MATHEMATICAL THINKING AND LEARNING
https://doi.org/10.1080/10986065.2020.1793055

The role of balance scales in supporting productive thinking about equations among diverse learners

Ana Stephens[a], Yewon Sung[a], Susanne Strachota[b], Ranza Veltri Torres[a], Karisma Morton[c], Angela Murphy Gardiner[d], Maria Blanton[d], Eric Knuth[e], and Rena Stroud[f]

[a]Wisconsin Center for Education Research, University of Wisconsin-Madison, Madison, Wisconsin, USA; [b]Department of Teacher Education, Ohio University, Athens, Ohio, USA; [c]Department of Teacher Education and Administration, University of North Texas, Denton, Texas, USA; [d]TERC, Cambridge, Massachusetts, USA; [e]Department of Curriculum and Instruction, University of Texas at Austin, Austin, Texas, USA; [f]Department of Education, Merrimack College, North Andover, Massachusetts, USA

ABSTRACT
This research focuses on ways in which balance scales mediate students' relational understandings of the equal sign. Participants included 21 Kindergarten–Grade 2 students who took part in an early algebra classroom intervention focused in part on developing a relational understanding of the equal sign through the use of balance scales. Students participated in pre-, mid- and post-intervention interviews in which they were asked to evaluate true-false equations and solve open number sentences. Students often worked with balance scales while solving these tasks. Interview analyses revealed several categories of affordances of these tools for supporting students' productive thinking about equations.

ARTICLE HISTORY
Received 22 January 2020
Revised 29 June 2020
Accepted 4 July 2020

KEYWORDS
Early algebra; elementary grades; equal sign; mathematical tools; balance scales

In response to low secondary school mathematics achievement and the correlation between advanced mathematics course taking and mathematics achievement, many districts across the U.S. have increased mathematics graduation requirements or have implemented "algebra for all" policies that promote or require the enrollment of all eighth-grade or ninth-grade students in algebra courses (Clotfelter et al., 2015; Plunk et al., 2014). While such policies are generally enacted with good intentions, negative impacts have been felt by many students. Clotfelter et al. (2015), for example, found that district-wide initiatives to greatly increase the number of eighth-grade students in Algebra I (a typical first-year algebra course in the U.S.) significantly weakened performance in that course as well as Geometry, the typical follow-up course. Inequities between low and high performing students were exacerbated as lower and moderately performing students experienced increased failure rates and were more apt to retake Algebra I in high school. Carolan (2014) likewise found negative academic consequences for Black male and female special education students, Black males of low-socio-economic status (SES), Hispanic females, Hispanic males who were English language learners, White females (especially those eligible for Special Education services), and Special Education students of all races and genders following a district-wide ninth-grade "Algebra for Everyone" initiative. These negative consequences included fewer mathematics credits earned, a reduction in the level of the highest mathematics class taken, lower mathematics and overall GPAs, lower college entrance exam scores, and an increased high school drop-out rate. Increased high school drop-out rates across demographic groups – but especially for Black and Hispanic men – have likewise been documented as a consequence of increased math and science graduation requirements (Plunk et al., 2014).

Why have "algebra for all" policies, often implemented with the intention of increasing the achievement of historically marginalized students, done just the opposite? We believe the answer

CONTACT Ana Stephens acstephens@wisc.edu Wisconsin Center for Education Research, Madison, WI 53706

lies in what comes before. That is, we find it unreasonable to assume that an elementary and middle school mathematics education that does not build students' informal intuitions about arithmetic into more formalized ways of reasoning algebraically can be followed by "the most pernicious curricular element of today's school mathematics – late, abrupt, isolated, and superficial high school algebra courses" (Kaput, 1998, p. 25) with an expectation of success.

We prefer to interpret "algebra for all" as a call to provide all students with elementary school experiences in which they are allowed the time and space necessary to develop algebraic ways of reasoning in the context of their arithmetic experiences so that the transition to secondary school algebra is *not* abrupt. Indeed, in reflecting on their findings, Clotfelter et al. (2015) suggested, "It is quite possible that a more thoroughgoing reform of the math curriculum, by way of promoting readiness for algebra by eighth grade, could well prove beneficial" (p. 161).

What is needed, then, is research that explores young students' algebraic thinking and the instruction, tasks, and tools that support such thinking. Our program of research, Project LEAP (Learning through an Early Algebra Progression) has involved developing and testing an early algebra intervention framed by core algebraic ideas and thinking practices in order to produce a curriculum and assessments from which we can better understand early algebra's impact on students' algebraic thinking. In a series of projects aimed at testing the effectiveness of our Grades 3–5 early algebra intervention and documenting progressions in students' algebraic thinking, we found that students who experienced the intervention outperformed their control counterparts on measures of algebra understanding and used more algebraic strategies in problem-solving (Blanton, Isler-Baykal et al., 2019; Blanton et al., 2015; Blanton, Stroud et al., 2019).

In our current work, we are extending our Grades 3–5 program of research while also drawing on additional work in Grades K–2 by Blanton and colleagues (Blanton et al., 2017, 2018) to develop and test a Grades K–2 intervention with the goal of establishing a research-vetted Grades K–5 program of early algebra education. In the work described here, we focus on one core aspect of early algebraic thinking – students' understandings of mathematical equivalence and reasoning with equations – and explore Grades K–2 students' moment-to-moment shifts in thinking as they engaged in tasks designed to help them explore this core concept. Importantly, we focus on the experiences of traditionally marginalized students – students of color, students of low SES, and students who have mathematics difficulties. As it is these students who have so often suffered the consequences of ill-conceived policy initiatives, they must be centered in research efforts that aim to ameliorate "the algebra problem" (Kaput, 2008).

Conceptual framework

Mathematical equivalence: a foundation for algebraic thinking

The concept of mathematical equivalence is foundational to arithmetic and algebraic thinking (Baroody & Ginsburg, 1983; Carpenter et al., 2003; Kieran, 1981; Knuth et al., 2006). Central to this concept is the understanding that the equal sign is a relational symbol denoting the equivalence or "sameness" of the quantities or expressions on either side of an equation. Unfortunately, decades of research have demonstrated that elementary and middle school students consistently struggle to develop a robust understanding of the equal sign as a relational symbol indicating an equivalence relationship (Behr et al., 1980; Matthews et al., 2012; McNeil & Alibali, 2005). This is evidenced in the definitions students give for the equal sign, their abilities to encode (i.e., reproduce from memory) equations, their solutions to missing-value items, and even the equation forms they are willing to accept as possible. Students who lack a robust understanding of the equal sign often assign an "operational" definition to the symbol, stating that it means "give the answer" or "the total" (Knuth et al., 2008; McNeil & Alibali, 2005). Such students often solve open number sentences such as $8 + 4 = \_\_\_ + 5$ by writing 12 or 17 in the blank (Carpenter et al., 2003) and have difficulty accepting equations of forms other than $a + b = c$ because they are in conflict with their operational interpretations of the equal sign (Falkner et al., 1999). Moreover, research suggests that operational thinking is partially rooted in children's informal mathematical experiences prior to kindergarten and formal instruction on equations, underscoring the need to understand how to support children's relational thinking at the start of formal schooling (e.g., Blanton et al., 2018; Seo & Ginsburg, 2003).

