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FATOU'S LEMMA FOR WEAKLY CONVERGING MEASURES UNDER THE UNIFORM INTEGRABILITY CONDITION*

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Abstract. This paper describes Fatou's lemma for a sequence of measures converging weakly to a finite measure and for a sequence of functions whose negative parts are uniformly integrable with respect to these measures. The paper also provides new formulations of uniform Fatou's lemma, uniform Lebesgue's convergence theorem, the Dunford–Pettis theorem, and the fundamental theorem for Young measures based on the equivalence of uniform integrability and the apparently weaker property of asymptotic uniform integrability for sequences of functions and finite measures.

Key words. Fatou lemma, weak convergence of measures, uniform integrability, asymptotic uniform integrability

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1. Introduction. The Fatou lemma states that under appropriate conditions the integral of the lower limit of a sequence of functions is not greater than the lower limit of the integrals. This inequality holds under one of the following conditions: (i) Each function is nonnegative; (ii) the sequence of functions is bounded below by an integrable function; (iii) the sequence of negative parts of the functions is uniformly integrable; see [30, Chap. II, section 6, Theorem 2 and Chap. II, section 13, Problem 10]. Serfozo [29, Lemma 3.2] established the Fatou lemma for a sequence of measures converging vaguely on a locally compact metric space and for nonnegative functions. Feinberg, Kasyanov, and Zadoianchuk [15, Theorem 4.2] provided the Fatou lemma for a sequence of measures converging weakly and for functions bounded below by a sequence of functions satisfying a certain minorant condition, which is satisfied for nonnegative functions. In this paper, we establish the Fatou lemma for a sequence of measures converging weakly and for functions whose negative parts satisfy the uniform integrability condition.

Uniform integrability of a family of functions plays an important role in probability theory and analysis. The relevant notion is the asymptotic uniform integrability of a sequence of random variables [34, p. 17]. In this paper, we introduce the definitions of uniformly integrable (u.i.) and asymptotically uniformly integrable (a.u.i.) functions with respect to (w.r.t.) a sequence of finite measures, and we show that these definitions are equivalent. This equivalence provides alternative formulations and proofs for some facts that involve uniform integrability or asymptotic uniform integrability assumptions. For the case of a single probability measure, this equivalence is established in [25, p. 180].

The Fatou lemmas for weakly converging measures have significant applications to various areas and fields, including stochastic processes [5], [7], [21], [26], statistics [31], [32], [19], control [6], [12], [14], [17], [35], game theory [22], functional analysis [20], optimization [37], and electrical engineering [28]. Our initial impetus for

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studying the Fatou lemma for variable probabilities came from its usefulness in the proof of the validity of optimality inequalities and the existence of stationary optimal policies for infinite-horizon, Borel-state, average-cost Markov decision process with noncompact action sets and unbounded costs [14]. These results have significant applications to inventory control [11], [18].

Other versions of the Fatou lemmas for variable measures are also important for applications. The recently discovered uniform Fatou lemma and uniform Lebesgue convergence theorems [16] play the central role in establishing sufficient optimality conditions for partially observable Markov decision processes with Borel state and action spaces [17]. Unlike the classical Fatou lemma, which provides sufficient conditions for the Fatou inequality, the uniform Fatou lemma states necessary and sufficient condition for the uniform version of the Fatou inequality. If all the functions are absolutely integrable, these necessary and sufficient conditions are more general than the conditions in the classic Fatou lemma. One of two necessary conditions in the uniform Fatou lemma [16, Theorem 2.1] is that the sequence of negative parts of functions is uniformly integrable w.r.t. the measures; see (2.1) below. The central result of this paper, Theorem 2.4, states that this condition is sufficient for the validity of the Fatou inequality for weakly converging measures. The examples in section 3 show that this condition and the sufficient condition in Assumption 2.5, which was introduced in [15], do not imply each other. In particular, Theorem 2.4 is useful for studying Markov decision processes and stochastic games with cost functions unbounded from above and from below; see [9], [13], [23], where such problems were examined.

The Fatou lemma and the Lebesgue convergence theorems for probabilities are classical facts in probability theory. However, their versions for finite measures are also important for probability theory and its applications. This is why this paper and [16] are concerned with finite measures rather than probabilities. For example, the theory of optimization of Markov decision processes with multiple criteria is based on considering occupancy (also often called occupation) measures, which typically are not probability measures [4]. Another example is [26], where the Fatou lemma for nonnegative functions and finite measures is used.

Though uniform integrability and asymptotic uniform integrability properties of a sequence of functions w.r.t. a sequence of finite measures are equivalent, it is typically easier to verify the asymptotic uniform integrability. This is important for applications. For this reason we provide in section 4 alternative formulations of the uniform Fatou lemma and Lebesgue convergence theorem from [16] and two classical facts important for applications. In these formulations, the uniform integrability is replaced by the asymptotic uniform integrability.

Section 2 of this paper provides definitions, describes the equivalence of uniform integrability and asymptotic uniform integrability, and states the Fatou lemma and the Lebesgue dominated convergence theorem for weakly converging measures. In particular, the Fatou lemma is formulated in section 2 for weakly converging measures and for a.u.i. sequences of functions, which is equivalent to the assumption that the sequence of functions is u.i. Section 3 illustrates by means of examples that the uniform integrability condition stated in Theorem 2.4 neither implies nor is implied by the minorant condition; see Assumption 2.5 and Corollary 2.7 below. Example 3.3 demonstrates that $\limsup$ in inequalities (2.7) in Assumption 2.5 cannot be relaxed to $\liminf$. By employing the equivalence of the uniform integrability and the asymptotic uniform integrability, section 4 provides alternative formulations of the uniform Fatou lemma, the uniform Lebesgue dominated convergence theorem, the Dunford–Pettis

theorem, and the Ball fundamental theorem for Young measures. Section 5 provides proofs of the Fatou lemma for weakly converging measures and relevant results formulated in section 2.

2. Main results. Let $(\mathbb{S},\Sigma)$ be a measurable space, let $\mathscr{M}(\mathbb{S})$ be the *family of all finite measures on* $(\mathbb{S},\Sigma)$, and let $\mathbf{P}(\mathbb{S})$ be the *family of all probability measures* on $(\mathbb{S},\Sigma)$. When $\mathbb{S}$ is a topological space, as a rule we consider $\Sigma := \mathscr{B}(\mathbb{S})$, where $\mathscr{B}(\mathbb{S})$ is the *Borel σ-field* on $\mathbb{S}$. Let $\mathbf{R}$ be the real line, and let $\overline{\mathbf{R}} := [-\infty,+\infty]$. We write $\mathbf{I}A$ for the indicator of an event A.

Throughout this paper, we deal with integrals of functions that can take both positive and negative values. The *integral* $\int_{\mathbb{S}} f(s)\,\mu(ds)$ of a measurable $\overline{\mathbf{R}}$-valued function f on $\mathbb{S}$ w.r.t. a measure μ is *defined* and equal to

$$\int_{\mathbb{S}} f(s)\,\mu(ds) = \int_{\mathbb{S}} f^{+}(s)\,\mu(ds) - \int_{\mathbb{S}} f^{-}(s)\,\mu(ds)$$

if

$$\min\biggl\{\int_{\mathbb{S}} f^{+}(s)\,\mu(ds),\ \int_{\mathbb{S}} f^{-}(s)\,\mu(ds)\biggr\} < \infty,$$

where $f^{+}(s) = \max\{f(s),0\}$, $f^{-}(s) = -\min\{f(s),0\}$, $s\in\mathbb{S}$. All the integrals in the assumptions of the theorems and corollaries throughout this paper are assumed to be defined.

