

Robust Adaptive Quadratic Programs using Control Lyapunov and Barrier Functions

Pan Zhao, Yanbing Mao, Chuyuan Tao, Naira Hovakimyan, Xiaofeng Wang

Abstract—This paper presents robust adaptive quadratic program (QP) based control using control Lyapunov and barrier functions for nonlinear systems subject to time-varying and state-dependent uncertainties. An adaptive estimation law is proposed to estimate the pointwise value of the uncertainties with pre-computable estimation error bounds. The estimated uncertainty and the error bounds are then used to formulate a robust QP, which ensures that the actual uncertain system will not violate the safety constraints defined by the control barrier function. Additionally, the accuracy of the uncertainty estimation can be systematically improved by reducing the estimation sampling time, leading subsequently to reduced conservatism of the formulated robust QP. The proposed approach is validated in simulations on an adaptive cruise control problem and through comparisons with existing approaches.

I. INTRODUCTION

Control Lyapunov functions (CLFs) provide a rigorous approach to analyze the closed-loop stability and synthesize stabilizing control signals for nonlinear systems without resorting to an explicit feedback control law, [1], [2]. They also facilitate optimization based control, e.g. via Quadratic Programs (QPs) [3], which could explicitly consider input constraints. On the other hand, inspired by the barrier functions that are used to certify the forward invariance of a set, control barrier functions (CBFs) are proposed to design feedback control laws to ensure that the system states stay in a safe set [4]. Unification of CLF and CBF conditions into a single QP was studied in [5], which allows compromising the CLF-defined control objectives to enforce safety.

Due to reliance on dynamic models, the performance of CLF and/or CBF based control is deteriorated in the presence of model uncertainties and disturbances. As an example, the safety constraints defined using CBF and a nominal model may be violated in the presence of uncertainties and disturbances. The paper [6] studied the robustness of CBF based control in the presence of bounded disturbance and established formal bounds on violation of the CBF constraints. Input-to-state safety in the presence of input disturbance was studied in [7] to ensure system states stay in a set that is close to the original safe set. However, in practice, safety constraints may often need to be strictly enforced. Towards this end, adaptive CBF approaches were proposed in [8], [9] for systems with parametric uncertainties. Robust CLF

and CBF based control was explored in [10], where the state-dependent uncertainties were assumed to have known uncertainty bounds that were used to formulate robust constraints. This approach can be rather conservative, as the uncertainty bounds need to hold in the entire set of admissible states, which can be overly large. Bayesian learning based approaches using Gaussian process regression (GPR) were also proposed to learn state-dependent uncertainties and subsequently use the learned model to enforce probabilistic safety constraints [11], [12]. Nevertheless, the expensive computation associated with GPR prohibits the use of these approaches for real-time control; additionally, the predicted uncertainty for a state far away from the collected data points can be quite poor, leading to overly conservative performance or even infeasible QP problems.

We present an adaptive estimation based approach to design of CLF and CBF based controllers via QPs in the presence of time-varying and state-dependent nonlinear uncertainties, while strictly enforcing the safety constraints. With the proposed estimation scheme, the pointwise value of the uncertainties can be estimated with pre-computable error bounds. The estimated uncertainty and the error bounds are then used to formulate a QP with robust constraints. We also show that by reducing the estimation sampling time, the estimated uncertainty can be arbitrarily accurate after a *single* sampling interval, which implies that the conservatism of the robust constraints can be arbitrarily small. The effectiveness of the approach is demonstrated on an automotive cruise control (ACC) problem in simulations.

We denote the n -dimensional real vector space by $\mathbb{R}^n$. Let $\mathbb{Z}_i$ and $\mathbb{Z}_1^n$ denote the integer sets $\{i, i+1, i+2, \dots\}$ and $\{1, 2, \dots, n\}$, respectively. We use $\|\cdot\|$ ($\|\cdot\|_\infty$) to denote the 2-norm (∞ -norm) of a vector or a matrix.

II. PRELIMINARIES

Consider a nonlinear control-affine system

$$\dot{x} = f(x) + g(x)u + d(t, x), \quad x(0) = x_0, \quad (1)$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are known and locally Lipschitz continuous functions, $d(t, x)$ represents the time-varying and state-dependent uncertainties. Suppose X is a compact set, and the control constraint set U is defined as $U \triangleq \{u \in \mathbb{R}^m : \underline{u} \leq u \leq \bar{u}\}$, where $\underline{u}, \bar{u} \in \mathbb{R}^m$ denote the lower and upper bounds of all control channels, respectively. Hereafter, we sometimes write $d(t, x)$ as d for brevity.