Given the well-documented difficulties students have with the equal sign and the genesis of these difficulties in early childhood experiences, we wondered how concrete tools might mediate a relational understanding of the equal sign before an operational conception becomes more entrenched in students' thinking. One model often used to teach elementary and middle grades students about mathematical equivalence and equation solving is a balance scale (e.g., Alibali, 1999; Fyfe et al., 2015; Linchevski & Herscovics, 1996). Rationales for the use of such a model include the analogy that can be made between balancing a scale and balancing an equation (e.g., Warren & Cooper, 2005), the model's grounding in physical experience (e.g., Alibali, 1999; Araya et al., 2010) and the link such a model provides between concrete experience and abstract representations (e.g., Fyfe et al., 2015). In a review of 34 research articles in which the use of a balance model was reported, Otten et al. (2019) found great diversity in terms of the specific types of models used, the purposes of their use, and learning outcomes for students. While their analysis identified some trends, the authors conclude that more research is necessary on the effects of using balance models of different types with students with varying levels of algebra experience.

This paper explores how diverse Kindergarten through second-grade students used physical balance scales to reason about equations. We use the term "diverse" in the way it is often used – to refer to heterogeneity in terms of race, SES, and language status – but we also use it to refer to the fact that many students from one of our (majority White and higher SES) school sites were identified as having mathematics difficulties (see Participants). We focus here on the affordances of physical balance scales for strengthening students' understandings of the equal sign as a relational symbol and their flexibility in thinking about equations. We also share misconceptions or difficulties related to equations that were revealed in students' thinking as students used these tools.

Affordances and the use of tools to shape students' thinking

Vygotsky (1978) argued that our thoughts are impacted by the tools that we use. These tools externally mediate actions that can make explicit processes that come to be internalized by individuals. We draw from Vygotsky in framing our study around the idea that mathematical tools can shape students' thinking. As Hiebert et al. (1997) state, "tools should help students do things more easily or help students do things they could not do alone" (p. 53) and in fact "can enable some thoughts that would hardly be possible without them" (p. 53). While tools are defined very broadly to include oral language, physical materials, written symbols, and existing skills (Hiebert et al., Vygotsky), we focus more narrowly on the use of balance scales. We are interested in exploring the ways in which these tools externally mediate action and influence students' conceptions about the meaning of the equal sign and equations.

In sharing our results, we invoke the notion of "affordances" of the balance scales. Our conception of affordance aligns with that of Greeno (1994) who used the term to refer to "whatever it is about the environment that contributes to the kind of interaction that occurs" (p. 338). Like Greeno, we also draw from Gibson (1977; as cited in Brown & Stillman, 2014) in stressing the importance of the relationship between environment and actor. That is, an affordance is neither solely in the environment (in our case, a balance scale) nor in the mind of an individual but rather arises in the interaction of the two.

Much of the work on affordances in mathematics education has taken place in the context of technological tools. At times, affordances are treated as inherent properties of the technology whereas in other cases (e.g., Brown, 2013; as cited in Brown & Stillman, 2014) the interaction between the technology and learner are considered. Like Brown and Stillman, we do not believe reducing the notion of affordance to something inherent in an object is particularly useful given our goal of

Figure 1. Pan balance representing an inequality relationship.

observing how students interact with tools (in this case, balance scales) to extend their understanding (in this case, of mathematical equivalence).

While Otten et al. (2019) found various justifications for the use of balance scales in teaching students about linear equation solving, the overarching justification concerned the concept of equality and strategies that invoke the idea of balance. The notion of balance was likewise central to our work with students as we investigated their thinking about mathematical equivalence and equations as they worked with pan balances (see Figure 1) and number balances (see Figure 2). Our goal was to identify how students reasoned about the equal sign and equations using these particular tools and, in turn, how these tools shaped students' thinking about equivalence. This also included characterizing misconceptions students demonstrated about equations or the equal sign that these tools helped reveal. Our research question was the following: *How does students' use of balance scales mediate their relational understanding of the equal sign?*

Method

The data we report in this paper come from a larger cross-sectional study in which, building on our work in Grades 3–5, we are developing an instructional intervention designed to engage Grades K–2 students in early algebraic thinking. The study reported here occurred as part of the development of the intervention. We focus here on data collected from students who participated in individual, task-based interviews concerning the meaning of the equal sign and work with equations. The data we share here come from interviews conducted before, during, and after the implementation of our early algebra intervention during our testing phase and in which balance scales were used as part of instruction. However, our focus here is not on pre-to-post growth in students' understandings. Rather, we focus on observations made across the interviews that shed light on how the balance models mediated students' relational thinking.

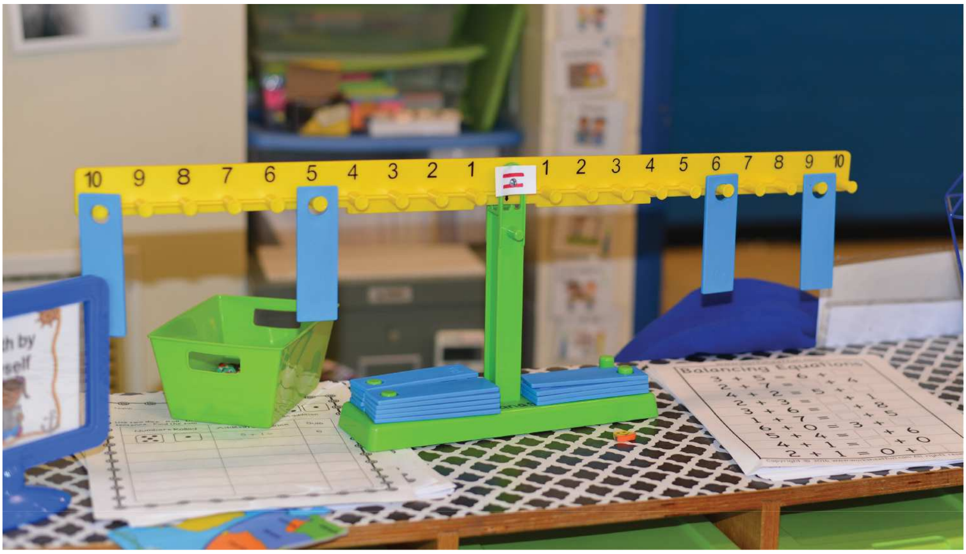

Figure 2. Number balance representing the equation 10 + 5 = 6 + 9.

