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Diagrammatic categorification of the Chebyshev polynomials of the second kind

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ABSTRACT

We develop a diagrammatic categorification of the polynomial ring Z[x], based on a geometrically defined graded algebra. This construction generalizes to categorification of some special functions, such as Chebyshev polynomials. Diagrammatic algebras featured in these categorifications lead to the first topological interpretations of the Bernstein-Gelfand-Gelfand reciprocity property.

1. Introduction

When categorifying a vector space with an additional data, such as the structure of a ring or a module over a ring, it's useful to have a bilinear form on the space. Upon categorification, it may turn into the form coming from the dimension of hom spaces between projective objects of the category or from the Euler characteristic of the Ext groups between arbitrary objects.

Categorification of rings are monoidal categories, with an underlying structure of an abelian or a triangulated category, so one can form the Grothendieck group and then equip it with the multiplication coming from the tensor product on the category.

One of the simplest rings to consider for a categorification is the polynomial ring $\mathbb{Z}[x]$ in one variable x. One can try to categorify various inner product on this ring. In [1] the authors considered an example of a categorification for the inner product $(x^n, x^m) = \binom{n+m}{m}$.

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In this paper we look at a categorification of $\mathbb{Z}[x]$ for the inner product corresponding to the Chebyshev polynomials of the second kind (13). Monomials x^n will become objects P_n of an additive category, while the inner product (x^n, x^m) will turn into the vector space $\mathrm{Hom}(P_n, P_m)$. We choose a field $\mathbf{k}$ and define $\mathrm{Hom}(P_n, P_m)$ as the $\mathbf{k}$-vector space with the basis of crossingless matching diagrams in the plane with n points on the left and m points on the right.

To construct a category out of these vector spaces one needs associative compositions

$$\mathrm{Hom}(P_n, P_m) \otimes \mathrm{Hom}(P_m, P_k) \longrightarrow \mathrm{Hom}(P_n, P_k).$$

The standard composition of this kind describes the Temperley-Lieb category. The composition is then given by concatenating diagrams, allowing isotopies rel boundary and removing a closed circle simultaneously with multiplying the diagram by $-q - q^{-1}$. As a special case when $n = m$ one recovers the n-stranded the Temperley-Lieb algebra TL_n [2]. The Temperley-Lieb category allows to extend the Jones polynomial (which coincides with the one-variable Kauffman bracket) to tangles. The Temperley-Lieb (TL) algebra has many idempotents. If we adopt the composition rules of TL algebra, the objects P_n will have many idempotent endomorphisms and will decompose into a direct sum of smaller objects if the ambient category is abelian (at least if quantum $[n]!$ is invertible in the ground ring). Either way, they will split into direct summands in the Karoubian envelope of the original additive category. In the Temperley-Lieb case and for generic q, the Grothendieck ring of the resulting category can also be identified with $\mathbb{Z}[x]$, with symbols on P_n corresponding to x^n. For generic q the category of semisimple, with simple objects corresponding to the Chebyshev polynomials.

In this paper we consider a different case, which can be viewed as a sort of frozen limit of the Temperley-Lieb category, where diagrams that contain a circle or a pair of U-turns that normally can be straightened up evaluate to zero. Section 2.4 discusses this two-parameter family of categories, isomorphic to the Temperley-Lieb categories at nonzero values of the parameters. Specialization $t = d = 0$ produces a monoidal category that we consider in the present paper. This category and its representations lead to a new categorification of the Chebyshev polynomials of the second kind described in Section 3, see Theorem 3.5. In Section 4 we further analyze the Chebyshev monoidal category and prove that it provides a topological interpretation for the Bernstein-Gelfand-Gelfand reciprocity property in that case, see Corollary 4.6. We also show in Corollary 4.10 that the corresponding algebra is Koszul. Finally, Section 2 provides necessary background on the Temperley-Lieb category, Chebyshev polynomials, and idempotented rings and Grothendieck rings.

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2. Background and motivation

2.1. Idempotented rings and Grothendieck groups

An idempotented ring $(A, \{1_i\}_{i\in S})$ is a ring A, non-unital in general, equipped with a set of pairwise orthogonal idempotents $\{1_i\}_{i\in S}$ $(1_i 1_j = \delta_{i,j} 1_i)$ such that $A = \bigoplus_{i,j\in S} 1_i A 1_j$. One can visualize an idempotented ring as a generalized matrix algebra, with rows and columns enumerated by elements of S, and the abelian group $1_i A 1_j$ sitting on the intersection of the i-th row and j-th column.

$$A = \begin{pmatrix} & \cdot & \\ & \cdot & \\ & \cdot & \\ \cdot \quad \cdot & 1_i A 1_j & \cdot \quad \cdot \\ & \cdot & \end{pmatrix}$$

Without loss of generality, one can impose the condition that the idempotents are non-zero (switching between the two versions of the definition amounts to discarding zero idempotents $1_i = 0$).

A is a unital ring if and only if the set of non-zero idempotents in S is finite; then $1 = \sum_{i \in S} 1_i$. We call $\{1_i\}_{i \in S}$ an *idempotent system*. Forgetting the actual system of idempotents leads to the notion of a *ring with enough idempotents*, that is, a ring admitting such a system, see [3] for instance.

Idempotented rings can be encoded by preadditive categories. A category is preadditive if for any two objects i, j the set $\mathrm{Hom}(i, j)$ is an abelian group, with bilinear composite maps

$$\mathrm{Hom}(i, j) \times \mathrm{Hom}(j, k) \to \mathrm{Hom}(i, k).$$

An idempotented ring A gives rise to a small preadditive category $\mathcal{A}$ with objects i, for $i \in S$, morphisms from i to j being $1_i A 1_j$, and composition of morphisms coming from the multiplication in A:

$$1_i A 1_j \times 1_j A 1_k \xrightarrow{m} 1_i A 1_k. \tag{1}$$

Vice versa, a small preadditive category $\mathcal{A}$ gives rise to the idempotented ring

$$A = \bigoplus_{i,j \in Ob(\mathcal{A})} \mathrm{Hom}_{\mathcal{A}}(i, j).$$

A (left) module M over idempotented ring A is called unital if $M = \bigoplus_{i \in S} 1_i M$. In the rest of the paper by a module we mean a unital left module unless specified otherwise. A (left) A-module M is called finitely-generated if there exist finitely many $m_1, \ldots, m_n \in M$ such that $M = Am_1 + Am_2 + \ldots + Am_n$. Idempotented ring A is called (left) Noetherian if any submodule of a finitely-generated (left) module is finitely generated. Viewed as a left module over itself, A is finitely generated if and only if it is unital, that is, if the system of idempotents is finite, $|S| < \infty$.

A module is called projective if it is a projective object in the category of A-modules. The notions of unital, projective, finitely-generated module do not depend on the choice of system of idempotents $\{1_i\}_{i \in S}$.

An idempotent $e \in A$ gives rise to a finitely-generated projective module Ae. For any $x \in A$ there exists a finite subset $T \subset S$ such that $x \in 1_T A 1_T$, where $1_T := \sum_{i \in T} 1_i$. Equivalently, $1_T x = x = x 1_T$. A minimal such T is unique; we can denote it by $T(x)$.

Definition 2.1. Given an algebra A its Grothendieck group $K_0(A)$ is a free abelian group with generators – symbols $[P]$ of finitely-generated (left) projective A-modules P and defining relations $[P] = [P_1] + [P_2]$ whenever $P \cong P_1 \oplus P_2$.

There is a canonical isomorphism between the Grothendieck group of an idempotented ring A and the direct limit of Grothendieck groups of unital rings $1_T A 1_T$, where T ranges over finite subsets of S:

$$K_0(A) \cong \lim_{\substack{T \subset S \\ |T| < \infty}} K_0(1_T A 1_T), \tag{2}$$

with inclusions $T_1 \subset T_2$ giving rise to homomorphisms $1_{T_1} A 1_{T_1} \to 1_{T_2} A 1_{T_2}$ and induced maps $K_0(1_{T_1} A 1_{T_1}) \to K_0(1_{T_2} A 1_{T_2})$.

For an idempotent e, $1_{T(e)} - e$ is an idempotent too, orthogonal to e,

$$e + (1_{T(e)} - e) = 1_{T(e)} = \sum_{i \in T(e)} 1_i,$$

and $Ae \oplus A(1_{T(e)} - e) \cong A1_{T(e)} \cong \bigoplus_{i \in T(e)} A1_i$.

For $i \in S$ define $P_i := A1_i$ and note that modules P_i are projective.

Proposition 2.2. *A projective A-module P is finitely-generated iff it is isomorphic to a direct summand of a finite direct sum of projective modules of the form $A1_i$, for some idempotents $i_1, \dots, i_n \in S$:*

$$P \oplus Q \cong \bigoplus_{k=1}^{n} A1_{i_k}, \quad i_1, \dots, i_n \in S. \tag{3}$$

Equivalently, P is a direct summand of $(A1_T)^m$, for some m and a finite subset $T \subset S$.

Proof. P has finitely many generators, and each generator is a finite sum of terms in $1_i P$, over various i's. We can thus assume that each generator p_i of P is in $1_i P$, for some i. There is a map $P_i \longrightarrow P$ taking $1_i \in P_i$ to p_i. The sum of these maps over all generators of P is a surjective map from $\oplus_i P_i$ to P. Since P is projective, this map splits. We can now take as T the union of i's and for m the number of generators. □

Left, right, and 2-sided ideals in idempotented rings are defined in the same way as for usual rings. A left ideal $I \subset A$ is an abelian subgroup, closed under the left action of A, $AI \subset A$. Note that ideals in A respect idempotent decomposition. For instance, a 2-sided ideal I satisfies:

$$I = \bigoplus_{i,j \in S} 1_i I 1_j.$$

The center $Z(A)$ of A is defined as the commutative ring of additive natural transformations of the identity functor

$$Id : A\text{−mod} \longrightarrow A\text{−mod}$$

(on either the category of left or right A-modules). Elements of $Z(A)$ are in bijection with collections $\{x_i | x_i \in 1_i A 1_i,\ i \in S,$ and $\forall i, j \in S, \forall y \in 1_i A 1_j,\ x_i y = y x_j\}$.

Given a field $\mathbf{k}$, we say that an idempotented ring A is a $\mathbf{k}$-algebra if the abelian groups $1_i A 1_j$ are naturally $\mathbf{k}$-vector spaces, over all $i, j \in S$, and multiplications (1) are $\mathbf{k}$-bilinear for all i, j, k. Such A will be called an idempotented $\mathbf{k}$-algebra.

