A Geometric Approach to Linear Ensemble Control Analysis and Design

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***Abstract*— In this paper, we tackle the long-standing problem of ensemble control design and analysis with a geometric approach in a Hilbert space setting. Specifically, we formulate the control of linear ensemble systems as a convex feasibility problem that can be solved using the techniques of iterative weighted projections. Such a non-trivial geometric interpretation enables a systematic design procedure for constructing feasible and optimal ensemble control signals, and, further, enables the implementation of numerical schemes to examine ensemble reachability. We conduct numerical experiments to validate the theoretical developments and demonstrate the robustness of the iterative projection methods.**

I. Introduction

The ensemble control problem, concerning the manipulation of a parameterized family of nearly identical systems by using a broadcast control signal, has recently received much attention due to its new theoretical structures and strong practical relevance to a broad spectrum of fields, ranging from quantum physics [1], [2], [3], and neuroscience [4], [5], [6] to other emerging engineering problems [7], [8].

In the past decade, extensive studies of ensemble control have been conducted, and the focus has been placed on the investigation of fundamental properties, such as ensemble controllability and ensemble observability [9], [10], [11], [12], [13], and on the design of optimal ensemble controls. Various computational and explicit control design approaches have also been proposed, such as pseudospectral methods [14], [15], polynomial approximation and expansion [16], [17], [18], iterative methods [19], and moment-based control designs [20].

To date, the developed theoretical methods for ensemble control analysis have made good use of the algebraic structure of an ensemble system to quantify its reachable set or, its observable space defined by the parameter-dependent vector fields, so as to understand controllability or observability [9], [21]. These methods are generally not suitable for designing ensemble controls, so that, independent of control-theoretic analysis, numerical methods [14], [17] or customized computational algorithms [19] are often the ultimate avenue toward tackling the challenge ensemble control design problems.

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In this paper, we develop a unified geometric approach to not only analyze fundamental properties of ensemble systems as well as but also to synthesize feasible and optimal ensemble controls. Our main idea is to cast the problem of designing a feasible, or, further, an optimal, control for an ensemble system as a 'convex feasibility problem', i.e., finding a feasible point in the intersection of a collection of convex sets defined by the admissible control sets of the individual systems in the ensemble. This novel interpretation leverages the iterative projections to synthesize a geometric method for systematic and efficient design of an ensemble control law and for rigorous numerical evaluation of ensemble controllability in a Hilbert space setting.

The paper is organized as follows. In Section II, we introduce the geometric interpretation of ensemble control and demonstrate that such an important interpretation leads to a systematic approach for ensemble control analysis and design. In Section III, we further cast the design of minimum-energy control as an 'orthogonal projection' problem and show that the minimum-energy control design problem can be integrated into the same geometric approach as designing feasible controls. In Section IV, we present several numerical experiments substantiating the claimed performance of the proposed approach.

II. Geometric Interpretation of Ensemble Control

In this section, we present the idea of casting the ensemble control design and controllability analysis as a 'convex feasibility problem'. Leveraging this novel interpretation, we develop methods based on the techniques of iterative projections to systematically construct feasible and, further, optimal controls for linear ensemble systems. In addition, the developed methods offer a rigorous numerical evaluation for reachability between ensemble states of interest, which has not been explored in the literature.

A. Feasibility Problem and Ensemble Control Design

Consider a time-varying linear ensemble system defined on a Hilbert space,

$$\frac{d}{dt}X(t,\beta) = A(t,\beta)X(t,\beta) + B(t,\beta)u(t), \qquad (1)$$

indexed by the parameter β taking values on a compact set $K \subset \mathbb{R}$, where $X(t,\cdot) \in L^2(K,\mathbb{R}^n)$ is the state, an n-tuple of L^2-functions over K for each $t \in [0,T]$ with $T \in (0,\infty)$; $u \in L^2([0,T],\mathbb{R}^m)$ is an L^2 control function; $A \in L^\infty(D,\mathbb{R}^{n\times n})$, and $B \in L^2(D,\mathbb{R}^{n\times m})$ are matrices whose elements are real-valued L^∞- and L^2-functions, respectively, defined on the compact set $D = [0, T] \times K$. A typical goal for the control of such an ensemble system is to design the 'broadcast' control signal u that steers the entire ensemble from an initial state $X(0, \beta)$ to, or to be within a desired neighborhood of, a target state $X_F(\beta)$ at a finite time T.