Definition 2.1. *A sequence of measurable $\overline{\mathbf{R}}$-valued functions $\{f_n\}_{n=1,2,\dots}$ is called*

— *uniformly integrable (u.i.) w.r.t. a sequence of measures $\{\mu_n\}_{n=1,2,\dots}\subset\mathscr{M}(\mathbb{S})$ if*

$$\lim_{K\to+\infty}\sup_{n=1,2,\dots}\int_{\mathbb{S}}|f_n(s)|\,\mathbf{I}\{s\in\mathbb{S}\colon |f_n(s)|\geqslant K\}\,\mu_n(ds)=0; \tag{2.1}$$

— *asymptotically uniformly integrable (a.u.i.) w.r.t. a sequence of measures $\{\mu_n\}_{n=1,2,\dots}\subset\mathscr{M}(\mathbb{S})$ if*

$$\lim_{K\to+\infty}\limsup_{n\to\infty}\int_{\mathbb{S}}|f_n(s)|\,\mathbf{I}\{s\in\mathbb{S}\colon |f_n(s)|\geqslant K\}\,\mu_n(ds)=0. \tag{2.2}$$

We remark that the limit as $K\to+\infty$ in (2.1) (resp., (2.2)) exists because the function

$$K\mapsto \sup_{n=1,2,\dots}\,(\limsup_{n\to\infty})\int_{\mathbb{S}}|f_n(s)|\,\mathbf{I}\{s\in\mathbb{S}\colon |f_n(s)|\geqslant K\}\,\mu_n(ds) \tag{2.3}$$

is nonincreasing in $K>0$.

If $\mu_n=\mu\in\mathscr{M}(\mathbb{S})$ for each $n=1,2,\dots$, then an (a.)u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$ sequence $\{f_n\}_{n=1,2,\dots}$ is called *(a.)u.i.* For a single finite measure μ, the definition of an a.u.i. sequence of functions (random variables in the case of a probability measure μ) coincides with the corresponding definition broadly used in the literature; see, e.g., [34, p. 17]. Also, for a single fixed finite measure, the definition of a u.i. sequence of functions is consistent with the classical definition of a family $\mathcal{H}$ of u.i. functions. We say that a function f is (a.)u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$ if the sequence $\{f,f,\dots\}$ is (a.)u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$. A function f is u.i. w.r.t. a family $\mathscr{N}$ of measures if

$$\lim_{K\to+\infty}\sup_{\mu\in\mathscr{N}}\int_{\mathbb{S}}|f(s)|\,\mathbf{I}\{s\in\mathbb{S}\colon |f(s)|\geqslant K\}\,\mu(ds)=0.$$

The following theorem states the equivalence of the uniform and asymptotic uniform integrability properties introduced in Definition 2.1. The proof of Theorem 2.2 is presented in section 6. Several examples of applications of Theorem 2.2 are provided in section 4. As mentioned in the introduction, for $\mu_n = \mu$, where μ is a probability measure, $n = 1, 2, \ldots$, Theorem 2.2 was presented in [25, p. 180].

THEOREM 2.2 (equivalence of u.i. and a.u.i.; cf. [25, p. 180]). *Let $(\mathbb{S}, \Sigma)$ be a measurable space, let $\{\mu_n\}_{n=1,2,\ldots}$ be a sequence of measures from $\mathscr{M}(\mathbb{S})$, and let $\{f_n\}_{n=1,2,\ldots}$ be a sequence of measurable $\overline{\mathbf{R}}$-valued functions on $\mathbb{S}$. Then there exists $N = 0, 1, \ldots$ such that $\{f_{n+N}\}_{n=1,2,\ldots}$ is u.i. w.r.t. $\{\mu_{n+N}\}_{n=1,2,\ldots}$ if and only if $\{f_n\}_{n=1,2,\ldots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$.*

We recall that the Fatou lemma claims that, for a sequence of nonnegative measurable functions $\{f_n\}_{n=1,2,\ldots}$ defined on a measurable space $(\mathbb{S}, \Sigma)$ and for a measure μ on this space,

$$\int_{\mathbb{S}} \liminf_{n\to\infty} f_n(s)\, \mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\, \mu(ds). \tag{2.4}$$

Although a sequence of functions is u.i. if and only if it is a.u.i., in many cases it is easier to verify that the sequence of functions is a.u.i. rather than it is u.i.

DEFINITION 2.3. *A sequence of measures $\{\mu_n\}_{n=1,2,\ldots}$ on a metric space $\mathbb{S}$ converges weakly to a finite measure μ on $\mathbb{S}$ if for each bounded continuous function f on $\mathbb{S}$,*

$$\lim_{n\to\infty} \int_{\mathbb{S}} f(s)\, \mu_n(ds) = \int_{\mathbb{S}} f(s)\, \mu(ds). \tag{2.5}$$

The following theorem is the main result of this section. We provide the proof of this theorem in section 5.

THEOREM 2.4 (the Fatou lemma for weakly converging measures). *Let $\mathbb{S}$ be a metric space, let $\{\mu_n\}_{n=1,2,\ldots}$ be a sequence of measures on $\mathbb{S}$ converging weakly to $\mu \in \mathscr{M}(\mathbb{S})$, and let $\{f_n\}_{n=1,2,\ldots}$ be a sequence of measurable $\overline{\mathbf{R}}$-valued functions on $\mathbb{S}$ such that $\{f_n^-\}_{n=1,2,\ldots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$. Then*

$$\int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f_n(s')\, \mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\, \mu_n(ds). \tag{2.6}$$

Consider the following assumption introduced in [15]; this is a sufficient condition for the validity of the Fatou lemma for weakly converging measures.

ASSUMPTION 2.5. *Let $\mathbb{S}$ be a metric space, let $\{\mu_n\}_{n=1,2,\ldots}$ be a sequence of measures on $\mathbb{S}$ that converges weakly to $\mu \in \mathscr{M}(\mathbb{S})$, and let $\{f_n, g_n\}_{n=1,2,\ldots}$ be a sequence of measurable $\overline{\mathbf{R}}$-valued functions on $\mathbb{S}$ such that $f_n(s) \geqslant g_n(s)$ for each $n = 1, 2, \ldots$ and $s \in \mathbb{S}$, and such that*

$$-\infty < \int_{\mathbb{S}} \limsup_{n\to\infty,\, s'\to s} g_n(s')\, \mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\, \mu_n(ds). \tag{2.7}$$

We note that Assumption 2.5 implies under certain conditions that the sequence of functions $\{f_n^-\}_{n=1,2,\ldots}$ is u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$; see Theorem 2.6 below. In general, these two conditions do not imply each other; see Examples 3.1 and 3.2. The following theorem, whose proof is provided in section 5, describes a sufficient condition for

the case when the uniform integrability is more general than Assumption 2.5. In addition, according to Example 3.1, these two assumptions are not equivalent under the sufficient condition stated in Theorem 2.6.

