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Quantum Topology

How to categorify the ring of integers localized at two

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Abstract. We construct a triangulated monoidal Karoubi closed category with the Grothendieck ring naturally isomorphic to the ring of integers localized at two.

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1. Introduction

Natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rationals $\mathbb{Q}$, and real numbers $\mathbb{R}$ are basic structures that belong to the foundations of modern mathematics. The category $\mathbf{k}$-vect of finite-dimensional vector spaces over a field $\mathbf{k}$ can be viewed as a categorification of $\mathbb{N} = \{0, 1, 2, \dots\}$. The Grothendieck monoid of $\mathbf{k}$-vect is naturally isomorphic to the semiring $\mathbb{N}$, via the map that sends the image $[V]$ of a vector space V in the monoid to its dimension, an element of $\mathbb{N}$. Direct sum and tensor product of vector spaces lift addition and multiplication in $\mathbb{N}$. Additive monoidal category $\mathbf{k}$-vect, which linear algebra studies, is indispensable in modern mathematics and its applications, even more so when the field $\mathbf{k}$ is $\mathbb{R}$ or $\mathbb{C}$.

The ring of integers $\mathbb{Z}$ is categorified via the category $\mathcal{D}(\mathbf{k}\text{-vect})$ of complexes of vector spaces up to chain homotopies, with finite-dimensional total cohomology groups. The Grothendieck ring K_0 of the category $\mathcal{D}(\mathbf{k}\text{-vect})$ is isomorphic

to $\mathbb{Z}$ via the map that takes a symbol $[V] \in K_0$ of a complex V in $\mathcal{D}(\mathbf{k}\text{-vect})$ to its Euler characteristic $\chi(V)$. The notion of cohomology and the use of category $\mathcal{D}(\mathbf{k}\text{-vect})$ is ubiquitous in modern mathematics as well.

The category $\mathcal{D}(\mathbf{k}\text{-vect})$ is triangulated monoidal, with the Grothendieck ring $\mathbb{Z}$. Given the importance of the simplest categorifications of $\mathbb{N}$ and $\mathbb{Z}$, which are behind the subjects of linear algebra and cohomology, it is natural to ask if the two objects next in complexity on the original list, rings $\mathbb{Q}$ and $\mathbb{R}$, can be categorified. More precisely, are there monoidal triangulated categories with Grothendieck rings isomorphic to $\mathbb{Q}$ and $\mathbb{R}$? A natural additional requirement is for the categories to be idempotent complete (Karoubi closed).

This question was recently asked in [17, Problem 2.3] for $\mathbb{Q}$ and without the idempotent completeness requirement, together with a related and potentially simpler problem [17, Problem 2.4] to categorify the ring $\mathbb{Z}\left[\frac{1}{n}\right]$, the localization of $\mathbb{Z}$ given by inverting n.

In the present paper we present an idempotent complete categorification of the ring $\mathbb{Z}\left[\frac{1}{2}\right]$. In Section 3 we construct a DG (differential graded) monoidal category $\mathcal{C}$. The full subcategory $D^c(\mathcal{C})$ of compact objects of the derived category $D(\mathcal{C})$ is an idempotent complete triangulated monoidal category. We then establish a ring isomorphism

$$K_0(D^c(\mathcal{C})) \cong \mathbb{Z}\left[\tfrac{1}{2}\right]. \tag{1}$$

The category $\mathcal{C}$ is generated by the unit object $\mathbf{1}$ and an additional object X and has a diagrammatic flavor. Morphisms between tensor powers of X are described via suitable planar diagrams modulo local relations. Diagrammatic approach to monoidal categories is common in quantum algebra and categorification, and plays a significant role in the present paper as well.

Let us provide a simple motivation for the construction. To have a monoidal category with the Grothendieck ring $\mathbb{Z}\left[\frac{1}{2}\right]$ one would want an object X whose image $[X]$ in the Grothendieck ring is $\frac{1}{2}$. The simplest way to try to get $[X] = \frac{1}{2}$ is via a direct sum decomposition $\mathbf{1} \cong X \oplus X$ in the category, since the symbol $[\mathbf{1}]$ of the unit object is the unit element 1 of the Grothendieck ring.

The relation $\mathbf{1} \cong X \oplus X$ would imply that the endomorphism algebra of $\mathbf{1}$ is isomorphic to $\mathrm{M}_2(\mathrm{End}(X))$, the 2×2 matrix algebra with coefficients in the ring of endomorphisms of X. This contradicts the property that the endomorphism ring of the unit object $\mathbf{1}$ in an additive monoidal category is a commutative ring (or a super-commutative ring, for categories enriched over super-vector spaces), since matrix algebras are far from being commutative.

Our idea is to keep some form of the direct sum relation but rebalance to have one X on each side. Specifically, assume that the category is triangulated and objects can be shifted by $[n]$. If there is a direct sum decomposition

$$X \cong X[-1] \oplus \mathbf{1}, \tag{2}$$

then in the Grothendieck ring there is a relation $[X] = -[X] + 1$, equivalent to $2[X] = 1$, that is, X descends to the element $\frac{1}{2}$ in the Grothendieck ring of the category.

Equation (2) can be implemented by requiring mutually-inverse isomorphisms between its two sides. Each of these isomorphisms has two components: a morphism between X and its shift $X[-1]$, and a morphism between X and $\mathbf{1}$, for the total of four morphisms, see Figure 15. Two morphisms between X and $X[-1]$ of cohomological degrees zero give rise to two endomorphisms of X of cohomological degrees 1 and -1. It results in a DG (differential graded) algebra structure on the endomorphism ring of $1 \oplus X$, but with the trivial action of the differential. The four morphisms and relations on them are represented graphically in Figures 11 and 12 at the beginning of Section 3. These generators and relations give rise to a monoidal DG category $\mathcal{C}$ with a single generator X (in addition to the unit object $\mathbf{1}$). This category is studied in Section 3, where it is also explained how it gives rise to a triangulated monoidal idempotent complete category $D^c(\mathcal{C})$.

Our main result is Theorem 3.19 in Section 3.5 stating that the Grothendieck ring of $D^c(\mathcal{C})$ is isomorphic to $\mathbb{Z}\left[\frac{1}{2}\right]$, when the ground field $\mathbf{k}$ has characteristic two. To prove this result, we compute the Grothendieck group of the DG algebra A_k of endomorphisms of $X^{\otimes k}$ for all k. Our computation requires determining the first K-groups of certain DG algebras and ends up being quite tricky. Higher K-theory of DG algebras and categories appears to be a subtle and difficult subject, where even definitions need to be chosen carefully for basic computational goals. This is the perception of the authors of the present paper, who only dipped their toes into the subject.

In the proof we eventually need to specialize to working over a field $\mathbf{k}$ of characteristic two, but the category is defined over any field and the isomorphism (1) is likely to hold over any field as well.

Section 2 is devoted to preliminary work, used in later sections, to construct a pre-additive monoidal category generated by a single object X (and the unit object $\mathbf{1}$) subject to additional restrictions that the endomorphism ring of $\mathbf{1}$ is the ground field $\mathbf{k}$ and the composition map

$$\mathrm{Hom}(X, \mathbf{1}) \otimes_{\mathbf{k}} \mathrm{Hom}(\mathbf{1}, X) \longrightarrow \mathrm{Hom}(X, X)$$

is injective. Such a structure can be encoded by the ring A of endomorphisms of $\mathbf{1} \oplus X$ and the idempotent $e \in A$ corresponding to the projection of $\mathbf{1} \oplus X$ onto $\mathbf{1}$ subject to conditions that $eAe \cong \mathbf{k}$ and the multiplication $(1-e)Ae \otimes eA(1-e) \to (1-e)A(1-e)$ is injective. We show that this data does generate a monoidal category, give a diagrammatical presentation for this category, and provide a basis for the hom spaces $\mathrm{Hom}(X^{\otimes n}, X^{\otimes m})$. From the diagrammatic viewpoint, categories of these type are not particularly complicated, since the generating morphisms are given by labels on single strands and labels on strands ending or appearing inside the diagrams. No generating morphisms go between tensor products of generating objects, which may give rise to complicated networks built out of generating morphisms.

Our categorification of $\mathbb{Z}\left[\frac{1}{2}\right]$ in Section 3 and a conjectural categorification of $\mathbb{Z}\left[\frac{1}{n}\right]$ in Section 4 both rely on certain instances of (A, e) data and on the monoidal categories they generate. The construction of Section 3 requires us to work in the DG setting, with the associated triangulated categories and with the complexity of the structure mostly happening on the homological side.

In Section 4 we propose another approach to categorification of $\mathbb{Z}\left[\frac{1}{2}\right]$ and $\mathbb{Z}\left[\frac{1}{n}\right]$, based on an alternative way to stabilize the impossible isomorphism $\mathbf{1} \cong X \oplus X$. One would want an isomorphism

$$X \cong \mathbf{1} \oplus X \oplus X \oplus X, \tag{3}$$

which does not immediately contradict $\mathrm{End}(\mathbf{1})$ being commutative or super-commutative. We develop this approach in Section 4. Object X can be thought of as categorifying $-\frac{1}{2}$. There are no shift functors present in isomorphism (3) and it is possible to work here with the usual K_0 groups of algebras. Interestingly, we immediately encounter *Leavitt path algebras*, that have gained wide prominence in ring theory, operator algebras and related fields over the last decade, see [1] and references therein.

The Leavitt algebra $L(1, n)$ is a universal ring R with the property that $R \cong R^n$ as a left module over itself [20], that is, the rank one free R-module is isomorphic to the rank n free module. Such an isomorphism is encoded by the entries of an $n \times 1$ matrix $(x_1, \ldots, x_n)^T$, giving a module map $R \to R^n$, and the entries of the $1 \times n$ matrix $(y_1, \ldots, y_n)$, giving a map $R^n \to R$, with x_i, y_i's elements of R. These maps being mutually-inverse isomorphisms produces a system of equations on x_i's and y_i's, and the Leavitt algebra $L(1, n)$ is the quotient of the free algebra on the x_i's and y_i's by these relations. Leavitt algebras have exponential growth and are not noetherian. They satisfy many remarkable properties and have found various applications [1]. The relation to equation (3) is that, when ignoring the

unit object (by setting any morphism that factors through $\mathbf{1}$ to zero), one would need an isomorphism $X \cong X^3$, and the six endomorphisms of X giving rise to such an isomorphism satisfy the Leavitt algebra $L(1,3)$ defining relations.

Natural generalizations of the Leavitt algebras include Leavitt path algebras [1] and Cohn-Leavitt algebras [5, 1] which can be encoded via oriented graphs. One can think of these algebras as categorifying certain systems of homogeneous linear equations with non-negative integer coefficients, see [1, Proposition 9], [3].

For the category in Section 4, the algebra of endomorphisms of the direct sum $\mathbf{1} \oplus X$ is a particular Leavitt path algebra $L(Q)$ associated to the graph Q given by

$$\begin{array}{ccccc} & & 1 & & \\ & & \circlearrowright & & \\ 2 & \circlearrowright & a & \longrightarrow & b. \\ & & \circlearrowleft & & \\ & & 3 & & \end{array} \tag{4}$$

The Leavitt path algebra $L(Q)$ categorifies the linear equation $x = 3x + y$. Quotient of this algebra by the two-sided ideal generated by the idempotent of projecting $X \oplus Y$ onto Y is isomorphic to the Leavitt algebra $L(1,3)$.

The construction of Section 4 can be viewed as forming a *monoidal envelope* of the Leavitt path algebra $L(Q)$, where Y is set to be the unit object $\mathbf{1}$. Endomorphisms of $\mathbf{1} \oplus X$ are encoded by $L(Q)$, and these endomorphism spaces are then extended to describe morphisms between arbitrary tensor powers of X. Passing from certain Leavitt path algebras to monoidal categories can perhaps be viewed as categorifications of quotients of free algebras by certain systems of inhomogeneous linear equations with non-negative integer coefficients imposed on generators of free algebras.

We come short of proving that the Grothendieck ring of the associated idempotent completion is indeed $\mathbb{Z}\left[\frac{1}{2}\right]$. The obstacle is in not knowing K-groups $K_i(L(1,3)^{\otimes k})$ of tensor powers of $L(1,3)$ for $i = 0, 1$, see Conjecture 4.2. This problem is discussed in [3], but the answer is not known for general k.

Equation (3) admits a natural generalization to

$$X \cong \mathbf{1} \oplus X^{n+1}, \tag{5}$$

where the right hand side contains $n + 1$ summands X. Now X plays the role of categorified $-\frac{1}{n}$. In Section 4.3 we construct an additive monoidal Karoubi closed category in which the isomorphism above holds and conjecture that its Grothendieck ring is isomorphic to $\mathbb{Z}\left[\frac{1}{n}\right]$, see Conjecture 4.3.

The DG ring A of endomorphisms of $\mathbf{1}\oplus X$ that appears in our categorification of $\mathbb{Z}\left[\frac{1}{2}\right]$ in Section 3 is also a Leavitt path algebra $L(T)$, of the Toeplitz graph T given by

$$\circlearrowright X \longrightarrow Y, \tag{6}$$

see [1, Example 7]. Considering A as a Leavitt path algebra, one should ignore the grading of A and its structure as a DG algebra. Leavitt path algebra of the Toeplitz graph T is isomorphic to the Jacobson algebra [12], with generators a, b and defining relation $ba = 1$, making a and b one-sided inverses of each other. The linear equation categorified by this algebra is $x = x + y$, lifted to an isomorphism of projective modules $X \cong X \oplus Y$. In the Grothendieck group of this Leavitt path algebra $[Y] = 0$. In the monoidal envelope where Y is the unit object $\mathbf{1}$, an isomorphism $X \cong X \oplus \mathbf{1}$ would imply the Grothendieck ring is zero, since $1 = [\mathbf{1}] = 0$. These problems are avoided by introducing a shift into the isomorphism as in equation (2), and consequently working in the DG framework, as explained earlier. Choice of $[-1]$ over $[1]$ is inessential, see Remark 3.20.

Having a monoidal structure or some close substitute is a natural requirement for a categorification of $\mathbb{Z}\left[\frac{1}{n}\right]$ and $\mathbb{Q}$, emphasized in [17]. The direct limit $D = M_{n^\infty}(\mathbf{k})$ of matrix algebras $M_{n^k}(\mathbf{k})$ under unital inclusions $M_{n^k}(\mathbf{k}) \subset M_{n^{k+1}}(\mathbf{k})$ has the Grothendieck group K_0 of finitely-generated projective modules naturally isomorphic to the abelian group $\mathbb{Z}\left[\frac{1}{n}\right]$, see [28, Exercise 1.2.7]. The isomorphism is that of groups, not rings. Similar limits give algebras with K_0 isomorphic to any subgroup of $\mathbb{Q}$.

Phillips [26] shows that the algebra D is algebraically strongly selfabsorbing, that is, there is an isomorphism $D \cong D \otimes_{\mathbf{k}} D$ which is algebraically approximately similar to the inclusion $D \cong D \otimes 1 \subset D \otimes_{\mathbf{k}} D$. This isomorphism allows to equip $K_0(D)$ with a ring structure, making $K_0(D)$ isomorphic to $\mathbb{Z}\left[\frac{1}{n}\right]$ as a ring, and likewise for the other subrings of $\mathbb{Q}$, see [26]. We are not aware of any monoidal structure or its close substitute on the category of finitely-generated projective D-modules that would induce the Phillips ring structure on $K_0(D)$.

Barwick et al. [8] construct triangulated categories (and stable ∞-categories) with Grothendieck groups isomorphic to localizations $S^{-1}\mathbb{Z}$ of $\mathbb{Z}$ along any set S of primes, as well as more general localizations. For these localizations a monoidal or some tensor product structure on the underlying categories does not seem to be present, either, to turn Grothendieck groups into rings.

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2. A family of monoidal categories via arc diagrams

A monoidal category from (A, e). For a field $\mathbf{k}$, let A be a unital $\mathbf{k}$-algebra and $e \in A$ an idempotent such that $eAe \cong \mathbf{k}$ and the multiplication map

$$(1-e)Ae \otimes eA(1-e) \xrightarrow{m'} (1-e)A(1-e),$$

denoted m', is injective. Another notation for $m'(b \otimes c)$ is simply bc. Let

$$A' = \mathrm{im}(m') \subset (1-e)A(1-e)$$

and choose a $\mathbf{k}$-vector subspace A'' of A such that $(1-e)A(1-e) = A' \oplus A''$.

We fix bases $\mathbb{B}_{1,0}$, $\mathbb{B}_{0,1}$, and $\mathbb{B}_{1,1}(1)$ of vector spaces $(1-e)Ae$, $eA(1-e)$, and A'', respectively. The subscript 0 corresponds to the idempotent e, and the subscript 1 corresponds to the complementary idempotent $1-e$. The notation $\mathbb{B}_{1,1}(1)$ will be explained later in (11). We also choose $\mathbb{B}_{0,0}$ to be a one-element set consisting of any nonzero element of $\mathbf{k} \cong eAe$ (element 1 is a natural choice). The set $\mathbb{B}_{1,0} \times \mathbb{B}_{0,1}$ of elements bc, over all $b \in \mathbb{B}_{1,0}$ and $c \in \mathbb{B}_{0,1}$, is naturally a basis of A'. The union $\mathbb{B}_{1,1}(1) \sqcup (\mathbb{B}_{1,0} \times \mathbb{B}_{0,1})$ gives a basis of $(1-e)A(1-e)$. Choices of A'' and $\mathbb{B}_{1,0}$, $\mathbb{B}_{0,1}$, $\mathbb{B}_{1,1}(1)$ are not needed in the definition of category $\mathcal{C}$ below.

To a pair (A, e) as above we will assign a $\mathbf{k}$-linear pre-additive strict monoidal category $\mathcal{C} = \mathcal{C}(A, e)$. Objects of $\mathcal{C}$ are tensor powers $X^{\otimes n}$ of the generating object X. The unit object $\mathbf{1} = X^{\otimes 0}$. The algebra A describes the ring of endomorphisms of the object $\mathbf{1} \oplus X$. Slightly informally, we write A in the matrix notation

$$A = \begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}. \tag{7}$$

meaning, in particular, that, as a $\mathbf{k}$-vector space, A is the direct sum of the four matrix entries, and the multiplication $A \otimes A \xrightarrow{m'} A$ in A reduces to matrix-like tensor product maps between the entries. The two diagonal entries are subalgebras, via nonunital inclusions. We declare (7) to be the matrix of homs

between the summands of the object $\mathbf{1} \oplus X$ in $\mathcal{C}$:

$$\begin{pmatrix} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}) & \mathrm{Hom}_{\mathcal{C}}(\mathbf{1},X) \\ \mathrm{Hom}_{\mathcal{C}}(X,\mathbf{1}) & \mathrm{Hom}_{\mathcal{C}}(X,X) \end{pmatrix} = \begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}.$$

The algebra $eAe = \mathbf{k}$ is the endomorphism ring of the unit object $\mathbf{1}$. The $\mathbf{k}$-vector space $(1-e)Ae$ is $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X)$, while $\mathrm{Hom}_{\mathcal{C}}(X,\mathbf{1}) = eA(1-e)$, and the ring $\mathrm{End}_{\mathcal{C}}(X) = (1-e)A(1-e)$.

