Editors' Suggestion

Majorana end states in an interacting quantum wire

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We propose and investigate a simple one-dimensional model for a single-channel quantum wire hosting electrons that interact repulsively and are subject to a significant spin-orbit interaction. We show that an external Zeeman magnetic field, applied at the right angle to the Rashba spin-orbit axis, drives the wire into a correlated spin-density wave state with gapped spin and gapless charge excitations. By computing the ground-state degeneracies of the model with either (anti)periodic or open boundary conditions, we conclude that the correlated spin-density state realizes a gapless symmetry-protected topological phase, as the ground state is unique in the ring geometry while it is twofold degenerate in the wire with open boundaries. Microscopically the twofold degeneracy is found to be protected by the conservation of the *magnetization parity*. Open boundaries induce localized zero-energy (midgap) states which are described, at the special Luther-Emery point of the model, by Majorana fermions. We find that spin densities at the open ends of the wire exhibit unusual long-ranged correlations despite the fact that all correlations in the bulk of the wire decay in a power-law or exponential fashion. Our study exposes the crucial importance of the long-ranged string operator needed to implement the correct commutation relations between spin densities at different points in the wire. Along the way we rederive the low-energy theory of Galilean-invariant electron systems in terms of current operators.

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I. INTRODUCTION

The search for condensed matter realization of Majorana fermions has been at the center of intense theoretical and experimental efforts in the last decade. It is strongly motivated by the promise of topological quantum computing as well as by its fundamental importance to our current understanding of numerous topological phases of matter [1,2]. Topological superconducting wires represent one of the most promising platforms for realizing Majorana end states [3–6]. By now, several experimental groups have reported transport and STM tunneling data consistent with Majorana physics [7–11], and many more studies are currently under way.

A topological superconducting wire is obtained by bringing a semiconducting quantum wire with significant spin-orbit interaction into close contact with an *s*-wave superconductor and then applying an external (Zeeman) magnetic field in the direction orthogonal to the Rashba spin-orbit axis of the wire [5,6,12]. Provided that the chemical potential lies within the gap induced by the Zeeman field, the wire effectively realizes a one-dimensional *p*-wave superconductor which features localized Majorana states at the open ends of the wire (more generally, at the boundaries between topological and trivial phases). A single-channel topological superconducting wire is the simplest realization and can be extended to more complex/other geometries such as multichannel wires [13,14], wires with periodic modulation of the spin-orbit potential [15], and chains of magnetic adatoms on the surface of a superconductor [16,17]. Realistic modeling of semiconductor-superconductor heterostructures has been developed [18,19].

Electron interactions are very important in zero- and one-dimensional systems [20,21] and their effect on the topological properties of the suggested quantum wire setup was investigated early on [22–24], and some exact/rigorous results were obtained [25,26]. In parallel, a search for strongly interacting wires with *algebraic* superconducting correlations [27] which would remove the need for close proximity of the wire to the macroscopic superconductor has begun. Kitaev's toy *p*-wave superconductor model is characterized by the twofold degeneracy of the ground states with *different fermion parities*, i.e., between the ground states with an even and an odd number of fermions in the wire with open ends [1,28]. This degeneracy makes it clear that in a wire with a fixed total number of electrons the conservation of the *subband parity* acquires crucial importance. One-dimensional models with superconducting interband interactions conserving subband parity [27,29–33] are found to possess twofold degeneracy in their ground state and thus represent one-dimensional topological states with exponentially localized Majorana modes at their open ends. Interestingly, their topological nature is preserved despite the presence of the critical center-of-mass fluctuations in the bulk.

Subband parity, in the form of a specific *magnetization parity*, plays the key role in our work as well. Unlike most of the previous studies, however, we present a physical realization of the parity-conserving system in a realistic quantum wire with purely *repulsive* electron-electron interaction. Our key finding is that localized Majorana end states can be realized in a simpler setting which does not require proximity to an *s*-wave superconductor. All that is needed is a single-channel

quantum wire with significant spin-orbit and strong repulsive interactions between electrons. Applying an external magnetic field in the direction orthogonal to the spin-orbit axis of the wire drives its many-electron state into a correlated spin-density wave (SDW) phase with a finite spin gap in the bulk and nontrivial magnetic correlations [34] and transport properties [35,36]. No superconductivity or fine tuning of the chemical potential is required. We show below that these Majorana zero-energy states live in the particle-hole sector of the many-body problem and can be thought of as spin density operators localized near the wire's ends.

Our paper is rather technical and is based on the bosonization technique as developed in Refs. [37–40] and designed to account for the periodic, antiperiodic, and open boundary conditions. It is organized as follows. In Sec. II we formulate the Hamiltonian of the problem and show, with the help of renormalization group (RG) arguments, that the wire flows to strong coupling describing an interesting correlated SDW state. Focusing first on the wire in the ring geometry, which depending on the parity of the magnetization corresponds to either periodic boundary conditions (PBC) or antiperiodic boundary conditions (anti-PBC), we show that its ground state is unique. In Sec. III we consider the wire with open boundary conditions (OBC) and derive its effective fermion Hamiltonian at the special Luther-Emery point. The effective Hamiltonian is solved in Sec. IV, where we find that the ground state is twofold degenerate. The physical importance of the magnetization parity and the physical manifestations of the discovered Majorana modes are analyzed in Sec. V. Our findings and physical insights derived from them are summarized in Sec. VI. Numerous technical details of our calculations are described in three extended appendices.

II. HAMILTONIAN

We consider a single-channel quantum wire, the Hamiltonian of which consists of three main contributions, $\mathcal{H}_{\rm wire} = \mathcal{H}_{\rm e} + \mathcal{V}_{\rm so} + \mathcal{V}_{\rm z}$. Here $\mathcal{H}_{\rm e} = \mathcal{H}_{\rm e,0} + \mathcal{H}_{\rm int}$ describes an ideal quantum wire,

$$\mathcal{H}_{\rm e,0} = \sum_s \int dx\, \Psi_s^\dagger(x)\Big(-\frac{\partial_x^2}{2m} - \mu\Big)\Psi_s(x), \tag{1a}$$

$$\mathcal{H}_{\rm int} = \frac{1}{2}\sum_{s,s'}\int dx dx'\, U(x-x')\Psi_s^\dagger(x)\Psi_{s'}^\dagger(x')\Psi_{s'}(x')\Psi_s(x), \tag{1b}$$

where $\mathcal{H}_{\rm e,0}$ is the kinetic energy with chemical potential μ, $U(x)$ is the screened Coulomb electron-electron (e-e) interaction, and $s = \uparrow, \downarrow$ is the spin index. The electrons are perturbed by the spin-orbit interaction

$$\mathcal{V}_{\rm so} = \sum_{s,s'}\int dx\, \Psi_s^\dagger(x)\big(-i\alpha_R\sigma^y_{ss'}\partial_x\big)\Psi_{s'}(x) \tag{2}$$

etic field which we take to be directed along the z axis, $\vec{B} = B\hat{z}$,

$$\mathcal{V}_{\rm z} = \sum_{s,s'}\int dx\, \Psi_s^\dagger(x)\Big(-\frac{g\mu_B}{2}\sigma^z_{ss'}B\Big)\Psi_{s'}(x). \tag{3}$$

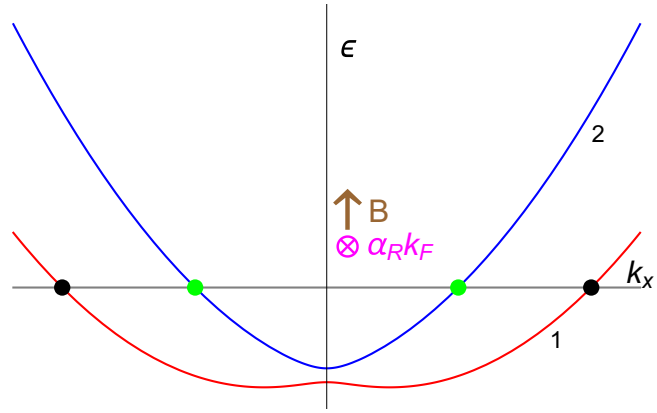

FIG. 1. Schematics of the Cooper scattering process. Spin-orbit interaction is directed along the $\hat{y}$ axis; Zeeman magnetic field is applied along the $\hat{z}$ axis. Two electrons in, say, subband 1 (represented by black points) are scattered into the opposite Fermi points in subband 2 (shown by green points). In the conjugate process two electrons initially in band 2 are scattered into band 1.

The spin-orbit interaction (2) is obtained from the standard Rashba interaction, $\alpha_R\hat{z}\cdot\vec{p}\times\vec{\sigma}$, by replacing the transverse component of the electron momentum $\vec{p}$ by its zero expectation value, $p_y \to \langle p_y\rangle = 0$. Corrections to this approximations are known to be very small [35,36].

Obviously $\mathcal{H}_{\rm wire}$ does not conserve spin—this fact is of crucial importance for our investigation. The key consequence of this can be understood by considering a limit of strong magnetic field $\mu \gg b \equiv g\mu_B B \gg 2\alpha_R k_F$, where k_F denotes the Fermi momentum of the unperturbed Hamiltonian $\mathcal{H}_{\rm e,0}$, Eq. (1a). In this limit $\mathcal{H}_{\rm wire}$ describes the standard problem of a partially magnetized quantum wire with two Zeeman-split subbands. In the absence of the spin-orbit interaction no scattering processes between these subbands are possible, simply because their spin wave functions are described by the orthogonal spinors, spin up ($\uparrow$) and spin down ($\downarrow$) states. This is just the consequence of the spin conservation. However, any finite spin-orbit interaction $\alpha_R \neq 0$ breaks spin conservation and immediately allows for a new scattering process: the Cooper scattering [41]. This momentum- and energy-conserving process describes scattering of the pair of electrons at $\pm k_F$ Fermi points of, say, the majority subband ($\uparrow$) into a similar pair of electrons in the minority ($\downarrow$) subband, and vice versa, see Fig. 1. That is, a pair of electrons with spin $S^z = +1$ is converted into that with spin $S^z = -1$ and vice versa. This superconducting, or Josephson-like scattering (hence the name Cooper), conserves fermion parity of each of the subbands and plays a crucial role in the following discussion.

Such a two-subband description can be straightforwardly extended to the physically most important regime of comparable Zeeman and spin-orbit energies, $b \approx 2\alpha_R k_F$, see for example Ref. [36]. We, however, will follow a less cumbersome approach, based on the chiral rotation trick, as detailed below. The end result of these complimentary calculations is the same [36].

A. Spin current formulation of the quantum wire with periodic boundary conditions

1. Chiral fermions

Our approach to the problem consists in treating perturbations $\mathcal{V}_{\rm so}$ and $\mathcal{V}_{\rm z}$ on equal footing. Initially, we turn off the perturbations $\mathcal{V}_{\rm so}$ and $\mathcal{V}_{\rm z}$. We express fermion fields $\Psi_s(x)$ in terms of low-energy modes Ψ_{Rs} and Ψ_{Ls} that live near $+k_F$ and $-k_F$ Fermi points, correspondingly,

$$\Psi_s(x) = \Psi_{Rs}(x)e^{ik_F x} + \Psi_{Ls}(x)e^{-ik_F x}. \tag{4}$$

The Fermi-momentum $k_F = \sqrt{2m\mu}$ is determined by the electron density in the usual way, $k_F = \pi N_s^0/L$, where L is the length of the wire and $N_s^0 = N_{R,s}^0 + N_{L,s}^0 = 2N_{R,s}^0$ is the total number of fermions of spin projection s. It is written in terms of the numbers $N_{R/L,s}^0$ of chiral fermions in the wire. The choice of $N_{R,s}^0 = N_{L,s}^0$ made here corresponds to considering the state with *no* charge current in the ground state, $j_\rho^0 = \sum_s (N_{R,s}^0 - N_{L,s}^0) = 0$. In the absence of the external magnetic field the ground state magnetization is zero as well, $M^0 = (N_\uparrow^0 - N_\downarrow^0)/2 = 0$.

Consider the wire in the closed loop geometry, with the chiral fermions (4) obeying the periodic boundary conditions (PBC) such that $\Psi_{R/L,s}(0) = \Psi_{R/L,s}(L)$. Note that $e^{ik_F L} = 1$ due to our choice $k_F = 2\pi N_{R,s}^0/L$ explained below (4).

In terms of the chiral fermion fields Ψ_{Rs} and Ψ_{Ls}, the kinetic energy is simply

$$\mathcal{H}_{\rm e,0} = \sum_s \int dx(-iv_F \Psi_{Rs}^\dagger \partial_x \Psi_{Rs} + iv_F \Psi_{Ls}^\dagger \partial_x \Psi_{Ls}), \tag{5}$$

where $v_F = k_F/m$ is the Fermi velocity. It is useful at this stage to write the kinetic energy as a sum of commuting *charge* and *spin* parts (Sugawara construction), $\mathcal{H}_{\rm e,0} = \mathcal{H}_\rho^0 + \mathcal{H}_\sigma^0$, where

$$\mathcal{H}_\rho^0 = \frac{\pi v_F}{2} \int dx \left(J_R^2 + J_L^2\right), \tag{6}$$

$$\mathcal{H}_\sigma^0 = \frac{2\pi v_F}{3} \sum_{a=x,y,z} \int dx \left(J_R^a J_R^a + J_L^a J_L^a\right). \tag{7}$$

Here we introduced normal-ordered charge currents

$$J_R = \sum_s :\Psi_{Rs}^\dagger \Psi_{Rs}:, \quad J_L = \sum_s :\Psi_{Ls}^\dagger \Psi_{Ls}:, \tag{8}$$

and spin currents ($a = x, y, z$)

$$J_R^a = \sum_{s,s'} :\Psi_{Rs}^\dagger \frac{\sigma_{ss'}^a}{2} \Psi_{Rs'}:, \quad J_L^a = \sum_{s,s'} :\Psi_{Ls}^\dagger \frac{\sigma_{ss'}^a}{2} \Psi_{Ls'}:. \tag{9}$$

As described in Appendix A, the interaction part of the Hamiltonian (1a) separates into charge and spin parts as well, $\mathcal{H}_{\rm int} = \mathcal{H}_{{\rm int},\rho} + \mathcal{H}_{{\rm int},\sigma}$, where

$$\mathcal{H}_{{\rm int},\rho} = \frac{1}{4}(U_0 - U_{2k_F}) \int dx\, (J_R + J_L)^2, \tag{10a}$$

$$\mathcal{H}_{{\rm int},\sigma} = -g \int dx\, \vec{J}_R \cdot \vec{J}_L. \tag{10b}$$

Here U_q denotes the qth component of the Fourier transform of the e-e interaction $U(x)$, and $g = 2U_{2k_F}$ denotes the magnitude of the spin backscattering interaction.

We now turn on the perturbations $\mathcal{V}_{\rm so}$ and $\mathcal{V}_{\rm z}$. The Zeeman magnetic field $b = g\mu_B B$ couples to the sum of spin currents (magnetization)

$$\mathcal{V}_{\rm z} = -b \int dx \left(J_R^z + J_L^z\right), \tag{11}$$

while the spin-orbit interaction couples to their *difference*, since the Rashba term (2) is odd under spatial inversion ($x \to -x$) which interchanges right- and left-moving excitations [35],

$$\mathcal{V}_{\rm so} = 2\alpha_R k_F \int dx \left(J_R^y - J_L^y\right). \tag{12}$$

2. Chiral rotations

It is crucial to notice now that $\mathcal{H}_\sigma^0$ possesses an extended $SU(2) \times SU(2)$ symmetry of independent rotations of the right- and left-moving currents. Our solution of the problem [36,42] exploits this extended symmetry. Namely, we next rotate $\vec{J}_R$ and $\vec{J}_L$ about the $\hat{x}$ axis in opposite directions so as to bring "vectorial" perturbation $\mathcal{V} = \mathcal{V}_{\rm so} + \mathcal{V}_{\rm z}$ into the standard Zeeman form, with *total* field $h = \sqrt{b^2 + (2\alpha_R k_F)^2}$ along the $\hat{z}$ axis

$$\mathcal{V} = -h \int dx \left(M_R^z + M_L^z\right). \tag{13}$$

The required chiral rotation is given by

$$\vec{J}_R = \mathcal{R}_x(\beta_R)\vec{M}_R, \quad \vec{J}_L = \mathcal{R}_x(\beta_L)\vec{M}_L, \tag{14}$$

where the rotation matrix $\mathcal{R}_x$ is

$$\mathcal{R}_x(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}. \tag{15}$$

The rotation angles are given by

$$\beta_R = -\beta_L = \beta = \arctan(2\alpha_R k_F/b). \tag{16}$$

These rotations do not affect $\mathcal{H}_\sigma^0$ (7), which retains its form in the rotated M basis

$$\mathcal{H}_\sigma^0 = \frac{2\pi v_F}{3} \sum_{a=x,y,z} \int dx \left(M_R^a M_R^a + M_L^a M_L^a\right), \tag{17}$$

where v_F here is understood as the one including the shift $-U_{2k_F}/3$ found in (A16).

In terms of the right- and left-moving fermions, the rotation (14) corresponds to the rotation of spinors $\Psi_{R/L} = (\Psi_{R/L\uparrow}, \Psi_{R/L\downarrow})^T$ and $\psi_{R/L} = (\psi_{R/L\uparrow}, \psi_{R/L\downarrow})^T$,

$$\Psi_R = e^{-i\beta\sigma^x/2}\psi_R, \quad \Psi_L = e^{i\beta\sigma^x/2}\psi_L. \tag{18}$$

This observation makes clear that the charge currents (8) do not transform under the rotations (14) and (18)—the Hamiltonian of the charge sector $\mathcal{H}_\rho^0 + \mathcal{H}_{{\rm int},\rho}$ is not affected. The new fermions parametrize the rotated currents $\vec{M}_{R/L}$ in the same way as the old ones parametrize the currents $\vec{J}_{R/L}$. For example, under the right rotation $\mathcal{R}_x(\beta)$

$$\vec{J}_R = :\Psi_R^\dagger \frac{\vec{\sigma}}{2} \Psi_R: \to \vec{M}_R = :\psi_R^\dagger \frac{\vec{\sigma}}{2} \psi_R:. \tag{19}$$

The interaction in the spin sector $\mathcal{H}_{\text{int},\sigma}$ (10b) is strongly modified by the rotation and changes to

$$\begin{aligned}\mathcal{H}_{\text{int},\sigma} &= -g\int dx\,\vec{M}_R\mathcal{R}_x^T(\beta_R)\mathcal{R}_x(\beta_L)\vec{M}_L\\ &= -g\int dx\big[M_R^xM_L^x+\cos\chi\left(M_R^yM_L^y+M_R^zM_L^z\right)\\ &\quad+\sin\chi\left(M_R^yM_L^z-M_R^zM_L^y\right)\big],\end{aligned}\tag{20}$$

where $\chi=\beta_R-\beta_L=2\beta$ is the *relative* rotation angle.