An idempotented $\mathbf{k}$-algebra A is *locally finite-dimensional* (lfd, for short) if $1_i A 1_j$ are finite-dimensional $\mathbf{k}$-vector spaces for all $i, j \in S$.

Proposition 2.3. *Finitely-generated modules over an idempotented lfd $\mathbf{k}$-algebra have the Krull-Schmidt property.*

Proof. The Krull-Schmidt property for a module M is the uniqueness of a decomposition of M into a direct sum of indecomposable modules. Let A be an idempotented lfd $\mathbf{k}$-algebra. A sufficient condition for this property to hold is for $\mathrm{End}_A(M)$ to be a finite-dimensional $\mathbf{k}$-algebra. Any finitely-generated A-module M is a quotient of a finite direct sum of modules P_i. Let P be such a finite sum surjecting onto M. Then

$\mathrm{End}_A(M)$ is a subspace in $\mathrm{Hom}_A(P, M)$. Since P is projective, the natural map $\mathrm{End}_A(P) \longrightarrow \mathrm{Hom}_A(P, M)$ is surjective, and finite-dimensionality of $\mathrm{End}_A(M)$ follows from finite-dimensionality of $\mathrm{End}_A(P)$. Algebra $\mathrm{End}_A(P)$ is finite dimensional, since it is isomorphic to a finite direct sum of vector spaces of the form $1_i A 1_j$, which are finite-dimensional. □

This result, restricted to projective finitely-generated modules, shows that any such module is a direct summand of $P_i = A1_i$, for some i.

Corollary 2.4. *The Grothendieck group $K_0(A)$ of finitely-generated projective modules over a locally finite dimensional idempotented* **k***-algebra A is a free abelian group with a basis given by symbols $[P]$ of indecomposable projective A-modules, one for each isomorphism class.*

Through the rest of the paper we assume that the idempotented **k**-algebra A is lfd.

The ring $1_i A 1_i$ is finite-dimensional, hence Artinian. Its Jacobson radical $J(1_i A 1_i)$ is nilpotent, and the quotient algebra $1_i A 1_i / J(1_i A 1_i)$ is semisimple. Any idempotent decomposition of the unit element 1 in the quotient ring lifts to a decomposition of 1_i in $1_i A 1_i$. For each i choose a decomposition $1_i = 1_{i,1} + 1_{i,2} + \cdots + 1_{i,r_i}$ of this idempotent into the sum of primitive mutually-orthogonal idempotents $1_{i,j} \in 1_i A 1_i$, $1 \leq j \leq r_i$. We refine the idempotent system $\{1_i\}_{i \in S}$ into the idempotent system made of $1_{i,j}$ over all such i, j, and denote this system by $\widetilde{S}$. The idempotented **k**-algebra $(A, \widetilde{S})$ is also lfd.

Recall that idempotents e_1, e_2 in a ring B are called equivalent if there are elements $x, y \in B$ such that $e_1 = xy$, $e_2 = yx$. Idempotents e_1, e_2 are equivalent iff projective B-modules Be_1, Be_2 are isomorphic. This notion of equivalence trivially generalizes to idempotented rings. In particular, some of the idempotents in A might be equivalent. Idempotents $1_{\widetilde{i}}$, over $\widetilde{i} \in \widetilde{S}$, decompose into equivalence classes. Choose one representative i' for each equivalence class, denote the set of such i''s by S', and define

$$A' = \bigoplus_{i', j' \in S'} 1_{i'} A 1_{j'}. \tag{4}$$

Idempotented **k**-algebra A' is a subalgebra of A. Its idempotented system is $\{1_{i'}\}_{i' \in S'}$, and (A', S') is lfd. Algebras A and A' are Morita equivalent. In particular, their categories of representations $A-$mod and $A'-$mod are equivalent and their Grothendieck groups are isomorphic.

Idempotented lfd **k**-algebra (A', S') has the property that all rings $1_{i'} A' 1_{i'}$ are local, and multiplication (1) takes tensor product $1_{i'} A' 1_{j'} \otimes 1_{j'} A' 1_{i'}$ into the Jacobson radical of $1_{i'} A' 1_{i'}$ for all $i', j' \in S'$, $i' \neq j'$. We call such an idempotented lfd algebra *basic* and shorten *basic lfd* to blfd. Equivalently, an idempotented lfd **k**-algebra (A', S') is basic if projective modules $A' 1_{i'}$ are indecomposable and pairwise non-isomorphic. Moreover, any idempotented lfd **k**-algebra is Morita equivalent to a basic one.

Corollary 2.5. *The Grothendieck group $K_0(A)$ of an idempotented blfd* **k***-algebra (A, S) is free abelian, with a basis consisting of symbols $[P_i]$ of indecomposable projective modules $P_i = A1_i$.*

Modules P_i are pairwise non-isomorphic, have a unique maximal proper submodule, and a unique simple quotient, denoted L_i. Module L_i is concentrated in position i, in the sense that $1_i L_i = L_i$, so that $1_j L_i = 0$ for $j \neq i$. Any simple A-module L is isomorphic to some L_i, for a unique i. $\mathrm{End}_A(L_i)$ is a finite-dimensional division algebra over **k**.

If (A, S) is an idempotented lfd **k**-algebra (not necessarily basic), there is still a bijection between indecomposable projective A-modules P_u, labeled by elements of some index set U, and simple modules L_u, labeled by elements of the same set, with the property that $\mathrm{Hom}_A(P_u, L_v) = 0$ unless $u = v$ and $\mathrm{Hom}_A(P_u, L_u) \cong \mathrm{End}_A(L_u, L_u)$. Thus, L_u is the unique simple quotient of P_u. For nonbasic algebras it is

possible for a simple module to be infinite dimensional over $\mathbf{k}$. For instance, this is true for the idempotented algebra of $S \times S$-matrices with coefficients in $\mathbf{k}$, with the column module being simple, where S is any infinite set.

For a ring or idempotented ring A denote by $A-$fl the abelian category of finite-length left A-modules. Denote by $G_0(A) = G_0(A-\text{fl})$ the Grothendieck group of the category of finite-length left A-modules. In general, the Grothendieck group $G_0(\mathcal{A})$ of an abelian category $\mathcal{A}$ has generators $[M]$, over all objects M of $\mathcal{A}$, and defining relations $[M] = [M_1] + [M_2]$ over all exact sequences $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$.

To an abelian category $\mathcal{A}$ we can associate at least three different versions of the Grothendieck group. The Grothendieck group $G_0(\mathcal{A})$ has generators - symbols $[M]$ of objects of $\mathcal{A}$, with short exact sequences as above giving defining relations. The Grothendieck group of projective objects $K_0(\mathcal{A})$ has generators $[P]$, over projective objects $P \in \mathcal{A}$, and defining relations $[P] = [P_1] + [P_2]$ whenever there is an isomorphism $P \cong P_1 \oplus P_2$. The split Grothendieck group, which we denote $SG(\mathcal{A})$, has with generators $[M]$, over all objects M and defining relations $[M] = [M_1] + [M_2]$ whenever $M \cong M_1 \oplus M_2$.

There are obvious homomorphisms

$$K_0(\mathcal{A}) \longrightarrow SG(\mathcal{A}) \longrightarrow G_0(\mathcal{A}).$$

The composition $K_0(\mathcal{A}) \longrightarrow G_0(\mathcal{A})$ is, in general, neither surjective nor injective.

For an idempotented lfd $\mathbf{k}$-algebra (A, S), there is a bilinear pairing

$$(\,,\,) \;:\; K_0(A) \otimes_{\mathbb{Z}} G_0(A) \longrightarrow \mathbb{Z} \tag{5}$$

given by

$$([P], [M]) = \dim_{\mathbf{k}} \mathrm{Hom}_A(P, M) \tag{6}$$

for a finitely generated projective module P and a finite length module M. If every simple A-module is absolutely irreducible, that is, $\mathrm{End}_A(L_u) = \mathbf{k}$ for all $u \in U$, then this pairing is perfect, and the bases $\{[P_u]\}_{u\in U}$ and $\{[L_u]\}_{u\in U}$ are dual with respect to this pairing. In the absence of absolute irreducibility the pairing becomes perfect upon tensoring the two Grothendieck groups and $\mathbb{Z}$ with $\mathbb{Q}$.

Let A be an idempotented $\mathbf{k}$-algebra. A left A-module M is called *locally finite-dimensional* (lfd, for short) if $1_i M$ is a finite-dimensional $\mathbf{k}$-vector space, for any $i \in S$.

Denote by $A-$lfd the abelian category of lfd A-modules (furthermore, it's a thick subcategory of $A-$mod). For each i in S there is a homomorphism

$$\rho_i \;:\; G_0(A-\text{lfd}) \longrightarrow \mathbb{Z} \tag{7}$$

taking $[M]$ to $\dim_{\mathbf{k}}(1_i M)$. Assume now that A is a basic lfd idempotented $\mathbf{k}$-algebra. Then the image of this homomorphism is spanned by $\dim_{\mathbf{k}}(L_i) \in \mathbb{Z}$, and the homomorphism is surjective iff L_i is absolutely irreducible.

Taking the product of ρ_i over all $i \in S$ gives a homomorphism

$$\rho \;:\; G_0(A-\text{lfd}) \longrightarrow \prod_{i\in S} \mathbb{Z}. \tag{8}$$

If (A, S) is a basic lfd idempotented $\mathbf{k}$-algebra, the image of ρ is the product $\prod_{i\in S} \dim_{\mathbf{k}}(L_i)\mathbb{Z}$ (consider the image of the object $\oplus_{i\in S} L_i^{n_i}$ of $A-$lfd for arbitrary $n_i \in \mathbb{N}$).

If, in addition, all L_i's are absolutely irreducible, ρ is surjective. It's not clear whether ρ is injective for various natural examples of lfd idempotented $\mathbf{k}$-algebras, including the ones considered in this paper.

If A is an idempotented lfd $\mathbf{k}$-algebra then any finitely generated A-module is lfd. In particular, simple A-modules and finite-length A-modules are lfd, and there are inclusions of categories

$$A\text{−fl} \subset A\text{−fg} \subset A\text{−lfd}. \tag{9}$$

In summary:

- A−fl is the abelian category of finite-length modules,
- A−fg is the additive category of finitely-generated modules (abelian category if A is a Noetherian idempotented algebra),
- A−lfd is the abelian category of locally finite-dimensional modules.