The algebra A can be used to generate a vector space of morphisms between tensor powers of X, by tensoring and composing morphisms between the objects $\mathbf{1}$ and X, and imposing only the relations that come from the axioms of a strict monoidal pre-additive category. The only nontrivial part, as explained below, is to check that the category does not degenerate, that is, the hom spaces in the resulting category have the expected sizes (bases).

Before considering diagrammatics for morphisms between tensor powers of X, we start with diagrams that describe homs between $\mathbf{1}$ and X. We draw the diagrams inside a strip $\mathbb{R} \times [0,1]$. An endomorphism a of X (a is an element of $(1-e)A(1-e)$) is depicted by a vertical line that starts and ends on the boundary of the strip and in the middle carries a dot labeled a, see Figure 1. We call such a line a *long strand* labeled by a. An element $b \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X) = (1-e)Ae$ is depicted by a *short top strand* labeled by b. The top endpoint of a short top strand is at the boundary. An element $c \in \mathrm{Hom}_{\mathcal{C}}(X,\mathbf{1}) = eA(1-e)$ is depicted by a *short bottom strand* labeled by c. Its bottom endpoint is at the boundary of the strip. Each short strand has two endpoints: the boundary endpoint (either at the top or bottom of the strand), and the *floating* endpoint, which is a labeled dot. An endomorphism h of the identity object $\mathbf{1}$ is depicted by a dot, labeled by h, in the middle of the plane (in our case, these endomorphisms are elements of the ground field $\mathbf{k}$). These four types of diagrams are depicted in Figure 1.

Vertical concatenation of diagrams corresponds to the composition of morphisms, as depicted in Figure 2. For instance, if an element of $(1-e)A(1-e)$ factors as bc, for $b \in (1-e)Ae$ and $c \in eA(1-e)$, we can depict it as a composition of a top strand with label b and a bottom strand with label c, see the lower right equality above.

Addition of alike diagrams is given by adding their labels, see examples in Figure 3 for adding elements of $(1-e)A(1-e)$ and $(1-e)Ae$.

Likewise, scaling a diagram by an element of $\mathbf{k}$ corresponds to multiplying its label by that element. An element $a \in (1-e)A(1-e)$ decomposes uniquely $a = a' + a''$, where $a' \in A' = \mathrm{im}(m')$, $a'' \in A''$. Furthermore, a' admits a (non-unique) presentation $a' = \sum_{i=1}^{k} b_i c_i$, $b_i \in (1-e)Ae$, $c_i \in eA(1-e)$, see Figure 4 for diagrammatic expression.

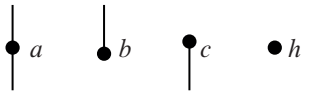

Figure 1. Presentation of morphisms, boundary of $\mathbb{R} \times [0, 1]$ not shown.

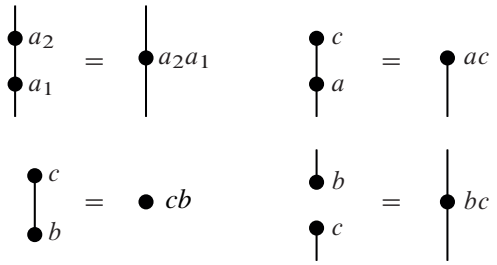

Figure 2. Graphical presentation of composition in A.

Figure 3. Adding diagrams.

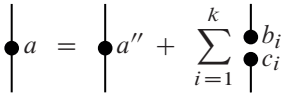

Figure 4. Decomposition of an element in $(1 - e)A(1 - e)$.

The algebra A has a basis given by diagrams in Figure 1 over all $a \in \mathbb{B}_{1,1}(1) \sqcup (\mathbb{B}_{1,0} \times \mathbb{B}_{0,1})$, $b \in \mathbb{B}_{1,0}$, $c \in \mathbb{B}_{0,1}$, and $h \in \mathbb{B}_{0,0}$ (recall that $\mathbb{B}_{0,0}$ has cardinality one). Vertical line without a label denotes the idempotent $1 - e$. This idempotent does not have to lie in A'', but we usually choose A'' to contain $1 - e$ and a basis $\mathbb{B}_{1,1}(1)$ of A'' to contain $1 - e$ as well (also see Example 2 below).

These diagrammatics for A extend to diagrammatics for a monoidal category with the generating object X and algebra A describing the endomorphisms of $\mathbf{1} \oplus X$. Morphisms from $X^{\otimes n}$ to $X^{\otimes m}$ are $\mathbf{k}$-linear combinations of diagrams with n bottom and m top endpoints which are concatenations of labeled long and short strands, as in the figure below (where $n = 5$ and $m = 4$).

Defining relations are isotopies of these labeled diagrams rel boundary and the relations coming from the algebra A as shown in Figures 2–4. Any floating strand (of the form cb as in Figure 2) reduces to a constant and can be removed by rescaling the coefficient of the diagram.

A basis for homs. We now describe what is obviously a spanning set of $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$. Ignoring labels, an isotopy class of a diagram of long and short strands as in Figure 5 corresponds to a partial order-preserving bijection

$$f\colon [1,n] \longrightarrow [1,m]. \tag{8}$$

Here $[1,n] = \{1,\dots,n\}$, viewed as an ordered set with the standard order. A partial bijection $f\colon X \longrightarrow Y$ is a bijection from a subset L_f of X to a subset of Y, and order-preserving means X, Y are ordered sets, and $f(i) < f(j)$ if $i < j$ and $i, j \in L_f$. Let $L_f = \{i_1,\dots,i_{|f|}\} \subset [1,n]$, where $i_1 < i_2 < \dots i_{|f|}$. Here $|f|$ denotes the cardinality of L_f, which we also call the width of f. Denote by $\mathrm{PB}_{m,n}$ the set of all partial order-preserving bijections (8).

A Long strand, counting from left to right, connects the k-th element $i_k \in [1,n]$ of L_f, viewed as a point on the bottom edge of the strip, to $f(i_k) \in [1,m]$, viewed as a point on the top edge. Elements in $[1,n] \setminus L_f$ are the lower endpoints of the short bottom strands. Elements of $[1,m] \setminus f(L_f)$ are the upper endpoints of the short top strands.

In Figure 5 partial bijection $f\colon [1,5] \to [1,4]$ has

$$L_f = \{1,2,5\}$$

and

$$f(1) = 2, \quad f(2) = 3, \quad f(5) = 4.$$

For each bijection f choose a diagram in the isotopy class of diagrams representing this bijection and add a dot to each long strand, see Figure 6. Each short strand already has a dot at its floating endpoint. Denote this diagram D_f.

Partial bijection f has $|f|$ long strands, $n - |f|$ bottom strands and $m - |f|$ top strands. Let $\mathbb{B}_f$ be the following set of elements of $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$. To the floating endpoint of each bottom strand assign an element of $\mathbb{B}_{0,1}$ and denote these elements $c_1,\dots,c_{n-|f|}$ from left to right. To the floating endpoint of each top strand assign an element of $\mathbb{B}_{1,0}$ and denote these elements $b_1,\dots,b_{m-|f|}$ from left to right. To the dot at each long strand assign an element of $\mathbb{B}_{1,1}(1)$ (recall that $\mathbb{B}_{1,1}(1)$ is a basis of A'') and denote them $a_1,\dots,a_{|f|}$. Figure 7 depicts an example with $n = 6, m = 7$ and $|f| = 3$.

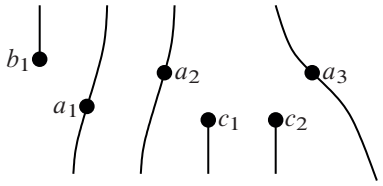

Figure 5. A diagram of long and short strands.

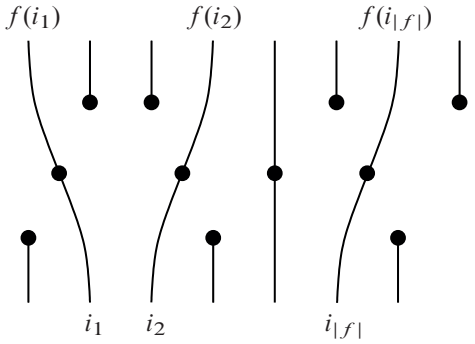

Figure 6. Partial bijection diagram D_f.

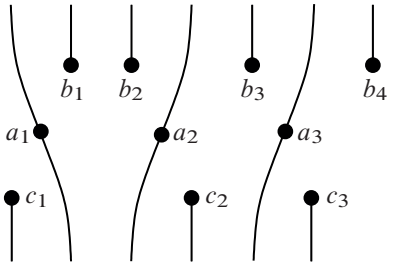

Figure 7. A labeled partial bijection diagram d.

The set $\mathbb{B}_f$ of labeled diagrams for a given f is naturally parametrized by the set

$$(\mathbb{B}_{1,0})^{m-|f|} \times (\mathbb{B}_{1,1}(1))^{|f|} \times (\mathbb{B}_{0,1})^{n-|f|},$$

and each labeled diagram d in $\mathbb{B}_f$ gives rise to an element of $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$, also denoted d.

Theorem 2.1. *The* $\mathbf{k}$*-vector space* $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$ *has a basis of labeled diagrams*

$$\mathbb{B}_{m,n} = \bigsqcup_{f \in \mathrm{PB}_{m,n}} \mathbb{B}_f$$

and is naturally isomorphic to the space

$$\bigoplus_{f \in \mathrm{PB}_{m,n}} ((1-e)Ae)^{\otimes(m-|f|)} \otimes (A'')^{\otimes |f|} \otimes (eA(1-e))^{\otimes(n-|f|)}.$$

The notation $\mathbb{B}_{m,n}$ is compatible for $(m,n) = (1,0), (0,1), (0,0)$ with the notation at the beginning of Section 2.

Proof. We first observe that vector spaces $\mathbf{k}\mathbb{B}_{m,n}$ admit multiplications

$$\mathbf{k}\mathbb{B}_{k,m} \otimes \mathbf{k}\mathbb{B}_{m,n} \longrightarrow \mathbf{k}\mathbb{B}_{k,n} \tag{9}$$

that turn the direct sum

$$\mathbf{k}\mathbb{B} := \bigoplus_{n,m \in \mathbb{N}} \mathbf{k}\mathbb{B}_{m,n} \tag{10}$$

into an idempotented (nonunital) associative algebra, where $\mathbb{N} = \{0, 1, 2, \dots\}$. To compute the product $h_2 h_1 \in \mathbf{k}\mathbb{B}_{k,n}$ for $h_2 \in \mathbb{B}_{k,m}$, $h_1 \in \mathbb{B}_{m,n}$ we concatenate the diagrams $h_2 h_1$ into a single diagram. Every floating strand in $h_2 h_1$ evaluates to a scalar in $\mathbf{k}$. Long strands in $h_2 h_1$ are concatenations of pairs of long strands in h_2, h_1, each carrying a label, say $a_2, a_1 \in \mathbb{B}_{1,1}(1)$. The concatenation carries the label $a_2 a_1 \in (1-e)A(1-e)$ and is simplified as in Figure 4, with $a = a_2 a_1$ on the left hand side. The right hand side term a'' in Figure 4 further decomposes into a linear combination of elements of $\mathbb{B}_{1,1}(1)$, and the terms in the sum into linear combinations of elements of $\mathbb{B}_{1,0} \times \mathbb{B}_{0,1}$.

Concatenation of a long strand and a short (top or bottom) strand results in a short (top or bottom) strand that carries the product label, see the right half of Figure 2. That label is a linear combination of elements in $\mathbb{B}_{1,0}$, in the top strand case, and elements of $\mathbb{B}^{0,1}$, in the bottom strand case.

The simplification procedure is consistent and results in a well-defined element $h_2 h_1$ of $\mathbf{k}\mathbb{B}_{k,n}$. Associativity of multiplications (9), resulting in well-defined maps

$$\mathbf{k}\mathbb{B}_{r,k} \otimes \mathbf{k}\mathbb{B}_{k,m} \otimes \mathbf{k}\mathbb{B}_{m,n} \longrightarrow \mathbf{k}\mathbb{B}_{k,n}$$

for all r, k, m, n, follows from the observation that the computation of $h_2 h_1$ can be localized along each concatenation point. Simplification of each pair of strands along their concatenation point can be done independently, and the resulting elements of A, interpreted as diagrams, can then be tensored (horizontally composed)

to yield h_2h_1. In this way associativity of (9) follows from associativity of multiplication in A. Since the multiplication is consistent, $\mathbf{k}\mathbb{B}$ carries an associative non-unital algebra structure.

A substitute for the unit element is a system of idempotents in $\mathbf{k}\mathbb{B}$. Denote by 1_n the element of $\mathbf{k}\mathbb{B}_{n,n}$ given by n parallel vertical lines without labels. In particular, $1_0 = e$. The diagram 1_n is an idempotent corresponding to the identity endomorphism of $X^{\otimes n}$, and for any $a \in \mathbb{B}_{n,m}, b \in \mathbb{B}_{m,n}$ the products $1_n a = a$, $b1_n = b$. The diagram 1_n is the horizontal concatenation of n copies of $1-e \in A$. In all cases considered in this paper $1 - e \in \mathbb{B}_{1,1}(1)$, but this is not necessary in general: $1 - e$ might not be in the basis $\mathbb{B}_{1,1}(1)$ of A'', or even in A''.

Elements 1_n, over all $n \geq 0$, constitute a local system of mutually-orthogonal idempotents in $\mathbf{k}\mathbb{B}$. For any finitely many elements $z_1, \ldots, z_m$ of $\mathbf{k}\mathbb{B}$ there exists n such that $z_i 1'_n = 1'_n z_i = z_i$ for $1 \leq i \leq m$, where $1'_n = 1_0 + 1_1 + \cdots + 1_n$. Non-unital algebra $\mathbf{k}\mathbb{B}$ can be written as a direct limit of unital algebras $1'_n \mathbf{k}\mathbb{B} 1'_n$ under non-unital inclusions

$$1'_n \mathbf{k}\mathbb{B} 1'_n \subset 1'_{n+1} \mathbf{k}\mathbb{B} 1'_{n+1}.$$

We also refer to an algebra with a local system of idempotents as an *idempotented algebra*.

Multiplication in $\mathbf{k}\mathbb{B}$ corresponds to vertical concatention of labeled diagrams, and is compatible with the horizontal concatenation (tensor product) of diagrams, giving us maps

$$\mathbf{k}\mathbb{B}_{m,n} \otimes \mathbf{k}\mathbb{B}_{m',n'} \longrightarrow \mathbf{k}\mathbb{B}_{m+m',n+n'}$$

and producing a monoidal category, denoted $\underline{\mathcal{C}}$, with a single generating object $\underline{X}$ and $\mathbf{k}\mathbb{B}_{m,n}$ the space of homs from $\underline{X}^{\otimes n}$ to $\underline{X}^{\otimes m}$. Injectivity of multiplication in $\underline{\mathcal{C}}$ implies that $\underline{\mathcal{C}}$ is equivalent (and even isomorphic) to $\mathcal{C}$, and that sets $\mathbb{B}_{m,n}$ are indeed bases of homs in $\mathcal{C}$.

The subalgebra

$$\mathbf{k}\mathbb{B}_{\leq 1} = \mathbf{k}\mathbb{B}_{0,0} \oplus \mathbf{k}\mathbb{B}_{0,1} \oplus \mathbf{k}\mathbb{B}_{1,0} \oplus \mathbf{k}\mathbb{B}_{1,1}$$

of $\mathbf{k}\mathbb{B}$ is naturally isomorphic to A. □

Thus, the space $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$ is a direct sum over all order-preserving partial bijections $f : [1, n] \longrightarrow [1, m]$ of vector spaces

$$((1-e)Ae)^{\otimes(m-|f|)} \otimes (A'')^{\otimes|f|} \otimes (eA(1-e))^{\otimes(n-|f|)}.$$

Denote by $\mathbb{B}_{m,n}(\ell)$ the subset of $\mathbb{B}_{m,n}$ corresponding to partial bijections f with $|f| = \ell$:

$$\mathbb{B}_{m,n}(\ell) = \bigsqcup_{|f|=\ell} \mathbb{B}_f. \tag{11}$$

These are all basis diagrams with ℓ long strands. The notation $\mathbb{B}_{m,n}(\ell)$ is compatible with the notation $\mathbb{B}_{1,1}(1)$ at the beginning of Section 2. We have $\mathbb{B}_{1,1} = \mathbb{B}_{1,1}(0) \sqcup \mathbb{B}_{1,1}(1)$, and there is a natural bijection $\mathbb{B}_{1,1}(0) \cong \mathbb{B}_{1,0} \times \mathbb{B}_{0,1}$.

Additive, idempotent complete extension. From $\mathcal{C}$ we can form its additive closure $\mathcal{C}^{\mathrm{add}}$, with objects—finite directs sums of objects of $\mathcal{C}$. Category $\mathcal{C}^{\mathrm{add}}$ is a $\mathbf{k}$-linear additive strict monoidal category. Furthermore, let $\mathrm{Ka}(\mathcal{C})$ be the Karoubi closure of $\mathcal{C}^{\mathrm{add}}$. Category $\mathrm{Ka}(\mathcal{C})$ is an idempotent complete $\mathbf{k}$-linear additive strict monoidal category. Its Grothendieck group $K_0(\mathrm{Ka}(\mathcal{C}))$ is naturally a unital associative ring under the tensor product operation, and there is a natural homomorphism

$$\mathbb{Z}[x] \longrightarrow K_0(\mathrm{Ka}(\mathcal{C}))$$

from the ring of polynomials in a variable x to its Grothendieck group, taking x to $[X]$, the symbol of the generating object X in the Grothendieck group. The homomorphism may not be injective or surjective.

By an inclusion $\mathcal{A} \subset \mathcal{B}$ of categories we mean a fully faithful functor $\mathcal{A} \to \mathcal{B}$. There is a sequence of categories and inclusion functors

$$\mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{add}} \longrightarrow \mathrm{Ka}(\mathcal{C}).$$

Category $\mathcal{C}$ is preadditive. Category $\mathcal{C}^{\mathrm{add}}$ is additive and contains $\mathcal{C}$ as a full subcategory. Category $\mathrm{Ka}(\mathcal{C})$ is additive, idempotent complete, and contains $\mathcal{C}^{\mathrm{add}}$ as a full subcategory. All three categories are monoidal.

Examples. We now provide some examples for the above construction.