Observe that the net field h, (13), points along the $\hat{z}$ axis. The magnetic field induces incommensurate fluctuations in the system which make some of the terms in (20) oscillate fast with the coordinate. It is easy to see that h can be absorbed into the kinetic energy of fermions $\psi_{R/L}$ by a simple x-dependent transformation

$$\psi_R\to e^{it_\varphi x\sigma^z/2}\psi_R,\quad \psi_L\to e^{-it_\varphi x\sigma^z/2}\psi_L,\quad t_\varphi=h/v_F.\tag{21}$$

As a result of this shift the transverse components $M^x_{R/L}\pm iM^y_{R/L}=M^\pm_{R/L}$ of the rotated spin current acquire oscillating position-dependent factors, $M_R^+\to M_R^+e^{-it_\varphi x}$ and $M_L^+\to M_L^+e^{it_\varphi x}$. The immediate consequence of this is that many terms in $\mathcal{H}_{\text{int},\sigma}$ (20) acquire x-dependent oscillations,

$$\begin{aligned}\mathcal{H}_{\text{int},\sigma} &= -g\int dx\Big\{\cos\chi M_R^zM_L^z+\frac{\sin^2\frac{\chi}{2}}{2}(M_R^+M_L^++\text{H.c.})\\ &\quad+\frac{\cos^2\frac{\chi}{2}}{2}(M_R^+M_L^-e^{-i2t_\varphi x}+\text{H.c.})\\ &\quad+i\frac{\sin\chi}{2}\left[\left(M_L^zM_R^-+M_R^zM_L^+\right)e^{it_\varphi x}-\text{H.c.}\right]\Big\}.\end{aligned}\tag{22}$$

Provided that the running backscattering coupling constant g/v_F is small, all oscillating terms, which represent momentum-nonconserving two-particle scattering processes, average out to zero. Assuming this, we are allowed to drop all oscillating terms in (22) and obtain the nonoscillating part of the spin-interaction Hamiltonian as

$$\begin{aligned}\mathcal{H}_{\text{int},\sigma} &= -\int dx\left[g_c\left(M_R^xM_L^x-M_R^yM_L^y\right)+g_zM_R^zM_L^z\right]\\ &= -\int dx\sum_{a=x,y,z}g_aM_R^aM_L^a,\end{aligned}\tag{23}$$

where

$$g_x=-g_y=g_c=g\frac{1-\cos\chi}{2}=\frac{g(2\alpha_Rk_F)^2}{b^2+(2\alpha_Rk_F)^2},\tag{24}$$

$$g_z=g\cos\chi=g\frac{b^2-(2\alpha_Rk_F)^2}{b^2+(2\alpha_Rk_F)^2}.\tag{25}$$

Note that at this point the complete Hamiltonian of the spin sector is given by the sum of equations (17) and (23). Importantly, the magnetic field is absent from it; it is absorbed into renormalization of the Fermi momenta $k_F\to k_{Fs}$. In fact, the coupling constants g_a in (23) have implicit dependence on h through renormalization-group transformation from the original energy scale (of the order of the band width) to the effective magnetic field h.

The meaning of (21) is simple. It represents splitting of the Fermi momentum k_F into the spin-dependent ones $k_{Fs}=k_F+st_\varphi/2$. Given that k_F is determined by the particle density, $k_F=\pi N_0/L$, the development of the spin-dependent Fermi momenta $k_{Fs}=\pi N_s/L$ describes the appearance of the finite magnetization with $N_\uparrow>N_\downarrow$. Therefore, $\Delta k_F=t_\varphi/2=\pi(N_\uparrow-N_\uparrow^0)/L$, so that $t_\varphi L=2\pi(N_\uparrow-N_\uparrow^0)=2\pi M$ is an integer multiple of 2π, since $N_\uparrow$ and $N_\uparrow^0$ are integers describing the number of spin-$\uparrow$ electrons in the system with finite h and zero h, respectively. The magnetization $M=(N_\uparrow-N_\downarrow)/2$ is also an integer.

After making the transformations, the fermions ψ_{Rs} and ψ_{Ls} obey the boundary conditions

$$\psi_R(0)=e^{i\sigma_zt_\varphi L/2}\psi_R(L)=(-1)^M\psi_R(L),\tag{26a}$$

$$\psi_L(0)=e^{-i\sigma_zt_\varphi L/2}\psi_L(L)=(-1)^M\psi_L(L).\tag{26b}$$

The boundary conditions depend on the parity of the magnetization M: periodic for even M and antiperiodic for odd M. It is appropriate to note here that even though our subsequent analysis will show that the magnetization M is not a conserved quantity in the ground state of the interacting wire, the magnetization parity $(-1)^M$ is conserved in the ground state. Therefore the boundary condition (26) is well defined.

The antiperiodic boundary condition for odd M can be implemented by introducing a magnetic flux threading the ring under the periodic boundary condition. Thus we replace ∂_x with $\partial_x-i\pi\lambda/L$ in the kinetic energy $\mathcal{H}_{\text{e},0}$ in (5) or equivalently we add

$$\mathcal{H}_{\text{flux}}=\frac{\pi\lambda v_F}{L}\int dx\,(J_L-J_R)\tag{27}$$

to the charge part of the kinetic energy $\mathcal{H}_\rho^0$. Here we demand the integer parameter λ to be

$$\lambda=\begin{cases}0 & \text{for } (-1)^M=1,\\ 1 & \text{for } (-1)^M=-1.\end{cases}\tag{28}$$

3. RG analysis

Equations (17) and (23) represent a nontrivial interacting problem, analysis of which requires renormalization group (RG) treatment. Let us assume for the moment that the wire length L is large so that finite-size effects are negligible. The couplings g_a obey the famous Berezinskii-Kosterlitz-Thouless (BKT) RG flow [20,21],

$$\frac{dg_x}{d\ell}=-\frac{g_yg_z}{2\pi v_F},\quad \frac{dg_y}{d\ell}=-\frac{g_xg_z}{2\pi v_F},\quad \frac{dg_z}{d\ell}=-\frac{g_xg_y}{2\pi v_F},\tag{29}$$

where $\ell=\log(\alpha'/\alpha)$ describes increase of the short-distance cutoff from α to α'. As discussed in detail in Ref. [43], the solution to the RG equations (29) depends on the initial values [(24) and (25)] of the couplings involved. Noting that $d(g_x^2-g_y^2)/d\ell=0$ and the fact that for $\ell=0$ $g_x+g_y=0$, we conclude that $g_x(\ell)=-g_y(\ell)=g_c(\ell)$ for all ℓ. Equation (29) then reduces to the two coupled equations

$$\frac{dg_z}{d\ell}=\frac{g_c^2}{2\pi v_F},\quad \frac{dg_c}{d\ell}=\frac{g_cg_z}{2\pi v_F},\tag{30}$$

which too are characterized by the integral of motion $Y=g_z^2(\ell)-g_c^2(\ell)$.

It turns out that the solution is towards *strong coupling*, meaning that $g_{z,c}(\ell) \to +\infty$ for sufficiently large ℓ, for all possible angles $\chi \in (0, \pi)$ [43]. This diverging solution implies an instability towards a correlated spin state with a nonvanishing excitation gap in the spin sector. The spin gap can be estimated as $\Delta_\sigma \sim (v_F/\alpha)e^{-\ell_0}$ where ℓ_0 is the RG scale at which the dimensionless coupling constants diverge. The *minimal* value of ℓ_0, corresponding to the strongest instability, occurs for $Y = 0$. This corresponds to $\cos\chi = 1/3$ and implies $b = 2\sqrt{2}\alpha_R k_F$. Therefore, the correlated spin state is strongest when Zeeman energy is comparable to the spin-orbit energy.

At $Y = 0$ RG equations (30) simplify to a single equation,

$$\frac{dg_z}{d\ell} = \frac{g_z^2}{2\pi v_F}, \tag{31}$$

whose solution is given by $g_z(\ell) = g_z(0)/[1 - g_z(0)\ell/(2\pi v_F)]$, and $g_c(\ell)$ is described by the same equation. Thus $\ell_0 = 2\pi v_F/g_z(0) = 6\pi v_F/g$. The corresponding gap is exponentially small, $\Delta_\sigma \sim (v_F/\alpha)\exp(-6\pi v_F/g)$.

An important clarification is in order here. Reference [44] has showed that quadratic in spin-orbit interaction terms affect the RG flow significantly, via the change of the initial values of the coupling constants, in the limit $b \ll \alpha_R k_F$. Under these conditions the ground state is actually an anisotropic Luttinger liquid [43,44]. This, however, does not affect the conclusion of the flow towards the strong coupling in the optimal case of $b \approx \alpha_R k_F$, on which we are focusing here.

4. Bosonized form

The physics of the spin gap phase is conveniently discussed with the help of *Abelian* bosonization, a brief description of which is summarized in Appendix B 1. With this powerful technique the charge Hamiltonian $\mathcal{H}_\rho = \mathcal{H}_\rho^0 + \mathcal{H}_{\text{int},\rho} + \mathcal{H}_{\text{flux}}$ turns into

$$\mathcal{H}_\rho = \int dx \frac{1}{2}\Big[\frac{v_\rho}{K_\rho} :(\partial_x\phi_\rho)^2: + v_\rho K_\rho :(\partial_x\theta_\rho)^2: + \frac{2\sqrt{2\pi}\lambda v_F}{L}\partial_x\theta_\rho\Big], \tag{32}$$

where $[\phi_\rho(x), \partial_y\theta_\rho(y)] = i\delta(x-y)$, and

$$K_\rho = \left(1 + \frac{2U_0 - U_{2k_F}}{\pi v_F}\right)^{-1/2}, \tag{33a}$$

$$v_\rho = v_F\left(1 + \frac{2U_0 - U_{2k_F}}{\pi v_F}\right)^{1/2}. \tag{33b}$$

Note that the relation $v_\rho K_\rho = v_F$ holds, which is a consequence of Galilean invariance and guarantees the 2π-flux periodicity irrespective to the Coulomb interaction [41,45,46]. The noninteracting spin Hamiltonian (17) turns into that of a pair of bosons, ϕ_σ and θ_σ,

$$\mathcal{H}_\sigma^0 = \int dx \frac{v_F}{2}[(\partial_x\phi_\sigma)^2 + (\partial_x\theta_\sigma)^2], \tag{34}$$

where the bosonic fields obey the commutation relation $[\phi_\sigma(x), \partial_y\theta_\sigma(y)] = i\delta(x-y)$. The bosonized form of the interaction (23) is obtained as

$$\mathcal{H}_{\text{int},\sigma} = \int dx\Big\{-\frac{g_z}{8\pi}[(\partial_x\phi_\sigma)^2 - (\partial_x\theta_\sigma)^2] - \frac{g_c}{(2\pi\alpha)^2}\cos(\sqrt{8\pi}\theta_\sigma)\Big\}, \tag{35}$$

where α is a short-distance cutoff. From (34) and (35) we obtain

$$\mathcal{H}_\sigma = \int dx\Big[\frac{v_\sigma}{2K_\sigma}(\partial_x\phi_\sigma)^2 + \frac{v_\sigma K_\sigma}{2}(\partial_x\theta_\sigma)^2 - \frac{g_c}{(2\pi\alpha)^2}\cos(\sqrt{8\pi}\theta_\sigma)\Big], \tag{36}$$

where we introduced dimensionless Luttinger parameter K_σ and renormalized spin velocity v_σ,

$$K_\sigma = \sqrt{\frac{1 + g_z/4\pi v_F}{1 - g_z/4\pi v_F}}, \tag{37a}$$

$$v_\sigma = v_F\sqrt{1 - (g_z/4\pi v_F)^2}, \tag{37b}$$

and we set $\eta_{R\uparrow}\eta_{R\downarrow}\eta_{L\uparrow}\eta_{L\downarrow} = 1$. The nonlinear cosine term $\cos(\sqrt{8\pi}\theta_\sigma)$ in (36) describes intersubband pair-tunneling processes, the Cooper scattering in Fig. 1, and is responsible for the opening of the spin gap. We note in passing that the above analysis can be easily extended to the case of nonorthogonal spin-orbit and Zeeman field directions [36,44,47]. In this case $M_R^+M_L^+$ in (22) [or, equivalently, $\cos(\sqrt{8\pi}\theta_\sigma)$ term in (36)] also acquire oscillating factors and therefore "average out" of the Hamiltonian. Physically, this corresponds to momentum-nonconserving pair tunneling between the two subbands [36] and results in the restoration of the critical Luttinger liquid ground state.

We see that the growth of g_c under RG flow leads to the growth of K_σ. According to the standard Tomonaga-Luttinger liquid phenomenology [20,21], this indicates the development of attractive interactions in the spin sector and associated superconductinglike behavior of various physical observables. A large positive value of g_c implies the development of the correlated state with $\cos(\sqrt{8\pi}\theta_\sigma) = +1$. This state is twofold degenerate, with $\sqrt{2\pi}\theta_\sigma^{(1)} = 0$ and $\sqrt{2\pi}\theta_\sigma^{(2)} = \pi \pmod{2\pi}$ describing two equivalent spin states.

The physical meaning of the obtained spin correlated state can be inferred from the behavior of spin density, as was done previously in Refs. [36,44], and corresponds to the Ising-type algebraic spin density wave (SDW) order. Specifically, we are interested in the $2k_F$ component of the spin density $S^a_{2k_F}(x) = N^a(x)e^{-i2k_F x} + \text{H.c.}$, where

$$N^a(x) = \tfrac{1}{2}\Psi_R^\dagger(x)\sigma^a\Psi_L(x), \quad a = x, y, z. \tag{38}$$

With the help of (18) and (21), N^a reduces to the form

$$N^x = \cos\beta\widetilde{N}^x + i\sin\beta\cos(t_\varphi x)\widetilde{N}^0 + \sin\beta\sin(t_\varphi x)\widetilde{N}^z, \tag{39}$$

$$N^y = \widetilde{N}^y, \tag{40}$$

$$N^z = \cos(t_\varphi x)\widetilde{N}^z - i\sin(t_\varphi x)\widetilde{N}^0, \tag{41}$$

where we have defined $\widetilde{N}^{a=0,x,y,z} = \frac{1}{2}\psi_R^\dagger \sigma^a \psi_L$, and σ^0 denotes the identity matrix. Using the standard bosonization (described in Appendix B 1), we obtain

$$\widetilde{N}^0 = \frac{\eta_{R\uparrow}\eta_{L\uparrow}}{2\pi\alpha} e^{-i\sqrt{2\pi}\phi_\rho - 2\pi i x/L} \cos(\sqrt{2\pi}\phi_\sigma), \tag{42a}$$

$$\widetilde{N}^x = \frac{i\eta_{R\uparrow}\eta_{L\downarrow}}{2\pi\alpha} e^{-i\sqrt{2\pi}\phi_\rho - 2\pi i x/L} \sin(\sqrt{2\pi}\theta_\sigma), \tag{42b}$$

$$\widetilde{N}^y = \frac{-i\eta_{R\uparrow}\eta_{L\downarrow}}{2\pi\alpha} e^{-i\sqrt{2\pi}\phi_\rho - 2\pi i x/L} \cos(\sqrt{2\pi}\theta_\sigma), \tag{42c}$$

$$\widetilde{N}^z = \frac{-i\eta_{R\uparrow}\eta_{L\uparrow}}{2\pi\alpha} e^{-i\sqrt{2\pi}\phi_\rho - 2\pi i x/L} \sin(\sqrt{2\pi}\phi_\sigma). \tag{42d}$$

It is now easy to observe that in the ground state of the sine-Gordon Hamiltonian (36) only the spin part of $\widetilde{N}^y$ acquires a nonvanishing expectation value—the spin density 'wants' to line up along the $\hat{y}$ axis, which is the spin-orbit axis; see (2). Therefore we can write, choosing the gauge $\eta_{R\uparrow}\eta_{L\downarrow} = i$ [36],

$$\begin{pmatrix} S^x \\ S^y \\ S^z \end{pmatrix}_{2k_F} \propto \cos\left[\sqrt{2\pi}\phi_\rho(x) + 2k_F x + \frac{2\pi x}{L}\right] \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \tag{43}$$

up to the "short-ranged" corrections involving field ϕ_σ, correlation functions of which decay exponentially on the scale v_σ/Δ_σ. The ± 1 part of the above equation corresponds to the choice of degenerate ground states $\theta_\sigma^{(1,2)}$. Gapless charge fluctuations, however, prevent the true symmetry breaking from happening. Equations (39), (42), and (43) show that spin correlations in the obtained SDW state are highly anisotropic in spin space and their spatial decay is controlled by the gapless charge sector of the wire.

It is also useful to consider the $2k_F$ component of the charge density, $\rho_{2k_F}(x) = \rho(x)e^{-i2k_F x} + \text{H.c.}$, where

$$\rho = \Psi_R^\dagger \Psi_L = \psi_R^\dagger e^{-it_\varphi x\sigma^z/2} e^{i\beta\sigma^x} e^{-it_\varphi x\sigma^z/2} \psi_L. \tag{44}$$

We find

$$\rho = 2\cos\beta\cos(t_\varphi x)\widetilde{N}^0 - 2i\cos\beta\sin(t_\varphi x)\widetilde{N}^z + 2i\sin\beta\widetilde{N}^x. \tag{45}$$

We see that $\rho_{2k_F}(x)$ is nullified by the SDW ground state. This means that *weak* scalar impurity, the potential of which couples to ρ_{2k_F}, renormalizes to zero—electron backscattering is suppressed [36].

Finally, it is interesting to note that $\mathcal{H}_\sigma$ (36) at $K_\sigma = 2$ is just a bosonized Hamiltonian of a one-dimensional p-wave superconductor, which is known to be a topological superconductor of class D [1,48] having a zero-energy Majorana mode at each end. Hence we can anticipate that our quantum wire model may also host a localized Majorana-like zero mode at the end of the wire, even though no superconducting order is present in the ground state. This is indeed the case as shown in Sec. III.

5. Ground state of a finite ring

From the RG analysis explained above, we have found that the ground state of the spin gap phase has Ising-type SDW quasi-long-range order. Here, however, we show that the ground state of the wire is *unique* under the ring geometry.

To this end, we need to pay close attention to zero modes in the low-energy Hamiltonian $H_\rho + H_\sigma$ [46,49]. As shown in Appendix B 1, the zero modes obey the selection rules (B30) and (B31). We restrict ourselves to the even particle number parity case when

$$(-1)^{N_\rho} = (-1)^{J_\rho} = (-1)^{N_\sigma} = (-1)^{J_\sigma} = 1 \tag{46}$$

and reproduce (B31) here for completeness

$$(-1)^{\frac{1}{2}(N_\rho + J_\rho)} = (-1)^{\frac{1}{2}(N_\sigma + J_\sigma)}. \tag{47}$$

Substituting (B1) into (32) and keeping only the zero-mode terms, we find

$$H_\rho^0 = \frac{\pi}{4L}\left(\frac{v_\rho}{K_\rho}N_\rho^2 + v_F N_\rho + v_F J_\rho^2 - 4\lambda v_F J_\rho\right), \tag{48}$$

which is minimized when $J_\rho = 2\lambda$. We note that λ is related to the magnetization; see Eq. (28).

