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AN ASYMPTOTICALLY TIGHT BOUND ON THE NUMBER OF RELEVANT VARIABLES IN A BOUNDED DEGREE BOOLEAN FUNCTION

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We prove that there is a constant $C \leq 6.614$ such that every Boolean function of degree at most d (as a polynomial over $\mathbb{R}$) is a $C\cdot 2^d$-junta, i.e., it depends on at most $C\cdot 2^d$ variables. This improves the $d\cdot 2^{d-1}$ upper bound of Nisan and Szegedy [Computational Complexity 4 (1994)].

The bound of $C\cdot 2^d$ is tight up to the constant C, since a read-once decision tree of depth d depends on all 2^d-1 variables. We slightly improve this lower bound by constructing, for each positive integer d, a function of degree d with $3\cdot 2^{d-1}-2$ relevant variables. A similar construction was independently observed by Shinkar and Tal.

1. Introduction

The *degree* of a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, denoted $\deg(f)$, is the degree of the unique multilinear polynomial in $\mathbb{R}[x_1,\dots,x_n]$ that agrees with f on all inputs from $\{0,1\}^n$. Minsky and Papert [4] initiated the study of combinatorial and computational properties of Boolean functions based on their representation by polynomials. We refer the reader to the excellent book of O'Donnell [6] on analysis of Boolean functions, and surveys [1,3] discussing relations between various complexity measures of Boolean functions.

An input variable x_i is *relevant to f* if x_i appears in a monomial having nonzero coefficient in the multilinear representation of f. Let $R(f)$ denote the number of relevant variables of f. Nisan and Szegedy [5] proved that $R(f) \leq \deg(f)\cdot 2^{\deg(f)-1}$.

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Let R_d denote the maximum of $R(f)$ over Boolean functions f of degree at most d, and let $C_d = R_d 2^{-d}$. By the result of Nisan and Szegedy, $C_d \le d/2$. On the other hand, $R_d \ge 2R_{d-1}+1$, since if f is a degree d–1 Boolean function with R_{d-1} relevant variables, and g is a copy of f on disjoint variables, and z is a new variable then $zf+(1-z)g$ is a degree d Boolean function with exactly $2R_{d-1}+1$ relevant variables. Thus $C_d \ge C_{d-1}+2^{-d}$, and so $C_d \ge 1-2^{-d}$. Since C_d is an increasing function of d it approaches a (possibly infinite) limit $C^* \ge 1$.

In this paper we prove:

Theorem 1.1. *There is a positive constant C so that $R(f)2^{-\deg(f)} \le C$ for all Boolean functions f, and thus $C_d \le C$ for all $d \ge 0$. In particular C^* is finite.*

Throughout this paper we use $[n] = \{1, \dots, n\}$ for the index set of the variables to a Boolean function f. A *maxonomial* of f is a set $S \subseteq [n]$ of size $\deg(f)$ for which $\prod_{i \in S} x_i$ has a nonzero coefficient in the multilinear representation of f. A *maxonomial hitting set* is a subset $H \subseteq [n]$ that intersects every maxonomial. Let $h(f)$ denote the minimum size of a maxonomial hitting set for f, and let h_d denote the maximum of $h(f)$ over Boolean functions of degree d. In Section 2 we prove:

Lemma 1.2. *For every $d \ge 1$, $C_d - C_{d-1} \le h_d 2^{-d}$.*

Through telescoping, this implies:

Corollary 1.3. *For every $d \ge 0$, $C_d \le \sum_{i=1}^{d} h_i 2^{-i}$.*

The next lemma is a simple combination of previous results.

Lemma 1.4. *For any Boolean function f, $h(f) \le \deg(f)^3$, and so for all $i \ge 1$, $h_i \le i^3$.*

Proof. Nisan and Smolensky (see Lemma 6 of [1]) proved $h_i \le \deg(f) bs(f)$, where $bs(f)$ is the block sensitivity of f. Combining with $bs(f) \le \deg(f)^2$ (proved by Tal [8], improving on $bs(f) \le 2\deg(f)^2$ of Nisan and Szegedy [5]) yields $h(f) \le \deg(f)^3$. ∎

Using Lemma 1.4, the infinite sum in Corollary 1.3 converges, and Theorem 1.1 follows.