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Assumption 1. There exist positive constants l_d , l_t , b_d such that for any $x, y \in X$ and $t, \tau \geq 0$, the following inequalities hold:

$$\|d(t, x) - d(\tau, y)\| \leq l_d \|x - y\| + l_t |t - \tau|, \quad (2)$$

$$\|d(t, 0)\| \leq b_d. \quad (3)$$

Moreover, the constants l_d , l_t and b_d are known.

Remark 1. This assumption essentially indicates that the growth rate of the uncertainty d is bounded so that it can be estimated by the estimation law (to be presented in Section III-A) with quantifiable error bounds.

Lemma 1. Given Assumption 1, for any $t \geq 0$, $x \in X$ and $u \in U$, we have

$$\|d(t, x)\| \leq \theta, \quad \|\dot{x}(t)\| \leq \phi, \quad (4)$$

where

$$\theta \triangleq l_d \max_{x \in X} \|x\| + b_d, \quad (5)$$

$$\phi \triangleq \max_{x \in X, u \in U} \|f(x) + g(x)u\| + \theta. \quad (6)$$

Proof. Note that

$$\begin{aligned} \|d(t, x)\| &= \|d(t, x) - d(t, 0) + d(t, 0)\| \\ &\leq l_d \|x\| + \|d(t, 0)\| \leq l_d \|x\| + b_d = \theta, \end{aligned}$$

where the two inequalities hold due to (2) and (3) in Assumption 1. Therefore, the dynamics (1) implies that

$$\begin{aligned} \|\dot{x}\| &\leq \|f(x) + g(x)u\| + \|d(t, x)\| \\ &\leq \max_{x \in X, u \in U} \|f(x) + g(x)u\| + \theta = \phi. \end{aligned}$$

The proof is complete. $\square$

A. Control Lyapunov Function

Control Lyapunov function (CLF) provides a way to analyze the closed-loop stability and synthesize a stabilizing control signal without constructing an explicit control law [1], [2]. A formal definition of CLF is given as follows.

Definition 1. A continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a CLF for the system (1), if it is positive definite and there exists a class $\mathcal{K}^1$ function $\alpha(\cdot)$ such that

$$\inf_{u \in U} \{L_f V(x) + L_g V(x)u + V_x(x)d\} \leq -\alpha(V(x)), \quad (7)$$

for all $t \geq 0$ and all $x \in X$, where $V_x(x) \triangleq \frac{\partial V(x)}{\partial x}$, $L_f V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x)$ and $L_g V(x) \triangleq \frac{\partial V(x)}{\partial x} g(x)$.

Definition 1 allows us to consider the set of all stabilizing control signals for every point $x \in X$ and $t \geq 0$:

$$\begin{aligned} K_{\text{clf}}(t, x) &\triangleq \{u \in U : L_f V(x) + L_g V(x)u + V_x(x)d(t, x) \\ &\leq -\alpha(V(x))\}. \end{aligned} \quad (8)$$

¹A function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ for some $a > 0$, if it is strictly increasing and $\alpha(0) = 0$.

B. Control Barrier Function

CBFs are introduced to ensure *forward invariance* (often termed as *safety* in the literature) of a set, defined as some superlevel set of a function: $\mathcal{C} \triangleq \{x \in X : h(x) \geq 0\}$, where $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuously differentiable function. A formal definition of CBF is stated as follows [5].

Definition 2. A continuously differentiable function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a (zeroing) CBF for the system (1), if there exists an extended class $\mathcal{K}$ function² $\beta(\cdot)$ such that

$$\sup_{u \in U} \{L_f h(x) + L_g h(x)u + h_x(x)d\} \geq -\beta(h(x)), \quad (9)$$

for all $x \in X$ and all $t \geq 0$, where $h_x(x) \triangleq \frac{\partial h(x)}{\partial x}$, $L_f h(x) \triangleq \frac{\partial h(x)}{\partial x} f(x)$ and $L_g h(x) \triangleq \frac{\partial h(x)}{\partial x} g(x)$.

The existence of a CBF satisfying (9) ensures that if $x(0) \in \mathcal{C}$, i.e. $h(x) \geq 0$, then there exists a control law $u(t) \in U$ such that for all $t \geq 0$, $x(t) \in \mathcal{C}$. Similarly, Definition 2 allows us to consider all control signals for each $x \in X$ and $t \geq 0$ that render $\mathcal{C}$ forward invariant:

$$\begin{aligned} K_{\text{cbf}}(t, x) &\triangleq \{u \in U : L_f h(x) + L_g h(x)u + h_x(x)d \\ &\geq -\beta(h(x))\}. \end{aligned} \quad (10)$$

Remark 2. Definitions 1 and 2 utilize the true uncertainty d , which is not accessible in real applications. Therefore, it is impossible to verify whether a given function is a CLF (CBF) according to Definition 1 (Definition 2).