THEOREM 2.6. *Let Assumption* 2.5 *hold. If a sequence of functions* $\{g_n\}_{n=1,2,\ldots}$ *is uniformly bounded from above, then there exists* $N = 0, 1, \ldots$ *such that* $\{f^-_{n+N}\}_{n=1,2,\ldots}$ *is u.i. w.r.t.* $\{\mu_{n+N}\}_{n=1,2,\ldots}$.

For weakly converging probability measures, the Fatou lemma was introduced in [29] and generalized in [15, Theorem 4.2]. The following corollary extends [15, Theorem 4.2] to finite measures. The proof of Corollary 2.7 is provided in section 5. Example 3.3 demonstrates that Corollary 2.7 is incorrect if Assumption 2.5 is weakened by replacing lim sup with lim inf in formula (2.7).

COROLLARY 2.7 (the Fatou lemma for weakly converging measures; cf. [15, Theorem 4.2]). *Inequality* (2.6) *holds under Assumption* 2.5.

The following corollary is the Lebesgue dominated convergence theorem for weakly converging measures. A similar result was given in [29, Theorem 3.5] in the form of a necessary and sufficient condition for nonnegative functions and for locally compact spaces. Though local compactness is not used in the proof of [29, Theorem 3.5], there is a difference between the cases of nonnegative and general functions. If the functions can take both positive and negative values, the converse to Corollary 2.8 does not hold. This can be seen from Example 3.2.

COROLLARY 2.8 (the Lebesgue dominated convergence theorem for weakly converging measures; cf. [29, Theorem 3.5]). *Consider a metric space* $\mathbb{S}$. *Let* $\{\mu_n\}_{n=1,2,\ldots}$ *be a sequence of measures on* $\mathbb{S}$ *that converges weakly to* $\mu \in \mathscr{M}(\mathbb{S})$, *and let* $\{f_n\}_{n=1,2,\ldots}$ *be a sequence of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$ *such that* $\lim_{n\to\infty,\, s'\to s} f_n(s')$ *exists for* μ*-a.e.* $s \in \mathbb{S}$. *If* $\{f_n\}_{n=1,2,\ldots}$ *is a.u.i. w.r.t.* $\{\mu_n\}_{n=1,2,\ldots}$, *then*

$$\lim_{n\to\infty} \int_{\mathbb{S}} f_n(s)\, \mu_n(ds) = \int_{\mathbb{S}} \lim_{n\to\infty,\, s'\to s} f_n(s')\, \mu(ds). \tag{2.8}$$

Proof. The corollary follows directly from Theorem 2.4, as applied to the sequences $\{f_n\}_{n=1,2,\ldots}$ and $\{-f_n\}_{n=1,2,\ldots}$.

The following assumption provides a sufficient condition for a sequence of measurable functions $\{f_n\}_{n=1,2,\ldots}$ to be u.i. w.r.t. a sequence of finite measures $\{\mu_n\}_{n=1,2,\ldots}$.

ASSUMPTION 2.9. *Let* $\mathbb{S}$ *be a metric space, let* $\{\mu_n\}_{n=1,2,\ldots}$ *be a sequence of measures on* $\mathbb{S}$ *that converges weakly to* $\mu \in \mathscr{M}(\mathbb{S})$, *and let* $\{f_n, g_n\}_{n=1,2,\ldots}$ *be a sequence of pairs of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$ *such that* $|f_n(s)| \leqslant g_n(s)$ *for each* $n = 1, 2, \ldots$ *and* $s \in \mathbb{S}$, *and such that*

$$\limsup_{n\to\infty} \int_{\mathbb{S}} g_n(s)\, \mu_n(ds) \leqslant \int_{\mathbb{S}} \liminf_{n\to\infty, s'\to s} g_n(s')\, \mu(ds) < +\infty. \tag{2.9}$$

COROLLARY 2.10 (the Lebesgue dominated convergence theorem for weakly converging measures; cf. [29, Theorem 3.3]). *If Assumption* 2.9 *holds and* $\lim_{n\to\infty,\, s'\to s} f_n(s')$ *exists for* μ*-a.e.* $s \in \mathbb{S}$, *then equality* (2.8) *holds.*

Proof. According to Theorem 2.6, as applied to

$$\mathtt{f}_n(s) := -|f_n(s)| \quad \text{and} \quad \mathtt{g}_n(s) := -g_n(s), \quad n = 1, 2, \ldots, \quad s \in \mathbb{S},$$

Assumption 2.9 implies that $\{f_n\}_{n=1,2,\ldots}$ is u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$. In view of Theorem 2.2, the rest of the proof follows from Corollary 2.8.

3. Counterexamples. The following two examples illustrate that the uniform integrability of $\{f^-_{n+N}\}_{n=1,2,\dots}$ for some $N=0,1,\dots$ neither implies nor is implied by Assumption 2.5. In Example 3.1, $\{f^-_n\}_{n=1,2,\dots}$ is u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$, but Assumption 2.5 does not hold.

Example 3.1. Consider $\mathbb{S}:=[0,1]$ endowed with the standard Euclidean metric, and consider the probability measures

$$\begin{aligned} \mu_n(C) &:= \int_C n\,\mathbf{I}\{s\in[0,n^{-1}]\}\,\nu(ds), \\ \mu(C) &:= \mathbf{I}\{0\in C\}, \quad C\in\mathscr{B}(\mathbb{S}), \quad n=1,2,\dots, \end{aligned} \tag{3.1}$$

where ν is the Lebesgue measure on $[0,1]$. Then μ_n converges weakly to μ as $n\to\infty$. Let $f_n\colon \mathbb{S}\mapsto\mathbf{R}$, $n=1,2,\dots$, be

$$f_n(s)=\begin{cases} -i & \text{if } s\in[n^{-1}(1-2^{-(i-1)}),\, n^{-1}(1-2^{-i})),\ i=1,2,\dots, \\ 0 & \text{otherwise.}\end{cases} \tag{3.2}$$

Then

$$\int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_n(s)\leqslant -K\}\,\mu_n(ds)=\sum_{i=\lceil K\rceil}^{\infty}\frac{-i}{2^i}=-\frac{\lceil K\rceil+1}{2^{\lceil K\rceil-1}}$$

for each $K>0$ and for all $n=1,2,\dots$. Since $(\lceil K\rceil+1)/2^{\lceil K\rceil-1}\to 0$ as $K\to+\infty$, the sequence $\{f^-_n\}_{n=1,2,\dots}$ is u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$. Now, we show that Assumption 2.5 does not hold. Consider an arbitrary sequence of measurable functions $\{g_n\}_{n=1,2,\dots}$ such that $g_n(s)\leqslant f_n(s)$ for all $n=1,2,\dots$ and for all $s\in\mathbb{S}$. Let us prove that (2.7) does not hold. Assume, on the contrary, that (2.7) holds. Let $G:=\limsup_{n\to\infty,\,s'\to 0} g_n(s')$. Since $g_n(s)\leqslant f_n(s)\leqslant 0$, we have $G\leqslant 0$. In view of (3.1), inequalities (2.7) become

$$-\infty<G\leqslant\liminf_{n\to\infty}\int_{\mathbb{S}} g_n(s)\,\mu_n(ds). \tag{3.3}$$