Similarly, the zero-mode part of the spin Hamiltonian H_σ is

$$H_\sigma^0 = \frac{\pi v_\sigma}{4L}\left(\frac{1}{K_\sigma}N_\sigma^2 + K_\sigma J_\sigma^2\right) - \frac{g_c\gamma}{(2\pi\alpha)^2}\int_0^L dx\, \cos\left(\sqrt{8\pi}\theta_\sigma^0 - 2\pi J_\sigma x/L\right), \tag{49}$$

where θ_σ^0 is defined by

$$\theta_\sigma^0 = \frac{1}{\sqrt{2}}\left(\phi_{L\uparrow}^0 - \phi_{R\uparrow}^0 - \phi_{L\downarrow}^0 + \phi_{R\downarrow}^0\right), \tag{50}$$

and the renormalization factor γ from finite-frequency modes is

$$\gamma = \left(\frac{2\pi\alpha}{L}\right)^{2/K_\sigma}. \tag{51}$$

Assuming that g_c is renormalized to strong coupling, we find that H_σ^0 is minimized when $(J_\sigma, e^{i\sqrt{2\pi}\theta_\sigma^0}) = (0, 1)$ or $(0, -1)$. It follows from the commutation relation

$$\left[\theta_\sigma^0, N_\sigma\right] = i\sqrt{\frac{2}{\pi}} \tag{52}$$

that

$$e^{i\sqrt{8\pi}\theta_\sigma^0} N_\sigma e^{-i\sqrt{8\pi}\theta_\sigma^0} = N_\sigma - 4, \quad \left\{e^{i\sqrt{2\pi}\theta_\sigma^0}, e^{i\pi N_\sigma/2}\right\} = 0. \tag{53}$$

We see that N_σ is not conserved but the *parity* $(-1)^{N_\sigma/2}$ is conserved.

Let us introduce eigenstates of $e^{i\sqrt{2\pi}\theta_\sigma^0}$:

$$e^{i\sqrt{2\pi}\theta_\sigma^0}|a\rangle = |a\rangle, \quad e^{i\sqrt{2\pi}\theta_\sigma^0}|b\rangle = -|b\rangle. \tag{54}$$

Since the two states $|a\rangle$ and $|b\rangle$ minimize the potential $-g_c\cos(\sqrt{8\pi}\theta_\sigma^0)$, they are candidates for ground states of H_σ^0. However, they are not eigenstates of a parity operator $(-1)^{N_\sigma/2}$.

Let us define

$$|+\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle). \tag{55}$$

We find from (53) that $|+\rangle$ and $|-\rangle$ are, respectively, the even and odd parity state,

$$e^{i\pi N_\sigma/2}|+\rangle = |+\rangle, \quad e^{i\pi N_\sigma/2}|-\rangle = -|-\rangle. \tag{56}$$

The discussion above follows that in Ref. [27] closely. It follows from (47) with $J_\sigma = 0$ that

$$(-1)^{\frac{1}{2}(N_\rho+J_\rho)} = \begin{cases} +1 & \text{for } |+\rangle, \\ -1 & \text{for } |-\rangle. \end{cases} \tag{57}$$

We are now ready to see that the ground state of $H_\rho^0 + H_\sigma^0$ is unique in the ring geometry. Since we seek the ground state for a fixed number of electrons, we can set $N_\rho = 0$.

(i) Suppose $(-1)^{\frac{1}{2}(N_\rho+J_\rho)} = +1$. With $N_\rho = 0$ we have that $J_\rho/2 =$ even and can set $J_\rho = 0$ to minimize the charge Hamiltonian (48). By (47) we have $(-1)^{\frac{1}{2}(N_\sigma+J_\sigma)} = +1$ too. Now, (49) [and also (36)] is minimized by $J_\sigma = 0$ when the field configuration in the argument of the cosine in (49) and (36) is uniform, i.e., does not have *kinks*. This is easiest to see by thinking of the full field θ_σ in (36) and following its definition in Appendix B 1, see (B32). Then we find $\theta_\sigma(0) = \theta_\sigma(L)$ and the kink-free configuration of θ_σ satisfies this. With $J_\sigma = 0$ we have $(-1)^{N_\sigma/2} = (-1)^M = +1$ and hence the ground state is the state $|+\rangle$. Note also that $(-1)^M = 1$ means $\lambda = 0$, see (28), and therefore the choice of $J_\rho = 0$ indeed corresponds to the energy minimum.

Let us now ask what is the lowest energy for the state $|-\rangle$? In this state $(-1)^{N_\sigma/2} = -1$ but then our initial assumption $(-1)^{\frac{1}{2}(N_\rho+J_\rho)} = +1$ and (47) require that $(-1)^{\frac{1}{2}(N_\sigma+J_\sigma)} = +1$. This is only possible if $J_\sigma = \pm 2$. (More generally, $J_\sigma = \pm 2 + 4n$, but this will lead to a multikink spin sector configuration with yet higher energy.) But then the field θ_σ must obey $\theta_\sigma(L) = \theta_\sigma(0) - \sqrt{\pi/2}J_\sigma$ so that it experiences discontinuity (kink) at $x = 0$ (which is the same as $x = L$ in the ring geometry). This boundary condition forces θ_σ to have another kink somewhere on the ring, at some $0 < x_k < L$. The lowest energy of the state with such a two-kink configuration (one at x_k and another at $x = 0 = L$) is *higher* than that of the kink-free configuration. Calculating this energy difference is not easy but the relevance of the cosine potential in (36) means that it is of the order of g_c/ξ, where $\xi = \Delta/v_\sigma$ is the correlation length of the correlated SDW state. The energy difference remains finite in the limit $L \to \infty$.

We therefore see that in the case of $(-1)^{\frac{1}{2}(N_\rho+J_\rho)} = +1$ the ground state of the wire is given by $|+\rangle$, i.e., the state with the positive magnetization parity. The state with negative magnetization parity $|-\rangle$ has much higher energy.

(ii) Next consider $(-1)^{\frac{1}{2}(N_\rho+J_\rho)} = -1$, which for $N_\rho = 0$ means $J_\rho/2 =$ odd. Now the identity (47) requires $(-1)^{\frac{1}{2}(N_\sigma+J_\sigma)} = -1$. Therefore the kink-free configuration of the spin sector, one with $J_\sigma = 0$, requires $(-1)^{N_\sigma/2} = (-1)^M = -1$. By (28) this means that $\lambda = 1$ and hence the charge sector energy is minimized by $J_\rho = 2$. The odd-parity state $|-\rangle$ is the lowest-energy state.

The other, positive magnetization parity state $|+\rangle$ must have finite spin current $J_\sigma = \pm 2$ which therefore forces the spin sector into a two-kink configuration and results in the higher energy for it.

We note that the energy difference between the lowest-energy state $|+\rangle$ under $(-1)^{\frac{1}{2}(N_\rho+J_\rho)} = +1$ and the lowest-energy state $|-\rangle$ under $(-1)^{\frac{1}{2}(N_\rho+J_\rho)} = -1$ is of order $1/L$ due to the charge Hamiltonian (48).

The presented arguments establish that the ground state of the wire in the correlated SDW state is unique in the *ring geometry*. It is worth noting that the gapless charge sector has played an important role in this conclusion, via the "superselection" rules (46) and especially (47). We'll see below that this is not the case in the case of the open wire, i.e., the wire with two open ends.

III. FINITE WIRE WITH OPEN BOUNDARIES

Now we turn to the case of our main interest, i.e., a finite wire with open boundaries at $x = 0$ and $x = L$, where $\Psi_s(x = 0) = 0 = \Psi_s(x = L)$. Equation (4) shows that OBC for the original fermions means $\Psi_{Rs}(0) = -\Psi_{Ls}(0)$ and $\Psi_{Rs}(L) = -\Psi_{Ls}(L)$. The last relation follows from $e^{ik_F L} = 1$, as explained below (4).

In order for the *rotated* fermions to obey simple boundary conditions which do not mix components with different spin indices s, it proves very convenient to change the direction of the external magnetic field to be along the $\hat{x}$ axis, $\vec{B} = B\hat{x}$, and not along the $\hat{z}$ axis as written in (3). This choice does not change the physics of the problem because the magnetic field and spin-orbit interaction remain orthogonal to each other and, therefore, the correlated SDW phase is preserved. Detailed arguments in Appendix C show how this chiral rotation about the $\hat{z}$ axis is done and, following the steps described there, one finds that the rotated fermions $\psi_{L/R,s}(x)$ introduced in (C15) satisfy the boundary condition (C19). In terms of s components it is just

$$\psi_{L,s}(x_o) = -e^{is\beta}\psi_{R,s}(x_o), \tag{58}$$

where up/down spin projection $s = \uparrow = +1, s = \downarrow = -1$ in the *rotated* basis and $x_o = 0, L$ denotes the two open ends of the wire.

After the chiral rotation (C4), the total magnetic field $h = \sqrt{b^2 + (2\alpha_R k_F)^2}$ experienced by electrons is pointing along the $\hat{x}$ axis. Subsequent manipulations [summarized as steps (1)–(3) in Appendix C] are needed in order to absorb h into the redefined Fermi momenta. Therefore up- and down-pointing spins $s = \uparrow, \downarrow$ in (58) and elsewhere in this section actually represent spins pointing along the positive and negative $\hat{x}$ axis in the rotated frame.

The resulting Hamiltonian is split into charge Hamiltonian and spin Hamiltonian and is written in terms of charge currents $J_{R/L}(x)$ (8), which are not affected by the performed rotations, and spin currents $K^a_{R/L}(x)$, which are related to $\psi_{R/L}$ by (C20). Both types of currents are expressed in terms of the rotated fermions $\psi_{R/L}(x)$.

We are now ready to write down the spin Hamiltonian of the wire of finite length L with open boundaries at $x_o = 0, L$. It is formulated in terms of right-moving current $\vec{K}_R(x)$ and reads [see (C36) and (C37)]

$$\mathcal{H}_\sigma = \mathcal{H}^0_\sigma + \mathcal{H}_{\text{int},\sigma}, \quad \mathcal{H}^0_\sigma = 2\pi v_F \int_{-L}^{L} dx \left[K^z_R(x)\right]^2, \tag{59}$$

$$\mathcal{H}_{\text{int},\sigma} = -\frac{1}{2}\int_{-L}^{L} dx \left(g_c K^z_R(x) K^z_R(-x) + \frac{g_x + g_c}{4}[e^{-i2\beta}K^+_R(x)K^+_R(-x) + \text{H.c.}] \right). \tag{60}$$

Our next task is to bosonize $\mathcal{H}_\sigma$. The first line of (60) represents quadratic correction to (59). Using (C35) we collect quadratic boson terms of $\mathcal{H}_\sigma$,

$$\mathcal{H}_\sigma^{(2)} = \int_{-L}^{L} dx \left\{ v_F [\partial_x \Phi_{R\sigma}(x)]^2 + \frac{g_c}{4\pi} \partial_x \Phi_{R\sigma}(x) \partial_x \Phi_{R\sigma}(-x) \right\} + \frac{\pi v_\sigma M^2}{L K_\sigma}. \tag{61}$$

This part can be diagonalized with the help of another chiral boson field $\tilde{\Phi}$ (see Chap. 27 of [21])

$$\Phi_{R\sigma}(x) = \frac{\sqrt{K_\sigma}}{2} [\tilde{\Phi}(x) - \tilde{\Phi}(-x)] + \frac{1}{2\sqrt{K_\sigma}} [\tilde{\Phi}(x) + \tilde{\Phi}(-x)], \tag{62}$$

where Luttinger parameter K_σ is introduced in (37a). Observe that under this transformation,

$$\Phi_{R\sigma}(x) + \Phi_{R\sigma}(-x) = \frac{1}{\sqrt{K_\sigma}} [\tilde{\Phi}(x) + \tilde{\Phi}(-x)]. \tag{63}$$

The nonlinear operator in the second line of (60) is found, with the help of Baker-Hausdorff identity $e^A e^B = e^{A+B} e^{[A,B]/2}$, to be

$$\begin{aligned} e^{-i2\beta} K_R^+(x) K_R^+(-x) &= \frac{(F_\uparrow^\dagger F_\downarrow)^2}{(2\pi\alpha)^2} e^{-i\sqrt{8\pi}[\Phi_{R\sigma}(x) + \Phi_{R\sigma}(-x)]} \\ &\quad \times e^{-2\pi i x/L} e^{-4\pi \Upsilon(x)} \\ &= -\frac{(F_\uparrow^\dagger F_\downarrow)^2}{(2\pi\alpha)^2} e^{-i\sqrt{\frac{8\pi}{K_\sigma}}[\tilde{\Phi}(x) + \tilde{\Phi}(-x)]}. \end{aligned} \tag{64}$$

Here

$$\Upsilon(x) = [\Phi_{R\sigma}(x), \Phi_{R\sigma}(-x)] = \frac{1}{4\pi} \ln \left(\frac{e^{\pi\alpha/L} - e^{-i2\pi x/L}}{e^{\pi\alpha/L} - e^{i2\pi x/L}} \right) \to \frac{i}{2\pi} \tan^{-1} \left[\cot \left(\frac{\pi x}{L} \right) \right], \tag{65}$$

which is obtained from (C32). The last line represents the limit $\alpha/L \to 0$. This leads to $e^{-i\frac{2\pi x}{L}} e^{-4\pi\Upsilon(x)} = -1$ in (64).

Putting everything together, we find

$$\widetilde{\mathcal{H}}_\sigma = \frac{\pi v_\sigma M^2}{L K_\sigma} + \int_{-L}^{L} dx\, \widetilde{H}_\sigma, \tag{66a}$$

where

$$\widetilde{H}_\sigma = \frac{g_x + g_c}{8(2\pi\alpha)^2} \left[(F_\uparrow^\dagger F_\downarrow)^2 e^{-i\sqrt{\frac{8\pi}{K_\sigma}}[\tilde{\Phi}(x) + \tilde{\Phi}(-x)]} + \text{H.c.} \right] + v_\sigma [\partial_x \tilde{\Phi}(x)]^2. \tag{66b}$$

Equation (66a) is the chiral version of (36), with θ_σ rescaled by $\sqrt{K_\sigma}$.

It is worth noting here that in the open wire geometry the charge J_ρ and the spin J_σ currents are necessarily absent [39], and as a result the zero-mode part of (66a) consists of a single term $\propto M^2/L$. This also means that the "super-selection" rule $(-1)^{N_\rho} = (-1)^{N_\sigma}$ while (47), with $J_\rho = J_\sigma = 0$, becomes its natural consequence. Altogether, this means that global zero-mode constraints (46) and (47), which played a crucial role in the ring geometry in Sec. II A 5, largely lose their importance in the open wire geometry.

We now observe that at a special value $K_\sigma = 2$, which defines the Luther-Emery point [40,50], the cosine term in (66b) is proportional to the product of $e^{-i\sqrt{4\pi}\tilde{\Phi}(x)}$ and $e^{-i\sqrt{4\pi}\tilde{\Phi}(-x)}$, suggesting, by comparison with (C32), that it can be written as a bilinear form of fermionlike operators. Therefore, at $K_\sigma = 2$ a refermionization is possible. To that end, we introduce the new spinless fermion operator via

$$f(x) = \frac{1}{\sqrt{2\pi\alpha}} \mathcal{F} e^{i\pi x M/L} e^{i\sqrt{4\pi}\tilde{\Phi}(x)}, \tag{67}$$

where $\mathcal{F} \equiv F_\downarrow^\dagger F_\uparrow$ is a new Klein factor. Observe that it satisfies all requirements of being the M-changing operator,

$$[M, \mathcal{F}] = -\mathcal{F}, \quad [M, \mathcal{F}^\dagger] = \mathcal{F}^\dagger, \quad \mathcal{F}^\dagger \mathcal{F} = \mathcal{F} \mathcal{F}^\dagger = 1. \tag{68}$$

The exponential in (66b) can now be rewritten, at $K_\sigma = 2$, as

$$\begin{aligned} (\mathcal{F}^\dagger)^2 e^{-i\sqrt{4\pi}[\tilde{\Phi}(x) + \tilde{\Phi}(-x)]} &= e^{-i\sqrt{4\pi}\tilde{\Phi}(x)} \mathcal{F}^\dagger e^{-i\sqrt{4\pi}\tilde{\Phi}(-x)} \\ &\quad \times \mathcal{F}^\dagger e^{2\pi\Upsilon(x)} \\ &= e^{-i\sqrt{4\pi}\tilde{\Phi}(x)} \mathcal{F}^\dagger e^{-i\frac{\pi x M}{L}} \\ &\quad \times e^{i\frac{\pi x M}{L}} e^{-i\sqrt{4\pi}\tilde{\Phi}(-x)} \mathcal{F}^\dagger e^{2\pi\Upsilon(x)} \\ &= 2\pi\alpha f^\dagger(x) f^\dagger(-x)\, e^{i\frac{\pi x}{L}} e^{2\pi\Upsilon(x)}. \end{aligned} \tag{69}$$

To obtain the last equality above, we used $\mathcal{F}^\dagger e^{-i\pi x M/L} = e^{-i\pi x M/L} \mathcal{F}^\dagger e^{i\pi x/L}$. Observe that $[\tilde{\Phi}(x), \tilde{\Phi}(-x)] = \Upsilon(x)$ and that for $|x| \gg \alpha$ (65) gives

$$e^{2\pi\Upsilon(x)} e^{i\pi x/L} = i s(x), \tag{70}$$

where $s(x) = \text{sgn}[\sin \pi x/L]$ is a $2L$-periodic sign function

$$s(x) = \begin{cases} 1, & x \in (0, L), \\ -1, & x \in (-L, 0), \\ 0, & x = 0, \pm L. \end{cases} \tag{71}$$

Keeping small but finite α in (65) rounds discontinuities of $s(x)$ in finite intervals of order α around endpoints $x = x_o$. Equations (64) and (69) show that at $K_\sigma = 2$

$$e^{-i2\beta} K_R^+(x) K_R^+(-x) = \frac{-i s(x)}{2\pi\alpha} f^\dagger(x) f^\dagger(-x). \tag{72}$$

The final ingredient is the kinetic energy which we, following Ref. [40] and using formalism developed in Appendix B 2, find to be

$$\int_{-L}^{L} dx\, f^\dagger(x)(-i v_\sigma \partial_x) f(x) = \frac{\pi v_\sigma}{2L} M(M+1) + \int_{-L}^{L} dx\, v_\sigma [\partial_x \tilde{\Phi}(x)]^2. \tag{73}$$

Hence at $K_\sigma = 2$ the spin Hamiltonian (66a) can be written in terms of new fermion operators (67) as

$$\widetilde{\mathcal{H}}_\sigma = \int_{-L}^{L} dx \left[f^\dagger(x)(-i v_\sigma \partial_x) f(x) + \frac{i\Delta}{2} s(x) f(x) f(-x) + \frac{i\Delta}{2} s(x) f^\dagger(x) f^\dagger(-x) \right] - \frac{\pi v_\sigma M}{2L}, \tag{74}$$

where the spin gap is given by $\Delta = (g_x + g_c)/(8\pi\alpha)$. Equation (74) describes a one-dimensional p-wave superconductor

with the pairing potential changing sign at $x = x_o = 0, \pm L$ [51], i.e., at the open boundaries of the wire in our original problem. The sign function in (74) is required because fermions anticommute, $\{f(x), f(-x)\} = 0$. The kink in Δ is of profound importance to the low-energy excitations. We show below that it induces a zero-energy self-conjugate state, the Majorana mode, which is exponentially localized near the boundary.

Note that although the Hamiltonian (74) does not conserve M due to the presence of the Klein factors $\mathcal{F}$ in $f(x)$, it does conserve $e^{i2\pi M} = e^{i\pi(N_\uparrow - N_\downarrow)}$, which follows from $[e^{i2\pi M}, \mathcal{F}] = 0$.