Due to the nonstandard, under-actuated nature of ensemble systems, it is opaque to realize how to assemble the right toolkit from classical systems theory for ensemble control-theoretic analysis and design. To our surprise, such a challenging task may become transparent from a delicate geometric perspective. To fix ideas, let us now consider a finite sample of systems, $X(t, \beta_i)$, $i = 1, \ldots, N$, from the ensemble in (1), with the parameter β_i taking values in K. Without loss of generality, we assume that β_i are all distinct; otherwise, they represent identical systems. In this way, each sampled system is a finite-dimensional time-varying linear system in $\mathbb{R}^n$, following the dynamics

$$\frac{d}{dt}X(t, \beta_i) = A(t, \beta_i)X(t, \beta_i) + B(t, \beta_i)u(t), \qquad (2)$$

where $X(t, \beta_i) \in \mathbb{R}^n$ for each β_i and for all $t \in [0, T]$. From linear systems theory [22], the control input $u(t)$ driving the system in (2) with a given β_i from an initial state $X(0, \beta_i) \in \mathbb{R}^n$ to a target state $X_F(\beta_i) \in \mathbb{R}^n$ at time T satisfies the following integral equation

$$L_i u = \xi_i,$$

where $L_i : L^2([0, T], \mathbb{R}^m) \to \mathbb{R}^n$ is defined by

$$L_i(u) = \int_0^T \Phi(T, \sigma, \beta_i)B(\sigma, \beta_i)u(\sigma)d\sigma, \qquad (3)$$

and $\xi_i \in \mathbb{R}^n$ is formed by the initial and target states,

$$\xi_i = X_F(\beta_i) - \Phi(T, 0, \beta_i)X(0, \beta_i), \qquad (4)$$

and Φ is the transition matrix associated with the system in (2).

We first observe that the linearity of the operator L_i with respect to the control u in (3) gives convexity of the admissible control set.

Lemma 1. *The admissible control set of the system in* (2) *associated with a given pair of initial and target states* $(X(0, \beta_i), X_F(\beta_i))$*, defined by*

$$C_i = \{u \in L^2([0, T], \mathbb{R}^m) \mid L_i u = \xi_i\}, \qquad (5)$$

is a convex and closed affine subspace, where L_i *and* ξ_i *are defined as in* (3) *and* (4)*, respectively.*

Proof. For any two controls $u_1, u_2 \in C_i$ and any constant $\lambda \in [0, 1]$, it holds that

$$\begin{aligned} L_i(\lambda u_1 + (1 - \lambda)u_2) &= \lambda L_i u_1 + (1 - \lambda)L_i u_2 \\ &= \lambda \xi_i + (1 - \lambda)\xi_i = \xi_i, \end{aligned}$$

and hence C_i is convex. Because L_i is continuous, C_i is closed since it is the inverse image of $\{\xi_i\}$, which is a closed set in $\mathbb{R}^n$. In addition, since $L_i(u_1 - u_2) = 0$ holds for any $u_1, u_2 \in C_i$, C_i is an affine subspace, as shown by the linearity of L_i. □

This result, though straightforward to show, inspires a new interpretation of ensemble control design. That is, if we consider two systems in (2) characterized by β_i and β_j for $i \neq j$, then a common control law that simultaneously steers the two systems to achieve the respective desired transfers must lie in the intersection of the convex admissible control sets C_i and C_j, which is also a convex set. In this case, we have $u \in C_i \cap C_j$ such that $L_i u = \xi_i$ and $L_j u = \xi_j$. The same logic applies to an arbitrary number of systems, as illustrated in Figure 1. Therefore, the design of a broadcast ensemble control input is equivalent to a 'convex feasibility problem' over a Hilbert space, namely, a problem of finding a point in the intersection of convex sets. This can be formulated as an optimization problem of the form

$$\begin{aligned} \min \quad & 0 \\ \text{s.t.} \quad & u \in \bigcap_{i=1}^{N} C_i, \end{aligned} \qquad (6)$$

where C_i are defined in (5) for $i = 1, \ldots, N$.

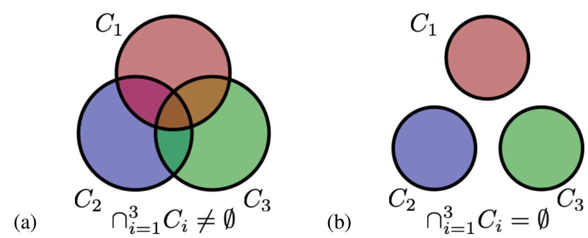

Fig. 1: Illustration of ensemble control design as a feasibility problem. (a) The case in which the admissible control sets C_i, $i = 1, 2, 3$, for three linear systems, have a nonempty intersection indicating the existence of a feasible ensemble control law that will steer these systems to the desired states. (b) The case in which C_i do not intersect, so that a control law for simultaneously steering the three systems as desired does not exist.

