Synergy and Redundancy Duality Between Gaussian Multiple Access and Broadcast Channels

1st Xueyan Niu
School of Industrial Engineering
Purdue University
West Lafayette, USA
niu26@purdue.edu

2nd Christopher J. Quinn
Department of Computer Science
Iowa State University
Ames, USA
cjquinn@iastate.edu

***Abstract*—We investigate a novel duality for scalar Gaussian multiple access channels and broadcast channels. The duality we explore is based on shared partial information quantities (e.g. synergy and redundancy). Using lattice theory, we establish a crossover correspondence of the synergistic and redundant components between these two channels. The dual channels are similar to the traditional pairs based on capacity regions, though the pairs we identify have equal transmission powers instead of a sum constraint relating transmission powers.**

I. Introduction

The information possessed by multiple agents in a system may present different interactions. For example, two users may have synergistic, redundant, and unique information about a signal. Conversely, an information source can send information to different users such that they perceive it in synergistic, redundant, and unique manners. The partial information decomposition framework proposed in [1] can be used to decouple unique, redundant, and synergistic informations that combine to form (joint) mutual and conditional mutual informations.

Gaussian Multiple-Input and Multiple-Output (MIMO) communication channels are important examples of systems of multiple interacting agents. One significant result in this area is the duality relationship between the Gaussian multiple access channel (MAC) and the Gaussian broadcast channel (BC). The capacity region of one channel type of each can be directly characterized by the capacity region of the other channel type [2]–[4].

In this paper, we identify a different type of duality between Gaussian MACs and BCs. Instead of shared achievable rates, we identify dual pairs that share partial informations (e.g. synergy and redundancy) between the transmitters in a MAC and the receivers in a corresponding BC. In the (traditional) duality based on capacity regions, MAC and BC channel pairs that share channel gains, share uniform noise powers, but have different transmission powers (with a sum constraint). In contrast, for the duality based on partial informations that we investigate, the channel pairs share channel gains, uniform noise powers, and uniform transmission powers.

A number of partial information measures have been proposed, including [1], [5]–[9]. These measures have been applied to various research areas. For example, the operational decomposition [6] was used to quantify information modification in developing neural networks [10]; net synergy was used to study how the correlations between neurons are related to the stimulus [11]; Rauh [12] associated secret sharing with partial information decomposition; and secret key agreement rates were studied as an operational measure [8]. To our knowledge, no other work has explored invariant partial informations in MIMO channels.

The remainder of this paper is organized as follows. We briefly review the partial information decomposition in Section II and the duality of the achievable rate regions in Section III. In Section IV, we study the synergy and redundancy duality for MAC and BC with two senders (receivers). In Section V, we generalize the duality in the general case involving n senders (receivers).

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II. Background and Notation

A. Lattice and Set Notation

We first briefly describe lattice and set notation (see, for example, [13]). Let $[n]$ denote the set $\{1, 2, \dots, n\}$ for a positive natural number n. Let $2^{[n]}$ to denote the power set of $[n]$. A partial order on a set is a binary relation satisfying reflexivity, antisymmetry, and transitivity. An anti-chain is a set where any two elements are incomparable under the partial order. We use $\mathcal{A}(2^{[n]})$ to denote the set of all the anti-chains of the power set $2^{[n]}$ under inclusion order. Given a set S and a partial order $\leq$, the lower set $\downarrow x$ and the upper set $\uparrow x$ of an element $x \in S$ is defined as

$$\downarrow x := \{y \in S : \ y \leq x\} \text{ and } \uparrow x := \{y \in S : \ y \geq x\}.$$

B. Partial Information Decomposition

Consider two (interacting) agents X_1 and X_2, and a target agent Y. The information possessed by the pair $\{X_1, X_2\}$ about Y is measured by the mutual information $I(X_1, X_2; Y)$. It is commonly conjectured [1], [5]–[9] that $I(X_1, X_2; Y)$ should include the redundant information, R, which is available from either X_1 or X_2 alone; the synergistic information, S, which is only available from the pair $\{X_1, X_2\}$; the unique information of X_1, U_1, which is available from X_1 alone but not X_2; and similarly the unique information of X_2, U_2. Following this,

$$\begin{aligned} I(X_1,X_2;Y) &= U_1+U_2+R+S \\ I(X_i;Y) &= U_i+R \qquad \text{for } i\in\{1,2\}. \end{aligned} \tag{1}$$