Participants

Twenty-one students (ten kindergarten students, six first-grade students, and five second-grade students) from two schools participated in task-based interviews concerning the meaning of the equal sign. Given the importance of centering the experiences of students from historically marginalized populations who are most apt to experience negative consequences from an abrupt introduction of algebra that does not take into account the quality of their prior experiences, we were purposeful in our selection of research sites. We selected one of the schools based on its diversity in terms of student ethnicity (64% students of color), socioeconomic status (63% of students qualifying for free or reduced-price lunch), and language status (27% English Language Learners). The second school was selected based on the fact that it serves a high percentage of students with learning difficulties, including learning differences and disabilities.[1]

Early algebra lessons

Interview participants took part in the whole-class early algebra intervention along with their classmates. Eighteen lessons of approximately 30 minutes each were taught in each of Kindergarten, first grade, and second grade. These lessons were spaced throughout the school year (October–May), took place during students' regular mathematics instruction time and were taught by a member of our research team. They addressed a range of early algebra concepts identified and developed in our previous Grades 3–5 work, including mathematical equivalence and equations, generalizations about arithmetic properties and the properties of even and odd numbers, the use of variable, and functional thinking (see Blanton et al., 2015; Fonger et al., 2018 for details regarding the development of the curricular framework used in the previous and current studies).

Nine lessons in Kindergarten, six lessons in first grade, and five lessons in second grade primarily engaged students in thinking about mathematical equivalence and equations. Kindergarten lessons first had students compare the weights of unspecified quantities with a pan balance, using this as a context to develop the notion of balance and introduce the equal sign and equations.

For example, given two opaque bags containing an unknown number of equally weighted small plastic bears (a common mathematics manipulative), students were asked to use the scale to determine if the quantities in the bags were equal or not equal and eventually wrote equations of the form $a = a$ to represent the relationship between equal quantities. Lessons then examined simple true-false and open equations, using pan balances and number balances to explore the relational meaning of the equal sign. The number balance was introduced by having students place one weight on the left side of the scale and then think about where they would need to place a weight on the right side to make the scale balance. This progressed to placing one weight on the left side of the scale and then asking students if they could place *two* weights on the right side to make the scale balance. The resulting balanced scales were represented with equations (e.g., at first 3 = 3 or 5 = 5 and then 5 = 1 + 4 or 5 = 2 + 3).

First-grade students worked with somewhat more challenging true-false and open number sentences (e.g., equations of the form $a + b = c + d$) using number balances, while second-grade students extended this work to include discussions about properties of equations. For example, they were asked to explore equations that exemplified the Commutative Property of Addition.

When not participating in our early algebra intervention (i.e., outside of the 18 lessons), students were taught mathematics by their regular classroom teachers. Teachers at the first school had no prescribed curricula but were expected to follow district pacing guidelines and had access to resources aligned with the *Common Core State Standards* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) provided by their state's department of public instruction and district personnel. Teachers at the second school taught using the *enVision Math* and *GO Math!* curricula.

Data collection

Students participated in teaching experiment interviews in the spirit of Vygotsky's (1978) laboratory experiments that "provide maximum opportunity for the subject to engage in a variety of activities that can be observed, not just rigidly controlled" (p. 12) and furnish data that are "not performance level as such but the methods by which the performance is achieved" (p. 13). These interviews addressed a wide range of early algebra content and took place prior to the early algebra intervention, mid-way through the intervention, and after the conclusion of the intervention. The particular tasks that are the focus of this paper are those that engaged students in solving true-false and open number sentences (see Table 1 for specific tasks included in the interview protocol).

Students were often encouraged to show their thinking using balance scales, while at other times the scales were simply available for students' use at their discretion. While these tools were initially unfamiliar at the time of the pre-intervention interview, students comfortably engaged with them after a brief introduction and continued to do so during the mid- and post-intervention interviews. To introduce the balance scales, the interviewer used a "think aloud" strategy that involved talking through her thinking about how she would determine if the amounts on both sides of a pan balance were the same, or how she would place two weights to balance a number balance or solve a particular

Table 1. Equation tasks posed to students.

Kindergarten		Grades 1–2	
True/False equations	5 = 5	*True/False equations*	7 = 3 + 2
	7 = 3 + 2		8 = 8
	4 = 4		3 + 5 = 7
	5 = 1 + 4		9 = 4 + 3
	4 + 2 = 3		4 + 5 = 9 + 3
	2 + 3 = 5 + 1		3 + 6 = 2 + 7
	3 = 1 + 2		
	4 = 1 + 2	*Open number sentences*	4 = 3 + ___
	2 + 4 = 3 + 3		6 = ___
	6 = 3 + 2		3 + ___ = 5
	4 = 2 + 2		3 + 2 = ____ +3
			4 + 2 = ____ + 4
			4 + 1 = 1 + ___
			4 + ___ = 8 + 4

open number sentence. The think aloud instructional strategy has been identified as a useful approach for struggling learners (Gersten et al., 2009).

The number of tasks in which a particular student was asked to engage depended on the time available and the degree to which students were able to engage with the tasks (with interviews prior to the intervention tending to be shorter than those that took place during or after the intervention). Of the 21 participating students, 13 completed equal sign tasks during all three interviews (pre, mid, and post), seven completed these tasks only during the mid-interview, and one completed these tasks only during the pre- and post-interviews.[2] Interviews were conducted by a member of the research team and were videotaped.

Data analysis

Given that the goals of the project include understanding how instructional tools help a diverse population of learners think about early algebraic concepts, our analysis of the videotaped interviews started with a grounded approach in which coders freely noted what they observed in students' responses about mathematical equivalence and the instructional prompts, discussion, and tools that may have led to such responses. At least two coders watched each video and discussed observations with each other. After this first pass, discussion among all coders focused on the identification of particular categories that would provide the focus for subsequent analyses. These categories were first posited from the pre-intervention interview data and then extended and refined with the subsequent mid- and post-interview data. Subsequent analyses involved a primary coder watching the interview video and populating the categories with interview excerpts while a second coder watched the video and noted agreement or disagreement with the first coder. Any disagreements were discussed and resolved. The majority of categories that emerged from the data – and what we focus on here – concerned the affordances of the balance scales in mediating students' relational understandings of the equal sign as well as the mathematical difficulties and misconceptions these tools helped reveal.