B. Searching Feasible Points using Iterative Projections

Because all of the admissible control sets, C_i, $i = 1, \ldots, N$, are closed and convex, and so is their intersection, a systematic and powerful approach to solve the feasibility problem as in (6) is then to utilize the techniques of iterative projections, as presented in the theorem on the iterative weighted projection algorithm.

Theorem 1 (Iterative Weighted Projection)**.** *Let* $C_1, \ldots, C_N$ *be a collection of closed and convex subsets in a Hilbert space* $\mathcal{U}$ *and let* P_{C_i} *be the projection operator onto* C_i *for* $i = 1, \ldots, N$*. Consider the sequence* $\{u^{(k)}\}$ *generated by the convex combination of projections,*

$$u^{(k+1)} = \sum_{i=1}^{N} \lambda_i P_{C_i} u^{(k)}, \quad k = 0, 1, 2, \ldots, \qquad (7)$$

with any $u^{(0)} \in \mathcal{U}$*, where* $\lambda_1, \ldots, \lambda_N \in (0, 1)$ *and* $\sum_{i=1}^{N} \lambda_i = 1$*.*

(i) *If* $\bigcap_{i=1}^{N} C_i \neq \emptyset$*, then* $\{u^{(k)}\}$ *converges to a point* $u^* \in \mathbb{C}$ *weakly. Specifically, if* C_i*'s are closed affine subspaces of* $\mathcal{U}$*, then* $\{u^{(k)}\}$ *converges in norm.*

(ii) *If* $\bigcap_{i=1}^N C_i = \emptyset$, *then if* $\inf_{u\in\mathcal{U}} \sum_{i=1}^N d^2(C_i, u)$ *is attainable, where* $d(C_i, u)$ *is the distance between* u *and* C_i *induced by the inner product, then* $u^{(k)}$ *converges weakly; otherwise* $\|u^{(k)}\| \to \infty$.

Proof. The proof can be facilitated by considering the product space $\Omega = \mathcal{U}\times\cdots\times\mathcal{U}$ constituted by a Cartesian product of N copies of $\mathcal{U}$ equipped with the inner product $\langle\cdot,\cdot\rangle_\Omega : \Omega\times\Omega \to \mathbb{R}$ defined by $\langle U, V\rangle_\Omega = \sum_{i=1}^N \lambda_i \langle u_i, v_i\rangle_{\mathcal{U}}$, where $U = (u_1, \ldots, u_N), V = (v_1, \ldots, v_N)$ and $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ is the inner product in $\mathcal{U}$. Now, let us construct two closed and convex sets $\mathcal{C}$ and $\mathcal{D}$ in Ω, defined by

$$\mathcal{C} = C_1 \times \cdots \times C_N, \tag{8}$$

$$\mathcal{D} = \{U \in \Omega \mid u_1 = \cdots = u_N\}. \tag{9}$$

Let us associate each $u^{(k)} \in \mathcal{U}$ with a unique element $U^{(k)} \in \mathcal{D}$ defined by $U^{(k)} = (u^{(k)}, \ldots, u^{(k)})$, such that if $\{U^{(k)}\}$ converges to $U^* := (u^*, \ldots, u^*) \in \mathcal{D}$, then $\{u^{(k)}\}$ converges to $u^* \in \mathcal{U}$.

For the sequence $\{u^{(k)}\}$ generated by (7), we can prove that the associated sequence $\{U^{(k)}\}$ satisfies $U^{(k+1)} = P_{\mathcal{D}} P_{\mathcal{C}} U^{(k)}$ (see Appendix I). So the sequence

$$\{U^{(k)} = P_{\mathcal{D}} P_{\mathcal{C}} \cdots P_{\mathcal{D}} P_{\mathcal{C}} U^{(0)}\}$$

is an alternating projection onto $\mathcal{C}$ and $\mathcal{D}$. By the von Neuman alternating projection algorithm in Hilbert space [23], we have the following results:

(i) If $\mathcal{C}\cap\mathcal{D} \neq \emptyset$, then $U^{(k)}$ converges to $U^* \in \mathcal{C}\cap\mathcal{D}$ weakly [24]. Equivalently, if $\bigcap_{i=1}^N C_i \neq \emptyset$, then $u^{(k)}$ converges to $u^* \in \bigcap_{i=1}^N C_i$ weakly. Furthermore, in the case that $\mathcal{C}$ and $\mathcal{D}$ are closed affine subspaces, then $U^{(k)} \to U^*$ in norm when $\mathcal{C}\cap\mathcal{D} \neq \emptyset$ [23]. Equivalently, if C_i's are closed affine subspaces, then $u^{(k)} \to u^*$ in norm when $\bigcap_{i=1}^N C_i \neq \emptyset$.