Equation (1) is under-determined. Using minimum mutual information, Williams and Beer [1] were the first to give a non-negative decomposition. In [5], a solution is derived using projective information. In [6], an operational approach is used based on max entropy. Other operational approaches were proposed using the secret key agreement rate [7], [8]. Using information geometry, [9] derived a measure that satisfies smoothness properties for the exponential family distributions. Among these measures, [1], [5]–[7], satisfy the following property discovered in [14].

Property 1: For any (scalar) jointly Gaussian distribution $P_{X_1,X_2,Y}$,

$$\begin{aligned} R &= \min\{I(X_1;Y), I(X_2;Y)\}, \\ S &= \min\{I(X_1;Y|X_2), I(X_2;Y|X_1)\}. \end{aligned}$$

In this work we do not focus on a specific partial information decomposition measure, but instead use this common property.

Although the setting of $n=2$, resulting in (1), is simple to describe, the general case of $n\geq 2$ requires more care. For example, a pair of variables can have synergistic and redundant information with another variable. Suppose we have a set of $n\in\mathbf{N}_+$ interacting agents $\boldsymbol{X}=\{X_1,X_2,\ldots,X_n\}$, and a target agent Y. To fully describe all the different types of interactions among $\boldsymbol{X}$ with respect to Y, Williams and Beer [1] propose using a lattice $(\mathcal{A}(2^{[n]}),\preccurlyeq)$ that consists of all the anti-chains of the power set of $[n]$, and the partial order $\preccurlyeq$ is defined such that $\alpha\preccurlyeq\beta$ if and only if $\forall B\in\beta, \exists A\in\alpha$, such that $A\subseteq B$. As an example, consider the redundancy lattice for $n=3$ depicted in Figure 3. Suppose $\alpha=\{\{2\}\{1,3\}\}$ and $\beta=\{\{1,2\}\}$. We have $\{\{2\}\{1,3\}\}\preccurlyeq\{1,2\}$ since the information that $\{2\}$ and $\{1,3\}$ have in common about Y is part of what $\{2\}$ can provide and consequently its superset $\{1,2\}$.

Let $I_\cap(\cdot)$ denote the function measuring the information in each anti-chain is defined on $\mathcal{A}(2^{[n]})$. It should satisfy that for any single subset $S\subseteq[n]$, $I_\cap(\{S\})=I(X_S;Y)$. Using the Möbius transformation, we can break $I_\cap$ into the sum of information quantities measuring the "partial" informations at each element in the lower set: $I_\cap(\alpha)=\sum_{\beta\preccurlyeq\alpha} I_\partial(\beta)$, where $I_\partial(\cdot)$ is non-negative on the lattice. We define a function $I_\cup(\cdot)$ on the lattice analogously as $I_\cup(\alpha)=\sum_{\alpha\preccurlyeq\beta} I_\partial(\beta)$. For example, consider the lattice $\mathcal{A}(2^{[2]})$, i.e., $n=2$,

$$\begin{aligned} I(X_2;Y) &= I_\cap(\{2\}) = I_\partial(\{2\}) + I_\partial(\{1\}\{2\}) = U_2+R, \\ I(X_1;Y|X_2) &= I(X_1,X_2;Y) - I(X_2;Y) \\ &= I_\cap(\{1,2\}) - I_\cap(\{2\}) \\ &= I_\cup(\{1\}) = I_\partial(\{1\}) + I_\partial(\{1,2\}) = U_1+S. \end{aligned}$$

It was shown in [15] that the redundancy lattice is isomorphic to the lattice of upper sets of $\boldsymbol{X}$, $(\mathcal{F}(\boldsymbol{X}),\supseteq)$ with the set inclusion order.