Example 1. A special case of the monoidal category $\mathcal{C}$ appeared in [18], with the sets $\mathbb{B}_{1,0}$, $\mathbb{B}_{0,1}$, and $\mathbb{B}_{1,1}(1)$ all of cardinality one. Denoting elements of these sets by b, c, and (1), respectively, the algebra A can be identified with the Figure 8 quiver algebra subject to the relation $cb = (0)$, where (j), for $j \in \{0, 1\}$, denotes the idempotent path of length zero at vertex j. Thus, the composition cb equals the idempotent path (0) at the vertex 0. Algebra A has a basis $\{(0), (1), b, c, bc\}$. It's a semisimple algebra isomorphic to the direct product $\mathrm{M}_2(\mathbf{k}) \times \mathbf{k}$, where the second factor is spanned by the idempotent $(1) - bc$. The first factor has a basis $\{(0), b, c, bc\}$.

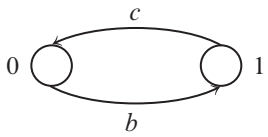

Figure 8. Quiver with vertices 0, 1 and arrows b, c between them.

A basis of the hom space $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$ is given by partial bijections from $[1,n]$ to $[1,m]$, with no additional decorations necessary. The idempotent completion $\mathrm{Ka}(\mathcal{C})$ of the additive closure $\mathcal{C}^{\mathrm{add}}$ of $\mathcal{C}$ is semisimple, and $X \cong \mathbf{1} \oplus X_1$, where the object $X_1 = (X, 1_X - bc)$ of the idempotent completion comes from the above idempotent $1_X - bc$. The simple objects, up to isomorphism, are all tensor powers of X_1. For this pair (A, e) the natural homomorphism $\mathbb{Z}[x] \to K_0(\mathrm{Ka}(\mathcal{C}))$ is an isomorphism, see [18].

Example 2. Consider a special case when $1 - e \in A' = \mathrm{im}(m')$. Then $A' = (1-e)A(1-e)$ and $A'' = 0$. Choose a presentation $1 - e = \sum_{i=1}^{n} b_i c_i$ with $b_i \in (1-e)Ae$, $c_i \in eA(1-e)$, and the smallest n. Multiplying this equality by b_j on the right implies $c_i b_j = \delta_{i,j} \in \mathbf{k}$. Multiplying on the right by any element of $(1-e)Ae$ shows that b_i's are a basis of $(1-e)Ae$. Likewise, elements c_i are a basis of $eA(1-e)$, and the pair (A, e) is isomorphic to the pair $(\mathrm{M}_{n+1}(\mathbf{k}), e_{11})$ of a matrix algebra of size $n+1$ and a minimal idempotent in it. In the additive closure $\mathcal{C}^{\mathrm{add}}$ of $\mathcal{C}$ (and in the idempotent completion $\mathrm{Ka}(\mathcal{C})$), the object X is isomorphic to n copies of the unit object $\mathbf{1}$. This degenerate case is of no interest to us.

Otherwise, $1-e$ is not in the subspace A' of $(1-e)A(1-e)$, and we can always choose A'' and $\mathbb{B}_{1,1}(1)$ to contain $1-e$, ensuring that the vertical line diagram is in the basis $\mathbb{B}_{1,1}(1)$.

Case when A is a super algebra. We now discuss a generalization when A is an algebra in the category of super-vector spaces. In that category the objects are $\mathbb{Z}/2$-graded and degree one summands are called odd components. Algebra A must be $\mathbb{Z}/2$-graded, $A = A_0 \oplus A_1$, with the idempotent $e \in A_0$ such that $eAe = \mathbf{k}$. Then $1-e$ is also in A_0. Vector spaces $eA(1-e)$, $(1-e)Ae$, eAe, and $(1-e)A(1-e)$ are then each a direct sum of its homogeneous components. For instance $eA(1-e) = eA_0(1-e) \oplus eA_1(1-e)$, with $eA_i(1-e)$ being the degree i component of $eA(1-e)$ for $i = 0, 1$.

We continue to require injectivity of m'. Subspace $A' = \mathrm{im}(m')$ is $\mathbb{Z}/2$-graded, and we select its complement A'' to be graded as well. All basis elements of $\mathbb{B}_{0,0}, \mathbb{B}_{0,1}, \mathbb{B}_{1,0}, \mathbb{B}_{1,1}(1)$ should be homogeneous (which is automatic for $\mathbb{B}_{0,0}$).

This will result in a $\mathbb{Z}/2$-graded idempotented algebra $\mathbf{k}\mathbb{B}$ with homogeneous basis elements in $\mathbb{B}$.

The monoidal category $\mathcal{C}$ that we assign to (A, e) is enriched over the category of super-vector spaces. For this reason, homogeneous morphisms in $\mathcal{C}$ supercommute when their relative height order changes during an isotopy, see Figure 9, with the coefficient $(-1)^{|a|\cdot|b|}$, where $|a| \in \{0, 1\}$ is the $\mathbb{Z}/2$-degree of the generator a.

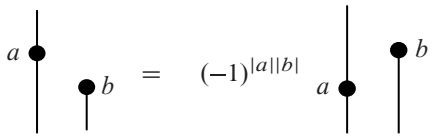

Figure 9. Super-commutativity of morphisms.

In our construction of a basis of hom spaces $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m}) = \mathbf{k}\mathbb{B}_{m,n}$, we need to choose a particular order of heights when dealing with decorated basis diagrams, to avoid sign indeterminancy. The order that we follow is shown in Figure 10. Lengths of top short arcs increase going from left to right, so that b_1 label is at the highest position, followed by b_2, and so on. Highest long strand label a_1 is below the lowest b-label $b_{m-|f|}$. It's followed by a_2 to the right and below, all the way to $a_{|f|}$, which has the lowest height of all a labels. Leftmost bottom arc label c_1 is lower than $a_{|f|}$ label, and the remaining bottom arc labels $c_2, \ldots, c_{n-|f|}$ continue with the lower heights. The lowest label in the diagram is $c_{n-|f|}$.

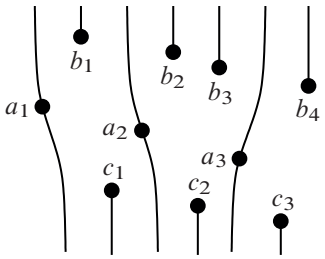

Figure 10. Keeping track of heights of labels, left to right and top to bottom.

Proof of Theorem 2.1 extends without any changes, simply by confirming the consistency of signs in several places. The theorem implies that there is no collapse in the size of homs between tensor powers of X and gives a basis for the space of morphisms from $X^{\otimes n}$ to $X^{\otimes m}$.

Case when A is a DG algebra with the trivial differential. A mild generalization of our construction to DG (differential graded) algebras will be needed in Section 3. A DG algebra A is a $\mathbb{Z}$-graded algebra, $A = \bigoplus_{i\in\mathbb{Z}} A^i$, with a differential d of degree one, such that $d(ab) = d(a)b + (-1)^{|a|}ad(b)$, where $|a| \in \mathbb{Z}$ is the degree of a, for homogeneous a. Each DG algebra is viewed as a superalgebra by reducing the degree modulo two and forgetting the differential.

In this paper we will only encounter the simplest case, when the differential is trivial on A. We assume this to be the case. With the differential zero on A, no additional conditions on A or e are needed, and the generalization from the super algebras to such DG algebras is straightforward. The grading is now by $\mathbb{Z}$ and not just $\mathbb{Z}/2$, with idempotent e in degree zero.

We choose A'' to be a subspace that is the direct sum of its intersections with the homogenous summands of $(1-e)A(1-e)$. Likewise, bases $\mathbb{B}_{0,0}$, $\mathbb{B}_{0,1}$, $\mathbb{B}_{1,0}$, and $\mathbb{B}_{1,1}(1)$ are chosen to consist of homogeneous elements.

The DG category $\mathcal{C}$ is constructed from this data just as in the super-algebra case. Due to $\mathbb{Z}$-grading, one introduces enlarged morphism spaces,

$$\mathrm{HOM}_{\mathcal{C}}(M,N) = \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(M, N[m]).$$

Morphism spaces $\mathrm{HOM}_{\mathcal{C}}$ between tensor powers of X have bases as described in Theorem 2.1, with heights in the basis diagrams tracked as in Figure 10. The differential acts by zero on all morphism spaces $\mathrm{HOM}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$. Diagrams super-commute, with the super grading given by reducing the $\mathbb{Z}$-grading modulo two.

A chain of ideals $J_{n,k}$. Now assume A is an algebra, or a super-algebra, or a DG algebra with the trivial differential. The ring $A_k = \mathrm{End}_{\mathcal{C}}(X^{\otimes k})$ is spanned by diagrams of decorated long and short strands, with each diagram having ℓ long strands and $2(k-\ell)$ short strands, an equal number $k-\ell$ at both top and bottom. Composing two such diagram D_1, D_2 with ℓ_1 and ℓ_2 long arcs, correspondingly, results in the product D_2D_1, which is also an endomorphism of $X^{\otimes k}$, that decomposes into a linear combination of diagrams, each with at most $\min(\ell_1,\ell_2)$ long strands. The number of long strands in a diagram cannot increase upon composition with another diagram.

Therefore, there is a two-sided ideal $J_{n,k}$ of A_k whose elements are linear combinations of diagrams with at most n long strands. Here $0 \le n \le k$. It's also convenient to define $J_{-1,k}$ to be the zero ideal. There is a chain of inclusions of

two-sided ideals

$$0 = J_{-1,k} \subset J_{0,k} \subset \cdots \subset J_{k-1,k} \subset J_{k,k} = A_k. \tag{12}$$

Basis $\mathbb{B}_{k,k}$ of A_k respects this ideal filtration, and restricts to a basis in each $J_{n,k}$.

Corollary 2.2. *Two-sided ideal $J_{n,k}$ has a basis*

$$\mathbb{B}_{k,k}(\leq n) = \bigsqcup_{0\leq \ell \leq n} \mathbb{B}_{k,k}(\ell).$$

In the special case $k = 1$, we denote by J the ideal $J_{0,1}$ and by L the quotient ring $A_1/J_{0,1}$. Thus, there is an exact sequence

$$0 \longrightarrow J \longrightarrow A_1 \longrightarrow L \longrightarrow 0,$$

where

$$J = J_{0,1} = (1-e)AeA(1-e),$$

$$A_1 = (1-e)A(1-e),$$

$$L = A_1/J \cong (1-e)A(1-e)/(1-e)AeA(1-e).$$

$A_1 = \mathrm{End}_{\mathcal{C}}(X)$ is a subring of A, with the unit element $1-e$, and L is isomorphic to the quotient of A_1 by the two-sided ideal of maps that factors through $\mathbf{1}$.

The quotient ring

$$L_k = A_k/J_{k-1,k} \tag{13}$$

is naturally isomorphic to $L^{\otimes k}$, the k-th tensor power of L. Graphically, we quotient the space of linear combinations of decorated diagrams with k endpoints at both bottom and top by the ideal of diagrams with at least one short strand (necessarily at least one at the top and the bottom). Elements in the quotient by this ideal will be represented by linear combinations of diagrams of k decorated long strands, modulo diagrams where a long strand simplifies into a linear combination of a pair of decorated short strands. The quotient is isomorphic to $L = L_1$, defined above for $k = 1$, and to the k-th tensor power of L for general k. In the super-case, the tensor power is understood correspondingly, counting signs.

3. A categorification of $\mathbb{Z}\left[\frac{1}{2}\right]$

The goal of this section is to describe a monoidal DG category $\mathcal{C}$, and its associated monoidal triangulated Karoubi closed category $D^c(\mathcal{C})$. We show that the Grothendieck ring $K_0(D^c(\mathcal{C}))$ is isomorphic to $\mathbb{Z}\left[\frac{1}{2}\right]$.

3.1. A diagrammatic category $\mathcal{C}$. As before, we work over a field $\mathbf{k}$. Consider a pre-additive monoidal category $\mathcal{C}$ with one generating object X, enriched over the category of $\mathbb{Z}$-graded super-vector spaces over $\mathbf{k}$, with the supergrading given by reducing the $\mathbb{Z}$-grading modulo 2. In a pre-additive category, homomorphisms between any two objects constitute an abelian group (in our case, a $\mathbb{Z}$-graded $\mathbf{k}$-vector space), but direct sums of objects are not formed.

A set of generating morphisms together with their degrees is given in Figure 11.

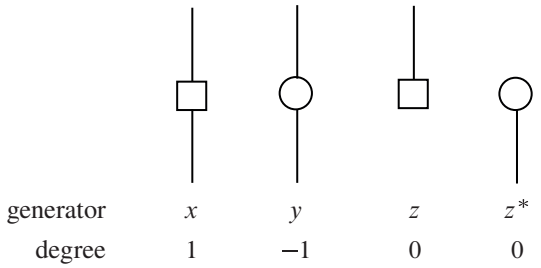

Figure 11. Generating morphisms.

Two of these four generating morphisms are endomorphisms of X, of degrees 1 and -1, correspondingly, one is a morphism from $\mathbf{1}$ to X of degree 0, and the fourth morphism goes from X to $\mathbf{1}$ and has degree 0. We denote these generators x, y, z, z^*, from left to right, so that x, y are endomorphisms of X, z a morphism from $\mathbf{1}$ to X, and z^* a morphism from X to $\mathbf{1}$. We draw x as a long strand decorated by a box, y as a long strand decorated by a circle, z as a short top strand decorated by a box, and z^* as a short bottom strand decorated by a circle, respectively. A pair of far away generators super-commute. The first two generators x and y have odd degrees, while z and z^* have even degrees.

Local relations are given in Figure 12. They are

$$\begin{aligned} z^*z = 1_{\mathbf{1}}, \quad z^*x = 0, \quad yz = 0, \\ yx = 1_X, \quad xy + zz^* = 1_X. \end{aligned} \tag{14}$$

The identity map $1_{\mathbf{1}}$ of the object $\mathbf{1}$ is represented by the empty diagram. Figure 13 shows our notation for powers and some compositions of the generating morphisms.

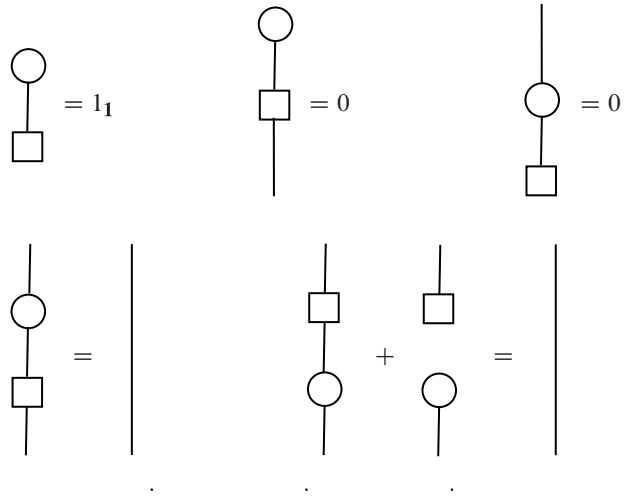

Figure 12. Defining local relations.

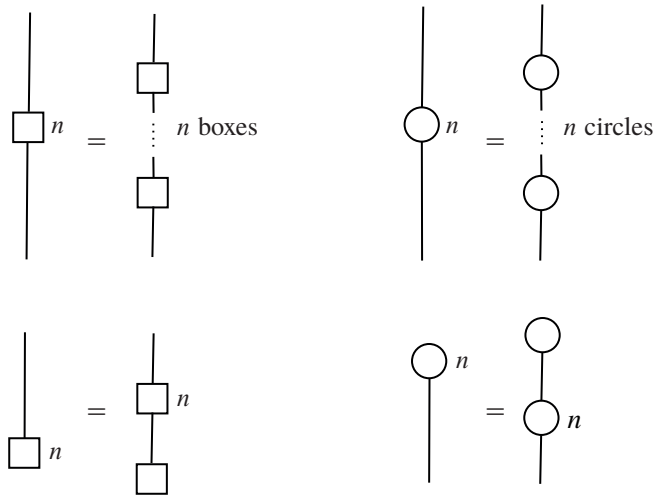

Figure 13. Notations for compositions, left to right: x^n, y^n, $x^n z$ and $z^* y^n$.

We write $\mathrm{Hom}_{\mathcal{C}}(M, N)$ for the vector space of degree 0 morphisms, and $\mathrm{HOM}_{\mathcal{C}}(M, N)$ for the graded vector space with degree components–homogeneous maps of degree m:

$$\mathrm{HOM}_{\mathcal{C}}(M, N) = \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(M, N[m]).$$

If $\mathrm{char}(\mathbf{k}) \neq 2$ we choose an order of heights of decorations as follows. For any pair of strands, the height of decorations on the left strand is above the height of decorations on the right strand. Let $f \otimes g$ denote the horizontal composition of two diagrams f and g, where the height of f is above that of g, see Figure 14.

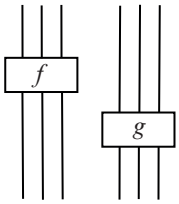

Figure 14. Convention for $f \otimes g$.

Bases of morphism spaces. We observe that the category $\mathcal{C}$ is generated from the suitable data (A, e) as described in Section 2, where A is a DG algebra with the trivial differential. To see this, we restrict the above diagrams and defining relations on them to the case when there is at most one strand at the top and at most one strand at the bottom. In other words, we consider generating morphisms in Figure 11 and compose them only vertically, not horizontally, with the defining relations in Figure 12. The idempotent e is given by the empty diagram, while $1 - e$ is the undecorated vertical strand diagram.

Ignoring the grading and the (zero) differential on A, it follows from the defining relations (14) that A is isomorphic to the Jacobson algebra [12] and to the Leavitt path algebra $L(T)$ of the Toeplitz graph T in (6), see [1, Example 7]. In particular, as a $\mathbf{k}$-vector space, algebra A has a basis

$$\{1_{\mathbf{1}}\} \cup \{x^n z \mid n \geq 0\} \cup \{z^* y^n \mid n \geq 0\} \cup \{x^n z z^* y^m \mid n, m \geq 0\} \cup \{1_X, x^n, y^n \mid n > 0\}$$

by [2, Corollary 1.5.12] or by a straightforward computation. Our notations for some of these basis elements are shown in Figure 13.

The basis of A can be split into the following disjoint subsets:

(1) $eAe \cong \mathbf{k}$ has a basis $\mathbb{B}_{0,0} = \{1_{\mathbf{1}}\}$ consisting of a single element which is the empty diagram;

(2) $(1-e)Ae$ has a basis $\mathbb{B}_{1,0} = \{x^n z \mid n \geq 0\}$. Element $x^n z$ is depicted by a short top strand decorated by a box with label n, see Figure 13;

(3) $eA(1-e)$ has a basis $\mathbb{B}_{0,1} = \{z^* y^n \mid n \geq 0\}$. Element $z^* y^n$ is depicted by a short bottom strand decorated by a circle with label n (lollipop in Figure 13);

(4) $(1-e)A(1-e)$ has a basis $\mathbb{B}_{1,1}(0) \sqcup \mathbb{B}_{1,1}(1)$, where

$$\mathbb{B}_{1,1}(0) = \{x^n z z^* y^m \mid n, m \geq 0\}$$

consists of pairs (short top strand with a labelled box, short bottom strand with a labelled circle), and $\mathbb{B}_{1,1}(1) = \{1_X, x^n, y^n \mid n > 0\}$ consists of long strand diagrams which may carry either circles or boxes, but not both.