One way to resolve this issue is to derive a sufficient and *verifiable* condition for (7) or (9) using the worst-case bound on the uncertainty d in (4). The following lemma gives such conditions. The proof is straightforward considering the bound on d in (4) and subsequently the inequalities $V_x(x)d \leq \|V_x(x)\| \theta$ and $h_x(x)d \leq \|h_x(x)\| \theta$.

Lemma 2. A continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a CLF for the uncertain system (1) under Assumption 1, if it is positive definite and there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that

$$\inf_{u \in U} \{L_f V(x) + L_g V(x)u + \|V_x(x)\| \theta\} \leq -\alpha(V(x)), \quad (11)$$

for all $t \geq 0$ and all $x \in X$. Similarly, a continuously differentiable function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a (zeroing) CBF for the uncertain system (1) under Assumption 1, if there exists an extended class $\mathcal{K}$ function $\beta(\cdot)$ such that

$$\sup_{u \in U} \{L_f h(x) + L_g h(x)u - \|h_x(x)\| \theta\} \geq -\beta(h(x)), \quad (12)$$

for all $x \in X$ and all $t \geq 0$.

C. Standard QP Formulation

The admissible sets of control signals defined in (8) and (10) inspire optimization based control. Recent work shows

²A function $\beta : (-b, a) \rightarrow (-\infty, \infty)$ is said to belong to extended class $\mathcal{K}$ for some $a, b > 0$, if it is strictly increasing and $\beta(0) = 0$.

that CLF and CBF conditions can be unified into a QP [5]:

$$\begin{aligned} u^*(t, x) &= \underset{(u, \delta) \in \mathbb{R}^{m+1}}{\operatorname{argmin}} \frac{1}{2} u^T H(x) u + p \delta^2 \quad (\mathbf{QP}) \\ \text{s.t. } L_f V(x) + L_g V(x) u + V_x(x) d + \alpha(V(x)) &< \delta, \quad (13) \\ L_f h(x) + L_g h(x) u + h_x(x) d + \beta(h(x)) &> 0, \quad (14) \\ u &\in U, \quad (15) \end{aligned}$$

where $H(x)$ is a (pointwise) positive definite matrix and δ is a positive constant to relax the CLF constraint that is penalized by $p > 0$. In this formulation, the CBF condition is often associated with safety and therefore is imposed as a *hard* constraint. In contrast, the CLF constraint is related to control objective (e.g. tracking a reference) and could be relaxed to ensure the feasibility of the QP when safety is a major concern. Therefore, it is imposed as a *soft* constraint.

Remark 3. The constraints (13) and (14) in $(\mathbf{QP})$ depend on the true uncertainty d , which makes $(\mathbf{QP})$ not implementable in practice. Although the worst-case bound (4) can be used to derive a robust version of (13) and (14) as done in [10], the resulting constraints can be overly conservative, as shown in Section IV.

III. ROBUST ADAPTIVE QP CONTROL USING CLBFs

In this section, we first introduce adaptive estimation laws to estimate the uncertainty with pre-computable error bounds, which can be systematically improved by reducing the estimation sampling time. We then show how the estimated uncertainty, as well as the error bounds, can be used to formulate a robust QP, while the introduced conservatism can be arbitrarily reduced, subject to only hardware limitations.

A. Adaptive Estimation of the Uncertainty

We extend the piecewise-constant adaptive (PWCA) law in [13, Section 3.3] to estimate the pointwise value of the uncertainty $d(t, x(t))$ at each time instant t . Similar results are available in [14] which considers control non-affine dynamics under slightly more sophisticated assumptions about the uncertainty, and in [15] which considers the nominal dynamics described by a linear parameter-varying model. The PWCA law consists of two elements, namely, a state predictor and an adaptive law, which are explained as follows. The state predictor is defined as:

$$\dot{\hat{x}} = f(x) + g(x)u + \hat{d}(t) - a\tilde{x}, \quad \hat{x}(0) = x_0, \quad (16)$$

where $\tilde{x} \triangleq \hat{x} - x$ is the prediction error, a is an arbitrary positive constant. A discussion about the role of a available in [15]. The adaptive estimation, $\hat{d}(t)$, is updated in a piecewise-constant way:

$$\begin{cases} \hat{d}(t) = \hat{d}(iT), & t \in [iT, (i+1)T), \\ \hat{d}(iT) = -\frac{a}{e^{aT} - 1} \tilde{x}(iT), \end{cases} \quad (17)$$

where T is the estimation sampling time, and $i = 0, 1, 2, \dots$.

