasi-minimizing. If both $S$ and $S_{p,0}$ had controlled density, we could conclude directly from Theorem 5.1 (Morse Lemma I) that the support of $S$ is within uniformly bounded distance from $\mathrm{spt}(S_{p,0})$ and hence from $\mathrm{C}_p(\Lambda(S))$. The following result, which subsumes Theorem 1.5, provides a sublinear bound for the general case. As indicated in the introduction, this may be viewed as an analog of the visibility axiom from [32].

**Theorem 8.1** (visibility property) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with $\mathrm{asrk}(X) = n \ge 2$. Then for all $Q \ge 1$, $C > 0$, $a \ge 0$, and $\epsilon > 0$ there exists $r_\epsilon > 0$ such that the following holds. Suppose that $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is $(Q, a)$-quasi-minimizing and satisfies $\Theta_{p,r}(S) \le C$ for some $p \in X$ and for all $r > a$. If $x \in \mathrm{spt}(S)$ is a point with $d(p, x) \ge r_\epsilon$, then*

*(1) for every $\lambda \in (0, 1)$ there is an $x_\lambda \in \mathrm{spt}(S)$ such that $d(x, h_{p,\lambda}(x_\lambda)) < \epsilon\, d(p, x)$;*
*(2) there exists a ray $\varrho \in \mathrm{R}_1^\sigma X$ with $\varrho(0) = p$ and $[\varrho] \in \Lambda(S)$ such that $d(x, \mathrm{im}(\varrho)) < \epsilon\, d(p, x)$.*

We prove (1) and (2) in a unified way by bounding the distance of $x$ from $\mathrm{spt}(S_{p,\lambda}) = \mathrm{spt}(h_{p,\lambda\#}S)$ for $\lambda \in (0, 1)$ and from $\mathrm{spt}(S_{p,0})$, respectively.

*Proof* Let $\lambda \in [0, 1)$. We know from Theorem 7.3 (conical representative) and its proof that $F_\infty(S - S_{p,\lambda}) = 0$ and $\Theta_{p,r}(S_{p,\lambda}) \le C$ for all $r > \lambda a$. In particular, $\Theta_{p,r}(S - S_{p,\lambda}) \le 2C$ for all $r > a$. Suppose now that $x \in \mathrm{spt}(S)$ and $s > 0$ are such that $B_x(s) \cap \mathrm{spt}(S_{p,\lambda}) = \emptyset$, and put $r_x := d(p, x)$. Proposition 4.5 (partial filling) shows that for every $\delta > 0$ there is a constant $a'_\delta = a'_\delta(X, C, a) \ge 0$ such that if $r_x + s > a'_\delta$, there exists $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ with $\mathrm{spt}(S - S_{p,\lambda} - \partial V) \cap B_p(r_x + s) = \emptyset$ and

$$\mathbf{M}(V) < \delta(r_x + s)^{n+1}.$$

Since $B_x(s)$ is disjoint from $\mathrm{spt}(S_{p,\lambda})$ and contained in $B_p(r_x + s)$, it follows that $\mathrm{spt}(S - \partial V) \cap B_x(s) = \emptyset$. Now Lemma 3.4 (filling density) shows that if $s > 4a$, then

$$\mathbf{M}(V) \geq c s^{n+1}$$

for some constant $c > 0$ depending only on $X$ and $Q$. Hence,

$$s < \left(c^{-1}\delta\right)^{1/(n+1)} (r_x + s)$$

whenever $r_x = d(p, x) > a'_\delta$ and $4a < s < d(x, \mathrm{spt}(S_{p,\lambda}))$. By choosing $\delta$ sufficiently small, in dependence of $n$, $c$, $a$ and $\epsilon > 0$, we infer that

$$d(x, \mathrm{spt}(S_{p,\lambda})) < \epsilon\, d(p, x)$$

for all $x \in \mathrm{spt}(S)$ with $d(p, x) > a'_\delta$.

From this, (1) and (2) follow easily. Note first that if $\lambda \in (0, 1)$, then $\mathrm{spt}(S_{p,\lambda}) = \mathrm{spt}(h_{p,\lambda\#} S) \subset h_{p,\lambda}(\mathrm{spt}(S))$; it thus follows that there is a point $x_\lambda \in \mathrm{spt}(S)$ such that $d(x, h_{p,\lambda}(x_\lambda)) < \epsilon\, d(p, x)$. Similarly, if $\lambda = 0$, then $\mathrm{spt}(S_{p,0}) \subset \mathrm{C}_p(\Lambda(S))$ by Theorem 7.3 (conical representative), thus there exists a ray $\varrho \in \mathrm{R}^\sigma_1 X$ emanating from $p$ such that $[\varrho] \in \Lambda(S)$ and $d(x, \mathrm{im}(\varrho)) < \epsilon\, d(p, x)$. □

As a by-product of this argument we obtain the following supplement to Theorem 7.3 (conical representative).

**Proposition 8.2** (equal limit sets) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with* $\mathrm{asrk}(X) = n \geq 2$. *If $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is quasi-minimizing, then $\Lambda(S_{p,0}) = \Lambda(S)$ for every $p \in X$.*

*Proof* Let $p \in X$. We already know that $\Lambda(S_{p,0}) \subset \Lambda(S)$. On the other hand, given $v \in \Lambda(S)$, it follows from the proof of Theorem 8.1 that there exist sequences of points $x_i \in \mathrm{spt}(S)$ and $y_i \in \mathrm{spt}(S_{p,0})$ such that $x_i \to v$ and $d(x_i, y_i) < (1/i)\, d(p, x_i)$. This implies that $y_i \to v$, thus $v \in \Lambda(S_{p,0})$. □

We now consider an asymptotic Plateau problem.

**Theorem 8.3** (minimizer with prescribed asymptotics) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with* $\mathrm{asrk}(X) = n \geq 2$. *Suppose that $S_0 \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is conical with respect to some point $p \in X$. Then there exists a minimizing $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ that is $F$-asymptotic to $S_0$; thus $S_{p,0} = S_0$. Every such $S$ satisfies $\Theta_\infty(S) = \Theta_\infty(S_0)$ and $\Lambda(S) = \Lambda(S_0)$. Furthermore, if $S' \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is another minimizer $F$-asymptotic to $S_0$, then $d_{\mathrm{H}}(\mathrm{spt}(S), \mathrm{spt}(S')) \leq b$ for some constant $b \geq 0$ depending only on $X$ and $\Theta_\infty(S_0)$.*

*Proof* By Theorem 5.6 (constructing minimizers) there exists a minimizing $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ with $F_\infty(S - S_0) = 0$, and every such $S$ satisfies $\Theta_\infty(S) \le \Theta_\infty(S_0)$. Then $S_{p,0} = S_0$ by the uniqueness assertion of Theorem 7.3 (conical representative), and since $\Theta_\infty(S_{p,0}) \le \Theta_\infty(S)$, it follows that $\Theta_\infty(S) = \Theta_\infty(S_0)$. By Proposition 8.2, $\Lambda(S) = \Lambda(S_{p,0}) = \Lambda(S_0)$.

If $S' \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is another minimizer $F$-asymptotic to $S_0$, then $F_\infty(S - S') = 0$, and by Proposition 7.4 (controlled density) $S - S'$ has $(2C, a)$-controlled density for some constants $C, a$ depending only on $X$ and $\Theta_\infty(S_0)$. Since $S - S'$ is $(1, 0)$-quasi-minimizing mod $\mathrm{spt}(S)$ as well as mod $\mathrm{spt}(S')$, it follows from Theorem 5.1 (Morse Lemma I) that $d_{\mathrm{H}}(\mathrm{spt}(S), \mathrm{spt}(S')) \le b$ for some constant $b$ as claimed. □

Proposition 8.2 and Theorem 8.3 show in particular that the following three classes of compact subsets of $\partial_\infty X$ agree.

**Definition 8.4** (canonical class of limit sets) Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with $\mathrm{asrk}(X) = n \ge 2$. We put

$$\begin{aligned}\mathscr{L}X &:= \{\Lambda(S) : S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X) \text{ is conical}\}\\ &= \{\Lambda(S) : S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X) \text{ is minimizing}\}\\ &= \{\Lambda(S) : S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X) \text{ is quasi-minimizing}\}.\end{aligned}$$

We now prove Theorem 1.7, reformulated for spaces with a convex bicombing.

**Theorem 8.5** (dense orbit) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with $\mathrm{asrk}(X) = n \ge 2$, and suppose that $\Gamma$ is a cocompact group of isometries of $X$. Then, for every non-empty set $\Lambda \in \mathscr{L}X$, the orbit of $\Lambda$ under the action of $\Gamma$, extended to $\overline{X} = X \cup \partial_\infty X$, is dense in $\partial_\infty X$ (with respect to the cone topology).*

*Proof* Suppose that $\Lambda = \Lambda(S)$, where $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is minimizing. By Proposition 7.4 (controlled density), $S$ has $(C, a)$-controlled density for some constants $C, a$ depending only on $X$ and $\Theta_\infty(S)$. Let $p \in X$, and let $\varrho_0 \in \mathrm{R}^\sigma_1 X$ be a ray emanating from $p$. Since $\Gamma$ acts cocompactly, there is a constant $b > 0$ such that for every $t \ge 0$ there exist an isometry $\gamma_t \in \Gamma$ and a point $x_t \in \gamma_t(\mathrm{spt}(S)) = \mathrm{spt}(\gamma_{t\#}S)$ such that $d(\varrho_0(t), x_t) \le b$. Note that $\gamma_{t\#}S$ is minimizing, and $\Theta_{p,r}(\gamma_{t\#}S) = \Theta_{\gamma_t^{-1}(p),r}(S) \le C$ for all $r > a$. Hence, given $\epsilon > 0$, if $t$ is sufficiently large, then by Theorem 8.1 there is a ray $\varrho \in \mathrm{R}^\sigma_1 X$ with $\varrho(0) = p$ such that $[\varrho] \in \Lambda(\gamma_{t\#}S)$ and $d(x_t, \mathrm{im}(\varrho)) < \epsilon\, d(p, x_t) \le \epsilon(t + b)$. Then

$$d(\varrho_0(t), \mathrm{im}(\varrho)) < b + \epsilon(t + b).$$

Note that $[\varrho] \in \partial_\infty(\gamma_t(\mathrm{spt}(S))) = \overline{\gamma}_t(\Lambda)$ for the extension $\overline{\gamma}_t$ of $\gamma_t$ to $\overline{X}$. Since $\epsilon > 0$ was arbitrary, this shows that every neighborhood of $[\varrho_0]$ in $\partial_\infty X$ contains a point of the orbit of $\Lambda$. □

Theorem 8.1 shows that the support of a quasi-minimizer $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ lies within sublinear distance from $\mathrm{C}_p(\Lambda(S))$, in terms of the distance to $p$. Next we show that, conversely, the entire cone $\overline{\mathrm{C}}_p(\mathrm{spt}(S))$ is within sublinear distance from $\mathrm{spt}(S)$; however, the estimate now depends on $S$ and $p$ rather than just on the data of $S$. The proof relies on Theorem 8.1 and a ball packing argument.

