Finding duality for Riesz bases of exponentials on multi-tiles

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Abstract

It is known [6, 14, 19] that if $\Omega \subset \mathbb{R}^d$ belongs to a class of multi-tiling domains when translated by a lattice Λ, there exists a Riesz basis of exponentials for $L^2(\Omega)$ constructed using k translates of the dual lattice Λ^*. In this paper, we give an explicit construction of the corresponding biorthogonal dual Riesz basis. We also extend the iterative reconstruction algorithm introduced in [11] to this setting.

1 Introduction

This paper centers on Riesz bases of exponentials $\{e_l(x) := e^{2\pi i l\cdot x}\}_{l\in L}$ for the space $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a set of positive and finite Lebesgue measure, and $L \subset \mathbb{R}^d$ is a countable set. The set $\{e_l(x)\}_{l\in L} \subset L^2(\Omega)$ is a Riesz basis for $L^2(\Omega)$ if each $f \in L^2(\Omega)$ has the unique representation

$$f(x) = \sum_{l\in L} c_l e_l(x), \tag{1}$$

where the coefficients $\{c_l\}_{l\in L} \in \ell^2(L)$ satisfy

$$A\|f\|^2_{L^2(\Omega)} \le \sum_{l\in L} |c_l|^2 \le B\|f\|^2_{L^2(\Omega)} \tag{2}$$

for some constants $0 < A \le B < \infty$. In this case, there exists a (unique) dual Riesz basis $\{g_l(x)\}_{l\in L} \subset L^2(\Omega)$ that satisfies the biorthogonality condition

$$\langle e_l, g_{l'}\rangle = \begin{cases} |\Omega| & \text{if } l = l' \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and the coefficients in (1) are given by $c_l = \frac{1}{|\Omega|}\langle f, g_l\rangle$.

Although no general proof of the statement exists, there are many cases where it is known that a set Ω admits a Riesz basis of exponential functions . These cases include when Ω is a finite union of co-measurable intervals in $\mathbb{R}$ or multi-rectangles in $\mathbb{R}^d$ [22, 8] and when Λ is a stable set of sampling for the Paley-Wiener space PW_Ω [5, 16, 25, 27]. It was recently established in [9] that any convex polytope that is centrally symmetric and whose faces of all dimensions are also centrally symmetric admits a Riesz basis of exponentials. For more details on properties of families of exponentials, we refer to [28].

However, to the best knowledge of the authors, no explicit algorithms or formulas for the corresponding dual Riesz bases are available in the literature. As seen from (1), knowing the biorthogonal dual is important for the reconstruction of any function in $L^2(\Omega)$. One of the goals of this paper is to construct biorthogonal Riesz bases for $L^2(\Omega)$ for a class of domains $\Omega \subset \mathbb{R}^d$ of finite, positive Lebesgue measure.

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The interest in Riesz bases of exponentials stems partially from the Fuglede Conjecture [12], which asserts that the set of exponentials $\{e_l(x)\}_{l\in L}$ is an orthogonal basis for $L^2(\Omega)$ if and only if Ω tiles $\mathbb{R}^d$ with respect to the discrete set $L \subset \mathbb{R}^d$. In this case, the system of exponentials is self-dual, that is $e_l \equiv g_l$ for $l \in L$ and $A = B = \frac{1}{|\Omega|}$ in (2). The Fuglede conjecture has been disproved in both directions when $d \geq 3$ [12, 26, 21, 18] but remains open when $d = 1, 2$. It has recently been proved in [24] that the Fuglede conjecture does hold for convex domains in all dimensions. Removing the rigidity imposed by orthonormality leads naturally to the problem of obtaining Riesz bases of exponentials.

Let Λ be a full lattice in $\mathbb{R}^d$ and k be a natural integer. We say that a Lebesgue measurable set $\Omega \subset \mathbb{R}^d$ of finite positive measure is a k-tile for Λ (or multi-tiling subset of $\mathbb{R}^d$) if

$$\sum_{l\in\Lambda} \chi_\Omega(x - l) = k \text{ for almost all } x \in \mathbb{R}^d.$$

When $\Omega \subset \mathbb{R}^d$ is an *admissible* k-tile for a full lattice $\Lambda \subset \mathbb{R}^d$ (defined in Remark 1), then it admits a Riesz basis of exponentials [6, 14, 19]. More specifically, there exists a set of vectors $\{a_s\}_{s=1}^k \subset \mathbb{R}^d$ such that the exponentials

$$\{e_l(x) = e^{2\pi i l\cdot x}\}_{l\in L}, \qquad L = \bigcup_{s=1}^{k} \Lambda^* + a_s \tag{4}$$

form a Riesz basis for $L^2(\Omega)$, where Λ^* is the dual lattice of Λ.

The first goal of the paper, accomplished in Section 2, is to introduce a procedure for constructing the biorthogonal dual Riesz basis for any multi-tiling domain that is known to admit a Riesz basis of exponentials of the form (4). The second goal of this paper is to derive an iterative (and adaptive) algorithm for performing the pointwise reconstruction of functions $f \in L^2(\Omega)$ given *data* of the form $\{\langle f, e_l\rangle\}_{l\in L}$. The algorithm is deterministic, it only involves inverting 1D Vandermonde systems, and it establishes a new framework for finding a set of vectors $\{a_s\}_{s=1}^k \subset \mathbb{R}^d$ for which (4) is guaranteed to be a Riesz basis with Riesz bounds that are quantifiable using analytical formulas for 1D Vandermonde matrices.

As we will show in Section 3, this algorithm extends the results that recently appeared in [11] to more general multi-tiling sets. Compared to the iterative procedure in [2, 3], our algorithm solves invertible 1D Vandermonde systems at each iteration. A related algorithm in [10] relies on the existence of solutions to a linear system. In contrast, the algorithm presented here chooses the lattice shifts to ensure invertibility. The algorithm introduced here more closely compares to the construction in [25], however instead of creating a p-th order sampling procedure at each iteration, a first-order sampling is used at each iteration. Another line of investigation that compares to the present work is considered in [15, 17] where numerical sampling algorithms for multivariate trigonometric polynomials are used to derive approximations of infinite-dimensional bandlimited functions from nonuniform sampling. Recent results in [17] employ numerical methods for sampling along random rank-1 lattices for a given frequency set to approximate multivariate periodic functions, focusing on the error in approximation. In contrast, we formulate a deterministic algorithm and provide guarantees for exact reconstruction in cases where the domain Ω is fully known.

2 Finding duals

The main result in this section offers an explicit construction of biorthogonal systems of exponentials corresponding to a class of multi-tiling sets. Throughout the paper, we let $\Lambda = M\mathbb{Z}^d$ be a full lattice generated by the basis vectors $M = [m_1, \ldots, m_n] \in \mathbb{R}^{d\times d}$. The canonical dual lattice is $\Lambda^* = \{M^{-T}z, z \in \mathbb{Z}^d\}$ and Π_Λ denotes the fundamental domain $\Pi_\Lambda = M\mathbb{T}^d$.