(ii) If $\mathcal{C}\cap\mathcal{D} = \emptyset$, then when $d(\mathcal{C}, \mathcal{D})$ is attainable, $U^{(k)}$ converges weakly;otherwise $U^{(k)} \to \infty$ [25]. Equivalently when $\inf_{u\in\mathcal{U}} \sum_{i=1}^N d^2(C_i, u)$ attainable, then $u^{(k)}$ converges weakly; otherwise $u^{(k)} \to \infty$.

□

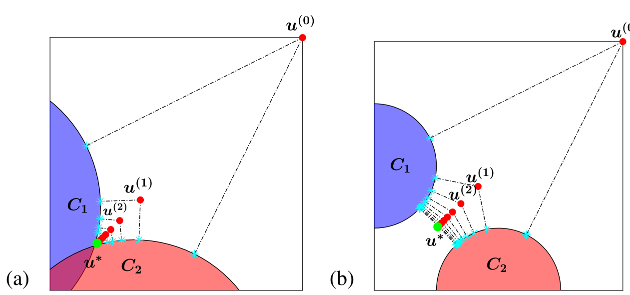

Fig. 2: Illustration of the iterative weighted projections. (a) If $C_1 \cap C_2 \neq \emptyset$, then the weighted projection, defined by $u^{(k+1)} = \frac{1}{2}(P_{C_1} u^{(k)} + P_{C_2} u^{(k)})$, converges to a point u^* in $C_1 \cap C_2$, where C_1, C_2 are closed and convex sets. (b) If $C_1 \cap C_2 = \emptyset$, then the iterative weighted projection algorithm still converges, but is convergent to a point outside C_1 and C_2.

Figures 2a and 2b schematically depicts using weighted iterative projections in (7) to solve the convex feasibility problem formulated in (6). Figure 2a illustrates the case in which a point in the intersection of two intersected convex sets is found from an arbitrary initial point $u^{(0)}$ by applying $u^{(k+1)} = \frac{1}{2}(P_{C_1} u^{(k)} + P_{C_2} u^{(k)})$. The process generates a sequence of points $\{u^{(k)}\}$, and the iterations finally converge to a point in the intersection, say $u^{(k)} \to u^* \in C_1 \cap C_2$, as $k \to \infty$. On the other hand, if $C_1 \cap C_2 = \emptyset$, the procedure will still be convergent, but not to a point of interest, i.e., $u^* \notin C_1$ and $u^* \notin C_2$, as displayed in Figure 2b.

C. Computing Ensemble Control Laws and Controllability

Theorem 1 provides a systematic approach and a powerful means to compute a feasible point in the intersection of finitely many closed and convex sets in a Hilbert space by iterative weighted projections. This feasible point corresponds to a feasible ensemble control law for steering the ensemble system in (2) from $X(0, \beta_i)$ to $X_F(\beta_i)$. Most importantly, a distinct feature of the iterative weighted projection algorithm is its capability to verify the existence of a nonempty intersection among the given convex sets by computing $\{u^{(k)}\}$ using (7). This validation is of particular importance in the context of ensemble control since it informs whether the ensemble is controllable or not. In particular, for a linear ensemble, because the admissible control set defined in (5) is an affine subspace of $\mathcal{U}$, by the contraposition of statement (i) in Theorem 1, if $\{u^{(k)}\}$ does not converge to a point in $\mathcal{U}$ in norm (either with $\|u^{(k)}\|_{\mathcal{U}} \to \infty$ or with $u^{(k)}$ oscillating because of weak convergence), then it must hold that $\bigcap_{i=1}^N C_i = \emptyset$. Hence the system in (2) is not ensemble controllable because there exists no common control u that will simultaneously steer the entire ensemble to $X_F(\beta_i)$.

On the other hand, if $\{u^{(k)}\}$ converges in norm with $u^{(k)} \to u^*$, then there are two possible cases, as shown in Figures 2a and 2b. To distinguish them, one can simply apply the control law u^* to the linear ensemble (2). If u^* steers the ensemble to the desired target state, then we have the case $\bigcap_{i=1}^N C_i \neq \emptyset$ (Figure 2a); otherwise, it must hold that $\bigcap_{i=1}^N C_i = \emptyset$ (Figure 2b). As a result, this convergence property renders a rigorous and tractable numerical approach to examining the reachability of an ensemble system and to systematically designing an ensemble control law, as described in the following theorem.