**Theorem 8.6** (asymptotic conicality) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with $\mathrm{asrk}(X) = n \geq 2$. Suppose that $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is quasi-minimizing, and $p \in X$. Then for all $\epsilon > 0$ there exists $r > 0$ such that*

$$d(y, \mathrm{spt}(S)) < \epsilon\, d(p, y)$$

*whenever $y \in \overline{\mathrm{C}}_p(\mathrm{spt}(S))$ and $d(p, y) \geq r$.*

*Proof* We consider the family of the compact sets

$$K_s := \mathrm{spt}(S) \cap B_p(s)$$

for $s > 0$. Let $\mu > 0$. It follows from Theorem 8.1 that there exists an $r > 0$ such that for all $s \geq r$, $x \in K_s$, and $\lambda \in (0, 1]$, there is a point $x' \in \mathrm{spt}(S)$ such that

$$d(x, h_{p,\lambda}(x')) < \mu s.$$

Then $\lambda\, d(p, x') = d(p, h_{p,\lambda}(x')) \leq d(p, x) + \mu s \leq (1 + \mu)s$. Hence, given any $t \geq s$, by choosing $\lambda := \min\{1, (1 + \mu)s/t\}$ and $x' := x$ in the case that $\lambda = 1$, we get that $d(p, x') \leq t$. Then

$$d(h_{p,\lambda}(x'), h_{p,s/t}(x')) = \left(\lambda - \frac{s}{t}\right) d(p, x') \leq \lambda t - s \leq \mu s.$$

We conclude that for every $x \in K_s$ and every $t \geq s$ there exists an $x' \in K_t$ such that $d(x, h_{p,s/t}(x')) < 2\mu s$. Furthermore, if $(y, y') \in K_s \times K_t$ is another such pair with $d(y, h_{p,s/t}(y')) < 2\mu s$, then

$$\frac{1}{t}\, d(x', y') \geq \frac{1}{s}\, d(h_{p,s/t}(x'), h_{p,s/t}(y')) > \frac{1}{s}\, d(x, y) - 4\mu$$

by convexity.

Let now $\epsilon > 0$. For $\delta > 0$, denote by $\mathcal{N}_{\delta,s}$ the maximal possible cardinality of a $\delta s$-separated set $N \subset K_s$ (see Sect. 2.1). By the assumptions on $S$ and Lemma 3.3 (density), $\bar{\mathcal{N}}_\delta := \limsup_{s\to\infty} \mathcal{N}_{\delta,s}$ is finite. Using the monotonicity of $\bar{\mathcal{N}}_\delta \in \mathbb{Z}$ in $\delta$ we now fix $\delta$, $\mu > 0$ such that $\bar{\mathcal{N}}_{\delta+8\mu} = \bar{\mathcal{N}}_\delta$ and $\delta + 2\mu \le \epsilon$. Then we choose $r > 0$ so large that the result of the first part of the proof holds, $\mathcal{N}_{\delta+8\mu,r} = \bar{\mathcal{N}}_{\delta+8\mu}$, and $\mathcal{N}_{\delta,t} \le \bar{\mathcal{N}}_\delta$ for all $t \ge r$. Let $N_r \subset K_r$ be a $(\delta+8\mu)r$-separated set with maximal cardinality $|N_r| = \mathcal{N}_{\delta+8\mu,r} = \bar{\mathcal{N}}_{\delta+8\mu} = \bar{\mathcal{N}}_\delta$. For all $t \ge s \ge r$, it follows from the first part of the proof that there exists a bijection $f$ from $N_r$ to a $(\delta+4\mu)s$-separated set $N_s \subset K_s$ as well as a bijection $g$ from $N_s$ to a $\delta t$-separated set $N_t \subset K_t$ such that $d(x, h_{p,s/t}(x')) < 2\mu s$ for all $x \in N_s$ and $x' := g(x) \in N_t$. Now $|N_t| \le \mathcal{N}_{\delta,t} \le \bar{\mathcal{N}}_\delta = |N_r| = |N_t|$, thus $N_t$ is in fact maximal and forms a $\delta t$-net in $K_t$.

Finally, suppose that $y$ is a point in $\mathrm{C}_p(\mathrm{spt}(S))$ with $s := d(p, y) \ge r$. Then $y = h_{p,s/t}(y')$ for some $y' \in \mathrm{spt}(S)$, where $t := d(p, y') \ge s$. As we have just shown, there exist $x \in K_s$ and $x' \in K_t$ such that $d(y', x') \le \delta t$ and $d(h_{p,s/t}(x'), x) < 2\mu s$, thus $d(y, h_{p,s/t}(x')) \le (s/t)\, d(y', x') \le \delta s$ and

$$d(y, x) \le d(y, h_{p,s/t}(x')) + d(h_{p,s/t}(x'), x) < (\delta + 2\mu)s \le \epsilon s.$$

Hence, $d(y, \mathrm{spt}(S)) < \epsilon\, d(p, y)$. This yields the result. □

We now turn to the Tits geometry. As a first application of Theorem 8.6 we will show that the limit sets $\Lambda \in \mathscr{L}X$ are compact with respect to the Tits topology.

For a proper metric space $X$ with a convex bicombing $\sigma$, the *Tits cone* of $(X, \sigma)$ is defined as the set

$$\mathscr{C}_\mathrm{T} X := \mathrm{R}^\sigma X / \sim$$

(see Sect. 7), equipped with the metric given by

$$d_\mathrm{T}([\varrho], [\varrho']) := \lim_{t\to\infty} \frac{1}{t}\, d(\varrho(t), \varrho'(t)).$$

Note that $t \mapsto d(\varrho(t), \varrho'(t))$ is convex, thus $t \mapsto d(\varrho(t), \varrho'(t))/t$ is non-decreasing if $\varrho, \varrho'$ are chosen such that $\varrho(0) = \varrho'(0)$. From this it is easily seen that $\mathscr{C}_\mathrm{T}X$ is complete. On $\mathscr{C}_\mathrm{T}X$, multiplication by a scalar $\lambda \in \mathbb{R}_+$ is defined by $\lambda[\varrho(\,\cdot\,)] := [\varrho(\lambda\,\cdot\,)]$. This yields a homothety

$$h_\lambda \colon \mathscr{C}_\mathrm{T}X \to \mathscr{C}_\mathrm{T}X,$$

thus $h_\lambda(v) = \lambda v$ and $d_\mathrm{T}(h_\lambda(v), h_\lambda(v')) = \lambda\, d_\mathrm{T}(v, v')$. The *cone vertex* $o$ of $\mathscr{C}_\mathrm{T}X$ is the class of the constant rays. For every base point $p \in X$ there exists

a canonical 1-Lipschitz map

$$\mathrm{can}_p \colon \mathscr{C}_{\mathrm{T}} X \to X$$

such that $\mathrm{can}_p([\varrho]) = \varrho(1)$ for all $\varrho \in \mathrm{R}^\sigma X$ with $\varrho(0) = p$. The *Tits boundary* of $(X, \sigma)$ is the unit sphere

$$\partial_{\mathrm{T}} X := S_o(1) = \mathrm{R}^\sigma_1 X / \sim$$

in $\mathscr{C}_{\mathrm{T}} X$, endowed with the topology induced by $d_{\mathrm{T}}$. This topology is finer than the cone topology on the visual boundary $\partial_\infty X$, which agrees with $\partial_{\mathrm{T}} X$ as a set. However, the following holds.

**Proposition 8.7** (compact limit sets) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with $\mathrm{asrk}(X) = n \geq 2$. Then every $\Lambda \in \mathscr{L} X$ is still compact when viewed as a subset of $\partial_{\mathrm{T}} X$.*

*Proof* Suppose that $\Lambda = \Lambda(S)$, where $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is quasi-minimizing. Fix $p \in X$, and let $\epsilon > 0$. Let $N \subset \Lambda$ be a finite $3\epsilon$-separated set; thus $d_{\mathrm{T}}(u, u') > 3\epsilon$ for distinct $u, u' \in N$. For $r > 0$ sufficiently large, $\mathrm{can}_p(rN)$ is $3\epsilon r$-separated, and every point in this set is at distance less than $\epsilon r$ from $\mathrm{spt}(S)$ by Theorem 8.6. This yields an $\epsilon r$-separated subset of $\mathrm{spt}(S) \cap B_p(r + \epsilon r)$ of the same cardinality as $N$. For $r$ sufficiently large, it then follows from Lemma 3.3 (density) that the cardinality of such sets is bounded from above by a constant depending on $\epsilon$ but not on $r$. We conclude that $\Lambda$ is totally bounded. Since $\mathscr{C}_{\mathrm{T}} X$ is complete and $\Lambda$ is closed in the Tits topology, this gives the result. □

## 9 Cycles at infinity

In this section we show that if $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is conical with respect to $p \in X$, then the cone $\mathbb{R}_+\Lambda \subset \mathscr{C}_{\mathrm{T}} X$ over the limit set $\Lambda = \Lambda(S) \in \mathscr{L} X$ is the support of a unique local $n$-cycle $\Sigma$ in $\mathscr{C}_{\mathrm{T}} X$ satisfying $\mathrm{can}_{p\#}\, \Sigma = S$. We then complete the proofs of Theorems 1.4 and 1.6.

In general, Tits cones are not locally compact, therefore the theory of local currents from [59], which depends on the supply of compactly supported Lipschitz functions, is not directly applicable to $\mathscr{C}_{\mathrm{T}} X$. However, by Proposition 8.7 above, $\mathbb{R}_+\Lambda$ is proper, and $\Sigma$ will be constructed as a current in its own support $\mathrm{spt}(\Sigma) = \mathbb{R}_+\Lambda$. Thus, we (re-)define $\mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}} X)$ as the collection of all local cycles $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(K_\Sigma)$ such that $K_\Sigma \subset \mathscr{C}_{\mathrm{T}} X$ is proper and $\mathrm{spt}(\Sigma) = K_\Sigma$ (compare the discussion after Proposition 3.3 in [59]). The sum of two elements $\Sigma, \Sigma' \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}} X)$ may be formed by viewing them temporarily as currents in $K_\Sigma \cup K_{\Sigma'}$; thus $\mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}} X)$ is an abelian group. The complexes $\mathbf{I}_{*,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}} X)$, $\mathbf{I}_{*,\mathrm{c}}(\mathscr{C}_{\mathrm{T}} X)$ and $\mathbf{I}_{*,\mathrm{c}}(\partial_{\mathrm{T}} X)$ are understood similarly.

We start with a basic fact.