Remark 4. Note that the PWCA law is not trying to estimate the analytic expression of the uncertainty; instead it estimates the pointwise value of the uncertainty at each time instant.

Next, we will show in Section III that the estimated pointwise value of the uncertainty can be used to formulate a new QP with robust constraints.

Let us first define:

$$\gamma(T) \triangleq 2\sqrt{n}\eta T + \sqrt{n}(1 - e^{-aT})\theta, \quad (18)$$

$$\eta \triangleq l_t + l_d \phi, \quad (19)$$

where θ and ϕ are defined in (5) and (6), respectively. We next establish the estimation error bounds associated with the estimation scheme in (16) and (17).

Lemma 3. *Given the dynamics (1), and the estimation law in (16) and (17), subject to Assumption 1, the estimation error can be bounded as*

$$\|\hat{d}(t) - d(t, x(t))\| \leq \begin{cases} \theta, & \forall 0 \leq t < T, \\ \gamma(T), & \forall t \geq T, \end{cases} \quad (20)$$

Moreover, $\lim_{T \rightarrow 0} \gamma(T) = 0$.

Proof. See Appendix.

Remark 5. Lemma 3 implies that the uncertainty estimation can be made arbitrarily accurate for $t \geq T$, by reducing T , the latter only subject to hardware limitations. Additionally, the estimation cannot be arbitrarily accurate for $t \in [0, T)$. This is because the estimate in $[0, T)$ depends on $\tilde{x}(0)$ according to (17). Considering that $\tilde{x}(0)$ is purely determined by the initial state of the system, x_0 , and the initial state of the predictor, $\hat{x}_0$, it does not contain any information of the uncertainty. Since T is usually very small in practice, lack of a *tight* estimation error bound for the interval $[0, T)$ will not cause an issue from a practical point of view.

B. Robust Adaptive QP Formulation

Let

$$\Psi_V(t, x, u) \triangleq L_f V(x) + L_g V(x) u + V_x(x) \hat{d}(t) + \|V_x(x)\| \gamma(T), \quad (21)$$

$$\Psi_h(t, x, u) \triangleq L_f h(x) + L_g h(x) u + h_x(x) \hat{d}(t) - \|h_x(x)\| \gamma(T). \quad (22)$$

We are ready to present the main result in the following theorem.

Theorem 1. *For any $t \geq T$,*

1) *the condition*

$$\inf_{u \in U} \Psi_V(t, x, u) \leq -\alpha(V(x)) \quad (23)$$

is a sufficient condition for (7), and also a necessary condition for (7) when $T \rightarrow 0$.

2) *the condition*

$$\sup_{u \in U} \Psi_h(t, x, u) \geq -\beta(h(x)) \quad (24)$$

is a sufficient condition for (9), and also a necessary condition for (9) when $T \rightarrow 0$.

Proof. Due to space limit, we only prove 2), while 1) can be proved analogously. For proving that (24) is sufficient for

(9), comparison of (9) and (24) indicates that we only need to show $h_x(x)d(t, x) \geq h_x(x)\hat{d}(t) - \|h_x(x)\| \gamma(T)$ for any $t \geq T$. The preceding inequality actually holds since

$$\begin{aligned} h_x(x)d(t, x) &= h_x(x) \left(\hat{d}(t) + d(t, x) - \hat{d}(t) \right) \\ &\geq h_x(x)\hat{d}(t) - \|h_x(x)\| \|d(t, x) - \hat{d}(t)\| \\ &\geq h_x(x)\hat{d}(t) - \|h_x(x)\| \gamma(T), \end{aligned}$$

where the last inequality is due to the bound in (20). For proving the necessity, we notice that when $T \rightarrow 0$, $\hat{d}(t) \rightarrow d(t, x)$ for any $t \geq T$ according to Lemma 3, which implies that (LHS) of (9) equates (LHS) of (24) considering (22). Therefore, (24) is also a necessary condition for (9) for $t \geq T$ when $T \rightarrow 0$. $\square$

Remark 6. Theorem 1 indicates that the condition (23) ((24)) that depends on the adaptive estimation is always a sufficient condition for (7) ((9)) that depends on the true uncertainty for $t \geq T$. Additionally, the conservatism of the condition (23) or (24) can be arbitrarily reduced by reducing T .