Moreover, (74) also conserves the *magnetization parity* $e^{i\pi M}$,

$$e^{i\pi M} = e^{i\frac{\pi}{2}(N_\uparrow - N_\downarrow)}. \tag{75}$$

This is because (74) contains *squares* of Klein factors $\mathcal{F}$ and $\mathcal{F}^\dagger$. As a result, we have to look on $e^{i\pi M}\mathcal{F}^2$. However, the commutation relation (68) implies that $[M, \mathcal{F}^2] = -2\mathcal{F}^2$ and therefore $e^{i\pi M}\mathcal{F}^2 = \mathcal{F}^2 e^{i\pi M} e^{-i2\pi} = \mathcal{F}^2 e^{i\pi M}$. That is, $[e^{i\pi M}, \widetilde{\mathcal{H}}_\sigma] = 0$, the magnetization parity (75) is conserved by the Hamiltonian (74). The charge sector of the open wire is described in Appendix C 6.

IV. SOLUTION OF THE REFERMIONIZED HAMILTONIAN

One-dimensional superconductor (74) is solved by the Bogolyubov transformation [52]

$$f(x) = \sum_{n\geqslant 0}[\gamma_n u_n(x) + \gamma_n^\dagger v_n^*(-x)], \tag{76}$$

which diagonalizes (74) into the form

$$\widetilde{\mathcal{H}}_\sigma = E_{\text{g.s.}} + \sum_n \epsilon_n \gamma_n^\dagger \gamma_n. \tag{77}$$

Here $\epsilon_n \geqslant 0$ are the excitation energy, and γ_n are Fermi operators satisfying $\{\gamma_n, \gamma_m^\dagger\} = \delta_{n,m}$ and $\{\gamma_n, \gamma_m\} = 0$. Functions $u(x)$, $v(x)$ are found with the help of the equation of motion

$$\begin{aligned} i\partial_t f(x) &= [f(x), \widetilde{\mathcal{H}}_\sigma] \\ &= -\frac{\pi v_\sigma}{2L} f(x) - i v_\sigma \partial_x f(x) - i\Delta \text{sgn}(x) f^\dagger(-x) \end{aligned} \tag{78}$$

by expressing both sides of the last equality in terms of fermion operators γ_n, $\gamma_n^\dagger$ with the help of (77) and (78). This leads to the Bogolyubov-de Gennes equation

$$\begin{pmatrix} -iv_\sigma\partial_x - \frac{\pi v_\sigma}{2L} & i\Delta s(x) \\ -i\Delta s(x) & iv_\sigma\partial_x + \frac{\pi v_\sigma}{2L} \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \epsilon \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}. \tag{79}$$

What are boundary conditions for $f(x)$ and, as a result, for $u(x)$ and $v^*(-x)$? Equation (67) shows that $f(x + 2L) = e^{i2\pi M} f(x)$, where we used $2L$ periodicity of $\tilde{\Phi}(x)$ and the commutation relations (68). Correspondingly, the vector $\begin{pmatrix} u(L) \\ v(L) \end{pmatrix}$ has the same boundary conditions as $f(x)$. It follows that

$$\begin{pmatrix} u(L) \\ v(L) \end{pmatrix} = e^{i\Theta} \begin{pmatrix} u(-L) \\ v(-L) \end{pmatrix}, \quad \Theta = 2\pi M. \tag{80}$$

Note that at $x = 0$ the vector $(u(x), v(x))^T$ is continuous.

As noted below (74), even though M is not conserved by $\widetilde{\mathcal{H}}_\sigma$, the exponential $e^{i2\pi M}$ remain unchanged and is conserved, because anomalous ff and $f^\dagger f^\dagger$ terms in $\widetilde{\mathcal{H}}_\sigma$ change M by ± 2. Therefore we can treat Θ in (80) as a real (non-operator) phase, but distinguish the cases of $M =$ integer and $M =$ half-integer,

$$\begin{aligned} &\Theta = 0 \text{ for } M \in \mathbb{Z}, \\ &\Theta = \pi \text{ for } M \in \mathbb{Z} + 1/2. \end{aligned} \tag{81}$$

Particle-hole symmetry of (79) ensures that vector $(v^*(-x), u^*(-x))^{\text{T}}$ describes states with energy $-\epsilon$ and satisfies (80). Full solution of (79) consists of scattering states $\tilde{f}_\epsilon$ with energies ϵ above the gap Δ and localized in-gap states f_0, f_L with nearly zero energy,

$$f(x) = f_0(x) + f_L(x) + \int_\Delta^\infty d\epsilon\, \tilde{f}_\epsilon(x). \tag{82}$$

We focus on the localized modes which, for $0 < x < L$, are described by

$$\begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} = \frac{A_1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{i\phi} \end{pmatrix} e^{(-\kappa + i\frac{\pi}{2L})x} + \frac{B_1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{-i\phi} \end{pmatrix} e^{(\kappa + i\frac{\pi}{2L})x}. \tag{83}$$

The corresponding energy is $\epsilon = \sqrt{\Delta^2 - v_\sigma^2\kappa^2}$, and we introduced $e^{i\phi} = (v_\sigma\kappa + i\epsilon)/\Delta$. Solution on the negative half of the wire, $-L < x < 0$, is given by the similar combination with amplitudes A_2, B_2 and $\Delta \to -\Delta$ due to the oddness of the function $s(x)$. Boundary condition (80) and continuity of $(u(x), v(x))^T$ at $x = 0$ can be written in the form of 4×4 matrix equation, acting on the vector $(A_1, B_1, A_2, B_2)^T$, with zero right-hand side. Setting determinant of that matrix to zero produces the relation between ϵ and κ,

$$\tan^2(\phi)\sinh^2(\kappa L) = \cos^2(\Theta/2), \tag{84}$$

which can be used to express everything in terms of κ as

$$\begin{aligned} \Delta &= v_\sigma\kappa\sqrt{1 + \frac{\cos^2\Theta/2}{\sinh^2(\kappa L)}}, \\ \epsilon &= v_\sigma\kappa\frac{\cos\Theta/2}{\sinh(\kappa L)} \approx 2\Delta\cos(\Theta/2)e^{-\Delta L/v_\sigma}. \end{aligned} \tag{85}$$

We used (81) which guaranties that $\cos\Theta/2 \geqslant 0$. It is worth pointing out the surprising feature of the vanishing splitting ϵ between the first excited state, localized at the opposite ends of the wire, and the ground state of the wire for the special value of the phase difference $\Theta = \pi \pmod{2\pi}$, when $M \in \mathbb{Z} + 1/2$ is half integer. The energy splitting ϵ is maximal when $M \in \mathbb{Z}$, which corresponds to *even* $2M = N_\uparrow - N_\downarrow$. Similar oscillatory dependence on the phase difference Θ has been previously studied in Ref. [53] in a different context.

The most important physical message of (85) is that the first excited state is exponentially close to the ground state. For $L \gg \xi = v_\sigma/\Delta$ it is essentially degenerate with it.

Straightforward algebra leads to

$$\begin{aligned} &A_1 = e^{2\kappa L}(1 - ie^{-\kappa L + i\Theta/2})\mathcal{C}, \quad B_1 = i(e^{\kappa L + i\Theta/2} + i)\mathcal{C}, \\ &A_2 = B_1^*, \quad B_2 = A_1^*. \end{aligned} \tag{86}$$

Here $\mathcal{C}$ is the real normalization constant. To the leading order $\mathcal{C} = \sqrt{\kappa/2}e^{-2\kappa L}$. Therefore, for $0 < x < L$,

$$\begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} \approx \frac{\sqrt{\kappa}}{2}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\kappa x + i\pi x/2L} + ie^{i\Theta/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\kappa(x-L)+i\pi x/2L}\right], \tag{87}$$

where we neglected exponentially small corrections $\propto e^{-\kappa L}$ to the two end contributions describing exponentially localized in-gap states near the left, $x = 0$, and the right, $x = L$, ends of the wire. We also used (85) to replace $e^{\pm i\phi}$ by 1 up to exponentially small terms. For $-L < x < 0$ we similarly find

$$\begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} \approx \frac{\sqrt{\kappa}}{2}\left[-ie^{-i\Theta/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\kappa(x+L)+i\pi x/2L} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\kappa x + i\pi x/2L}\right]. \tag{88}$$

Equations (76), (87), and (88) finally allow us to express exponentially localized end modes as

$$f_0(x) \approx \frac{\sqrt{\kappa}}{2} e^{-\kappa x}(\gamma_0 - \gamma_0^\dagger) \quad \text{for } 0 < x \ll L, \tag{89}$$

and

$$f_L(x) \approx -\frac{\sqrt{\kappa}}{2} e^{-\kappa(L-x)} e^{i\Theta/2}(\gamma_0 + \gamma_0^\dagger) \quad \text{for } L - x \ll L. \tag{90}$$

Note that (89) and (90) describe self-conjugate Majorana modes, $f_0 = -f_0^\dagger$ and $f_L^\dagger = e^{-i\Theta} f_L$. Namely, f_0 and f_L are proportional to the independent Majorana modes, d_1 and d_2 defined via

$$\gamma_0 = \frac{1}{\sqrt{2}}(d_1 + id_2) \tag{91}$$

with

$$\{d_1, d_2\} = 0, \quad (d_1)^2 = (d_2)^2 = \tfrac{1}{2}, \tag{92}$$

such that

$$f_0(0) = i\sqrt{\frac{\kappa}{2}} d_2, \quad f_L(L) = -e^{i\Theta/2}\sqrt{\frac{\kappa}{2}} d_1. \tag{93}$$

The appearance of the independent Majorana modes at the opposite ends of the quantum wire agrees with the original proposal of Kitaev [1]. The ground state of the wire is doubly degenerate since states $|0\rangle$ and $|1\rangle = \gamma_0^\dagger|0\rangle$ have the same energy [1], up to exponentially small energy difference ϵ given by (85). Here $|0\rangle$ is the vacuum state of γ_n, $\gamma_n|0\rangle = 0$ for $n \geqslant 0$.

Finally, we comment on the applicability of the present analysis away from $K_\sigma = 2$. It is likely from (37a) that the [illegible] of the Luttinger parameter K_σ is smaller than 2 [illegible] ing ($g_z/4\pi v_F < 3/5$). However, K_σ is [illegible] according to the RG equation (30) and reaches $K_\sigma = 2$ at some length scale, at which we can apply the refermionization. In this sense the analysis above is applicable to a broader range of parameters. Physically, the twofold ground-state degeneracy is a direct consequence of the SDW order.

V. PHYSICAL MEANING OF THE MAJORANA MODE

A. Spin correlations in the bulk of the wire

Spin excitations of the open wire consist of massive propagating modes $\tilde{f}_\epsilon$, with energy $\epsilon \geqslant \Delta$, and zero-energy modes $f_{0,L}$ which are exponentially localized on the scale $\xi = \kappa^{-1} = v_\sigma/\Delta$ near $x_o = 0$ and $x_o = L$, correspondingly. We therefore expect that spin correlations inside the open wire, for $\xi \ll x \ll L - \xi$, should coincide with those in the ring geometry, see Sec. II A 4.

To see how this comes about, we start with $K_R^+(x)$, (C38),

$$K_R^+(x) \propto e^{-i\sqrt{4\pi}[\tilde{\Phi}(x)-\tilde{\Phi}(-x)]-i\sqrt{\pi}[\tilde{\Phi}(x)+\tilde{\Phi}(-x)]} \tag{94}$$

and observe that according to (66a) the SDW ordered state is characterized by the ordered, or "frozen," symmetric combination of spin fields $\tilde{\Phi}(x) + \tilde{\Phi}(-x)$. Importantly, the antisymmetric combination $\tilde{\Phi}(x) - \tilde{\Phi}(-x)$ does not commute with the symmetric one. Indeed, a simple calculation, similar to the one in (65), shows that for $-L < x, y < L$

$$[\tilde{\Phi}(x) + \tilde{\Phi}(-x), \tilde{\Phi}(y) - \tilde{\Phi}(-y)] = \frac{i}{2}\left(\mathrm{sgn}(x-y) - \mathrm{sgn}(x+y) + \frac{2y}{L}\right). \tag{95}$$

Therefore the ordering (freezing) of the symmetric combination $\tilde{\Phi}(x) + \tilde{\Phi}(-x)$ makes correlations of operators involving the antisymmetric one $\tilde{\Phi}(x) - \tilde{\Phi}(-x)$ short ranged, e.g., decaying exponentially with distance.

Now we turn to the $2k_F$ component of the spin density N^a (38) and its rotated version (C39), (C40). Analysis in Appendix C 5 shows that $\widetilde{N}^{x,y}$ fields involve the symmetric spin mode $\tilde{\Phi}(x) + \tilde{\Phi}(-x)$ as well as an antisymmetric charge one $\Phi_{R\rho}(x) - \Phi_{R\rho}(-x)$. In the SDW phase the symmetric spin mode is frozen, but charge excitations remain critical. Moreover, Eq. (C43) and discussion around it shows that, when the frozen value of the symmetric mode is substituted, the spin part of $\widetilde{N}^x$ vanishes while that of $\widetilde{N}^y$ approaches a constant value. Correspondingly, correlations of $\widetilde{N}^y$ field inside the wire decay algebraically with the exponent which is controlled by the Luttinger constant of the charge mode K_ρ, in agreement with expressions (42c) and (43) for the closed wire case. At the same time, correlations of components $\widetilde{N}^{0,z}$ decay exponentially with the distance, because they involve quantum-disordered antisymmetric spin combination $\tilde{\Phi}(x) - \tilde{\Phi}(-x)$. The described correspondence also shows that $\tilde{\Phi}(x) + \tilde{\Phi}(-x)$ plays the role of θ_σ, while $\tilde{\Phi}(x) - \tilde{\Phi}(-x)$ is analogous to ϕ_σ in Sec. II A 4.

B. End-to-end correlations

Correlations between the end regions of the wire, $0 < x < \xi$ and $L - \xi < x < L$, are very different. Observe that for $x \approx 0$ the quantum-disordered combination vanishes, $\tilde{\Phi}(x) - \tilde{\Phi}(-x) \approx 0$. The same is true for $x \approx L$ due to the $2L$ periodicity of the field $\tilde{\Phi}(x)$. At the same time the symmetric combination simplifies to $\tilde{\Phi}(x) + \tilde{\Phi}(-x) \approx 2\tilde{\Phi}(0)$. [Obviously, for $x \approx L$ we have $\tilde{\Phi}(x) + \tilde{\Phi}(-x) \approx 2\tilde{\Phi}(L)$.] Using (C38) and the definition of the fermion $f(x)$ (67), we observe

that

$$K_R^+(0) = \frac{e^{i\beta}}{\sqrt{2\pi\alpha}} f^\dagger(0),$$
$$K_R^+(L) = -\frac{e^{i\beta}}{\sqrt{2\pi\alpha}} e^{-i\pi M} f^\dagger(L). \tag{96}$$

Equation (93) shows that fermion operators at the ends of the wire reduce to the Majorana modes $d_{1,2}$, and therefore the same is true for the spin currents K_R^+ at the ends of the chain. Note, however, the appearance of the "string" operator $e^{-i\pi M}$ in $K_R^+(L)$ in (96). This string operator is in fact the *magnetization parity*, introduced in Eq. (75). It represents a key integral of motion of the problem and plays a very important role in the subsequent analysis of the open wire.

It is this string operator that makes sure that spin densities at $x \sim 0$ and $x \sim L$ actually commute, as they must do (and not anticommute, as they would if it was absent). At this point it is important to observe that $e^{\pm i\pi M}$ and $f(x)$, introduced in (67), anticommute for all x. This is easy to see with the help of identity (C9) of Ref. [40] and (68). Hence

$$\{e^{\pm i\pi M}, f(x)\} = 0 = \{e^{\pm i\pi M}, f^\dagger(x)\}. \tag{97}$$

Next, Eq. (76) implies that, for all n,

$$\{e^{i\pi M}, \gamma_n\} = 0 = \{e^{i\pi M}, \gamma_n^\dagger\}. \tag{98}$$

Therefore we can establish an operator identity

$$e^{i\pi M} = e^{i\Theta/2} \prod_\ell e^{i\pi\gamma_\ell^\dagger\gamma_\ell} = e^{i\Theta/2} \prod_\ell (1 - 2\gamma_\ell^\dagger\gamma_\ell), \tag{99}$$

which enforces Eqs. (97) and (98) and also insures that $e^{i2\pi M} = e^{i\Theta} = \pm 1$, since $(1 - 2\gamma_\ell^\dagger\gamma_\ell)^2 = 1$ for every ℓ. Here phase Θ is the c number introduced in (80). Note that (99) *does not* mean that $M = \Theta/(2\pi) + \sum_\ell \gamma_\ell^\dagger\gamma_\ell$.

At very low energies $\epsilon \ll \Delta$

$$e^{i\pi M} \approx e^{i\Theta/2}(1 - 2\gamma_0^\dagger\gamma_0) = -2ie^{i\Theta/2} d_1 d_2. \tag{100}$$

Therefore, while $f^\dagger(L) \sim d_1$, the magnetization parity acting on it changes it into the Majorana mode d_2, $e^{-i\pi M} f^\dagger(L) \sim (d_1 d_2) d_1 = -d_2$. More accurately, we obtain

$$K_R^+(L) = -\frac{e^{i\beta}}{\sqrt{2\pi\alpha}} e^{-i\Theta} f^\dagger(0) = -e^{-i\Theta} K_R^+(0). \tag{101}$$

Therefore spin currents at the opposite ends of the wire are equal, up to a complex prefactor. We see that they commute, $[K_R^+(L), K_R^+(0)] = 0$, as they should. Note that without the string operator $e^{-i\pi M}$ in (96) the spin currents in question would anticommute, $\{K_R^+(L), K_R^+(0)\} = 0$, just as fermion operators do. The string operator is crucial for obtaining the correct result. From the quantum computing point of view, the string operator spoils braiding statistics of the localized end modes [54].

We also observe that $K_R^+(-L) = K_R^+(L)$, thanks to $e^{i2\pi M} = 1$ for the integer/half-integer M. The last line in (C35) allows us to write the total spin current in terms of the right-moving one,

$$K^+(L) = K_R^+(L) + K_L^+(L) = (1 + e^{-i2\beta}) K_R^+(L)$$
$$= -e^{-i\Theta} K^+(0). \tag{102}$$

Therefore

$$K^+(L)K^-(0) = -e^{-i\Theta} K^+(0)K^-(0)$$
$$= -4\cos^2(\beta)\, e^{-i\Theta} K_R^+(0) K_R^-(0)$$
$$= -\cos^2(\beta)\, e^{-i\Theta} \frac{\kappa}{2\pi\alpha}. \tag{103}$$

This shows unusual long-ranged end-to-end correlations between spin currents at the opposite ends of the wire.

Equation (103) is to be contrasted with exponentially decaying correlations of K_R^+ in the bulk of the wire, as discussed in Sec. V A above. The strength of the correlation between the opposite edges is determined by the spin gap, $\kappa \approx \Delta/v_\sigma$, see (85).