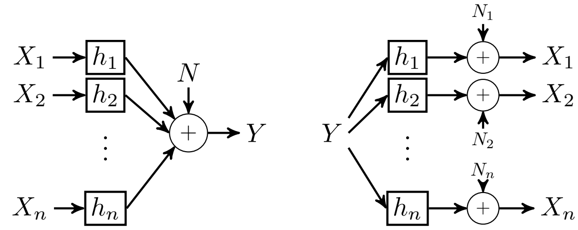


Fig. 1: The channel models of the MAC (left) and the BC (right) defined in section III.

III. MAC AND BC RATE DUALITY

A. Channel Models

We first briefly describe system models for the scalar Gaussian MAC and Gaussian BC. We then review the capacity region based duality for scalar Gaussian MACs and BCs. Diagrams of the models are shown in Figure 1.

For the MAC (Fig 1; left), there are $n\geq 2$ agents who wish to communicate independent messages to a common receiver. Let $N\sim\mathcal{N}(0,\sigma^2)$ denote the noise and for each agent $i\in[n]$ let $X_i\sim\mathcal{N}(0,P_i)$ be independent of the others and noise, with P_i denoting the individual power constraint on agent i. Let Y denote the received signal. Let $H=(\sqrt{h_1},\sqrt{h_2},\ldots,\sqrt{h_n})^T$ denote the constant channel gain matrix and suppose that it is known perfectly at the transmitters and the receiver. Then the MAC model is

$$Y = H^T(X_1,X_2,\ldots,X_n)^T + N. \tag{2}$$

For the Gaussian BC (Fig 1; right), there is one transmitter that wishes to communicate independent messages to $n\geq 2$ receivers. Each receiver obtains the transmitted signal plus noise. Let $N_i \overset{i.i.d.}{\sim} \mathcal{N}(0,\sigma)$ for $i\in[n]$ denote the receiver noises and let $Y\sim\mathcal{N}(0,\overline{P})$ denote the transmitted signal, with power constraint $\overline{P}$. Similar to the MAC, consider a constant channel gain matrix $H=(\sqrt{h_1},\sqrt{h_2},\ldots,\sqrt{h_n})^T$ that is known perfectly at the transmitters and the receivers. Following [2], the BC which is dual to the MAC (2) is defined as

$$(X_1,X_2,\ldots,X_n)^T = HY + (N_1,N_2,\ldots,N_n)^T. \tag{3}$$

B. Achievable Rate Duality

The achievable rate region of the MAC in general can be characterized using a set of mutual information and conditional mutual information quantities with different subsets of the $n\geq 2$ senders. For $S\subseteq[n]$, let $\overline{S}=[n]\backslash S$ denote the complement. Let $\mathcal{C}_{MAC}$ and $\mathcal{C}_{BC}$ denote capacity regions for the MAC and the BC respectively.

Theorem 3.1 ([16]): For the MAC channel (2),

$$\begin{aligned} &\mathcal{C}_{MAC}((\overline{P}_1,\ldots,\overline{P}_n);H) \\ &\quad = \{(r_1,r_2,\ldots,r_n)\colon 0\leq\sum_{i\in S} r_i \leq I(X_S;Y|X_{\overline{S}}),\ \forall S\subseteq[n]\}. \end{aligned}$$

The duality of the capacity regions between the scalar Gaussian MAC and BC was established in [2].

Theorem 3.2 ([2]): For the channels defined in (2) and (3),

$$\mathcal{C}_{BC}(\overline{P}, H^T) = \bigcup_{\substack{\{(\overline{P}_1,\ldots,\overline{P}_n):\\ \sum_{i=1}^n \overline{P}_i=\overline{P}\}}} \mathcal{C}_{MAC}((\overline{P}_1,\ldots,\overline{P}_n); H) \quad (4)$$

$$\mathcal{C}_{MAC}((\overline{P}_1,\ldots,\overline{P}_n); H) \quad (5)$$

$$= \bigcap_{\substack{\{(\alpha_1,\ldots,\alpha_n):\\ \alpha_i>0\ \forall\ i\in[n]\}}} \mathcal{C}_{BC}\left(\sum_{i=1}^n \frac{\overline{P}_i}{\alpha_i}; \sqrt{\alpha}^T \times H^T\right), \quad (6)$$

where the α_i's are scale factors and $\sqrt{\alpha}^T \times H^T$ denotes $(\sqrt{\alpha_1 h_1}, \ldots, \sqrt{\alpha_n h_n})^T$.
Mathematically, the duality between the capacity regions can also be viewed as the Lagrangian duality in minimax optimization [3], [4].