Results

We share the categories of affordances and difficulties observed in the interviews with the intention of highlighting the ways in which students' use of balance scales mediated their understandings of equivalence and the ways in which they interpreted equations. Like Otten et al. (2019) reported in their meta-analysis, we found that the overall benefit of these tools was that they helped students link the notion of balance on a scale with the property of equivalence in equations. Here we discuss more specifically how we observed this benefit play out in the ways in which students interacted with the scales. We identified one overarching affordance that was revealed in students' work with both types of balance scales and three related, more narrowly defined, affordances that were revealed by students' work with the number balance. We report on these here. We also report on a student difficulty that work with both types of balance scales helped to reveal. We exemplify these categories with excerpts from the interviews.

Observed affordances of the balance scales

Recall that only Kindergarten students worked with the pan balance while students from all three grades worked with the number balance. We identified one broad and three more specific "subset" categories of affordances of these tools in terms how they mediated students' thinking about equations and the meaning of the equal sign.

Interactions with balance scales challenged students' initial interpretations of equations

Students' interactions with both the pan balance and the number balance led the majority of students to question their initial interpretations of multiple equations. Importantly, this included expanding the forms of equations students accepted as legitimate.

For example, while a Kindergarten student initially viewed the equation 5 = 1 + 4 as false, his exploration of this equation on the pan balance led him to change his mind.

Interviewer [I] How could we test [the equation 5 = 1 + 4]? Do you think it's true or false? Any ideas?

Student [S] If we do that [represent the equation on the scale] they will not balance.

I You don't think it's going to balance?

[S places five cubes on one side and one and four cubes on the other side.]

I What do you think?

S It did.

I Yeah, it did balance, didn't it? So, is this [equation] right or wrong?

S It's right, cause, look, 1 [holding up one finger on left hand] and 2, 3, 4, 5 [holding up four fingers on right hand]!

Here, the student counted one and four fingers to show that the sum 1 + 4 [the right side of the given equation] is equal to 5.

I So, how many cubes do you have on each side?

S So, this is five [pointing to right side], and this is five, too [pointing to left side].

I That's a great observation.

S That's why it's balancing.

The student counted with his fingers to provide evidence of why the balanced scale showed the equation's truth value ("It's right, cause, look … "). Importantly, it was his interaction with the pan balance that caused him to rethink his initial position and prompted him to construct the justification. In this, the pan balance functioned as a mediating tool by which he explored a mathematical idea embodied in a concrete representation (that is, the idea of equivalence embodied in a balanced scale) as a proxy for reasoning about its abstract, symbolic counterpart (that is, the truth of the equation 5 = 4 + 1). Enabling such connections is an important way to support struggling learners (Gersten et al., 2009).

Given 2 + 3 = 5 + 1, another kindergartner correctly read the equation aloud but did not know whether the equation was true or false. She used the pan balance to explore. As she explained, she placed "two and three" on the left side and "five and one" on the right side. After observing that the scale did not balance, she stated, "not true." Note that this student was successful in determining the equation's truth value by relying completely on the scale's status without needing to compute. This example underscores that students can use balance as a metaphor to reason about equivalence even before they fully understand symbolized expressions and equations such as 2 + 3, 5 + 1 and 2 + 3 = 5 + 1 and can perform these operations. In this way, a balance scale where quantities are not specified can mediate students' intuitive, pre-symbolic thinking about equivalence in ways that potentially prepare them for reasoning with symbolized operations.

Like the pan balance, the number balance served as a tool to introduce unfamiliar equation forms to students as well as challenge students' initial conceptions of equations. As an example of the former, when first introducing the number balance to a first grader during the pre-interview, the interviewer employed a think aloud strategy to consider how she could make the scale balance by placing one weight on each side of the scale. Describing her actions and thinking aloud for the student, she placed a weight on the "5" on each side and wrote "5 = 5" to show the student what the scale was representing, pointing out the "5 s" on each side of the scale and on each side of the equation. The student said he had never seen that kind of equation before, but that it made sense. That he did not reject the equation outright (as young students who are unfamiliar with equations of the form $a = a$ often do) suggests that introducing equivalence through the concrete representation of the balance scale mediated the student's thinking about the validity of its abstract representation (the equation 5 = 5).

As an example of the latter, many students initially said that 8 = 8 was false but were challenged to reconsider after working with the number balance. One first grader explained that 8 = 8 was false because "they didn't put a plus and a zero there [pointing to the blank space to the left of the equation]." Such a response is not unusual among students first learning about the equal sign, as many expect equations to involve operations (e.g., Behr et al., 1980). When asked to represent the equation on the scale, this student correctly placed the weights and said, "It says it is true." When asked, "Did we have to do a plus zero here for this to be true?" the student shook his head. When this same student was later asked to solve 6 = ___, he initially exclaimed, "What?! There are only two numbers." (We believe he assumed the blank represented one number, which was our intention.) He placed a weight at six on the left side of the scale and after thinking for a moment placed a weight at six on the right side as well. Upon seeing the scale balance, he wrote "6" in the blank.

I Is six the same as six? Is six equal to six?

S Well, but if you put a zero, then it's six.

I You mean like a plus zero?

S Yeah.

I You want to put a plus zero?

S A zero and then a plus.

I Okay. You can write that in.

[S writes "0 +" before the equation, making it 0 + 6 = 6].

I Was it still balanced even without that?

S Yes, still because there is no zero on it, and . . .

I Well, I guess we could think of the very middle as zero. Let's see what happens if we add zero [placing a weight in the center of the scale].

S It still is because if you put it right here [moving the center weight to "1" on the left side] it goes in another way.

While this student still clearly preferred that equations have operations, the number balance was acting to challenge this thinking and, in doing so, provided a context for discussing whether equations such as 8 = 8 and 6 = 6 are legitimate given that they contain no operations.

Other students were aided by the scale in accepting what they initially deemed as "backwards" equations. A first grader, for example, initially said that 5 = 3 + 2 was false for this reason but, after representing the equation on the scale, changed her mind, and decided it was true.

We focus next on a series of tasks presented to a first grader to illustrate how the number balance mediated his thinking about the validity of equations in different forms, a critical component in understanding equivalence. This student initially said 7 = 3 + 2 was false "because these numbers [indicating the right side of the equation] need to go right here [indicating the left side of the equation]." While the student was correct that the equation is false, and confirmed this using the scale, his reasoning was not correct and indicated difficulty accepting equations of forms other than *a* + *b* = *c*. When next asked whether 3 + 2 = 7 was true or false, the student reasoned correctly that it was false "because three plus two is not seven, it's five."

The student was next posed the equation 8 = 8 and said it was false. The interviewer asked him to try the equation on the number balance. The student correctly placed the weights on the scale and then, looking a little surprised, said "It's right." He noted that when he put a weight on the same number on each side, the scale balanced. He explained that he initially thought the equation was false "because I thought that eight plus eight was 16," indicating his difficulty with equations lacking an operation. Research shows that students with misconceptions about the equal sign can interpret an equation such as 8 = 8 as 8 + 8 = 16, just as this student did (Blanton et al., 2018). That is, even though the equation 8 = 8 does not contain a "16" or a "+," students who have an operational view of the equal sign might still "see" it as 8 + 8 = 16. Moreover, there is nothing inherent in the symbolism itself that challenges their thinking. However, as with this student, a concrete tool such as a number balance can prompt students to begin to rethink previously held misconceptions.