Theorem 2. *Consider the linear ensemble system in* (2)*. Let* $\{u^{(k)}\}$ *be a control sequence generated according to the iterative weighted projections in* (7)*, given by the explicit form*

$$u^{(k+1)} = \big(I_d - \sum_{i=1}^N \lambda_i L_i^* (L_i L_i^*)^{-1} L_i\big) u^{(k)} + \sum_{i=1}^N \lambda_i L_i^* (L_i L_i^*)^{-1} \xi_i, \tag{10}$$

starting with an arbitrary initialization $u^{(0)} \in L^2([0,T], \mathbb{R}^m)$*, where* I_d *is the identity operator in*

$L^2([0,T],\mathbb{R}^m)$; L_i and $\xi_i = X_F(\beta_i) - \Phi(T,0,\beta_i)X(0,\beta_i)$ are defined in (3) and (4), respectively; L^* denotes the adjoint operator of L; and $\lambda_i \in (0,1)$ with $\sum_{i=1}^N \lambda_i = 1$. If $\{u^{(k)}\}$ converges in norm to $u^* \in L^2([0,T],\mathbb{R}^m)$ and u^* satisfies $L_i u^* = \xi_i$ for $i = 1,\ldots,N$, then the system is ensemble reachable with respect to ξ_i, and u^* is a feasible ensemble control law. Otherwise, the given state $X_F(\beta_i)$ is not ensemble reachable from $X(0,\beta_i)$.

Proof. The proof directly follows Lemma 1, Theorem 1, and the analysis above Theorem 1. What remains to be shown here is to derive the explicit expression of the projection operator P_{C_i} in (7).

It is known that finding the projection of a vector $u \in L^2([0,T],\mathbb{R}^m)$ onto a closed subspace C_i, denoted $P_{C_i}u$, is equivalent to solving the least-squares problem

$$\begin{aligned} \min_{v\in L^2([0,T],\mathbb{R}^m)} \quad & \|u-v\|_2 \\ \text{s.t.} \quad & L_i v = \xi_i. \end{aligned} \tag{11}$$

For the ensemble control problem, we are interested in the case when the system indexed by each β_i is controllable. Then we have $\mathcal{R}(L_i) = \mathbb{R}^n$ (see [22]). It is well-known in functional analysis that $\mathcal{R}(L_i L_i^*) = \mathcal{R}(L_i) = \mathbb{R}^n$, and the solution of (11) can be written as

$$P_{C_i}u = (I_d - L_i^*(L_iL_i^*)^{-1}L_i)u + L_i^*(L_iL_i^*)^{-1}\xi_i. \tag{12}$$

Substituting (12) into (7) yields the update rule in (10). □

III. Minimum-energy control for Linear ensemble systems

In Section II, we formulated ensemble control design as a convex feasibility problem, through which we were able to calculate a feasible ensemble control law. In this section, we take a step further and extend the developed method to find optimal ensemble controls for linear ensemble systems. Here, we study the minimum-energy control for the ensemble system as in (2).

A. A Formulation of Minimum-Energy Ensemble Control as an Optimization Problem

Adopting the same idea of formulating the ensemble control design as a feasibility problem, the minimum-energy control of the ensemble system in (2) can be cast as an optimization problem, given by

$$\begin{aligned} \min \quad & \|u\|_2^2 \\ \text{s.t.} \quad & u \in \bigcap_{i=1}^N C_i, \end{aligned} \tag{13}$$

where $\|u\|_2^2 = \int_0^T u'u\,dt$ and C_i are defined as in (5).

An intriguing fact in this optimization is that the objective function represents the distance between u and the origin, i.e., the zero function in $L^2([0,T],\mathbb{R}^m)$. Therefore, minimizing the energy of u simply involves finding the point in $\bigcap_{i=1}^N C_i$ that is closest to the origin in $L^2([0,T],\mathbb{R}^m)$, which is the 'orthogonal projection' of the origin onto the set $\bigcap_{i=1}^N C_i$.

B. Orthogonal Projection onto Intersection of Convex Sets

Let $\mathcal{U}$ be a Hilbert space and C_i be closed and convex sets in $\mathcal{U}$. It is in general difficult to directly compute the projection of a vector $u \in \mathcal{U}$ onto $\bigcap_{i=1}^N C_i$. Fortunately, when the projection onto each C_i, i.e., $P_{C_i}u$, is easy to compute, the projection onto the intersection $\bigcap_{i=1}^N C_i$ can then be obtained by cyclic projections onto each individual set. Dykstra's algorithm [26] is a notable method of performing such computations, described in Algorithm 1.