Remark 1: Unfortunately, although there is an elegant duality between MAC and BC in the Gaussian MIMO setting, and there are some extensions [17], [18], the duality does not generalize to arbitrary MAC and BC [19]. We will also restrict our attention to the Gaussian setting, where Property 1 is known to hold for several different PID measures.

IV. Synergy and Redundancy Duality for $n = 2$

In this section, we explore an alternative duality relation between scalar Gaussian MACs (2) and BCs (3) for $n = 2$ agents (transmitters and receivers respectively) based on partial information terms. We use superscripts for the partial information terms, such as R^{MAC} and U_i^{BC} to distinguish which channels they are associated with. Similar to duality arising from capacity (i.e. Theorem 3.2), the channel gain matrix H is shared and the noise powers are uniform. In contrast to duality arising from capacity regions, the dual channels will have uniform transmission power $\overline{P}^{BC} = \overline{P}_1^{MAC} = \overline{P}_2^{MAC}$, denoted by $\overline{P}$, instead of the sum-power constraint.

Theorem 4.1: For the MAC and BC defined in (2) and (3) respectively with equal transmission powers $\overline{P}$,

$$U_i^{MAC} + S^{MAC} = U_i^{BC} + R^{BC}, \quad i \in \{1, 2\}.$$

Proof: Using the chain rule of mutual information and equation (1), we can see that with respect to the channel models and the input distributions, for $i = 1$ in the MAC ($i = 2$ is symmetric),

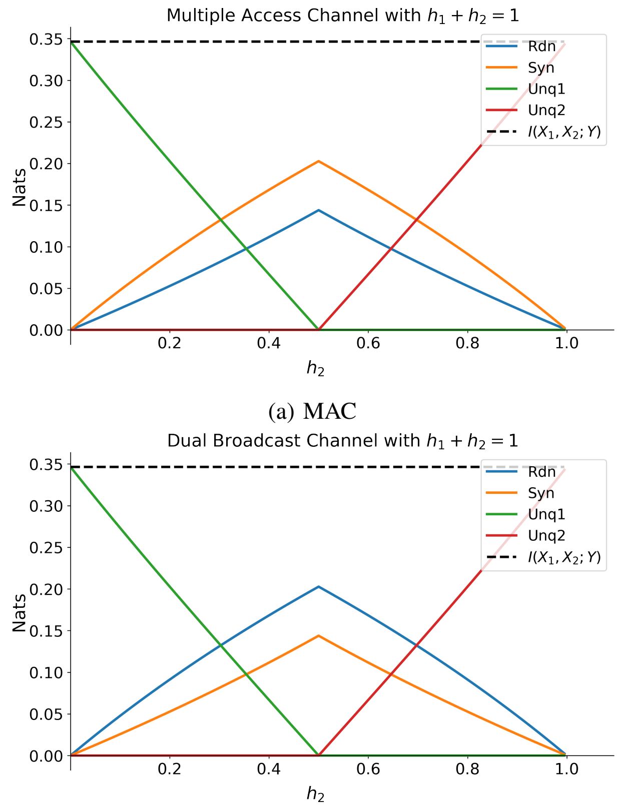
$$U_1^{MAC} + S^{MAC} = I(X_1, X_2; Y) - I(X_2; Y)$$
$$= I(X_1; Y|X_2) = \frac{1}{2}\log\left(1 + \frac{h_1 \overline{P}_1^{MAC}}{\sigma^2}\right),$$

and for the BC,

$$U_1^{BC} + R^{BC} = I(X_1; Y) = \frac{1}{2}\log\left(1+\frac{h_1 \overline{P}^{BC}}{\sigma^2}\right).$$

With $\overline{P}^{BC} = \overline{P}_1^{MAC} = \overline{P}_2^{MAC}$, the two sets of equations are equal. ■

Theorem 4.1 relates sums of partial information terms. This can be further refined to equality of individual terms.