Let $\Omega \subset \mathbb{R}^d$ be a measurable domain with $0 < |\Omega| < \infty$ that is a k-tile for Λ. Then, there exists a partition $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_k \cup E$, where E is a set of measure zero, and Ω_j are mutually disjoint measurable sets that are each a fundamental domain of Λ (Lemma 1 in [19]). Let $\Lambda_x(\Omega) := \Lambda_x = \{\lambda \in \Lambda \mid x + \lambda \in \Omega\}$. Denote the cardinality of the finite set $\mathcal{S}$ by $\#\mathcal{S}$. Then $\#\Lambda_x = k$ for almost every $x \in \Pi_\Lambda$ and we define the points $\{\lambda_r(x)\} \subset \Lambda$ to be the unique lattice points that satisfy

$$x + \lambda_r(x) \in \Omega_r, \qquad 1 \leq r \leq k. \tag{5}$$

The mapping $\omega_r : \Pi_\Lambda \to \Omega_r$ given by $x \to x + \lambda_r(x)$ is then invertible.

Our first main result is the following.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a k-tile for a full lattice Λ, and define $\{\lambda_r(x)\}_{r=1}^k$ and $\{\omega_r(x)\}_{r=1}^k$ by (5). If there exists a set of vectors $\{a_s\}_{s=1}^k \subset \mathbb{R}^d$ and positive constants α and β such that the matrix function $V = V(x) \in \mathbb{C}^{k\times k}$ with entries*

$$(V(x))_{sr} = e^{-2\pi i \lambda_r(x)\cdot a_s}, \qquad 1 \le s, r \le k, \tag{6}$$

has uniformly bounded singular values, $0 < \alpha < \sigma_i(V(x)) < \beta < \infty$, $1 \le i \le d$, then the following two families of functions form a pair of biorthogonal Riesz bases for $L^2(\Omega)$:

$$\{e_l(x)\}_{l\in L}, \qquad \{g_l(x)\}_{l\in L}, \qquad L = \bigcup_{s=1}^k \Lambda^* + a_s \tag{7}$$

where, for $\lambda^* \in \Lambda^*$ and $1 \le s \le k$,

$$g_{\lambda^*+a_s}(x) = e_{\lambda^*+a_s}(x)\left(k\sum_{r=1}^k (V(\omega_r^{-1}(x)))_{sr}(V(\omega_r^{-1}(x))^{-1})_{rs}\chi_{\Omega_r}(x)\right). \tag{8}$$

Here χ_{Ω_r} denotes the indicator function of Ω_r.

Proof. Let $f \in L^2(\Omega)$. Then the mappings from $\ell^2 \to L^2(\Pi_\Lambda)$ given by

$$\{\langle f, e_l\rangle\}_{l\in\Lambda^*+a_s} \to \sum_{l\in\Lambda^*+a_s} \langle f, e_l\rangle e_l(x)$$

are well-defined (in fact, isometries) for all $1 \le s \le k$ by the orthogonality of the exponentials $\{e_l\}_{l\in\Lambda^*+a_s}$ in the space $L^2(\Pi_\Lambda)$. It follows from the Poisson summation formula that the sequences $\{\langle f, e_l\rangle\}_{l\in\Lambda^*+a_s}$ and $\{f(\omega_r(x))\}_{r=1}^k$ are related by the following set of linear equations for almost every $x \in \Pi_\Lambda$:

$$\sum_{l\in\Lambda^*+a_s} \langle f, e_l\rangle e_l(x) = \mathrm{vol}(\Lambda)\sum_{r=1}^k (V(x))_{sr} f(\omega_r(x)), \qquad 1 \le s \le k. \tag{9}$$

By assumption, the lattice shifts $\{a_s\}$ are chosen so that $V(x)$ is invertible, leading to the following ℓ_2 estimate

$$\|V(x)^{-1}\|^{-2}\sum_{r=1}^k |f(\omega_r(x))|^2 \le \sum_{s=1}^k |\sum_{l\in\Lambda^*+a_s} \langle f, e_l\rangle e_l(x)|^2 \le \|V(x)\|^2 \sum_{r=1}^k |f(\omega_r(x))|^2.$$

Since both $\|V(x)\| = \max_{1\le i\le d}\sigma_i(V(x)) < \beta$ and $\|V(x)^{-1}\| = 1/\min_{1\le i\le d}\sigma_i(V(x)) > 1/\alpha$ are uniformly bounded, integrating over Π_Λ produces

$$A\|f\|^2_{L^2(\Omega)} \le \sum_{l\in L} |\langle f, e_l\rangle|^2 \le B\|f\|^2_{L^2(\Omega)}, \qquad L = \bigcup_{s=1}^k \Lambda^* + a_s, \tag{10}$$

where $0 < A \le B < \infty$ are constants.

To show (1), let $l = \eta^* + a_s$ for a fixed lattice point $\eta^* \in \Lambda^*$ and $1 \le s \le k$ and define the functions $G_{l;r}(x), 1 \le r \le$ to be the unique solutions to the system

$$\begin{pmatrix} 0 \\ \vdots \\ e_l(x) \\ \vdots \\ 0 \end{pmatrix} = \frac{1}{k}V(x)\begin{pmatrix} G_{l;1}(x) \\ \vdots \\ G_{l;r}(x) \\ \vdots \\ G_{l;k}(x)) \end{pmatrix}. \tag{11}$$

Since $\|(V(x))^{-1}\|$ is also uniformly bounded, $G_{l;r}$ are functions in $L^2(\Pi_\Lambda)$. Then the function uniquely defined by $g_l(\omega_r(x)) = G_{l;r}(x)$ is in $L^2(\Omega)$ and solves (9) for almost every $x \in \Pi_\Lambda$:

$$\frac{1}{|\Omega|}\begin{pmatrix} e_{a_1}(x) & & & & 0 \\ & \ddots & & & \\ & & e_{a_s}(x) & & \\ & & & \ddots & \\ 0 & & & & e_{a_k}(x) \end{pmatrix}\begin{pmatrix} \sum_{\lambda^*\in\Lambda^*}\langle g_l, e_{\lambda^*+a_1}\rangle e_{\lambda^*}(x) \\ \vdots \\ \sum_{\lambda^*\in\Lambda^*}\langle g_l, e_{\lambda^*+a_s}\rangle e_{\lambda^*}(x) \\ \vdots \\ \sum_{\lambda^*\in\Lambda^*}\langle g_l, e_{\lambda^*+a_k}\rangle e_{\lambda^*}(x) \end{pmatrix} = \frac{1}{k}V(x)\begin{pmatrix} g_l(\omega_1(x)) \\ \vdots \\ g_l(\omega_r(x)) \\ \vdots \\ g_l(\omega_k(x)) \end{pmatrix}. \tag{12}$$