Algorithm 1 Dykstra's algorithm

function DYKSTRA($C_1,\ldots,C_N,u^{(0)}$)
Initialize:
(0) Assign $u_0^{(1)} = u^{(0)}$.
(1) Project $u_0^{(1)}$ onto C_1 to obtain $u_1^{(1)}$.
Compute $I_1^{(1)} = u_1^{(1)} - u_0^{(1)}$.
⋮
(N) Project $u_{N-1}^{(1)}$ onto C_N to obtain $u_N^{(1)}$.
Compute $I_N^{(1)} = u_N^{(1)} - u_{N-1}^{(1)}$.
for $k \leftarrow 2,3,\ldots$ **do**:
(0) Assign $u_0^{(k)} = u_N^{(k-1)}$.
(1) Project $u_0^{(k)} - I_1^{(k-1)}$ onto C_1 to obtain $u_1^{(k)}$.
Compute $I_1^{(k)} = u_1^{(k)} - (u_0^{(k)} - I_1^{(k-1)})$.
⋮
(N) Project $u_{N-1}^{(k)} - I_N^{(k-1)}$ onto C_N to obtain $u_N^{(k)}$.
Compute $I_N^{(k)} = u_N^{(k)} - (u_{N-1}^{(k)} - I_N^{(k-1)})$.
end for
return $\{u_1^{(k)}\}_{k=1}^\infty,\ldots,\{u_{N-1}^{(k)}\}_{k=1}^\infty$
end function

Lemma 2. *Let $C_1,\ldots,C_N$ be closed and convex in a Hilbert space $\mathcal{U}$, and let u^* be the orthogonal projection of $u^{(0)}$ onto $\bigcap_{i=1}^N C_i$. Then the sequence $\{u_i^{(k)}\}$ obtained by Algorithm 1 converges to u^* in norm for any $1 \le i \le N$. Furthermore, if C_i is affine for all $i = 1,\ldots,N$, then the offset $I_i^{(k-1)}$ in Algorithm 1 can be set as 0 for all $k = 2,3,\ldots$, with the same guarantee that $\{u_i^{(k)}\}$ converges to u^* in norm for all $i = 1,\ldots,N$.*

Proof. See the proof of Theorem 2 in [26]. □

Remark 1. (*Dykstra's algorithm v.s. iterative weighted projection algorithm*) It is worth pointing out that Dykstra's algorithm involves an offset $I_i^{(k)}$ when projecting the next iteration $u_i^{(k+1)}$ onto C_i, which eventually finds in the point in $\bigcap_{i=1}^N C_i$ that is the closest to $u^{(0)}$; while the iterative weighted projection algorithm returns only one feasible point in $\bigcap_{i=1}^N C_i$, but not necessarily the one closest to $u^{(0)}$.

C. Minimum-Energy Control for Linear Ensembles by Iterative Projections

From the discussion in Section III-A, we have interpreted the minimum-energy ensemble control problem as the projection of the zero function, $0 \in L^2([0,T],\mathbb{R}^m)$, onto the intersection of the admissible control sets, formulated as in (13). Here, we show that computing this projection using Dykstra's algorithm, for linear ensemble, is equivalent to conducting the iterative weighted projections as in (10), as stated by the following theorem.

Theorem 3. (Minimum-energy control for linear ensemble systems) *Consider the linear ensemble system in* (2)*, with* L_i*,* ξ_i*, and* C_i *defined as in* (3)*,* (4) *and* (5)*, respectively. If* $\bigcap_{i=1}^{N} C_i \neq \emptyset$*, then the control sequence* $\{u^{(k)}\}$*, with* $u^{(0)} = 0 \in L^2([0,T],\mathbb{R}^m)$*, generated by the iterations,*

$$u^{(k+1)} = (I_d - \sum_{i=1}^{N} \lambda_i L_i^*(L_i L_i^*)^{-1} L_i) u^{(k)} + \sum_{i=1}^{N} \lambda_i L_i^*(L_i L_i^*)^{-1} \xi_i,$$

for $k = 0, 1, 2, \ldots,$ *converges to the minimum-energy ensemble control in norm, where* I_d *is the identity operator in* $L^2([0,T],\mathbb{R}^m)$*,* L_i^* *denotes the adjoint operator of* L_i*, and* $\lambda_i \in (0,1)$ *with* $\sum_{i=1}^{N} \lambda_i = 1$*.*

Proof. We first define the product space $\mathcal{H}$ and the augmented sets $\mathcal{C}$ and $\mathcal{D}$ in the same way as in (8) and (9) in the proof of Theorem 1. Then we associate each $u \in \bigcap_{i=1}^{N} C_i$ with an element $U \in \mathcal{D}$ defined by $U = (u, \ldots, u)$, so that we have $U \in \mathcal{C} \cap \mathcal{D}$, and $\|U\|_{\mathcal{H}}^2 = \sum_{i=1}^{N} \lambda_i \|u\|_{\mathcal{U}}^2 = \|u\|_{\mathcal{U}}^2$. This reformulates the optimization problem in (13) into a new optimization problem over the product space $\mathcal{H}$ to minimize $\|U\|_2^2$ subject to $U \in \mathcal{C} \cap \mathcal{D}$. The solution of the new optimization problem is provided by the orthogonal projection of $0 \in \mathcal{H}$ that can be computed by the Dykstra's algorithm. Since $\mathcal{C}$ and $\mathcal{D}$ are both closed affine subspaces of $\mathcal{H}$, Dykstra's algorithm for $\mathcal{C} \cap \mathcal{D}$ with $U^{(0)} = 0 \in \mathcal{H}$ boils down to the iterative weighted projections on $\mathcal{C}$ and $\mathcal{D}$, which leads to the projection of $u^{(0)} = 0 \in \mathcal{U}$ onto $\bigcap_{i=1}^{N} C_i$. The convergence of the iterative weighted projections on $\mathcal{C}$ and $\mathcal{D}$ has already been fully analyzed in Theorem 1. Hence simply changing the initial condition of Theorem 2 provides a sequence $\{u^{(k)}\}$ converging to the minimum-energy control law of the ensemble. ☐