**Lemma 9.1** (uniform convergence) *Let $X$ be a proper metric space with a convex bicombing $\sigma$. Suppose that $K$ is a compact subset of $\mathscr{C}_{\mathrm{T}}X$, and $p \in X$. Then for every $\epsilon > 0$ there is an $r_\epsilon > 0$ such that*

$$d_{\mathrm{T}}(u, v) - \epsilon \le r^{-1} d(\mathrm{can}_p(ru), \mathrm{can}_p(rv)) \le d_{\mathrm{T}}(u, v)$$

*for all $r \ge r_\epsilon$ and $u, v \in K$. In particular, $K$ is an asymptotic subset of $X$ as defined in Definition 4.2.*

*Proof* For every $r > 0$, the map $u \mapsto \mathrm{can}_p(ru)$ is $r$-Lipschitz on $\mathscr{C}_{\mathrm{T}}X$. It follows that the function $\varrho_r \colon (u, v) \mapsto r^{-1} d(\mathrm{can}_p(ru), \mathrm{can}_p(rv))$ is 1-Lipschitz with respect to the $l_1$ product metric on $\mathscr{C}_{\mathrm{T}}X \times \mathscr{C}_{\mathrm{T}}X$. Moreover, as $r \to \infty$, $\varrho_r \to d_{\mathrm{T}}$ pointwise on $\mathscr{C}_{\mathrm{T}}X \times \mathscr{C}_{\mathrm{T}}X$ by the definition of $d_{\mathrm{T}}$. Hence the convergence is uniform on $K \times K$ for every compact set $K \subset \mathscr{C}_{\mathrm{T}}X$.

In particular, the rescaled sets $(\mathrm{can}_p(rK), r^{-1}d)$ converge in the Gromov–Hausdorff topology to $K$. □

*Remark 9.2* It follows from Lemma 9.1 and Remark 4.3 that if $\mathrm{asrk}(X) = n$ and $m > n$, then $\mathscr{C}_{\mathrm{T}}X$ contains no set bi-Lipschitz homeomorphic to a compact subset of $\mathbb{R}^m$ with positive Lebesgue measure, and this implies that $\mathbf{I}_{m,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}}X) = \{0\}$ and $\mathbf{I}_{m-1,\mathrm{c}}(\partial_{\mathrm{T}}X) = \{0\}$.

As a consequence, if $\mathrm{asrk}(X) = n$, then *every* local cycle $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}}X)$ is conical with respect to the cone vertex $o$, that is, $h_{\lambda\#}\Sigma = \Sigma$ for all $\lambda > 0$. To see this, consider the radial homotopy $H \colon (t, v) \mapsto (1 - t + \lambda t)v$ of $\mathscr{C}_{\mathrm{T}}X$; then $h_{\lambda\#}\Sigma - \Sigma$ equals the boundary of $H_\#(\llbracket 0, 1 \rrbracket \times \Sigma) = 0 \in \mathbf{I}_{n+1,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}}X)$.

We now prove the following general result, which is independent of the asymptotic rank. However, the assumption $\mathrm{asrk}(X) = n$ will guarantee that $\Lambda(S) \subset \partial_{\mathrm{T}}X$ is compact.

**Theorem 9.3** (lifting cones) *Let $X$ be a proper metric space with a convex bicombing $\sigma$. Suppose that $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is conical with respect to some point $p \in X$, and $\Lambda := \Lambda(S)$ is compact in the Tits topology. Then there is a unique local cycle $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}}X)$ such that $\mathrm{can}_{p\#}\Sigma = S$. Moreover, $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}}X)$ is conical with respect to $o$, $\mathbf{M}(\Sigma \llcorner B_o(1)) = \Theta_\infty(S)$, $\mathrm{spt}(\Sigma) = \mathbb{R}_+\Lambda$, and $\mathrm{spt}(\partial(\Sigma \llcorner B_o(1))) = \Lambda$.*

Note that since $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_{\mathrm{T}}X)$ is conical, $\Sigma \llcorner B_o(\lambda) \in \mathbf{I}_{n,\mathrm{c}}(\mathscr{C}_{\mathrm{T}}X)$ for all $\lambda > 0$ (compare Lemma 7.2 (conical)).

To construct $\Sigma$ we will consider the family of all $S_r := S \llcorner B_p(r) \in \mathbf{I}_{n,\mathrm{c}}(X)$ for $r > 0$. First we embed each $S_r$ by a map that dilates all distances by the factor $1/r$ into a fixed compact metric space $Y$. The embedded family converges, as $r \to \infty$, to an integral current in $Y$ with support in an isometric

copy of the cone $K := [0, 1]\Lambda \subset \mathscr{C}_{\mathrm{T}}X$, and this yields $\Sigma \llcorner B_o(1)$. As regards $Y$, we will use the following general fact. Given any compact metric space $(K, d_K)$ with diameter $D$, the set $Y$ of all 1-Lipschitz functions $y\colon K \to [0, D]$, endowed with the metric defined by

$$d_Y(y, y') := \sup_{v\in K} |y(v) - y'(v)|,$$

is a compact convex subspace of $l_\infty(K)$, and the map $u \mapsto d_K(u, \cdot)$ is an isometric embedding of $K$ into $Y$. Furthermore, $Y$ is an injective metric space; that is, every 1-Lipschitz map $\varrho\colon A \to Y$ defined on a subset $A$ of a metric space $B$ extends to a 1-Lipschitz map $\bar{\varrho}\colon B \to Y$. In fact, such an extension is given by

$$\bar{\varrho}(b)(v) := \sup_{a\in A} \max\{\varrho(a)(v) - d(a, b), 0\}$$

for all $b \in B$ and $v \in K$.

*Proof* For $s > r > 0$, we put $\pi_r := \mathrm{can}_p \circ h_r\colon \mathscr{C}_{\mathrm{T}}X \to X$ and $\pi_{s,r} := h_{p,r/s}\colon X \to X$. Note that $\pi_r$ is $r$-Lipschitz, $\pi_{s,r}$ is $(r/s)$-Lipschitz, and $\pi_r = \pi_{s,r} \circ \pi_s$.

Let first $K \subset \mathscr{C}_{\mathrm{T}}X$ be an arbitrary compact set, and put $K_r := \pi_r(K)$. Let $(Y, d_Y)$ be the compact convex subspace of $l_\infty(K)$ as described before the proof, and let

$$f\colon K \to Y, \quad f(u) := d_{\mathrm{T}}(u, \cdot),$$

denote the canonical isometric embedding of $K$ into $Y$. Similarly, since $\pi_r$ is $r$-Lipschitz, there is a map

$$f_r\colon K_r \to Y, \quad f_r(x) := r^{-1}d(x, \pi_r(\cdot)),$$

and since $\pi_r$ maps $K$ onto $K_r$, it follows that

$$d_Y(f_r(x), f_r(x')) = r^{-1} \sup_{v\in K} \big|d(x, \pi_r(v)) - d(x', \pi_r(v))\big| = r^{-1}d(x, x')$$

for all $x, x' \in K_r$. Note also that $f_r(p) = f(o) =: y_0 \in Y$.

Let $\epsilon > 0$. By Lemma 9.1 there is an $r_\epsilon > 0$ such that if $s > r \geq r_\epsilon$, then $s^{-1}d(\pi_s(u), \pi_s(v)) \leq d_{\mathrm{T}}(u, v) \leq r^{-1}d(\pi_r(u), \pi_r(v)) + \epsilon$ for all $u, v \in K$. We infer that

$$\begin{aligned} &d_Y(f_s(\pi_s(u)), f_r(\pi_r(u))) \\ &\quad = \sup_{v\in K} \big|s^{-1}d(\pi_s(u), \pi_s(v)) - r^{-1}d(\pi_r(u), \pi_r(v))\big| \leq \epsilon \end{aligned}$$

for all $u \in K$ and hence

$$d_Y(f_s(x), (f_r \circ \pi_{s,r})(x)) \le \epsilon$$

for all $x \in K_s$. Similarly,

$$d_Y(f(u), (f_r \circ \pi_r)(u)) \le \epsilon$$

for all $u \in K$. Thus $f_r(K_r)$ lies within distance $\epsilon$ of $f(K)$.

We now apply this construction for the cone $K := [0,1]\Lambda \subset \mathscr{C}_{\mathrm{T}}X$. Let $C := \Theta_\infty(S)$. By Lemma 7.2 (conical), for all $s > r > 0$, $S_r := S \llcorner B_p(r) \in \mathbf{I}_{n,\mathrm{c}}(X)$, $\pi_{s,r\#}S_s = S_r$, $\mathbf{M}(S_r) \le Cr^n$, and $\mathrm{spt}(S) \subset \mathrm{C}_p(\Lambda)$; thus $\mathrm{spt}(S_r) \subset K_r = \pi_r(K)$. Since $\pi_{s,r}$ is $r/s$-Lipschitz and $\pi_{s,r\#}(\partial S_s) = \partial S_r$, it follows that $s^{n-1}\mathbf{M}(\partial S_r) \le r^{n-1}\mathbf{M}(\partial S_s)$, and integration over $s$ yields

$$\frac{R^n - r^n}{n}\mathbf{M}(\partial S_r) \le r^{n-1}\mathbf{M}(S_R) \le Cr^{n-1}R^n$$

for all $R > r$; thus $\mathbf{M}(\partial S_r) \le nCr^{n-1}$. Since $f_r\colon K_r \to Y$ is $(1/r)$-Lipschitz, we get the uniform bounds

$$\mathbf{M}(f_{r\#}S_r) \le C, \quad \mathbf{M}(\partial(f_{r\#}S_r)) = \mathbf{M}(f_{r\#}(\partial S_r)) \le nC.$$

For $\epsilon > 0$ and $s > r \ge r_\epsilon$, let $H\colon [0,1] \times K_s \to Y$ be the affine homotopy from $f_s$ to $f_r \circ \pi_{s,r}$ in $Y \subset l_\infty(K)$. Then $H(t,\cdot)$ is $(1/s)$-Lipschitz for every $t \in [0,1]$, and $H(\cdot, x)$ is a segment of length at most $\epsilon$ for every $x \in K_s$. It follows that the family $(f_{r\#}S_r)_{r>0}$ is Cauchy with respect to the flat distance $\mathscr{F}$ on $\mathbf{I}_{n,\mathrm{c}}(Y)$ (see Sects. 2.7 and 2.8), and by Theorem 2.3 (compactness) there exists a current $\bar{\Sigma}_1 \in \mathbf{I}_{n,\mathrm{c}}(Y)$ such that

$$\lim_{r\to\infty} \mathscr{F}(f_{r\#}S_r - \bar{\Sigma}_1) = 0.$$

Note that $\mathbf{M}(\bar{\Sigma}_1) \le C$ and $\mathrm{spt}(\bar{\Sigma}_1) \subset f(K)$, furthermore $\mathrm{spt}(\partial\bar{\Sigma}_1) \subset S_{y_0}(1)$ because $\mathrm{spt}(\partial(f_{r\#}S_r)) \subset f_r(\mathrm{spt}(\partial S_r)) \subset S_{y_0}(1)$ for all $r > 0$. Via the isometric embedding $f^{-1}\colon f(K) \to \mathscr{C}_{\mathrm{T}}X$ we get a current $\Sigma_1 := (f^{-1})_\#\bar{\Sigma}_1 \in \mathbf{I}_{n,\mathrm{c}}(\mathscr{C}_{\mathrm{T}}X)$ with $\mathrm{spt}(\Sigma_1) \subset K$, $\mathrm{spt}(\partial\Sigma_1) \subset \Lambda \subset S_o(1)$, and $\mathbf{M}(\Sigma_1) \le C = \Theta_\infty(S)$.