Note that if (2.7) is true for $\{g_n\}_{n=1,2,\dots}$, then it is true for $\{\widetilde{g}_n\}$ such that $\widetilde{g}_n(s):=g_n(s)-C$, where $C\geqslant 0$. Therefore, we can select $\{g_n\}_{n=1,2,\dots}$ such that $G\in\{-2,-3,\dots\}$. Then we show that $\liminf_{n\to\infty}\int_{\mathbb{S}} g_n(s)\,\mu_n(ds)<G$. Observe that the definition of G implies

$$\lim_{m\to\infty}\sup_{n\geqslant m,\,s\in[0,1/m]} g_n(s)\leqslant G;$$

in fact, the equality takes place, but we do not need it. Then, for every $\varepsilon>0$, there exists $N(\varepsilon)>0$ such that $g_n(s)\leqslant G+\varepsilon$ for all $n\geqslant N(\varepsilon)$ and for all $s\in[0,1/N(\varepsilon)]$. Therefore, $g_n(s)\leqslant\min\{G+\varepsilon, f_n(s)\}$ for all $s\in[0,1/N(\varepsilon)]$ and for all $n\geqslant N(\varepsilon)$, which implies

$$\int_{\mathbb{S}} g_n(s)\,\mu_n(ds)\leqslant\int_{\mathbb{S}}\min\{G+\varepsilon, f_n(s)\}\,\mu_n(ds), \qquad n\geqslant N(\varepsilon). \tag{3.4}$$

Let $\varepsilon \in (0,1)$. For $n \geqslant N(\varepsilon)$,

$$\begin{aligned}
&\int_{\mathbb{S}} \min\{G+\varepsilon, f_n(s)\}\,\mu_n(ds) \\
&\quad = \int_0^{(1-2^{G+1})/n} (G+\varepsilon)\,\mu_n(ds) + \int_{(1-2^{G+1})/n}^{1/n} f_n(s)\,\mu_n(ds) \\
&\quad = (G+\varepsilon)(1-2^{G+1}) + \sum_{i=-G}^{\infty} \frac{-i}{2^i} = (G+\varepsilon)(1-2^{G+1}) - \frac{-G+1}{2^{-G-1}} \\
&\quad = G+\varepsilon-(1+\varepsilon)\cdot 2^{G+1}, \qquad (3.5)
\end{aligned}$$

where, as follows from (3.2), the first equality holds because

$$\begin{aligned}
f_n(s) &\geqslant G+1 > G+\varepsilon \quad \text{for} \quad s \in [0, n^{-1}(1-2^{G+1})), \\
f_n(s) &\leqslant G < G+\varepsilon \quad \text{for} \quad s \in [n^{-1}(1-2^{G+1}), n^{-1}).
\end{aligned}$$

As follows from (3.4) and (3.5), for every $\varepsilon \in (0,1)$,

$$\liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\,\mu_n(ds) < G+\varepsilon-2^{G+1},$$

and therefore $\liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\,\mu_n(ds) \leqslant G-2^{G+1} < G$, which contradicts the second inequality in (3.3). Hence, Assumption 2.5 does not hold.

In addition, the example in Kamihigashi [24, Example 5.1] of a sequence of functions, which is not u.i., demonstrates that Assumption 2.5 does not imply that $\{f_{n+N}^-\}_{n=1,2,\ldots}$ is u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$ for some $N=0,1,\ldots$. The following example is a slight modification of [24, Example 5.1].

Example 3.2 (cf. [24, Example 5.1]). Consider $\mathbb{S} := [-1,1]$ endowed with the standard Euclidean metric. Let $\mu_n = \mu$, $n=1,2,\ldots$, be the Lebesgue measure on $\mathbb{S}$, and for $n=1,2,\ldots$ and $s \in \mathbb{S}$ let

$$f_n(s) = \begin{cases} -n & \text{if } s \in [l-n^{-1}, 0), \\ n & \text{if } s \in (0, n^{-1}], \\ 0 & \text{otherwise.} \end{cases}$$

Then $\liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\mathbf{I}\{s \in \mathbb{S}\colon f_n(s) \leqslant -K\}\,\mu(ds) = -1$ for each $K>0$, which implies that $\{f_n^-\}_{n=1,2,\ldots}$ is not a.u.i. Hence $\{f_{n+N}^-\}_{n=1,2,\ldots}$ is not u.i. for each $N=0,1,\ldots$; see Theorem 2.2. For each $n=1,2,\ldots$, since $\int_{\mathbb{S}} \limsup_{n\to\infty, s'\to s} f_n(s')\,\mu(ds) = \int_{\mathbb{S}} f_n(s)\,\mu(ds) = 0$, it follows that (2.7) holds for $\mu_n = \mu$ with $g_n = f_n$.

The following example demonstrates that Corollary 2.7 fails if inequalities (2.7) in Assumption 2.5 are replaced by

$$-\infty < \int_{\mathbb{S}} \liminf_{n\to\infty, s'\to s} g_n(s')\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\,\mu_n(ds). \qquad (3.6)$$

Example 3.3 (inequalities (3.6) hold, but inequality (2.6) and the second inequality in (2.7) do not hold). Let $\mathbb{S} := [0,+\infty)$, $\mu_n(S) = \mu(S) := \int_S 2^{-s} ds$, $S \in \mathscr{B}(\mathbb{S})$,

$$f_n(s) := -2^n \mathbf{I}\{s \in [n, n+1)\},$$

and

$$g_n(s) := f_n(s) - \frac{2^{s-1}}{\ln 2} \sum_{k=0}^{2^n-1} \mathbf{I}\left\{ s \in \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right) \right\}$$

for all $s \in [0, +\infty)$ and $n = 1, 2, \ldots$. Note that $f_n(s) \geqslant g_n(s)$ for all $s \in [0, +\infty)$ and $n = 1, 2, \ldots$. Also,

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') = \limsup_{n\to\infty,\, s'\to s} g_n(s') = 0, \qquad s \in [0, +\infty). \tag{3.7}$$

Indeed, $\lim_{n\to\infty, s'\to s} f_n(s') \equiv 0$ because $f_n(s') = 0$ for $s' \in [0, s+1)$, when $n \geqslant \lfloor s \rfloor + 2$ and $s \in [0, +\infty)$, where $\lfloor a \rfloor$ is the integer part of a real number $a \in \mathbf{R}$. Since $g_n(s) \leqslant 0$ for all $s \in [0, +\infty)$ and $n = 1, 2, \ldots$, the second equality in (3.7) follows from

$$0 \geqslant \limsup_{n\to\infty, s'\to s} g_n(s') \geqslant \lim_{n\to\infty} g_n\left(\frac{2\lfloor 2^{n-1}s \rfloor + 3/2}{2^n}\right) = 0, \qquad s \in [0, +\infty), \tag{3.8}$$

where the second inequality in (3.8) holds because

$$s - \frac{1}{2^{n+1}} < \frac{2\lfloor 2^{n-1}s \rfloor + 3/2}{2^n} \leqslant s + \frac{3}{2^{n+1}}$$

for each $n = 1, 2, \ldots$ and because

$$\lim_{n\to\infty}\left(s - \frac{1}{2^{n+1}}\right) = \lim_{n\to\infty}\left(s + \frac{3}{2^{n+1}}\right) = s,$$

and the equality in (3.8) holds because

$$\frac{2\lfloor 2^{n-1}s \rfloor + 3/2}{2^n} \notin \bigcup_{k=0}^{2^n-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)$$

for each $n = 1, 2, \ldots$. Therefore, equalities (3.7) hold.