IV. Numerical experiments

In this section, we provide numerical experiments to show the performance of the presented geometric approach in designing control laws for linear ensemble systems. All the numerical experiments are implemented in Matlab R2017b on a single workstation with a Xeon Gold 6144 3.5 GHz processor and 192 GB memory.

We consider controlling an ensemble of harmonic oscillators modeled by

$$\frac{d}{dt}\begin{bmatrix} x_1(t,\omega_i) \\ x_2(t,\omega_i) \end{bmatrix} = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}\begin{bmatrix} x_1(t,\omega_i) \\ x_2(t,\omega_i) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad (14)$$

where the frequencies of the oscillators are uniformly sampled over $[-1, 1]$ with a step size of 0.1, resulting in 21 systems in the ensemble. All the systems in the ensemble are steered from the same initial state $X(0,\omega_i) = (1,0)'$ to the same target state $X_F(\omega_i) = (0,1)'$ at time $T = 1$.

Figure 3a shows the feasible control law and final state errors under the designed feasible ensemble control. The update rule (10) is applied by discretizing $u^{(k)}$ and the operator L_i in time. It is demonstrated that more iterations lead to smaller final state errors.

Figure 3b provides the minimum-energy control and the corresponding final state error by initializing (10) with $u^{(0)} = 0$. In this experiment, the Hilbert space $L^2([0,T],\mathbb{R}^m)$ is expanded using Legendre polynomials of orders up to 20. Then the update rule (10) is applied to iterate the coordinates of control law $u^{(k)}$ under the Legendre basis. It is observed that the control law computed after 1×10^7 iterations works well and is same as the one generated by existing control synthesis methods such as [27].

V. Conclusion

In this paper, we propose a geometric method for ensemble control analysis and design. Our main idea is to cast these challenging tasks as a convex feasibility problem in a Hilbert space. This novel geometric interpretation allows us to numerically evaluate reachability between specified ensemble states, and to design a feasible or an optimal ensemble control. We present numerical experiments to illustrate the theoretical development and applicability of the presented iterative projection methods.

Appendix I
Proof of the iterative weighted projections algorithm

Since $C_1, \ldots, C_N$ is closed and convex, $\mathcal{C}$ is also closed and convex. $\mathcal{D}$ is a subspace of $\mathcal{H}$, which is automatically closed and convex. Therefore the projections onto $\mathcal{C}$ and $\mathcal{D}$, denoted as $P_{\mathcal{C}}$ and $P_{\mathcal{D}}$, are well-defined. We associate each $u^{(k)} \in \mathcal{U}$ with $U^{(k)} := (u^{(k)}, \ldots, u^{(k)}) \in \mathcal{H}$. Then

Lemma 3. *The update rule* (7) *yields*

$$U^{(k+1)} = P_{\mathcal{D}} P_{\mathcal{C}} U^{(k)}.$$

Proof. We first show that

$$P_{\mathcal{C}} U^{(k)} = (P_{C_1} u^{(k)}, \ldots, P_{C_N} u^{(k)}).$$

By the definition of projection, $P_{\mathcal{C}} U^{(k)} = \underset{V \in \mathcal{C}}{\operatorname{argmin}} \|U^{(k)} - V\|_{\mathcal{H}}$. Instead of minimizing $\|U^{(k)} - V\|_{\mathcal{H}}$, we are seeking to minimize

$$\|U^{(k)} - V\|_{\mathcal{H}}^2 = \langle U^{(k)} - V, U^{(k)} - V \rangle_{\mathcal{H}}, \forall V \in \mathcal{C}. \quad (15)$$