(a) MAC

(b) BC

Fig. 2: Partial information decomposition using measures based on marginal distributions such as [1], [5], [6] of the MAC and dual BC defined in section IV. The mutual information $I(X_1, X_2; Y)$ (dashed) is decomposed into the sum of four partial information components. The synergy of the MAC (yellow in 2a) corresponds to the redundancy of the dual BC (blue in 2b), and the redundancy of the MAC (blue in 2a) corresponds to the synergy of the dual BC (yellow in 2b).

Theorem 4.2: For the Gaussian MAC (2) and BC (3) with equal transmission powers $\overline{P} = \overline{P}^{BC} = \overline{P}_i^{MAC}$ with $n = 2$ agents

$$S^{MAC} = R^{BC} \quad \text{and} \quad R^{MAC} = S^{BC}.$$

Proof: The proof follows immediately from Theorem 4.1 and Property 1. ■

In Figure 2, we plot the partial information decomposition for varying channel gains satisfying $h_1 + h_2 = 1$ when all transmitters have unit power $\overline{P} = 1$. The duality of the synergy and redundancy between the MAC and BC is visually manifested by the exchange of the yellow and blue curves in Figure 2a and Figure 2b.

V. Partial Information Duality for $n \geq 2$

In this section, we generalize the dual channel models in Section IV to the case with $n \geq 2$ interacting agents.

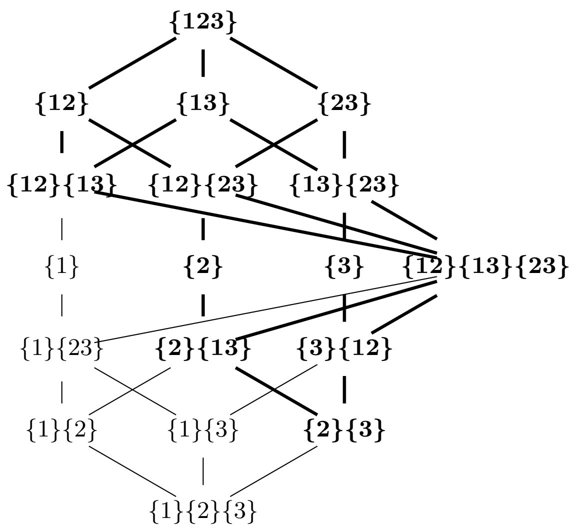


Fig. 3: The redundancy lattice $(\mathcal{A}(2^{[3]}), \preccurlyeq)$. For simplicity, we omit the outer brackets of the anti-chains. As an example, consider the subset $S = \{2, 3\} \subseteq [3]$. Then, $\downarrow \overline{S} = \downarrow \{1\}$ is the lattice lower set shown in normal font in the graph. The complement of the principal lower set $\downarrow \overline{S}$ is exactly the principal upper set $\uparrow \{2\}\{3\}$ drawn in boldface.

A. Transformation on the Redundancy Lattice

As a preparation, we prove a relation between certain principal lower sets and upper sets. In the following theorem, we pick an element $S \in 2^{[n]}$ of the power set, and consider its complement $\overline{S}$ with respect to the power set $2^{[n]}$. So for $S = \{2, 3\}$, $\overline{S} = \{1\}$. Both S and $\overline{S}$ correspond to elements in the anti-chain lattice, namely $\{S\} \in \mathcal{A}(2^{[n]})$ and $\{\overline{S}\} \in \mathcal{A}(2^{[n]})$. Note further that the element $S' = \{\{2\}\{3\}\} \in \mathcal{A}(2^{[n]})$ has the interesting property that its upper set is the complement of the lower set of $\overline{S} = 1$. That is, $\mathcal{A}(2^{[n]}) \setminus \downarrow \{\{1\}\} = \uparrow \{\{2\}\{3\}\}$. As another example, that relation holds for $S = \{2\}$, $\overline{S} = \{1, 3\}$ and $S' = \{\{2\}\}$. Not all complements of lower sets are upper sets; for instance $\mathcal{A}(2^{[n]}) \setminus \downarrow \{\{1\}\{3\}\}$ is not an upper set of any element in $\mathcal{A}(2^{[n]})$.

