Theory Probab. Appl.
Vol. 65, No. 2, pp. 270–291

FATOU'S LEMMA IN ITS CLASSICAL FORM AND LEBESGUE'S CONVERGENCE THEOREMS FOR VARYING MEASURES WITH APPLICATIONS TO MARKOV DECISION PROCESSES*

E. A. FEINBERG†, P. O. KASYANOV‡, AND Y. LIANG§

Abstract. The classical Fatou lemma states that the lower limit of a sequence of integrals of functions is greater than or equal to the integral of the lower limit. It is known that Fatou's lemma for a sequence of weakly converging measures states a weaker inequality because the integral of the lower limit is replaced with the integral of the lower limit in two parameters, where the second parameter is the argument of the functions. In the present paper, we provide sufficient conditions when Fatou's lemma holds in its classical form for a sequence of weakly converging measures. The functions can take both positive and negative values. Similar results for sequences of setwise converging measures are also proved. We also put forward analogies of Lebesgue's and the monotone convergence theorems for sequences of weakly and setwise converging measures. The results obtained are used to prove broad sufficient conditions for the validity of optimality equations for average-cost Markov decision processes.

Key words. Fatou's lemma, measure, weak convergence, setwise convergence, Markov decision process

DOI. 10.1137/S0040585X97T989945

1. Introduction.

For a sequence of nonnegative measurable functions $\{f_n\}$, Fatou's lemma states the inequality

$$\int_{\mathbb{S}} \liminf_{n\to\infty} f_n(s)\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu(ds). \tag{1.1}$$

Many problems in probability theory and its applications deal with sequences of probabilities or measures converging in some sense rather than with a single probability or measure μ. Examples of areas of applications include limit theorems [2], [15], [21, Chap. III], continuity properties of stochastic processes [16], and stochastic control [5], [8], [10], [14].

If a sequence of measures $\{\mu_n\}$ converging setwise to a measure μ is considered instead of a single measure μ, then (1.1) holds with the measure μ replaced in its right-hand side with the measures μ_n (see [18, p. 231]). However, for a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ converging weakly to a measure μ, the weaker inequality

$$\int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f_n(s')\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds) \tag{1.2}$$

holds. Studies of Fatou's lemma for weakly converging probabilities were started by Serfozo [20] and continued in [4], [6]. For a sequence of measures converging in total

*Received by the editors October 7, 2018. This paper was presented at the conference "Innovative Research in Mathematical Finance" (September 3–7, 2018, Marseille, France). The first and third authors were partially supported by NSF grant CMMI-1636193. Originally published in the Russian journal *Teoriya Veroyatnostei i ee Primeneniya*, 65 (2020), pp. 338–367.
https://doi.org/10.1137/S0040585X97T989945

†Department of Applied Mathematics and Statistics, Stony Brook University, Stony Brook, NY 11794-3600 (eugene.feinberg@stonybrook.edu).

‡Institute for Applied System Analysis, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute," Peremogy pr., 37, build 35, 03056, Kyiv, Ukraine (kasyanov@i.ua).

§Rotman School of Management, University of Toronto, 105 St. George Street, Toronto, ON M5S 3E6, Canada (yan.liang@rotman.utoronto.ca).

variation, Feinberg, Kasyanov, and Zgurovsky [9] obtained the uniform Fatou lemma, which is a more general fact than Fatou's lemma.

This paper describes sufficient conditions ensuring that Fatou's lemma holds in its classical form for a sequence of weakly converging measures. In other words, we provide sufficient conditions for the validity of inequality (2.6)—this is inequality (1.2) with its left-hand side replaced by the left-hand side of (1.1). We consider the sequence of functions that can take both positive and negative values. In addition to the results for weakly converging measures, we provide parallel results for setwise converging measures. We also investigate the validity of Lebesgue's and the monotone convergence theorems for sequences of weakly and setwise converging measures. The results are applied to Markov decision processes (MDPs) with long-term average costs per unit time, and we provide general conditions for the validity of optimality equations for such processes.

Section 2 describes the three types of convergence of measures: weak convergence, setwise convergence, and convergence in total variation, and it provides the known formulations of Fatou's lemmas for these types of convergence modes. Section 3 introduces conditions for the double lower limit of a sequence of functions on the left of (1.2) to be equal to the standard lower limit. Section 4 gives sufficient conditions for the validity of Fatou's lemma in its classical form for a sequence of weakly converging measures. This section also provides results for sequences of measures converging setwise. Sections 5 and 6 describe Lebesgue's and the monotone convergence theorems for weakly and setwise converging measures. Section 7 deals with applications.

2. Known formulations of Fatou's lemmas for varying measures. Let $(\mathbb{S},\Sigma)$ be a measurable space, $\mathcal{M}(\mathbb{S})$ the *family of all finite measures on* $(\mathbb{S},\Sigma)$, and $\mathbf{P}(\mathbb{S})$ the *family of all probability measures* on $(\mathbb{S},\Sigma)$. When $\mathbb{S}$ is a metric space, we always consider $\Sigma := \mathcal{B}(\mathbb{S})$, where $\mathcal{B}(\mathbb{S})$ is the *Borel σ-field* on $\mathbb{S}$. Let $\mathbf{R}$ be the real line, $\overline{\mathbf{R}} := [-\infty,+\infty]$, and $\mathbf{N}^* := \{1,2,\dots\}$. We denote by $\mathbf{I}\{A\}$ the *indicator* of the event A.