With the inequalities (23) and (24) that depend on the adaptive estimation of the uncertainty, we can formulate a robust QP, which we call Ra-QP:

$$\begin{aligned} u^*(t, x) &= \underset{(u, \delta) \in \mathbb{R}^{m+1}}{\operatorname{argmin}} \frac{1}{2} u^T H(x) u + p \delta^2 \quad (\mathbf{Ra-QP}) \\ \text{s.t. } \Psi_V(t, x, u) + \alpha(V(x)) &< \delta, \quad (25) \\ \Psi_h(t, x, u) + \beta(h(x)) &> 0, \quad (26) \\ u &\in U. \quad (27) \end{aligned}$$

Here, Ψ_V and Ψ_h are defined in (21) and (22), respectively.

Remark 7. The estimation sampling time T can be different from the sampling time T_{qp} for solving the (Ra-QP) problem. Considering the effect of T on the estimation accuracy, as well as the relatively low computational cost of the adaptive estimation compared to solving the (Ra-QP) problem, T can be set much smaller than T_{qp} .

IV. SIMULATION RESULTS

In this section, we validate the theoretical development using the adaptive cruise control (ACC) problem from [5].

A. ACC Problem Setting

The lead and following vehicles are modeled as point masses moving on a straight road. The following vehicle is equipped with an ACC system, and its objective is to cruise at a given constant speed, while maintaining a safe following distance as specified by a time headway. Let v_l and v_f denote the speeds (in m/s) of the lead car and the following car, respectively, and D be the distance (in m) between the two vehicles. Denoting $x = [v_l, v_f, D]$ as the system state, the dynamics of the system can be described as

$$\begin{bmatrix} \dot{v}_l \\ \dot{v}_f \\ \dot{D} \end{bmatrix} = \underbrace{\begin{bmatrix} a_l \\ 0 \\ v_l - v_f \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 1/m \\ 0 \end{bmatrix}}_{g(x)} u + \underbrace{\begin{bmatrix} 0 \\ -F_r/m + d_0(t) \\ 0 \end{bmatrix}}_{d(t, x)}, \quad (28)$$

TABLE I: Parameter Settings

g	9.81 Ns ² /m	f_2	0.25 Ns ² /m	τ_d	1.8 s	p	100
m	1650 kg	v_d	22 m/s	T_{qp}	0.01 s	$\alpha(V)$	5V
f_0	0.1 N	$x(0)$	[18 12 80]	T	1 ms	$\beta(h)$	h
f_1	5 Ns/m	$u_{\max}$	0.4 mg	a	1		

TABLE II: $\gamma(T)$ versus T

T	10 ms	1 ms	0.1 ms	0.01 ms
$\gamma(T)$	2.98	0.298	0.0298	0.00298

where u and m are the control input and the mass of the following car, respectively, a_l is the acceleration of the lead car, $F_r = f_0 + f_1 v_f + f_2 v_f^2$ is the aerodynamic drag term with unknown constants f_0, f_1 and f_2 , $d_0(t)$ is an external disturbance term (reflecting the unmodeled road condition or aerodynamic force).

The safety constraint requires the following car to keep a safe distance from the lead car, which can be expressed as $D/v_f \geq \tau_d$ with τ_d being the desired time headway. Defining the function $h = D - \tau_d v_f$, the safety constraint can be expressed as $h \geq 0$. In terms of control objective, the following car should achieve a desired speed v_d when adequate headway is ensured. This objective naturally leads to the CLF, $V = (v_f - v_d)^2$.

The parameters used in the simulation are shown in Table I, where m, f_0, f_1, f_2, τ_d are selected following [5]. The disturbance is set to $d_0(t) = 0.2g \sin(2\pi 10t)$ following [6]. The input constraints are set to $-u_{\max} \leq u \leq u_{\max}$, where $u_{\max} = 0.4 \, mg$ with g being the gravitational constant.

B. Uncertainty Estimation Setting

According to (18), given the estimation sampling time T , one would like to obtain the smallest values of the constants in Assumption 1 to get the tightest estimation error bound, $\gamma(T)$. For this ACC problem, the constants in Assumption 1 are selected as

$$l_t = (0.2g(2\pi)10)\xi, \quad l_d = (f_1 + 2f_2 v_{\max})\xi, \quad b_d = (0.2g)\xi,$$

where $v_{\max} = 160$ km/h is the maximum speed considered, $\xi \geq 1$ is a constant to reflect the conservatism in estimating the constants l_t, l_d and b_d that satisfy (2) and (3) in Assumption 1. We set $\xi = 2$. We further set $a = 1$ in (16). With this setting, the estimation error bounds under different T are computed according to (18) and listed in Table II. For the simulation results in Section IV-D, $T = 1$ ms is selected.