Given that C^* is finite, it is interesting to obtain upper and lower bounds on C^*. The bounds that we will show in this paper are $3/2 \le C^* \le \frac{13545}{2048} \le 6.614$; we discuss these bounds in Section 3. (Recently, Wellens [9] refined our argument to obtain an improved upper bound of $C^* \le 4.416$.)

Filmus and Ihringer [2] recently considered an analog of the parameter $R(f)$ for the family of *level k slice functions*, which are Boolean functions whose domain is restricted to the set of inputs of Hamming weight exactly k. They showed that, provided that $\min(k, n-k)$ is sufficiently large, every level k slice function on n variables of degree at most d depends on at most R_d variables. As a result, our improved upper bound on R_d applies also to the number of relevant variables of slice functions.

Proof Overview

Similar to Nisan and Szegedy [5], we upper bound $R(f)$ by assigning a weight to each variable, and bounding the total weight of all variables. The weight assigned to a variable by Nisan and Szegedy was its *influence* on f; the novelty of our approach is to use a different weight function.

We assign to a variable x_i of a Boolean function f a weight $w_i(f)$ that is 0 if f does not depend on x_i and otherwise equals $2^{-\deg_i(f)}$ where $\deg_i(f)$ is the degree of the maximum degree monomial of f containing x_i. We then upper and lower bound the total weight $W(f)$ of a degree d Boolean function f. It follows from the definition that for a degree d Boolean function f, $W(f) \geq 2^{-d} \cdot R(f)$. Hence, to upper bound $R(f)$ it suffices to upper bound $W(f)$. Let W_d be the maximum of $W(f)$ among degree d Boolean functions f. We prove that

$$W_d \leq h_d 2^{-d} + W_{d-1}.$$

We show this by considering a minimum size maxonomial hitting set H for a $W(f)$ maximizing f. We argue that for such an f, all variables in H have maximum degree d, and hence their total weight adds up to $2^{-d} \cdot |H|$. Additionally, we show that the remaining variables have total weight at most W_{d-1}, by considering degree $d-1$ restrictions of f that are achieved by fixing variables in H. See proof of Proposition 2.3 for more details.

Combining above with Lemma 1.4 we have shown that $R(f) \leq 2^d \cdot \sum_{i=1}^{d} i^3 2^{-i}$, which readily implies $R(f) \leq 26 \cdot 2^d$. However, the same argument as above also implies

$$R(f) \leq 2^d \cdot (W_k + \sum_{i=k+1}^{d} i^3 2^{-i}).$$

Finally, plugging a bound of $W_k \leq k/2$ which follows from previous works and optimizing the right hand side, we obtain an improved bound of $R(f) \leq 6.614 \cdot 2^d$.

2. Proof of Lemma 1.2

For a variable x_i, let $\deg_i(f)$ be the maximum degree among all monomials that contain x_i and have nonzero coefficient in the multilinear representation of f. Let $w_i(f):=0$ if x_i is not relevant to f, and $w_i(f):=2^{-\deg_i(f)}$ otherwise. Note that if x_i is a relevant variable of the degree d function f, then $w_i(f)=2^{-\deg_i(f)}\geq 2^{-\deg(f)}=2^{-d}$.

The weight of f, $W(f)$, is defined to be $\sum_i w_i(f)$, and W_d denotes the maximum of $W(f)$ over all Boolean functions f of degree at most d; this maximum is well defined since, by the Nisan-Szegedy upper bound of R_d, it is taken over a finite set of functions. A function f of degree at most d for which $W_d=W(f)$ is *W_d-maximizing.*

Lemma 1.2 will follow as an immediate consequence of $W_d=C_d$ (Corollary 2.2) and $W_d\leq W_{d-1}+h_d 2^{-d}$ (Proposition 2.3).