Throughout this paper, we deal with integrals of functions that can take both positive and negative values. The *integral* $\int_{\mathbb{S}} f(s)\,\mu(ds)$ of a measurable $\overline{\mathbf{R}}$-valued function f on $\mathbb{S}$ with respect to a measure μ is *defined* if

$$\min\left\{\int_{\mathbb{S}} f^+(s)\,\mu(ds), \int_{\mathbb{S}} f^-(s)\,\mu(ds)\right\} < +\infty, \tag{2.1}$$

where $f^+(s) = \max\{f(s),0\}$, $f^-(s) = -\min\{f(s),0\}$, $s\in\mathbb{S}$. If (2.1) holds, then the integral is defined as $\int_{\mathbb{S}} f(s)\,\mu(ds) = \int_{\mathbb{S}} f^+(s)\,\mu(ds) - \int_{\mathbb{S}} f^-(s)\,\mu(ds)$. All of the integrals in the assumptions of the following lemmas, theorems, and corollaries are assumed to be defined. For $\mu\in\mathcal{M}(\mathbb{S})$ consider the vector space $L^1(\mathbb{S};\mu)$ of all absolutely integrable measurable functions $f\colon \mathbb{S}\mapsto\overline{\mathbf{R}}$, that is, $\int_{\mathbb{S}} |f(s)|\,\mu(ds) < +\infty$.

We recall the definitions of the following three types of convergence of measures: weak convergence, setwise convergence, and convergence in total variation.

DEFINITION 2.1 (weak convergence). *A sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ on a metric space $\mathbb{S}$ converges weakly to a finite measure μ on $\mathbb{S}$ if, for each bounded continuous function f on $\mathbb{S}$,*

$$\int_{\mathbb{S}} f(s)\,\mu_n(ds) \to \int_{\mathbb{S}} f(s)\,\mu(ds) \quad as \quad n\to\infty. \tag{2.2}$$

Remark 2.1. Definition 2.1 implies that $\mu_n(\mathbb{S}) \to \mu(\mathbb{S}) \in \mathbf{R}$ as $n \to \infty$. Therefore, if $\{\mu_n\}_{n\in\mathbf{N}^*}$ converges weakly to $\mu \in \mathcal{M}(\mathbb{S})$, then there exists $N \in \mathbf{N}^*$ such that $\{\mu_n\}_{n=N,N+1,\dots} \subset \mathcal{M}(\mathbb{S})$.

DEFINITION 2.2 (setwise convergence). *A sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ on a measurable space $(\mathbb{S},\Sigma)$ converges setwise to a measure μ on $(\mathbb{S},\Sigma)$ if, for each $C \in \Sigma$,*

$$\mu_n(C) \to \mu(C) \quad \text{as} \quad n \to \infty.$$

DEFINITION 2.3 (convergence in total variation). *A sequence of finite measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ on a measurable space $(\mathbb{S},\Sigma)$ converges in total variation to a measure μ on $(\mathbb{S},\Sigma)$ if*

$$\sup\left\{\left|\int_{\mathbb{S}} f(s)\,\mu_n(ds) - \int_{\mathbb{S}} f(s)\,\mu(ds)\right| : f\colon \mathbb{S}\mapsto[-1,1] \text{ is measurable}\right\} \to 0$$

as $n \to \infty$.

Remark 2.2. As follows from Definitions 2.1–2.3, if a sequence of finite measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ on a measurable space $(\mathbb{S},\Sigma)$ converges in total variation to a measure μ on $(\mathbb{S},\Sigma)$, then $\{\mu_n\}_{n\in\mathbf{N}^*}$ converges setwise to μ as $n \to \infty$, and the measure μ is finite. This fact follows from the inequality $|\mu_n(\mathbb{S}) - \mu(\mathbb{S})| < +\infty$ when $n \geqslant N$ for some $N \in \mathbf{N}^*$. Furthermore, if a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ on a metric space $\mathbb{S}$ converges setwise to a finite measure μ on $\mathbb{S}$, then this sequence converges weakly to μ as $n \to \infty$.

Recall the following definitions of the uniform and asymptotic uniform integrability of sequences of functions.