Here we used that $\mathrm{vol}(\Lambda) = \frac{|\Omega|}{k}$ because Ω is a k-tile for Λ. Since the right hand side of the equations (11) and (12) are equal, we can equate the left hand side of both equations to obtain

$$\frac{1}{|\Omega|}\sum_{\lambda^*\in\Lambda^*}\langle g_l, e_{\lambda^*+a_{s'}}\rangle e_{\lambda^*}(x) = \begin{cases} e_{\eta^*}(x), & s = s' \\ 0, & s \neq s' \end{cases}.$$

Since $\{e_{\lambda^*}\}_{\lambda^*\in\Lambda^*}$ form an orthogonal basis for $L^2(\Pi_\Lambda)$, this implies that $\langle g_{\eta^*+a_s}, e_{\lambda^*+a_{s'}}\rangle = 0$ for $\lambda^* + a_{s'} \neq \eta^* + a_s$ and $|\Omega|$ otherwise. Therefore there is a unique sequence $\{g_l\}_{l\in L} \subset L^2(\Omega)$ satisfying (3) and subsequently all functions $f \in L^2(\Omega)$ have the unique representation (1), with $c_l = \langle f, g_l\rangle$. Then $\sum_{l\in L}|\langle f, g_l\rangle|^2 = \sum_{l\in L}|\langle f, \sum_{l'\in l}\langle g_l, e_l'\rangle e_l'\rangle|^2 = \sum_{l\in L}|\langle f, e_l\rangle|^2$. By (10), the family $\{e_l\}_{l\in L} \subset L^2(\Omega)$ satisfies the frame condition (2). We have shown that $\{e_l\}_{l\in L}$ is a Riesz basis of exponentials.

To find an explicit formula for the dual Riesz basis $\{g_l\}_{l\in L}$, the system (11) can be explicitly solved using only the s^{th} column of the matrix $(V(x))^{-1}$:

$$k(V(x)^{-1})_{rs}e_l(x) = g_l(\omega_r(x)), \quad 1 \le r \le k, \text{ a.e. } x \in \Pi_\Lambda.$$

Since $e_l(\omega_r(x)) = e^{2\pi i l\cdot(\lambda_r(x)+x)} = e_l(x)e^{2\pi i l\cdot\lambda_r(x)} = e_l(x)\overline{(V(x))_{sr}}$, we can write this as

$$k(V(x))_{sr}(V(x)^{-1})_{rs}e_l(\omega_r(x)) = g_l(\omega_r(x)), \quad 1 \le r \le k, \text{ a.e. } x \in \Pi_\Lambda. \tag{13}$$

The functions in (8) are obtained by extending the domain in (13) to all $x \in \Omega$ using the inverse functions $\omega_r^{-1}(x) = x - \lambda_r(x)$ for $x \in \Omega_r$.

□

Remark 1.
1. This result holds for all multi-tiling sets for which there exists a Riesz basis of exponentials of the form (4), including *admissible* multi-tiling sets. A multi-tiling set for a full lattice Λ is admissible if there exists an element of the dual lattice $v \in \Lambda^*$ and an integer $n \in \mathbb{Z}_+$ such that $v\cdot\lambda_1(x), \dots, v\cdot\lambda_k(x)$ are distinct integers *modulo* n for almost every x [14, 19, 6, 7].

2. For $k = 1$ the theorem implies that $g_l(x) = e_l(x)\chi_\Omega(x) = e_l(x)$ and the system is an orthogonal basis.

3. If the system is self-dual, that is $g_l = e_l$, then (13) implies that $(V(x)^{-1})_{rs} = \frac{1}{k}e^{2\pi i\lambda_r(x)\cdot a_s}$. Therefore, Theorem 2.1 shows that the system (7) forms an orthogonal basis if and only if $V(x)^*V(x) = kI$ meaning that $V(x)$ is a (log) Hadamard matrix [20].

3 Finding Riesz bases of exponentials

Suppose $\Omega \subset \mathbb{R}^d$ is a k-tile for the full lattice $\Lambda = M\mathbb{Z}^d$. Theorem 2.1 provides sufficient conditions on the set of *lattice shifts* $\{a_s\}_{s=1}^k \subset \mathbb{R}^d$ so that the two families of functions defined in (7) form a pair of biorthogonal Riesz bases for $L^2(\Omega)$. These conditions are based on the uniform bound on the singular values of the $k \times k$ matrices $V(x)$ defined in (6). For admissible domains Ω, the lattice shifts can be chosen so that the matrices $V(x)$ are square Vandermonde matrices $(V(x))_{rs} = w_r^{s-1}$ in which the nodes $w_r = e^{2\pi i(\lambda_r(x)\cdot v)/n}$ are a subset of the n^{th} roots of unity $\{w_r\}_{r=1}^k \subset \{e^{2\pi ij/n}\}_{j=1}^n$ for almost every x. Since the distance between any two

Table 1: List of Symbols used in §3.

$\mathcal{M}^d$	A given set of k vectors $m_1, \dots m_k$ in $\mathbb{R}^d$
$\mathcal{M}^l$	$= \{m_i \in \mathbb{R}^l \mid (m_i, m') \in \mathcal{M}^d \text{ for some } m' \in \mathbb{R}^{d-l}\}^1$
$\mathcal{Z}_i^l$	$= \begin{cases} \mathcal{M}^1 & l = i = 1 \\ \{z_l \in \mathbb{R} \mid (m_i, z_l) \in \mathcal{M}^l\} & l \geq 2, \end{cases}$
$\mathcal{M}_i^{l-1}$	$= \{m_j' \in \mathcal{M}^l \mid j \geq i\} \qquad l \geq 2$
$\mathcal{Q}_i(\mathcal{M}^l)$	$= \begin{cases} \{0, \dots, \#\mathcal{Z}_i^l - 1\} & i = 1 \\ \{\#\mathcal{Z}_{i-1}^l, \dots, \#\mathcal{Z}_i^l - 1\} & i \geq 2 \end{cases}$
$\mathcal{K}^l(\mathcal{M}^l)$	$= \begin{cases} \{0, \dots, \#\mathcal{M}^1 - 1\}, & l = 1 \\ \bigcup_{i=1}^{\#\mathcal{M}^{l-1}} \mathcal{K}^{l-1}(\mathcal{M}_i^{l-1}) \times \mathcal{Q}_i(\mathcal{M}^l) & l \geq 2 \end{cases}$

[1] $\mathcal{M}^{l-1} = \{m_i\}$ is enumerated so that $\#\mathcal{Z}_i^l \leq \#\mathcal{Z}_j^l$ for $i \leq j$.

nodes has a uniform lower bound, that is, $\max_{r \neq r'} |w_r - w_{r'}| \geq |1 - e^{2\pi i/n}| > 0$, it follows from [13] that the matrices $V(x)$ have uniformly bounded singular values for almost every $x \in \Pi_\Lambda$.