The student then indicated that 3 + 5 = 7 was false "because it's three and then you put five more, it's eight" and confirmed this on the scale. Next, 9 = 4 + 3 was determined to be false because "it's the same but with different numbers." In retrospect, we are not sure if this student was referring to the earlier 7 = 3 + 2 and was indicating that the operations should be on the left side of the equation or was referring to the immediately prior 3 + 5 = 7 and was indicating that the sum was incorrect. The equation was confirmed to be false using the scale.

The equation 4 + 5 = 9 + 3 was next deemed false because "if it says equal [pointing to equal sign], and right here is not equal [pointing to the space after the three]."

I So there is no equal sign after the three? Why would you want there to be an equal sign after the three?

S Because if there's no equal [sign] we don't know what the answer is.

The student confirmed the equation was false on the scale, but again, the fact that it is false allowed the faulty reasoning to go unchallenged.

The next equation posed to the student, 3 + 6 = 2 + 7, did challenge his thinking. The student initially said the equation was false because it had the same problem as the last one (i.e., there needed to be an equal sign after the seven). The student represented the equation correctly on the scale and was surprised to see that, unlike the previous equation, it balanced. The interviewer tried to help him make sense of this surprise:

I What is 3 + 6? Can you figure that out for me?

S [Counting on fingers] Nine.

[I writes 9 above "3 + 6"]

I How about 2 + 7?

S [Counting on fingers] Nine.

[I writes 9 above "2 + 7"]

I This [pointing to equation] is saying that nine [pointing to the left nine] is the same as [pointing to the equal sign] nine [pointing to the right nine]. Do you agree with that?

S Yeah.

I Yeah. So, does it make sense that our scale would balance?

S [Hesitantly] Yeah.

This student's thinking was clearly challenged, and he was beginning to make progress toward accepting equations in unfamiliar forms. He was then posed a series of open number sentences, starting with 4 = 3 + ___.

S One.

I How did you already know that?

S Because four [holding out four fingers on his right hand] and three [holding out three fingers on his left hand] and these are the same [matching up three fingers from each hand and showing one unmatched finger] and if you take the three you have one.

The student confirmed this solution with the scale, accepting an equation form that was recently rejected as backwards. He was next posed 6 = ___.

S [After thinking for 15 seconds] Six?

I What should the scale do if this is true?

S If you put it on the same one [placing weights on six on either side], it will go like that [indicating a balanced scale].

As illustrated by this extended example, the number balance served as a tool to challenge this student's conception that equations must be in the form $a + b = c$. True equations of the form $a = a$, $c = a + b$, and $a + b = c + d$ proved especially fruitful for creating a contradiction between this student's initial conception and the feedback he received after placing weights on the number balance. This was the case for several students when they were asked to use the pan and number balances to represent true equations that they initially deemed to be false. This is not to suggest that all difficulties were eliminated, but the balance scales served as a tool to mediate students' thinking about equations that they had initially rejected because of their form.

We next share three related affordances of the number balance with respect to mediating students' understandings of the equal sign. These affordances are related in the sense that they, too, lead students' initial conceptions of equations to be challenged. They might be considered "subset" affordances in the sense that they are more narrowly defined than those described thus far.

Interactions with the number balance encouraged trial-and-error equation solving

One helpful feature of the number balance is that it allowed for a "trial and error" approach when solving open equations. Students could move weights from number to number on either side of the scale until balance was achieved. Achieving balance was often followed by reflection on the solution and sometimes a rethinking of students' initial understandings of the equations. This "trial and error" affordance of the number balance was demonstrated by over one-third of students across the interviews. That is, rather than serving as a way for students to simply show an already existing form of (correct or incorrect) thinking about equivalence, the number balance actually prompted a sequence of refinements in problem-solving as they engaged with the tool.

As an example of this "trial and error" affordance, consider the approach taken by a first grader just being introduced to the number balance. While the interviewer began a think aloud with the equation 4 = 3 + ___, placing a weight on four on the left side and three on the right, the student took the remaining weight and tested it in various places on the right side of the number balance, trying four, eight, and nine before finding the balance at one. She then immediately wrote "1" in the equation's blank, indicating an understanding of the relationship between the scale's status and the equation's solution.

Another student demonstrated operational thinking about the equal sign that was challenged by her experiences with the number balance. Given 4 + 2 = ___ + 4, a second grader initially wrote "6" in the blank "because four plus two equals six" (a common "add to the equal sign" response among students with an operational view of the equal sign). The interviewer asked her to test her answer on the number balance, at which point she noticed the scale did not balance and the equation as she completed it was thus not true.

I Should we put another number there [indicating the blank]?

S Yeah.

I What do you think? What number would make this true?

S Ten?

I Tell me how you got ten.

S Because six plus four equals ten.

While the student did not gesture to the equation, it appeared she was reading the right side of the equation with her inserted "6" in the blank. The interviewer encouraged her to test this on the number balance. She moved the weight currently on six to ten and found that it again did not balance. She then engaged in trial and error, moving the weight to seven, five, four, and three before ultimately finding the balance at two. The interviewer then asked the student to read the completed equation aloud and think about why replacing the blank with two made the equation true. While the student did not note the Commutative Property of Addition relationship, she did state that "they both equal six." The fact that the equal sign indicates an equivalence relationship between the quantities on either side of the symbol is an important understanding that was mediated by this student's use of the number balance. This tool allowed this student to reconsider her thinking in a way that simply working with an equation on paper might not.

Interactions with the number balance encouraged productive "tinkering" and play

Closely related to the *trial and error* affordance is the notion that the number balance can encourage students to "tinker" and play to explore questions of their own. While less frequently observed than the aforementioned categories, we noted that occasionally students veered from the specific task at hand to do some exploring of their own. This typically occurred with false equations, when students sometimes wanted to figure out how the equation could be changed so that it would be true. Two first graders did this after representing 3 + 5 = 7 on the number balance and finding it to be unbalanced. One first grader moved the weight that was placed on five to four to balance the scale, representing the equation 3 + 4 = 7. The other first grader moved the weight that was placed on seven to eight to balance the scale, representing the equation 3 + 5 = 8. A second grader similarly represented 9 = 4 + 3 and then sought to make the scale balance by moving the weight placed on nine to eight and then to seven, representing the equation 7 = 4 + 3. In this, the scale encouraged students to modify and extend the original task (from "decide if the equation is true or false" to "find a way to make the equation true"), something that equations in their abstract (symbolic) form generally do not. That is, in our experience students who evaluate an equation such as 9 = 4 + 3 as false in a paper/pencil format rarely, if ever, step spontaneously further into the task to imagine how they might make a false equation true.