2.2. Chebyshev polynomials

The Chebyshev polynomials of the second kind $U_n(x)$ are defined by the recurrence relation and initial conditions

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad U_0(x) = 1, \quad U_1(x) = 2x.$$

We use their rescaled counterparts $\mathcal{U}_n = U_n(\frac{x}{2})$, which are sometimes called *the Chebyshev polynomials of the second kind on the interval* $[-2, 2]$, see [4, Section 1.3.2]. In this paper we simply call $\mathcal{U}_n$'s Chebyshev polynomials. The are determined by the recurrence relation

$$\mathcal{U}_{n+1}(x) = x\mathcal{U}_n(x) - \mathcal{U}_{n-1}(x) \tag{10}$$

and initial conditions

$$\mathcal{U}_0(x) = 1, \quad \mathcal{U}_1(x) = x. \tag{11}$$

For later use, we rewrite the recurrence as

$$x\mathcal{U}_n(x) = \mathcal{U}_{n+1}(x) + \mathcal{U}_{n-1}(x) \tag{12}$$

Chebyshev polynomials $\{\mathcal{U}_n\}_{n\geq 0}$ form an orthogonal set on the interval $[-2, 2]$ with respect to the weighting function $\sqrt{4-x^2}$. If we define the inner product on polynomials by

$$(f, g) = \frac{1}{2\pi} \int_{-2}^{2} f(x)g(x)\sqrt{4-x^2}dx, \tag{13}$$

then

$$(\mathcal{U}_n(x), \mathcal{U}_m(x)) = \frac{1}{2\pi} \int_{-2}^{2} \mathcal{U}_n(x)\mathcal{U}_m(x)\sqrt{4-x^2}dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Chebyshev polynomials for small values of n are $\mathcal{U}_0(x) = 1, \mathcal{U}_1(x) = x, \mathcal{U}_2(x) = x^2 - 1, \mathcal{U}_3(x) = x^3 - 2x, \mathcal{U}_4(x) = x^4 - 3x^2 + 1, \mathcal{U}_5(x) = x^5 - 4x^3 + 3x, \mathcal{U}_6(x) = x^6 - 5x^4 + 6x^2 - 1, \mathcal{U}_7(x) = x^7 - 6x^5 + 10x^3 - 4x, \mathcal{U}_8(x) = x^8 - 7x^6 + 15x^4 - 10x^2 + 1$.

Chebyshev polynomials satisfy the multiplication rule

$$\mathcal{U}_n\mathcal{U}_m = \mathcal{U}_{|n-m|} + \mathcal{U}_{|n-m|+2} + \cdots + \mathcal{U}_{n+m}. \tag{15}$$

$\mathcal{U}_n$ is a monic polynomial of degree n with n real roots $2\cos(\frac{\pi k}{n+1})$, $k = 1, \ldots, n$. Notice that $2\cos(\frac{\pi k}{n+1}) = \zeta^k + \zeta^{-k}$, where $\zeta = e^{\frac{\pi i}{n+1}}$. The n-th Chebyshev polynomial $\mathcal{U}_n$ has integral coefficients, which alternate with each change in the exponent by two.

Chebyshev polynomials can be described via the determinantal formula

$$\mathcal{U}_n(x) = \begin{vmatrix} x & 1 & 0 & \ldots & 0 & 0 \\ 1 & x & 1 & \ldots & 0 & 0 \\ 0 & 1 & x & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & x & 1 \\ 0 & 0 & 0 & \ldots & 1 & x \end{vmatrix} \tag{16}$$

The Chebyshev polynomial has the following evaluations $\mathcal{U}_n(1) = 0$ and $\mathcal{U}_n(-1) = (-1)^{n-1}(n-2)2^{(n-1)}$ for $n \geq 2$. Chebyshev polynomials have the generating function

$$\sum_{n\geq 0} \mathcal{U}_n(x)t^n = \frac{1}{1 - xt + t^2}.$$

2.3. A categorification of the Chebyshev polynomials via sl_2 representations

The Lie algebra $sl(2)$ has one irreducible representation in each dimension. Let V_n denote the irreducible $(n+1)$ dimensional representation of $sl(2)$. Representation V_0 is trivial, while V_1 is the defining (vector) representation of $sl(2)$, and $V_n \simeq S^n(V_1)$. The category $sl(2)-$mod of finite-dimensional $sl(2)$ representations is a semisimple tensor category with the Grothendieck ring $K_0(sl(2))$ being a free abelian group with the basis $\{[V_0], [V_1], \ldots\}$ in the symbols of all irreducible modules. The multiplication in the Grothendieck ring is defined by:

$$[V][W] := [V \otimes W]$$

Direct sum decomposition for the tensor product

$$V_n \oplus V_m \simeq V_{|n-m|} \oplus V_{|n-m|+2} \oplus \ldots \oplus V_{n+m} \tag{17}$$

categorifies the equation (15) and gives the multiplication in the basis of irreducibles

$$[V_n][V_m] = [V_{|n-m|}] + [V_{|n-m|+2}] + \ldots + [V_{n+m}].$$

We identify the Grothendieck ring with the polynomial ring

$$K_0(sl(2)) \simeq \mathbb{Z}[x] \tag{18}$$

in one variable x by taking $[V_1]$ to x and, correspondingly, $[V_1^{\otimes n}]$ to x^n. Under this isomorphism symbols of irreducibles go to Chebyshev polynomials, $[V_n] \leftrightarrow \mathcal{U}_n(x)$. Thus, irreducible $sl(2)$ modules offer a categorification of Chebyshev polynomials.

The Temperley-Lieb category, denoted by TL, is a monoidal $\mathbb{C}$–linear category with objects non-negative integers $n \in \mathbb{Z}_+$, $n \otimes m = n + m$, and $\mathrm{Hom}_{TL}(n, m) = \mathrm{Hom}_{sl(2)}(V_1^{\otimes n}, V_1^{\otimes m})$. A basis in $\mathrm{Hom}_{TL}(n, m)$ is

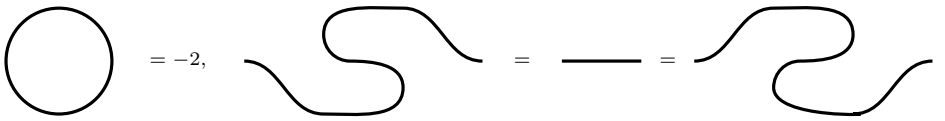

Fig. 1. Defining relations in TL: on the left, the value of the circle is set to minus two, and the isotopy relations are shown on the right.

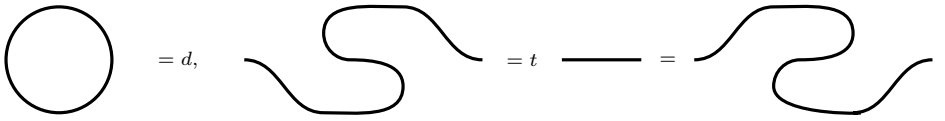

Fig. 2. Defining relations in $T_{d,t}$.

given by the diagrams of crossingless matchings in the plane between n points on the left and m points on the right. As a $\mathbb{C}$-linear monoidal category, TL can be described by a single generating object 1, corresponding to representation V_1, and two generating morphisms between the unit object 0 and the tensor square $2 = 1 \otimes 1$ of the generating object, corresponding to the intertwiners $V_0 \to V_1^{\otimes 2}$, $V_1^{\otimes 2} \to V_0$. These generating morphisms can be depicted by the U-turn diagrams $\subset$ and $\supset$, with defining relations in TL, viewed as a monoidal category, shown in Fig. 1, including the isotopy relations.

The category $sl(2)-$mod of finite-dimensional $sl(2)$-representations is equivalent to the Karoubian envelope C of the additive closure of TL. To define the latter we first allow finite direct sums of objects of TL, than pass to the Karoubian envelope $C = Kar(Add(TL))$. Quantum deformation C_d of C is obtained by changing the value of a circle to $d = -q - q^{-1}$, $q \in \mathbb{C}$, see Fig. 2 left.

This results in the Temperley-Lieb category TL_d. This category has non-negative numbers n as objects and k-linear combinations of diagrams of planar arcs with n left and m right endpoints as morphisms, modulo the isotopy relations and evaluating a circle to d. Allowing finite direct sums of objects and passing to the Karoubi envelope results in the monoidal category $C_d = Kar(Add(TL_d))$.

For q not a root of unity, C_d is equivalent to the category of finite-dimensional representations of the quantum group $U_q(sl(2))$ on which the Cartan generator K acts with eigenvalues powers of q.

We refer the reader to Queffelec-Wedrich [5] for a similar categorification of the Chebyshev polynomials of the first kind.

2.4. A 2-parameter family of Temperley-Lieb type monoidal categories

One-parameter family of Temperley-Lieb categories can be viewed as a limit of a two-parameter family $TL_{d,t}$ of monoidal categories, where we evaluate a circle to d and a squiggle to t times the identity, see Fig. 2.

If t is a nonzero element of the ground field $\mathbf{k}$, one can rescale by dividing either left or right turns by t to reduce to $t = 1$ case and the relations shown in Fig. 3.

These are the relations in the Temperley-Lieb category with the value of the circle dt^{-1}. Consequently, there's an equivalence (even an isomorphism) of categories

$$TL_{d,t} \;\cong\; TL_{dt^{-1}}$$

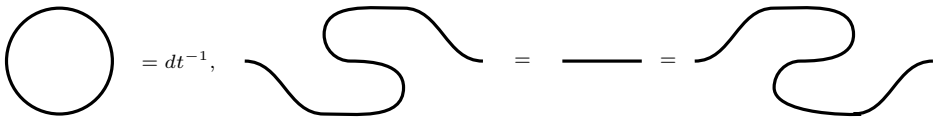

Fig. 3. Rescaling relations in $T_{d,t}$ for $t \neq 0$ to those of $T_{dt^{-1}}$.

between $TL_{d,t}$ and the original Templerley-Lieb category with a circle evaluating to dt^{-1}.

The only case when we cannot divide by t over a field is when $t = 0$. This is a degenerate case, which further splits into two cases $d \neq 0$ and $d = 0$.

If $d \neq 0$, rescaling a left or a right return by d^{-1} reduces to the case $d = 1$, giving an equivalence of categories

$$TL_{d,0} \;\cong\; TL_{1,0}, \quad \text{if} \quad d \neq 0.$$

In the present paper we deal with the last case, $d = t = 0$, which is the maximally degenerate one, in some sense.