In general, determining the invertibility of Fourier matrices is challenging, especially when d is large. Even when invertibility is guaranteed, directly solving the linear system (9) becomes increasingly difficult as k and d grow. In this section, we will discuss a strategy for reconstructing functions using a family of exponentials and finding a Riesz basis of exponentials for a class of multi-tiles.

To keep the notations simple, we collect in Table 1 a list of symbols used in this section.

For $x \in \Pi_\Lambda$ define the *frequency set* $\mathcal{M}^d = \mathcal{M}^d(x) = \Lambda_x$. Since the *shift index set* $\mathcal{K}^d(\mathcal{M}^d)$ is defined recursively, we first prove the following lemma.

Lemma 3.1. *For each $l = 1, \dots d$, $\#\mathcal{K}^l(\mathcal{M}^l) = \#\mathcal{M}^l$.*

Proof. By induction. This is true for $l = 1$ by definition. Consider $1 < l \leq d$. The induction assumption asserts that there are $\#\mathcal{M}^{l'}$ elements in $\mathcal{K}^{l'}(\mathcal{M}^{l'})$ for $1 \leq l' < l$. Therefore there are $\#\mathcal{M}^{l-1}$ elements in $\mathcal{K}^{l-1}(\mathcal{M}^{l-1})$ and for $i > 1$, $\#\mathcal{M}^{l-1} - i + 1 = \#\mathcal{M}_i^{l-1} = \#\mathcal{K}^{l-1}(\mathcal{M}_i^{l-1})$. Then, there are $\#\mathcal{Z}_1^l$ in $\mathcal{Q}_1(\mathcal{M}^l)$ and for $i > 1$, $(\#\mathcal{Z}_i^l - \#\mathcal{Z}_{i-1}^l)$ elements in $\mathcal{Q}_i^l(\mathcal{M}^l)$. Summing this up,

$$\begin{aligned} \#\mathcal{K}^l(\mathcal{M}^l) &= \#\mathcal{M}^{l-1}\#\mathcal{Z}_1^l + \sum_{i=2}^{\#\mathcal{M}^{l-1}} (\#\mathcal{M}^{l-1} - i + 1)(\#\mathcal{Z}_i^l - \#\mathcal{Z}_{i-1}^l) \\ &= \sum_{i=1}^{\#\mathcal{M}^{l-1}} \#\mathcal{Z}_i^l = \# \bigcup_{i=1}^{\#\mathcal{M}^l} \{(m_i, z_l) \mid z_l \in \mathcal{Z}_i^l\} = \#\mathcal{M}^l. \end{aligned}$$

□

The construction of the shift index set $\mathcal{K}^d(\mathcal{M}^d)$ is based on a tree structure admitted by the frequency set $\mathcal{M}^d$. The sets $\mathcal{M}^l$ correspond to the collection of vectors in each parent node in the l^{th} level of the tree. The ordering $\mathcal{M}^{l-1} = \{m_i\}$ corresponds to nodes in increasing order of the number of immediate children. The sets $\mathcal{Z}_p^l$ correspond to the last coordinate of the children of the p^{th} vector in $\mathcal{M}^{l-1}$.

Example 1. Let $k = 10$ and consider the frequency set

$$\mathcal{M}^4 = \left\{ \begin{array}{cccc} (1,1,1,1), & (2,1,1,1), & (3,1,1,1), & (4,1,1,1), \\ & (2,2,1,1), & (3,2,1,1), & (4,2,1,1), \\ & (2,2,1,2), & (3,2,2,1), & (4,3,1,1) \end{array} \right\}.$$

The tree diagram produced by $\mathcal{M}^4$ is illustrated in Figure 1.

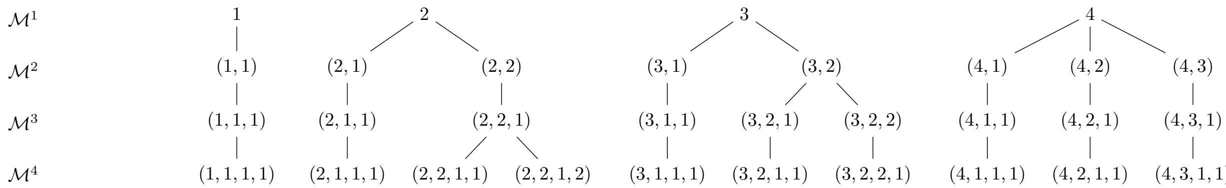

Figure 1: Tree structure produced by $\mathcal{M}^4$ in Example 1.

3.1 Finding Riesz bases of exponentials

We first define a weaker notion of admissibility.

Definition 3.1. Let $\Omega \subset \mathbb{R}^d$ be a k-tile for a full lattice Λ. For $v \in \mathbb{R}^d$ and $q \in \mathbb{Z}_+^d$ we say that Ω is *weakly (v, q)-admissible* if for almost every $x \in \Pi_\Lambda$ and $\mathcal{M}^d = \mathcal{M}^d(x) = \Lambda_x$, the following condition is satisfied:

$$\#\{v_l z_l \mod q_l \mid z_l \in \mathcal{Z}_p^l\} = \#\mathcal{Z}_p^l, \quad 1 \le p \le N_{l-1}, \quad 1 \le l \le d,$$

where $N_0 = 1$ and $N_l = \#\mathcal{M}^l$ if $1 \le l \le d$.

These conditions state that the numbers $v_l z_l$ are distinct modulo q_l. In particular, any bounded k-tile for a lattice Λ is weakly admissible for some pair (v, q). Weakly admissible domains are not necessarily admissible as defined in Remark 1.

Theorem 3.2. *Suppose that $\Omega \subset \mathbb{R}^d$ is a k-tile for a full lattice Λ that is weakly (v, q)-admissible for some $q \in \mathbb{Z}_+^d$ and $v \in \mathbb{R}^d$. Then, for almost every $x \in \Omega$, $f(x)$ can be uniquely determined from the data*

$$\{\langle f, e_l \rangle\}_{l \in L}, \qquad L = \bigcup_{s=1}^{k} \Lambda^* + a_s,$$

where the lattice shifts are given by

$$\{a_s(x) = \delta j_s + \eta \mid j_s \in \mathcal{K}^d(\mathcal{M}^d(x))\}, \tag{14}$$

for any diagonal matrix $\delta \in \mathbb{R}^{d \times d}$ of the form $\delta_{ll} = v_l/q_l$, $1 \le l \le d$ and η is any point in the dual lattice Λ^*.