Proposition 2.1. *If f is W_d-maximizing, then every relevant variable of f belongs to a degree d monomial.*

Proof. Let the relevant variables of f be $x_1,\dots,x_n$. Assume for contradiction that there are $l\geq 1$ variables that do not belong to any degree d monomial, and that these variables are $x_1,\dots,x_l$. We now construct a function g of degree at most d such that $W(g)>W(f)$, contradicting that f is W_d-maximizing. Let g be the $n+l+1$-variate function given by:

$$\begin{aligned} &g(x_1,\dots,x_{n+l+1})\\ &\quad := x_{n+l+1}f(x_1,\dots,x_n)+(1-x_{n+l+1})f(x_{n+1},\dots,x_{n+l},x_{l+1},\dots,x_n). \end{aligned}$$

This function is Boolean since it is equal to $f(x_{n+1},\dots,x_{n+l},x_{l+1},\dots,x_n)$ if $x_{n+l+1}=0$ and to $f(x_1,\dots,x_n)$ if $x_{n+l+1}=1$. It clearly has no monomials of degree larger than $d+1$. Since x_i appears in no degree d monomials of f for any $i\leq l$, $f(x_1,\dots,x_n)$ and $f(x_{n+1},\dots,x_{n+l},x_{l+1},\dots,x_n)$ have the same set of degree d monomials. Thus, the degree $d+1$ monomials of $x_{n+l+1}f(x_1,\dots,x_n)$ cancel out the degree $d+1$ monomials of $(1-x_{n+l+1})f(x_{n+1},\dots,x_{n+l},x_{l+1},\dots,x_n)$, and g has degree at most d. Furthermore, all of the degree d monomials involving $x_{l+1},\dots,x_n$ appear with the same coefficient in g as in f so $w_i(g)=w_i(f)=2^{-d}$ for all $i\in\{l+1,\dots,n\}$. Also, for each $i\in\{1,\dots,l\}$, any monomial $m=x_i m'$ containing x_i gives rise to monomials $x_{n+l+1}x_i m'$ and $-x_{n+1+i}x_{n+i}m'$ in g and so

$w_i(g) = w_{n+i}(g) = \frac{1}{2} w_i(f)$. Thus we have:

$$\begin{aligned} W(g) &= \sum_{i=1}^{n+l+1} w_i(g) = \sum_{i=1}^{l} (w_i(g) + w_{n+i}(g)) + \sum_{i=l+1}^{n} w_i(g) + w_{n+l+1}(g) \\ &= \sum_{i=1}^{l} w_i(f) + \sum_{i=l+1}^{n} w_i(f) + w_{n+l+1}(g) \\ &= W(f) + w_{n+l+1}(g) > W(f), \end{aligned}$$

where the final inequality holds since x_{n+l+1} is a relevant variable of g (which is true since for any monomial m of f containing x_1, mx_{n+l+1} is a monomial of g). Thus, g is a function of degree d with $W(g) > W(f)$, which gives us the desired contradiction to complete the proof. ∎

Corollary 2.2. *For all* $d \geq 1$, $W_d = C_d$.

Proof. For any function f of degree at most d, we have $W(f) \geq R(f)2^{-d}$. Thus $W_d \geq C_d$. If f is W_d-maximizing, then by Proposition 2.1, $W(f) = R(f)2^{-d} \leq C_d$.[1] ∎

Therefore, to prove Lemma 1.2 it suffices to prove:

Proposition 2.3. $W_d - h_d 2^{-d} \leq W_{d-1}$.