The following provides a simple summary of two-parameter family $TL_{d,t}$ based on possible values of d and t:

- $t \neq 0$: one can rescale to get the Temperley-Lieb category $TL_{dt^{-1}}$,
- $t = 0$ provides two options:
  1. $d \neq 0$: rescale to $d = 1$, which yields the category $TL_{1,0}$, or
  2. $d = 0$ the category analyzed in this paper.

Note that $d = 1, t = 0$ case is somewhat reminiscent of the categorification of the polynomial ring in [1]. Since the value of the circle is one, maps of objects $n \longrightarrow n + 2$ given by inserting a return somewhere inside the diagram of n parallel lines has a splitting given by the reflected diagram, so that n becomes a direct summand of $n + 2$ in the Karoubi additive closure of $TL_{1,0}$. There are $n + 1$ such maps, one for each position of a return relative to n parallel lines. Due to squiggles being 0, these maps and their duals are mutually orthogonal, and allow to split off $n + 1$ copies of the object n from the object $n + 2$ in $C_{1,0} = Kar(Add(TL_{1,0}))$. Iterating, object $2n$ contains the n-th Catalan number of copies of object 0 as direct summands.

Additive Karoubi envelope of $TL_{d,t}$ is semisimple for generic $d, t \in \mathbf{k}$. Category $TL_{0,0}$ is on the opposite end of this spectrum, providing the *most nonsemisimple* extreme behavior in this family of categories. Nevertheless, there are interesting families of objects even in this extreme case, BGG reciprocity and categorification properties, considered below.

3. Diagrammatic categorification of the Chebyshev polynomials of the second kind

Let ${}_nB^c_m$, for $n, m \geq 0$ denote the set of isotopy classes of plane diagrams consisting of n vertices on the line $x = 0$ and m vertices on the line $x = 1$, and crossingless connections between them (no intersections or self-intersections are allowed). Crossingless connections fall into three types: through arcs that connect points on lines $x = 0$ and $x = 1$, and left and right returns that connect pairs of points on the line $x = 0$ and line $x = 1$, respectively, see Fig. 4. The number of through arcs is called the *width* of a diagram. We let ${}_nB^c_m(w)$ and ${}_nB^c_m(\leq w)$ denote all diagrams in ${}_nB^c_m$ of width exactly w and at most w, respectively. Denote by B^c_m the disjoint union of sets ${}_nB^c_m$ over all $n \geq 0$, likewise for ${}_nB^c$.

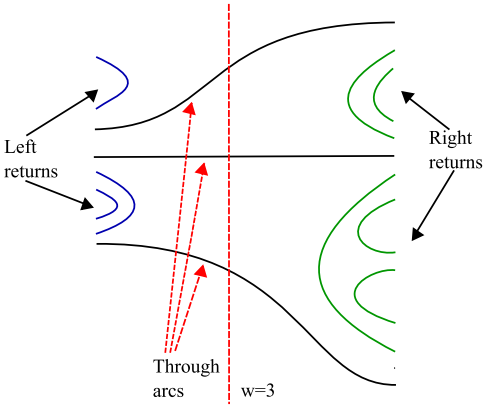

Fig. 4. A diagram in ${}_9B^c_{13}(3)$ of width $w = 3$ (3 through arcs connecting left and right side) with 3 left returns (blue), and 5 right returns (red). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Moreover, through arcs are not allowed to have critical points with respect to the orthogonal projection onto the x–axis, and returns need to have exactly one critical point, see Fig. 1. Diagrams are considered up to isotopies that preserve these conditions.

The cardinality of the set ${}_nB^c_m$ is the k-th Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$ for $k = \frac{n+m}{2}$.
Let $A^c = \bigoplus_{n,m\geq 0} {}_nA^c_m$, where ${}_nA^c_m$ denotes a vector space over a base field $\mathbf{k}$ with the basis of diagrams ${}_nB^c_m$. In addition, A^c is given $\mathbf{k}$-linear multiplication by the horizontal concatenation of diagrams. More precisely, the product $xy \in {}_nB^c_s$ of diagrams $x \in {}_nB^c_m$ and $y \in {}_lB^c_s$ is zero unless $m = l$, and also equals zero if the resulting diagram contains one of the diagrams (other then the horizontal line) shown in Fig. 1. In other words, the product is zero if there is a circle in the concatenated diagram, or a twist which is a composition of two returns. Equivalently, it is zero if some connected component of the concatenation has more than one critical point under the projection to the x-axis. Diagrammatic relations in this category are shown in Fig. 15, where the category is treated as a $\mathbf{k}$-linear category ignoring the monoidal structure, and a non-local version of defining relation is shown, with key regions where skein relations are nontrivial surrounded by identity morphisms on both sides. Monoidally, the category is given by the relations in Fig. 2 with $d = t = 0$.

Algebra A^c equipped with this multiplication becomes an associative non-unital $\mathbf{k}$–algebra. This algebra contains an idempotent 1_n for each $n \in \mathbb{Z}_+$, consisting of n through arcs. These elements $\{1_n\}_{n\geq 0}$ satisfy the following equalities:

$$1_n x = x, \text{ for } x \in {}_nB^c_m,$$
$$y 1_n = y, \text{ for } y \in {}_lB^c_n,$$
$$1_m 1_n = \delta_{m,n}\, 1_n.$$

$\{1_n\}_{n\geq 0}$ are mutually-orthogonal idempotents. For any $x \in A^c$, there exists $k \in \mathbb{N}$ such that $\sum_{n=0}^{k} 1_n x = x = x\sum_{n=0}^{k} 1_n$. Thus, A^c is an idempotented $\mathbf{k}$-algebra. Alternatively, this structure can be viewed as a preadditive category with objects $n \in \mathbb{Z}_+$, morphisms $n \to m$ being ${}_nA^c_m$, and composition–the product in A^c.

Next we define types of modules of interest to us. Let $P_n = A^c 1_n$ denote the module over A^c generated (as a vector space) by all diagrams in B^c with n right endpoints. P_n's are indecomposable projective modules and $A^c \cong \bigoplus_{n\geq 0} P_n$. Any projective A^c-module is a direct sum of P_n's, with multiplicities being invariants of the module. The Grothendieck group $K_0(A^c)$ is free abelian with the basis $\{[P_n]\}_{n\geq 0}$.

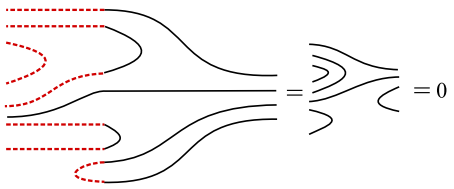

Fig. 5. Action of algebra A^c on standard modules M_n: if a resulting diagram contains a right return it is equal to zero.

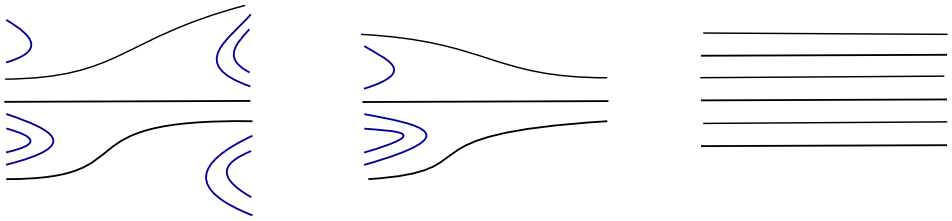

Fig. 6. Typical basis elements of projective, standard and simple A^c–modules, respectively.

Theorem 3.1. *Any projective left locally finite–dimensional A^c-module P is isomorphic to a finite direct sum of indecomposable projective modules P_n, $P \cong \bigoplus_{n\geq 0} P_n^{a_n}$, where the multiplicities $a_n \in \mathbb{Z}_+$ are invariants of P.*

Proof. The theorem follows from more general results about projective modules over idempotented lfd algebras discussed in Section 2.1 and the observation that $1_n A^c 1_n$ is a local algebra, with the maximal nilpotent ideal J_n spanned by diagrams other than the identity diagram. The quotient $1_n A^c 1_n / J_n$ is the ground field $\mathbf{k}$. Furthermore, the multiplication map $1_n A_c 1_m \otimes 1_m A_c 1_n \longrightarrow 1_n A_c 1_n$ for $m \neq n$ has the image in J_n. □

Indecomposable projective module P_n has a unique simple quotient module L_n. This module is one-dimensional. Denote the basis vector generating L_n by 1_n and note that, with some informality, we use the same notation for this basis vector as for the corresponding idempotent in A^c. Any diagram from B_n^c other than 1_n acts by zero on L_n. Any simple A^c-module is isomorphic to L_n, for a unique n. The idempotented $\mathbf{k}$-algebra A^c with this set of idempotents $\{1_n\}_{n\geq 0}$ is basic locally finite-dimensional, providing an example to the theory discussed in Section 2.1. All of its simple modules are absolutely irreducible. The bilinear pairing (5), in the case of A^c, is perfect. The bases $\{[P_n]\}_{n\geq 0}$ and $\{[L_n]\}_{n\geq 0}$ of $K_0(A^c)$ and $G_0(A^c)$, respectively, are dual to each other.

Let M_n, for $n \geq 0$ denote the standard module, a quotient of P_n by the submodule I_n spanned by diagrams with n right endpoints and at least one right return. While P_n has a basis of all diagrams in B_n^c, diagrams with no right returns give a basis in M_n. Any diagram which contains a right return acts by zero on M_n (irregardless of the number of right endpoints of this diagram), see Fig. 5, for instance.

Examples of diagrams corresponding to basis elements in projective, standard and simple modules are shown in Fig. 6.

A^c is not Noetherian, since the submodule Q of the projective module P_0 generated by the diagrams b_i, $i > 0$, shown in Fig. 7 is infinitely generated: b_i does not belong to the submodule of Q generated by diagrams b_j, $j < i$, since the number of left returns parallel to the outermost return can not be increased by the left action of A^c.