The multiplication map $(1-e)Ae \otimes eA(1-e) \to (1-e)A(1-e)$ sends the basis $\mathbb{B}_{1,0} \times \mathbb{B}_{0,1}$ of $(1-e)Ae \otimes eA(1-e)$ bijectively to $\mathbb{B}_{1,1}(0)$ so that the multiplication map is injective.

We see that the conditions on (A, e) from the beginning of Section 2 are satisfied, and we can indeed form the monoidal category $\mathcal{C}$ as above with objects $X^{\otimes n}$, over $n \geq 0$. Algebra A can then be described as the direct sum

$$A \;\cong\; \mathrm{END}_{\mathcal{C}}(\mathbf{1}) \oplus \mathrm{HOM}_{\mathcal{C}}(\mathbf{1}, X) \oplus \mathrm{HOM}_{\mathcal{C}}(X, \mathbf{1}) \oplus \mathrm{END}_{\mathcal{C}}(X),$$

which is a DG algebra with the trivial differential. Therefore, a basis of the space $\mathrm{HOM}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$ is given in Theorem 2.1.

3.2. DG extensions of $\mathcal{C}$. We turn the category $\mathcal{C}$ into a DG category by introducing a differential ∂ which is trivial on all generating morphisms. Necessarily, ∂ is trivial on the space of morphisms between any two objects of $\mathcal{C}$. The resulting DG category is still denoted by $\mathcal{C}$.

We refer the reader to [15] for an introduction to DG categories. For any DG category $\mathcal{D}$ we write $\mathrm{Hom}_{\mathcal{D}}(Y, Y')$ for the vector space of degree 0 morphisms, and $\mathrm{HOM}_{\mathcal{D}}(Y, Y')$ for the chain complex of vector spaces with degree components $\mathrm{Hom}_{\mathcal{D}}(Y, Y'[m])$ of homogeneous maps of degree m. A *right DG $\mathcal{D}$-module M* is a DG functor $M\colon \mathcal{D}^{\mathrm{op}} \to Ch(\mathbf{k})$ from the opposite DG category $\mathcal{D}^{\mathrm{op}}$ to the DG category of chain complexes of $\mathbf{k}$-vector spaces. For each object Y of $\mathcal{D}$, there is a right module $Y^\wedge$ *represented by* Y

$$Y^\wedge = \mathrm{HOM}_{\mathcal{D}}(-, Y).$$

Unless specified otherwise, all DG modules are right DG modules in this paper.

We use the notations from [30, Section 3.2.21]. For any DG category $\mathcal{D}$, there is a canonical embedding $\mathcal{D} \subset \mathcal{D}^{\mathrm{pre}}$ of $\mathcal{D}$ into the pre-triangulated DG category $\mathcal{D}^{\mathrm{pre}}$ associated to $\mathcal{D}$. It's obtained from $\mathcal{D}$ by formally adding iterated shifts, finite direct sums, and cones of morphisms. The homotopy category $\mathrm{Ho}(\mathcal{D}^{\mathrm{pre}})$ of $\mathcal{D}^{\mathrm{pre}}$ is triangulated. It is equivalent to the full triangulated subcategory of the derived category $D(\mathcal{D})$ of DG $\mathcal{D}$-modules which is generated by $\mathcal{D}$. Each object Y of $\mathrm{Ho}(\mathcal{D}^{\mathrm{pre}})$ corresponds to a module $Y^\wedge$ of $D(\mathcal{D})$ under the equivalence.

The idempotent completion $\widetilde{\mathrm{Ho}}(\mathcal{D}^{\mathrm{pre}})$ of $\mathrm{Ho}(\mathcal{D}^{\mathrm{pre}})$ is equivalent to the triangulated category $D^c(\mathcal{D})$ of compact objects in $D(\mathcal{D})$ by [22, Lemma 2.2].

To summarize, there is a chain of categories

$$\mathcal{D} \subset \mathcal{D}^{\mathrm{pre}} \dashrightarrow \mathrm{Ho}(\mathcal{D}^{\mathrm{pre}}) \subset \widetilde{\mathrm{Ho}}(\mathcal{D}^{\mathrm{pre}}) \simeq D^c(\mathcal{D}) \subset D(\mathcal{D}).$$

The first two categories are DG categories, and $\mathcal{D} \subset \mathcal{D}^{\mathrm{pre}}$ is fully faithful. The last four categories are triangulated. The dashed arrow between $\mathcal{D}^{\mathrm{pre}}$ and $\mathrm{Ho}(\mathcal{D}^{\mathrm{pre}})$ is not a functor. More precisely, $\mathrm{Ho}(\mathcal{D}^{\mathrm{pre}})$ has the same objects as $\mathcal{D}^{\mathrm{pre}}$, and morphism spaces as subquotients of morphism spaces of $\mathcal{D}^{\mathrm{pre}}$. It is a full subcategory of its idempotent completion $\widetilde{\mathrm{Ho}}(\mathcal{D}^{\mathrm{pre}})$. The category $D^c(\mathcal{D})$ of compact objects in $D(\mathcal{D})$ is a full triangulated subcategory of $D(\mathcal{D})$.

Definition 3.1. For a unital DG algebra R, let $\mathcal{B}(R)$ be a DG category with a single object $*$ such that $\mathrm{END}_{\mathcal{B}(R)}(*) = R$. Let $D(R)$ and $D^c(R)$ denote $D(\mathcal{B}(R))$ and $D^c(\mathcal{B}(R))$, respectively.

Remark 3.2. If R is an ordinary unital algebra viewed as a DG algebra concentrated in degree 0 with the trivial differential, then $D(\mathcal{B}(R))$ is equivalent to the derived category of R-modules, and $D^c(\mathcal{B}(R))$ is equivalent to the triangulated category of perfect complexes of R-modules, see [19, Section 6.5] and [31, Proposition 70.3].

Since the DG category $\mathcal{C}$ is monoidal, it induces a monoidal structure on $\mathcal{C}^{\mathrm{pre}}$ which preserves homotopy equivalences. There are induced monoidal structures on the triangulated categories $\mathrm{Ho}(\mathcal{C}^{\mathrm{pre}})$ and $\widetilde{\mathrm{Ho}}(\mathcal{C}^{\mathrm{pre}})$. We are interested in the Grothendieck ring of $\widetilde{\mathrm{Ho}}(\mathcal{C}^{\mathrm{pre}}) \simeq D^c(\mathcal{C})$.

Isomorphisms in $D^c(\mathcal{C})$. Each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$ with $\partial f = 0$ induces a morphism in $\mathrm{Hom}_{D^c(\mathcal{C})}(Y^\wedge, Y'^\wedge)$, denoted f by abuse of notation. The generating morphisms in Figure 11 and the local relations in Figure 12 induce an isomorphism in $D^c(\mathcal{C})$

$$X^\wedge \cong \mathbf{1}^\wedge \oplus X^\wedge[-1], \tag{15}$$

given by $(z^*, y)^T \in \mathrm{Hom}_{D^c(\mathcal{C})}(X^\wedge, \mathbf{1}^\wedge \oplus X^\wedge[-1])$, and $(z, x) \in \mathrm{Hom}_{D^c(\mathcal{C})}(\mathbf{1}^\wedge \oplus X^\wedge[-1], X^\wedge)$, see Figure 15. Tensoring with $(X^\wedge)^{\otimes(k-1)}$ in $D^c(\mathcal{C})$ on either side of isomorphism (15) results in isomorphisms in $D^c(\mathcal{C})$

$$(X^\wedge)^{\otimes k} \cong (X^\wedge)^{\otimes(k-1)} \oplus (X^\wedge)^{\otimes k}[-1]. \tag{16}$$

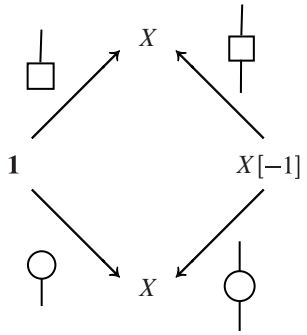

Figure 15. The isomorphism $X^\wedge \cong \mathbf{1}^\wedge \oplus X^\wedge[-1]$.

3.3. DG algebras of endomorphisms. Part of the structure of $\mathcal{C}$ can be encoded into an idempotented DG algebra B with the trivial differential, which has a complete system of mutually orthogonal idempotents $\{1_k\}_{k\geq 0}$, so that

$$B = \bigoplus_{m,n\geq 0} 1_m B 1_n,$$

and

$$1_m B 1_n = \mathrm{HOM}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m}).$$

Multiplication in B matches composition of morphisms in $\mathcal{C}$.

The tensor structure of $\mathcal{C}$ induces a tensor structure on B. Given $f, g \in B$ represented by some diagrams in $\mathcal{C}$, let $f \otimes g$ be an element of B represented by the horizontal composition of the two diagrams for f and g, where the diagram for f is on the left whose height is above the height of the diagram for g. There is the super-commutativity relation

$$(f \otimes g)(f' \otimes g') = (-1)^{\deg(g)\deg(f')} ff' \otimes gg'$$

for homogeneous elements $f, f', g, g' \in B$.

We also define

$$B_k = \bigoplus_{m,n\leq k} 1_m B 1_n,$$

which is a DG algebra with the trivial differential and the unit element $\sum_{n\leq k} 1_n$. The inclusions $B_k \subset B_{k+1}$ and $B_k \subset B$ are nonunital. Define

$$A_k = 1_k B 1_k = \mathrm{END}_{\mathcal{C}}(X^{\otimes k}), \tag{17}$$

which is a DG algebra with the trivial differential and the unit element 1_k. For $k = 0$, the DG algebras

$$A_0 = B_0 \cong \mathbf{k}. \tag{18}$$

The inclusion $A_k \subset B_k$ is nonunital for $k > 0$.

Let $\alpha_k\colon A_{k-1} \hookrightarrow A_k$ be an inclusion of DG algebras given by tensoring with zz^* on the left

$$\alpha_k(f) = (zz^*) \otimes f, \tag{19}$$

for $f \in A_{k-1}$. Note that α_k is nonunital.

Definition 3.3. For $k \geq 1$, let J_k be the two-sided DG ideal of A_k generated by diagrams with at most $k-1$ long strands. The quotient $L_k = A_k/J_k$ is naturally a unital DG algebra with the trivial differential.

By Theorem 2.1 and Proposition 2.2 in Section 2, we know that A_k has a $\mathbf{k}$-basis $\bigsqcup_{0\leq\ell\leq k} \mathbb{B}_{k,k}(\ell)$, and J_k has a $\mathbf{k}$-basis $\bigsqcup_{0\leq\ell\leq k-1} \mathbb{B}_{k,k}(\ell)$. So L_k has a $\mathbf{k}$-basis given by the images of elements of $\mathbb{B}_{k,k}(k)$ under the quotient map $A_k \to L_k$.

The ideal $J = J_1$ has a $\mathbf{k}$-basis $\mathbb{B}_{1,1}(0) = \{x^i zz^* y^j \mid i, j \geq 0\}$. The unital DG algebra $L = L_1$ is generated by $\bar{x}, \bar{y}$, which are the images of $x, y \in A_1$ under the quotient map $A_1 \to L$. There is an exact sequence

$$0 \longrightarrow J \longrightarrow A_1 \longrightarrow L \longrightarrow 0$$

of DG algebras with the trivial differentials. For $n \geq 0$, let $\mathrm{M}_n(\mathbf{k})$ be the $(n+1)\times(n+1)$ matrix DG algebra with the trivial differential and a standard basis $\{e_{ij} \mid 0 \leq i, j \leq n\}$ of elementary matrices, with $\deg(e_{ij}) = i - j$.

Proposition 3.4. *There are isomorphisms of DG* $\mathbf{k}$*-algebras with trivial differentials*:

$$J \cong M_{\mathbb{N}}(\mathbf{k}),$$

$$L \cong \mathbf{k}[a, a^{-1}], \quad \deg(a) = 1,$$

$$L_k \cong \mathbf{k}\langle a_1^{\pm 1}, \ldots, a_k^{\pm 1}\rangle/(a_i a_j = -a_j a_i, i \neq j), \quad \deg(a_i) = 1.$$

Proof. Define the isomorphism $\mathrm{M}_{\mathbb{N}}(\mathbf{k}) \to J$ by $e_{ij} \mapsto x^i zz^* y^j$ for $i, j \in \mathbb{N}$. The nonunital DG algebra J is isomorphic to the direct limit $\mathrm{M}_{\mathbb{N}}(\mathbf{k})$ of unital DG algebras $\mathrm{M}_n(\mathbf{k})$ under non-unital inclusions $\mathrm{M}_n(\mathbf{k}) \subset \mathrm{M}_{n+1}(\mathbf{k})$ taking e_{ij} to e_{ij}.

Define a map of algebras $L \to \mathbf{k}[a, a^{-1}]$ by $\bar{x} \mapsto a, \bar{y} \mapsto a^{-1}$. It is an isomorphism since

$$\bar{y}\,\bar{x} = \overline{yx} = 1 \in L, \qquad \bar{x}\,\bar{y} = \overline{xy} = \overline{1 - zz^*} = 1 \in L,$$

by the local relations (14).

For $1 \le i \le k$, let $x_i = 1 \otimes \cdots \otimes x \otimes \cdots 1$ and $y_i = 1 \otimes \cdots \otimes y \otimes \cdots 1 \in A_k$ whose ith factors are x and $y \in A_1$, respectively. Then L_k is generated by images $\overline{x_i}$, $\overline{y_i}$, and subject to relations

$$\overline{x_i}\ \overline{y_i} = \overline{y_i}\ \overline{x_i} = 1, \quad \overline{x_i}\ \overline{x_j} = -\overline{x_j}\ \overline{x_i}, \quad \text{for } i \ne j.$$

Define the isomorphism $L_k \to \mathbf{k}\langle a_1^{\pm 1}, \dots, a_k^{\pm 1}\rangle/(a_i a_j = -a_j a_i, i \ne j)$ by $\overline{x_i} \mapsto a_i$, $\overline{y_i} \mapsto a_i^{-1}$ for $1 \le i \le k$. □

We fix the isomorphisms for J, L, L_k in Proposition 3.4. See Figure 16 for J and L.

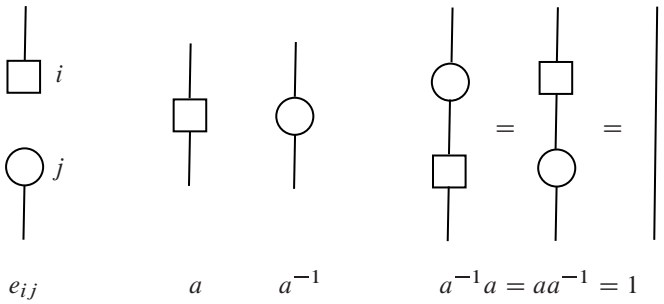

Figure 16. Basis element e_{ij} of the ideal J and elements a, a^{-1} of the quotient L, where generators of L are represented by the same diagrams as for A_1 by abuse of notation.

3.4. Approximation of $D^c(\mathcal{C})$ by $D^c(A_k)$. Let $\mathcal{C}_k$ be the smallest full DG subcategory of $\mathcal{C}$ which contains the objects $X^{\otimes n}$, $0 \le n \le k$. Note that $\mathcal{C}_k$ is not monoidal. There is a family of inclusions $\mathcal{C}_{k-1} \subset \mathcal{C}_k$ of DG categories. They induce a family of functors $\imath_k \colon D(\mathcal{C}_{k-1}) \to D(\mathcal{C}_k)$. For $0 \le n \le k-1$, $\imath_k(X^{\wedge \otimes n}) = X^{\wedge \otimes n}$ is compact in $D(\mathcal{C}_k)$, and

$$\mathrm{END}_{D(\mathcal{C}_{k-1})}(X^{\wedge \otimes n}) \cong A_n \cong \mathrm{END}_{D(\mathcal{C}_k)}(X^{\wedge \otimes n}) \cong \mathrm{END}_{D(\mathcal{C}_k)}(\imath_k(X^{\wedge \otimes n})).$$

The functor $\imath_k \colon D(\mathcal{C}_{k-1}) \to D(\mathcal{C}_k)$ is fully faithful by [13, Lemma 4.2 (a, b)]. The restriction to the subcategory $\imath_k^c \colon D^c(\mathcal{C}_{k-1}) \to D^c(\mathcal{C}_k)$ of compact objects is also fully faithful. Similarly, there is a family of inclusions $g_k^c \colon D^c(\mathcal{C}_k) \to D^c(\mathcal{C})$ of triangulated categories.

Recall that the Grothendieck group $K_0(\mathcal{T})$ of an essentially small triangulated category $\mathcal{T}$ is the abelian group generated by symbols $[Y]$ for every object Y of $\mathcal{T}$, modulo the relation $[Y_2] = [Y_1] + [Y_3]$ for every distinguished triangle $Y_1 \to Y_2 \to Y_3 \to Y_1[1]$ in $\mathcal{T}$. In particular, $[Y_1] = [Y_2]$ if Y_1 and Y_2 are

isomorphic. The functors $\imath_k^c$ and g_k^c send distinguished triangles to distinguished triangles and induce maps of abelian groups

$$\imath_{k*}^c \colon K_0(D^c(\mathcal{C}_{k-1})) \longrightarrow K_0(D^c(\mathcal{C}_k)), \quad g_{k*}^c \colon K_0(D^c(\mathcal{C}_k)) \longrightarrow K_0(D^c(\mathcal{C})).$$

Let $\varinjlim K_0(D^c(\mathcal{C}_k))$ denote the direct limit of $K_0(D^c(\mathcal{C}_k))$ with respect to $\imath_{k*}^c$.