Proof. Consider the system of equations (9) given by $(F_{j_s}(x))_{s=1}^k = V(x)(F^{\lambda_r}(x))_{r=1}^k$ for almost every $x \in \Pi_\Lambda$, where $V(x) = V$ has the form (6) and

$$F_{j_s}(x) = F_{j_s} = \frac{1}{\mathrm{vol}(\Lambda)} \sum_{\lambda^* \in \Lambda^*} \langle f, e_{\lambda^* + a_s} \rangle e_{\lambda^* + a_s}(x), \tag{15}$$

$$F^{\lambda_r}(x) = F^{\lambda_r} = f\left(\omega_r(x)\right). \tag{16}$$

Notice that $e^{2\pi i \lambda_r(x) \cdot (\delta j_s + \eta)} = e^{2\pi i \lambda_r(x) \cdot \delta j_s}$ so it suffices to consider shifts of the form $a_s = \delta j_s$. Since for almost every $y \in \Omega$, $y = \omega_r(x)$ for some $x \in \Pi_\Lambda$ and $1 \le r \le k$, we will prove the unique recovery of $f(x)$ by showing the invertiblity of the matrices $V^l(x) = V^l$, $1 \le l \le d$, defined by

$$(V^l)_{sp} = e^{-2\pi i \delta'' j_s'' \cdot m_p}, \qquad j_s'' \in \mathcal{K}^l(\mathcal{M}^l), \quad m_p \in \mathcal{M}^l. \tag{17}$$

Here, $\mathcal{K}^l(\mathcal{M}^l)$ is the set of $\#\mathcal{M}^l$ shift indices constructed in Lemma 3.1, and we define the diagonal matrices $\delta' = (\delta_{ij})_{l+1 \le i,j \le d}$ and $\delta'' = (\delta_{ij})_{1 \le i,j \le l}$. This is equivalent to proving the uniqueness of solutions of the linear systems

$$\left(F_{(j_s'', j')}\right)_{j_s'' \in \mathcal{K}^l(\mathcal{M}^l)} = V^l \left(F_{j'}^{m_p}\right)_{m_p \in \mathcal{M}^l} \tag{18}$$

for all $1 \le l \le d$ where for a fixed $j' \in \mathbb{R}^{d-l}$ we define

$$F_{j'}^{m_p} = \sum_{\substack{m' \in \mathbb{R}^{d-l}: \\ (m_p, m') \in \mathcal{M}^d}} F^{(m_p, m')} e^{-2\pi i \delta' j' \cdot m'}.$$

In the case $l = 1$, $V^1 = W_1^1$, where W_1^1 is a $\#\mathcal{M}^1 \times \#\mathcal{M}^1$ Vandermonde matrix $(W_1^1)_{st} = w_t^{s-1}$ with nodes $w_t = e^{-2\pi i \delta_{11} z_t}$, $z_t \in \mathcal{M}^1$. By the admissibility condition, the set $\{v_1 z_1 \mod q_1 \mid z_1 \in \mathcal{Z}_1^1\}$ contains $\#\mathcal{Z}_1^1$ distinct numbers and therefore the nodes w_t are distinct, and W_1^1 is invertible for $\delta_{11} = v_1/q_1$.

For $l \ge 2$ and $p = 0, \ldots, \#\mathcal{M}^{l-1}$ define the 1D Vandermonde matrices $\tilde{W}_{(q,p)}^l \in \mathbb{C}^{\#\mathcal{Z}_p^l \times \#\mathcal{Z}_q^l}$ for $q < p$ and $W_{(p,p)}^l \in \mathbb{C}^{\#\mathcal{Z}_p^l \times \#\mathcal{Z}_p^l}$ by

$$\begin{cases} (\tilde{W}_{(q,p)}^l)_{st} = e^{2\pi i (s-1) z_t \delta_{ll}} & z_t \in \mathcal{Z}_q^l \\ (W_{(p,p)}^l)_{st} = e^{2\pi i (s-1) z_t \delta_{ll}} & z_t \in \mathcal{Z}_p^l, \end{cases}, \qquad 1 \le s \le \#\mathcal{Z}_p^l.$$

Define the matrices $\tilde{V}_p^{l-1} \in \mathbb{C}^{\#\mathcal{M}_p^{l-1} \times \#\mathcal{M}_q^{l-1}}$ and $V_p^{l-1} \in \mathbb{C}^{\#\mathcal{M}_p^{l-1} \times \#\mathcal{M}_p^{l-1}}$

$$\begin{cases} (\tilde{V}_p^{l-1})_{sq} = e^{-2\pi i m_q \cdot \delta'' j_s''}, & q < p \\ (V_p^{l-1})_{s(q-p+1)} = e^{-2\pi i m_q \cdot \delta'' j_s''}, & q \ge p \end{cases}, \quad j_s'' \in \mathcal{K}^{l-1}(\mathcal{M}_p^{l-1}), \qquad m_q \in \mathcal{M}_p^{l-1}.$$

Then, we can form the linear systems from (18) as

$$\left(F_{(j'', j_l, j')}\right)_{j'' \in \mathcal{K}^{l-1}(\mathcal{M}_1^{l-1})} = V_1^{l-1} \left(F_{(j_l, j')}^{m_q}\right)_{m_q \in \mathcal{M}_1^{l-1}}, \qquad j_l \in \mathcal{Q}_1^l, \tag{19}$$

$$\left(F_{(j'', j_l, j')}\right)_{j'' \in \mathcal{K}^{l-1}(\mathcal{M}_p^{l-1})} - \tilde{V}_p^{l-1} \left(F_{(j_l, j')}^{m_q}\right)_{q<p} = V_p^{l-1} \left(F_{(j_l, j')}^{m_q}\right)_{m_q \in \mathcal{M}_p^{l-1}}, \qquad j_l \in \mathcal{Q}_p^l, \quad p \ge 2. \tag{20}$$

The vector $\left(F_{(j_l, j')}^{m_p}\right)_{j_l \in \mathcal{Q}_i^l, 1 \le i \le p}$, created from solutions to (19) and (20) when V_p^{l-1} are all invertible, solves the system

$$\left(F_{(j_l, j')}^{m_p}\right)_{j_l \in \mathcal{Q}_i^l, 1 \le i \le p} = W_{(p,p)}^l \left(F_{j'}^{(m_p, z_l)}\right)_{z_l \in \mathcal{Z}_p^l}, \tag{21}$$

where the square matrices $W_{(p,p)}^l$ are invertible again by weak admissibility.