Next we show that for each $r > 0$, $\pi_{r\#}\Sigma_1 = S_r$. We know that if $\epsilon > 0$ and $s \ge r_\epsilon$, then $d_Y(f(u), (f_s \circ \pi_s)(u)) \le \epsilon$ for all $u \in K$. Since $f_\#\Sigma_1 = \bar{\Sigma}_1$, this yields

$$\lim_{s\to\infty} \mathscr{F}(f_{s\#}(\pi_{s\#}\Sigma_1) - \bar{\Sigma}_1) = 0.$$

Putting $T_s := f_{s\#}(\pi_{s\#}\Sigma_1 - S_s)$, we get that $\lim_{s\to\infty} \mathscr{F}(T_s) = 0$. For every $s > r$, the 1-Lipschitz map $\varrho_s := f_r \circ \pi_{s,r} \circ f_s^{-1} \colon f_s(K_s) \to f_r(K_r)$ satisfies $\varrho_{s\#} T_s = T_r$ and possesses a 1-Lipschitz extension $\bar{\varrho}_s \colon Y \to Y$. It follows that $\mathscr{F}(T_r) \le \mathscr{F}(T_s)$ for all $s > r$, thus $\mathscr{F}(T_r) = 0$ and therefore $T_r = 0$. Hence, $\pi_{r\#}\Sigma_1 - S_r = (f_r^{-1})_\# T_r = 0$, as claimed.

As a consequence, $\mathbf{M}(S_r) \le r^n \mathbf{M}(\Sigma_1)$ and $\operatorname{spt}(\partial S_r) \subset \pi_r(\operatorname{spt}(\partial \Sigma_1))$ for all $r > 0$, thus $\Theta_\infty(S) \le \mathbf{M}(\Sigma_1)$ and $\Lambda \subset \operatorname{spt}(\partial \Sigma_1)$. Hence, in view of the relations shown above, $\mathbf{M}(\Sigma_1) = \Theta_\infty(S)$ and $\operatorname{spt}(\partial \Sigma_1) = \Lambda$.

Finally, consider the family $\{\Sigma_\lambda\}_{\lambda>0}$ in $\mathbf{I}_{n,\mathrm{c}}(\mathscr{C}_\mathrm{T} X)$ such that $\Sigma_\lambda = h_{\lambda\#}\Sigma_1$ for every $\lambda > 0$. Then $\pi_{r\#}\Sigma_\lambda = S_{\lambda r}$ for all $r > 0$, and we claim that $\Sigma_\lambda$ is the unique element of $\mathbf{I}_{n,\mathrm{c}}(\mathscr{C}_\mathrm{T} X)$ with this property. Let $\Sigma'$ be any non-zero element of $\mathbf{I}_{n,\mathrm{c}}(\mathscr{C}_\mathrm{T} X)$. It suffices to show that $\pi_{r\#}\Sigma' \ne 0$ for some $r > 0$. Put $K := \operatorname{spt}(\Sigma')$, $K_r := \pi_r(K)$, and define $Y$, $f$, and $f_r$ as above. Then it follows that $\lim_{r\to\infty} \mathscr{F}(f_{r\#}(\pi_{r\#}\Sigma') - f_\#\Sigma') = 0$. Since $f_\#\Sigma' \ne 0$, this implies the claim. Now if $0 < \lambda < \lambda'$, then $B_o(\lambda) = \pi_r^{-1}(B_p(\lambda r))$ and hence

$$\pi_{r\#}(\Sigma_{\lambda'} \llcorner B_o(\lambda)) = (\pi_{r\#}\Sigma_{\lambda'}) \llcorner B_p(\lambda r) = S_{\lambda' r} \llcorner B_p(\lambda r) = S_{\lambda r}$$

for all $r > 0$; therefore $\Sigma_{\lambda'} \llcorner B_o(\lambda) = \Sigma_\lambda$ by uniqueness. It follows that the family $\{\Sigma_\lambda\}_{\lambda>0}$ determines a local cycle $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_\mathrm{T} X)$ such that $\Sigma \llcorner B_o(\lambda) = \Sigma_\lambda$ for all $\lambda > 0$, and it is easily verified that $\Sigma$ has the desired properties. Note that $\lambda\Lambda = \operatorname{spt}(\partial\Sigma_\lambda) \subset \operatorname{spt}(\Sigma) \subset \mathbb{R}_+\Lambda$ for all $\lambda > 0$, thus $\operatorname{spt}(\Sigma) = \mathbb{R}_+\Lambda$. □

From Theorem 9.3 we obtain the following result which, in conjunction with Theorem 7.3 (conical representative) and Proposition 8.2 (equal limit sets), establishes Theorem 1.4 stated in the introduction.

**Theorem 9.4** (Tits boundary) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with $\operatorname{asrk}(X) = n \ge 2$. Then for every $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ there exists a unique local cycle $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_\mathrm{T} X)$ such that $\operatorname{can}_{p\#}\Sigma = S_{p,0}$ for all $p \in X$; furthermore $\Sigma$ is conical with respect to $o$, and the slice $\partial(\Sigma \llcorner B_o(1))$ defines an element $\partial_\mathrm{T} S = \partial_\mathrm{T}[S] \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X)$ with $\operatorname{spt}(\partial_\mathrm{T} S) = \Lambda(S_{p,0})$ for all $p \in X$. This yields an isomorphism*

$$\partial_\mathrm{T} \colon \mathscr{Z} X \to \mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X).$$

*Proof* Let $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$, and let $p, p' \in X$. By Theorem 7.3, $S_{p,0}$ and $S_{p',0}$ are the unique representatives of $[S]$ that are conical with respect to $p$ and $p'$, respectively. Theorem 9.3 together with Proposition 8.7 (compact limit sets) then shows that there exist unique elements $\Sigma, \Sigma' \in \mathbf{Z}_{n,\mathrm{loc}}(\mathscr{C}_\mathrm{T} X)$ such that $\operatorname{can}_{p\#}\Sigma = S_{p,0}$ and $\operatorname{can}_{p'\#}\Sigma' = S_{p',0}$; furthermore $\Sigma$, $\Sigma'$ are conical with respect to $o$, and $\operatorname{spt}(\partial(\Sigma \llcorner B_o(1))) = \Lambda(S_{p,0})$. Now $\operatorname{can}_{p\#}\Sigma' \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ is

conical with respect to $p$, and using the $\sigma$-homotopy $H\colon [0,1]\times \mathscr{C}_{\mathrm{T}}X \to X$ from $\mathrm{can}_p$ to $\mathrm{can}_{p'}$ one can easily check that $\mathrm{can}_{p\#}\,\Sigma'$ is $F$-asymptotic to $S_{p',0}$ and hence to $S$. It follows from the above uniqueness assertions that $\mathrm{can}_{p\#}\,\Sigma' = S_{p,0}$ and $\Sigma' = \Sigma$. This shows that $\mathrm{can}_{p'\#}\,\Sigma = S_{p',0}$ for all $p' \in X$. In particular, $\Sigma$ depends only on $[S]$. Viewing $\partial(\Sigma \llcorner B_o(1)) \in \mathbf{Z}_{n-1,\mathrm{c}}(\mathscr{C}_{\mathrm{T}}X)$ as an element $\partial_{\mathrm{T}}S = \partial_{\mathrm{T}}[S] \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}}X)$, we get a map $\partial_{\mathrm{T}}\colon \mathscr{Z}X \to \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}}X)$, and it is easily verified that this is an isomorphism. □

Returning to the asymptotic Plateau problem, we may now reformulate Theorem 8.3 as follows.

**Theorem 9.5** (minimizer with prescribed Tits data) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with* $\mathrm{asrk}(X) = n \geq 2$. *Then for every cycle $R \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}}X)$ there exists an area-minimizing local cycle $S \in \mathbf{Z}^{\infty}_{n,\mathrm{loc}}(X)$ with $\partial_{\mathrm{T}}S = R$. Every such $S$ satisfies $\Lambda(S) = \mathrm{spt}(R)$ and $\mathbf{M}(R)/n \leq \Theta_\infty(S) \leq \mathbf{M}(R)$.*

*Proof* Let $R \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}}X)$. By Theorems 9.4 and 7.3 there is a conical local cycle $S_0 \in \mathbf{Z}^{\infty}_{n,\mathrm{loc}}(X)$ with Tits boundary $\partial_{\mathrm{T}}S_0 = R$, and $\Lambda(S_0) = \mathrm{spt}(R)$. By Theorem 8.3 there exists a minimizing $S \in [S_0]$, and every such $S$ satisfies $\Lambda(S) = \Lambda(S_0) = \mathrm{spt}(R)$ and $\Theta_\infty(S) = \Theta_\infty(S_0)$. Note that $S \in [S_0]$ if and only if $\partial_{\mathrm{T}}S = R$. By Theorem 9.3 and the coarea inequality,

$$\Theta_\infty(S_0) = \mathbf{M}(\Sigma \llcorner B_o(1)) \geq \int_0^1 \lambda^{n-1}\mathbf{M}(R)\,d\lambda = \mathbf{M}(R)/n,$$

and since $\Sigma \llcorner B_o(1)$ agrees with the cone over $R$, $\mathbf{M}(\Sigma \llcorner B_o(1)) \leq \mathbf{M}(R)$. □

When $X$ is a CAT(0) space, the last inequality holds with $\mathbf{M}(R)/n$ in place of $\mathbf{M}(R)$; then $\Theta_\infty(S) = \mathbf{M}(R)/n$ for every minimizing $S \in \mathbf{Z}^{\infty}_{n,\mathrm{loc}}(X)$ with $\partial_{\mathrm{T}}S = R$. Furthermore, $\Theta_{p,r}(S) \leq \Theta_\infty(S)$ for all $p \in X$ and $r > 0$ by monotonicity (see Remark 7.5), and $\lim_{r\to 0}\Theta_{p,r}(S) \geq \omega_n$ for $\|S\|$-almost every $p$ (see [81], (4.28)); thus $\Theta_\infty(S) \geq \omega_n$ whenever $R \neq 0$. This proves Theorem 1.6.

*Remark 9.6* By the above results, we may rephrase Theorem 6.3 (visual metrics) in terms of cycles at infinity. For a reference point $p \in X$ and $[S] \in \mathscr{Z}X$, we put $\langle \partial_{\mathrm{T}}[S]\rangle_p := \langle [S]\rangle_p$, thus

$$\langle R\rangle_p = \inf\{d(p, \mathrm{spt}(\tilde{S})) : \tilde{S} \in \mathbf{Z}^{\infty}_{n,\mathrm{loc}}(X) \text{ is minimizing}, \partial_{\mathrm{T}}\tilde{S} = R\}$$

for all $R \in \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}}X)$. Let $C > 0$ and $a \geq 0$. Then, for every sufficiently small $b > 1$, there exist a constant $c \geq 1$ and a metric $\nu$ on $\partial_{\mathrm{T}}(\mathscr{Z}_{C,a}X) \subset \mathbf{Z}_{n-1,\mathrm{c}}(\partial_{\mathrm{T}}X)$ satisfying

$$c^{-1}b^{-\langle R-R'\rangle_p} \leq \nu(R,R') \leq c\,b^{-\langle R-R'\rangle_p}$$

for all $R, R' \in \partial_T(\mathscr{Z}_{C,a}X)$; furthermore $\partial_T(\mathscr{Z}_{C,a}X)$ is compact with respect to any such metric. Note that if $[S] \in \mathscr{Z}_{C,a}X$ and $\tilde{S} \in [S]$ is minimizing, then $\Theta_\infty(\tilde{S}) \le C$ by Theorem 5.6 (constructing minimizers) and thus

$$\partial_T(\mathscr{Z}_{C,a}X) \subset \{R \in \mathbf{Z}_{n-1,c}(\partial_T X) : \mathbf{M}(R) \le nC\}$$

by Theorem 9.5. When $X$ is CAT(0), these two sets agree for each $a \ge 0$.