Proposition 3.5. *There is an isomorphism of abelian groups*

$$K_0(D^c(\mathcal{C})) \cong \varinjlim K_0(D^c(\mathcal{C}_k)).$$

Proof. The family of maps g_{k*}^c induces a map $g_* \colon \varinjlim K_0(D^c(\mathcal{C}_k)) \to K_0(D^c(\mathcal{C}))$ since functors g_{k-1}^c and $g_k^c \circ \imath_k^c$ are isomorphic. The map g_* is surjective since any object of $D^c(\mathcal{C})$ is contained in $D^c(\mathcal{C}_k)$ for some k up to isomorphism. The map g_* is injective since any distinguished triangle in $D^c(\mathcal{C})$ is contained in $D^c(\mathcal{C}_k)$ for some k up to isomorphism. □

The category $\mathcal{C}_k$ contains a full DG subcategory $\mathcal{C}_k'$ of a single object $X^{\otimes k}$ whose endomorphism DG algebra $\mathrm{END}_{\mathcal{C}_k}(X^{\otimes k}) = A_k$ by (17). Thus, the category $\mathcal{C}_k'$ is isomorphic to $\mathcal{B}(A_k)$, and $D(A_k) = D(\mathcal{B}(A_k))$, see Definition 3.1. There is an inclusion $\mathcal{B}(A_k) \subset \mathcal{C}_k$ of DG categories. The induced functors

$$h_k \colon D(A_k) \longrightarrow D(\mathcal{C}_k), \quad h_k^c \colon D^c(A_k) \longrightarrow D^c(\mathcal{C}_k) \tag{20}$$

of triangulated categories are fully faithful by [13, Lemma 4.2 (a, b)]. A set $\mathcal{H}$ of objects of a triangulated category $\mathcal{T}$ is a *set of generators* if $\mathcal{T}$ coincides with its smallest strictly full triangulated subcategory containing $\mathcal{H}$ and closed under infinite direct sums, see [13, Section 4.2]. In particular, $\{X^{\wedge\otimes n}, 0 \leq n \leq k\}$ forms a set of generators for $D(\mathcal{C}_k)$. Equation (16) implies that $X^{\wedge\otimes n}$ is isomorphic to a direct summand of $X^{\wedge\otimes k}$ for $0 \leq n \leq k$. Let $p_n \in \mathrm{End}_{D(\mathcal{C}_k)}(X^{\wedge\otimes k})$ denote the idempotent of projection onto the direct summand $X^{\wedge\otimes n}$. Then $X^{\wedge\otimes n}$ is isomorphic to a DG $\mathcal{C}_k$-module given by a complex

$$\cdots \xrightarrow{1-p_n} X^{\wedge\otimes k} \xrightarrow{p_n} X^{\wedge\otimes k} \xrightarrow{1-p_n} X^{\wedge\otimes k}.$$

Thus, $\{X^{\wedge\otimes k}\}$ forms a set of compact generators for $D(\mathcal{C}_k)$. The functor h_k is an equivalence of triangulated categories by [13, Lemma 4.2 (c)]. It is clear that $h_k^c \colon D^c(A_k) \to D^c(\mathcal{C}_k)$ is also an equivalence and thus induces an isomorphism of Grothendieck groups $h_{k*}^c \colon K_0(D^c(A_k)) \cong K_0(D^c(\mathcal{C}_k))$. By Proposition 3.5, there is a canonical isomorphism of abelian groups:

$$K_0(D^c(\mathcal{C})) \cong \varinjlim K_0(D^c(A_k)). \tag{21}$$

3.5. K-theory computations. For a DG category $\mathcal{D}$, let $K_0(\mathcal{D})$ denote the Grothendieck group of the triangulated category $D^c(\mathcal{D})$.

If R is an ordinary unital algebra viewed as a DG algebra concentrated in degree 0 with the trivial differential, then there is a canonical isomorphism $K_0(D^c(R)) \cong K_0(R)$ by Remark 3.2, where $K_0(R)$ is the Grothendieck group of the ring R.

Without ambiguity let $K_0(R)$ denote $K_0(D^c(R))$ for a unital DG algebra R. The isomorphism (21) can be rewritten as

$$K_0(\mathcal{C}) \cong \varinjlim K_0(A_k).$$

In order to compute $K_0(A_k)$, we need higher K-theory of DG algebras and DG categories. We briefly recall the definition of higher K-theory of DG categories following [30, Section 3.2.21]. Schlichting [30, Section 3.2.12] introduces the notion of *complicial exact category with weak equivalences* whose higher K-theory is defined. For a DG category $\mathcal{D}$, its pre-triangulated envelope $\mathcal{D}^{\mathrm{pre}}$ can be made into an exact category whose morphisms are maps of degree 0 which commute with the differential. A sequence is exact if it is a split exact sequence when ignoring the differential. Then $(\mathcal{D}^{\mathrm{pre}}, w) = (\mathcal{D}^{\mathrm{pre}}, \text{homotopy equivalences})$ is a complicial exact category with homotopy equivalences as weak equivalences. The K-theory of the DG category $\mathcal{D}$ is defined as the K-theory of the complicial exact category with weak equivalences $(\mathcal{D}^{\mathrm{pre}}, w)$. This definition is equivalent to Waldhausen's definition of K-theory of a DG category according to [30, Remark 3.2.13].

We introduce the following notations. For a DG category $\mathcal{D}$,

$$K_1(\mathcal{D}) = K_1(\mathcal{D}^{\mathrm{pre}}, w), \quad K_0'(\mathcal{D}) = K_0(\mathcal{D}^{\mathrm{pre}}, w). \tag{22}$$

For a unital DG algebra A,

$$K_1(A) = K_1(\mathcal{B}(A)), \quad K_0'(A) = K_0'(\mathcal{B}(A)). \tag{23}$$

Note that $K_0'(\mathcal{D}) \cong K_0(\mathrm{Ho}(\mathcal{D}^{\mathrm{pre}}))$ by [30, Proposition 3.2.22]. Recall that

$$K_0(\mathcal{D}) = K_0(D^c(\mathcal{D})) \cong K_0(\widetilde{\mathrm{Ho}}(\mathcal{D}^{\mathrm{pre}})).$$

By [32, Corollary 2.3], $K_0'(\mathcal{D}) \to K_0(\mathcal{D})$ is injective.

Exact sequences of derived categories. The main tool to compute $K_0(A_k)$ is the Thomason-Waldhausen Localization Theorem specialized to the case of DG categories.

A sequence of triangulated categories and exact functors $\mathcal{T}_1 \overset{F_1}{\to} \mathcal{T}_2 \overset{F_2}{\to} \mathcal{T}_3$ is called *exact* if $F_2F_1 = 0$, F_1 is fully faithful, and F_2 induces an equivalence $\mathcal{T}_2/F_1(\mathcal{T}_1) \to \mathcal{T}_3$, see [14, Section 2.9] and [30, Section 3.1.5].

A sequence of triangulated categories $\mathcal{T}_1 \overset{F_1}{\to} \mathcal{T}_2 \overset{F_2}{\to} \mathcal{T}_3$ is called *exact up to factors* if $F_2F_1 = 0$, F_1 is fully faithful, and F_2 induces an equivalence $\mathcal{T}_2/F_1(\mathcal{T}_1) \to \mathcal{T}_3$ up to factors, see [30, Definition 3.1.10]. An inclusion $F\colon \mathcal{A} \to \mathcal{B}$ of triangulated categories is called an *equivalence up to factors* [30, Definition 2.4.1] if every object of $\mathcal{B}$ is a direct summand of an object in $F(\mathcal{A})$.

Given a sequence of DG categories $\mathcal{A} \to \mathcal{B} \to \mathcal{D}$, if the sequence

$$D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \longrightarrow D(\mathcal{D})$$

of derived categories of DG modules is exact, then the associated sequence

$$D^c(\mathcal{A}) \longrightarrow D^c(\mathcal{B}) \longrightarrow D^c(\mathcal{D})$$

of derived categories of compact objects is exact up to factors by Neeman's result [22, Theorem 2.1]. According to [30, Theorem 3.2.27], the Thomason-Waldhausen Localization Theorem implies that there is an exact sequence of K-groups:

$$K_1(\mathcal{D}) \longrightarrow K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B}) \longrightarrow K_0(\mathcal{D}).$$

We will fit $D(A_k)$ into an exact sequence of derived categories and then make use of the localization theorem.

Definition 3.6. For a DG algebra A, a DG A-module Q is (*finitely generated*) *relatively projective* if it is a direct summand of a (finite) direct sum of modules of the form $A[n]$.

We refer the reader to [14, Section 3.1] for the definition of *Property* (P) for a DG module. Any relatively projective module has Property (P). For any object $M \in D(A)$ there exists $P(M) \in D(A)$ which is isomorphic to M in $D(A)$ and has Property (P) [14, Theorem 3.1]. The object $P(M) \in D(A)$ is unique up to isomorphism. If A is an ordinary algebra viewed as a DG algebra concentrated in degree 0, then $P(M)$ is a projective resolution of M.

Let A, B be DG algebras, and X be a DG left A, right B bimodule. We call X a DG (A, B)-bimodule. The derived tensor product functor $-\otimes_A^{\mathbf{L}} X\colon D(A) \to D(B)$ is defined by $M \otimes_A^{\mathbf{L}} X \cong P(M) \otimes_A X$. Note that the derived tensor product commutes with infinite direct sums, see [14, Section 6.1].

Consider $\alpha_k\colon A_{k-1} \to A_k$ in (19). Let $e_k = \alpha_k(1_{k-1}) = (zz^*) \otimes 1_{k-1}$ which is an idempotent of A_k. It generates a right ideal $I_k = e_k A_k$ of A_k. The map α_k makes I_k a DG (A_{k-1}, A_k)-bimodule and induces a functor

$$i_k = - \otimes^{\mathbf{L}}_{A_{k-1}} I_k\colon D(A_{k-1}) \longrightarrow D(A_k).$$

Note that I_k is relatively projective as a right A_k-module.

It is clear that $h_k \circ i_k$ is isomorphic to $\imath_k \circ h_{k-1}$ as functors $D(A_{k-1}) \to D(\mathcal{C}_k)$, see (20). So

$$K_0(\mathcal{C}) \cong \varinjlim K_0(A_k) \tag{24}$$

with respect to i_{k*}.

The quotient map $A_k \to L_k$ makes L_k a DG (A_k, L_k)-bimodule, and induces a functor

$$j_k = - \otimes^{\mathbf{L}}_{A_k} L_k\colon D(A_k) \longrightarrow D(L_k).$$

A construction of $P(L_k)$ for $L_k \in D(A_k)$. For $k = 1$, let $P(L)$ be a complex

$$\bigoplus_{j\in\mathbb{N}} e_{jj} A_1 \longrightarrow A_1,$$

whose the differential is the sum of inclusions $\imath_j\colon e_{jj}A_1 \hookrightarrow A_1$, where A_1 is in degree 0, $\bigoplus_{j\in\mathbb{N}} e_{jj}A_1$ is in degree -1, and $e_{jj} \in J \subset A_1$ are idempotents. In other words, $P(L)$ is the DG A_1-module

$$\Big(\Big(\bigoplus_{j\in\mathbb{N}} e_{jj} A_1[1]\Big) \oplus A_1,\ \partial = \sum_{j\in\mathbb{N}} \imath_j\Big).$$

Since $\bigoplus_{j\in\mathbb{N}} e_{jj}A_1 = J$ as A_1-modules, $P(L) = (J \to A_1) \cong L \in D(A_1)$.

For $k > 1$, we take a product of k copies of $P(L)$, where the product corresponds to the monoidal structure on $\mathcal{C}$. More precisely, let $u(t,i)$ denote the idempotent of A_k whose diagram consists of $k-1$ vertical long arcs and one pair of short arcs e_{ii} as the t-th strand from the left, for $1 \le t \le k, i \in \mathbb{N}$, see Figure 17. They satisfy the commuting relations $u(t,i)u(t',i') = u(t',i')u(t,i)$ for $t \ne t'$. So their products are also idempotents of A_k, denoted by $u(T, \mathbf{i})$ for $T \subset \{1,\dots,k\}$ and $\mathbf{i} \in \mathbb{N}^{|T|}$. Here $u(\emptyset,\emptyset)$ is understood as the identity 1_k of A_k.

Let

$$P(T,\mathbf{i}) = u(T,\mathbf{i})A_k$$

which is a relatively projective DG A_k-module. For $P(T,\mathbf{i})$, $P(S,\mathbf{j})$ such that $T = S \sqcup \{r\}$ and $i_s = j_s$ for $s \in S$, there is an inclusion $\iota(T,\mathbf{i},r)\colon P(T,\mathbf{i}) \to P(S,\mathbf{j})$ of A_k-modules given by $\iota(T,\mathbf{i},r)(u(T,\mathbf{i})) = u(S,\mathbf{j})u(r,i_r)$.

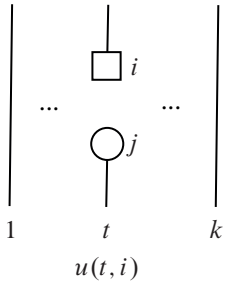

Figure 17. The idempotent $u(t,i) \in A_k$.

Consider a DG A_k-module $P(L_k)$ given by a complex of relatively projective DG A_k-modules of finite length:

$$\bigoplus_{\substack{|T|=k \\ \mathbf{i}\in\mathbb{N}^{|T|}}} P(T,\mathbf{i}) \longrightarrow \bigoplus_{\substack{|T|=k-1 \\ \mathbf{i}\in\mathbb{N}^{|T|}}} P(T,\mathbf{i}) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{|T|=1 \\ \mathbf{i}\in\mathbb{N}^{|T|}}} P(T,\mathbf{i}) \longrightarrow A_k$$

with the differential

$$\partial = \sum_{T,\mathbf{i},r\in T} (-1)^{c(T,r)} \iota(T,\mathbf{i},r),$$

where A_k is in degree 0, and $c(T,r) = \#\{t \in T \mid t < r\}$. The complex $P(L_k)$ is exact except at A_k. Let

$$\mathrm{pr}\colon P(L_k) \longrightarrow L_k \tag{25}$$

be the quotient map $A_k \to L_k$ on the summand A_k, and the zero map on the remaining summands of $P(L_k)$. Then pr is an isomorphism in $D(A_k)$.

Except for the last term A_k, each $P(T,\mathbf{i})$ is naturally a submodule of the ideal J_k of A_k, which is the kernel of the quotient map $A_k \to L_k$, see Definition 3.3. This implies

$$j_k(L_k) = L_k \otimes^{\mathbf{L}}_{A_k} L_k \cong P(L_k) \otimes_{A_k} L_k \cong A_k \otimes_{A_k} L_k \cong L_k \in D(A_k). \tag{26}$$

Lemma 3.7. *The sequence of derived categories* $D(A_{k-1}) \xrightarrow{i_k} D(A_k) \xrightarrow{j_k} D(L_k)$ *is exact.*

Proof. The image $i_k(A_{k-1})$ is isomorphic to the module $e_k A_k$ which is finitely generated relatively projective. In particular, $i_k(A_{k-1})$ is compact in $D(A_k)$, and

$$\mathrm{END}_{D(A_{k-1})}(A_{k-1}) \cong A_{k-1} \cong e_k A_k e_k \cong \mathrm{END}_{D(A_k)}(i_k(A_{k-1})).$$

The functor $i_k\colon D(A_{k-1}) \to D(A_k)$ is fully faithful by [13, Lemma 4.2].

The composition $j_k \circ i_k$ sends the free module A_{k-1} to

$$\begin{aligned} j_k \circ i_k(A_{k-1}) &= j_k(A_{k-1} \otimes_{A_{k-1}} I_k) \cong j_k(I_k) \\ &= e_k A_k \otimes^{\mathbf{L}}_{A_k} L_k \cong e_k A_k \otimes_{A_k} L_k = 0, \end{aligned}$$

where the last isomorphism holds since $e_k A_k$ is relatively projective, and the last equality holds since $e_k A_k$ is contained in the ideal J_k which is the kernel of the quotient map $A_k \to L_k$. The composition $j_k \circ i_k$ commutes with the infinite direct sums [14, Section 6.1]. Thus, $j_k \circ i_k = 0$ on the smallest full triangulated subcategory of $D(A_{k-1})$ containing the free module A_{k-1} and closed under infinite direct sums. This full subcategory coincides with $D(A_{k-1})$, see [14, Section 4.2]. It follows that $j_k \circ i_k = 0$ on $D(A_{k-1})$.

The algebras L_k and A_k act on L_k both from left and right via the map $A_k \to L_k$. In the following computation, we view L_k in one of the three ways: (1) as a right L_k-module, denoted L_k^L; (2) as a right A_k-module, denoted L_k^A; and (3) as a (A_k, L_k)-bimodule, denoted ${}^A L_k^L$.

The functor j_k admits a right adjoint functor $f_k\colon D(L_k) \to D(A_k)$ which is the restriction functor with respect to the quotient map $A_k \to L_k$. In particular, $f_k(L_k^L) = L_k^A$. The functor f_k is fully faithful if and only if the counit map

$$\delta_{L_k}\colon L_k^A \otimes^{\mathbf{L}}_{A_k} {}^A L_k^L \longrightarrow L_k^L \tag{27}$$

is an isomorphism of right L_k-modules, see [25, Lemma 4 (1,3)]. The counit map δ_{L_k} is the image of $1_{L_k} \in \mathrm{Hom}_{D(A_k)}(L_k^A, L_k^A) = \mathrm{Hom}_{D(A_k)}(L_k^A, f_k(L_k^L))$ under the adjunction isomorphism

$$\mathrm{ad}\colon \mathrm{Hom}_{D(A_k)}(L_k^A, f_k(L_k^L)) \cong \mathrm{Hom}_{D(L_k)}(L_k^A \otimes^{\mathbf{L}}_{A_k} {}^A L_k^L, L_k^L).$$

Replacing L_k^A by its resolution $P(L_k)$, there is a chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{D(A_k)}(L_k^A, f_k(L_k^L)) &\xrightarrow{f} \mathrm{Hom}_{D(A_k)}(P(L_k), f_k(L_k^L)) \\ &\xrightarrow{\mathrm{ad}} \mathrm{Hom}_{D(L_k)}(P(L_k) \otimes_{A_k} {}^A L_k^L, L_k^L) \\ &\xrightarrow{g} \mathrm{Hom}_{D(L_k)}(L_k^L, L_k^L) \\ &\xrightarrow{h} \mathrm{Hom}_{D(L_k)}(L_k^A \otimes^{\mathbf{L}}_{A_k} {}^A L_k^L, L_k^L). \end{aligned}$$

Here g and h are induced by (26). Recall pr: $P(L_k) \to L_k$ from (25), and let $m\colon L_k \otimes_{A_k} L_k \to L_k$ denote the multiplication map. Then

$$\delta_{L_k} = h \circ g \circ \mathrm{ad} \circ f(1_{L_k}) = h \circ g \circ \mathrm{ad}(\mathrm{pr}) = h \circ g(m \circ (\mathrm{pr} \otimes 1_{L_k})) = h(1_{L_k})$$

which is an isomorphism. It follows that $f_k\colon D(L_k) \to D(A_k)$ is fully faithful.

Let $\mathcal{T}_k = D(A_k)/i_k(D(A_{k-1}))$, and $q_k\colon D(A_k) \to \mathcal{T}_k$ denote the quotient functor. Since $j_k \circ i_k$ is zero, the functor j_k factors through $t_k\colon \mathcal{T}_k \to D(L_k)$. Let

$$s_k = q_k \circ f_k\colon D(L_k) \longrightarrow D(A_k) \longrightarrow \mathcal{T}_k.$$

It is clear that $t_k \circ s_k = t_k \circ q_k \circ f_k = j_k \circ f_k$ is an equivalence since the counit map $\delta_{L_k}\colon j_k \circ f_k(L_k) \to L_k$ in (27) is an isomorphism, and the conditions of Lemma 4.2(a,c) in [13] hold.