To write (19) – (21) in matrix form, we define $[V_p^{l-1}]$, $[\tilde{V}_p^{l-1}]$ to be block diagonal matrices with $\#\mathcal{Q}_p^l$ block entries all equal to V_p^{l-1} and $\tilde{V}_p^{l-1}$, respectively, and

$$[\tilde{W}_p^l] = \begin{pmatrix} \tilde{W}_{(1,p)}^l & & & 0 \\ & \tilde{W}_{(2,p)}^l & & \\ & & \ddots & \\ 0 & & & \tilde{W}_{(p-1,p)}^l \end{pmatrix}, \qquad [W_p^l] = \begin{pmatrix} W_{(1,1)}^l & & & 0 \\ & W_{(2,2)}^l & & \\ & & \ddots & \\ 0 & & & W_{(p-1,p-1)}^l \end{pmatrix}.$$

The block diagonal matrix $[W_p^l]$ is invertible, and for appropriate permutation matrices[1] $\tilde{U}_p$, and U_p, the matrix defined as

$$X_p = [\tilde{V}_p^{l-1}] \tilde{U}_p [\tilde{W}_p^l]([W_p^l])^{-1} U_p,$$

satisfies

$$X_p \left(\left(F_{(j_l, j')}^{m_q}\right)_{q \ge i, j_l \in \mathcal{Q}_i^l} \right)_{i<p} = \left(\tilde{V}_p^{l-1} \left(F_{(j_l, j')}^{m_q}\right)_{q<p} \right)_{j_l \in \mathcal{Q}_p^l}.$$

[1] U_p and $\tilde{U}_p$ correspond to the mappings $U_p((v)_{q \ge i, j_l \in \mathcal{Q}_i^l})_{i<p} = ((v)_{z_l \in \mathcal{Z}_q^l})_{q<p}$ and $\tilde{U}_p((v)_{j_l \in \mathcal{Q}_p^l})_{q<p} = ((v)_{q<p})_{j_l \in \mathcal{Q}_p^l}$.

Therefore (19) − (20) can be written as

$$\begin{pmatrix} \left(F_{(j'',j_l,j')}\right)_{j''\in\mathcal{K}^{l-1}(\mathcal{M}_1^{l-1}),j_l\in\mathcal{Q}_1^l} \\ \left(F_{(j'',j_l,j')}\right)_{j''\in\mathcal{K}^{l-1}(\mathcal{M}_2^{l-1}),j_l\in\mathcal{Q}_2^l} \\ \vdots \\ \left(F_{(j'',j_l,j')}\right)_{j''\in\mathcal{K}^{l-1}(\mathcal{M}_p^{l-1}),j_l\in\mathcal{Q}_p^l} \\ \vdots \end{pmatrix} = \begin{pmatrix} [V_1^{l-1}] & & \cdots & & 0 \\ X_2 & [V_2^{l-1}] & & & \\ & & \ddots & & \vdots \\ X_p & & & [V_p^{l-1}] & \\ \vdots & & & & \ddots \end{pmatrix} \begin{pmatrix} \left(F^{m_q}_{(j_l,j')}\right)_{q\geq 1,j_l\in\mathcal{Q}_1^l} \\ \left(F^{m_q}_{(j_l,j')}\right)_{q\geq 2,j_l\in\mathcal{Q}_2^l} \\ \vdots \\ \left(F^{m_q}_{(j_l,j')}\right)_{q\geq p,j_l\in\mathcal{Q}_p^l} \\ \vdots \end{pmatrix}. \tag{22}$$

The solutions to (22) can then be rearranged to form the linear system,

$$\left(F^{m_p}_{(j_l,j')}\right)_{m_p\in\mathcal{M}^{l-1},j_l\in\cup_{i=1}^p\mathcal{Q}_i^l} = W^l\left(F_{j'}^{(m_p,z)}\right)_{(m_p,z)\in\mathcal{M}^l} \tag{23}$$

where $W^l = [W^l_{(\#\mathcal{M}^{l-1},\#\mathcal{M}^{l-1})}]$ is an invertible matrix.

Putting it all together, for an appropriate permutation matrix U, and the block lower triangular matrix in (22) denoted by $[[V^{l-1}]]$, it follows that the matrix V^l in (17) can be expressed as $V^l = [[V^{l-1}]]UW^l$, and it is invertible provided that V_p^{l-1} are all invertible. The case $l = d$ is proved in the same way, with modifications in notation, implying that the terms $\{F^{\lambda_r}(x)\}_{\lambda_r=m\in\mathcal{M}^d(x)}$ are determined uniquely. Then, $f(x)$ can be uniquely constructed as

$$\begin{aligned} f(x) &= \sum_{r=1}^k f(x)\chi_{\Omega_r}(x) = \sum_{r=1}^k F^{\lambda_r}(\omega_r^{-1}(x))\chi_{\Omega_r}(x) \\ &= \frac{1}{\mathrm{vol}(\Lambda)}\sum_{s=1}^k\sum_{\lambda^*\in\Lambda^*}\langle f, e_{\lambda^*+a_s}\rangle\left(\sum_{r=1}^k (V(x)^{-1})_{rs}(V(x))_{sr}\chi_{\Omega_r}(x)\right)e_{\lambda^*+a_s}(x). \end{aligned}$$

Note that this reconstruction has the form $f(x) = \sum_{l\in L} c_l g_l(x)$ with $c_l = \frac{1}{|\Omega|}\langle f, e_l\rangle$ and $g_l(x)$ is defined as in Theorem 2.1, however, weak admissibility of Ω does not guarantee that g_l is a function in $L^2(\Omega)$.

□

The proof can be summarized in the following pointwise reconstruction algorithm for functions $f \in L^2(\Omega)$, where Ω is a weakly (v, q)-admissible domain. For almost every $x' \in \Omega$, there is a unique $r \in \{1, \dots k\}$ so that $x' \in \Omega_r$, so it suffices to state the algorithm for recovering $f(\omega_r(x)), x \in \Pi_\Lambda$.

Algorithm 1. Define $\mathcal{M}^d(x) = \mathcal{M}^d = \Lambda_x$. Construct the lattice shift set $\{a_s\}_{s=1}^k$ given by (14) for a fixed diagonal matrix $\delta \in \mathbb{R}^{d\times d}$ satisfying the assumptions of Theorem 3.2.

Step 1: For each $j \in \mathcal{K}^d(\mathcal{M}^d)$, define $F_j = F_j(x)$ by (15).

Step 2: If $d = 1$, solve the 1D Vandermonde system (18) for $l = 1$, obtaining $(F^{m_1})_{m_1\in\mathcal{M}^1}$. Then, skip to Step 4. If $d \geq 2$, solve (18) with $l = 1$ for all $j'' \in \mathcal{K}^1(\mathcal{M}^1)$ and $j'_1 = 0 \in \mathbb{Z}^{d-1}$, obtaining $\left(F^{m_1}_{j'_1}\right)_{m_1\in\mathcal{M}^1}$.

Step 3: For $l = 2, \dots, d-1$ and $\mathcal{M}^{l-1} = \{m_q\}$

(a) For $p = 1$ repeat the $l-1$ iteration (Step 2 for $l = 2$ or Step 3 for $l > 2$), substituting j'_{l-1} with (j_l, j'_l) where $j_l \in \mathcal{Q}_1^l$ and $j'_l = 0 \in \mathbb{R}^{d-l}$ to obtain $\left(F^{m_q}_{(j_l,j'_l)}\right)_{m_q\in\mathcal{M}_1^{l-1}}$. Then, determine $\left(F^{(m_1,z_l)}_{j'_l}\right)_{z_l\in\mathcal{Z}_1^l}$ by solving the 1D Vandermonde system (21).