This property holds more generally. For a given $S \subseteq [n]$, define an $S' \in \mathcal{A}(2^{[n]})$ as

$$S' = \{\cup_{i \in S}\{i\}\}. \tag{7}$$

Theorem 5.1: Given an $S \in 2^{[n]}$, with S' defined in (7), then

$$\mathcal{A}(2^{[n]}) \setminus \downarrow \overline{S} = \uparrow S'.$$

A sketch of the proof is in Appendix A.

This theorem will be useful to prove a relation between redundancy lattices for MACs and BCs.

B. Partial Information Duality for $n \geq 2$

We next present a generalization of Theorem 4.1 for $n \geq 2$. It shows how sums of partial information terms in the redundancy lattices for the MAC and BC are related. Specifically, for any $S \in 2^{[n]}$, that the lower set of $\{S\}$ for the BC lattice has the same sum as the upper set of the element $\{\cup_{i \in S}\{i\}\}$ in the MAC lattice. Recall from Section II.B the definitions of $I_\cap$ and $I_\cup$ as summations over lower and upper sets respectively.

Theorem 5.2: For the MAC and BC defined in (2) and (3) respectively with equal transmission powers $\overline{P}$, for any $S \subseteq [n]$, we have

$$I_\cap^{BC}(\{S\}) = I_\cup^{MAC}(\{\cup_{i \in S}\{i\}\}), \quad \text{and}$$
$$I_\cap^{MAC}(\{S\}) = I_\cup^{BC}(\{\cup_{i \in S}\{i\}\}).$$

The proof is in Appendix B.

The duality in Theorem 5.2 holds for each subset $S \subseteq [n]$, meaning that we can establish 2^n equations regarding the MAC and BC partial informations, exactly the number of conditional mutual informations in the form $I(\boldsymbol{X}_S; Y | \boldsymbol{X}_{\overline{S}})$ that define the boundaries of the MAC in Theorem 3.1. When $n = 2$, Theorem 4.2 provides a direct correspondence between synergy and redundancy. When $n > 2$, however, the number of synergy and redundancy elements in the lattice $|\mathcal{A}(2^{[n]})|$ is much more than $2^{[n]}$, so they are not uniquely determined. However, for specific partial decomposition measures (e.g., using more than Property 1), it may be possible more relations can be identified.

VI. Conclusion and Future Directions

In this work, we studied invariant partial information quantities in the Gaussian MIMO setting and identified a duality between synergistic and redundant terms. For simplicity, we focused on the same MAC (2) and BC (3) models as are used in the standard channel-capacity duality. In the future we will investigate how the informational duality we identified extends for more general models.

The standard channel-capacity duality provides insights into coding schemes (e.g. dirty paper decoding). Although the duality we derived in Theorem 5.2 is different, the partial information components are more expressive than the (conditional) mutual informations. In the future, we will investigate how the results presented here can provide additional insights into coding schemes.

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APPENDIX A
PROOF SKETCH OF THEOREM 5.1

Proof: Let $T = \mathcal{A}(2^{[n]}) \setminus \downarrow (\overline{S})_{\mathcal{A}(2^{[n]})}$. Define a $\{0,1\}$-homomorphism $\phi_S : \mathcal{A}(2^{[n]}) \longrightarrow \{0,1\}$ (as in page 43 remark 2.17 of [13]) such that $\forall \alpha \in T, \phi_S(T) = 1, \forall \alpha \in \overline{T}, \phi_S(\overline{T}) = 0$. Since $\overline{T} = \downarrow (\overline{S})_{\mathcal{A}(2^{[n]})}$ is an lower set in $\mathcal{A}(2^{[n]})$, according to 2.21(2) in page 45 of [13], $\phi_S^{-1}(1)$ is a upper set in $\mathcal{A}(2^{[n]})$.