It remains to show that s_k is an equivalence. By [13, Lemma 4.2 (a,c)], it is enough to show that $s_k(L_k)$ is a compact generator of $\mathcal{T}_k$, and

$$s_{k*}\colon \mathrm{Hom}_{D(L_k)}(L_k, L_k[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}_k}(s_k(L_k), s_k(L_k)[n])$$

is an isomorphism. The object

$$s_k(L_k) = q_k \circ f_k(L_k) \cong q_k(P(L_k)) \cong q_k(A_k),$$

since all other terms except for A_k in $P(L_k)$ lie in $i_k(D(A_{k-1}))$. Theorem 2.1 in [22] implies that $q_k(A_k)$ is a compact object of $\mathcal{T}_k$ since A_k is a compact object of $D(A_k)$. Moreover, $\{q_k(A_k)\}$ generates $\mathcal{T}_k$ since $\{A_k\}$ generates $D(A_k)$. We have $s_{k*} = q_{k*} \circ f_{k*}$, where

$$f_{k*}\colon \mathrm{Hom}_{D(L_k)}(L_k, L_k[n]) \longrightarrow \mathrm{Hom}_{D(A_k)}(f_k(L_k), f_k(L_k)[n]),$$

$$q_{k*}\colon \mathrm{Hom}_{D(A_k)}(f_k(L_k), f_k(L_k)[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}_k}(s_k(L_k), s_k(L_k)[n]).$$

The map f_{k*} is an isomorphism since f_k is fully faithful. The map q_{k*} is an isomorphism if $\mathrm{Hom}_{D(A_k)}(i_k(M), f_k(L_k)[n]) = 0$ for any $M \in D(A_{k-1})$ by [23, Definition 9.1.3, Lemma 9.1.5]. By adjointness $\mathrm{Hom}_{D(A_k)}(i_k(M), f_k(L_k)[n]) = \mathrm{Hom}_{D(L_k)}(j_k \circ i_k(M), L_k[n]) = 0$ since $j_k \circ i_k = 0$. We finally conclude that s_k is an equivalence. □

There is an exact sequence of K-groups

$$K_1(L_k) \xrightarrow{\partial} K_0(A_{k-1}) \xrightarrow{i_{k*}} K_0(A_k) \xrightarrow{j_{k*}} K_0(L_k), \tag{28}$$

induced by the exact sequence of the derived categories in Lemma 3.7.

To compute $K_0(A_k)$ we need $K_i(L_k)$ for $i = 0, 1$.

3.6. K-theory of L_k. We compute $K_i(L_k)$ for $i = 0, 1$ in this subsection. The key tool to compute $K_0(L_k)$ is a result of Keller [13, Theorem 3.1(c)]. Recall the notion of relatively projective DG modules from Definition 3.6. For any DG right L_k-module M, let $M[1]$ denote the shift of M, where $M[1]^i = M^{i+1}$, $d_{M[1]} = -d_M$, and $m[1] \cdot a = ma[1]$ for $m \in M, m[1] \in M[1]$ and $a \in L_k$. See Sections 10.3 and 10.6.3 in [6] for definitions of shifts of left and right DG modules, respectively.

Theorem 3.8 (Keller [13]). *Given any DG algebra A and a DG A-module M, let*

$$\cdots \longrightarrow \overline{Q_n} \longrightarrow \cdots \longrightarrow \overline{Q_0} \longrightarrow H^*(M) \longrightarrow 0$$

be a projective resolution of $H^*(M)$ viewed as graded $H^*(A)$-module such that $\overline{Q_n} \xrightarrow{\sim} H^*(Q_n)$ for a relatively projective DG A-module Q_n. Then M is isomorphic to a module $P(M)$ in the derived category $D(A)$ which admits a filtration F_n such that $\bigcup_{n=0}^{\infty} F_n = P(M)$, the inclusion $F_{n-1} \subset F_n$ splits as an inclusion of graded A-modules, and $F_n/F_{n-1} \xrightarrow{\sim} Q_n[n]$ as DG A-modules.

We specialize to the case $A = L_k$. There is an isomorphism of free right DG L_k-modules

$$h_k \colon L_k \simeq L_k[1] \tag{29}$$

given by $h_k(m) = a_k \cdot m$ for $m \in L_k$, where the multiplication is that of the algebra L_k and a_k is the invertible closed element of degree 1. So Q is relatively projective if it is a direct summand of a free module L_k^I, where I is the index set. Since the differential is trivial on L_k, $H^*(L_k) \cong L_k$ as graded algebras. So Q is a relatively projective DG L_k-module if and only if $Q \cong H^*(Q)$ is a direct summand of L_k^I as graded L_k-module. Given any projective resolution of $H^*(M)$ as in Theorem 3.8 we can take $Q_n = \overline{Q_n}$ viewed as a DG L_k-module.

We now consider projective resolutions of $N = H^*(M)$. Let $N = \bigoplus N^i$ be its decomposition into homogenous components. Let R_{k-1} denote the degree zero subalgebra of the graded algebra L_k. Then R_{k-1} is generated by $b_i = a_i a_k^{-1}$ for $1 \leq i \leq k-1$, and

$$R_{k-1} = \mathbf{k}\langle b_1^{\pm 1}, \ldots, b_{k-1}^{\pm 1}\rangle / (b_i b_j = -b_j b_i, i \neq j). \tag{30}$$

We fix the inclusion $R_{k-1} \to L_k$ from now on. There is an isomorphism of graded algebras

$$L_k \cong R_{k-1}\langle a_k^{\pm 1}\rangle / (b_i a_k = -a_k b_i). \tag{31}$$

For any graded L_k-module N, each component N^i is a R_{k-1}-module. Since a_k is invertible of degree 1, any graded L_k-module N is completely determined by N^0 as a R_{k-1}-module. More precisely, the action of a_k induces an isomorphism of R_{k-1}-modules

$$N^{i+1} \longrightarrow \alpha(N^i), \tag{32}$$

where $\alpha(N^i)$ is the abelian group N^i with the α-twisted action of R_{k-1} via an automorphism $\alpha\colon R_{k-1} \to R_{k-1}$ given by $\alpha(b_i) = -b_i$. Any projective resolution $P(N^0)$ of a R_{k-1}-module N^0 induces a projective resolution $P(N^i)$ of N^i. The direct sum $\bigoplus_{i\in\mathbb{Z}} P(N^i)$ is a projective resolution of the graded L_k-module N.

We recall the following results about R_{k-1} studied by Farrell and Hsiang [11]. The algebra $R_{k-1} \cong R_{k-2}[b_{k-1}^{\pm 1}]$ is an α-twisted finite Laurent series ring. According to [11, Theorem 25], R_{k-1} is right regular. So any finitely generated R_{k-1}-module admits a finite resolution by finitely generated projective R_{k-1}-modules. Furthermore, $K_0(R_{k-1}) \cong \mathbb{Z}$ with a generator $[R_{k-1}]$ by [11, Theorem 27].

Any isomorphism class of objects in $D^c(L_k)$ has a representative M which is isomorphic to a direct summand of $L_k^{\oplus r}$ for some finite r as graded L_k-modules (ignoring the differential). So M^0 and $H^0(M)$ are finitely generated R_{k-1}-modules since R_{k-1} is Noetherian. Then $H^0(M)$ admits a finite resolution by finitely generated projective R_{k-1}-modules. The graded L_k-module $H^*(M)$ admits a finite resolution by finitely generated projective L_k-modules. We have the following lemma by applying Keller's Theorem 3.8.

Lemma 3.9. *Any M in $D^c(L_k)$ is isomorphic to $P(M)$ which admits a finite filtration $F_n(M)$ such that $F_n(M)/F_{n-1}(M) \xrightarrow{\sim} Q_n(M)[n]$ is a finitely generated relatively projective DG L_k-module.*

Lemma 3.10. *There is a surjection of abelian groups $\eta_k\colon \mathbb{Z}/2 \to K_0(L_k)$.*

Proof. By Lemma 3.9 we have

$$[M] = \sum_n (-1)^n [Q_n(M)] \in K_0(L_k)$$

for M in $D^c(L_k)$, where the sum is a finite sum. The abelian group $K_0(L_k)$ is generated by classes $[Q]$ of finitely generated relative projective Q.

The inclusion $R_{k-1} \to L_k$ is a map of unital DG algebras, where R_{k-1} is viewed as a DG algebra concentrated in degree 0. It induces a functor $g_k\colon D^c(R_{k-1}) \to D^c(L_k)$ given by tensoring with the (R_{k-1}, L_k)-bimodule L_k.

Any Q is a direct summand of a finite free module $\bigoplus L_k$, and has the trivial differential. Its degree zero component Q^0 is a finitely generated projective R_{k-1}-module, and the action of a_k induces an isomorphism of R_{k-1}-modules $Q^{i+1} \to \alpha(Q^i)$. We have

$$g_k(Q^0) = Q^0 \otimes_{R_{k-1}} L_k = \bigoplus_{i\in\mathbb{Z}} Q^0 \otimes R_{k-1} a_k^i \cong \bigoplus_{i\in\mathbb{Z}} Q^i = Q,$$

by (31), where the direct sums are taking as R_{k-1}-modules. It follows that $g_{k*}\colon K_0(R_{k-1}) \to K_0(L_k)$ is surjective since $g_{k*}([Q^0]) = [Q]$. The group $K_0(L_k)$ is generated by $[L_k] = g_{k*}([R_{k-1}])$ since $K_0(R_{k-1}) \cong \mathbb{Z}$ with a generator $[R_{k-1}]$, see [11, Theorem 27]. Isomorphism (29) implies that $[L_k] = -[L_k]$. Hence the map $\eta_k\colon \mathbb{Z}/2 \to K_0(L_k)$ defined by $\eta_k(1) = [L_k]$ is surjective. □

According to [30, Section 3.2.12], the K-space $K(\mathcal{E}, w)$ of a complicial exact category $\mathcal{E}$ with weak equivalences w is the homotopy fiber of of $BQ(\mathcal{E}^w) \to BQ(\mathcal{E})$, where $\mathcal{E}^w \subset \mathcal{E}$ is the full exact subcategory of objects X in $\mathcal{E}$ for which the map $0 \to X$ is a weak equivalence. Here $BQ(\mathcal{E})$ is the classifying space of the category $Q(\mathcal{E})$ used in Quillen's Q-construction. By definition, there is a an exact sequence:

$$K_1(\mathcal{E}^w) \longrightarrow K_1(\mathcal{E}) \longrightarrow K_1(\mathcal{E}, w) \longrightarrow K_0(\mathcal{E}^w) \xrightarrow{i} K_0(\mathcal{E}) \longrightarrow K_0(\mathcal{E}, w). \quad (33)$$

Here, $K_i(\mathcal{E}^w)$ and $K_i(\mathcal{E})$ are K groups of the exact categories $\mathcal{E}^w$ and $\mathcal{E}$, respectively.

From now on let $\mathcal{E}$ denote the complicial exact category $\mathcal{B}(L_k)^{\mathrm{pre}}$, see Definition 3.1. A sequence $L \to M \to N$ is exact if it is split exact when forgetting the differential. The weak equivalences are the homotopy equivalences. So $\mathcal{E}^w \subset \mathcal{E}$ is the full subcategory of contractible objects in $\mathcal{E}$. By [30, Proposition 3.2.22] and the definition of K_1 in (22) and (23)

$$K_0(\mathcal{E}, w) \cong K_0'(L_k), \quad K_1(\mathcal{E}, w) = K_1(L_k). \quad (34)$$

Any $M \in \mathcal{E}$ is a finite direct sum of free modules $L_k[n]$ when forgetting the differential. Since $L_k \cong L_k[1] \in \mathcal{E}$, any M is isomorphic to $L_k^{\oplus r}$ for some $r \in \mathbb{N}$ as graded L_k-modules. Its degree zero component $M^0 \cong R_{k-1}^{\oplus r}$ as free R_{k-1}-modules. Since any exact sequence $L \to M \to N$ in $\mathcal{E}$ induces a split exact sequence $L^0 \to M^0 \to N^0$ of free R_{k-1}-modules, it induces a homomorphism $r\colon K_0(\mathcal{E}) \to K_0(R_{k-1}) \cong \mathbb{Z}$ defined by $r([M]) = [M^0] \in K_0(R_{k-1})$. It is clear that r is surjective.

Lemma 3.11. *The group* $K_0(\mathcal{E}) \cong \mathbb{Z}$ *with a generator* $[L_k]$.

Proof. Define a homomorphism $\mathbb{Z}[q, q^{-1}] \to K_0(\mathcal{E})$ by mapping q^n to the class $[L_k[n]]$ for $n \in \mathbb{Z}$. It is surjective since any object $M \in \mathcal{E}$ admits a finite filtration whose subquotients are finite direct sums of free modules $L_k[n]$. The map factors through $\mathbb{Z}[q, q^{-1}] \xrightarrow{q=1} \mathbb{Z}$ since $L_k \cong L_k[1] \in \mathcal{E}$. Let $\phi: \mathbb{Z} \to K_0(\mathcal{E})$ denote the induced map which is surjective. It is clear that r and ϕ are inverse to each other. □

We now consider $K_0(\mathcal{E}^w)$. Any object $M \in \mathcal{E}^w$ is contractible. There exists a degree -1 map $h: M \to M$ of graded L_k-modules such that $dh + hd = 1$ on M. Thus, $\operatorname{Ker} d_M = \operatorname{Im} d_M$. Moreover, $\operatorname{Ker} d_M$ and $h(\operatorname{Im} d_M)$ are graded L_k-submodules of M. For any $m \in \operatorname{Im} d_M \cap h(\operatorname{Im} d_M)$, $m = h(n)$ for some $n \in \operatorname{Im} d_M$ so that $n = dh(n) + hd(n) = d(m) = 0$ and $m = 0$. Thus $\operatorname{Im} d_M \cap h(\operatorname{Im} d_M) = \{0\}$. As graded L_k-modules, $M \cong (\operatorname{Im} d_M \oplus h(\operatorname{Im} d_M))$ since $m = dh(m) + hd(m)$. Moreover, $\operatorname{Im} d_M$ is a DG L_k-submodule of M with the trivial differential. As DG L_k-modules,

$$\begin{aligned} M &\cong (h(\operatorname{Im} d_M) \oplus \operatorname{Im} d_M, d = d_M: h(\operatorname{Im} d_M) \longrightarrow \operatorname{Im} d_M) \\ &\cong (\operatorname{Im} d_M[1] \oplus \operatorname{Im} d_M, d = \mathrm{id}: \operatorname{Im} d_M[1] \longrightarrow \operatorname{Im} d_M). \end{aligned} \tag{35}$$

The degree zero component $(\operatorname{Im} d_M)^0$ is a direct summand of a finitely generated free R_{k-1}-module M^0. Thus $(\operatorname{Im} d_M)^0$ is a finitely generated projective R_{k-1}-module. Recall from [11, Theorem 27] that $K_0(R_{k-1}) \cong \mathbb{Z}$ with a generator of the class $[R_{k-1}]$ of the free module R_{k-1}. It follows that every finitely generated projective R_{k-1}-module P is stably free, i.e. $P \oplus R_{k-1}^m \cong R_{k-1}^n$ for some $m, n \in \mathbb{N}$. Thus $(\operatorname{Im} d_M)^0$ is a finitely generated stably free R_{k-1}-module so that $\operatorname{Im} d_M$ is a stably free DG L_k-module. Let $C(L_k) = \operatorname{Cone}(L_k \xrightarrow{\mathrm{id}} L_k) \in \mathcal{E}^w$, where two L_k's are in degrees -1 and 0. There exists $m, n \in \mathbb{N}$ such that $M \oplus C(L_k)^{\oplus m} \cong C(L_k)^{\oplus n}$ by (35).

Define a homomorphism $\psi: \mathbb{Z} \to K_0(\mathcal{E}^w)$ by $\psi(1) = [C(L_k)]$. Then ψ is surjective.

For $M \in \mathcal{E}^w$, let

$$t(M) = [(\operatorname{Im} d_M)^0] \in K_0(R_{k-1}) \cong \mathbb{Z}. \tag{36}$$

For any exact sequence $L \xrightarrow{f} M \xrightarrow{g} N$ in $\mathcal{E}^w$, there is an induced sequence

$$\operatorname{Im} d_L \xrightarrow{f} \operatorname{Im} d_M \xrightarrow{g} \operatorname{Im} d_N.$$

We claim that it is a short exact sequence of graded L_k-modules.

(1) The first map is clearly injective.

(2) The last map is surjective. For any $n = d_N(n') \in \operatorname{Im} d_N$, $n' = g(m')$ for some $m' \in M$ since g is surjective. So $n = d_N(g(m')) = g(d_M(m')) \in g(\operatorname{Im} d_M)$.

(3) The middle term is exact. For any $m \in \operatorname{Im} d_M \cap \operatorname{Ker}(g)$, $m = f(l)$ for some $l \in L$. Then $d_M(m) = d_M(f(l)) = f(d_L(l)) = 0$ implies that $d_L(l) = 0$ since f is injective. So $l \in \operatorname{Ker} d_L = \operatorname{Im} d_L$ and $m = f(l) \in f(\operatorname{Im} d_L)$.

Then $(\operatorname{Im} d_L)^0 \xrightarrow{f} (\operatorname{Im} d_M)^0 \xrightarrow{g} (\operatorname{Im} d_N)^0$ is a short exact sequence of finitely generated stably free R_{k-1}-modules. Therefore, the map t given by (36) induces a homomorphism $t\colon K_0(\mathcal{E}^w) \to K_0(R_{k-1}) \cong \mathbb{Z}$ which maps $[C(L_k)]$ to 1. It is clear that t and ψ are inverse to each other. We have the following lemma.

Lemma 3.12. *The group $K_0(\mathcal{E}^w) \cong \mathbb{Z}$ with a generator $[C(L_k)]$.*

The map $i\colon K_0(\mathcal{E}^w) \to K_0(\mathcal{E})$ in the exact sequence (33) takes the generator $[C(L_k)]$ to $2[L_k]$, see Lemma 3.11. In particular, i is injective and not surjective.

Proposition 3.13. *The group $K_0(L_k) \cong \mathbb{Z}/2$ with a generator $[L_k]$.*

Proof. Recall from (22) that $K_0'(L_k) \cong K_0(\mathrm{Ho}(\mathcal{C}^{\mathrm{pre}}))$, and $K_0'(L_k) \to K_0(L_k)$ is injective by [32, Corollary 2.3]. The group $K_0'(L_k)$ is nonzero since i is not surjective in the exact sequence (33). It implies that $K_0(L_k)$ is nonzero. Hence the surjection $\eta_k\colon \mathbb{Z}/2 \to K_0(L_k)$ in Lemma 3.10 is an isomorphism. □

Define a map $\rho_k\colon \mathbb{Z} \to K_0(A_k)$ of abelian groups by $\rho_k(1) = [A_k]$.