Spin currents at the ends of the wire in the original basis are related to K^a via (C4), when position-dependent rotation (C22) reduces to matrix identity. Therefore at x_o we obtain

$$J_{R/L}^+(x_o) = e^{\mp i\beta} K_{R/L}^+(x_o) \tag{104}$$

and, correspondingly, find the original spin currents at the ends of the wire are proportional to Majorana mode d_2 as well,

$$J^+(0) = J_R^+(0) + J_L^+(0) = 2e^{-i\beta} K_R^+(0) = \frac{2}{\sqrt{2\pi\alpha}} f^\dagger(0),$$
$$J^+(L) = -2e^{-i\Theta} e^{-i\beta} K_R^+(0) = \frac{-2e^{-i\Theta}}{\sqrt{2\pi\alpha}} f^\dagger(0), \tag{105}$$

where

$$f^\dagger(0) = -i\sqrt{\frac{\kappa}{2}} d_2. \tag{106}$$

Since $\Theta = 0$ or π, the above equation means that $J^x(x_o) = 0$ while the $\hat{y}$ component of $\vec{J}(x_o)$ is finite, $J^y(x_o) \sim d_2$. The end-to-end correlation of the uniform part of the spin density is similarly long ranged,

$$J^+(L)J^-(0) = -e^{-i\Theta} \frac{\kappa}{2\pi\alpha} = -J^+(L)J^+(0). \tag{107}$$

It differs from the same-position correlation of the spin density at the end of the wire only by the sign, $J^+(0)J^-(0) = \kappa/(2\pi\alpha)$. The last equality in (107) reflects the fact that $J^+(x_o) = iJ^y(x_o)$, as noted above. This behavior is schematically sketched in Fig. 2.

Next we look at the correlations of the staggered part $\vec{N}$ of the spin density. Equation (C41) shows that near the ends of the wire staggered spin density is also proportional to Majorana modes

$$N^+(x_o) \propto e^{-i\sqrt{2\pi}[\Phi_{R\rho}(x_o) - \Phi_{R\rho}(-x_o)]} e^{-i\sqrt{\pi}[\tilde{\Phi}(x_o) + \tilde{\Phi}(-x_o)]}$$
$$= e^{-\sqrt{4\pi}\tilde{\Phi}(x_o)} \sim f^\dagger(x_o) \tag{108}$$

because both $\Phi_{R\rho}$ and $\tilde{\Phi}$ are $2L$ periodic.

As a matter of fact, it is easy to argue that end-to-end correlations of N^+ field must be identical to those of J^+ one, (107). This is because at the ends of the open wire the total spin density must vanish, and therefore

$$\vec{S}(x_o) = \vec{J}_R(x_o) + \vec{J}_L(x_o) + [\vec{N}(x_o) e^{-i2k_F x_o} + \text{H.c.}] = 0, \tag{109}$$

since in addition $e^{\pm i2k_F x_o} = 1$. Hence indeed, staggered components of the spin density possess the same end-to-end

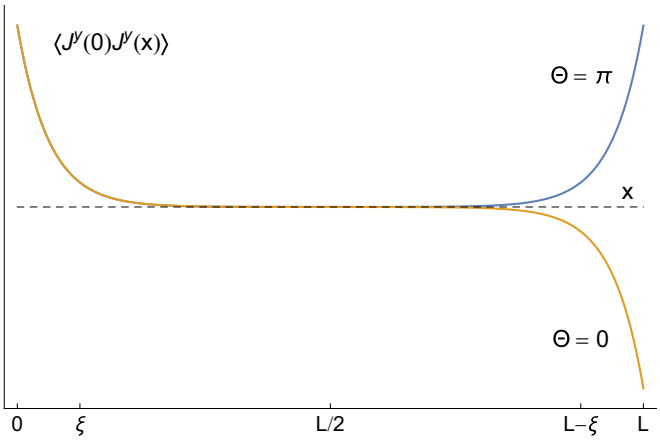

FIG. 2. Schematics of the spin correlations in the SDW phase of the open quantum wire. The $\hat{y}$ component of spin density exhibits long-ranged edge-to-edge correlations, Eq. (107), while the edge-to-bulk correlations decay exponentially on the scale $\xi = v_\sigma/\Delta$, as discussed in Sec. V A. Staggered components of the spin density, N^y, behave similarly.

correlations as the uniform ones, (107). Technically, this happens because near the wire's ends the charge exponential in (108) can be expanded as $1 - i\sqrt{8\pi}\alpha d\Phi_{R\rho}/dx + \ldots$ and be approximated by the unity. That is, near the wire's ends charge fluctuations are frozen out, while the spin part of N^+ reduces to the negative of the J^+ one at the same time. Explicit calculation based on full expressions given in Appendix C 5 confirms this natural conclusion. Once again, we see that spin density exhibits an unusual long-range end-to-end correlations despite the fact that in the bulk of the wire all correlations decay, some exponentially fast (such as spin currents and $N^{0,z}$) while others algebraically ($N^{\pm}$), as discussed in Sec. V A.

C. Twofold degeneracy and the magnetization parity

Proportionality of $\vec{J}$ and $\vec{N}$ to the fermion operators $f^\dagger(0)$ and $f^\dagger(L)$ merits additional discussion. Consider the wire with a fixed total number of electrons $N_\rho = N_\uparrow + N_\downarrow$. Magnetization is $M = (N_\uparrow - N_\downarrow)/2$. Therefore parities of spin-↑ and -↓ band are $(-1)^{N_\uparrow} = e^{i\pi N_\rho/2}e^{i\pi M}$ and $(-1)^{N_\downarrow} = e^{i\pi N_\rho/2}e^{-i\pi M}$. It is sufficient to discuss just one of them, say $P_1 = (-1)^{N_\uparrow}$. Let us assume, for definiteness, that N_ρ is *even*, so that the factor $e^{i\pi N_\rho/2}$ is real valued. Then $2M$ is also even and hence $e^{i\pi M}$ has eigenvalues ± 1. As discussed below (75), Hamiltonian $\widetilde{\mathcal{H}}_\sigma$ conserves magnetization parity $e^{i\pi M}$, and therefore the parity P_1 as well. Hence the ground state of $\widetilde{\mathcal{H}}_\sigma$ is characterized by the definite parity P_1. But anticommutation of $e^{i\pi M}$ and $f^\dagger(x)$, Eq. (97), implies that the expectation value of $f^\dagger(x_o)$ in the state $|\phi\rangle$ with definite fermion parity is zero. Indeed,

$$0 = \langle\phi|e^{i\pi\hat{M}}f^\dagger(x_o) + f^\dagger(x_o)e^{i\pi\hat{M}}|\phi\rangle = 2e^{i\pi M}\langle\phi|f^\dagger(x_o)|\phi\rangle, \tag{110}$$

where $i\pi\hat{M}$ on the right-hand side of (110) is the eigenvalue of the parity operator $e^{i\pi\hat{M}}$ in the state $|\phi\rangle$, where we have used $\hat{M}$ for the magnetization operator M. Therefore we conclude that the ground state expectation value of spin operators $\vec{J}$ and $\vec{N}$ near the ends of the wire is zero, $\langle\phi|J^+(x_o)|\phi\rangle = 0 = \langle\phi|N^+(x_o)|\phi\rangle$. As discussed above, the expectation value of operators $\vec{J}$ and $\vec{N}$ in the bulk of the wire is zero, too.

We thus see that the ground state of the Hamiltonian (74) is disordered but twofold degenerate. The degeneracy is topological; it comes from the degeneracy of many-body ground states $|0\rangle$ and $|1\rangle$, which have opposite magnetization parities. These states are defined via $\gamma_0|0\rangle = 0$ and $|1\rangle = \gamma_0^\dagger|0\rangle$. Let $|0\rangle$ be an eigenstate of the parity $e^{i\pi\hat{M}}$ with eigenvalue $e^{i\pi M} = \pm 1$, that is

$$e^{i\pi\hat{M}}|0\rangle = e^{i\pi M}|0\rangle. \tag{111}$$

Then state $|1\rangle$ has the opposite parity,

$$e^{i\pi\hat{M}}|1\rangle = e^{i\pi\hat{M}}\gamma_0^\dagger|0\rangle = -\gamma_0^\dagger e^{i\pi M}|0\rangle = -e^{i\pi M}|1\rangle. \tag{112}$$

Note that these two states share property (110), that is $\langle 0|f(x_o)|0\rangle = \langle 1|f(x_o)|1\rangle = 0$.

Conversely, we can construct states $|u/d\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, where the plus (minus) sign corresponds to the state $|u\rangle$ ($|d\rangle$), for which the expectation value of J^+, N^+ operators near the ends is finite,

$$\begin{aligned}\langle u/d|\gamma_0|u/d\rangle &= \tfrac{1}{2}(\langle 0| \pm \langle 1|)(\gamma_0|0\rangle \pm \gamma_0|1\rangle) = \pm\tfrac{1}{2}\langle 0|\gamma_0|1\rangle \\ &= \pm\tfrac{1}{2}, \\ \langle u/d|\gamma_0^\dagger|u/d\rangle &= \tfrac{1}{2}(\langle 0| \pm \langle 1|)\gamma_0^\dagger|0\rangle = \pm\tfrac{1}{2}.\end{aligned} \tag{113}$$

But for these states magnetization parity is not defined

$$e^{i\pi\hat{M}}|u/d\rangle = \frac{e^{i\pi M}}{\sqrt{2}}(|0\rangle - (\pm)|1\rangle) = e^{i\pi M}|d/u\rangle. \tag{114}$$

Rather, the parity operator $e^{i\pi\hat{M}}$ represents Pauli matrix σ^x in the subspace spanned by the states $|u\rangle$ and $|d\rangle$.

The physical states of the open wire are of the type $|0\rangle$, $|1\rangle$ from the above, simply because they are characterized by the definite magnetization parity. Fermi operator γ_0 is introduced in (91) and, according to the discussion above, can be constructed with the help of equations (89), (90), (96), and (104) as

$$\gamma_0 = -\sqrt{\frac{2\pi\alpha}{\kappa}}(J_R^+(0) - e^{i\Theta/2}e^{i\pi\hat{M}}J_R^+(L)), \tag{115}$$

which makes explicit its nonlocal nature.

It is also useful to notice now that the single fermion operator (C32) does not have simple expression in terms of f because

$$e^{i\sqrt{2\pi}\Phi_{R\sigma}(x)} \sim e^{i\sqrt{\pi}[\tilde{\Phi}(x)+\tilde{\Phi}(-x)]/2 + i\sqrt{\pi}[\tilde{\Phi}(x)-\tilde{\Phi}(-x)]} \tag{116}$$

and therefore reduces to the "square root" of Majorana in the $x \to x_o$ limit, e.g., $\psi_R(0) \sim e^{i\sqrt{\pi}\tilde{\Phi}(0)} \sim [f(0)]^{1/2}$. At the same time, in the bulk of the wire the presence of the dual combinations $\tilde{\Phi}(x) + \tilde{\Phi}(x)$ and $\tilde{\Phi}(x) - \tilde{\Phi}(-x)$ in (116) imply exponential decay of correlation functions of fermion operator ψ_R with distance and time. That is, gapped behavior, just as expected.

D. Instability of the twofold degeneracy to the parity-breaking perturbations

There are several physically-reasonable perturbations which violate the magnetization parity conservation [30].

(i) Local magnetic field near the end of the wire, such as, for example, due to the magnetic impurity $\delta H' = \vec{h}_1 \cdot \vec{S}(a)$, located at a short distance $a < \xi = v_\sigma/\Delta$ from, for example, the left end of the wire. Then $a/\xi < 1$ ensures that $\vec{h}_1$ couples the end modes of the wire. Using $\vec{S}(x) = \vec{J}(x) + [\vec{N}(x)e^{-i2k_F x} +$ H.c.] and the fact that $2\vec{N}(a) \approx -\vec{J}(a)$, we find that $\delta H_1 \approx [1 - \cos(2k_F a)]\vec{h}_1 \cdot \vec{J}(a)$. Now, the discussion around (105) and (107) shows that $\vec{J}(a) \approx i\sqrt{\kappa/(2\pi\alpha)}e^{-\kappa a}(\gamma_0 - \gamma_0^\dagger)\hat{y}$, so that we can write $\delta H_1 = i\tilde{h}_1(\gamma_0 - \gamma_0^\dagger)$ by absorbing all nonessential constants into $\tilde{h}_1$. In the low-energy subspace of definite parity states $\{|0\rangle, |1\rangle\}$ this perturbation is off-diagonal and reduces to $\delta H' = \tilde{h}_1\sigma^y$. Its eigenstates are those of the Pauli matrix σ^y, with energies $\pm\tilde{h}_1$, and they are *not* eigenstates of the magnetization parity $e^{i\pi\hat{M}}$.

Therefore this local perturbation breaks magnetization parity conservation and removes the twofold degeneracy of the ground state in favor of the unique state $(|0\rangle - i|1\rangle)/\sqrt{2}$ with energy $-\tilde{h}_1$.

An interesting consequence of the end-to-end correlation (107), which for $2M =$ even (see the beginning of this section where $N_\rho =$ even was set) can be written as $J^y(L)J^y(a) \approx -\kappa/(2\pi\alpha)$, is that $\delta H'$ acting near the left end of the wire causes finite polarization $J^y(L)$ at its opposite, right end. This kind of "long-distance rigidity" in the absence of rigid spin correlations in the bulk of the wire is unusual and represents a bosonic version of the teleportation phenomena previously suggested for fermion Majorana states [55,56].

(ii) Next, consider applying the local magnetic field somewhere in the middle of the wire, so that the perturbation still has the form $\delta H'$ but now with $\xi \ll a \ll L - \xi$. The uniform part of the spin density is exponentially suppressed there and the field couples to the staggered part, $\widetilde{N}(a)$. Since $\tilde{\Phi}(x) + \tilde{\Phi}(-x)$ is locked to the optimal value, $\delta H'$ reduces to

$$\delta H' \approx (h_{1x} - ih_{1y})\mathcal{F}^\dagger e^{-i\frac{\pi x}{L}(N_\rho+1)}e^{-i\sqrt{2\pi}[\Phi_{R\rho}(x)-\Phi_{R\rho}(-x)]} + \text{H.c.}, \tag{117}$$

see (C41). The expectation value of this operator in a finite-size system is proportional to $(\alpha/L)^{K_\rho/2}$. This is because projection of the charge-mode exponential to the ground state gives, after normal ordering it,

$$e^{-i\sqrt{2\pi}[\Phi_{R\rho}(x)-\Phi_{R\rho}(-x)]} \to \left(\frac{\pi\alpha}{2L|\sin(\pi x/L)|}\right)^{K_\rho/2} \tag{118}$$

for $\alpha \ll x \ll L$. Note also that fermion changing operator $\mathcal{F}$ in (105) violates the conservation of the magnetization parity. Therefore such a perturbation, which has the meaning of electron spin-flip backscattering off a magnetic impurity, also breaks the twofold degeneracy of the ground state. Here the breaking of the degeneracy due to the perturbation is smaller than in the previous example (i); it vanishes algebraically with the size of the system as $h_1(\alpha/L)^{K_\rho/2}$ and therefore is not [illegible] for sufficiently long wires.

VI. DISCUSSION

We found that ground states of the interacting wire in the correlated SDW phase have all the features of the symmetry-protected topological (SPT) state [28,57,58]. Indeed, in the closed (ring) geometry the ground state is unique and is an eigenstate of a definite magnetization parity $e^{i\pi N_\sigma/2}$. In the open wire geometry, however, the ground states corresponding to different magnetization parities $e^{i\pi M}$ are degenerate with exponential accuracy $e^{-L/\xi}$. The localized Majorana modes that appear in this geometry are found to describe spin density near the wire's ends. Importantly, the expectation value of the spin density in the state with definite magnetization parity is zero everywhere in the wire, including its ends, and cannot be used to distinguish the degenerate ground states. The correlated SDW state can therefore be classified as a SPT state which is protected by the magnetization parity.

In the topological SDW state the spin sector is Ising ordered along the y direction (the direction of the Rashba spin-orbit interaction) while the charge sector is a gapless Tomonaga-Luttinger liquid. The charge fluctuations weaken the correlation of the S^y spins and make it quasi-long-ranged in the bulk. However, at the ends of a wire the charge fluctuations are frozen so that the long-range Ising spin correlation between the end spins can manifest itself without being obscured by charge fluctuations. In some sense this long-range spin correlation is just the Ising order of the XYZ spin chain covered by the critical charge fluctuations in the bulk. Nevertheless, we regard the SDW state as a SPT state, because its ground state degeneracy is determined by the boundary conditions—the ground state is unique in the ring geometry with both PBC and anti-PBC conditions and becomes twofold degenerate in the open wire geometry. The degeneracy is protected by the magnetization parity. Indeed, it takes a parity-breaking perturbation to lift the ground state degeneracy, as we show in Sec. V C.

It is important to note that without critical charge mode our model would reduce to the transverse field Ising (TFI) chain the ground state of which is not an SPT phase. This is seen from the fact that in the ring geometry the TFI model retains twofold degeneracy (with exponentially small in the system size splitting) which is just the usual Ising Z_2 degeneracy. In our case it is the gapless charge mode which endows gapped quantum wire with the SPT properties.

This observation is in agreement with several previous studies of topological states of interacting quantum wires [30,31,59,60] which found that the presence of the gapless charge (more generally, center-of-mass) mode is crucial for the ground state degeneracy. It must be added here that as far as possible physical realization of the described physics goes, the model studied here appears to be the simplest one. Its realization requires only a quantum wire with significant spin-orbit coupling and strong repulsive e-e interactions, and all of these ingredients are readily available in the present-day experimental setups.

Another important lesson of our study follows from the fact that it is collective spin degrees of freedom, which are described by bosonlike operators that become 'topologically' correlated. The difference between a one-particle fermion operator and a two-particle boson operator is fundamental. Simple one-particle fermion operators at different points must anticommute, and naturally they do, as (91) and (93) show. The two-particle operators, which necessarily are bosonlike, such as the spin density here, on the other hand, must commute when taken at different points. This is achieved with the

help of the string operator $e^{-i\pi M}$, as explained in Sec. V B. Therefore two-particle operators at the opposite ends of the wire *must* be proportional to each other, up to unessential phase factors. This is the crucial difference between the fundamental degrees of freedom of the fermionic Kitaev chain (one-dimensional p-wave superconductor), which are single particles of BdG type [1,51], and the transverse field Ising chain, where they are two-particle excitations of magnetic kind, and we have rediscovered it here for the correlated SDW wire.

We therefore arrive at the logical conclusion that many-body states are not particularly good for realizing Majorana degrees of freedom as long as they are based on some kind of two-particle (boselike) operators. For single-particle based constructions, such as weakly-interacting semiconducting quantum wires in contact with a superconductor, the commutation requirement does not exist and therefore there are no fundamental restrictions to realizing the sought-after Majorana fermions in such platforms.

We nonetheless believe that our problem is interesting in its own way. It shows how fractionalized degrees of freedom emerge in a basic setup of a quantum wire with repulsive interactions only. The finding of the twofold degenerate ground state with unusual long-ranged correlations between the spin densities at the opposite ends of the open wire, while the correlations in the bulk of the wire decay, at best, as a power law, deserves further theoretical and experimental studies.