By the definition of inner product on $\mathcal{H}$, (15) is computed as

$$\begin{aligned} \|U^{(k)} - V\|_{\mathcal{H}}^2 &= \langle U^{(k)} - V, U^{(k)} - V \rangle_{\mathcal{H}} \\ &= \sum_{i=1}^{N} \lambda_i \langle u^{(k)} - v_i, u^{(k)} - v_i \rangle_{\mathcal{U}} = \sum_{i=1}^{N} \lambda_i \|u^{(k)} - v_i\|_{\mathcal{U}}^2 \end{aligned} \quad (16)$$

Since each $v_i \in C_i \subset X$, by the definition of projection on $\mathcal{U}$, $\|u^{(k)} - v_i\|_{\mathcal{U}}^2 \geq \|u^{(k)} - P_{C_i} u^{(k)}\|_{\mathcal{U}}^2$. Hence (16) can be bounded below by

$$\|U^{(k)} - V\|_{\mathcal{H}} \geq \sum_{i=1}^{N} \lambda_i \|u^{(k)} - P_{C_i} u^{(k)}\|_{\mathcal{U}}^2. \quad (17)$$

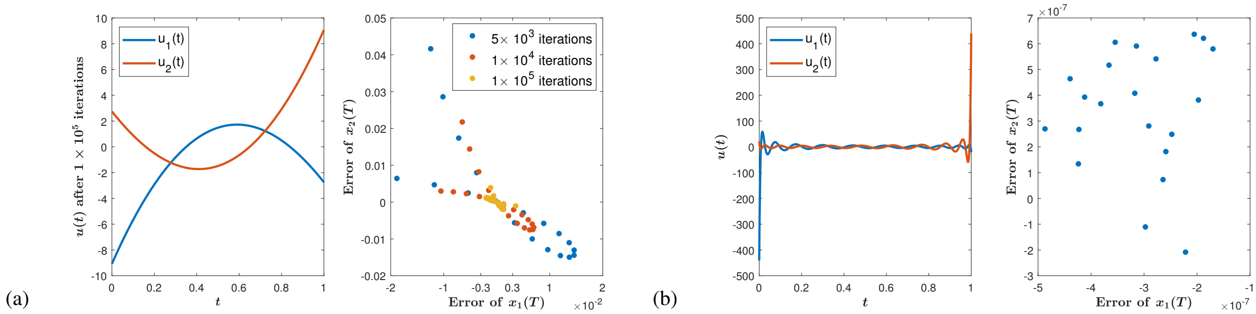

Fig. 3: Results of controlling an ensemble of 21 harmonic oscillators. Initial states and target states are set as $X(0,\omega_i)=(1,0)'$ and $X_F(\omega_i)=(0,1)'$, for all $i=1,\ldots,21$. (a) A feasible control law obtained by running iterative weighted projections for 1×10^5 iterations and the corresponding final state errors. The initial control law is $(u_1^{(0)}(t),u_2^{(0)}(t))\equiv(1,1)$. (b) Minimum-energy ensemble control law and the corresponding final state errors.

We observe that $P_{C_i}u^{(k)}\in C_i$ for all i, and the equality in (17) can be achieved when $V=(P_{C_1}u^{(k)},\ldots,P_{C_N}u^{(k)})$. Hence we conclude that $P_{\mathcal{C}}U^{(k)}=(P_{C_1}u^{(k)},\ldots,P_{C_N}u^{(k)})$.

Denote $W=(w,\ldots,w)=P_{\mathcal{D}}P_{\mathcal{C}}U^{(k)}$. We observe that $\mathcal{D}$ is a subspace in $\mathcal{H}$. Hence it holds that $\forall s\in\mathcal{H},\forall t\in\mathcal{D}, s-P_{\mathcal{D}}s\perp t$. Now taking $s=P_{\mathcal{C}}U^{(k)}$ and $t=rW$, where r is an arbitrary real number yields that

$$\langle P_{\mathcal{C}}U^{(k)}-W,rW\rangle_{\mathcal{H}}=0. \tag{18}$$

Substituting the definition of the inner product in $\mathcal{H}$ into (18) yields

$$\sum_{i=1}^{N}\lambda_i\langle P_{C_i}u^{(k)}-w,rw\rangle_{\mathcal{U}}=\langle\sum_{i=1}^{N}\lambda_i P_{C_i}u^{(k)}-w,rw\rangle_{\mathcal{U}}. \tag{19}$$

Since (19) holds for all $r\in\mathbb{R}$, $\sum_{i=1}^{N}\lambda_i P_{C_i}u^{(k)}-w$ must be 0, which implies that $w=\sum_{i=1}^{N}\lambda_i P_{C_i}u^{(k)}$. By (7), it holds that $u^{(k+1)}=\sum_{i=1}^{N}\lambda_i P_{C_i}u^{(k)}$, which implies $U^{(k+1)}=W=P_{\mathcal{D}}P_{\mathcal{C}}U^{(k)}$. □

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