Proposition 3.14. *The map ρ_k is surjective for all $k \geq 0$.*

Proof. There is a commutative diagram from the exact sequence (28) and Proposition 3.13:

$$\begin{array}{ccccccc} K_0(A_{k-1}) & \xrightarrow{i_{k*}} & K_0(A_k) & \xrightarrow{j_{k*}} & K_0(L_k) & & \\ \uparrow{\scriptstyle \rho_{k-1}} & & \uparrow{\scriptstyle \rho_k} & & \uparrow{\scriptstyle \cong} & & \\ \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \end{array}$$

The first square commutes because $[A_k] = i_{k*}[A_{k-1}] + [A_k[-1]]$ by an analogue of (16). The second square commutes since $j_{k*}[A_k] = [L_k]$. The map ρ_0 is an isomorphism since $A_0 \cong \mathbf{k}$ by (18). By induction on k one can prove that ρ_k is surjective using the snake lemma. □

Taking the direct limit of ρ_k, we have a surjective map $\rho\colon \varinjlim \mathbb{Z} \to \varinjlim K_0(A_k)$ of abelian groups, where $\varinjlim \mathbb{Z} \cong \mathbb{Z}[\frac{1}{2}]$, and $\varinjlim K_0(A_k)$ with respect to i_{k*} is naturally isomorphic to $K_0(\mathcal{C})$ as abelian groups, see (24). The monoidal structure on $D^c(\mathcal{C})$ induces a ring structure on $K_0(\mathcal{C})$, where $[A_k][A_{k'}] = [A_{k+k'}] \in K_0(\mathcal{C})$. Then $\rho\colon \mathbb{Z}[\frac{1}{2}] \to K_0(\mathcal{C})$ is a ring homomorphism such that $\rho(1) = [A_0]$, and $\rho(\frac{1}{2}) = [A_1]$.

Theorem 3.15. *There is a surjective homomorphism of rings*: $\rho\colon \mathbb{Z}[\frac{1}{2}] \to K_0(\mathcal{C})$.

To show that ρ is an isomorphism, we need to assume that $\mathrm{char}(\mathbf{k}) = 2$. The exact sequence (33) gives $K_1(\mathcal{E}^w) \to K_1(\mathcal{E}) \to K_1(L_k) \to 0$, since i is injective.

Proposition 3.16. *If* $\mathrm{char}(\mathbf{k}) = 2$*, then* $K_1(L_k)$ *is* 2*-torsion.*

Proof. It is enough to show that 2α is in the image of $K_1(\mathcal{E}^w) \to K_1(\mathcal{E})$ for any $\alpha \in K_1(\mathcal{E})$. We use Nenashev's presentation of $K_1(\mathcal{E})$ of the exact category $\mathcal{E}$, see [24]. Any $\alpha \in K_1(\mathcal{E})$ is represented by a double short exact sequence

$$M \underset{g_1}{\overset{f_1}{\rightrightarrows}} N \underset{g_1}{\overset{f_1}{\rightrightarrows}} L.$$

Consider the cone $C(M)$ of $\mathrm{id}\colon M \to M$, and morphisms $i_M\colon M \to C(M)$ and $j_M\colon C(M) \to M[1]$ in $\mathcal{E}$. Any morphism $f\colon M \to N$ induces two morphisms $f[1]\colon M[1] \to N[1]$ and $C(f)\colon C(M) \to C(N)$. Let $\alpha[1], C(\alpha) \in K_1(\mathcal{E})$ be the classes of double short exact sequences consisting of $M[1], N[1], L[1]$ and $C(M), C(N), C(L)$, respectively. The diagram

$$\begin{array}{ccccc}
M & \underset{g_1}{\overset{f_1}{\rightrightarrows}} & N & \underset{g_2}{\overset{f_2}{\rightrightarrows}} & L \\
{\scriptstyle i_M}\downarrow\downarrow{\scriptstyle i_M} & & {\scriptstyle i_N}\downarrow\downarrow{\scriptstyle i_N} & & {\scriptstyle i_L}\downarrow\downarrow{\scriptstyle i_L} \\
C(M) & \underset{C(g_1)}{\overset{C(f_1)}{\rightrightarrows}} & C(N) & \underset{C(g_2)}{\overset{C(f_2)}{\rightrightarrows}} & C(L) \\
{\scriptstyle j_M}\downarrow\downarrow{\scriptstyle j_M} & & {\scriptstyle j_N}\downarrow\downarrow{\scriptstyle j_N} & & {\scriptstyle j_L}\downarrow\downarrow{\scriptstyle j_L} \\
M[1] & \underset{g_1[1]}{\overset{f_1[1]}{\rightrightarrows}} & N[1] & \underset{g_2[1]}{\overset{f_2[1]}{\rightrightarrows}} & L[1]
\end{array}$$

satisfies Nenashev's condition in [24, Proposition 5.1]. Hence

$$\alpha - C(\alpha) + \alpha[1] = \beta_M - \beta_N + \beta_L \in K_1(\mathcal{E}),$$

where β_X is the class of the vertical double short exact sequence consisting of $X, C(X), X[1]$ for $X = M, N, L$. By [24, Lemma 3.1], $\beta_X = 0 \in K_1(\mathcal{E})$. So $\alpha + \alpha[1] = C(\alpha)$, which is in the image of $K_1(\mathcal{E}^w) \to K_1(\mathcal{E})$.

If $\mathrm{char}(\mathbf{k}) = 2$, then $L_k = \mathbf{k}\langle a_1^{\pm 1}, \dots, a_k^{\pm 1}\rangle/(a_i a_j = a_j a_i, i \neq j)$ is commutative as in Proposition 3.4. In particular, a_i is central, invertible, closed and of degree 1. Define $h_M \colon M \to M[1]$ by $h_M(m) = ma_1$ which is an endomorphism of M in $\mathcal{E}$. It is an isomorphism since a_1 is invertible. Moreover, $h_N \circ f = f[1] \circ h_M$ for any morphism $f \colon M \to N$ of $\mathcal{E}$. The diagram

$$\begin{array}{ccccc}
M & \overset{f_1}{\underset{g_1}{\rightrightarrows}} & N & \overset{f_2}{\underset{g_2}{\rightrightarrows}} & L \\
{\scriptstyle h_M}\downarrow\downarrow{\scriptstyle h_M} & & {\scriptstyle h_N}\downarrow\downarrow{\scriptstyle h_N} & & {\scriptstyle h_L}\downarrow\downarrow{\scriptstyle h_L} \\
M[1] & \overset{f_1[1]}{\underset{g_1[1]}{\rightrightarrows}} & N[1] & \overset{f_2[1]}{\underset{g_2[1]}{\rightrightarrows}} & L[1] \\
0\downarrow\downarrow 0 & & 0\downarrow\downarrow 0 & & 0\downarrow\downarrow 0 \\
0 & \overset{0}{\underset{0}{\rightrightarrows}} & 0 & \overset{0}{\underset{0}{\rightrightarrows}} & 0
\end{array}$$

satisfies Nenashev's condition in [24, Proposition 5.1]. It implies that $\alpha - \alpha[1] + 0 = 0 - 0 + 0$ by [24, Lemma 3.1]. Hence $\alpha = \alpha[1]$, and 2α is in the image of $K_1(\mathcal{E}^w) \to K_1(\mathcal{E})$ for any $\alpha \in K_1(\mathcal{E})$. □

Remark 3.17. For the field $\mathbf{k}$ of any characteristic, there is a central element $a_1 \cdots a_k$ of degree k if k is odd. So $\alpha = \alpha[k]$, and $K_1(L_k)$ is 2-torsion. But the same argument does not apply if k is even.

Remark 3.18. It can be computed from [29] that $K_0(L_1) \cong \mathbb{Z}/2[L_1]$ and $K_1(L_1) \cong \mathbf{k}^*/(\mathbf{k}^*)^2$ which is 2-torsion for the ground field $\mathbf{k}$ of any characteristic.

Theorem 3.19. *There is an isomorphism of rings* $K_0(\mathcal{C}) \cong \mathbb{Z}[\frac{1}{2}]$ *if* $\mathrm{char}(\mathbf{k}) = 2$.

Proof. We want to show that $\rho_k \colon \mathbb{Z} \to K_0(A_k)$ is an isomorphism by induction on k. It is true when $k = 0$ since $A_0 \cong \mathbf{k}$ by (18). Suppose that ρ_{k-1} is an isomorphism. If $\mathrm{char}(\mathbf{k}) = 2$, then $K_1(L_k)$ is 2-torsion so that the map $\partial \colon K_1(L_k) \to K_0(A_{k-1}) \cong \mathbb{Z}$ is zero. The commutative diagram in the proof of Proposition 3.14 becomes

$$\begin{array}{ccccccccc}
K_1(L_k) & \xrightarrow{\partial=0} & K_0(A_{k-1}) & \xrightarrow{i_{k*}} & K_0(A_k) & \xrightarrow{j_{k*}} & K_0(L_k) & & \\
 & & {\scriptstyle \rho_{k-1}}\uparrow & & {\scriptstyle \rho_k}\uparrow & & {\scriptstyle \cong}\uparrow & & \\
 & & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0
\end{array}$$

The snake lemma implies that ρ_k is an isomorphism, and $K_0(A_k)$ is freely generated by $[A_k]$.

The same argument before Theorem 3.15 shows that $K_0(\mathcal{C}) \cong \mathbb{Z}[\frac{1}{2}]$ as rings. □

Remark 3.20. If degrees of the generators x and y in Figure 11 are exchanged, i.e. $\deg(x) = -1, \deg(y) = 1$, then the resulting derived category will have an isomorphism $X^\wedge \cong \mathbf{1}^\wedge \oplus X^\wedge[1]$ as the analogue of equation (15). Since the quotient algebra L_k is unchanged under the exchange of the degrees, Theorem 3.19 also holds in this case.

If $\deg(x) = m, \deg(y) = -m$ for some odd $m \neq \pm 1$, then in the resulting derived category there is an isomorphism $X^\wedge \cong \mathbf{1}^\wedge \oplus X^\wedge[-m]$, leading to a homomorphism from $\mathbb{Z}[\frac{1}{2}]$ to the Grothendieck ring of that category. We don't know whether it is an isomorphism.

3.7. A p-DG extension. Witten–Reshetikhin–Turaev 3-manifold invariants, when extended to a 3-dimensional TQFT, require working over the ring $\mathbb{Z}\left[\frac{1}{N}, \xi\right] \subset \mathbb{C}$, where ξ is a primitive N-th root of unity. The space associated to a (decorated) surface in the TQFT is a free module over $\mathbb{Z}\left[\frac{1}{N}, \xi\right]$ and the maps associated to cobordisms are $\mathbb{Z}\left[\frac{1}{N}, \xi\right]$-linear. This ring contains the subring $\mathbb{Z}[\xi]$ of cyclotomic integers.

When N is a prime p, the ring $\mathbb{Z}[\xi] \cong \mathbb{Z}[q]/(1+q+\cdots+q^{p-1})$. In the notation, $\xi = e^{\frac{2\pi i}{p}}$ is an element of $\mathbb{C}$ while q is a formal variable, and the isomorphism takes q to ξ. Let us also denote this ring by R_p. Ring R_p admits a categorification, investigated in [16, 27]. One works over a field $\mathbf{k}$ of characteristic p and forms a graded Hopf algebra $H = \mathbf{k}[\partial]/(\partial^p)$, with $\deg(\partial) = 1$. The category of finitely-generated graded H-modules has a quotient category, called the stable category, where morphisms which factor through a projective module are set to 0. The stable category H-$\underline{\text{mod}}$ is triangulated monoidal and its Grothendieck ring $K_0(H\text{-}\underline{\text{mod}})$ is naturally isomorphic to the cyclotomic ring R_p. Multiplication by q corresponds to the grading shift $\{1\}$ in the category of graded H-modules. The shift functor $[1]$ in the triangulated category H-$\underline{\text{mod}}$ is different from the grading shift functor $\{1\}$.

We now explain a conjectural way to enhance this categorification of R_p using a version of isomorphism (2) from the introduction to categorify the ring $\mathbb{Z}\left[\frac{1}{p}, \xi\right]$ which contains both R_p and $\mathbb{Z}[\xi]$ as subrings. The point is that in $\mathbb{Z}[\xi]$ there is an equality of principal ideals

$$(p) = (1-\xi)^{p-1}$$

see [21, Proposition 6.2], so that subrings $\mathbb{Z}\left[\frac{1}{p}, \xi\right]$ and $\mathbb{Z}\left[\frac{1}{1-\xi}, \xi\right]$ of $\mathcal{C}$ coincide (equivalently, localizations $R_p[p^{-1}]$ and $R_p[(1-q)^{-1}]$ are isomorphic). Inverting p is equivalent to inverting $1-\xi$, and the latter can be inverted using a variation of isomorphism (2).

Namely, one would like to have a monoidal category over a field $\mathbf{k}$ of characteristic p with a generating object X and an isomorphism

$$X \cong X\{1\} \oplus \mathbf{1}, \tag{37}$$

where $\{1\}$ is degree shift by one. Having this isomorphism requires the four generating morphisms as in Figure 15, denoted x, y, z, z^* in Figure 11. The degrees are now the opposite, $\deg(x) = -1$, $\deg(y) = 1$, $\deg(z) = \deg(z^*) = 0$. The defining relations are the same, see Figure 12, but now all far-away generating morphisms commute rather than super-commute.

The construction gives rise to a graded pre-additive category $\mathcal{C}$ with objects – tensor powers of X and morphisms being planar diagrams built out of generators subject to defining relations. We make $\mathcal{C}$ into a p-DG category by equipping it with the derivation ∂ of degree 1 that acts by zero on all generating morphisms, hence on all morphisms.

We then extend category $\mathcal{C}$ to a triangulated category, as explained in Section 3.2 for the DG case, by substituting the p-DG version everywhere. We pass to the pre-triangulated p-DG category $\mathcal{C}^{\mathrm{pre}}$ by formally adding iterated tensor products with objects of H-$\underline{\mathrm{mod}}$, finite direct sums and cones of morphisms. Shifts of objects are included in this construction, since they are isomorphic to tensor products with one-dimensional graded H-modules. The homotopy category $\mathrm{Ho}(\mathcal{C}^{\mathrm{pre}})$ is triangulated, and we define $\widetilde{\mathcal{C}}$ to be its idempotent completion. The category $\widetilde{\mathcal{C}}$ is triangulated monoidal Karoubi closed, and there is a natural ring homomorphism

$$\mathbb{Z}\left[\frac{1}{p}, \xi\right] \longrightarrow K_0(\widetilde{\mathcal{C}}) \tag{38}$$

taking $(1-\xi)^{-1}$ to $[X]$.

Problem 3.21. *Is the map* (38) *an isomorphism*?

Beyond this problem, there is an open question whether category $\widetilde{\mathcal{C}}$ can be used to enhance known categorifications of quantum groups at prime roots of unity and to help with categorification of the Witten–Reshetikhin–Turaev 3-manifold invariants at prime roots.

4. A monoidal envelope of Leavitt path algebras

The goal of this section is to describe an additive monoidal Karoubi closed category $\mathcal{C}$ whose Grothendieck ring is conjecturally isomorphic to $\mathbb{Z}[\frac{1}{2}]$.

4.1. Category $\mathcal{C}$. Let $\mathbf{k}$ be a field. Consider a $\mathbf{k}$-linear pre-additive strict monoidal category $\mathcal{C}$ with one generating object X, in addition to the unit object $\mathbf{1}$. A set of generating morphisms is given in Figure 18.

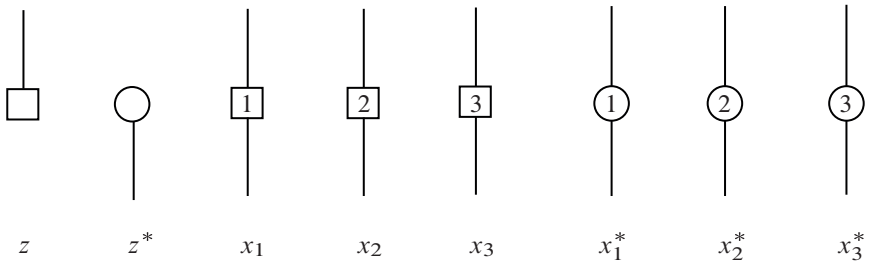

Figure 18. Generating morphisms.

Six of these eight generating morphisms are endomorphisms of X, one is a morphism from $\mathbf{1}$ to X, and the last morphism goes from X to $\mathbf{1}$. We denote these generators by z, z^*, x_i, x_i^* for $i = 1, 2, 3$, from left to right in Figure 18. In particular, x_i, x_i^* are endomorphisms of X, z a morphism from $\mathbf{1}$ to X, and z^* a morphism from X to $\mathbf{1}$. We draw x_i as a long strand decorated by a box with label i, x_i^* as a long strand decorated by a circle with label i, and z as a short top strand decorated by an empty box, and z^* as a short bottom strand decorated by an empty circle, respectively.

A pair of far away generators commute. Therefore, a horizontal composition of diagrams is independent of their height order. Given two diagrams f, g, let $f \otimes g$ denote the horizontal composition of f and g, where f is on the left of g.

Local relations are given in Figure 19, where the vertical line is 1_X, and the empty diagram is $1_{\mathbf{1}}$. The relations can be written as

$$\begin{aligned} &z^* z = 1_{\mathbf{1}}, \\ &x_i^* z = 0, \quad z^* x_i = 0, \quad \text{for } i = 1, 2, 3, \\ &x_i^* x_j = \delta_{i,j} 1_X, \qquad \text{for } i, j = 1, 2, 3, \\ &\sum_{i=1}^{3} x_i x_i^* + z z^* = 1_X. \end{aligned} \tag{39}$$

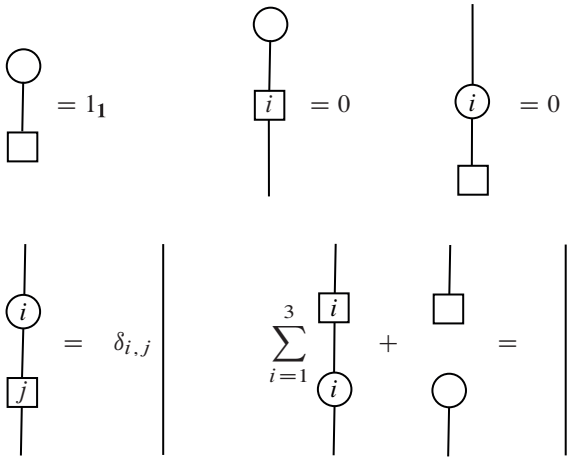

Figure 19. Defining local relations.