C. QP Setting

We consider several QP controllers:

- A standard QP controller defined by (QP) using the true uncertainty $d(t, x)$,
- A standard QP controller ignoring the uncertainty, obtained by setting $d(t, x) \equiv 0$ in (QP),
- A robust QP (R-QP) controller proposed in [10] using the worst-case bound on $d(t, x)$ in (4), which was obtained by setting $\hat{d}(t) \equiv 0$ in (Ra-QP), and
- A robust adaptive QP (Ra-QP) controller from (Ra-QP).

Note that the first controller is not implementable due to its reliance on the true uncertainty model (see Remark 3), and is included to merely show the *ideal* performance. The objectives in both (QP) and (Ra-QP) are set to be $\min \frac{1}{2}Hu^2 + \frac{1}{2}p\delta^2$, where $H = 1/m^2$, $p = 100$. We further set $\alpha(V) = 5V$ and $\beta(h) = h$. Under the above settings, the conditions (11) and (12) in Lemma 2 can be verified, which indicates that V and h are indeed a CLF and CBF for the uncertain system, respectively. We set $T_{qp} = 0.01$ second.

D. Simulation Results

The results are shown in Fig. 1–3. As expected, the QP controller using the true uncertainty $d(t, x)$ achieved good performance in tracking the desired speed when enforcing safety was not a major concern, while maintaining the safety (h was above 0) throughout the simulation with minimal conservatism (h was quite close to 0 when enforcing safety was a major concern). On the other hand, the QP controller ignoring the uncertainty did not provide satisfactory tracking performance (note the relatively large tracking error between 25 and 32 seconds in Fig. 1) even when there was adequate headway; it also failed to guarantee the safety throughout the simulation (h was below 0 during some intervals). Although the R-QP controller did provide safety guarantee as shown by the trajectory of h in Fig. 2, it yielded rather conservative performance: (1) h was constantly far away from 0; (2) speed tracking objective was often compromised more than necessary (note the speed decrease between 2 and 5 seconds).

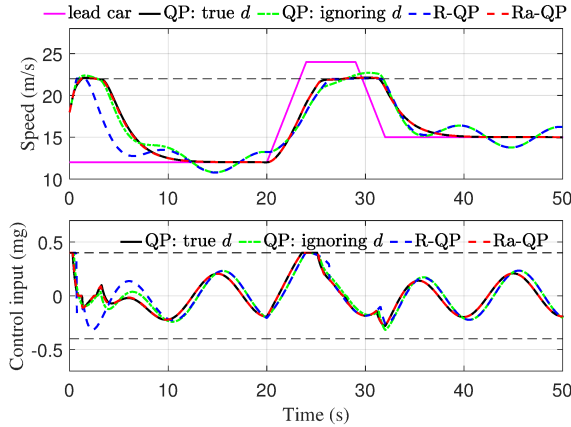


Fig. 1: Trajectories of the following car speed (top) and control input (bottom). Black dashed lines denote the desired speed (top) and input limits (bottom).

Finally, utilizing the estimated uncertainty, the Ra-QP controller almost recovered the performance of the QP controller using the true uncertainty, in terms of both the car speed and control input. It also maintained the safety throughout the simulation with slightly increased conservatism compared to the performance of the QP controller using the true uncertainty, as shown in Fig. 2. Figure 3 depicts the trajectories of the true and estimated uncertainties with error bounds also displayed. One can see that the estimated uncertainty overlaps the true uncertainty after a *single* sampling interval, T . Additionally, the true uncertainty always lies within a

tube determined by the estimated uncertainty and the error bounds defined in (20), which is consistent with Lemma 3.