How to circumvent the problem of A^c not being Noetherian, i.e., that the category of finitely generated A^c–modules is not abelian? One solution is to consider the abelian category A^c-fl of finite-length A^c-modules. Since simple A^c-modules are all one-dimensional, A^c-fl is also the category of finite-dimensional

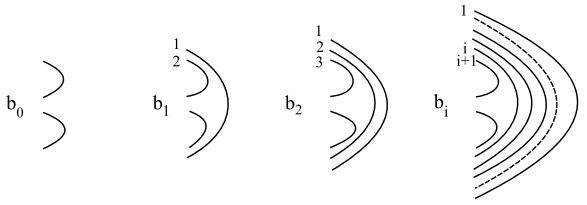

Fig. 7. Generators of the submodule Q of P_0 which can not be finitely generated.

A^c-modules A^c-fd. Also, by analogy with the slarc algebra [6,1] and Section 2.1, we use the category A^c-lfd of locally finite–dimensional A^c–modules and the category A^c-pfg of projective finitely-generated modules.

The Grothendieck group

$$G_0(A^c) := G_0(A^c-\text{fl}) \cong G_0(A^c-\text{fd}) \tag{19}$$

is free abelian with a basis given by symbols of simple modules $[L_n]$, $n \geq 0$. The category A^c-lfd is abelian as well, and contains modules L_n, M_n and P_n. However, the Grothendieck group of A^c-lfd is large, admitting a surjection

$$G_0(A^c-\text{lfd}) \xrightarrow{\rho} \prod_{n\geq 0} \mathbb{Z}$$

given by sending $[M] \mapsto (\dim 1_0 M, \dim 1_1 M, \ldots)$, see equation (8) in Section 2.1. Observe that $[P_n]$, over all $n \geq 0$, are linearly independent in $G_0(A^c-\text{lfd})$, so that the natural map $K_0(A^c) \to G_0(A^c-\text{lfd})$ is injective. Overall, there are natural injective maps of Grothendieck groups

$$K_0(A^c) \longrightarrow G_0(A^c-\text{lfd}) \longleftarrow G_0(A^c),$$

and a perfect pairing $K_0(A^c) \otimes_{\mathbb{Z}} G_0(A^c-\text{fd}) \to \mathbb{Z}$ with

$$([P],[M]) = \dim \text{Hom}_{A^c}(P, M),$$

making bases $\{[P_n]\}_{n\geq 0}$ and $\{[L_n]\}_{n\geq 0}$ dual to each other, $([P_n],[L_m]) = \delta_{n,m}$.

Theorem 3.2. *There is a natural isomorphism sending generators $[P_n]$ of $K_0(A^c)$ to $x^n \in \mathbb{Z}[x]$, categorifying $\mathbb{Z}[x]$ as an abelian group:*

$$K_0(A^c) \cong \mathbb{Z}[x]. \tag{20}$$

In order to categorify $\mathbb{Z}[x]$ as a ring we need to define a monoidal structure on our category. The Hom space $\text{Hom}(P_n, P_m)$ between projective modules P_n and P_m has a basis of diagrams in ${}_nB^c_m$. Stacking diagrams on top of each other defines bilinear maps:

$$\text{Hom}(P_{m_1}, P_{n_1}) \otimes \text{Hom}(P_{m_2}, P_{n_2}) \to \text{Hom}(P_{m_1+m_2}, P_{n_1+n_2}). \tag{21}$$

Define the tensor product bifunctor $A-\text{pmod} \otimes A-\text{pmod} \to A-\text{pmod}$ on objects by:

$$P_{n_1} \otimes P_{n_1} := P_{n_1+n_2} \tag{22}$$

and on morphisms $P_{m_1} \xrightarrow{\alpha} P_{n_1}$, $P_{m_2} \xrightarrow{\beta} P_{n_2}$, as in [6,1], by stacking basic morphisms one on top of another and extending using bilinearity $(\alpha, \beta) \mapsto \alpha \otimes \beta$ where $P_{m_1+m_2} \xrightarrow{\alpha\otimes\beta} P_{n_1+n_2}$.

On the level of Grothendieck groups, the tensor product descends to the multiplication, with $[P_n] = x^n$ and $[P_n \otimes P_m] = [P_{n+m}] = x^n x^m = x^{n+m}$. This gives us a categorification of the ring $\mathbb{Z}[x]$ via the category of finitely-generated projective A^c-modules. Notice that $\otimes$ is not symmetric i.e. $M \otimes N \ncong N \otimes M$.

The tensor product extends to the category of complexes of projective modules up to homotopies. Denote by $\mathcal{C}(A^c\text{−pfg})$ the category of bounded complexes of (finitely generated) projective A^c–modules modulo homotopies (given two chains, say $C_1, C_2 \in C(A^c\text{−pmod})$ then $C_1 \otimes C_2$ is $\bigoplus_{i,j\in\mathbb{Z}} C_1^i \otimes C_2^j$ and the differential $d = d_1 \otimes 1 + 1 \otimes d_2$), see [7,8]. The tensor product is a bifunctor

$$\otimes \ : \ \mathcal{C}(A^c\text{−pfg}) \times \mathcal{C}(A^c\text{−pfg}) \longrightarrow \mathcal{C}(A^c\text{−pfg}).$$

Denote by $\widetilde{X}_{n,n+2k}$ the subset of ${}_nB^c_{n+2k}$ consisting of diagrams without left returns, and denote by $X_{n,n+2k}$ the cardinality of $\widetilde{X}_{n,n+2k}$. Note that

$$X_{n,n+2k} = \frac{n+1}{n+k+1}\binom{n+2k}{k}.$$

We also let $X_{n,m} = 0$ unless $m - n$ is an even nonnegative integer.

Let $P_n(\leq w)$ denote a submodule of P_n generated by all diagrams in P_n of width less than or equal to w, for $w \geq 0$. P_n has a finite decreasing filtration

$$P_n = P_n(\leq n) \supset P_n(\leq n-2) \supset \ldots \supset P_n(\leq n-2k) \supset \ldots \supset P_n(\leq n - 2\lfloor\frac{n}{2}\rfloor).$$

The quotient module $P_n(\leq n-2k)/P_n(\leq n-2(k+1))$ has a natural basis consisting of diagrams in B^c_n whose width is exactly $n-2k$. Any such diagram can be uniquely presented as a composition of a diagram in B^c_{n-2k} with no right returns (corresponding to a basis element of the standard module M_{n-2k}), and a diagram in $\widetilde{X}_{n-2k,n}$ which has exactly k right and no left returns, see Fig. 8. For $k \leq \lfloor\frac{n}{2}\rfloor$ we have

$$X_{n-2k,n} = \frac{n-2k+1}{n-k+1}\binom{n}{k}. \tag{23}$$

Each diagram in $\widetilde{X}_{n-2k,n}$ (one is shown in Fig. 8 on the right) provides a generator for a copy of M_{n-2k}, viewed as a direct summand of the quotient module $P_n(\leq n-2k)/P_n(\leq n-2(k+1))$. This yields the following relation:

$$P_n(n-2k)/P_n(n-2(k+1)) \cong M_{n-2k}^{X_{n-2k,n}},$$

where M^I, for a module M and a set I, denotes the direct sum of copies of M, one for each element of the set I. For $i \in I$ we denote the corresponding copy of M in the summand by M^i.

We get a relation in any suitable Grothendieck group of A^c, for instance in $G_0(A^c\text{−lfd})$:

$$[P_n] \ = \ \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} X_{n-2k,n}[M_{n-2k}]. \tag{24}$$

Module M_m appears $X_{m,n}$ times in the above filtration of P_n by standard modules. This multiplicity number is independent of the choice of a filtration of P_n with subsequent quotients isomorphic to standard modules. We denote this multiplicity by $[P_n : M_m] = X_{m,n}$.

Diagrams in B^c can have nested returns, for example a diagram contains right (or left) nested returns if endpoints of one return lie between the endpoints of the other, outer return, see the right picture on Fig. 8.

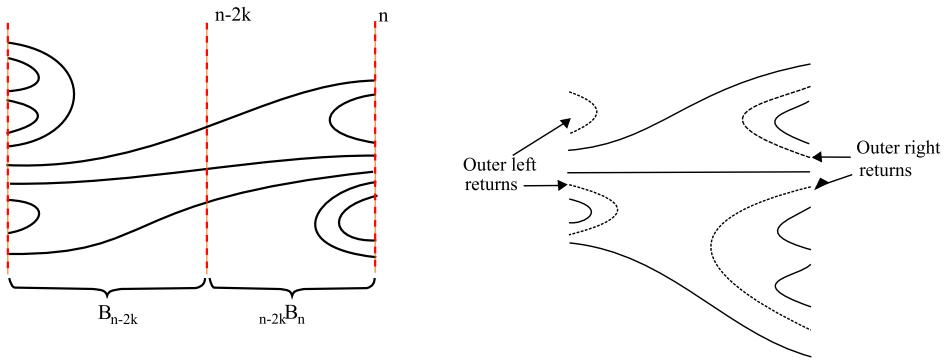

Fig. 8. Left: Decomposition of a diagram in B_n^c into a diagram in B_{n-2k}^c with no right returns (corresponding to a basis element of the standard module M_{n-2k}), and a diagram in $\widetilde{X}_{n-2k,k}$. Right: Diagram in B^c with nested returns; outer returns are represented by dashed lines.

Let X and Y be upper-triangular $\mathbb{N} \times \mathbb{N}$ matrices with nonzero entries

$$X_{n,n+2k} = \frac{n+1}{n+k+1}\binom{n+2k}{k}, \quad Y_{n,n+2k} = (-1)^k \binom{n+k}{k}, n, k \geq 0. \tag{25}$$

The entries of X count diagrams in $\widetilde{X}_{n,n+2k}$, as defined earlier. Let $\widetilde{Y}_{n,n+2k}$ be the set of diagrams in ${}_nB_{n+2k}^c$ with no left returns and only unnested right returns. Such diagrams necessarily has k right returns, and the absolute value of $Y_{n,n+2k}$ is the cardinality of the set $\widetilde{Y}_{n,n+2k}$.