Let

$$\Lambda = \bigsqcup_{k\geq 0} \Lambda^k = \bigsqcup_{k\geq 0} \{1,2,3\}^k$$

be the set of sequences of indices, where Λ^0 consists of a single element of the empty sequence. For $I = (i_1, \dots, i_k) \in \Lambda^k$, let $I^* = (i_k, \dots, i_1)$, and $|I| = k$. Let $x_I, x_I^* \in \mathrm{End}_{\mathcal{C}}(X)$ denote compositions $x_{i_1} \cdots x_{i_k}$ and $x_{i_1}^* \cdots x_{i_k}^*$. Let $z_I = x_I z \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X)$ and $z_I^* = z^* x_I^* \in \mathrm{Hom}_{\mathcal{C}}(X, \mathbf{1})$. If $|I| = |J|$, then $x_I^* x_J = \delta_{I,J^*} 1_X$, and subsequently $z_I^* z_J = \delta_{I,J^*} 1_{\mathbf{1}}$.

Figure 20 shows our notations for some vertical compositions of generating morphisms. We draw x_I as a long strand decorated by a box with label I, and x_I^* as a long strand decorated by a circle with label I, respectively. We draw z_I as a short top strand decorated by a box with label I, and z_I^* as a short bottom strand decorated by a circle with label I, respectively.

Bases of morphism spaces. We observe that the category $\mathcal{C}$ is generated from the suitable data (A, e) as described in Section 2, where A is a $\mathbf{k}$-algebra. To see this, we restrict the above diagrams and defining relations on them to the case when there is at most one strand at the top and at most one strand at the bottom. In other words, we consider generating morphisms in Figure 18 and compose them only vertically, not horizontally, with the defining relations in Figure 19. The idempotent e is given by the empty diagram, while $1-e$ is the undecorated vertical strand diagram.

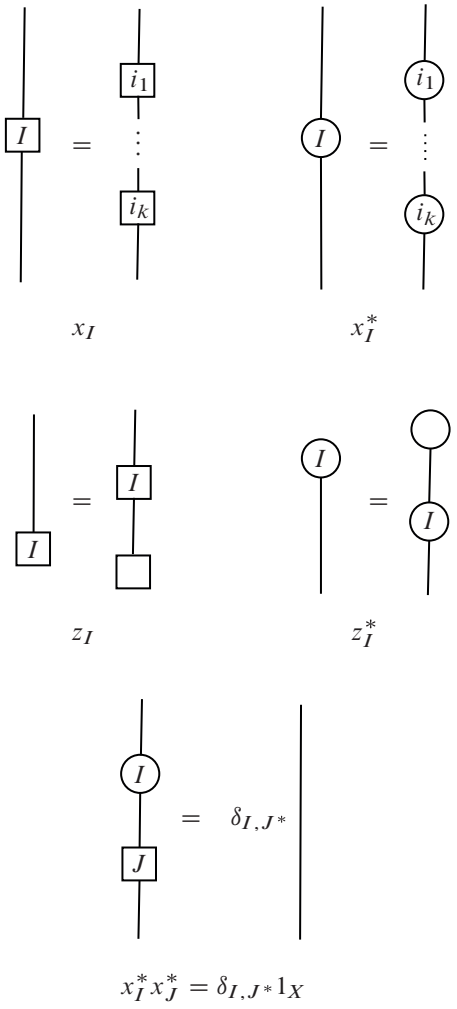

Figure 20. Notations for compositions x_I, x_I^*, z_I, z_I^* for $I = (i_1, \ldots, i_k)$, and the relation $x_I^* x_J = \delta_{I,J^*} 1_X$ if $|I| = |J|$.

It follows from the defining relations (39) that A is isomorphic to the Leavitt path algebra $L(Q)$ of the graph Q in (4). In particular, as a $\mathbf{k}$-vector space, algebra A has a basis

$$\begin{aligned}&\{1_{\mathbf{1}}\} \cup \{z_I \mid I \in \Lambda\}\\ &\quad \cup \{z_I^* \mid I \in \Lambda\} \cup \{z_I z_J^* \mid I, J \in \Lambda\}\\ &\quad \cup \{x_I\ x_J^* \mid I, J \in \Lambda, (i_{|I|}, j_1) \neq (3,3)\}\end{aligned}$$

by [2, Corollary 1.5.12]. Our notations for some of these basis elements are shown in Figure 20.

The basis of A can be split into the following disjoint subsets:

(1) $eAe \cong \mathbf{k}$ has a basis

$$\mathbb{B}_{0,0} = \{1_{\mathbf{1}}\}$$

consisting of a single element which is the empty diagram;

(2) $(1-e)Ae$ has a basis

$$\mathbb{B}_{1,0} = \{z_I \mid I \in \Lambda\}.$$

Element z_I is depicted by a short top strand decorated by a box with label I, see Figure 20;

(3) $eA(1-e)$ has a basis

$$\mathbb{B}_{0,1} = \{z_I^* \mid I \in \Lambda\}.$$

Element z_I^* is depicted by a short bottom strand decorated by a circle with label I (lollipop in Figure 20);

(4) $(1-e)A(1-e)$ has a basis

$$\mathbb{B}_{1,1}(0) \sqcup \mathbb{B}_{1,1}(1),$$

where

$$\mathbb{B}_{1,1}(0) = \{z_I z_J^* \mid I, J \in \Lambda\}$$

consists of pairs (short top strand with a labelled box, short bottom strand with a labelled circle), and

$$\mathbb{B}_{1,1}(1) = \{x_I\, x_J^* \mid I, J \in \Lambda, (i_{|I|}, j_1) \neq (3,3)\}$$

consists of long strands whose decoration satisfies that no circle is above any box, and no box with label 3 is next to a circle with label 3.

The multiplication map $(1-e)Ae \otimes eA(1-e) \to (1-e)A(1-e)$ sends the basis $\mathbb{B}_{1,0} \times \mathbb{B}_{0,1}$ of $(1-e)Ae \otimes eA(1-e)$ bijectively to $\mathbb{B}_{1,1}(0)$ so that the multiplication map is injective.

We see that the conditions on (A, e) from the beginning of Section 2 are satisfied, and we can indeed form the monoidal category $\mathcal{C}$ as above with objects $X^{\otimes n}$, over $n \geq 0$. Algebra A can then be described as the direct sum

$$A \;\cong\; \mathrm{End}_{\mathcal{C}}(\mathbf{1}) \oplus \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X) \oplus \mathrm{Hom}_{\mathcal{C}}(X, \mathbf{1}) \oplus \mathrm{End}_{\mathcal{C}}(X).$$

Therefore, a basis of $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$ is given in Theorem 2.1.

The idempotent completion of $\mathcal{C}$. Recall that $\mathcal{C}^{\mathrm{add}}$ denotes the additive closure of $\mathcal{C}$, and $\mathrm{Ka}(\mathcal{C})$ denotes the idempotent completion of $\mathcal{C}^{\mathrm{add}}$. Objects of $\mathcal{C}^{\mathrm{add}}$ are finite formal direct sums of nonnegative powers $X^{\otimes n}$, where $X^{\otimes 0} = \mathbf{1}$. The category $\mathrm{Ka}(\mathcal{C})$ is $\mathbf{k}$-linear additive strict monoidal.

Defining local relations are chosen to have an isomorphism in $\mathrm{Ka}(\mathcal{C})$:

$$X \cong \mathbf{1} \oplus X^3, \tag{40}$$

given by

$$(z^*, x_1^*, x_2^*, x_3^*)^T \in \mathrm{Hom}_{\mathrm{Ka}(\mathcal{C})}(X, \mathbf{1} \oplus X^3),$$

and

$$(z, x_1, x_2, x_3) \in \mathrm{Hom}_{\mathrm{Ka}(\mathcal{C})}(\mathbf{1} \oplus X^3, X),$$

see Figure 21. Tensoring with $X^{\otimes(k-1)}$ in $\mathrm{Ka}(\mathcal{C})$ on either side of isomorphism (40) results in isomorphisms in $\mathrm{Ka}(\mathcal{C})$

$$X^{\otimes k} \cong X^{\otimes(k-1)} \oplus (X^{\otimes k})^3. \tag{41}$$

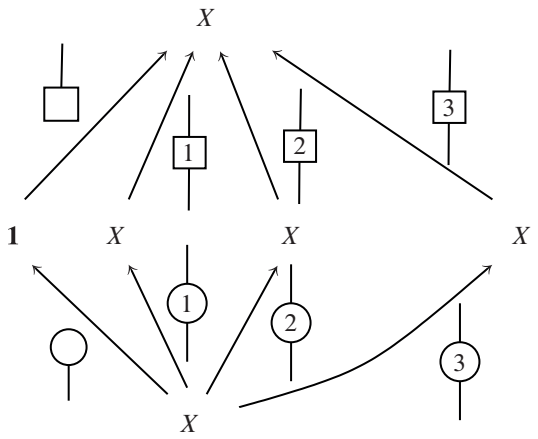

Figure 21. The isomorphism $X \cong \mathbf{1} \oplus X^3$ in $\mathrm{Ka}(\mathcal{C})$.

Algebras of endomorphisms. Part of the structure of $\mathrm{Ka}(\mathcal{C})$ can be encoded into an idempotented algebra B, which has a complete system of mutually orthogonal idempotents $\{1_n\}_{n\geq 0}$, so that

$$B = \bigoplus_{m,n\geq 0} 1_m B 1_n,$$

and

$$1_m B 1_n = \mathrm{Hom}_{\mathrm{Ka}(\mathcal{C})}(X^{\otimes n}, X^{\otimes m}).$$

Multiplication in B matches composition of morphisms in $\mathcal{C}$.

We also define

$$B_k = \bigoplus_{m,n \leq k} 1_m B 1_n,$$

which is an algebra with the unit element $\sum_{n \leq k} 1_n$. The inclusion $B_k \subset B_{k+1}$ is nonunital. The algebra $B_0 \cong \mathbf{k}$. Define

$$A_k = 1_k B 1_k = \mathrm{End}_{\mathrm{Ka}(\mathcal{C})}(X^{\otimes k}),$$

which is an algebra with the unit element 1_k. The inclusion $A_k \subset B_k$ is nonunital for $k > 0$.

Let $\alpha_k \colon A_{k-1} \hookrightarrow A_k$ be an inclusion of algebras given by tensoring with zz^* on the left

$$\alpha_k(f) = (zz^*) \otimes f, \tag{42}$$

for $f \in A_{k-1}$. Note that α_k is nonunital.

4.2. Towards computing $K_0(\mathrm{Ka}(\mathcal{C}))$. Let $\mathcal{C}_k$ be the smallest full subcategory of $\mathcal{C}$ which contains the objects $X^{\otimes n}$, $0 \leq n \leq k$. Let $\mathrm{Ka}(\mathcal{C}_k)$ be the idempotent completion of the additive closure $\mathcal{C}_k^{\mathrm{add}}$ of $\mathcal{C}_k$. There is a family of inclusions of additive categories $\mathrm{Ka}(\mathcal{C}_{k-1}) \subset \mathrm{Ka}(\mathcal{C}_k)$. Similarly, there is a family of inclusions $g_k \colon \mathrm{Ka}(\mathcal{C}_k) \to \mathrm{Ka}(\mathcal{C})$ of additive categories. We have the analogue of Proposition 3.5.

Proposition 4.1. *There is a natural isomorphism of abelian groups*

$$K_0(\mathrm{Ka}(\mathcal{C})) \cong \varinjlim K_0(\mathrm{Ka}(\mathcal{C}_k)).$$

For a unital algebra A, let $\mathcal{P}(A)$ denote the additive category of finitely generated projective right A-modules. Let $K_0(A)$ denote the split Grothendieck group of $\mathcal{P}(A)$. The category $\mathcal{C}_k$ contains a full subcategory $\mathcal{C}'_k$ with a single object $X^{\otimes k}$ whose endomorphism algebra $\mathrm{End}_{\mathcal{C}_k}(X^{\otimes k}) = A_k$. Let $\mathrm{Ka}(\mathcal{C}'_k)$ be the idempotent completion of the additive closure $\mathcal{C}'^{\,\mathrm{add}}_k$ of $\mathcal{C}'_k$. Thus, the category $\mathrm{Ka}(\mathcal{C}'_k)$ is isomorphic to $\mathcal{P}(A_k)$. There is an inclusion $h_k \colon \mathcal{P}(A_k) \subset \mathrm{Ka}(\mathcal{C}_k)$ of additive categories. Isomorphism (41) implies that h_k is an equivalence. Therefore,

$h_{k*}\colon K_0(A_k) \to K_0(\mathrm{Ka}(\mathcal{C}_k))$ is an isomorphism. By Proposition 4.1, there is a natural isomorphism of abelian groups:

$$K_0(\mathrm{Ka}(\mathcal{C})) \cong \varinjlim K_0(A_k). \tag{43}$$

Here A_k is just a $\mathbf{k}$-algebra without the grading and the differential, and $K_0(A_k)$ is the usual Grothendieck group of the ring. The major part of the complexity in this construction lies in dealing with algebras of exponential growth, including the endomorphism algebra A_k of the object $X^{\otimes k}$.

An approach to $K_0(A_k)$. Recall the chain of two-sided ideals $J_{n,k}$ from (12), and the quotient algebra $L_k = A_k/J_{k-1,k}$ from (13) in Section 2. The algebra L_k is naturally isomorphic to $L^{\otimes k}$ for $L = L_1$.

For $k = 1$, $A_1 = J_{1,1}$ has a basis $\mathbb{B}_{1,1} = \mathbb{B}_{1,1}(0) \sqcup \mathbb{B}_{1,1}(1)$, and $J = J_{0,1}$ has a basis $\mathbb{B}_{1,1}(0)$ with respect to the inclusion $J_{0,1} \subset A_1$, by Corollary 2.2. Under the quotient map $A_1 \to L$, the set $\mathbb{B}_{1,1}(1)$ is mapped bijectively to the *normal form* basis of $L = L_1$, see [9, Section 5]. The algebra L is naturally isomorphic to the Leavitt algebra $L(1,3)$. Thus,

$$L_k \cong L(1,3)^{\otimes k}.$$

If we view A_k as a DG algebra concentrated in degree 0 with the trivial differential, the analogue of Lemma 3.7 still holds. Therefore, there is an induced exact sequence of K-groups

$$K_1(L_k) \xrightarrow{\partial} K_0(A_{k-1}) \xrightarrow{i_{k*}} K_0(A_k) \xrightarrow{j_{k*}} K_0(L_k). \tag{44}$$

Conjecture 4.2. *For $k \geq 1$, $K_0(L_k)$ is isomorphic to $\mathbb{Z}/2$ with a generator $[L_k]$, and $K_1(L_k)$ is torsion.*

The Leavitt algebra L_1 is regular supercoherent by [4, Lemma 6.1]. The conjecture is known to be true for $k = 1, 2$, see [3, Theorem 7.6].

By an argument similar to that in the proof of Theorem 3.19, if Conjecture 4.2 is true, then there is a ring isomorphism

$$K_0(\mathrm{Ka}(\mathcal{C})) \cong \mathbb{Z}\left[\tfrac{1}{2}\right].$$

Categorical actions of $\mathrm{Ka}(\mathcal{C})$. There is an action $F_m\colon \mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}/(2m+1) \to \mathbb{Z}/(2m+1)$ of the ring $\mathbb{Z}[\frac{1}{2}]$ on the abelian group $\mathbb{Z}/(2m+1)$, where $-\frac{1}{2}$ acts as multiplication by m.

Recall that $L(m,n)$ is the $\mathbf{k}$ algebra generated by entries of x_{ij}, y_{ij} of matrices $X = (x_{ij}), Y = (y_{ij})$ of size $m \times n$ and $n \times m$ respectively, subject to the relation: $XY = I_m, YX = I_n$. The algebra $L(m,n)$ is the universal object with respect to the non-IBN (Invariant Basis Number) property: $R^m \cong R^n$. It is known [5] that $K_0(L(m,n)) \cong \mathbb{Z}/(n-m)$ generated by the class $[L(m,n)]$.

There is a family of categorical actions

$$\mathcal{F}_m \colon \mathrm{Ka}(\mathcal{C}) \times \mathcal{P}(L(m,3m+1)) \longrightarrow \mathcal{P}(L(m,3m+1))$$

of $\mathrm{Ka}(\mathcal{C})$ on the category of finitely generated projective right $L(m,3m+1)$-modules, where the generating object X of $\mathrm{Ka}(\mathcal{C})$ acts by tensoring with the $L(m,3m+1)$-bimodule $L(m,3m+1)^m$. Conjecturally, $\mathcal{F}_m$ categorifies the linear action F_m.

4.3. A possible categorification of $\mathbb{Z}\left[\frac{1}{n}\right]$. We consider a pre-additive $\mathbf{k}$-linear strict monoidal category $\mathcal{C}$ with a generating object X in addition to the unit object $\mathbf{1}$ and require the following isomorphism

$$X \cong \mathbf{1} \oplus X^{n+1}. \tag{45}$$

This isomorphism is a natural generalization of isomorphism (40). Generating morphisms that induce these mutually-inverse isomorphisms are denoted $z \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X), z^* \in \mathrm{Hom}_{\mathcal{C}}(X, \mathbf{1})$, and $x_i, x_i^* \in \mathrm{End}_{\mathcal{C}}(X)$ for $1 \le i \le n+1$. The defining relations, generalizing relations (39), are

$$\begin{aligned} &z^* z = 1_{\mathbf{1}}, \\ &x_i^* z = 0, \quad z^* x_i = 0, \quad \text{for } 1 \le i \le n+1, \\ &x_i^* x_j = \delta_{i,j} 1_X, \qquad \text{for } 1 \le i, j \le n+1, \\ &\sum_{i=1}^{n+1} x_i x_i^* + z z^* = 1_X. \end{aligned} \tag{46}$$

Let A denote the algebra generated by $1_{\mathbf{1}}, 1_X, z, z^*, x_i, x_i^*$, $1 \le i \le n+1$, subject to the relations above and the obvious compatibility relations between generators z, z^*, x_i, x_i^* and idempotents $1_{\mathbf{1}}, 1_X$, for instance, $z 1_{\mathbf{1}} = z = 1_X z$ and $z 1_X = 0 = 1_{\mathbf{1}} z$. . The data $(A, e = 1_{\mathbf{1}})$ satisfies the conditions described at the beginning of Section 2. Thus, we can form a pre-additive monoidal category $\mathcal{C}$ and recover bases of morphisms between tensor powers of X from suitable bases of A compatible with the idempotent decomposition $1 = e + (1-e)$, as explained in Section 2. Let $\mathrm{Ka}(\mathcal{C})$ denote the idempotent completion of the additive closure of $\mathcal{C}$. Category $\mathrm{Ka}(\mathcal{C})$ is an additive $\mathbf{k}$-linear Karoubi closed monoidal category.

Conjecture 4.3. *There is a ring isomorphism* $K_0(\mathrm{Ka}(\mathcal{C})) \cong \mathbb{Z}\left[\frac{1}{n}\right]$.

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