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APPENDIX A: SCREENED COULOMB INTERACTION AND SPIN-CHARGE SEPARATION

In this Appendix we derive the low-energy effective theory in terms of current operators $J_{R/L}(x)$ and $\vec{J}_{R/L}(x)$. The key idea of the derivation is similar to Refs. [45,61] although on a technical level we proceed by employing operator-product expansion (OPE) of fermion operators while these references apply it in the bosonic language. We assume the limit $L \to \infty$ in this Appendix. We use simplified notations

$$R_s(x) = \Psi_{Rs}(x), \quad L_s(x) = \Psi_{Ls}(x), \tag{A1}$$

and write the electron density as

$$\rho(x) = \sum_{s=\uparrow,\downarrow} :\Psi_s^\dagger(x)\Psi_s(x): = \rho_0(x) + \rho_{2k_F}(x), \tag{A2a}$$

where

$$\rho_0(x) = \sum_{s=\uparrow,\downarrow} [:R_s^\dagger(x)R_s(x): + :L_s^\dagger(x)L_s(x):] = J_R(x) + J_L(x), \tag{A2b}$$

$$\rho_{2k_F}(x) = \sum_{s=\uparrow,\downarrow} [\, e^{2ik_Fx}L_s^\dagger(x)R_s(x) + e^{-2ik_Fx}R_s^\dagger(x)L_s(x)]. \tag{A2c}$$

The density-density interaction is decomposed into two parts

$$H_{\rm int} = \frac{1}{2}\int dx \int dy\, \rho(x)U(x-y)\rho(y) \approx H_F + H_B, \tag{A3}$$

where forward- and backward-scattering interactions are

$$H_F = \frac{1}{2}\int dx \int dy\, \rho_0(x)U(x-y)\rho_0(y), \tag{A4}$$

$$H_B = \frac{1}{2}\int dx \int dy\, \rho_{2k_F}(x)U(x-y)\rho_{2k_F}(y). \tag{A5}$$

Here we have discarded rapidly oscillating cross terms $(\rho_0\rho_{2k_F})$.

The interaction potential $U(x-y)$ is short ranged. Introducing new variables $X = (x+y)/2$ and $\tilde{x} = x - y$, we can rewrite the forward-scattering interaction H_F as

$$H_F \approx \frac{1}{2}\int d\tilde{x}\, U(\tilde{x}) \int dX [\rho_0(X)]^2 = \frac{U_0}{2}\int dx\, [J_R(x) + J_L(x)]^2. \tag{A6}$$

The product of ρ_{2k_F} in H_B yields

$$\rho_{2k_F}(x)\rho_{2k_F}(y) \approx e^{2ik_F\tilde{x}} \sum_{\mu,\sigma} L_\mu^\dagger(x)R_\mu(x)R_\sigma^\dagger(y)L_\sigma(y) + e^{-2ik_F\tilde{x}} \sum_{\mu,\sigma} R_\mu^\dagger(x)L_\mu(x)L_\sigma^\dagger(y)R_\sigma(y), \tag{A7}$$

where we have discarded rapidly oscillating terms $(\propto e^{\pm 4ik_FX})$. The backward-scattering interaction H_B can be calculated using the operator-product expansions

$$\begin{aligned} &L_\mu^\dagger(x)R_\mu(x)R_\sigma^\dagger(y)L_\sigma(y) \\ &= \left[-\frac{i\delta_{\mu,\sigma}}{2\pi\tilde{x}} + :L_\mu^\dagger(x)L_\sigma(y): \right]\left[\frac{i\delta_{\mu,\sigma}}{2\pi\tilde{x}} - :R_\sigma^\dagger(y)R_\mu(x): \right] \\ &= \frac{\delta_{\mu,\sigma}}{(2\pi\tilde{x})^2} - :L_\mu^\dagger(X)L_\sigma(X)::R_\sigma^\dagger(X)R_\mu(X): \\ &\quad + \frac{i\delta_{\mu,\sigma}}{2\pi\tilde{x}}[:L_\sigma^\dagger(X)L_\sigma(X): + :R_\sigma^\dagger(X)R_\sigma(X):] \\ &\quad + \frac{i\delta_{\mu,\sigma}}{4\pi}\{:[\partial_X L_\sigma^\dagger(X)]L_\sigma(X): - :L_\sigma^\dagger(X)\partial_X L_\sigma(X): \\ &\quad - :[\partial_X R_\sigma^\dagger(X)]R_\sigma(X): + :R_\sigma^\dagger(X)\partial_X R_\sigma(X):\} \\ &\quad + \ldots, \end{aligned} \tag{A8}$$

and

$$
\begin{aligned}
&R^\dagger_\mu(x)L_\mu(x)L^\dagger_\sigma(y)R_\sigma(y)\\
&=\Big[-\frac{i\delta_{\mu,\sigma}}{2\pi\tilde{x}}-:L^\dagger_\sigma(y)L_\mu(x):\Big]\Big[\frac{i\delta_{\mu,\sigma}}{2\pi\tilde{x}}+:R^\dagger_\mu(x)R_\sigma(y):\Big]\\
&=\frac{\delta_{\mu,\sigma}}{(2\pi\tilde{x})^2}-:L^\dagger_\sigma(X)L_\mu(X)::R^\dagger_\mu(X)R_\sigma(X):\\
&\quad-\frac{i\delta_{\mu,\sigma}}{2\pi\tilde{x}}[:L^\dagger_\sigma(X)L_\sigma(X):+:R^\dagger_\sigma(X)R_\sigma(X):]\\
&\quad+\frac{i\delta_{\mu,\sigma}}{4\pi}\{:[\partial_X L^\dagger_\sigma(X)]L_\sigma(X):-:L^\dagger_\sigma(X)\partial_X L_\sigma(X):\\
&\quad-:[\partial_X R^\dagger_\sigma(X)]R_\sigma(X):+:R^\dagger_\sigma(X)\partial_X R_\sigma(X):\}\\
&\quad+\ldots.
\end{aligned}
\tag{A9}
$$

The second term in the above expansions can be written in terms of currents,

$$
\begin{aligned}
&\sum_{\mu,\sigma=\uparrow,\downarrow}:L^\dagger_\mu(x)L_\sigma(x)::R^\dagger_\sigma(x)R_\mu(x):\\
&\qquad=\frac{1}{2}J_L(x)J_R(x)+2\vec{J}_L(x)\cdot\vec{J}_R(x),
\end{aligned}
\tag{A10}
$$

and the fourth term (a kinetic energy density) can be written as

$$
\begin{aligned}
&-\frac{i}{2}\big\{:[\partial_x L^\dagger_\sigma(x)]L_\sigma(x):-:L^\dagger_\sigma(x)\partial_x L_\sigma(x):\\
&\qquad-:[\partial_x R^\dagger_\sigma(x)]R_\sigma(x):+:R^\dagger_\sigma(x)\partial_x R_\sigma(x):\big\}\\
&=\frac{\pi}{2}\big\{:[J_L(x)]^2:+:[J_R(x)]^2:\big\}\\
&\qquad+\frac{2\pi}{3}\big[:\vec{J}_L(x)\cdot\vec{J}_L(x):+:\vec{J}_R(x)\cdot\vec{J}_R(x):\big].
\end{aligned}
\tag{A11}
$$

Combining these contributions, we obtain

$$
\begin{aligned}
H_B=&\int dx\int d\tilde{x}\,U(\tilde{x})\frac{\cos(2k_F\tilde{x})}{2\pi^2\tilde{x}^2}\\
&-\int d\tilde{x}\,U(\tilde{x})\frac{\sin(2k_F\tilde{x})}{2\pi\tilde{x}}\int dx[J_L(x)+J_R(x)]\\
&-U_{2k_F}\int dx\Big\{\frac{1}{2}J_L(x)J_R(x)+2\vec{J}_L(x)\cdot\vec{J}_R(x)\\
&\quad+\frac{1}{4}\{:[J_L(x)]^2+[J_R(x)]^2:\}\\
&\quad+\frac{1}{3}[:\vec{J}_L(x)\cdot\vec{J}_L(x):+:\vec{J}_R(x)\cdot\vec{J}_R(x):]\Big\},
\end{aligned}
\tag{A12}
$$

where

$$
U_{2k_F}=\int d\tilde{x}\,U(\tilde{x})\cos(2k_F\tilde{x}).
\tag{A13}
$$

The first term on the right-hand side of (A12) is a constant, and the second term is a renormalization of the chemical potential. We thus keep the last two contributions proportional to U_{2k_F} and finally obtain

$$
H_F+H_B=H_c+H_s+\ldots,
\tag{A14}
$$

with the charge part

$$
H_c=\frac{1}{4}(2U_0-U_{2k_F})\int dx\,[J_R(x)+J_L(x)]^2,
\tag{A15}
$$

and the spin part

$$
\begin{aligned}
H_s=&-2U_{2k_F}\int dx\,\vec{J}_L(x)\cdot\vec{J}_R(x)\\
&-\frac{U_{2k_F}}{3}\int dx\,[:\vec{J}_L(x)\cdot\vec{J}_L(x):+:\vec{J}_R(x)\cdot\vec{J}_R(x):].
\end{aligned}
\tag{A16}
$$

The second line in (A16) gives renormalization of the velocity v_F in (7). We note that H_c is a functional of J_R+J_L, which is a consequence of Galilean invariance.

APPENDIX B: BOSONIZATION

1. Bosonization under periodic boundary condition

Here we summarize bosonization rules for the ring geometry [37,38,62]. We first define chiral bosonic fields

$$
\phi_{Rs}(x)=\phi^0_{Rs}+\frac{\sqrt{\pi}x}{L}N_{Rs}+\varphi_{Rs}(x),
\tag{B1a}
$$

$$
\phi_{Ls}(x)=\phi^0_{Ls}+\frac{\sqrt{\pi}x}{L}N_{Ls}+\varphi_{Ls}(x),
\tag{B1b}
$$

where $s=\uparrow,\downarrow$, and

$$
[\phi_{Rs}(x),\phi_{Rs'}(y)]=-[\phi_{Ls}(x),\phi_{Ls'}(y)]=\frac{i}{4}\delta_{s,s'}\mathrm{sgn}(x-y).
\tag{B2}
$$

The zero-mode operators satisfy the commutation relations

$$
\big[\phi^0_{Rs},N_{Rs'}\big]=-\big[\phi^0_{Ls},N_{Ls'}\big]=-\frac{i\delta_{s,s'}}{\sqrt{4\pi}},
\tag{B3}
$$

$$
\big[\phi^0_{Rs},N_{Ls'}\big]=\big[\phi^0_{Ls},N_{Rs'}\big]=[\phi_{Rs},\phi_{Ls'}]=[N_{Rs}.N_{Ls'}]=0.
\tag{B4}
$$

The fields φ_{Rs} and φ_{Ls} are periodic functions of x, $\varphi_{R/Ls}(x+L)=\varphi_{R/Ls}(x)$, and can be expanded as

$$
\varphi_{Rs}(x)=\sum_{n=1}^{\infty}\frac{e^{-\pi\alpha n/L}}{\sqrt{4\pi n}}(e^{iq_nx}a_{n,Rs}+e^{-iq_nx}a^\dagger_{n,Rs}),
\tag{B5}
$$

$$
\varphi_{Ls}(x)=\sum_{n=1}^{\infty}\frac{e^{-\pi\alpha n/L}}{\sqrt{4\pi n}}(e^{-iq_nx}a_{n,Ls}+e^{iq_nx}a^\dagger_{n,Ls}),
\tag{B6}
$$

where $q_n=2\pi/L$, α is a short-distance cutoff, and the boson annihilation/creation operators obey the commutation relations

$$
[a_{n,Rs},a^\dagger_{n',Rs'}]=[a_{n,Ls},a^\dagger_{Ls'}]=\delta_{n,n'}\delta_{s,s'}.
\tag{B7}
$$

The fields with different chiralities commute,

$$
[\phi_{Rs}(x),\phi_{Ls'}(y)]=0.
\tag{B8}
$$

The chiral fermion fields are written in terms of the chiral bosonic fields as follows:

$$\psi_{Rs}(x) = \frac{\eta_{Rs}}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi^0_{Rs}} e^{2\pi i N_{Rs} x/L + i\sqrt{4\pi}\varphi_{Rs}(x)} = \frac{\eta_{Rs}}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_{Rs}(x) + i\pi x/L}, \tag{B9}$$

$$\psi_{Ls}(x) = \frac{\eta_{Ls}}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi^0_{Ls}} e^{-2\pi i N_{Ls} x/L - i\sqrt{4\pi}\varphi_{Ls}(x)} = \frac{\eta_{Ls}}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_{Ls}(x) - i\pi x/L}, \tag{B10}$$

where $\eta_{R/Ls}$ obey the anticommutation relations

$$\{\eta_{Rs}, \eta_{Rs'}\} = \{\eta_{Ls}, \eta_{Ls'}\} = 2\delta_{s,s'}, \quad \{\eta_{Rs}, \eta_{Ls'}\} = 0. \tag{B11}$$

The fermion field operators ψ_{Rs} and ψ_{Ls} satisfy the standard anticommutation relations. The fermion density operators are given by

$$:\psi^\dagger_{Rs}(x)\psi_{Rs}(x): = \frac{1}{\sqrt{\pi}}\partial_x\phi_{Rs}(x), \tag{B12}$$

$$:\psi^\dagger_{Ls}(x)\psi_{Ls}(x): = \frac{1}{\sqrt{\pi}}\partial_x\phi_{Ls}(x), \tag{B13}$$

and therefore the fermion number operators N_{Rs} and N_{Ls}, defined by

$$N_{Rs} = \int_0^L dx \frac{1}{\sqrt{\pi}}\partial_x\phi_{Rs}(x), \tag{B14}$$

$$N_{Ls} = \int_0^L dx \frac{1}{\sqrt{\pi}}\partial_x\phi_{Ls}(x), \tag{B15}$$

are integer-valued operators. The charge current operators defined in (8) are thus given by

$$J_R(x) = \frac{1}{\sqrt{\pi}}\partial_x[\phi_{R\uparrow}(x) + \phi_{R\downarrow}(x)], \tag{B16}$$

$$J_L(x) = \frac{1}{\sqrt{\pi}}\partial_x[\phi_{L\uparrow}(x) + \phi_{L\downarrow}(x)]. \tag{B17}$$

One can show, using $e^{2\pi i N_{Rs}} = e^{2\pi N_{Ls}} = 1$, that the fermion fields satisfy the periodic boundary conditions, $\psi_{Rs}(x+L) = \psi_{Rs}(x)$ and $\psi_{Ls}(x+L) = \psi_{Ls}(x)$.

The linearized kinetic energy is given by [40]

$$\int_0^L dx : \psi^\dagger_{Rs}(-i\partial_x)\psi_{Rs} : = \frac{\pi}{L}\left(N^2_{Rs} + N_{Rs}\right) + \int_0^L dx : (\partial_x\varphi_{Rs})^2 :, \tag{B18}$$

$$\int_0^L dx : \psi^\dagger_{Ls}(i\partial_x)\psi_{Ls} : = \frac{\pi}{L}\left(N^2_{Ls} + N_{Ls}\right) + \int_0^L dx : (\partial_x\varphi_{Ls})^2 : . \tag{B19}$$

We define nonchiral bosonic fields

$$\phi_s(x) = \phi_{Ls}(x) + \phi_{Rs}(x), \tag{B20}$$

$$\theta_s(x) = \phi_{Ls}(x) - \phi_{Rs}(x), \tag{B21}$$

and then introduce a pair of charge field operators,

$$\phi_\rho(x) = \frac{1}{\sqrt{2}}[\phi_\uparrow(x) + \phi_\downarrow(x)], \tag{B22}$$

$$\theta_\rho(x) = \frac{1}{\sqrt{2}}[\theta_\uparrow(x) + \theta_\downarrow(x)], \tag{B23}$$

and a pair of spin field operators,

$$\phi_\sigma(x) = \frac{1}{\sqrt{2}}[\phi_\uparrow(x) - \phi_\downarrow(x)], \tag{B24}$$

$$\theta_\sigma(x) = \frac{1}{\sqrt{2}}[\theta_\uparrow(x) - \theta_\downarrow(x)]. \tag{B25}$$

Finally, we introduce charge/spin number and current operators,

$$N_\rho = N_{R\uparrow} + N_{R\downarrow} + N_{L\uparrow} + N_{L\downarrow}, \tag{B26}$$

$$J_\rho = N_{R\uparrow} + N_{R\downarrow} - N_{L\uparrow} - N_{L\downarrow}, \tag{B27}$$

$$N_\sigma = N_{R\uparrow} - N_{R\downarrow} + N_{L\uparrow} - N_{L\downarrow}, \tag{B28}$$

$$J_\sigma = N_{R\uparrow} - N_{R\downarrow} - N_{L\uparrow} + N_{L\downarrow}. \tag{B29}$$

By definition these operators must satisfy the following relations [46]:

$$(-1)^{N_\rho} = (-1)^{J_\rho} = (-1)^{N_\sigma} = (-1)^{J_\sigma}, \tag{B30}$$

$$(-1)^{\frac{1}{2}(N_\rho + J_\rho)} = (-1)^{\frac{1}{2}(N_\sigma + J_\sigma)}. \tag{B31}$$

It is easy to write down the explicit form of the spin boson θ_σ which will be useful for discussions in Sec. II A 5,

$$\theta_\sigma(x) = -\sqrt{\frac{\pi}{2}}\frac{x}{L}J_\sigma + \frac{1}{\sqrt{2}}\left(\phi^0_{L\uparrow} - \phi^0_{L\downarrow} - \phi^0_{R\uparrow} + \phi^0_{R\downarrow} + \varphi_{L\uparrow} - \varphi_{L\downarrow} - \varphi_{R\uparrow} + \varphi_{R\downarrow}\right). \tag{B32}$$

Observe that in the presence of a finite spin current $J_\sigma \neq 0$ the spin field acquires a kink at $x = 0 = L$ since then $\theta_\sigma(0) - \theta_\sigma(L) = \sqrt{\pi/2}J_\sigma$.