Interactions with the number balance helped students notice equation structure

Our first- and second-grade interviews included the open number sentences 4 + 2 = ___ + 4, 3 + 2 = ___ + 3, 4 + 1 = 1 + ___, and 4 + ___ = 8 + 4 to exemplify the Commutative Property of Addition. We found that the majority of first and second graders explicitly recognized equation structure (if not the Commutative Property explicitly) as a consequence of their number balance work on one or more of these equations.

For example, given the equation 4 + 2 = ___ + 4, a first grader used the number balance to place the four, the two, and the four correctly and then hung a weight at six on the right side before quickly finding that the scale balanced when the weight was placed at two. Upon seeing the balance, he immediately said "Oh! Because it's the same. Because this is two and this is four [pointing to weights on left side of scale] and this is two and this is four [pointing to weights on right side of scale]."

Another student initially said he would put six in the blank given 4 + 1 = 1 + ___, using the "add all" strategy previously mentioned. He placed weights on the number balance to confirm and found that the scale was unbalanced. He then moved the weight that was at six to four because "1 + 4 is on this side [pointing to right side of scale] and 1 + 4 is on this side [pointing to left side of scale]." The student then solved 4 + ___ = 8 + 4, immediately placing the weights in the correct places on the scale and writing eight in the blank. When asked how he knew so quickly, the student pointed to the earlier 4 + 1 = 1 + ___ equation on his paper. He had difficulty articulating his thinking, but it was clear he was seeing a parallel in these two equations. The student's work with the number balance given 4 + 1 = 1 + ___ mediated his thinking in a way that allowed him to solve 4 + ___ = 8 + 4 very quickly, only using the number balance to demonstrate his solution.

In sum, we found that the balance scales helped students engage with equations in ways that would not have been possible without these tools and that advanced their understandings of equivalence. In addition to promoting understanding, the balance scales also made visible some of the difficulties students faced when working with equations. We turn to this next.

Difficulty revealed through students' interactions with the balance scales

While we have focused thus far on the affordances of the pan and number balances in terms of encouraging students to think in new ways, another benefit we observed was that the ways that students used these tools often revealed difficulties in their thinking about the equal sign and the role that it plays in an equation. While many cases of students treating the equal sign as an operational symbol have already been highlighted in our discussion of tool affordances, we focus here on a difficulty that was revealed immediately in the ways in which students initially represented equations on the scales.

We observed that the majority of students, at times throughout the interviews, had difficulty viewing an equation as a comparison of two equivalent quantities. This difficulty was manifested in two ways. First, students at times did not attend to the "sides" of given equations when representing these equations on the balance scale, instead representing numbers from both sides of an equation on one side of the scale. Second, students at times did not attend to equations in their entirety when representing them on the balance scales.

For example, we observed a kindergartner putting two cubes and three cubes on the same side of the pan balance given the equation 4 + 2 = 3; a kindergartner placing weights at three on the left side and six and two on the right side of the number balance given the equation 6 = 3 + 2 and all weights on the left side given the equation 2 + 3 = 5; and a second grader putting two weights at four on the left side and one weight at 8 on the right side given the equation 4 + ___ = 8 + 4. These errors suggest a lack of understanding of the equal sign and its role in an equation of denoting a comparison of two separate quantities. A secondary issue could be in how students understand the tool itself.

As examples of students not attending to equations in their entirety, two kindergartners ignored the two in 7 = 3 + 2, representing this equation with a pan balance by placing seven cubes on one side and three cubes on the other. One of these same kindergartners represented 4 = 1 + 2 on the number balance by placing only the four and one on opposite sides. The other of these kindergartners approached 6 = 3 + 2 with an "add all" strategy, looking for an 11 on which to place his weight on the number balance.

While the fact that students do not always attend to the equal sign or to equations in their entirety has been previously documented (e.g., McNeil, 2014), we believe what is important here is the fact that these difficulties were revealed through students' engagement with the balance scales. In what follows, we discuss what the results shared above reveal about the potential of balance scales to mediate students' relational understanding of the equal sign.

Discussion

The importance of a robust understanding of the equal sign and its role in equations has been established as critical to students' success in algebra (Knuth et al., 2006). Mathematical equivalence has thus emerged as a core concept on which to focus early algebra efforts (e.g., Blanton et al., 2015) that seek to build formalized ways of thinking from students' experiences in arithmetic. Working from the stance that tools mediate thinking (Vygotsky, 1978), we sought to investigate the ways in which two particular tools – the pan balance and the number balance – could mediate students' understanding of equivalence, particularly the equal sign. It is in understanding how these tools mediate students' mathematical thinking with a diverse student participant group – including those with math difficulties – that our work makes a contribution.

We drew from Greeno (1994) in stressing that the "affordances" of a particular tool – in our case balances scales – concern the interaction of learners with the tool. The affordances of the balance scales we identified from our observations of Kindergarten through second-grade students had largely to do with helping students (begin to) build appropriate conceptions of the equal sign and equations in a way that may have been more difficult to do had the only tools available been a written equation and a pencil. We highlight here a few of these conceptions and discuss how students' interactions with balance scales helped shape their thinking.

The equal sign indicates the equivalence of two quantities

In their meta-analysis of research studies where balance scales were used to teach linear equation solving, Otten et al. (2019) found that the overarching reason these scales were used was to help students understand equivalence. In the studies they reviewed, "inherent properties of the balance were connected to the concept of equality and the strategies that can be applied while maintaining the balance" (p. 12). This equality-balance connection was evidenced in the responses of many of our participants who demonstrated an understanding that whether a scale balanced or not was "telling them" something about the equation. These students connected the concept of a balancing scale with a balancing equation and thus were able to allow the tool to impact their thinking (Vygotsky, 1978) about whether particular equations were true or false or what their solutions were. At times the feedback obtained from the balance scales was surprising; for example, in the case of the second grader who, given 4 + 2 = ___ + 4, thought a six should be placed in the blank and upon placing weights on the number balance realized that something was not right. Interacting with the scale allowed the student to revise her solution because she understood what the status of the tool (balanced or not balanced) indicated about the more abstract equation. At times the conflict between operational misconceptions of the equal sign and the expectation of the equality-balance connection were not resolved, but the scales clearly provided an opportunity for students to at least begin to develop this important conception.