(b) For $p = 2, \dots, \#\mathcal{M}^{l-1}$. Define

$$\tilde{F}_{(j'',j_l,j')} := \left(F_{(j'',j_l,j')}\right)_{j''\in\mathcal{K}^{l-1}(\mathcal{M}_p^{l-1})} - \tilde{V}_p^{l-1}\left(F_{(j_l,j')}^{m_q}\right)_{q<p}$$

and $\tilde{\mathcal{M}}^d = \{m \in \mathcal{M}^d \mid (m_1, \dots, m_l) \in \mathcal{M}_p^l\}$. Repeat Step 3 a) substituting F_j and $\mathcal{M}^d$ with $\tilde{F}_j$ and $\tilde{\mathcal{M}}^d$, obtaining $\left(F_{j_l'}^{(m_p,z_l)}\right)_{z_l\in\mathcal{Z}_p^l}$.

Step 4: For $l = d$, perform Step 3 omitting j_d' to obtain

$$f(\omega_r(x)) = f(\lambda_r(x) + x)\}_{\lambda_r\in\Lambda_x}.$$

Although Theorem 3.2 guarantees that the matrix $V(x)$ in (6) is invertible, Algorithm 1 does not directly find the inverse. Instead, the algorithm iteratively solves the system in a block-by-block fashion, and therefore only involves the inversion of 1D Vandermonde matrices. Equations (19) and (20) are only solved directly in the case $l = 1$.

Remark 2. The proof of Theorem 3.2 provides an explicit procedure for recovering functions that arise in, for example, [19], in which the existence of the set of vectors $\{a_s\}_{s=1}^k \subset \mathbb{R}^d$ is proved by showing the existence of Λ-periodic functions $\tilde{f}_s \in L^2(\Pi_\Lambda), 1 \le s \le k$, such that

$$f(\lambda_r(x) + x) = \sum_{s=1}^{k} e^{2\pi i a_s\cdot(x-\lambda_r(x))}\tilde{f}_s(x), \qquad 1 \le r \le k, \tag{24}$$

and showing that the determinant of the matrix for this system as a function of x has finitely many zeros. Theorem 3.2 provides one way to intuitively choose the lattice shifts for certain domains Ω by setting $\{a_s = \delta j_s\}_{s=1}^k$, $j_s \in \mathcal{K}^d(\mathcal{M}^d)$ for a suitable matrix $\delta \in \mathbb{R}^{d\times d}$.

Algorithm 1 provides a procedure for the pointwise (and adaptive) reconstruction of functions in $L^2(\Omega)$ that does not rely on a Riesz basis of exponentials. In fact, it may not be known *a priori* if V in (16) satisfies the assumptions of Theorem 2.1. The factorization of V in the proof of Theorem 3.2 provides a systematic procedure for choosing the lattice shifts $\{a_s\}$ and estimating the Riesz bounds (when they exist). The algorithm finds a Riesz basis of exponentials for the following subset of domains satisfying Definition 3.1.

Definition 3.2. Let $\Omega \subset \mathbb{R}^d$ be a k-tile for a full lattice Λ. For $v \in \mathbb{R}^d$ and $q \in \mathbb{Z}_+^d$ we say that Ω is *strongly* (v, q)*-admissible* if it is weakly (v, q)-admissible and for almost every $x \in \Pi_\Lambda$ the sets $\{v_l z_l \mod q_l \mid z_l \in \mathcal{Z}_p^l\}, 1 \le p \le N_{l-1}, 1 \le l \le d$ are all sets of integers.

For example, any bounded multi-rectangle in $\mathbb{R}^d$ of the form $\Omega = \cup_{r=1}^k(\Pi_\Lambda + \lambda_r)$ with $\{\lambda_r\} \subset \Lambda$ is strongly admissible. The advantage of strong admissibility is the uniform boundedness of $\|V(x)\|$ which we will show next leads to a Riesz basis of exponentials.

Corollary 3.2.1. *Any strongly* (v, q)*-admissible* k*-tile for a full lattice* Λ *admits a Riesz basis of exponentials* $\{e_l\}_{l\in L}$ *of the form given in Theorem 3.2.*

Proof. Let $\Omega \subset \mathbb{R}^d$ be a strongly (v, q)-admissible k-tile for a full lattice Λ. Since the matrix $V(x)$ given in (6) is piecewise constant on Π_Λ, there exists a normalized eigenvector of $V(x)$, denoted by $u(x) = (u_1(x), \dots, u_k(x))$, that corresponds to an eigenvalue with squared magnitude $\sigma(x)$. It follows that $u(x)$ is also piecewise constant and each of its entries $u_r(x)$ are functions in $L^2(\Pi_\lambda)$.

Define the measurable function $f(x) \in L^2(\Omega)$ by $f(\omega_r(x)) \equiv u_r(x)$ for $x \in \Pi_\Lambda$. Square-integrability follows immediately: $|\Pi_\lambda| = \int_{\Pi_\lambda}\sum_{r=1}^k |u_r(x)|^2dx = \int_{\Pi_\lambda}\sum_{r=1}^k |f(\omega_r(x))|^2dx = \int_\Omega |f(x)|^2dx$. By taking the ℓ^2-norm of the vector $(F_{j_s}(x))_{s=1}^k = V(x)\left(f(\omega_r(x))\right)_{r=1}^k = V(x)u(x)$,

$$\sigma(x) = \|V(x)u(x)\|^2 = \sum_{s=1}^{k} |F_{j_s}(x)|^2. \tag{25}$$

We will use the proof of Theorem 3.2 to show that $\sigma(x)$ is uniformly bounded. Taking the ℓ_2 norm of (18), for $l=1$ and $j' \in \mathbb{R}^{d-1}$, we obtain

$$\|V^1(x)^{-1}\|^{-2} \sum_{m_1 \in \mathcal{M}^1} |F_{j'}^{m_1}(x)|^2 \le \sum_{j_1 \in \mathcal{K}^1(\mathcal{M}^1)} |F_{(j_1,j')}(x)|^2 \le \|V^1(x)\|^2 \sum_{m_1 \in \mathcal{M}^1} |F_{j'}^{m_1}(x)|^2.$$

Applying the inequality (23) for $l = 2, \dots, d$,

$$\prod_{l=1}^{d} \|W^l(x)^{-1}\|^{-2} \sum_{m_p \in \mathcal{M}^d} |F^{m_p}(x)|^2 \le \sum_{j \in \mathcal{K}^d(\mathcal{M}^d)} |F_j(x)|^2 \le \prod_{l=1}^{d} \|W^l(x)\|^2 \sum_{m_p \in \mathcal{M}^d} |F^{m_p}(x)|^2.$$

By (25) this implies that $\alpha \le \sigma(x) \le \beta$, where

$$\alpha = \inf_{x \in \Pi_\Lambda} \left(\prod_{l=1}^{d} \|W^l(x)^{-1}\|^{-2} \right), \qquad \beta = \sup_{x \in \Pi_\Lambda} \left(\prod_{l=1}^{d} \|W^l(x)\|^2 \right).$$

The block matrices $W^l(x)$ in (23) all have uniformly bounded norm because the square Vandermonde matrices $W^l_{(p,p)}(x)$ have nodes that form a subset of the q_l roots of unity by the strong admissibility condition. Therefore $\alpha > 0$, and $\beta < \infty$, and Theorem 2.1 can then be applied to complete the proof.