To find S', consider the order-embedding defined in Theorem 3.1 of [15]:

$$
\begin{aligned}
g : 2^{[n]} &\longrightarrow \mathcal{A}(2^{[n]}) \\
\alpha &\longrightarrow \{\alpha\}.
\end{aligned}
$$

We have, $T = g(g^{-1}(\mathcal{A}(2^{[n]}) \setminus \downarrow \overline{S})) = g(2^{[n]} \setminus \downarrow \overline{S}) = \uparrow (\underline{2^{[n]} \setminus \downarrow \overline{S}})$. ∎

APPENDIX B
PROOF OF THEOREM 5.2

Proof: The partial information sum of the BC can be shown as follows.

$$
\begin{aligned}
&I_\cap^{BC}(\{S\}) \\
&\quad = I^{BC}(\boldsymbol{X}_S; Y) && (8)\\
&\quad = h(\boldsymbol{X}_S) - h(\boldsymbol{X}_S|Y) && (9)\\
&\quad = \frac{1}{2}\log\det \mathrm{E}\left[(H_S Y + N_S)^T (H_S Y + N_S)\right] \\
&\qquad - \frac{1}{2}\log\det \mathrm{E}\left[(N_S)^T (N_S)\right] && (10)\\
&\quad = \frac{1}{2}\log\det\left[H_S^T H_S \overline{P} + \sigma^2 \mathrm{I}_{|S|}\right] - \frac{1}{2}\log\det \sigma^2 \mathrm{I}_{|S|} && (11)\\
&\quad = \frac{1}{2}\log \sigma^{2n} \det\left[H_S H_S^T \frac{\overline{P}}{\sigma^2} + \mathrm{I}_1\right] - \frac{n}{2}\log\sigma^2 && (12)\\
&\quad = \frac{1}{2}\log\det\left[\frac{\overline{P}}{\sigma^2}\sum_{i\in S} h_i + 1\right] \\
&\quad = \frac{1}{2}\log\left[1 + \frac{\overline{P}\sum_{i\in S} h_i}{\sigma^2}\right],
\end{aligned}
$$

where (8) relates $I_\cap^{BC}$ to mutual information (as discussed in Section II-B), (9) uses differential entropy, (10) uses H_S and N_S to denote channel gains and noise terms for the subset $\boldsymbol{X}_S$, $|S|$ denotes the cardinality of set S in (11), (12) follows from the Weinstein-Aronszajn identity and cancels the $\log\sigma^2$ terms.

The partial information sum for the MAC can be shown as follows.

$$
\begin{aligned}
&I_\cup^{MAC}(\{\cup_{i\in S}\{i\}\}) \\
&\quad = I_\cup^{MAC}(S') \\
&\quad = \sum_{S' \preccurlyeq \beta} I_\partial^{MAC}(\beta) \\
&\quad = \sum_{\beta \in \uparrow S'} I_\partial^{MAC}(\beta) \\
&\quad = \sum_{\beta\in\mathcal{A}(2^{[n]})} I_\partial^{MAC}(\beta) - \sum_{\beta\in\mathcal{A}(2^{[n]})\setminus\uparrow S'} I_\partial^{MAC}(\beta) \\
&\quad = I^{MAC}(\boldsymbol{X}_{[n]}; Y) - \sum_{\beta\in\downarrow\overline{S}} I_\partial^{MAC}(\beta) && (13)\\
&\quad = I^{MAC}(\boldsymbol{X}_{[n]}; Y) - I^{MAC}(\boldsymbol{X}_{\overline{S}}; Y) \\
&\quad = I^{MAC}(\boldsymbol{X}_S; Y|\boldsymbol{X}_{\overline{S}}) \\
&\quad = \frac{1}{2}\log(1 + \frac{\overline{P}}{\sigma^2}\sum_{i\in S} h_i),
\end{aligned}
$$

where (13) follows from Theorem 5.1. Thus, both sides of Theorem 5.2 are equal to the same quantity. The second identity can be derived similarly, e.g., replacing equation (9) with $h(Y) - h(Y|\boldsymbol{X}_S)$. ∎