2. Bosonization under open boundary condition

Here we summarize bosonization rules for electrons in a wire of length L with open boundaries [37,38,62–64]. We first define chiral boson fields

$$\phi_{Ls}(x) = \frac{\sqrt{\pi}}{4} + \frac{\sqrt{\pi}x}{2L}N_s + \frac{\theta^0_s}{\sqrt{4\pi}} + \Phi_{Ls}(x), \tag{B33}$$

$$\phi_{Rs}(x) = \frac{\sqrt{\pi}}{4} + \frac{\sqrt{\pi}x}{2L}N_s - \frac{\theta^0_s}{\sqrt{4\pi}} + \Phi_{Rs}(x), \tag{B34}$$

where $s = \uparrow, \downarrow$,

$$\left[\theta^0_s, N_{s'}\right] = i\delta_{s,s'}, \tag{B35}$$

and Φ_{Ls} and Φ_{Rs} have mode expansions,

$$\Phi_{Ls}(x) = -\Phi_{Rs}(-x) = \sum_{n=1}^\infty \frac{e^{-\pi n\alpha/2L}}{\sqrt{4\pi n}}(e^{-i\pi nx/L}a_{n,s} + e^{i\pi nx/L}a^\dagger_{n,s}), \tag{B36}$$

which satisfy $\Phi_{Ls}(x+2L)=\Phi_{Ls}(x)$ and the same for $\Phi_{Rs}(x)$. In (B36) α is a short-distance cutoff. One can verify that the chiral boson fields introduced above satisfy the commutation relations

$$[\phi_{Rs}(x),\phi_{Rs'}(y)] = -[\phi_{Ls}(x),\phi_{Ls'}(y)] = \frac{i}{4}\delta_{s,s'}\mathrm{sgn}(x-y), \tag{B37}$$

and

$$[\phi_{Rs}(x),\phi_{Ls'}(y)] = \begin{cases} 0, & x=y=0, \\ -\frac{i}{4}\delta_{s,s'}, & 0<x,y<L \\ -\frac{i}{2}\delta_{s,s'}, & x=y=L. \end{cases} \tag{B38}$$

We define a pair of bosonic fields ($s=\uparrow,\downarrow$)

$$\phi_s(x) = \phi_{Ls}(x)+\phi_{Rs}(x), \tag{B39a}$$
$$\theta_s(x) = \phi_{Ls}(x)-\phi_{Rs}(x), \tag{B39b}$$

which satisfy the commutation relation,

$$[\phi_s(x),\theta_{s'}(y)] = -i\delta_{s,s'}\Theta(x-y) \tag{B40}$$

for $0<x,y<L$. The field $\phi_s(x)$ obeys the Dirichlet boundary conditions at $x=0,L$:

$$\phi_s(0)=\frac{\sqrt{\pi}}{2}, \quad \phi_s(L)=\sqrt{\pi}\left(N_s+\frac{1}{2}\right). \tag{B41}$$

We then introduce charge fields,

$$\phi_\rho(x) = \frac{1}{\sqrt{2}}[\phi_\uparrow(x)+\phi_\downarrow(x)], \tag{B42a}$$
$$\theta_\rho(x) = \frac{1}{\sqrt{2}}[\theta_\uparrow(x)+\theta_\downarrow(x)], \tag{B42b}$$

and spin fields,

$$\phi_\sigma(x) = \frac{1}{\sqrt{2}}[\phi_\uparrow(x)-\phi_\downarrow(x)], \tag{B43a}$$
$$\theta_\sigma(x) = \frac{1}{\sqrt{2}}[\theta_\uparrow(x)-\theta_\downarrow(x)]. \tag{B43b}$$

Fermion fields are written in terms of the chiral boson fields $\psi_s(x)=e^{ik_Fx}\psi_{Rs}(x)+e^{-ik_Fx}\psi_{Ls}(x)$, where [39]

$$\psi_{Rs}(x) = \frac{\eta_s}{\sqrt{2\pi\alpha}}e^{i\sqrt{4\pi}\phi_{Rs}(x)+i\pi x/2L} = \frac{i\eta_s}{\sqrt{2\pi\alpha}}e^{-i\theta_s^0}e^{i\pi N_s x/L}e^{i\sqrt{4\pi}\Phi_{Rs}(x)}, \tag{B44}$$

$$\psi_{Ls}(x) = \frac{\eta_s}{\sqrt{2\pi\alpha}}e^{-i\sqrt{4\pi}\phi_{Ls}(x)-i\pi x/2L} = \frac{i\eta_s}{\sqrt{2\pi\alpha}}e^{-i\theta_s^0}e^{-i\pi N_s x/L}e^{-i\sqrt{4\pi}\Phi_{Ls}(x)} = -\psi_{Rs}(-x). \tag{B45}$$

Here η_s obeys the anticommutation relations

$$\{\eta_s,\eta_{s'}\}=2\delta_{s,s'}. \tag{B46}$$

The electron density operator is given by

$$:\psi_{Rs}^\dagger(x)\psi_{Rs}(x): = \frac{1}{\sqrt{\pi}}\partial_x\phi_{Rs}(x), \tag{B47a}$$
$$:\psi_{Ls}^\dagger(x)\psi_{Ls}(x): = \frac{1}{\sqrt{\pi}}\partial_x\phi_{Ls}(x). \tag{B47b}$$

We define Klein factors [40]

$$F_s = \eta_s e^{-i\theta_s^0}, \tag{B48}$$

which satisfy

$$F_s^\dagger F_s = 1, \quad [F_s,N_s]=F_s. \tag{B49}$$

The operator N_s is integer valued and measures the number of electrons with spin s,

$$N_s = \int_0^L dx[:\psi_{Rs}^\dagger(x)\psi_{Rs}(x):+:\psi_{Ls}^\dagger(x)\psi_{Ls}(x):]. \tag{B50}$$

It follows that $\psi_{Rs}(x+2L)=\psi_{Rs}(x)$. The Fermi wave number is given by $k_F=\pi N_s^0/L$, where N_s^0 is another integer. We see from (B45) that the open boundary conditions are satisfied

$$\psi_s(0)=\psi_s(L)=0. \tag{B51}$$

APPENDIX C: DETAILS OF THE ANALYSIS FOR THE WIRE WITH OBC

1. Rotations

To treat the wire with open boundaries it is convenient to orient external magnetic field along the $\hat{x}$ axis, while the spin-orbit axis continues to point along the $\hat{y}$ axis. Such a choice leads to chiral rotations about the $\hat{z}$ axis, see below, and results in convenient boundary conditions for rotated fermions $\psi(x)$, as we demonstrate now. Thus, the Zeeman magnetic field $b=g\mu_B B$ couples to the magnetization along the $\hat{x}$ axis,

$$\mathcal{V}_\mathrm{x} = -b\int dx\left(J_R^x+J_L^x\right), \tag{C1}$$

while the spin-orbit interaction couples to the difference of the $\hat{y}$ components of the currents,

$$\mathcal{V}_\mathrm{so} = 2\alpha_R k_F\int dx\left(J_R^y-J_L^y\right). \tag{C2}$$

Using again extended $SU(2)\times SU(2)$ symmetry of the non-interacting spin Hamiltonian with respect to independent rotations of the right- and left-moving currents, we rotate spin currents $\vec{J}_R$ and $\vec{J}_L$ about the $\hat{z}$ axis in opposite directions so as to bring "vectorial" perturbation $\mathcal{V}=\mathcal{V}_\mathrm{so}+\mathcal{V}_\mathrm{x}$ into the standard Zeeman form, with *total* field $h=\sqrt{b^2+(2\alpha_R k_F)^2}$ along the $\hat{x}$ axis

$$\mathcal{V} = -h\int dx\left(M_R^x+M_L^x\right). \tag{C3}$$

Compare this with (13) where the field h is pointing along the $\hat{z}$ axis.

The required chiral rotation is given by

$$\vec{J}_R = \mathcal{R}_z(\beta_R)\vec{M}_R, \quad \vec{J}_L = \mathcal{R}_z(\beta_L)\vec{M}_L, \tag{C4}$$

where the rotation matrix is $\mathcal{R}$

$$\mathcal{R}_z(\beta) = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{C5}$$

The rotation angles are given by

$$\beta_R = -\beta_L = \beta = \arctan(2\alpha_R k_F/b). \tag{C6}$$

These rotations do not affect $\mathcal{H}^0_\sigma$ (7), which retains its form in the rotated M basis

$$\mathcal{H}^0_\sigma = \frac{2\pi v_F}{3} \sum_{a=x,y,z} \int dx \left(M^a_R M^a_R + M^a_L M^a_L\right). \tag{C7}$$

In terms of the right- and left-moving fermions, the rotation (C5) corresponds to the rotation of spinors $\Psi_{R/L} = (\Psi_{R/L\uparrow}, \Psi_{R/L\downarrow})^T$,

$$\Psi_R = e^{i\beta\sigma^z/2}\Psi'_R, \quad \Psi_L = e^{-i\beta\sigma^z/2}\Psi'_L. \tag{C8}$$

As before, the charge currents (8) do not transform under the rotations (C4) and (C8)—the Hamiltonian of the charge sector $\mathcal{H}^0_\rho + \mathcal{H}_{\mathrm{int},\rho}$ is not affected. The new (primed) fermions parametrize the rotated currents $\vec{M}_{R/L}$ in the same way as the old (unprimed) ones parametrize the currents $\vec{J}_{R/L}$. For example, under the right rotation $\mathcal{R}_z(\beta)$

$$\vec{J}_R =: \Psi^\dagger_R \frac{\vec{\sigma}}{2} \Psi_R : \to \vec{M}_R =: \Psi'^\dagger_R \frac{\vec{\sigma}}{2} \Psi'_R : . \tag{C9}$$

The interaction in the spin sector $\mathcal{H}_{\mathrm{int},\sigma}$ (10b) is strongly modified by the rotation and changes to

$$\begin{aligned} \mathcal{H}_{\mathrm{int},\sigma} &= -g \int dx\, \vec{M}_R \mathcal{R}^T(\beta_R)\mathcal{R}(-\beta_R)\vec{M}_L \\ &= -g \int dx \big[M^z_R M^z_L + \cos\chi \left(M^y_R M^y_L + M^x_R M^x_L\right) \\ &\qquad + \sin\chi \left(M^y_R M^x_L - M^x_R M^y_L\right)\big], \end{aligned} \tag{C10}$$

where $\chi = \beta_R - \beta_L = 2\beta$ is the *relative* rotation angle.

The net field h, (C3), pointing along the $\hat{x}$ axis, induces incommensurate fluctuations in the system which makes some of the terms in (C10) oscillate fast with the coordinate. To account for this important effect we proceed as follows:

(1) We do a *global* rotation of $\vec{M}_{R/L}$ about the $\hat{y}$ axis in order to make external field h (C3) point along the $\hat{z}$ axis. This is achieved by the following transformation to the new L basis, $(M^x, M^y, M^z)^T = \mathcal{R}_y(\pi/2)(L^x, L^y, L^z)^T = (L^z, L^y, -L^x)^T$. The corresponding rotation for fermions reads $\Psi'_{R/L} \to e^{-i\pi\sigma^y/4}\Psi''_{R/L}$. Here, similar to (C9), $L^a_{R/L} = \frac{1}{2} : \Psi''^\dagger_{R/L}\sigma^a\Psi''_{R/L} :$.

Noninteracting Hamiltonian (5) is invariant under constant-angle rotations (C8) and $\Psi'_{R/Ls} \to e^{i\pi\sigma^y/4}\Psi''_{R/Ls}$, while the field-dependent term (C3) is rotated into $\mathcal{V} = -h\int dx\,(L^z_R + L^z_L)$. It is then easy to see that h can be absorbed into fermions $\Psi''_{R/L}$ by a simple x-dependent

$$\Psi''_R \to e^{-it_\varphi x\sigma^z/2}\Psi''_R, \quad \Psi''_L \to e^{-it_\varphi x\sigma^z/2}\Psi''_L, \quad t_\varphi = h/v_F, \tag{C11}$$

under which kinetic energy (5) transforms into that of rotated Ψ'' fermions *plus* $\int dx\, h(L^z_R + L^z_L)$ term which exactly compensates the rotated $\mathcal{V}$ one.

(2) As a result of this shift the transverse components $L^x_{R/L} \pm iL^y_{R/L} = L^\pm_{R/L}$ of the rotated spin current acquire oscillating position-dependent factors, $L^+_R \to L^+_R e^{-it_\varphi x}$, $L^+_L \to L^+_L e^{it_\varphi x}$. The immediate consequence of this is that many terms in $\mathcal{H}_{\mathrm{int},\sigma}$ (C10) acquire x-dependent oscillations,

$$\begin{aligned} \mathcal{H}_{\mathrm{int},\sigma} = -g \int dx \Big\{ &\cos\chi L^z_R L^z_L - \frac{\cos\chi - 1}{4}(L^+_R L^+_L + \mathrm{H.c.}) \\ &+ \frac{\cos\chi + 1}{4}(L^+_R L^-_L e^{-i2t_\varphi x} + \mathrm{H.c.}) \\ &- i\frac{\sin\chi}{2}\left[(L^z_R L^+_L + L^-_R L^z_L)e^{it_\varphi x} - \mathrm{H.c.}\right]\Big\}. \end{aligned} \tag{C12}$$

Provided that the running backscattering coupling constant g/v_F is small, all oscillating terms, which represent momentum-nonconserving two-particle scattering processes, average out to zero. Assuming this, we are allowed to drop all oscillating terms in (12).

The meaning of (C11) is simple. It represents splitting of the Fermi momentum k_F into the spin-dependent ones $k_{Fs} = k_F + st_\varphi/2$. Given that k_F is determined by the particle density, $k_F = \pi N^0_s/L$, the development of the spin-dependent Fermi momenta $k_{Fs} = \pi N_s/L$ describes the appearance of the finite magnetization with $N_\uparrow > N_\downarrow$ in the *magnetized* ground state of the rotated system. Therefore, $\Delta k_F = t_\varphi/2 = \pi(N_\uparrow - N^0_\uparrow)/L$, so that $t_\varphi L = 2\pi(N_\uparrow - N^0_\uparrow) = -2\pi(N_\downarrow - N^0_\downarrow)$ is an integer multiple of 2π since N_s and N^0_s are integers describing number of spin-s electrons in the system with finite $h \neq 0$ and with zero $h = 0$, correspondingly.

(3) Having absorbed the h field (C3) in the preceding step, we now apply global rotation back, by $-\pi/2$ about the $\hat{y}$ axis, to the nonoscillating terms (first line) in (22). So that $(L^x, L^y, L^z)^T = \mathcal{R}_y(-\pi/2)(K^x, K^y, K^z)^T = (-K^z, K^y, K^x)^T$ and we obtain nonoscillating part of the spin-interaction Hamiltonian to be

$$\begin{aligned} \mathcal{H}_{\mathrm{int},\sigma} &= -\int_0^L dx \left[g_x K^x_R K^x_L + g_c\left(K^z_R K^z_L - K^y_R K^y_L\right)\right] \\ &= -\int_0^L dx\, g_a K^a_R K^a_L, \end{aligned} \tag{C13}$$

where $g_x = g\cos\chi$, $g_z = -g_y = g_c = g(1 - \cos\chi)/2$. Here the fermions rotate as $\Psi''_{R/L} \to e^{i\pi\sigma^y/4}\psi_{R/L}$. Note close similarity of (C13) with (23) as well as the fact that the roles of $\hat{x}$ and $\hat{z}$ axes are interchanged in these two expressions.

Under steps (1)–(3) the noninteracting spin Hamiltonian (C7) transforms into that in terms of spin currents $K^a_{R/L}$,

$$\mathcal{H}^0_\sigma = \frac{2\pi v_F}{3} \sum_{a=x,y,z} \int_0^L dx \left(K^a_R K^a_R + K^a_L K^a_L\right). \tag{C14}$$

At this stage the complete Hamiltonian of the spin sector is given by the sum of equations (C14) and (C13). The magnetic field is absent from the above Hamiltonian because it is absorbed into renormalization of the Fermi momenta $k_F \to k_{Fs}$.

Tracing the above steps (1)–(3) we find relation between $\Psi'_{R/L}$ and rotated fermions $\psi_{R/L}$, in terms of which the spin Hamiltonian, (C14) and (C13), and the charge Hamiltonian,

$\mathcal{H}^0_\rho$ in (7) and $\mathcal{H}_{\text{int},\rho}$ in (10a), are now formulated,

$$\Psi'_R(x) = e^{-i\pi\sigma^y/4}e^{it_\varphi x\sigma^z/2}e^{i\pi\sigma^y/4}\psi_R = \mathcal{A}(x)\psi_R(x),$$
$$\Psi'_L(x) = e^{-i\pi\sigma^y/4}e^{-it_\varphi x\sigma^z/2}e^{i\pi\sigma^y/4}\psi_L = \mathcal{A}(-x)\psi_L(x),$$
$$\mathcal{A}(x) = \sigma^0\cos\left(\frac{t_\varphi x}{2}\right) + i\sigma^x\sin\left(\frac{t_\varphi x}{2}\right). \tag{C15}$$

We are now in position to understand the boundary condition for the rotated fermions. For the original fermions the open boundary requires that $\Psi_s(x=0) = 0 = \Psi_s(x=L)$, which means that their right- and left-moving components are related as

$$\Psi_{Rs}(0) = -\Psi_{Ls}(0), \quad \Psi_{Rs}(L) = -\Psi_{Ls}(L). \tag{C16}$$

After the chiral rotation (C8) fermions $\Psi'_{R/L}$ obey

$$\Psi'_{Ls}(0) = -e^{is\beta}\Psi'_{Rs}(0), \quad \Psi'_{Ls}(L) = -e^{is\beta}\Psi'_{Rs}(L), \tag{C17}$$

where $s=+1$ for the up spin and $s=-1$ for the down spin. In matrix notations, (C17) is just $\Psi'_L(x_o) = -e^{i\beta\sigma^z}\Psi'_R(x_o)$, where $x_o = 0, L$ denotes the wire's open ends.

Next, Eq. (C15) shows how $\Psi'_{R/L}(x)$ transform as a result of global rotations in steps (1)–(3). Therefore the boundary condition (C17) actually reads $\mathcal{A}(-x_o)\psi_L(x_o) = -e^{i\beta\sigma^z}\mathcal{A}(x_o)\psi_R(x_o)$. Observing that $\mathcal{A}^{-1}(x_o) = \mathcal{A}(-x_o)$, we get

$$\psi_L(x_o) = -\mathcal{A}(x_o)e^{i\beta\sigma^z}\mathcal{A}(x_o)\psi_R(x_o) = -\mathcal{B}\psi_R(x_o),$$
$$\mathcal{B} = \begin{pmatrix} \cos(t_\varphi x_o)\cos\beta + i\sin\beta & i\cos\beta\sin(t_\varphi x_o) \\ i\cos\beta\sin(t_\varphi x_o) & \cos(t_\varphi x_o)\cos\beta - i\sin\beta \end{pmatrix}. \tag{C18}$$

The matrix $\mathcal{B}$ reduces to $e^{i\beta\sigma^z}$ when $t_\varphi x_o = \pi(N_\uparrow - N_\downarrow) = 2\pi M$, as discussed below (C12). Hence at the end of the day (C18) leads to

$$\psi_L(x_o) = -e^{i\beta\sigma^z}\psi_R(x_o), \tag{C19}$$

so that boundary conditions for spinors $\psi_{R/L}$ coincides with those for $\Psi'_{R/L}$, see (C17).

Observe that by construction spin current operators $K^a_{R/L}$ in (C13) and (C14) are given by

$$K^a_{R/L} =: \psi^\dagger_{R/L}\frac{\sigma^a}{2}\psi_{R/L} : . \tag{C20}$$

Relation between $M^a_{R/L}$ and $K^a_{R/L}$ currents is established with the help of equations (C9), (C15), and (C20),

$$M^a_R(x) =: \psi^\dagger_R(x)\mathcal{A}^\dagger(x)\frac{\sigma^a}{2}\mathcal{A}(x)\psi_R(x) : . \tag{C21}$$

We find that they are connected by an x-dependent rotation about the $\hat{x}$ axis,

$$\vec{M}_R = \mathcal{R}_x(-t_\varphi x)\vec{K}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t_\varphi x) & \sin(t_\varphi x) \\ 0 & -\sin(t_\varphi x) & \cos(t_\varphi x) \end{pmatrix}\vec{K}_R, \tag{C22}$$

$$\vec{M}_L = \mathcal{R}_x(t_\varphi x)\vec{K}_L .$$

Since at the boundary $e^{it_\varphi x_o} = 1$, we find that there the two operators coincide, $M^a_{R/L}(x_o) = K^a_{R/L}(x_o)$.