Proof. Again, let f be W_d-maximizing. Let H be a maxonomial hitting set for f of minimum size. Note that $\deg_i(f) = d$ for all $i \in H$, as otherwise $H - \{i\}$ would be a smaller maxonomial hitting set. Thus:

$$W(f) = \sum_i w_i(f) = 2^{-d}|H| + \sum_{i \notin H} w_i(f). \tag{1}$$

We will now show:

$$\sum_{i \notin H} w_i(f) \leq W_{d-1}, \tag{2}$$

which, combined with equation (1), yields the desired conclusion $W_d \leq 2^{-d} h_d + W_{d-1}$. We deduce equation (2) by bounding $w_i(f)$ by the average of $w_i(f')$ over a collection of *restrictions* f' of f. We recall some definitions. A *partial assignment* is a mapping $\alpha\colon [n] \longrightarrow \{0,1,*\}$, and $\mathrm{Fixed}(\alpha)$ is the set $\{i\colon \alpha(i) \in \{0,1\}\}$. For $J \subseteq [n]$, $\mathrm{PA}(J)$ is the set of partial assignments α with $\mathrm{Fixed}(\alpha) = J$. The *restriction* of f by α, f_α, is the function on variable set $\{x_i\colon i \in [n] - \mathrm{Fixed}(\alpha)\}$ obtained by setting $x_i = \alpha_i$ for each $i \in \mathrm{Fixed}(\alpha)$.

[1] In a previous version of this paper, our proof that $W_d \leq C_d$ was erroneous; this has been amended to its present form in this version. We thank Jake Lee Wellens for pointing out the error in the previous version.

Claim 2.4. *For every* $J \subseteq [n]$ *and* $i \notin J$,

$$w_i(f) \leq 2^{-|J|} \sum_{\alpha \in \mathrm{PA}(J)} w_i(f_\alpha).$$

Proof. Fix $j \in J$ and write $f = (1-x_j)f_0 + x_j f_1$ where f_0 is the restriction of f to $x_j = 0$ and f_1 is the restriction of f to $x_j = 1$.

We proceed by induction on $|J|$. We consider the base cases of $|J| \leq 1$. The $|J| = 0$ case is trivial. Let us now consider the $|J| = 1$ case where we have $J = \{j\}$.

- If f_0 does not depend on x_i, then $w_i(f) = w_i(f_1)/2 \leq (w_i(f_0) + w_i(f_1))/2$.
- If f_1 does not depend on x_i, then $w_i(f) = w_i(f_0)/2 \leq (w_i(f_0) + w_i(f_1))/2$.
- Suppose f_1 and f_0 both depend on x_i.
  - If $\deg_i(f_0) < \deg_i(f_1)$, let m be a monomial containing x_i of degree $\deg_i(f_1)$ that appears in f_1. Then $x_j m$ is a maxonomial of $f = x_j(f_0 - f_1) + f_0$. Therefore $\deg_i(f) = 1 + \deg_i(f_1)$. Thus $w_i(f) = \frac{1}{2} w_i(f_1) \leq \frac{1}{2}(w_i(f_0) + w_i(f_1))$.
  - If $\deg_i(f_0) \geq \deg_i(f_1)$, then $w_i(f_0) \leq w_i(f_1)$. It suffices that $w_i(f) \leq w_i(f_0)$, and this holds because each monomial that appears in f_0 appears with the same coefficient in $f = x_j(f_1 - f_0) + f_0$.

In every case, we have $w_i(f) \leq \frac{1}{2}(w_i(f_0) + w_i(f_1))$, as desired.

For the induction step, assume $|J| \geq 2$. We start with $w_i(f) \leq \frac{1}{2}(w_i(f_0) + w_i(f_1))$, and apply the induction hypothesis separately to f_0 and f_1 with the set of variables $J - \{j\}$:

$$\begin{aligned}
& w_i(f) \leq \frac{1}{2}(w_i(f_0) + w_i(f_1)) \\
& \leq \frac{1}{2}\left(2^{1-|J|} \left(\sum_{\beta \in \mathrm{PA}(J-\{j\})} w_i(f_{0,\beta}) \right) + 2^{1-|J|} \left(\sum_{\beta \in \mathrm{PA}(J-\{j\})} w_i(f_{1,\beta}) \right) \right) \\
& \leq 2^{-|J|} \sum_{\alpha \in \mathrm{PA}(J)} w_i(f_\alpha).
\end{aligned}$$