We observe that

$$\liminf_{n\to\infty,\, s'\to s} g_n(s') = -\frac{2^{s-1}}{\ln 2}\, \mathbf{I}\{s \in [0, 2]\}, \qquad s \in [0, +\infty). \tag{3.9}$$

Indeed, since $g_n(s) \geqslant f_n(s) - (2^{s-1}/\ln 2)\, \mathbf{I}\{s \in [0, 2]\}$ for each $s \in [0, +\infty)$ and $n = 1, 2, \ldots$, and the function $s \mapsto \liminf_{n\to\infty,\, s'\to s} g_n(s')$ is lower semicontinuous, equality (3.9) follows from

$$\liminf_{n\to\infty,\, s'\to s} g_n(s') = \lim_{n\to\infty} g_n\left(\frac{2\lfloor 2^{n-1}s \rfloor + 1/2}{2^n}\right) = -\frac{2^{s-1}}{\ln 2}, \qquad s \in [0, 2), \tag{3.10}$$

where the first equality in (3.10) holds because

$$s - \frac{3}{2^{n+1}} < \frac{2\lfloor 2^{n-1}s \rfloor + 1/2}{2^n} \leqslant s + \frac{1}{2^{n+1}}$$

for each $n = 1, 2, \ldots$ and because

$$\lim_{n\to\infty}\left(s - \frac{3}{2^{n+1}}\right) = \lim_{n\to\infty}\left(s + \frac{1}{2^{n+1}}\right) = s,$$

and the second equality in (3.10) holds because

$$\frac{2\lfloor 2^{n-1}s\rfloor + 1/2}{2^n} \in \bigcup_{k=0}^{2^n-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)$$

for $n \geqslant \max\{1, \lfloor -\log_2(2-s)\rfloor\}$. Therefore, equality (3.9) holds.

Equality (3.9) implies

$$\int_0^\infty \liminf_{n\to\infty,\, s'\to s} g_n(s')\,\mu(ds) = -\int_0^2 \frac{1}{2\ln 2}\,ds = -\frac{1}{\ln 2}. \tag{3.11}$$

For each $n = 1, 2, \ldots,$

$$\int_0^\infty f_n(s)\,\mu(ds) = -2^n \int_n^{n+1} 2^{-s}\,ds = \frac{2^n}{\ln 2}(2^{-n-1} - 2^{-n}) = -\frac{1}{2\ln 2} \tag{3.12}$$

and

$$\int_0^\infty g_n(s)\,\mu(ds) = \int_0^\infty f_n(s)\,\mu(ds) - \frac{1}{2\ln 2}\sum_{k=0}^{2^n-1}\frac{1}{2^n} = -\frac{1}{\ln 2}. \tag{3.13}$$

Inequalities (3.6) hold because, according to (3.11) and (3.13),

$$-\infty < -\frac{1}{\ln 2} = \int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} g_n(s')\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\,\mu_n(ds) = -\frac{1}{\ln 2}.$$

However, inequality (2.6) does not hold because, according to (3.12) and (3.7),

$$-\frac{1}{2\ln 2} = \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds) < \int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f_n(s')\,\mu(ds) = 0.$$

The second inequality in (2.7) does not hold either, because, according to (3.13) and (3.7),

$$-\frac{1}{\ln 2} = \liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\,\mu_n(ds) < \int_{\mathbb{S}} \limsup_{n\to\infty,\, s'\to s} g_n(s')\,\mu(ds) = 0.$$

Therefore, inequalities (3.6) hold, but inequality (2.6) and the second inequality in (2.7) do not hold.

4. Examples of applications of Theorem 2.2. The usefulness of these applications lies in the fact that it is typically easier to verify the asymptotic u.i. w.r.t. a sequence of measures than u.i.

4.1. The uniform Fatou lemma and the uniform Lebesgue dominated convergence theorem for measures converging in total variation. The following results are Theorem 2.1 and Corollary 2.9 of [16], with conditions (ii) replacing the conditions that $\{f_n^-\}_{n=1,2,\ldots}$ and $\{f_n\}_{n=1,2,\ldots}$ are u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$, respectively. As explained in [16], inequality (4.1) is stronger than the inequality in the Fatou lemma, and the sufficient condition in Proposition 4.1 can be viewed as the uniform version of the Fatou lemma. Since the convergence in (4.2) is a uniform version of convergence of integrals, the sufficient condition in Proposition 4.2 can be viewed as the uniform version of the Lebesgue dominated convergence theorem.

PROPOSITION 4.1 (the uniform Fatou lemma; cf. [16, Theorem 2.1]). *Let* $(\mathbb{S},\Sigma)$ *be a measurable space, let a sequence of measures* $\{\mu_n\}_{n=1,2,\ldots}$ *from* $\mathscr{M}(\mathbb{S})$ *converge in total variation to a measure* $\mu\in\mathscr{M}(\mathbb{S})$, *and let* $\{f,f_n\}_{n=1,2,\ldots}$ *be a sequence of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$. *Assume that* $f\in L^1(\mathbb{S};\mu)$ *and* $f_n\in L^1(\mathbb{S};\mu_n)$ *for each* $n=1,2,\ldots$. *Then the inequality*

$$\liminf_{n\to\infty}\inf_{C\in\Sigma}\left(\int_C f_n(s)\,\mu_n(ds)-\int_C f(s)\,\mu(ds)\right)\geqslant 0 \tag{4.1}$$

takes place if and only if the following two conditions hold:

(i) *For each* $\varepsilon>0$,

$$\mu\big(\{s\in\mathbb{S}\colon f_n(s)\leqslant f(s)-\varepsilon\}\big)\to 0 \quad \text{as} \quad n\to\infty;$$

(ii) $\{f_n^-\}_{n=1,2,\ldots}$ *is a.u.i. w.r.t.* $\{\mu_n\}_{n=1,2,\ldots}$.

Proof. The proposition follows from Theorem 2.1 of [16] and Theorem 2.2.

We notice that, since $\varnothing\in\Sigma$, the left-hand side of (4.1) is nonpositive. Therefore, inequality (4.1) takes place if and only if it holds in the form of the equality. Since the left-hand side of (4.1) is the lower limit of a sequence of nonpositive numbers, the lower limit in (4.1) is the limit.

PROPOSITION 4.2 (the uniform Lebesgue dominated convergence theorem; cf. [16, Corollary 2.9]). *Let* $(\mathbb{S},\Sigma)$ *be a measurable space, let* $\{\mu_n\}_{n=1,2,\ldots}$ *be a sequence of measures from* $\mathscr{M}(\mathbb{S})$ *converging in total variation to a measure* $\mu\in\mathscr{M}(\mathbb{S})$, *and let* $\{f,f_n\}_{n=1,2,\ldots}$ *be a sequence of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$. *Assume that* $f\in L^1(\mathbb{S};\mu)$ *and* $f_n\in L^1(\mathbb{S};\mu_n)$ *for each* $n=1,2,\ldots$. *Then*

$$\lim_{n\to\infty}\sup_{C\in\Sigma}\left|\int_C f_n(s)\,\mu_n(ds)-\int_C f(s)\,\mu(ds)\right|=0 \tag{4.2}$$

if and only if the following two conditions hold:

(i) $\{f_n\}_{n=1,2,\ldots}$ *converges to* f *w.r.t. the measure* μ;

(ii) $\{f_n\}_{n=1,2,\ldots}$ *is a.u.i. w.r.t.* $\{\mu_n\}_{n=1,2,\ldots}$.