Proposition 3.3. *Matrices X and Y are mutually inverse, $XY = YX = \mathrm{Id}$.*

Proof. Composing a diagram from $\widetilde{Y}_{n,n+2k}$ with a diagram from $\widetilde{X}_{n+2k,n+2m}$ for $0 \leq k \leq m$ results in a diagram in $\widetilde{X}_{n,n+2m}$. A given diagram $\gamma \in \widetilde{X}_{n,n+2m}$ has 2^r such presentations as a composition (with varying k), where r is the number of outer (right) returns of γ, that is, returns that are not nested inside other returns. For instance, diagram (a) in Fig. 9 has two outer returns and 2^2 possible decompositions, two of which are shown as diagrams (b) and (c). Each diagram γ contributes to the $(n, n+2m)$-entry of the matrix YX (the entry is the sum of contributions from all such diagrams). The contribution is the alternating sum of 2^r one's, with signs counting the number of outer returns of γ that appear in the Y-part of the decomposition. Clearly, the contribution is 0, unless $r = 0$, which is only possible with $m = 0$, in which case the entry is 1. We see that $YX = \mathrm{Id}$, while $XY = \mathrm{Id}$ follows now from upper-triangularity of X and Y. □

On the level of Grothendieck groups, inverting (24), we get

$$[M_n] \;=\; \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} Y_{n-2k,n}[P_{n-2k}] \;=\; \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} [P_{n-2k}]. \tag{26}$$

Based on this equation, one expects to have a projective resolution of the standard module M_n with the k-th term equal to the direct sum of $Y_{n-2k,n}$ copies of P_{n-2k}:

$$0 \to \ldots \to P_{n-2k}^{\binom{n-k}{k}} \xrightarrow{d_k} \ldots \xrightarrow{d_2} P_{n-2}^{n-1} \xrightarrow{d_1} P_n \to M_n \to 0. \tag{27}$$

Let us now construct such complex, parametrizing summands of the k-th term by elements of $\widetilde{Y}_{n-2k,n}$:

$$0 \to \ldots \to P_{n-2k}^{Y_{n-2k,n}} \xrightarrow{d_k} P_{n-2(k-1)}^{Y_{n-2(k-1),n}} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_2} P_{n-2}^{Y_{n-2,n}} \xrightarrow{d_1} P_n \to M_n \to 0, \tag{28}$$

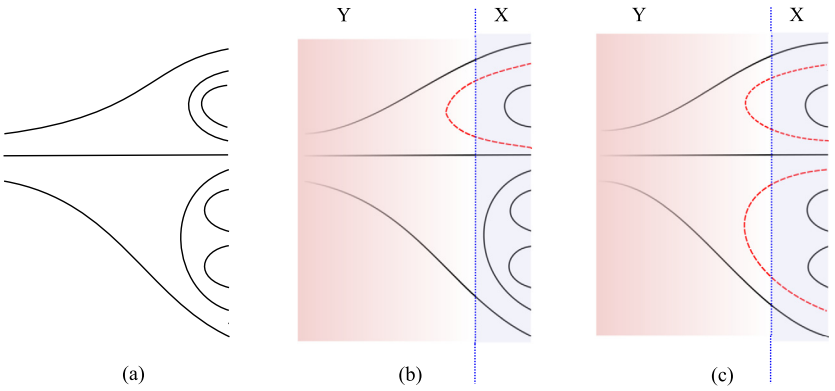

Fig. 9. Diagram (a) contributes to $(3, 13)$-entry of matrix YX. Diagrams (b) and (c) are two out of four of its decompositions.

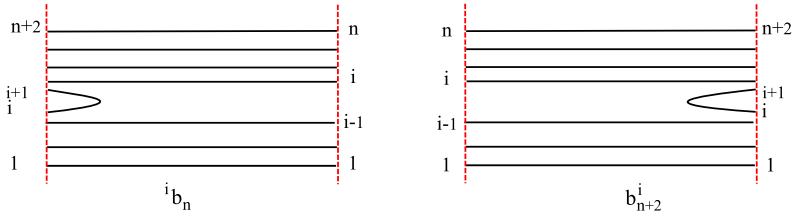

Fig. 10. Diagrams used in defining differentials in resolutions of standard and simple modules.

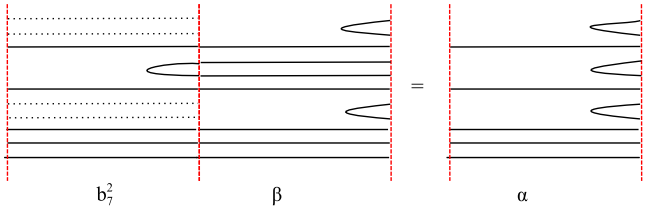

Fig. 11. One component of the differential sends P^{α}_{5} into P^{β}_{7} for these $\alpha \in {}_5B^c_{11}$ and $\beta \in {}_7B^c_{11}$ by composing with $-b^2_7$; we have $b^2_7\beta = \alpha$.

with the convention that $\widetilde{Y}_{n,n}$ is the one-element set consisting of the diagram 1_n.

The module map $P_{n-2k} \longrightarrow P_{n-2(k-1)}$ can be described uniquely by a linear combination c of diagrams in ${}_{n-2k}B^c_{n-2(k-1)}$. A linear combination c takes $a \in P_{n-2k}$ to $ac \in P_{n-2(k-1)}$. The differential d_k in (28) decomposes as the sum of its components

$$d_{k,\beta,\alpha} \ : \ P^{\alpha}_{n-2k} \longrightarrow P^{\beta}_{n-2(k-1)}$$

over all $\alpha \in \widetilde{Y}_{n-2k,n}$ and $\beta \in \widetilde{Y}_{n-2(k-1),n}$.

Composing the diagram $b^i_{n-2(k-1)}$ with β gives us a diagram $b^i_{n-2(k-1)}\beta \in {}_{n-2k}B^c_n$. This diagram has k right returns, no left returns, and lies in $\widetilde{Y}_{n-2k,n}$ iff it has no nested returns. The unique right return of $b^i_{n-2(k-1)}$ becomes a right return of the composition. Assuming that the composition has no nested returns, let j be the order of this right return, where we count right returns from top to bottom, see Fig. 11.

For each α and β as above there is at most one i such that $\alpha = b^i_{n-2(k-1)}\beta$. If there is no such i, we set $d_{k,\beta,\alpha} = 0$, otherwise set $d_{k,\beta,\alpha}(a) = (-1)^{j-1}ab^i_{n-2(k-1)}$ for $a \in P^{\alpha}_{n-2k}$. Differential d_k is the sum of $d_{k,\beta,\alpha}$ over all β, α as above. Thus, the differential is a signed sum of maps given by composing with diagrams with no left and one right return. The equation $d_{k-1}d_k = 0$ follows at once.

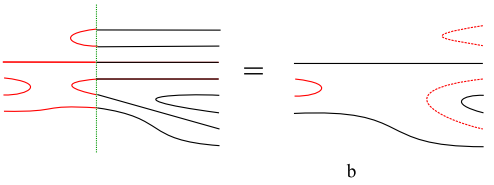

Fig. 12. Diagram b on the right gives rise to a 2-dimensional cube since it has 2 outermost right returns (represented by dotted lines).

Proposition 3.4. *The complex* (28) *is exact, giving the finite projective resolution of standard modules* M_n, $n > 0$.

Proof. Diagrams $a \in B^c_{n-2k}$ constitute a basis of the module P^{α}_{n-2k}. Composition $a\alpha$ is either 0 or a diagram in B^c_n. Components of the differential d_k take a to a signed sum of diagrams $ab^i_{n-2(k-1)}$, which are basis elements in $P^{\beta}_{n-2(k-1)}$, with the property that $a\alpha = ab^i_{n-2(k-1)}\beta$. Thus, the product $a\alpha$ is "preserved" by the differential, in the following sense. The complex (28) is a direct sum of complexes of vector spaces, over all $b \in B^c_n$, which have as basis pairs (α, a) over all α, a as above with $a\alpha = b$. Each of this complexes of vector spaces is isomorphic to the complex given by collapsing an anticommutative r-dimensional cube of one-dimensional vector spaces, one for each vertex of the cube, with all edge maps being isomorphisms, where r is the number of right returns of b, see Fig. 12.

If $r > 0$, the corresponding complex is contractible, while when $r = 0$ the complex has one-dimensional cohomology in degree 0. Thus, cohomology of (28) lives entirely in cohomological degree 0 and has a basis of diagram in B^c_n without right returns. These diagrams constitute a basis of M^c_n, implying that (28) is a resolution of the standard module M_n. □

Consider the category $C(A^c\text{−pfg})$ of bounded complexes of finitely-generated projective A^c-modules up to chain homotopy. This is a monoidal triangulated category, and the inclusion of monoidal categories

$$A^c\text{−pfg} \ \subset \ C(A^c\text{−pfg})$$

induces an isomorphism of Grothendieck rings

$$K_0(A^c\text{−pfg}) \ \subset \ K_0(C(A^c\text{−pfg})).$$

Thus, $K_0(C(A^c\text{−pfg})) \cong \mathbb{Z}[x]$ as a ring. The standard module M_n can be viewed as an object of $C(A^c\text{−pfg})$ due to the finite projective resolution (28).

Theorem 3.5 *(Categorification of the Chebyshev polynomials of the second kind* $\mathcal{U}_n(x)$*). Standard modules* M_n *admit finite resolutions by projectives* P_m *for* $m \leq n$*, see equation* (28)*, and can be converted into objects of the homotopy category of finitely-generated projective* A^c*-modules. Their symbols, viewed as elements of the Grothendieck ring* $K_0(A^c) \cong \mathbb{Z}[x]$*, give the Chebyshev polynomials* $\mathcal{U}_n(x)$*:*

$$[M_n] = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} [P_{n-2k}] = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} = \mathcal{U}_n(x).$$

4. Relations between two categorifications, BGG reciprocity and more

In Section 2.3 we briefly recalled the Temperley-Lieb category and its relation to the representation theory of Lie algebra $sl(2)$ and its quantum deformation. Resolution (28) has a counterpart in that theory, as a

resolution of the irreducible $(n+1)$-dimensional representation V_n of $sl(2)$ or quantum $sl(2)$ (for generic q) by multiples of tensor powers of the two-dimensional representation V_1:

$$0 \to \ldots \to (V_1^{\otimes(n-2k)})^{\widetilde{Y}_{n-2k,n}} \xrightarrow{d_k} (V_1^{\otimes(n-2(k-1))})^{\widetilde{Y}_{n-2(k-1),n}} \xrightarrow{d_{k-1}} \ldots$$
$$\ldots \xrightarrow{d_2} (V_1^{\otimes(n-2)})^{n-1} \xrightarrow{d_1} V_1^{\otimes n} \to V_n \to 0 \quad (29)$$

The differential is described by the same rules as in (28), via composing diagrams with only right returns. In this case the diagrams are viewed as describing maps between tensor powers of V_1 (equivalently, morphisms in the Temperley-Lieb category). For diagrams with both left and right returns the composition rules are different from those in A^c, but for diagrams with right returns only there is no change. That (29), for any q, is a resolution of V_n can be proved similar to Proposition 3.4, by using diagrammatics [9,10] for the Lusztig dual canonical basis of tensor powers of V_1 instead of the basis B_n^c of P_n. Notice that, unlike the quantum $sl(2)$ case, where V_n is irreducible for generic q, standard modules M_n do not even have finite length.