## 10 Quasi-isometries

We now turn to quasi-isometric embeddings of $X$ into another proper metric space $\bar{X}$ with a convex bicombing.

The following auxiliary result will be used in conjunction with Lemma 3.5 (doubling).

**Proposition 10.1** (Lipschitz extension) *Suppose that $X$ is a metric space, $\bar{X}$ is a metric space with a convex bicombing $\bar{\sigma}$, and $A \subset X$ is a non-empty closed set that is doubling. Then there is a constant $\mu \ge 1$, depending only on the doubling constant, such that for every $L$-Lipschitz map $f : A \to \bar{X}$ there is a $\mu L$-Lipschitz map $g : X \to \bar{X}$ with $g|_A = f$.*

This follows from Theorem 1.6 in [62] since $\bar{X}$ is Lipschitz $k$-connected for all $k \ge 0$ and doubling sets have finite Nagata dimension (in fact, according to Theorem 1.1 in [64], the latter is less than or equal to the Assouad dimension).

*Remark 10.2* The assumption in Proposition 10.1 that $A$ be doubling can be dropped if, for example, $\bar{X}$ is a homogeneous Hadamard manifold or a Euclidean building; the constant $\mu$ then depends (only) on $\bar{X}$. See Theorem 1.2 in [61]. It is still unknown whether every Hadamard manifold has this property.

By virtue of Lemma 3.5 and Proposition 10.1, given a quasi-isometric embedding $f : X \to \bar{X}$ and a quasi-minimizer $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ with controlled density, one can easily produce a Lipschitz map $g : X \to \bar{X}$ with $\sup_{x \in \mathrm{spt}(S)} d(f(x), g(x)) < \infty$ by extending $f|_A$ for a suitable separated net $A$ in $\mathrm{spt}(S)$. We now show that then $g_\# S \in \mathbf{Z}_{n,\mathrm{loc}}(\bar{X})$ is again a quasi-minimizer with controlled density.

**Proposition 10.3** (quasi-isometry invariance) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with $\mathrm{asrk}(X) = n \ge 2$. Then for all $L, Q \ge 1$, $C > 0$, and $a \ge 0$ there exist $\bar{Q} \ge 1$, $\bar{C} > 0$, and $\bar{a} \ge 0$ such that the following holds. Suppose that $\bar{X}$ is another proper metric space, $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ is a $(Q, a)$-quasi-minimizer with $(C, a)$-controlled density, and $g : \mathrm{spt}(S) \to \bar{X}$ is a map satisfying*

$$L^{-1} d(x, y) - a \le d(g(x), g(y)) \le L\, d(x, y)$$

*for all $x, y \in \operatorname{spt}(S)$. Then $\bar{S} := g_\# S \in \mathbf{Z}_{n,\mathrm{loc}}(\bar{X})$ is a $(\bar{Q}, \bar{a})$-quasi-minimizer with $(\bar{C}, \bar{a})$-controlled density, and $d(g(x), \operatorname{spt}(\bar{S})) \le \bar{a}$ for all $x \in \operatorname{spt}(S)$.*

*Proof* If $x \in \operatorname{spt}(S)$ and $r > a$, then $g^{-1}(B_{g(x)}(r)) \subset B_x(2Lr)$ and thus

$$\|\bar{S}\|(B_{g(x)}(r)) \le L^n \|S\|(B_x(2Lr)) \le L^n C_1 r^n$$

for $C_1 := (2L)^n C$. Hence, given any $\bar{p} \in \bar{X}$ and $r > a/2$ such that $B_{\bar{p}}(r) \cap \operatorname{spt}(\bar{S}) \ne \emptyset$, it follows that $\|\bar{S}\|(B_{\bar{p}}(r)) \le \|\bar{S}\|(B_{g(x)}(2r)) \le L^n C_1 (2r)^n$ for some $x \in \operatorname{spt}(S)$. This shows that $\bar{S}$ has $((2L)^n C_1, a/2)$-controlled density.

Next we show that there is a Lipschitz map $\bar{g}\colon \bar{X} \to X$ such that $h := \bar{g} \circ g$ is at finite distance from the identity on $\operatorname{spt}(S)$. Let $N \subset \operatorname{spt}(S)$ be a $4La$-separated $4La$-net in $\operatorname{spt}(S)$. By Lemma 3.5 (doubling), $N$ is doubling, and $g|_N\colon N \to g(N)$ is $(4L/3)$-bi-Lipschitz, so $g(N)$ is doubling as well. The doubling constant depends only on $n, L, C$. Then, by Proposition 10.1, $(g|_N)^{-1}$ admits an $\bar{L}$-Lipschitz extension $\bar{g}\colon \bar{X} \to X$ for some constant $\bar{L}$ depending on $n, L, C$. For every $x \in \operatorname{spt}(S)$ there is a $y \in N$ such that $d(x, y) \le 4La$. Then $h(y) = y$, and

$$d(h(x), x) \le d(h(x), h(y)) + d(y, x) \le (L\bar{L} + 1)\, d(x, y) \le b$$

for $b := 4(L\bar{L} + 1)La$.

Let again $x \in \operatorname{spt}(S)$ and $r > a$, and put $B_r := B_{g(x)}(r)$. For almost every such $r$, both $\bar{S}' := \bar{S} \llcorner B_r$ and $S' := S \llcorner g^{-1}(B_r)$ are integral currents, $g_\# S' = \bar{S}'$, and

$$\mathbf{M}(S') \le C_1 r^n, \quad \mathbf{M}(\bar{S}') \le L^n C_1 r^n, \quad \mathbf{M}(h_\# S') \le (L\bar{L})^n C_1 r^n.$$

Let $H\colon [0, 1] \times \operatorname{spt}(S) \to X$ denote the homotopy from $\mathrm{id}_{\operatorname{spt}(S)}$ to $h$ given by $H(t, x) = \sigma(x, h(x), t)$. The deformation chain $W := H_\#(\llbracket 0, 1 \rrbracket \times S') \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ satisfies

$$\mathbf{M}(W) \le C_2 r^n$$

for $C_2 := (n + 1)(L\bar{L})^n b\, C_1$. Furthermore, the support of the cylinder $R := H_\#(\llbracket 0, 1 \rrbracket \times \partial S') = h_\# S' - S' - \partial W$ lies in the closed $b$-neighborhood of $\operatorname{spt}(\partial S')$, and $\operatorname{spt}(\partial S') \subset \operatorname{spt}(S - S')$ is at distance at least $r/L$ from $x$ because $g$ is $L$-Lipschitz.

Suppose now that $\bar{T} \in \mathbf{I}_{n,\mathrm{c}}(\bar{X})$ and $\epsilon > 0$ are such that $\partial \bar{T} = \partial \bar{S}'$ and

$$\mathbf{M}(\bar{T}) \le \epsilon r^n.$$

Since $\mathbf{M}(\bar{g}_\# \bar{T}) \le \bar{L}^n \epsilon r^n$ and $\partial(\bar{g}_\# \bar{T}) = \bar{g}_\#(\partial \bar{S}') = h_\#(\partial S')$, Theorem 2.4 (minimizing filling) shows that there is a minimizing $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ with $\partial T =$

$h_\#(\partial S')$ and

$$\mathbf{M}(T) \le \bar{L}^n \epsilon r^n,$$

and if $\epsilon$ is sufficiently small, then $\mathrm{spt}(T)$ is within distance $r/(3L)$, say, from $\mathrm{spt}(h_\#(\partial S'))$. For $r > 3Lb$, it follows that $r/L - b - r/(3L) > r/(3L)$ and thus $\mathrm{spt}(T - R) \cap B_x(r/(3L)) = \emptyset$. Note that $\partial(T - R) = \partial S'$. By Lemma 3.4 there is a constant $c > 0$ such that $F_{x,r/(3L)}(S) > c$ for $r > 12La$. Put $Z := h_\# S' - T \in \mathbf{Z}_{n,\mathrm{c}}(X)$. It follows from Theorem 4.4 (sub-Euclidean isoperimetric inequality) that there is a constant $\bar{a} \ge 3Lb \ge 12La$ such that if $r > \bar{a}$, then $Z$ possesses a filling $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ with

$$\mathbf{M}(V - W) \le \mathbf{M}(V) + \mathbf{M}(W) < cr^{n+1}.$$

Since $\partial(V - W) = S' - (T - R)$ and $\mathrm{spt}(T - R) \cap B_x(r/(3L)) = \emptyset$, this contradicts the fact that $F_{x,r/(3L)}(S) > c$. Hence, there is an $\epsilon_0 > 0$ such that, for almost all $r > \bar{a}$, $\mathbf{M}(\bar{T}) \ge \epsilon_0 r^n$ and thus

$$\mathbf{M}(\bar{S}') \le L^n C_1 r^n \le Q\, \mathbf{M}(\bar{T})$$

for $Q := L^n C_1/\epsilon_0$. In the case that $g(x) \in \mathrm{spt}(\bar{S})$, this shows that $S$ is $(\bar{Q}, \bar{a})$-quasi-minimizing.

If $g(x) \notin \mathrm{spt}(\bar{S})$, the same argument for $\bar{T} := \bar{S}' = \bar{S} \llcorner B_{g(x)}(r)$ shows that $\|\bar{S}\|(B_{g(x)}(r)) \ge \epsilon_0 r^n > 0$ for almost all $r > \bar{a}$. Thus $d(g(x), \mathrm{spt}(\bar{S})) \le \bar{a}$. □

Our next goal is to prove Theorem 10.6 below. We need the following auxiliary results.