Proof. The proposition follows from Corollary 2.9 of [16] and Theorem 2.2.

4.2. On the Dunford–Pettis theorem. As follows from the Eberlein–Šmulian theorem, the Dunford–Pettis theorem implies that a sequence $\{f_n\}_{n=1,2,\ldots}\subset L^1(\mathbb{S};\mu)$ has a weakly convergent subsequence $\{f_{n_k}\}_{k=1,2,\ldots}$ to $f\in L^1(\mathbb{S};\mu)$ in $L^1(\mathbb{S};\mu)$ if and only if $\{f_n\}_{n=1,2,\ldots}$ is u.i.; see, for example, [1, Theorem 5.2.9, p. 109], [3, Theorem 4.7.18], [8, p. 93], [10, Theorem 4.21.2, p. 274], [27, Theorem 23, p. 20], [33, Theorem 46.1, p. 471], and [36, Theorem 12, p. 137].

The main result of this subsection has the following formulation.

PROPOSITION 4.3. *Let* $(\mathbb{S},\Sigma)$ *be a measurable space, let* $\mu\in\mathscr{M}(\mathbb{S})$, *and let* $\{f_n\}_{n=1,2,\ldots}\subset L^1(\mathbb{S};\mu)$ *be a sequence of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$. *Then the following conditions are equivalent*:

(i) *There exists* $\{f_{n_k}\}_{k=1,2,\ldots}\subset\{f_n\}_{n=1,2,\ldots}$ *such that* $f_{n_k}\to f$ *weakly in* $L^1(\mathbb{S};\mu)$ *for some* $f\in L^1(\mathbb{S};\mu)$;

(ii) *there exists* $N=0,1,\ldots$ *such that* $\{f_{n+N}\}_{n=1,2,\ldots}$ *is u.i.*;

(iii) $\{f_n\}_{n=1,2,\ldots}$ *is a.u.i.*

Proof. In view of the Eberlein–Šmulian theorem, conditions (i) and (ii) are equivalent due to the Dunford–Pettis theorem. The equivalence of conditions (ii) and (iii) directly follows from Theorem 2.2.

4.3. The fundamental theorem for Young measures. In this subsection we provide an equivalent formulation of the fundamental theorem for Young measures from [2]. Let $n, m = 1, 2, \dots$, and let $\Omega \subset \mathbf{R}^n$ be Lebesgue measurable and $C \subset \mathbf{R}^m$ be closed. Let meas denote the Lebesgue measure on $\mathbf{R}^n$. Consider the Banach spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ of all integrable and essentially bounded functions on Ω, respectively, endowed with the standard norms.

PROPOSITION 4.4. *Let $z^{(j)}\colon \Omega \mapsto \mathbf{R}^m$, $j = 1, 2, \dots$, be a sequence of Lebesgue measurable functions satisfying $z^{(j)}(\,\cdot\,) \to C$ in measure as $j \to \infty$, i.e., for every neighborhood U of C in $\mathbf{R}^m$*

$$\lim_{j\to\infty} \operatorname{meas}\{x \in \Omega\colon z^{(j)}(x) \notin U\} = 0.$$

Then there exist a subsequence $\{z^{(j_k)}\}_{k=1,2,\dots}$ of $\{z^{(j)}\}_{j=1,2,\dots}$ and a family (ν_x), $x \in \Omega$, of positive measures on $\mathbf{R}^m$, depending measurably on x, such that

(i) $\|\nu_x\|_M := \int_{\mathbf{R}^m} d\nu_x \leqslant 1$ *for a.e.* $x \in \Omega$;

(ii) $\operatorname{supp} \nu_x \subset C$ *for a.e.* $x \in \Omega$; *and*

(iii) $f(z^{(j_k)}) \to \langle \nu_x, f\rangle = \int_{\mathbf{R}^m} f(\lambda)\, d\nu_x(\lambda)$ *weakly star in* $L^\infty(\Omega)$ *for each continuous function* $f\colon \mathbf{R}^m \mapsto \mathbf{R}$ *satisfying* $\lim_{|\lambda|\to\infty} f(\lambda) = 0$.

Suppose further that $\{z^{(j_k)}\}_{k=1,2,\dots}$ satisfies the asymptotic boundedness condition

$$\lim_{K\to+\infty} \limsup_{k\to\infty} \operatorname{meas}\{x \in \Omega \cap B_R\colon |z^{(j_k)}(x)| \geqslant K\} = 0, \tag{4.3}$$

for every $R > 0$, where $B_R = B_R(\bar{0})$ is a ball of radius R and center $\bar{0}$ in the Euclidean n-space $\mathbf{R}^n$. Then $\|\nu_x\|_M = 1$ for a.e. $x \in \Omega$ (i.e., ν_x is a probability measure), and, given any measurable subset A of Ω,

$$f(z^{(j_k)}) \to \langle \nu_x, f\rangle \quad \text{weakly in } L^1(A) \tag{4.4}$$

for any continuous function $f\colon \mathbf{R}^m \mapsto \mathbf{R}$ such that $\{f(z^{(j_k)})\}_{k=1,2,\dots}$ is sequentially weakly relatively compact in $L^1(A)$.

Remark 4.5. Theorem from [2] is Proposition 4.4 with (4.3) replaced with

$$\lim_{K\to+\infty} \sup_{k=1,2,\dots} \operatorname{meas}\{x \in \Omega \cap B_R\colon |z^{(j_k)}(x)| \geqslant K\} = 0. \tag{4.5}$$

Remark 4.6. Condition (4.3) is equivalent to the following one: given any $R > 0$, there exists a continuous function $g_R\colon [0, +\infty) \mapsto \mathbf{R}$ satisfying the condition $\lim_{t\to\infty} g_R(t) = +\infty$, such that

$$\limsup_{k\to\infty} \int_{\Omega\cap B_R} g_R(|z^{(j_k)}(x)|)\, dx < \infty; \tag{4.6}$$

see [2, Remark 1]. Without loss of generality, we can also replace limit superior with limit inferior in (4.6) if condition (4.3) is replaced with the condition in this remark.

Remark 4.7. If A is bounded in Proposition 4.4, then Proposition 4.3 implies that the condition that $\{f(z^{(j_k)})\}_{k=1,2,\dots}$ is sequentially weakly relatively compact in $L^1(A)$ is satisfied if and only if $\{f(z^{(j_k)})\}_{k=1,2,\dots}$ is a.u.i.