Equations do not have to look like 5 + 3 = 8

Given the experiences of many students in arithmetic-focused elementary school mathematics classrooms, it should not be surprising that many students believe that equations should look something like 5 + 3 = 8, 2 + 2 = 4, or 5 + 0 = 5 (Seo & Ginsburg, 2003). The students we observed made many statements indicating they held this misconception; for example, the first grader who said 7 = 3 + 2 was false "because these numbers [indicating the right side of the equation] need to go right here [indicating the left side of the equation]." However, reasoning with balance scales created for students a cognitive conflict about this misconception which, in some cases, resolved to a complete, relational understanding. This again was possible because the students had developed an understanding of the way in which the concrete tool provided information about the more abstract equation. The first grader who said 7 = 3 + 2 was false (with faulty reasoning), for example, came to hesitantly agree that 3 + 5 = 2 + 7 was true because he understood that a balanced scale was an indication of a balanced (or true) equation. His thinking was shaped by his experience interacting with the tool (Greeno, 1994; Vygotsky, 1978), and he was later able to successfully and confidently solve 4 = 3 + ___.

Equations do not need to involve operations

The understanding that an equation does not need to involve an operation is a subset of the aforementioned "equations do not have to look like 5 + 3 = 8," but it is one that is worth mentioning given how frequently discomfort with 6 = 6 or 8 = 8 was observed among our first- and second-grade participants (who likely had more experiences with equations than our Kindergartners and thus more opportunities to develop entrenched operational conceptions of the equal sign). Exclamations such as, "What?! There are only two numbers." and the desire to insert a "plus zero" were not uncommon and not surprising given the aforementioned difficulties students often experience seeing the equal sign as a relational symbol. As was the case with equations of the form $c = a + b$ and $a + b = c + d$, students moved toward acceptance of equations of the form $a = a$ when their experiences with balance scales challenged their thinking and they were able to draw on their understanding of the equivalence-balance connection. We emphasize again that it is not simply that the balance scales "told" students they were wrong (e.g., about thinking that 6 = 6 or 8 = 8 are not true) but that their interaction with the scale and their understanding of the connection between a balancing scale and a balancing equation allowed their thinking about the abstract forms to change.

Equations can be reasoned about without computation

In the presentation of the results, we at times alluded to the fact that students working with the balance scales worked with individual numbers and did not immediately (or ever) engage in computation. For example, we described a kindergartner who placed "two and three" and "five and one" on the two sides of a pan balance rather than "five" and "six," respectively. While this may seem trivial, we observed that computations such as these were in fact difficult for some of our participants. Recall that half of our participants came from a school that served a high percentage of struggling students, including those with learning difficulties. Given our desire to help *all* students be successful in algebra, it is important to find creative entryways into algebraic thinking that do not require computational fluency as a prerequisite to engagement. The balance scales served as a tool to allow students to consider, for example, whether 2 + 3 = 5 + 1 was true or false, and thus engage in important thinking about equations, balance, and the meaning of the equal sign, without being denied these opportunities due to difficulties with computation. We believe this was possible because students' work with the scales encouraged them to think about equations holistically – in terms of whether the equations balanced or not – rather than as computations that needed to be performed.

Conclusion

An important goal in the shift toward introducing core algebraic ideas in the elementary grades is the engagement of *all* learners in algebraic thinking. Meeting this goal requires the use of instructional strategies, tasks, and mathematical tools that build on what students already know and help move their thinking forward. Future research might include implementing the early algebra intervention with a larger number of students and investigating the impact on students' understanding of mathematical equivalence; investigating the knowledge that students initially bring to balance scales or even the concept of balance itself and how we might build on these early conceptions; comparing the impact of different forms of balance scales (e.g., physical scales, pictures, or embodied experiences) on students' understandings; and exploring any possible connections between students' growing algebraic thinking and their computational fluency.

We purposefully included a diverse (broadly defined) student sample in this study knowing that it is these students who often feel the negative impact of algebra's gatekeeper status (Carolan, 2014; Clotfelter et al., 2015; Plunk et al., 2014) and for whom new approaches to algebra teaching and learning must be successful. The Kindergarten through second-grade students whom we interviewed showed that they were able to move forward in their understanding of equations and the meaning of the equal sign through the use of balance scales. Importantly, this included students who lacked fluency with computation, a skill that in many mathematics classrooms might be viewed as prerequisite to engagement in algebraic thinking. We argue that the balance scales did, in fact, "enable some thoughts that would hardly be possible without them" (Hiebert et al., p. 53) and in doing so allowed students to engage more deeply in algebraic thinking.

Notes

1. The population at this second school included 18% students of color and 17% who qualified for free or reduced-price lunch. We do not have data regarding the number of students in these classrooms with identified disabilities but can report that of the elementary schools in this district, this school is known for having a concentration of resources to support students with learning difficulties. Students struggling to meet grade-level expectations were identified by classroom teachers for participation in interviews.
2. Three different interview forms – two of which included equivalence tasks – were implemented in this study. As students had not yet encountered some of the early algebra content included in the intervention by the time of the mid-interview, more students were asked questions from interview forms that included equivalence tasks (i.e., content that had already been addressed in the intervention) at this time point.

Acknowledgments

The research reported here was supported in part by the National Science Foundation under Award # 1720129. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Notes on contributor

Ana Stephens is a Researcher at the Wisconsin Center for Education Research at the University of Wisconsin–Madison. Her research addresses the development of elementary students' algebraic reasoning. She is particularly interested in students' understanding of mathematical equivalence and the tasks that support the development of this understanding.

Yewon Sung is a doctoral student in Mathematics Education at the University of Wisconsin–Madison. She is interested in learners' ways of thinking and feeling when learning early algebra in the elementary grades.

Susanne Strachota is the Faculty-in-Residence at the Patton College of Education at Ohio University. She studies how students develop algebraic reasoning.

Ranza Veltri Torres is a doctoral student in Mathematics Education at the University of Wisconsin–Madison. Her current research concerns mathematics identity, particularly in community college students who are enrolled in non-credit-bearing remedial math coursework.

Karisma Morton is an Assistant Professor in the Department of Teacher Education and Administration at the University of North Texas. Her research explores gender and racial/ethnic inequity in mathematics education with a particular interest in the impact of well-designed and reform-based mathematics curriculum and pedagogy that promotes mathematical literacy for all students.

Angela Murphy Gardiner is a senior research associate at TERC, Inc., in Cambridge, MA. Her primary research interests include students' algebraic thinking and understanding of functions, methods for delivering effective teacher professional development, and understanding the relationship between characteristics of professional development and student outcomes.

Maria Blanton is a Senior Scientist at TERC, Inc., in Cambridge, MA. Her research focuses on the cognitive foundations of children's algebraic thinking and the impacts of early algebra education on children's algebra-readiness for middle grades.

Eric Knuth is a Professor and Director of the STEM Center at the University of Texas at Austin. His research concerns students' engagement in mathematical practices, particularly practices related to algebraic reasoning and mathematical argumentation (justifying and proving).