**Lemma 10.4** (mapping small fillings) *Let $(X, \sigma)$ be a proper metric space with a convex bicombing. Suppose that $n \ge 1$, $Z \in \mathbf{Z}_{n,\mathrm{loc}}(X)$, $p \in X$, and $g\colon X \to \bar{X}$ is an $L$-Lipschitz map into a proper metric space $\bar{X}$ such that $d(g(p), g(z)) \ge L^{-1} d(p, z) - a$ for all $z \in \mathrm{spt}(Z)$, for some constants $L \ge 1$ and $a \ge 0$. If $F_\infty(Z) = 0$, then $\bar{Z} := g_\# Z \in \mathbf{Z}_{n,\mathrm{loc}}(\bar{X})$ satisfies $F_\infty(\bar{Z}) = 0$.*

*Proof* Let $\epsilon > 0$. For every sufficiently large $r > 0$ there exists $V \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ such that $\mathrm{spt}(Z - \partial V) \cap B_p(r) = \emptyset$ and $\mathbf{M}(V) < (\epsilon r)^{n+1}$. By Theorem 2.4 (minimizing filling) we can assume that $V$ is minimizing and $d(x, \mathrm{spt}(\partial V)) < \epsilon c r$ for all $x \in \mathrm{spt}(V)$, where $c > 0$ depends only on $n$. Assuming that $\epsilon c < 1/2$, we find an $s > r/2$ such that $W := V \llcorner B_p(s) \in \mathbf{I}_{n+1,\mathrm{c}}(X)$, $\mathrm{spt}(Z - \partial W) \cap B_p(r/2) = \emptyset$, and $d(x, \mathrm{spt}(Z)) < \epsilon c r$ for all $x \in \mathrm{spt}(W)$. Put $\bar{W} := g_\# W \in \mathbf{I}_{n+1,\mathrm{c}}(\bar{X})$. Then, for any $x \in \mathrm{spt}(Z - \partial W) \subset \mathrm{spt}(Z) \cup \mathrm{spt}(W)$

and $z \in \operatorname{spt}(Z)$ with $d(x,z) < \epsilon c r$,

$$\begin{aligned} d(g(p),g(x)) &\geq d(g(p),g(z)) - d(g(x),g(z)) \\ &\geq L^{-1}(d(p,x) - d(x,z)) - a - L\,d(x,z) \\ &> (2L)^{-1}r - (L^{-1}+L)\epsilon c r - a =: \bar{r}. \end{aligned}$$

If $\epsilon$ is sufficiently small and $r$ is sufficiently large, so that $r \leq 3L\bar{r}$ say, then $\mathbf{M}(\bar{W}) < (\epsilon L r)^{n+1} \leq (3\epsilon L^2 \bar{r})^{n+1}$, and the support of $\bar{Z} - \partial \bar{W} = g_{\#}(Z - \partial W)$ is disjoint from $B_{g(p)}(\bar{r})$. This gives the result. □

The next lemma states a simple general fact about Lipschitz maps.

**Lemma 10.5** (combining Lipschitz maps) *Let $X$ be a proper metric space, and let $\bar{X}$ be a metric space with a convex bicombing $\bar{\sigma}$. Suppose that $A_1, A_2 \subset X$ are two closed non-empty sets, $L, a \geq 0$ are constants, and $g_1, g_2 \colon X \to \bar{X}$ are $L$-Lipschitz maps such that $d(g_1(x_1), g_2(x_2)) \leq L\,d(x_1,x_2) + a$ for all $(x_1,x_2) \in A_1 \times A_2$. Then there exists a $7L$-Lipschitz map $\hat{g} \colon X \to \bar{X}$ such that $d(\hat{g}(x), g_i(x)) \leq a/2$ for all $x \in A_i$, for $i = 1, 2$.*

*Proof* We assume that $a > 0$; the case $a = 0$ requires only minor modifications (note that then $g_1 = g_2$ on $A_1 \cap A_2$ by assumption).

For $i = 1, 2$, let $u_i \colon X \to \mathbb{R}$ be the $L$-Lipschitz function defined by $u_i(x) := L\,d(x, A_i) + a/4$. Put $w := u_1 + u_2$, $\lambda := u_1/w$, and define $\hat{g} \colon X \to \bar{X}$ by

$$\hat{g}(x) := \bar{\sigma}(g_1(x), g_2(x), \lambda(x)).$$

Let $x, y \in X$, and put $\bar{z} := \bar{\sigma}(g_1(x), g_2(x), \lambda(y))$. Then

$$\begin{aligned} d(\hat{g}(x), \hat{g}(y)) &\leq d(\hat{g}(x), \bar{z}) + d(\bar{z}, \hat{g}(y)), \\ d(\bar{z}, \hat{g}(y)) &\leq (1 - \lambda(y))\, d(g_1(x), g_1(y)) + \lambda(y)\, d(g_2(x), g_2(y)) \\ &\leq L\,d(x,y), \end{aligned}$$

and $d(\hat{g}(x), \bar{z}) = |\lambda(x) - \lambda(y)|\, d(g_1(x), g_2(x))$. Furthermore,

$$\begin{aligned} |\lambda(x) - \lambda(y)| &\leq \left|\lambda(x) - \frac{u_1(y)}{w(x)}\right| + \left|\frac{u_1(y)}{w(x)} - \lambda(y)\right| \\ &\leq \frac{1}{w(x)}|u_1(x) - u_1(y)| + \frac{\lambda(y)}{w(x)}|w(y) - w(x)| \\ &\leq \frac{3L}{w(x)}\, d(x,y), \end{aligned}$$

and if $x_1 \in A_1$ and $x_2 \in A_2$ are such that $d(x, x_i) = d(x, A_i)$, then

$$\begin{aligned} d(g_1(x), g_2(x)) &\le d(g_1(x), g_1(x_1)) + d(g_1(x_1), g_2(x_2)) + d(g_2(x_2), g_2(x)) \\ &\le L\, d(x, x_1) + \big(L\, d(x_1, x_2) + a\big) + L\, d(x_2, x) \\ &\le 2L\, d(x, x_1) + 2L\, d(x, x_2) + a \\ &= 2w(x). \end{aligned}$$

It follows that $\hat{g}$ is $7L$-Lipschitz. If $x \in A_1$, then $\lambda(x) = a/(4w(x))$, thus $d(g_1(x), \hat{g}(x)) \le \lambda(x)\, d(g_1(x), g_2(x)) \le a/2$. Similarly, $d(\hat{g}(x), g_2(x)) \le a/2$ for all $x \in A_2$. □

We now consider again the group $\mathscr{Z}X$ of $F$-asymptote classes from Definition 6.1.

**Theorem 10.6** (mapping asymptote classes) *Let $(X, \sigma)$ and $(\bar{X}, \bar{\sigma})$ be two proper metric spaces with convex bicombings and with* $\operatorname{asrk}(X) = \operatorname{asrk}(\bar{X}) = n \ge 2$*, and suppose that $f\colon X \to \bar{X}$ is a quasi-isometric embedding. Then there exists a unique monomorphism*

$$\mathscr{Z}f\colon \mathscr{Z}X \to \mathscr{Z}\bar{X}$$

*with the property that if $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ and $g\colon X \to \bar{X}$ is a Lipschitz map such that $\sup_{x\in\operatorname{spt}(S)} d(f(x), g(x)) < \infty$, then $\mathscr{Z}f\,[S] = [g_\# S]$. If $f$ is a quasi-isometry, then $\mathscr{Z}f$ is an isomorphism.*

Note that if $S$ and $g$ are as in the theorem, then $g_\# S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(\bar{X})$ by the argument in the first paragraph of the proof of Proposition 10.3, thus the class $[g_\# S] \in \mathscr{Z}\bar{X}$ is defined. Combining Theorem 10.6 with Theorem 9.4 (Tits boundary), we get a monomorphism $f_\mathrm{T}$ that makes the diagram

$$\begin{array}{ccc} \mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} X) & \xrightarrow{f_\mathrm{T}} & \mathbf{Z}_{n-1,\mathrm{c}}(\partial_\mathrm{T} \bar{X}) \\ {\scriptstyle\partial_\mathrm{T}}\uparrow & & \uparrow{\scriptstyle\partial_\mathrm{T}} \\ \mathscr{Z}X & \xrightarrow[\mathscr{Z}f]{} & \mathscr{Z}\bar{X} \end{array}$$

commutative. This yields Theorem 1.8 in the introduction.

*Proof* Due to Theorem 5.6 (constructing minimizers) and Proposition 7.4 (controlled density), every class in $\mathscr{Z}X$ is represented by a minimizer $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ with controlled density. It then follows from Lemma 3.5 (doubling) and Proposition 10.1 that a Lipschitz map $g\colon X \to \bar{X}$ with $\sup_{x\in\operatorname{spt}(S)} d(f(x), g(x)) < \infty$ exists. In particular, there is at most one map $\mathscr{Z}f\colon \mathscr{Z}X \to \mathscr{Z}\bar{X}$ with the property stated in the theorem.

Suppose now that $S_1, S_2 \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$ are arbitrary and $g_1, g_2 \colon X \to \bar{X}$ are Lipschitz maps with $\sup_{x\in\mathrm{spt}(S_i)} d(f(x), g_i(x)) < \infty$ for $i = 1, 2$. It follows from Lemma 10.5 that there exists a Lipschitz map $\hat{g} \colon X \to \bar{X}$ such that $\sup_{\mathrm{spt}(S_1)\cup\mathrm{spt}(S_2)} d(f(x), \hat{g}(x)) < \infty$. Using the $\bar{\sigma}$-homotopy from $g_i$ to $\hat{g}$ one can easily check that $g_{i\#}S_i \sim_F \hat{g}_\# S_i$. In the case that $S_1 \sim_F S_2$, Lemma 10.4 shows that $\hat{g}_\# S_1 \sim_F \hat{g}_\# S_2$, thus $g_{1\#}S_1 \sim_F g_{2\#}S_2$. This yields the existence of a unique map $\mathscr{Z} f \colon \mathscr{Z} X \to \mathscr{Z}\bar{X}$ with the property stated in the theorem. Furthermore, since

$$\begin{aligned}\mathscr{Z} f\,[S_1] + \mathscr{Z} f\,[S_2] &= [\hat{g}_\# S_1] + [\hat{g}_\# S_2] = [\hat{g}_\#(S_1 + S_2)] = \mathscr{Z} f\,[S_1 + S_2] \\ &= \mathscr{Z} f\,([S_1] + [S_2]),\end{aligned}$$

$\mathscr{Z} f$ is a homomorphism. To show that $\mathscr{Z} f$ is injective, suppose that $[S] \neq 0$, where $S$ is a minimizer with controlled density. Then it follows from Proposition 10.3 that $g_\# S$ is quasi-minimizing and non-zero for any Lipschitz map $g \colon X \to \bar{X}$ with $\sup_{x\in\mathrm{spt}(S)} d(f(x), g(x)) < \infty$. Lemma 3.4 (filling density) then shows that $F_\infty(g_\# S) \neq 0$, thus $\mathscr{Z} f\,[S] = [g_\# S] \neq 0$.