In terms of the original spin currents the OBC $\vec{J}_R(x_o) = \vec{J}_L(x_o)$ becomes $\vec{M}_R(x_o) = \mathcal{R}_z(-2\beta)\vec{M}_L(x_o)$, which means that $M^z_R(x_o) = M^z_L(x_o)$, $M^+_R(x_o) = e^{i2\beta}M^+_L(x_o)$. Given the relation $M^a_{R/L}(x_o) = K^a_{R/L}(x_o)$ derived above, we obtain that at the open boundaries the currents $K^a_{R/L}$ obey the same boundary condition as $M^a_{R/L}$

$$K^z_R(x_o) = K^z_L(x_o), \quad K^+_R(x_o) = e^{i2\beta}K^+_L(x_o), \tag{C23}$$

and, moreover, at $x_o = 0, L$ the original currents $J^a_{R/L}$ and $K^a_{R/L}$ are connected by chiral rotations (C5)

$$\vec{J}_R(x_o) = \mathcal{R}_z(\beta_R)\vec{K}_R(x_o),$$
$$\vec{J}_L(x_o) = \mathcal{R}_z(\beta_L)\vec{K}_L(x_o). \tag{C24}$$

2. Hamiltonian and the RG analysis

We are now ready to write down the Hamiltonian of the wire of finite length L with open boundaries at $x_o = 0, L$. The simplest way to derive the free part of the Hamiltonian is to go back to the original fermion formulation, equations (5), (13), and (C8), and observe that rotation (C8) leaves (5) invariant. The same is not true for the x-dependent rotation (C15) which, in addition to the kinetic energy of $\psi_{R/L}$ fermions, produces the *opposite* of (13) so as to cancel the field h term (13) completely. This, of course, is exactly the purpose of steps (1)–(3) and transformation (C15) as explained in Appendix C 1. In this way we arrive at $\mathcal{H}^0_\sigma$ in (C14).

It is useful to remark here that there is another, slightly more involved, way to derive this result: start with equation (C7) and apply rotations (C22) to it. Doing so requires one to implement a careful point-splitting procedure and treat $(M^a_R)^2$ as a limit of $M^a_R(x)M^a_R(y)$, with subsequent limit $x \to y$ at the end of the calculation. Then, using operator product expansion (OPE) of SU(2) currents and fermion bilinears [21]

$$\psi^\dagger_{R/Ls}(x)\psi_{R/Ls}(y) = \frac{\pm i}{2\pi(x-y)} + :\psi^\dagger_{R/Ls}(x)\psi_{R/Ls}(x):,$$
$$K^a_{R/L}(x)K^b_{R/L}(y) = -\frac{\delta^{ab}}{8\pi^2(x-y)^2} - \frac{\pm\epsilon^{abc}K^c_{R/L}(x)}{2\pi(x-y)}, \tag{C25}$$

where x, y are spatial coordinates and the limit $x \to y$ is implied. In particular, the first line above helps to establish that the field h produces a constant shift (magnetization) of $M^x_R = K^x_R + t_\varphi/4\pi$, which should be added to (C22). [The same shift of the along-the-field component of the spin current by $h/(4\pi v_F)$ is easily obtained in the Abelian bosonization, when one absorbs $-h\partial_x\phi_\sigma$ term by "completing the square."] Next, using (C36) we again arrive at the final result (C14) and also obtain the cancellation of the h-field term (C3).

Now we manipulate the interaction term. It is useful to observe that

$$g_x K^x_R K^x_L - g_c K^y_R K^y_L = \frac{g_x + g_c}{4}(K^+_R K^+_L + K^-_R K^-_L) + \frac{g_x - g_c}{4}(K^+_R K^-_L + K^-_R K^+_L), \tag{C26}$$

and therefore the interaction Hamiltonian (C13) can be written as

$$\mathcal{H}_{\mathrm{int},\sigma} = -\int_0^L dx\Big[g_c K_R^z K_L^z + \frac{g_x+g_c}{4}(K_R^+K_L^+ + K_R^-K_L^-) + \frac{g_x-g_c}{4}(K_R^+K_L^- + K_R^-K_L^+)\Big]. \tag{C27}$$

Equations (C14) and (C27) represent a nontrivial interacting problem, analysis of which requires renormalization group (RG) treatment. The couplings g_a obey the famous BKT RG flow,

$$\frac{dg_x}{d\ell} = -\frac{g_y g_z}{2\pi v_F}, \quad \frac{dg_y}{d\ell} = -\frac{g_x g_z}{2\pi v_F}, \quad \frac{dg_z}{d\ell} = -\frac{g_x g_y}{2\pi v_F}, \tag{C28}$$

where $\ell = \log(\alpha'/\alpha)$ describes increase of the short-distance cutoff from α to α'. As discussed in detail in Ref. [43], the solution to the RG equations (C28) depends on the initial values of the couplings involved,

$$g_x(0) = g\cos\chi, \quad g_z(0) = -g_y(0) = g_c = \frac{g}{2}(1-\cos\chi). \tag{C29}$$

Noting that $d(g_y^2 - g_z^2)/d\ell = 0$ and the fact that for $\ell = 0$ $g_z + g_y = 0$, we conclude that $g_z(\ell) = -g_y(\ell) = g_c(\ell)$ for all ℓ. Equations (C28) then reduce to the two coupled equations

$$\frac{dg_x}{d\ell} = \frac{g_c^2}{2\pi v_F}, \quad \frac{dg_c}{d\ell} = \frac{g_c g_x}{2\pi v_F}, \tag{C30}$$

which too is characterized by the integral of motion $Y = g_x^2(\ell) - g_c^2(\ell)$.

In the case of comparable spin-orbit and Zeeman energies ($\cos\chi \approx 1/3$), which is the focus of this paper, the combination $g_x + g_c$ towards positive infinity. This describes development of the correlated SDW state. This means that the combination $g_x - g_c = Y/(g_x+g_c)$ flows to zero in the same limit. As a result, (C27) simplifies to

$$\mathcal{H}_{\mathrm{int},\sigma} = -\int_0^L dx\Big[g_c K_R^z K_L^z + \frac{g_x+g_c}{4}(K_R^+K_L^+ + K_R^-K_L^-)\Big]. \tag{C31}$$

Equations (C14) and (C31) constitute the basis for the subsequent analysis.

3. Bosonization

Boundary conditions (C19) represent only a slight modification of the OBC considered in Appendix B 2. They are satisfied by the following representation of the fermion operators [37,38]

$$\psi_{Rs}(x) = \frac{ie^{i\pi/2}}{\sqrt{2\pi\alpha}}\eta_s e^{-i\tilde{\theta}_s} e^{i\frac{\pi x}{L}N_s} e^{i\sqrt{4\pi}\Phi_{Rs}(x)}, \tag{C32a}$$

$$\psi_{Ls}(x) = \frac{-ie^{i\pi/2}}{\sqrt{2\pi\alpha}}\eta_s e^{-i\tilde{\theta}_s} e^{-i\frac{\pi x}{L}N_s} e^{i\sqrt{4\pi}\Phi_{Rs}(-x)}, \tag{C32b}$$

$$\Phi_{Rs}(x) = \sum_{n=1}^{\infty}\frac{e^{-\alpha q_n/2}}{\sqrt{4\pi n}}(e^{iq_n x}b_{ns} + e^{-iq_n x}b_{ns}^\dagger), \tag{C32c}$$

where $q_n = \frac{\pi n}{L}$, b_{ns} is canonical boson with $[b_{ns}, b_{ms'}] = \delta_{n,m}\delta_{s,s'}$, $[\tilde{\theta}_s, N_s] = i$, η_s is the Majorana Klein factor satisfying $\{\eta_s, \eta_{s'}\} = 2\delta_{s,s'}$, and N_s is the (integer) number of particles relative to the equilibrium N_0 value. Note that following the constructive bosonization [40], $\eta_s e^{-i\tilde{\theta}_s} = F_s$ is the fermion number-changing operator, $[N_s, F_s] = -F_s$ and $F_s^\dagger F_s = 1$. Also notice that $\Phi_{Rs}(x)$ is $2L$ periodic.

As usual, we define commuting charge $\Phi_{R\rho}$ and spin $\Phi_{R\sigma}$ bosons

$$\Phi_{R\rho} = \frac{1}{\sqrt{2}}(\Phi_{R\uparrow} + \Phi_{R\downarrow}), \quad \Phi_{R\sigma} = \frac{1}{\sqrt{2}}(\Phi_{R\uparrow} - \Phi_{R\downarrow}). \tag{C33}$$

Observe that (C32) implies that in fact

$$\psi_{Ls}(x) = -e^{is\beta}\psi_{Rs}(-x) \tag{C34}$$

for *all* $x \in [0, L]$ and not only for the wire's endpoints $x_o = 0, L$ in (C19). This is a very general consequence of the chiral nature of one-dimensional fermions, see for example Fabrizio-Gogolin formulation [39] of the OBC. Using bosonization (C32) we obtain

$$K_R^+(x) = \frac{e^{i\beta}}{2\pi\alpha}e^{-i\frac{\pi x}{L}}\mathcal{F}^\dagger e^{-i\frac{2\pi x}{L}M}e^{-i\sqrt{8\pi}\Phi_{R\sigma}(x)},$$
$$K_R^z(x) = \frac{M}{2L} + \frac{1}{\sqrt{2\pi}}\partial_x\Phi_{R\sigma}(x), \quad K_L^z(x) = K_R^z(-x),$$
$$K_L^+(x) = e^{-i2\beta}K_R^+(-x), \tag{C35}$$

where $M = N_\sigma = (N_\uparrow - N_\downarrow)/2$ is the magnetization operator, $\Phi_{R\sigma} = (\Phi_{R\uparrow} - \Phi_{R\downarrow})/\sqrt{2}$ is the spin boson, and we used $e^{-i\frac{\pi x}{L}N_s}F_s^\dagger = F_s^\dagger e^{-i\frac{\pi x}{L}(N_s+1)}$.

4. Unfolding of the spin Hamiltonian

Next, relation (C35) allows us to write $[\vec{K}_L(x)]^2$ as $[\vec{K}_R(-x)]^2$, so that $\int_0^L dx[\vec{K}_R(-x)]^2 = \int_{-L}^0 dx[\vec{K}_R(x)]^2$ and (C14) can be *unfolded* onto $(-L, L)$ interval as

$$\mathcal{H}_\sigma^0 = \frac{2\pi v_F}{3}\int_{-L}^L dx\,[\vec{K}_R(x)]^2 = 2\pi v_F\int_{-L}^L dx\,\big[K_R^z(x)\big]^2. \tag{C36}$$

The interaction part (C31) can be written, with the help of (C35), as

$$\mathcal{H}_{\mathrm{int},\sigma} = -\int_{-L}^L\frac{dx}{2}\Big\{g_c K_R^z(x)K_R^z(-x) + \frac{g_x+g_c}{4}[e^{-i2\beta}K_R^+(x)K_R^+(-x) + \mathrm{H.c.}]\Big\}. \tag{C37}$$

Equations (C36) and (C37) constitute complete spin Hamiltonian of the open quantum wire, written in terms of the chiral (right) current $\vec{K}_R$. It is worth adding here that charge currents (8) and charge Hamiltonian $\mathcal{H}_\rho^0 + \mathcal{H}_{\mathrm{int},\rho}$ are not affected by the rotations.

5. Observables

Here we express spin operators in terms of boson field $\tilde{\Phi}$ (61). Uniform spin current is easy, using (C35) and setting

$K_\sigma = 2$,

$$K_R^+(x) = \frac{e^{i\beta}}{2\pi\alpha} e^{-i\frac{\pi x}{L}} \mathcal{F}^\dagger e^{-i\frac{2\pi x}{L}M} \times e^{-i\sqrt{4\pi}[\tilde\Phi(x)-\tilde\Phi(-x)]-i\sqrt{\pi}[\tilde\Phi(x)+\tilde\Phi(-x)]}. \quad \text{(C38)}$$

Observe that it does not contain charge fields. The original spin currents $\vec J_{R/L}$ and $\vec K_{R/L}$ are connected by (C4) and (C22).

The $2k_F$ component of the spin density (38) requires more work. First of all, by (C8) and (C15)

$$N^+(x) = \tfrac{1}{2}\Psi_R'^\dagger(x)(\sigma^x + i\sigma^y)\Psi_L'(x) = \tfrac{1}{2}\psi_R^\dagger(x)\mathcal{A}(-x)(\sigma^x + i\sigma^y)\mathcal{A}(-x)\psi_L(x). \quad \text{(C39)}$$

This gives

$$N^+(x) = \widetilde N^+(x) - i\sin(t_\varphi x)\widetilde N^0(x) + [\cos(t_\varphi x) - 1]\widetilde N^x(x), \quad \text{(C40)}$$

where $\widetilde N^+ = \psi_{R\uparrow}^\dagger \psi_{L\downarrow}$ reads

$$\widetilde N^+(x) = -\frac{1}{2\pi\alpha} e^{-i\frac{\pi x}{L}(N_\rho+1)} e^{-i\sqrt{2\pi}[\Phi_{R\rho}(x)-\Phi_{R\rho}(-x)]} \times e^{i\frac{\pi x}{L}M} e^{-i\sqrt{\pi}[\tilde\Phi(x)+\tilde\Phi(-x)]} e^{-i\frac{\pi x}{L}M}\mathcal{F}^\dagger. \quad \text{(C41)}$$

Here we defined $\widetilde N^a = \frac{1}{2}\psi_R^\dagger \sigma^a \psi_L$ ($a = 0, x, y, z$), and σ^0 denotes the identity matrix. The last three factors in the above equation combine into $f^\dagger(x)$ operator, see (67). We see that $\widetilde N^{x,y}$ depends on symmetric combination of the spin modes $\tilde\Phi(x) + \tilde\Phi(-x)$, similar to (C41), while $\widetilde N^{0,z}$ depends on the antisymmetric one $\tilde\Phi(x) - \tilde\Phi(-x)$. Also important is that (C41) and other components of $\widetilde N^a$ depend also on the critical charge mode via the antisymmetric charge combination $\Phi_{R\rho}(x) - \Phi_{R\rho}(-x)$.

Moreover, it is easy to see that the potential part of $\widetilde H_\sigma$ [the first line of (66b)] can be written as

$$\widetilde H_\sigma \propto \frac{g_x + g_c}{8}(\widetilde N^+(x)\widetilde N^+(-x) + \text{H.c.}). \quad \text{(C42)}$$

Therefore $\widetilde{\mathcal{H}}_\sigma$ is minimized when $\widetilde N^+(x)\widetilde N^+(-x) = -1$, which means that the spin part of $\widetilde N^+(x)$ is reduced $\pm i$. That is,

$$\widetilde N^+(x) \to \frac{\pm i}{2\pi\alpha} e^{-i\frac{\pi x}{L}(N_\rho+1)} e^{-i\sqrt{2\pi}[\Phi_{R\rho}(x)-\Phi_{R\rho}(-x)]}. \quad \text{(C43)}$$

Comparison with (42c) shows that similar to the ring geometry case, the open wire situation too is characterized by the finite expectation value of the *spin part* of $\widetilde N^y$ and, correspondingly, zero expectation value for the spin part of $\widetilde N^x$.

6. Charge sector Hamiltonian

With the help of (A15) the charge Hamiltonian is given by

$$\int_{x_0}^{L} dx \left\{ \frac{\pi v_F}{2}\left[J_R^2(x) + J_L^2(x)\right] + \frac{2U_0 - U_{2K_F}}{4}[J_R(x) + J_L(x)]^2 \right\}, \quad \text{(C44)}$$

where $J_R(x) = \frac{1}{2L}N_\rho + \sqrt{\frac{2}{\pi}}\partial_x \Phi_{R\rho}(x)$ and, in the open wire, $J_L(x) = J_R(-x)$. Therefore (C44) can be written as

$$H_\rho = \int_{-L}^{L} dx \left\{ \left(v_F + \frac{2U_0 - U_{2K_F}}{2\pi}\right)[\partial_x \Phi_{R\rho}(x)]^2 - \frac{2U_0 - U_{2K_F}}{2\pi}\partial_x \Phi_{R\rho}(x)\partial_x \Phi_{R\rho}(-x) \right\} + \frac{\pi}{4L}\left(v_F + \frac{2U_0 - U_{2K_F}}{\pi}\right)N_\rho^2 \quad \text{(C45)}$$

and can be diagonalized similarly to the spin Hamiltonian, see (62). We introduce $\Phi_\rho(x)$ via

$$\Phi_{R\rho}(x) = \Phi_\rho(x)\cosh\nu - \Phi_\rho(-x)\sinh\nu \quad \text{(C46)}$$

and find

$$H_\rho = \frac{\pi v_\rho}{4LK_\rho}N_\rho^2 + \int_{-L}^{L} dx\, v_\rho[\partial_x \Phi_\rho(x)]^2 \quad \text{(C47)}$$

provided that $e^{2\nu} = K_\rho$, as given by (33a), and $v_\rho = v_F/K_\rho$, see (33b).

Therefore

$$\Phi_{R\rho}(x) - \Phi_{R\rho}(-x) = \sqrt{K_\rho}[\Phi_\rho(x) - \Phi_\rho(-x)] \quad \text{(C48)}$$

and we can evaluate $Q_\rho = e^{-i\sqrt{2\pi}[\Phi_{R\rho}(x)-\Phi_{R\rho}(-x)]}$ from (105) by normal ordering it,

$$Q_\rho = \prod_{n=1}^{\infty} e^{g_n(x)a_{\rho,n}^\dagger} e^{-g_n(x)a_{\rho,n}} e^{-\frac{1}{2}g_n^2(x)}, \qquad g_n(x) = -\sqrt{\frac{2K_\rho}{n}} e^{-\frac{\pi\alpha n}{2L}} \sin\left(\frac{\pi x n}{L}\right). \quad \text{(C49)}$$

We used mode expansion [see (C32)]

$$\Phi_\rho(x) = \sum_{n=1}^{\infty} \frac{e^{-\alpha q_n/2}}{\sqrt{4\pi n}}(e^{iq_n x}a_{\rho,n} + e^{-iq_n x}a_{\rho,n}^\dagger) \quad \text{(C50)}$$

with $q_n = \pi n/L$. We are projecting (105) onto the state with no bosons, so that $a_{\rho,n}|\cdot\rangle = 0$. The presence of the fermion-number changing operator $\mathcal{F}$ in $\delta H'$ implies that the perturbation connects states with opposite magnetization parity, $\langle 0|\delta H'|1\rangle \neq 0$. Projecting Q_ρ (C49) onto the states $|0\rangle$ and $|1\rangle$, we find that exponentials of $a_{\rho,n}^\dagger$ and $a_{\rho,n}$ operators reduce to 1, and

$$Q_\rho = \prod_{n=1}^{\infty} e^{-\frac{1}{2}g_n^2(x)} = \left(\ln\frac{(1 - e^{-\frac{\pi\alpha}{L}})^2}{|1 - e^{-\frac{\pi\alpha}{L}}e^{i\frac{2\pi x}{L}}|}\right)^{\frac{K_\rho}{4}}. \quad \text{(C51)}$$

For $x = 0, L$ it reduces to 1, corresponding to the case (i) of spin-flip scattering near the open end of the wire. For $\alpha < x < L$ it gives $Q_\rho = \{\pi\alpha/[2L\sin(\pi x/L)]\}^{K_\rho/2}$, which is quoted in the main text, case (ii) in Sec. V D.

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