∎

To complete the proofs of equations (2) and Proposition 2.3 apply Claim 2.4 with J being a hitting set H of minimum size, and sum over $i \in [n] - H$ to get:

$$\begin{aligned}
\sum_{i \in [n]-H} w_i(f) &\leq 2^{-|H|} \sum_{i \in [n]-H} \sum_{\alpha \in \mathrm{PA}(H)} w_i(f_\alpha) \\
&= 2^{-|H|} \sum_{\alpha \in \mathrm{PA}(H)} W(f_\alpha) \leq W_{d-1},
\end{aligned}$$

where the last inequality follows since $\deg(f_\alpha) \leq d-1$ for all $\alpha \in \mathrm{PA}(H)$. ∎

As noted earlier Corollary 2.2 and Proposition 2.3 combine to prove Lemma 1.2.

3. Bounds on C^*

Lemma 1.2 implies $C_d \le \sum_{i=1}^{d} 2^{-i} h_i$. Combining with Lemma 1.4 yields $C_d \le \sum_{i=j}^{d} i^3 2^{-i}$, and thus $C^* \le \sum_{i=1}^{\infty} i^3 2^{-i}$, which equals 26 (since $\sum_{i\ge 0} \binom{i}{j} 2^{-i} = 2$ for all $j \ge 0$, and $i^3 = 6\binom{i}{3} + 6\binom{i}{2} + i$). As noted in the introduction, $R_d \ge 2^d - 1$, and so $C^* \ge 1$. We improve these bounds to:

Theorem 3.1. $\frac{3}{2} \le C^* \le \frac{13545}{2048}$.

Proof. For the upper bound, Lemma 1.2 implies that for any positive integer d,

$$C^* \le C_d + \sum_{i=d+1}^{\infty} 2^{-i} h_i.$$

Using $C_d \le d/2$ as proved by Nisan and Szegedy, we have

$$C^* \le \min_d \left(\frac{d}{2} + \sum_{i=d+1}^{\infty} i^3 2^{-i} \right).$$

The minimum occurs at the largest d for which $d^3 2^{-d} > 1/2$, which is 11. Evaluating the right hand side for $d = 11$ gives $C^* \le \frac{13545}{2048} \le 6.614$.

We lower bound C^* by exhibiting, for each d, a function Ξ_d of degree d with $l(d) = \frac{3}{2} 2^d - 2$ relevant variables. (A similar construction was found independently by Shinkar and Tal [7].) It is more convenient to switch our Boolean set to be $\{-1, 1\}$.

We define $\Xi_d \colon \{-1,1\}^{l(d)} \to \{-1,1\}$ as follows. $\Xi_1 \colon \{-1,1\} \to \{-1,1\}$ is the identity function, and for all $d > 1$, Ξ_d on $l(d) = 2l(d-1) + 2$ variables is defined recursively by:

$$\Xi_d(s,t,\vec{x},\vec{y}) = \frac{s+t}{2} \Xi_{d-1}(\vec{x}) + \frac{s-t}{2} \Xi_{d-1}(\vec{y})$$

for all $s, t \in \{-1,1\}$ and $\vec{x}, \vec{y} \in \{-1,1\}^{l(d-1)}$. It is evident from the definition that $\deg(\Xi_d) = 1 + \deg(\Xi_{d-1})$, which is d by induction (as for the base case $d = 1$, Ξ_1 is linear). It is easily checked that Ξ_d depends on all of its variables, and that $\Xi_d(s,t,\vec{x},\vec{y})$ equals $s \cdot \Xi_{d-1}(\vec{x})$ if $s = t$ and equals $s \cdot \Xi_{d-1}(\vec{(y)})$ if $s \ne t$, and is therefore Boolean. ∎

Jake Wellens [9] recently refined the arguments of this paper to improve the upper bound to $C^* \le 4.416$.

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