If $f$ is a quasi-isometry, then there is a quasi-isometric embedding $\bar{f} \colon \bar{X} \to X$ such that $\sup_{\bar{x}\in\bar{X}} d((f \circ \bar{f})(\bar{x}), \bar{x}) < \infty$, and it is not difficult to show that $\mathscr{Z} f \circ \mathscr{Z}\bar{f}$ is the identity on $\mathscr{Z}\bar{X}$. □

*Remark 10.7* Resuming the discussion of visual metrics, we note that when $f \colon X \to \bar{X}$ is an $(L, a_0)$-quasi-isometric embedding, the monomorphism $\mathscr{Z} f \colon \mathscr{Z} X \to \mathscr{Z}\bar{X}$ maps each of the subsets $\mathscr{Z}_{C,a} X \subset \mathscr{Z} X$ into $\mathscr{Z}_{\bar{C},\bar{a}}\bar{X}$, where $\bar{C}, \bar{a}$ depend on $X, L, a_0, C, a$. Furthermore, there is a constant $\bar{D}$, depending in addition on $\bar{X}$, such that if $S, S' \in \mathscr{Z}_{C,a} X$ and $Z \in [S - S']$, $\bar{Z} \in \mathscr{Z} f[S - S']$ are minimizing, then $\mathrm{spt}(\bar{Z})$ is at Hausdorff distance at most $\bar{D}$ from $f(\mathrm{spt}(Z))$. As a consequence, for every $p \in X$,

$$L^{-1}\langle [S - S']\rangle_p - a_0 - \bar{D} \leq \langle \mathscr{Z} f[S - S']\rangle_{f(p)} \leq L\,\langle [S - S']\rangle_p + a_0 + \bar{D}.$$

It follows readily that both the restriction of $\mathscr{Z} f$ to $\mathscr{Z}_{C,a} X$ and its inverse are Hölder continuous with exponent $1/L$ for any pair of visual metrics on $\mathscr{Z}_{C,a} X$ and $\mathscr{Z}_{\bar{C},\bar{a}}\bar{X}$ with the same parameter $b$.

Higher rank visual metrics will be further discussed elsewhere.

## 11 Mapping limit sets

We will now describe the effect of a quasi-isometric embedding $f \colon X \to \bar{X}$, or of the associated monomorphism $\mathscr{Z} f \colon \mathscr{Z} X \to \mathscr{Z}\bar{X}$, on the collection of limit sets $\mathscr{L} X$ introduced in Definition 8.4. We associate to every class

$[S] \in \mathscr{Z}X$ a limit set $\Lambda[S] \subset \partial_\infty X$ such that

$$\Lambda[S] = \Lambda(S')$$

for every $S' \in [S]$ that is quasi-minimizing or conical; in these cases the invariance of $\Lambda(S')$ is granted by Theorem 7.3 (conical representative) and Proposition 8.2 (equal limit sets). Thus $\Lambda[S] \in \mathscr{L}X$. For any $S \in \mathbf{Z}^\infty_{n,\mathrm{loc}}(X)$, the set $\Lambda[S]$ also agrees with $\mathrm{spt}(\partial_{\mathrm{T}} S)$; however, Theorems 9.3 (lifting cones) and 9.4 (Tits boundary) are not needed for the proof of Theorem 11.2 below.

The following preliminary result relies on Theorems 8.1 (visibility property) and 8.6 (asymptotic conicality).

**Proposition 11.1** (mapping cones) *Let $(X, \sigma)$ and $(\bar{X}, \bar{\sigma})$ be two proper metric spaces with convex bicombings and with* $\mathrm{asrk}(X) = \mathrm{asrk}(\bar{X}) = n \geq 2$*, and let $f\colon X \to \bar{X}$ be a quasi-isometric embedding. Suppose that $[S] \in \mathscr{Z}X$ and $\mathscr{Z}f\,[S] = [\bar{S}] \in \mathscr{Z}\bar{X}$. Choose base points $p \in X$ and $\bar{p} \in \bar{X}$, and consider the geodesic cones $K := \mathrm{C}_p(\Lambda[S]) \subset X$ and $\bar{K} := \mathrm{C}_{\bar{p}}(\Lambda[\bar{S}]) \subset \bar{X}$. Then for all $\epsilon > 0$ there exists an $r > 0$ such that*

$$d(f(x), \bar{K}) < \epsilon\, d(p, x)$$

*for all $x \in K$ with $d(p, x) \geq r$ and*

$$d(\bar{x},\, f(K)) < \epsilon\, d(\bar{p}, \bar{x})$$

*for all $\bar{x} \in \bar{K}$ with $d(\bar{p}, \bar{x}) \geq r$.*

*Proof* We assume that $f$ is an $(L, a)$-quasi-isometric embedding, $f(p) = \bar{p}$, $S$ is a quasi-minimizer with controlled density, and $\bar{S} = g_\# S$ for some Lipschitz map $g\colon X \to \bar{X}$ with $\bar{b} := \sup_{x \in \mathrm{spt}(S)} d(f(x), g(x)) < \infty$.

Let $\epsilon' \in (0, 1)$. If $r > 0$ is sufficiently large, then it follows from Theorem 8.6 that for every $x \in K$ with $d(p, x) \geq r$ there is a $y \in \mathrm{spt}(S)$ such that $d(x, y) < \epsilon' d(p, x)$, thus

$$d(f(x),\, f(y)) \leq L\epsilon' d(p, x) + a$$

and $(1-\epsilon')\, d(p, x) \leq d(p, y) \leq (1+\epsilon')\, d(p, x)$. By Proposition 10.3 (quasi-isometry invariance), $\bar{S}$ is a quasi-minimizer with controlled density, and there is a point $\bar{y} \in \mathrm{spt}(\bar{S})$ such that $d(g(y), \bar{y}) \leq \bar{a}$ for some constant $\bar{a} \geq 0$, thus

$$d(f(y), \bar{y}) \leq \bar{a} + \bar{b} =: \bar{c}$$

and $d(\bar{p}, \bar{y}) \geq d(f(p), f(y)) - \bar{c} \geq L^{-1}(1-\epsilon')r - a - \bar{c}$. Hence, if $r$ is sufficiently large, then by the second part of Theorem 8.1,

$$d(\bar{y}, \bar{K}) < \epsilon' d(\bar{p}, \bar{y}) \leq 2L\epsilon' d(p, x),$$

as $d(\bar{p}, \bar{y}) \leq d(f(p), f(y)) + \bar{c} \leq L(1+\epsilon') d(p, x) + a + \bar{c} \leq 2L\, d(p, x)$. Combining these estimates we get the first assertion, and the second is proved similarly. □

We now prove that $f$ induces an injective map $\mathscr{L} f \colon \mathscr{L} X \to \mathscr{L} \bar{X}$. If $\mathscr{L} f(\Lambda) = \bar{\Lambda}$, then the cones $\mathbb{R}_+\Lambda \subset \mathscr{C}_{\mathrm{T}} X$ and $\mathbb{R}_+\bar{\Lambda} \subset \mathscr{C}_{\mathrm{T}} \bar{X}$ are bi-Lipschitz homeomorphic.

**Theorem 11.2** (mapping limit sets) *Let $(X, \sigma)$ and $(\bar{X}, \bar{\sigma})$ be two proper metric spaces with convex bicombings and with* $\operatorname{asrk}(X) = \operatorname{asrk}(\bar{X}) = n \geq 2$, *and suppose that $f \colon X \to \bar{X}$ is an $(L, a)$-quasi-isometric embedding. Then there exists an injective map*

$$\mathscr{L} f \colon \mathscr{L} X \to \mathscr{L} \bar{X}$$

*such that $\mathscr{L} f(\Lambda[S]) = \Lambda[\bar{S}]$ whenever $\mathscr{Z} f\,[S] = [\bar{S}]$. For every finite union $M := \bigcup_{i=1}^k \Lambda_i$ of sets $\Lambda_i \in \mathscr{L} X$ and the corresponding union $\bar{M} := \bigcup_{i=1}^k \bar{\Lambda}_i$ of the sets $\bar{\Lambda}_i := \mathscr{L} f(\Lambda_i)$, there is a pointed $L$-bi-Lipschitz homeomorphism $\Phi \colon \mathbb{R}_+ M \to \mathbb{R}_+ \bar{M}$ such that $\Phi(\mathbb{R}_+\Lambda_i) = \mathbb{R}_+\bar{\Lambda}_i$ for $i = 1, \ldots, k$. If $f$ is a quasi-isometry, then $\mathscr{L} f$ is a bijection.*

Here $\Phi$ is said to be *pointed* if $\Phi(o) = \bar{o}$, where $o$ and $\bar{o}$ are the cone vertices of $\mathscr{C}_{\mathrm{T}} X$ and $\mathscr{C}_{\mathrm{T}} \bar{X}$, respectively.

*Proof* Choose base points $p \in X$ and $\bar{p} := f(p) \in \bar{X}$. Suppose that $\mathscr{Z} f\,[S] = [\bar{S}]$ and $\mathscr{Z} f\,[T] = [\bar{T}]$. We use Proposition 11.1. If $\Lambda[S] = \Lambda[T]$, then for every $\epsilon \in (0, 1)$ and every $\bar{x} \in \mathrm{C}_{\bar{p}}(\Lambda[\bar{S}])$ with sufficiently large distance to $\bar{p}$ there exists an $x \in C_p(\Lambda[S]) = \mathrm{C}_p(\Lambda[T])$ such that

$$d(\bar{x}, f(x)) < \epsilon\, d(\bar{p}, \bar{x})$$

and $(2L)^{-1} d(\bar{p}, \bar{x}) \leq d(p, x) \leq 2L\, d(\bar{p}, \bar{x})$; then there is also a point $\bar{y} \in C_{\bar{p}}(\Lambda[\bar{T}])$ such that

$$d(f(x), \bar{y}) < \epsilon\, d(p, x) \leq 2L\epsilon\, d(\bar{p}, \bar{x}).$$

It follows that $\Lambda[\bar{S}] \subset \Lambda[\bar{T}]$, and the reverse inclusion holds by symmetry. Conversely, if $\Lambda[\bar{S}] = \Lambda[\bar{T}]$, then a similar argument shows that $\Lambda[S] = \Lambda[T]$. This yields the existence of an injective map $\mathscr{L} f \colon \mathscr{L} X \to \mathscr{L} \bar{X}$ such that $\mathscr{L} f(\Lambda[S]) = \Lambda[\bar{S}]$ whenever $\mathscr{Z} f\,[S] = [\bar{S}]$.