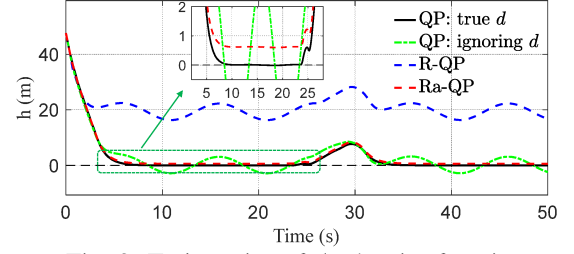


Fig. 2: Trajectories of the barrier function

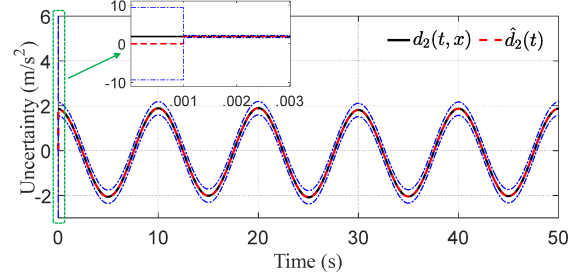


Fig. 3: Trajectories of the estimated and true uncertainties. Blue dash-dotted lines denote the estimation error bounds computed according to (20). $d_2(t, x)$ ($\hat{d}_2(t)$) is the second element of $d(t, x)$ ($\hat{d}(t)$).

V. CONCLUSIONS

This paper summarizes an adaptive estimation inspired approach to control Lyapunov and barrier functions based control via QPs in the presence of time-varying and state-dependent uncertainties. An adaptive estimation law is proposed to estimate the pointwise value of the uncertainties with pre-computable error bounds. The estimated uncertainty and the error bounds are then used to formulate a robust quadratic program, which ensures that the actual uncertain system will not violate the safety constraints. It is also shown both theoretically and numerically that the estimation error bound and the conservatism of the robust constraints can be systematically reduced by reducing the estimation sampling time. The proposed approach is validated on an adaptive cruise problem through comparisons with existing approaches.

APPENDIX

Proof of Lemma 3: From (1) and (16), the prediction error dynamics are obtained as

$$\dot{\tilde{x}} = -a\tilde{x} + \hat{d}(t) - d(t, x), \quad \tilde{x}(0) = 0. \quad (29)$$

Therefore, $\hat{d}(t) = 0$ for any $t \in [0, T)$ according to (17). Further considering the bound on d in (4), we have

$$\|\hat{d}(t) - d(t, x)\| \leq \theta, \quad \forall t \in [0, T). \quad (30)$$

We next derive the bound on $\|\hat{d}(t) - d(t, x)\|$ for $t \geq T$. For notation brevity, we often write $d(t, x(t))$ as $d(t)$ hereafter.

For any $t \in [iT, (i+1)T)$ ($i \in \mathbb{Z}_0$), we have

$$\tilde{x}(t) = e^{-a(t-iT)}\tilde{x}(iT) + \int_{iT}^t e^{-a(t-\tau)}(\hat{d}(\tau) - d(\tau))d\tau.$$

Since $\tilde{x}(t)$ is continuous, the preceding equation implies

$$\begin{aligned}\tilde{x}((i+1)T) &= e^{-aT} \tilde{x}(iT) + \int_{iT}^{(i+1)T} e^{-a((i+1)T-\tau)} d\tau \hat{d}(iT) \\ &\quad - \int_{iT}^{(i+1)T} e^{-a((i+1)T-\tau)} d(\tau) d\tau. \\ &= e^{-aT} \tilde{x}(iT) + \frac{1}{a} (1 - e^{-aT}) \hat{d}(iT) \\ &\quad - \int_{iT}^{(i+1)T} e^{-a((i+1)T-\tau)} d(\tau) d\tau \\ &= - \int_{iT}^{(i+1)T} e^{-a((i+1)T-\tau)} d(\tau) d\tau, \quad (31)\end{aligned}$$

where the first and last equalities are due to the estimation law (17).

Since $x(t)$ is continuous, $d(t, x)$ is also continuous given Assumption 1. Furthermore, considering that $e^{-a((i+1)T-\tau)}$ is always positive, we can apply the first mean value theorem in an element-wise manner³ to (31), which leads to

$$\begin{aligned}\tilde{x}((i+1)T) &= - \int_{iT}^{(i+1)T} e^{-a((i+1)T-\tau)} d\tau [d_j(\tau_j^*)] \\ &= - \frac{1}{a} (1 - e^{-aT}) [d_j(\tau_j^*)], \quad (32)\end{aligned}$$

for some $\tau_j^* \in (iT, (i+1)T)$ with $j \in \mathbb{Z}_1^n$ and $i \in \mathbb{Z}_0$, where $d_j(t)$ is the j -th element of $d(t)$, and