For generic q the category of finite-dimensional representations of quantum $sl(2)$ is semisimple, and resolving objects seems to carry little sense from the homological viewpoint. Nevertheless, essentially this resolution was used in [11] to categorify the colored Jones polynomial, also see [12,13]. Resolutions of irreducible representations of $sl(n)$ by tensor products of fundamental representations have been studied by Akin, Buchsbaum, Weyman [14,15] and others, often in the dual Schur-Weyl context, where simple symmetric group representations are resolved via induced ones, with applications to algebraic geometry [16].

Projective resolution (28) provides some homological information about standard modules, observed in the two propositions below. To establish Proposition 4.1 below one forms the above projective resolution of M_n leading to an easy computation of the Ext groups.

Proposition 4.1. *Given two standard modules M_n, M_m, the k-th Ext group for $k \leq \lfloor \frac{n}{2} \rfloor$ is*

$$\mathrm{Ext}^k(M_n, M_m) \cong (1_{n-2k} M_m)^{\binom{n-k}{k}} \text{ and}$$
$$\dim_{\mathbf{k}} \mathrm{Ext}^k(M_n, M_m) = \begin{cases} \binom{n-k}{k}\binom{\frac{m+n}{2}-k}{m} & \text{if } m \leq n-2k \text{ and } n+m \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.2. *The k-th Ext group for standard and simple modules M_n, L_m has dimension*

$$\dim_{\mathbf{k}} \mathrm{Ext}^k(M_n, L_m) = \begin{cases} \binom{n-k}{k} & \text{if } m \leq n, k = \frac{n-m}{2}, \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

Proposition 4.3. *Homological dimension of the standard module M_n is $\lfloor \frac{n}{2} \rfloor$.*

A finite–dimensional A^c–module M has a finite filtration by simple modules L_n. Due to one-dimensionality of L_n the multiplicity of L_n in M, denoted by $[M : L_n]$, equals $\dim_{\mathbf{k}}(1_n M)$. With this observation in mind, we give the following definition.

Definition 4.4. For a locally finite-dimensional A^c-module M define the multiplicity of a simple module L_n in M by

$$[M : L_n] := \dim_{\mathbf{k}}(1_n M). \quad (31)$$

Proposition 4.5. *The multiplicity $[M_m : L_n]$ of L_n in the standard module M_m equals the number of diagrams in ${}_nB_m^c$ with no right returns:*

$$[M_m : L_n] = X_{m,n} = \begin{cases} \frac{2(m+1)}{n+m+2}\binom{n}{\frac{n-m}{2}} & \text{if } n \geq m \text{ and } n-m \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

The last proposition implies the Bernstein-Gelfand-Gelfand (BGG) reciprocity property [17] for our category. The BGG reciprocity property is the equality of suitably defined multiplicities for indecomposable projective, standard and simple modules: $[P_n : M_m] = [M_m : L_n]$ for all m, n assuming the notation for the multiplicity as in Definition 4.4.

Corollary 4.6. *Indecomposable projective, standard, and simple A^c-modules satisfy the Bernstein-Gelfand-Gelfand (BGG) reciprocity property:*

$$[P_n : M_m] = [M_m : L_n]. \tag{33}$$

Resolution of a simple module L_n by standard modules M_m, which we now describe, is, in a sense, dual to the projective resolution (27) of a standard module by projective modules. This is an infinite to the left resolution, with the k-th term consisting of the standard module M_{n+2k} with multiplicity $Y_{n,n+2k}$:

$$\ldots \to M_{n+2k}^{Y_{n,n+2k}} \xrightarrow{d_k} M_{n+2(k-1)}^{Y_{n,n+2(k-1)}} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_1} M_{n+2}^{n+1} \to M_n \to L_n \to 0. \tag{34}$$

The differential d_k is a sum of its components

$$d_{k,\beta,\alpha} \ : \ M_{n+2k}^{\alpha} \longrightarrow M_{n+2(k-1)}^{\beta},$$

over all $\alpha \in \widetilde{Y}_{n,n+2k}$ and $\beta \in \widetilde{Y}_{n,n+2(k-1)}$. For each such pair (α, β) there exists at most one i, $1 \leq i \leq n-1$, such that $\alpha = \beta b_n^i$, see Fig. 10 for the latter notation. If that's the case, define j to be the order (counting from the bottom) of the right return of b_n^i when viewed as a right return of α upon composing with β. If such i does not exist, we set $d_{k,\beta,\alpha} = 0$. If it does, we let $d_{k,\beta,\alpha}(x) = (-1)^{j-1} x \ {}^i b_{n-2}$, for $x \in M_{n+2k}^{\alpha}$.

Notice that any diagram $b \in {}_n B_m^c$ without right returns induces a homomorphism of standard modules $M_n \longrightarrow M_m$. Diagram ${}^i b_{n-2}$ has no right returns and induces a map from M_{n+2k}^{α} to $M_{n+2(k-1)}^{\beta}$. The summand $d_{k,\beta,\alpha}$ of the differential d_k is just this map, with a sign.

The diagram ${}^i b_{n-2}$ is a reflection of b_n^i about the y-axis. In fact, it would also have been natural to label copies of M_{n+2k} in the resolution by reflections of diagram in $\widetilde{Y}_{n,n+2k}$, but we did not do this, to avoid an additional notation.

Proposition 4.7. *The complex (34) describing the resolution of simple by standard modules is exact.*

Proof. M_{n+2k}^{α} has a basis of diagrams $\gamma \in B_{n+2k}^c$ without right returns. Composition $\gamma\alpha'$, where α' is the reflection of α about a vertical axis, is a diagram in B_n^c without right returns. The element $d_{k,\beta,\alpha}(\gamma)$, when nonzero, is, up to a sign, a diagram without right returns in $B_{n+2(k-1)}^c$, and diagrams $\pm d_{k,\beta,\alpha}\beta'$ and $\gamma\alpha'$ are equal. Consequently, the complex (34), with L_n removed, decomposes into the direct sum of complexes of vector spaces, over all diagrams $b \in B_n^c$ without right returns, with basis elements of the underlying vector space corresponding to pairs (α, γ) as above with $\gamma\alpha' = b$. Each such direct summand is a complex isomorphic to the r-th tensor power of the contractible complex $0 \to \mathbf{k} \xrightarrow{\cong} \mathbf{k} \to 0$, where r is the number of left returns of b. Only the diagram 1_n leads to a summand with nontrivial cohomology, which maps isomorphically onto L_n, implying that (34) is exact. □

To build a projective resolution of a simple module L_n, we start with the resolution (34) of L_n by standard modules and then convert each standard module M_{n+2k} into its projective resolution (28). These resolutions combine into a bicomplex in the second quadrant of the plane; Fig. 13 shows one square of the bicomplex.

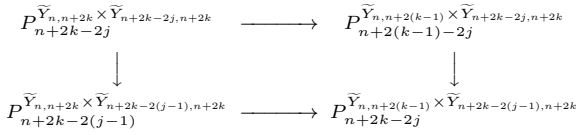

Fig. 13. An anticommutative square in the bicomplex for a simple module.

Horizontal and vertical differentials are defined identically to those in complexes (34) and (28), correspondingly. Differential applied to a single term in the direct summand in the upper left corner is a signed sum of maps in commutative squares

$$\begin{array}{ccc} P^{\alpha_1\times\alpha_2}_{n+2k-2j} & \longrightarrow & P^{\beta_1\times\alpha_2}_{n+2(k-1)-2j} \\ \downarrow & & \downarrow \\ P^{\alpha_1\times\beta_2}_{n+2k-2j+2} & \longrightarrow & P^{\beta_1\times\beta_2}_{n+2k-2j} \end{array}$$

defined in the same way as for complexes (34) and (28), where $\alpha_1 \in \widetilde{Y}_{n,n+2k}$, $\beta_1 \in \widetilde{Y}_{n,n+2(k-1)}$, etc.

The projective resolution of simple module L_n is obtained by forming the total complex of this bicomplex. Due to finite-dimensionality of homs between projective modules P_m, this is the unique minimal resolution of L_n. Any other projective resolution is isomorphic to the direct sum of the minimal one with contractible complexes of the form $0 \to P \xrightarrow{1} P \longrightarrow 0$ in various homological degrees.

Proposition 4.8. $\mathbf{k}$*-vector space* $\mathrm{Ext}^k(L_n, L_m)$ *has dimension* $\binom{\frac{3n+2k+m}{4}}{n}\binom{\frac{n+2k+3m}{4}}{m}$ *if in each of the two binomials* $\binom{a}{b}$ *above both* $b, a-b$ *are nonnegative integers. Otherwise* $\mathrm{Ext}^k(L_n, L_m) = 0$.

Corollary 4.9. *Simple* A^c*-modules* L_n *have infinite homological dimension.*

4.1. Koszul algebra structure and relations

Algebra A^c has a structure of a graded algebra where grading is given by the total number of left and right returns in a diagram. The zeroth degree part of A^c is semisimple, being the direct sum of ground fields $\mathbf{k}1_n$. The projective resolution of L_n is naturally graded, with the m-th term generated by degree m elements, since the differential is a sum of maps over diagrams with a single return, all of which have degree one.

The referee has pointed out that A^c is the associated graded of the filtered algebra $TL_{q,t}$ for the filtration given by the minimal number of U-turns in a diagram.