Let now $M$ and $\bar{M}$ be given as in the theorem. By Proposition 8.7 (compact limit sets), the cones $\mathbb{R}_+M$ and $\mathbb{R}_+\bar{M}$ are proper and thus separable. For $r > 0$, let $\pi_r\colon \mathscr{C}_{\mathrm{T}}X \to X$ and $\bar{\pi}_r\colon \mathscr{C}_{\mathrm{T}}\bar{X} \to \bar{X}$ denote the $r$-Lipschitz maps defined by

$$\pi_r(u) := \mathrm{can}_p(ru), \quad \bar{\pi}_r(\bar{u}) := \mathrm{can}_{\bar{p}}(r\bar{u}).$$

Let first $N \subset \mathbb{R}_+M$ be a finite set containing $o$, and let $\epsilon > 0$. It follows from Proposition 11.1 that if we pick $r > 0$ sufficiently large, then for every $u \in N$ and $i \in I(u) := \{i : u \in \mathbb{R}_+\Lambda_i\}$ there is a point $\bar{u}_{r,i} \in \mathbb{R}_+\bar{\Lambda}_i$ such that

$$d(f(\pi_r(u)), \bar{\pi}_r(\bar{u}_{r,i})) \le \epsilon r,$$

where $\bar{o}_{r,i} := \bar{o}$ for $i = 1, \dots, k$. Then, for all $u, v \in N$ and $i \in I(u)$, $j \in I(v)$,

$$\begin{aligned} L^{-1}d(\pi_r(u), \pi_r(v)) - a - 2\epsilon r &\le d(\bar{\pi}_r(\bar{u}_{r,i}), \bar{\pi}_r(\bar{v}_{r,j})) \\ &\le L\, d(\pi_r(u), \pi_r(v)) + a + 2\epsilon r \end{aligned}$$

and $d(\bar{p}, \bar{\pi}_r(\bar{u}_{r,i})) \le L\, d(p, \pi_r(u)) + a + \epsilon r$, thus $d_{\mathrm{T}}(\bar{o}, \bar{u}_{r,i}) \le L\, d_{\mathrm{T}}(o, u) + r^{-1}a + \epsilon$. We infer from Lemma 9.1 (uniform convergence) that if $r > a/\epsilon$ is sufficiently large, then

$$L^{-1}d_{\mathrm{T}}(u, v) - 4\epsilon \le d_{\mathrm{T}}(\bar{u}_{r,i}, \bar{v}_{r,j}) \le L\, d_{\mathrm{T}}(u, v) + 4\epsilon.$$

For $u = v$, this also shows that the set $\{\bar{u}_{r,i} : i \in I(u)\}$ associated to $u$ has diameter at most $4\epsilon$. Let $s \ge 2\epsilon$. It follows again from Proposition 11.1 that if $r > a/\epsilon$ is sufficiently large, then for every $\bar{w} \in [\epsilon, s]\bar{\Lambda}_i$ there is a $w \in \mathbb{R}_+\Lambda_i$ such that

$$d(f(\pi_r(w)), \bar{\pi}_r(\bar{w})) \le \epsilon r$$

and $d_{\mathrm{T}}(o, w) \le L(d_{\mathrm{T}}(\bar{o}, \bar{w}) + 2\epsilon) \le 2Ls$. Then, for $u \in N \cap \mathbb{R}_+\Lambda_i$, we can conclude as above that $d_{\mathrm{T}}(\bar{u}_{r,i}, \bar{w}) \le L\, d_{\mathrm{T}}(u, w) + 4\epsilon$, provided $r$ is large enough. Hence, if we assume that $N \cap [0, 2Ls]\Lambda_i$ is an $\epsilon$-net in $[0, 2Ls]\Lambda_i$, then $\{\bar{u}_{r,i} : u \in N \cap \mathbb{R}_+\Lambda_i\}$ forms an $(L+4)\epsilon$-net in $[0, s]\bar{\Lambda}_i$. Repeating this construction for some sequences $\epsilon_l \to 0$ and $s_l \to \infty$ and a suitable sequence $N_1 \subset N_2 \subset \dots$ of subsets of $\mathbb{R}_+M$, we get the desired map $\Phi\colon \mathbb{R}_+M \to \mathbb{R}_+\bar{M}$ via a diagonal sequence argument.

Finally, if $f$ is a quasi-isometry, then $\mathscr{Z}f\colon \mathscr{Z}X \to \mathscr{Z}\bar{X}$ is an isomorphism by Theorem 10.6. Hence, for every $\bar{\Lambda} = \Lambda[\bar{S}] \in \mathscr{L}\bar{X}$ there exists a $\Lambda = \Lambda[S] \in \mathscr{L}X$ such that $\mathscr{Z}f\,[S] = [\bar{S}]$ and thus $\mathscr{L}f(\Lambda) = \bar{\Lambda}$. □

This result readily implies Theorem 1.9 in the introduction. Note that if $P = \bigcap_{i=1}^{j} \Lambda_i$, $Q = \bigcap_{i=j+1}^{k} \Lambda_i$, and $\bar{P}$, $\bar{Q}$ are the corresponding intersections of the sets $\bar{\Lambda}_i := \mathscr{L} f(\Lambda_i)$, then the existence of a map $\Phi$ as in Theorem 11.2 guarantees that $P \subset Q$ if and only if $\bar{P} \subset \bar{Q}$.

If $X$ and $\bar{X}$ are symmetric spaces of non-compact type and of rank $n \geq 2$, then their Tits boundaries have the structure of thick $(n-1)$-dimensional spherical buildings, and every Weyl chamber is the intersection of the limit sets of two $n$-flats. It then follows from Theorem 1.9 that every quasi-isometry $f\colon X \to \bar{X}$ induces an isomorphism (order preserving bijection) between the two buildings, which must carry apartments to apartments. This shows that the map $\mathscr{L} f\colon \mathscr{L} X \to \mathscr{L} \bar{X}$ takes limit sets of $n$-flats to limit sets of $n$-flats, and it follows from the case $k = 1$ of Theorem 11.3 below or Theorem 1.10 that for every $n$-flat $F \subset X$ there is an $n$-flat $\bar{F} \subset \bar{X}$ at uniformly bounded Hausdorff distance from $f(F)$. This constitutes a major step in the proof of the quasi-isometric rigidity theorem for symmetric spaces of non-compact type without rank one de Rham factors; compare Corollary 7.1.5 in [54] and Lemma 8.6 in [34]. The proof may then be completed along the lines in these papers, using Tits' work [78].

**Theorem 11.3** (structure of quasiflats) *Let $X$ be a proper metric space with a convex bicombing $\sigma$ and with* $\operatorname{asrk}(X) = n \geq 2$. *Let $f\colon \mathbb{R}^n \to X$ be an $(L, a_0)$-quasi-isometric embedding with limit set $\Lambda := \partial_\infty(f(\mathbb{R}^n))$. Then the cone $K := \mathbb{R}_+\Lambda \subset \mathscr{C}_{\mathrm{T}} X$ is $L$-bi-Lipschitz equivalent to $\mathbb{R}^n$. Suppose that $K$ is the union of closed sets $K_1, \dots, K_k$ such that, for some point $p \in X$, $\operatorname{can}_p|_{K_i}$ is a (1-Lipschitz) $(L, a_0)$-quasi-isometric embedding for $i = 1, \dots, k$. Then $f(\mathbb{R}^n)$ is within distance at most $b$ from $\mathrm{C}_p(\Lambda) = \operatorname{can}_p(K)$ for some constant $b$ depending only on $X$, $L$, $a_0$ and $k$. In the case $k = 1$, $f(\mathbb{R}^n)$ is at Hausdorff distance at most $b$ from $\mathrm{C}_p(\Lambda)$.*

*Proof* Let $E := [\![\mathbb{R}^n]\!] \in \mathbf{Z}_{n,\mathrm{loc}}(\mathbb{R}^n)$. By Proposition 3.6 (Lipschitz quasi-flats) there are constants $Q, C, a$, depending only on $n, L, a_0$, such that $\mathscr{Z} f[E] = [S]$ for some $(Q, a)$-quasi-minimizer $S \in \mathbf{Z}_{n,\mathrm{loc}}(X)$ with $(C, a)$-controlled density and $d_{\mathrm{H}}(\operatorname{spt}(S), f(\mathbb{R}^n)) \leq a$. Then $\Lambda = \Lambda(S) = \Lambda[S]$, and Theorem 11.2 shows that there exists an $L$-bi-Lipschitz homeomorphism $\phi\colon \mathbb{R}^n \to K = \mathbb{R}_+\Lambda$.

Suppose now that the additional assumption in the theorem holds for some $p \in X$. By Theorem 7.3 (conical representative), $\operatorname{spt}(S_{p,0}) \subset \mathrm{C}_p(\Lambda)$ and $\Theta_\infty(S_{p,0}) \leq \Theta_\infty(S) \leq C$. Our aim is to show that $S_{p,0}$ has controlled density. By Theorem 9.3 (lifting cones) there exists a local cycle $\Sigma \in \mathbf{Z}_{n,\mathrm{loc}}(K)$ in $K$ such that $\operatorname{can}_{p\#}\Sigma = S_{p,0}$ and $\Theta_\infty(\Sigma) \leq C$. Note, however, that if $k = 1$ and $\operatorname{can}_p|_K$ is bi-Lipschitz or even isometric (for example, if $X$ is CAT(0) and $\mathrm{C}_p(\Lambda)$ is a flat), then one can simply put $\Sigma := (\operatorname{can}_p|_K^{-1})_\# S_{p,0}$ and the theorem is not needed. Now $(\phi^{-1})_\#\Sigma$ is an element of $\mathbf{Z}_{n,\mathrm{loc}}(\mathbb{R}^n)$ and hence

of the form $mE$ for some constant integer multiplicity $m$. Since $\phi^{-1}$ is $L$-bi-Lipschitz, it follows that $|m|$ is bounded in terms of $C$ and $L$ (there is no need to show that in fact $|m| = 1$). For $i = 1, \ldots, k$, let $\psi_i$ denote the restriction of $\mathrm{can}_p \circ \phi$ to $\phi^{-1}(K_i)$. Note that $\psi_i$ is $L$-Lipschitz and $(L^2, a_0)$-quasi-isometric by the assumption on $\mathrm{can}_p \,|_{K_i}$. Choose Borel sets $B_i \subset K_i$ such that the union $\bigcup_{i=1}^k B_i = K$ is disjoint. Since $\phi_\#(mE) = \Sigma$,

$$\psi_{i\#}(mE \llcorner \phi^{-1}(B_i)) = \mathrm{can}_{p\#}\big(\phi_\#(mE \llcorner \phi^{-1}(B_i))\big) = \mathrm{can}_{p\#}(\Sigma \llcorner B_i).$$

If $q \in X$ and $r > a_0$, then ${\psi_i}^{-1}(B_q(r))$ has diameter at most $L^2(2r + a_0) \leq 3L^2 r$, and it follows that

$$\| \mathrm{can}_{p\#}(\Sigma \llcorner B_i)\|(B_q(r)) \leq C_0 r^n$$

for some constant $C_0$ depending only on $m, n, L$. Since $\sum_{i=1}^k \mathrm{can}_{p\#}(\Sigma \llcorner B_i) = \mathrm{can}_{p\#} \Sigma = S_{p,0}$, we conclude that $S_{p,0}$ has $(kC_0, a_0)$-controlled density. Now Theorem 5.1 (Morse Lemma I) yields the first conclusion of the theorem.

If $k = 1$, then $\psi_1 = \mathrm{can}_p \circ \phi$ is a Lipschitz quasiflat, hence $S_{p,0} = \psi_{1\#}(mE)$ is quasi-minimizing and $\mathrm{spt}(S_{p,0})$ is at finite Hausdorff distance from $\psi_1(\mathbb{R}^n) = \mathrm{C}_p(\Lambda)$ by Proposition 3.6 (which extends to higher multiples of $E = [\![\mathbb{R}^n]\!]$). The desired estimate follows again from Theorem 5.1. □

Theorem 1.10 stated in the introduction follows as a special case. In the case $k = 1$, this applies in particular to CAT(0) spaces with isolated flats; compare Lemma 3.1 in [75] (the case $\mathbf{F} = \mathbb{R}$) and Theorem 4.1.1 in [47]. Furthermore, it follows easily that every $n$-dimensional quasiflat in a nonpositively curved symmetric space of rank $n \geq 2$ lies within uniformly bounded distance from the union of a finite, uniformly bounded number of $n$-flats; compare Theorem 1.2.5 in [54] and Theorem 1.1 in [34]. We also refer to [10,13,48,50,63] for various similar statements.

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