Rena Stroud is an Assistant Professor in the Department of Education at Merrimack College. She is a quantitative

methodologist with research interests in teaching and learning in early mathematics.

References

Alibali, M. W. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental Psychology*, *35*(1), 127–145. https://doi.org/10.1037/0012-1649.35.1.127

Araya, R., Calfucura, P., Jiménez, A., Aguirre, C., Palavicino, M. A., Lacourly, N., … Dartnell, P. (2010). The effect of analogies on learning to solve algebraic equations. *Pedagogies: An International Journal*, *5*(3), 216–232. https://doi.org/10.1080/1554480X.2010.486160

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *Elementary School Journal*, *84*(2), 199–212. https://doi.org/10.1086/461356

Behr, M. J., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching*, *92*, 13–15.

Blanton, M., Brizuela, B., Gardiner, A., & Sawrey, K. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics*, *95*(2), 181–202. https://doi.org/10.1007/s10649-016-9745-0

Blanton, M., Isler-Baykal, I., Stroud, R., Stephens, A., Knuth, E., & Gardiner, A. M. (2019). Growth in children's understanding of generalizing and representing mathematical structure and relationships. *Educational Studies in Mathematics*, *102*(2), 193–219. https://doi.org/10.1007/s10649-019-09894-7

Blanton, M., Otalora Sevilla, Y., Brizuela, B., Gardiner, A., Sawrey, K., & Gibbons, A. (2018). Exploring kindergarten students' early understandings of the equal sign. *Mathematical Thinking and Learning*, *20*(3), 167–201. https://doi.org/10.1080/10986065.2018.1474534

Blanton, M., Stephens, A., Knuth, E., Gardiner, A., Isler, I., & Kim, J. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education*, *46*(1), 39–87. https://doi.org/10.5951/jresematheduc.46.1.0039

Blanton, M., Stroud, R., Stephens, A., Gardiner, A., Stylianou, D., Knuth, E., Isler, I., & Strachota, S. (2019). Does early algebra matter? The effectiveness of an early algebra intervention in grades 3–5. *American Educational Research Journal*, *56*(5), 1930–1972. https://doi.org/10.3102/0002831219832301

Brown, J. P. (2013). *Perceiving and enacting the affordances of Technology-Rich Teaching and Learning Environments (TRTLE's) for student understanding of function* [Unpublished doctoral dissertation]. University of Melbourne.

Brown, J. P., & Stillman, G. (2014). Affordances: Ten years on. In J. Anderson, M. Cavanagh, & A. Prescott (Eds.), *Proceedings of the 37th annual conference of the mathematics education research group of Australasia* (pp. 111–118). MERGA.

Carolan, T. (2014). Algebra for all: The hidden cost. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th conference of the international group for the psychology of mathematics education and the 36th conference of the north american chapter of the psychology of mathematics education* (Vol. 2, pp. 241–248). PME.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Heinemann.

Clotfelter, C. T., Ladd, H. F., & Vigdor, J. L. (2015). The aftermath of accelerating algebra: Evidence from district policy initiatives. *The Journal of Human Resources*, *50*(1), 159–188. https://doi.org/10.3368/jhr.50.1.159

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, *6*(4), 56–60. https://www.nctm.org/Publications/Teaching-Children-Mathematics/1999/Vol6/Issue4/Early-Childhood-Corner_-Children_s-Understanding-of-Equality_-A-Foundation-for-Algebra/

Fonger, N. L., Stephens, A., Blanton, M., Isler, I., Knuth, E., & Gardiner, A. M. (2018). Developing a learning progression for curriculum, instruction, and student learning: An example from early algebra research. *Cognition and Instruction*, *36*(1), 30–55. https://doi.org/10.1080/07370008.2017.1392965

Fyfe, E. R., McNeil, N. M., & Borjas, S. (2015). Benefits of "concreteness fading" for children's mathematics understanding. *Learning and Instruction*, *35*, 104–120. https://doi.org/10.1016/j.learninstruc.2014.10.004

Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., & Witzel, B. (2009). *Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle schools (NCEE 2009-4060)*. National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. http://ies.ed.gov/ncee/wwc/publications/practiceguides/

Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, acting, and knowing: Toward an ecological psychology* (pp. 67–82). Lawrence Erlbaum.

Greeno, J. G. (1994). Gibson's affordances. *Psychological Review*, *101*(2), 336–342. https://doi.org/10.1037/0033-295X.101.2.336

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., … Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Heinemann.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In S. Fennel (Ed.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium* (pp. 25–26). National Research Council, National Academy Press.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. In J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). Lawrence Erlbaum.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317–326. https://doi.org/10.1007/BF00311062

Knuth, E. J., Alibali, M. W., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2008). Equal sign understanding in the middle grades. *Mathematics Teaching in the Middle School*, *13*(9), 514–519. https://www.nctm.org/Publications/Mathematics-Teaching-in-Middle-School/2008/Vol13/Issue9/The-Importance-of-Equal-Sign-Understanding-in-the-Middle-Grades/

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*(4), 297–312. https://www.nctm.org/Publications/journal-for-research-in-mathematics-education/2006/Vol37/Issue4/Does-Understanding-the-Equal-Sign-Matter_-Evidence-from-Solving-Equations/

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics*, *30*(1), 39–65. https://doi.org/10.1007/BF00163752

Matthews, P. G., Rittle-Johnson, B., McEldoon, K. L., & Taylor, R. S. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, *43*(3), 316–350. https://doi.org/10.5951/jresematheduc.43.3.0316

McNeil, N. M. (2014). A change–resistance account of children's difficulties understanding mathematical equivalence. *Child Development Perspectives*, *8*(1), 42–47. https://doi.org/10.1111/cdep.12062

McNeil, N. M., & Alibali, M. W. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development*, *76*(4), 883–899. https://doi.org/10.1111/j.1467-8624.2005.00884.x

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Otten, M., Van den Heuvel-panhuizen, M., & Veldhuis, M. (2019). The balance model for teaching linear equations: A systematic literature review. *International Journal of STEM Education*, *6(*30). https://doi.org/10.1186/s40594-019-0183-2

Plunk, A. D., Tate, W. F., Bierut, L. J., & Grucza, R. A. (2014). Intended and unintended effects of state-mandated high school science and mathematics course graduation requirements on educational attainment. *Educational Researcher*, *43*(5), 230–241. https://doi.org/10.3102/0013189X14540207

Seo, K.-H., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence … ": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Recent research and theory* (pp. 161–187). Lawrence Erlbaum Associates.

Vygotsky, L. S. (1978). *Mind in Society*. Harvard University Press.

Warren, E., & Cooper, T. J. (2005). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal*, *17*(1), 58–72. https://doi.org/10.1007/BF03217409