Corollary 4.10. *The Chebyshev algebra* A^c *is Koszul.*

We now write A^c via its generators and homogeneous relations. The basis of A^c in degree zero is $\{1_n\}_{n\geq 0}$. Basis of degree one part of $1_nA^c1_{n+2}$ is given by diagrams b^i_{n+2}, $1 \leq i \leq n+1$, see Fig. 10. Basis of degree one part of $1_{n+2}A^c1_n$ is given by diagrams ${}^ib_n$, $1 \leq i \leq n+1$, which are reflections of b^i_{n+2} relative to a vertical axis, see Fig. 10. For $m \neq n \pm 2$, the degree one subspace of $1_nA^c1_m$ is trivial. The defining relations, are quadratic, except for the relations involving idempotents 1_n. The latter relations have degree zero and one:

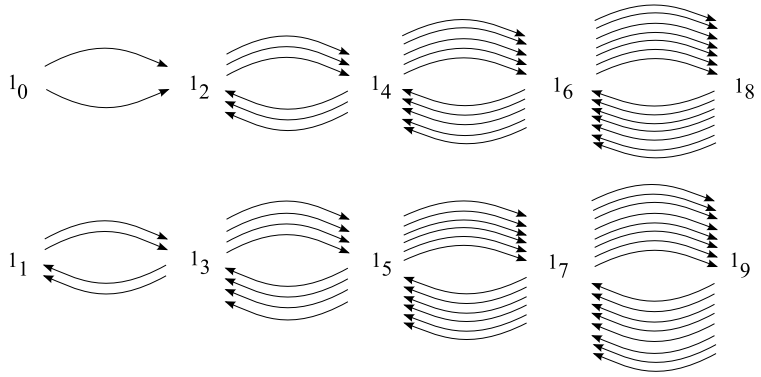

Fig. 14. Quiver describing homogeneous generators of the algebra A^c, with $n+1$ arrows in both directions between 1_n and 1_{n+2}.

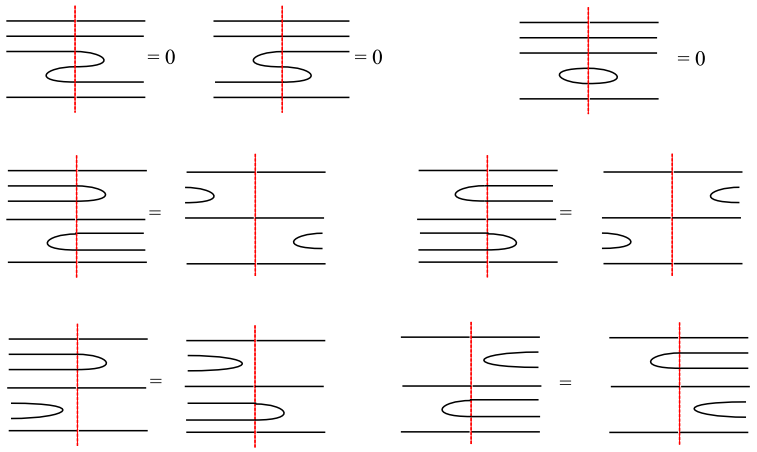

Fig. 15. Chebyshev quadratic relations from (35)-(39): from left to right and top to bottom.

$$
\begin{aligned}
1_n 1_m &= \delta_{nm} 1_n, \\
b^i_{n+2} 1_m &= \delta_{n+2,m} b^i_{n+2}, \\
1_m b^i_{n+2} &= \delta_{m,n} b^i_{n+2}, \\
{}^i b_n 1_m &= \delta_{n,m} {}^i b_n, \quad 1_m {}^i b_n = \delta_{n+2,m} {}^i b_n,
\end{aligned}
\qquad
\begin{aligned}
b^i_n \, b^j_m &= 0 \text{ if } m \neq n+2, \\
b^i_{n+2} \, {}^j b_m &= 0 \text{ if } m \neq n, \\
{}^i b_n \, {}^j b_m &= 0 \text{ if } n \neq m+2, \\
{}^i b_n b^j_{m+2} &= 0 \text{ if } n \neq m.
\end{aligned}
$$

Diagrammatically, these relations come from the conditions that the product of diagrams is zero if the number of endpoints does not match. We can represent these generators as arrows in the quiver which is a disjoint union of quivers for even and odd values of n in 1_n, see Fig. 14.

The following genuine quadratic relations are shown schematically in Fig. 15:

$$b^i_{n+2} \, {}^{i\pm 1} b_n = 0, \tag{35}$$

$$b^i_{n+2} \, {}^i b_n = 0, \tag{36}$$

$$b^i_{n+2} \, {}^j b_n = \begin{cases} {}^j b_{n-2} \, b^{i-2}_n & \text{if } i \geq j+2 \\ {}^j b_{n-2} \, b^{i-2}_n & \text{if } j \geq i+2, \end{cases} \tag{37}$$

$${}^i b_{n+2} \, {}^j b_n = {}^{j+2} b_{n+2} \, {}^i b_n \text{ if } j \geq i+2, \tag{38}$$

$$b^i_{n+2} \, b^j_n = b^{j-2}_{n+2} \, b^i_n \text{ if } j \geq i+2. \tag{39}$$

4.2. Approximations of the identity via truncation functors

Following the notation ${}_nB^c_m(k)$ for the diagrams of width k, let $A^c(\leq k) \subset A^c$ denote the two-sided ideal of A^c generated by diagrams with at most k through strands. Notice that $A^c(\leq k) \subset A^c(\leq k+1)$ and that $\cup_{k\geq 0} A^c(\leq k) = A^c$. For $k \geq 0$, define a right exact functor $F_k : A^c\text{−mod} \to A^c\text{−mod}$ by $F_k(M) = A^c(\leq k) \otimes_{A^c} M$.

Proposition 4.11. *Functor F_k acts on indecomposable projective modules by the identity, $F_k(P_n) = P_n$ for $n \leq k$, and $F_k(P_n) = P_n(\leq k) := A^c(\leq k)1_n$ for $n > k$. For standard modules $M_n \in A^c\text{−mod}$ the action is $F_k(M_n) = M_n$ for $n \leq k$ and $F_k(M_n) = 0$ for $n > k$.*

Proof. The proof is straightforward. Note, that M_n has a basis of diagrams with exactly n through strands. Modules $F_k(P_n) = P_n(\leq k)$ for $n > k$ have finite homological dimension, since they admit finite filtrations with successive quotients isomorphic to standard modules M_{n-2m} for $n-2m \leq k$. In particular, $P_n(\leq k)$ has a finite length resolution by finitely-generated projective, and F_k is a well-defined functor on the category of bounded complexes of finitely-generated projective A^c-modules. □

Proposition 4.12. *Derived functors of the functor F_k applied to a standard module are $L^iF_k(M_n) = M_n$ if $n \leq k$ and $i = 0$, otherwise $L^iF_k(M_n) = 0$.*

Proof. To compute the derived functor LF_k on M_n we apply F_k to the terms of the projective resolution (28), removing non-projective term M_n on the far right of the diagram. If $n \leq k$, F_k acts as identity on all terms of the resolution. Consequently $L^0F_k(M_n) \cong M_n$ and $L^iF_k(M^n) = 0$ for $i > 0$ in this case. If $n > k$, applying F_k to all terms of the projective resolution (28) results in an exact complex. In the standard diagram bases of projective modules in this resolution, applying F_k removes all diagrams of width greater than k. The remaining diagrams, of width at most k, constitute an exact complex of A^c-modules. Note that only diagrams of width n (modulo the span of those of smaller width) constitute a non-exact complex, whose homology is, naturally, M_n. Thus, for $n > k$, the derived functor $LF_k(M_n) = 0$. □

Recall that the Grothendieck group $K_0(A^c)$ of finitely-generated projective A^c-modules has a basis $\{[P_n]\}_{n\geq 0}$ of symbols of indecomposable projective modules. Monoidal structure of A^c takes products of projectives to projectives and induces multiplication on $K_0(A^c)$. The latter can naturally be identified with $\mathbb{Z}[x]$, with $x^n = [P_n]$.

Thus, symbols of indecomposable projective modules correspond to the monomials $[P_n] = x^n$, while symbols of standard modules correspond to Chebyshev polynomials, $[M_n] = \mathcal{U}_n$.

Functor F_k descends to an operator on the Grothendieck group $K_0(A^c)$, denoted by $[F_k]$. This operator acts by

$$[F_k](x^n) = [F_k]([P_n]) = x^n, \quad \text{if} \quad n \leq k,$$

and

$$[F_k](\mathcal{U}_n) = [F_k]([M_n]) = [LF_k(M_n)] = 0 \quad \text{for} \quad n > k.$$

Thus, $[F_k]$ acts by identity on the subspace of polynomials of degree at most k and by zero on the linear span of Chebyshev polynomials $\mathcal{U}_n$ for $n > k$. It's a reproducing kernel, and the projection operator onto the subspace spanned by the first $k+1$ Chebyshev polynomial orthogonally to the subspace spanned by $\mathcal{U}_n$ for $n > k$. We can think of $[F_k]$ as an approximation to the identity operator, which gets better as k goes to infinity. Likewise, the truncation functor F_k and its derived functors can be thought of as approximations to

the identity functor. In particular F_k acts as the identity functor on the full subcategory of the triangulated category generated by modules P_n for $n \leq k$, while annihilating the full subcategory of complexes of projectives generated by resolutions of M_n for $n > k$.

4.3. Restriction and induction functors

For a unital inclusion $\iota : B \subset A$ of arbitrary rings the induction functor $Ind : B-\text{mod} \to A-\text{mod}$, defined by $Ind(M) = A \otimes_B M$, is left adjoint to the restriction functor $Res : A-\text{mod} \to B-\text{mod}$

$$\text{Hom}_A(Ind(M), N) \cong \text{Hom}_B(M, Res(N)).$$

If the inclusion is non-unital, i.e. ι takes the unit element of B to an idempotent $e \neq 1$ of A, the restriction functor needs to be redefined. In this case, to a B-module N the restriction functor assigns an eAe-module $N/(1-e)N$ and then restricts the action to B.

The induction functor is defined as before:

$$Ind(M) = A \otimes_B M \cong Ae \otimes_B M \oplus A(1-e) \otimes_B M = Ae \otimes_B M$$

and the induction is still left adjoint to the restriction. A similar construction works for non-unital B and A equipped with systems of idempotents.

We consider the map from Chebyshev diagrams in ${}_nB^c_m$ to those in ${}_{n+1}B^c_{m+1}$ given by adding a horizontal line above a diagram. This map of diagrams respects composition and induces an inclusion of idempotented algebras $\iota : A^c \hookrightarrow A^c$ such that $\iota(1_n) = 1_{n+1}$. The inclusion gives rise to induction and restriction endofunctors of $A^c-\text{mod}$, which we denote Ind and Res. Note that Res is exact and Ind is right exact. The induction functor takes P_n to P_{n+1}. On the level of Grothendieck group, the induction functor descends to the operator of multiplication by x. The action of the restriction functor Res is more complicated and does not seem to admit an elegant description, since $\text{Res}(P_n)$ is neither projective nor finitely generated.

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