Proof of Proposition 4.4. All assertions follow from the theorem of [2] and Theorem 2.2, as applied to $\mathbb{S} := \Omega \cap B_R$ endowed with the Lebesgue σ-algebra Σ on

$\Omega \cap B_R$,

$$f_k(s) := |z^{(j_k)}(s)|\,\mathbf{I}\{s \in \mathbb{S}\colon |z^{(j_k)}(s)| \geqslant 1\},$$
$$\mu_k(S) := \int_S \frac{ds}{|z^{(j_k)}(s)|\,\mathbf{I}\{s \in \mathbb{S}\colon |z^{(j_k)}(s)| \geqslant 1\}},$$

for each $S \in \Sigma$ and sufficiently large $k \geqslant 1$. Indeed, since

$$\int_{\mathbb{S}} |f_k(s)|\,\mathbf{I}\{s \in \mathbb{S}\colon |f_k(s)| \geqslant K\}\mu_k(ds) = \text{meas}\{x \in \Omega \cap B_R\colon |z^{(j_k)}(x)| \geqslant K\},$$

for each $K > 1$ and sufficiently large $k \geqslant 1$, we see that Theorem 2.2 implies that (4.3) and (4.5) are equivalent, and, therefore, the conclusions of Theorem from [2] and Proposition 4.4 are equivalent. Proposition 4.4 is proved.

5. Proofs of Theorems 2.4, 2.6 and Corollary 2.7. This section contains the proofs of Theorems 2.4, 2.6 and Corollary 2.7.

Proof of Theorem 2.4. Let us fix an arbitrary $K > 0$. Then

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds) \geqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s \in \mathbb{S}\colon f_n(s) > -K\}\,\mu_n(ds)\\ + \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s \in \mathbb{S}\colon f_n(s) \leqslant -K\}\,\mu_n(ds).\end{aligned} \tag{5.1}$$

The following inequality holds:

$$\liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s \in \mathbb{S}\colon f_n(s) > -K\}\,\mu_n(ds) \geqslant \int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f_n(s')\,\mu(ds). \tag{5.2}$$

Indeed, if $\mu(\mathbb{S}) = 0$, then

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s \in \mathbb{S}\colon f_n(s) > -K\}\,\mu_n(ds)\\ \geqslant -K \lim_{n\to\infty} \mu_n(\mathbb{S}) = 0 = \int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f_n(s')\,\mu(ds),\end{aligned}$$

where the equalities hold because $\mu_n(\mathbb{S}) \to \mu(\mathbb{S}) = 0$ as $n \to \infty$. Otherwise, if $\mu(\mathbb{S}) > 0$, then Theorem 4.2 of [15], as applied to $\{\widetilde{f}_n\}_{n=1,2,\ldots} := \{f_{n+N}\}_{n=1,2,\ldots}$, $\widetilde{g}_n \equiv -K$, $\widetilde{\mu}_n(C) := \mu_{n+N}(C)/\mu_{n+N}(\mathbb{S})$, and $\widetilde{\mu}(C) := \mu(C)/\mu(\mathbb{S})$, for each $n = 1, 2, \ldots$ and $C \in \mathscr{B}(\mathbb{S})$, where $N = 1, 2, \ldots$ is sufficiently large, implies

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s \in \mathbb{S}\colon f_n(s) > -K\}\,\mu_n(ds)\\ \geqslant \int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f_n(s')\,\mathbf{I}\{s' \in \mathbb{S}\colon f_n(s') > -K\}\,\mu(ds).\end{aligned} \tag{5.3}$$

Here we note that $\{\widetilde{\mu}_n\}_{n=1,2,\ldots} \subset \mathbf{P}(\mathbb{S})$ converges weakly to $\widetilde{\mu} \in \mathbf{P}(\mathbb{S})$. We also note that

$$f_n(s)\,\mathbf{I}\{s \in \mathbb{S}\colon f_n(s) > -K\} \geqslant f_n(s) \tag{5.4}$$

for all $s \in \mathbb{S}$ because $K > 0$. Thus, (5.2) follows from (5.3) and (5.4).

Inequalities (5.1) and (5.2) imply

$$\liminf_{n\to\infty}\int_{\mathbb{S}} f_n(s)\,\mu_n(ds) \geqslant \int_{\mathbb{S}} \liminf_{n\to\infty,\,s'\to s} f_n(s')\,\mu(ds) + \lim_{K\to+\infty}\liminf_{n\to\infty}\int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_n(s)\leqslant -K\}\,\mu_n(ds),$$

which is equivalent to (2.6) because $\{f_n^-\}_{n=1,2,\ldots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$. Theorem 2.4 is proved.

Proof of Theorem 2.6. Let Assumption 2.5 hold. According to Theorem 2.2, it is sufficient to prove that $\{f_n^-\}_{n=1,2,\ldots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$, i.e.,

$$\lim_{K\to+\infty}\liminf_{n\to\infty}\int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_n(s)\leqslant -K\}\,\mu_n(ds)=0. \tag{5.5}$$

Let us prove (5.5). Indeed, since $f_n(s)\geqslant g_n(s)$,

$$\mathbf{I}\{s\in\mathbb{S}\colon f_n(s)\leqslant -K\}\leqslant \mathbf{I}\{s\in\mathbb{S}\colon g_n(s)\leqslant -K\},$$

for all $n=1,2,\ldots$, $K>0$, and $s\in\mathbb{S}$. Therefore,

$$\begin{aligned}\lim_{K\to+\infty}\liminf_{n\to\infty}&\int_{\mathbb{S}} f_n(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_n(s)\leqslant -K\}\,\mu_n(ds)\\ &\geqslant \lim_{K\to+\infty}\liminf_{n\to\infty}\int_{\mathbb{S}} g_n(s)\,\mathbf{I}\{s\in\mathbb{S}\colon g_n(s)\leqslant -K\}\,\mu_n(ds).\end{aligned} \tag{5.6}$$

Inequalities (2.7) imply

$$\begin{aligned}\lim_{K\to+\infty}\liminf_{n\to\infty}&\int_{\mathbb{S}} g_n(s)\,\mathbf{I}\{s\in\mathbb{S}\colon g_n(s)\leqslant -K\}\,\mu_n(ds)\\ &\geqslant \int_{\mathbb{S}} \limsup_{n\to\infty,\,s'\to s} g_n(s')\,\mu(ds)\\ &\qquad+\lim_{K\to+\infty}\liminf_{n\to\infty}\int_{\mathbb{S}}(-g_n(s))\,\mathbf{I}\{s\in\mathbb{S}\colon g_n(s)>-K\}\,\mu_n(ds).\end{aligned} \tag{5.7}$$

Since the functions $\{g_n\}_{n=1,2,\ldots}$ are bounded from above by the same constant, Theorem 2.4, as applied to the sequence of the functions $\{\mathtt{f}_n\}_{n=1,2,\ldots}$, which are uniformly bounded from below, where $\mathtt{f}_n(s):=-g_n(s)\,\mathbf{I}\{s\in\mathbb{S}\colon g_n(s)>-K\}$, $s\in\mathbb{S}$, $n=1,2,\ldots$, implies

$$\begin{aligned}\liminf_{n\to\infty}&\int_{\mathbb{S}} -g_n(s)\,\mathbf{I}\{s\in\mathbb{S}\colon g_n(s)>-K\}\,\mu_n(ds)\\ &\geqslant -\int_{\mathbb{S}} \limsup_{n\to\infty,\,s'\to s} g_n(s')\,\mathbf{I}\{s'\in\mathbb{S}\colon g_n(s')>-K\}\,\mu(ds)\end{aligned} \tag{5.8}$$

for each $K>0$. If for each $s\in\mathbb{S}$

$$\limsup_{n\to\infty,\,s'\to s} g_n(s')\,\mathbf{I}\{s'\in\mathbb{S}\colon g_n(s')>-K\}\downarrow \limsup_{n\to\infty,\,s'\to s} g_n(s') \tag{5.9}$$

as $K\uparrow+\infty$, then (5.6)–(5.8) directly imply (5.5), i.e., $\{f_n^-\}_{n=1,2,\ldots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$.