□

The Riesz bounds A and B give a sense of the stability of the reconstruction (1). For the choice of lattice shifts $\{a_s\}$ given (14), these constants depend on the conditioning of the matrices $W^l(x)$. There are infinitely many feasible choices for $\delta \in \mathbb{R}^{d \times d}$ and $\eta \in \Lambda^*$ that can produce a Riesz basis of exponentials of the form (4), however, for $|\delta_{ll}| \ll 1$ the condition numbers of these matrices grow large. It is, in general, a hard problem to estimate the condition number of a Vandermonde matrix [23, 1, 13], and although the question of determining the optimal choice of δ is an important one, we consider this direction out of the scope of the present work. However, in special cases, we can show that the factorization of $V(x)$ produced by Algorithm 1 can be used to derive optimally conditioned matrices.

Definition 3.3. We say that a k-tile for a full lattice is *perfectly admissible* if there exists a vector v such that it is strongly (v, q^*)-admissible, where $(q^*)_l = \#\mathcal{Z}^l_1 = \ldots = \#\mathcal{Z}^l_{N_{l-1}}$.

For example, any bounded multi-tiling for the full lattice $\Lambda = M\mathbb{Z}^d$ of the form $\Omega = \cup_{z \in \mathcal{M}^d}(\Pi_\Lambda + Mz)$ with $\mathcal{M}^d = \{Mz \mid z = (z_1, \dots, z_d)^T \mid\in \mathbb{Z}^d, \mid |z_i| \le K, 1 \le i \le d\}$ for any $K \ge 1$ is perfectly admissible. We show next that perfect admissibility guarantees the existence of an orthogonal basis of exponentials.

Corollary 3.2.2. *Any perfectly admissible k-tile for a full lattice Λ admits an orthogonal basis of exponentials $\{e_l\}_{l \in L}$ of the form (4).*

Proof. Define the lattice shifts $\{a_s\}_{s=1}^k$ by (14) for the perfectly admissible domain Ω with (v, q^*) as in Definition 3.3. Then for $1 < l \le d$ and $1 \le p \le \#\mathcal{M}^{l-1}$ the sets $\{v_l z \mid z \in \mathcal{Z}^l_p\}$ form a complete residue set modulo $\#\mathcal{Z}^l_p$, and it follows that each set $\mathcal{M}^l$ can be ordered so that $W^l_1 = W^l_2 = \ldots = W^l_p$ and the matrix V in (6) has the form $V = W^1_1 \otimes W^2_1 \otimes \ldots \otimes W^d_1$. For the choice of $\delta_{ll} = v_l / \#\mathcal{Z}^l_1$, the nodes of W^l_1 form the $\#\mathcal{Z}^l_1$ roots of unity. Since the singular values of V are products of the singular values of each factor, it holds that $\kappa(V(x)) = \prod_l \kappa(W^l(x)) = 1$, where $\kappa(V(x)) = \frac{\sigma_{\max}(V(x))}{\sigma_{\min}(V(x))}$ is the condition number of the matrix $V(x)$ for $x \in \Pi_\Lambda$, defined as the ratio of the maximum singular value σ_{max} and minimum singular value σ_{min}.

Since $V(x)^*V(x) = kI \iff \kappa(V(x)) = 1$, Remark 1 implies that the family $\{e_l\}_{l \in L}$ in (7) forms an orthogonal basis of exponentials for $L^2(\Omega)$.

□

The following proves a partial converse result in one dimension.

Corollary 3.2.3. *Let Ω be k-tile for $\Lambda = \mathbb{Z}$. Suppose that there exists an orthogonal basis of exponentials for $L^2(\Omega)$ of the form $\{e_{\lambda^* + s\delta}\}_{\lambda^* \in \Lambda^*, 1 \le s \le k}$ for some $\delta \in \mathbb{R}$. If, in addition, there exists a number $Q \in \mathbb{Z}$ so that $Q = gcd(\Lambda_x = \{z_1, \dots, z_k\})$ for almost every $x \in \Pi_\Lambda$, then Ω is perfectly admissible.*

Proof. For any set $z_1, \ldots, z_k \in \mathbb{Z}$, it is known that there exists a $\tau \in \mathbb{R}$ so that the Vandermonde matrix V with entries $(V)_{st} = e^{\frac{2\pi i \tau s z_t}{k}}$ is perfectly conditioned, that is, $\kappa(V) = 1$, if and only if $\{\frac{z_r}{Q} \bmod k\}_{r=1}^{k}$ is a complete residue system, where $Q = \gcd\{z_r\}_{r=1}^{k}$ [4]. The choice $\tau = \pm\frac{1}{Q} + nk$ for an integer $n \in \mathbb{Z}$ produces a perfectly conditioned matrix. For the choice $v = \tau/k$, and letting $q^* = k$, Ω satisfies the conditions of perfect admissibility.

□

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Listing 1: MATLAB code for finding $\mathcal{K}^d(\mathcal{M}^d)$

```
function K=K_l(Ml)
[n_Ml,l]=size(Ml);
%  If $l=1$, return $\{0, \hdots, \#\mathcal{M}^1-1\}$, otherwise, append the sets $K_i^l\times Q_i^l$
if l==1
    K = (0:(n_Ml-1))';
    return
else
    [Ml_old, n_Zl]=Ml_sort(Ml,l-1);
    [n_Ml, ~]=size(Ml_old);
    K=[];
    for i=1:n_Ml
        if i==1
            Qi_l = (0:(n_Zl(i)-1))';
        else
            Qi_l = (n_Zl(i-1):(n_Zl(i)-1))';
        end

        Ml_i=Ml_old(i:end,:);
        Kl_i=K_l(Ml_i);
        for ii=1:length(Kl_i(:,1))
            for jj=1:length(Qi_l)
                K=[K; Kl_i(ii,:) Qi_l(jj)];
            end

        end
    end
end

%  Frequencies in tiling lattice $\Lambda$
function [Ml n_Zl]=Ml_sort(Md,l)
if l==0
    Ml=Md;
    n_Zl=0;
else
    [Ml ia, ic] = unique(Md(:,1:l),'rows');
    n_Zl = histc(ic,unique(ic));
    [n_Zl, idx] = sort(n_Zl);
    Ml=Ml(idx,:);
end
```