$$[d_j(\tau_j^*)] \triangleq [d_1(\tau_1^*), \dots, d_n(\tau_n^*)]^\top.$$

The adaptive law (17) indicates that for any t in $[(i+1)T, (i+2)T)$, we have $\dot{\hat{d}}(t) = -\frac{a}{e^{aT}-1} \tilde{x}((i+1)T)$. The preceding equality and (32) imply that for any t in $[(i+1)T, (i+2)T)$ with $i \in \mathbb{Z}_0$, there exist $\tau_j^* \in (iT, (i+1)T)$ ($j \in \mathbb{Z}_1^n$) such that

$$\hat{d}(t) = e^{-aT} [d_j(\tau_j^*)]. \quad (33)$$

Note that

$$\begin{aligned}\|d(t) - [d_j(\tau_j^*)]\| &\leq \sqrt{n} \|d(t) - [d_j(\tau_j^*)]\|_\infty \\ &= \sqrt{n} |d_{\bar{j}_t}(t) - d_{\bar{j}_t}(\tau_{\bar{j}_t}^*)| \leq \sqrt{n} \|d(t) - d(\tau_{\bar{j}_t}^*)\|, \quad (34)\end{aligned}$$

where $\bar{j}_t = \arg \max_{j \in \mathbb{Z}_1^n} |d_j(t) - d_j(\tau_j^*)|$. Similarly, we have

$$\|[d_j(\tau_j^*)]\| \leq \sqrt{n} \|[d_j(\tau_j^*)]\|_\infty = \sqrt{n} |d_{\bar{j}}(\tau_{\bar{j}}^*)| \leq \sqrt{n} \|d(\tau_{\bar{j}}^*)\|, \quad (35)$$

where $\bar{j} = \arg \max_{j \in \mathbb{Z}_1^n} |d_j(\tau_j^*)|$. Therefore, for any $t \in [(i+1)T, (i+2)T)$ ($i \in \mathbb{Z}_0$), we have

$$\begin{aligned}\|d(t) - \hat{d}(t)\| &= \|d(t) - e^{-aT} [d_j(\tau_j^*)]\| \\ &\leq \|d(t) - [d_j(\tau_j^*)]\| + (1 - e^{-aT}) \|[d_j(\tau_j^*)]\| \\ &\leq \sqrt{n} \|d(t) - d(\tau_{\bar{j}_t}^*)\| + (1 - e^{-aT}) \sqrt{n} \|d(\tau_{\bar{j}}^*)\|, \quad (36)\end{aligned}$$

³Note that the mean value theorem for definite integrals does not hold for vector-valued functions.

for some $\tau_{\bar{j}_t}^* \in (iT, (i+1)T)$ and $\tau_{\bar{j}}^* \in (iT, (i+1)T)$, where the equality is due to (33), and the last inequality is due to (34) and (35). The inequality (4) implies that

$$\|x(t) - x(\tau_{\bar{j}_t}^*)\| \leq \int_{\tau_{\bar{j}_t}^*}^t \|\dot{x}(\tau)\| d\tau \leq \int_{\tau_{\bar{j}_t}^*}^t \phi d\tau = \phi(t - \tau_{\bar{j}_t}^*).$$

The preceding inequality and (2) indicate that

$$\begin{aligned}\|d(t, x(t)) - d(\tau_{\bar{j}_t}^*, x(\tau_{\bar{j}_t}^*))\| &\leq l_t(t - \tau_{\bar{j}_t}^*) + l_d \|x(t) - x(\tau_{\bar{j}_t}^*)\| \\ &\leq (l_t + l_d \phi)(t - \tau_{\bar{j}_t}^*) = \eta(t - \tau_{\bar{j}_t}^*) \leq 2\eta T, \quad (37)\end{aligned}$$

where η is defined in (19), and the last inequality is due to the fact that $t \in [(i+1)T, (i+2)T)$ and $\tau_{\bar{j}_t}^* \in (iT, (i+1)T)$.

Finally, plugging (37) into (36) leads to

$$\begin{aligned}\|d(t, x(t)) - \hat{d}(t)\| &\leq 2\sqrt{n}\eta T + \sqrt{n}(1 - e^{-aT}) \|d(\tau_{\bar{j}}^*, x(\tau_{\bar{j}}^*))\| \\ &\leq 2\sqrt{n}\eta T + \sqrt{n}(1 - e^{-aT})\theta = \gamma(T), \quad \forall t \geq T, \quad (38)\end{aligned}$$

where the second inequality is due to (4). From (30) and (38), we arrive at (20). Considering that X and U are compact, the constants θ (defined in (5)), ϕ (defined in (6)) and η (defined in (19)) are all finite, the definition of $\gamma(T)$ in (18) immediately implies $\lim_{T \rightarrow 0} \gamma(T) = 0$. $\square$

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