Let us prove (5.9). Since for each $s' \in \mathbb{S}$ and $n = 1, 2, \dots$

$$g_n(s')\,\mathbf{I}\{s' \in \mathbb{S}\colon g_n(s') > -K\} \downarrow g_n(s')$$

as $K \uparrow +\infty$, we have

$$\sup_{m \geqslant n,\, s' \in B_\delta(s)} g_m(s')\,\mathbf{I}\{s' \in \mathbb{S}\colon g_m(s') > -K\} \downarrow \sup_{m \geqslant n,\, s' \in B_\delta(s)} g_m(s')$$

as $K \uparrow +\infty$, for each $n = 1, 2, \dots$ and $\delta > 0$, where $B_\delta(s)$ is the ball in the metric space $\mathbb{S}$ of radius δ centered at s. Therefore,

$$\inf_{n \geqslant 1,\, \delta > 0} \sup_{m \geqslant n,\, s' \in B_\delta(s)} g_m(s')\,\mathbf{I}\{s' \in \mathbb{S}\colon g_m(s') > -K\}$$
$$\downarrow \inf_{n \geqslant 1,\, \delta > 0} \sup_{m \geqslant n,\, s' \in B_\delta(s)} g_m(s')$$

as $K \uparrow +\infty$, i.e., (5.9) holds for each $s \in \mathbb{S}$. Thus, $\{f_n^-\}_{n=1,2,\dots}$ is u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$.

Proof of Corollary 2.7. Theorem 2.4, as applied to the sequence of nonnegative functions $\{f_n - g_n\}_{n=1,2,\dots}$, implies

$$\begin{aligned}
&\int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f_n(s')\,\mu(ds) - \int_{\mathbb{S}} \limsup_{n\to\infty,\, s'\to s} g_n(s')\,\mu(ds) \\
&\qquad \leqslant \int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} (f_n(s') - g_n(s'))\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} (f_n(s) - g_n(s))\,\mu_n(ds) \\
&\qquad \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds) - \liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\,\mu_n(ds), \qquad (5.10)
\end{aligned}$$

where the first and third inequalities follow from the basic properties of infima and suprema. Inequality (2.6) follows from (5.10) and Assumption 2.5. Corollary 2.7 is proved.

6. Appendix. Proof of Theorem 2.2. This appendix contains the proof of Theorem 2.2. This proof is close to that of a similar result in [25, p. 180] for the case of a single probability measure. The main reason for providing this appendix is that reference [25] may not be available to the majority of the readers. We were not aware of [25] for a long time. Once the first version of our paper, which contained a direct proof of Theorem 2.2, was published on arxiv.org, Professor G. M. Shevchenko informed us about the book by M. V. Kartashov [25].

LEMMA 6.1 (Kartashov [25, p. 134]). *Consider a sequence of real-valued functions* $\varepsilon_n(K)$, *where* $K > 0$, *such that*

(a) $\varepsilon_n(K) \downarrow 0$ *as* $K \to +\infty$ *for each* $n = 1, 2, \dots$; *and*

(b) $\lim_{K\to+\infty} \limsup_{n\to\infty} \varepsilon_n(K) = 0$.

Then (c) $\lim_{K\to+\infty} \sup_{n=1,2,\dots} \varepsilon_n(K) = 0$.

Proof. Assume on the contrary that (c) does not hold. In this case, the limit in (c) is equal to δ for some $\delta > 0$. Observe that $\sup_{n=1,2,\dots} \varepsilon_n(K) \geqslant \delta$ for all $K > 0$, since this function does not increase w.r.t. K. For each $m = 1, 2, \dots$ there is a natural number n_m such that $\varepsilon_{n_m}(m) \geqslant \delta/2$. If the sequence $\{n_m\}_{m=1,2,\dots}$ is bounded, then it is possible to choose $n_m = k$ for some $k \in \{1, 2, \dots\}$ and for an infinite subset of integer numbers m. Therefore, $\varepsilon_k(m) \geqslant \delta/2$ for these numbers, which contradicts

assumption (a). For $m > K$, from the monotonicity in (a), we have $\varepsilon_n(K) \geqslant \varepsilon_n(m)$, which implies

$$\limsup_{n\to\infty} \varepsilon_n(K) \geqslant \limsup_{m\to\infty} \varepsilon_{n_m}(K) \geqslant \limsup_{m\to\infty} \varepsilon_{n_m}(m) \geqslant \frac{\delta}{2} > 0, \tag{6.1}$$

where $K > 0$ is an arbitrary real number. Now (6.1) contradicts condition (b). Lemma 6.1 is proved.

Proof of Theorem 2.2. The uniform integrability w.r.t. $\{\mu_{n+N}\}_{n=1,2,\dots}$ of the sequence $\{f_{n+N}\}_{n=1,2,\dots}$ for some $N = 0, 1, \dots$ implies the asymptotic uniform integrability w.r.t. $\{\mu_n\}_{n=1,2,\dots}$ of the sequence $\{f_n\}_{n=1,2,\dots}$.

Vice versa, let $\{f_n\}_{n=1,2,\dots}$ be a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$. Then there exists $N = 0, 1, \dots$ such that $f_n \in L^1(\mathbb{S}; \mu_n)$ for each $n = N+1, N+2, \dots$. Indeed, if there were a subsequence $\{f_{n_k}\}_{k=1,2,\dots}$ of $\{f_n\}_{n=1,2,\dots}$ such that $\int_{\mathbb{S}} |f_{n_k}(s)|\, \mu_{n_k}(ds) = \infty$, then we would have

$$\int_{\mathbb{S}} |f_{n_k}(s)|\, \mathbf{I}\{s \in \mathbb{S}\colon |f_{n_k}| \geqslant K\}\, \mu_{n_k}(ds) = \infty$$

for each $K > 0$ and $k = 1, 2, \dots$. Therefore,

$$\lim_{K\to+\infty} \limsup_{n\to\infty} \int_{\mathbb{S}} |f_n(s)|\, \mathbf{I}\{s \in \mathbb{S}\colon |f_n(s)| \geqslant K\}\, \mu_n(ds) = \infty,$$

contradicting the assumption that $\{f_n\}_{n=1,2,\dots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$.

Consider $\varepsilon_n(K) := \int_{\mathbb{S}} |f_{n+N}(s)|\, \mathbf{I}\{s \in \mathbb{S}\colon |f_{n+N}(s)| \geqslant K\}\, \mu_{n+N}(ds)$, $n = 1, 2, \dots$. Since $f_{n+N} \in L^1(\mathbb{S}; \mu_{n+N})$ for each $n = 1, 2, \dots$, condition (a) in Lemma 6.1 holds. Condition (b) holds because $\{f_n\}_{n=1,2,\dots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\dots}$. Then Lemma 6.1 implies the validity of (2.1) for $\{f_{n+N}\}_{n=1,2,\dots}$ w.r.t. $\{\mu_{n+N}\}_{n=1,2,\dots}$. Theorem 2.2 is proved.

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