DEFINITION 2.4. *A sequence $\{f_n\}_{n\in\mathbf{N}^*}$ of measurable $\overline{\mathbf{R}}$-valued functions is called*
– uniformly integrable (u.i.) w.r.t. a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ if

$$\lim_{K\to+\infty} \sup_{n\in\mathbf{N}^*} \int_{\mathbb{S}} |f_n(s)|\,\mathbf{I}\{s\in\mathbb{S}\colon |f_n(s)| \geqslant K\}\,\mu_n(ds) = 0; \tag{2.3}$$

– asymptotically uniformly integrable (a.u.i.) w.r.t. a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ if

$$\lim_{K\to+\infty} \limsup_{n\to\infty} \int_{\mathbb{S}} |f_n(s)|\,\mathbf{I}\{s\in\mathbb{S}\colon |f_n(s)| \geqslant K\}\,\mu_n(ds) = 0. \tag{2.4}$$

If $\mu_n = \mu \in \mathcal{M}(\mathbb{S})$ for each $n \in \mathbf{N}^*$, then an (a.)u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$ sequence $\{f_n\}_{n\in\mathbf{N}^*}$ is called *(a.)u.i.* For $\mu \in \mathcal{M}(\mathbb{S})$, a sequence $\{f_n\}_{n\in\mathbf{N}^*}$ of functions from $L^1(\mathbb{S};\mu)$ is u.i. if and only if it is a.u.i. (see [17, p. 180]). For a single finite measure μ, the definition of an a.u.i. sequence of functions (random variables in the case of a probability measure μ) coincides with the corresponding definition broadly used in the literature; see, e.g., [22, p. 17]. Also, for a single fixed finite measure, the definition of a u.i. sequence of functions is consistent with the classical definition of a family $\mathcal{H}$ of u.i. functions. We say that a function f is (a.)u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$ if the sequence $\{f, f, \dots\}$ is (a.)u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$. A function f is u.i. w.r.t. a family $\mathcal{N}$ of measures if

$$\lim_{K\to+\infty} \sup_{\mu\in\mathcal{N}} \int_{\mathbb{S}} |f(s)|\,\mathbf{I}\{s\in\mathbb{S}\colon |f(s)| \geqslant K\}\,\mu(ds) = 0.$$

THEOREM 2.1 (equivalence of u.i. and a.u.i. [4, Theorem 2.2]). *Let* $(\mathbb{S}, \Sigma)$ *be a measurable space,* $\{\mu_n\}_{n\in\mathbf{N}^*} \subset \mathcal{M}(\mathbb{S})$, *and let* $\{f_n\}_{n\in\mathbf{N}^*}$ *be a sequence of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$. *Then there exists* $N \in \mathbf{N}^*$ *such that* $\{f_n\}_{n=N,N+1,\ldots}$ *is u.i. w.r.t.* $\{\mu_n\}_{n=N,N+1,\ldots}$ *if and only if* $\{f_n\}_{n\in\mathbf{N}^*}$ *is a.u.i. w.r.t.* $\{\mu_n\}_{n\in\mathbf{N}^*}$.

Fatou's lemma (FL) for weakly converging probabilities was introduced in Serfozo [20] and generalized in [4], [6].

THEOREM 2.2 (FL for weakly converging measures [4, Theorem 2.4 and Corollary 2.7]). *Let* $\mathbb{S}$ *be a metric space, let* $\{\mu_n\}_{n\in\mathbf{N}^*}$ *be a sequence of measures on* $\mathbb{S}$ *converging weakly to* $\mu \in \mathcal{M}(\mathbb{S})$, *and let* $\{f_n\}_{n\in\mathbf{N}^*}$ *be a sequence of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$. *Assume that one of the following two conditions holds*:

(i) $\{f_n^-\}_{n\in\mathbf{N}^*}$ *is a.u.i. w.r.t.* $\{\mu_n\}_{n\in\mathbf{N}^*}$;

(ii) *there exists a sequence of measurable real-valued functions* $\{g_n\}_{n\in\mathbf{N}^*}$ *on* $\mathbb{S}$ *such that* $f_n(s) \geqslant g_n(s)$ *for all* $n \in \mathbf{N}^*$ *and* $s \in \mathbb{S}$, *and*

$$-\infty < \int_{\mathbb{S}} \limsup_{n\to\infty,\, s'\to s} g_n(s')\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\,\mu_n(ds). \tag{2.5}$$

Then inequality (1.2) *holds.*

Recall that FL for setwise converging measures is stated in [18, p. 231] for nonnegative functions. FL for setwise converging probabilities is stated in [6, Theorem 4.1] for functions taking positive and negative values.

THEOREM 2.3 (FL for setwise converging probabilities [6]). *Let* $(\mathbb{S}, \Sigma)$ *be a measurable space, let a sequence of measures* $\{\mu_n\}_{n\in\mathbf{N}^*} \subset \mathbf{P}(\mathbb{S})$ *converge setwise to* $\mu \in \mathbf{P}(\mathbb{S})$, *and let* $\{f_n\}_{n\in\mathbf{N}^*}$ *be a sequence of measurable real-valued functions on* $\mathbb{S}$. *Then the inequality*

$$\int_{\mathbb{S}} \liminf_{n\to\infty} f_n(s)\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds) \tag{2.6}$$

holds if there exists a sequence of measurable real-valued functions $\{g_n\}_{n\in\mathbf{N}^*}$ *on* $\mathbb{S}$ *such that* $f_n(s) \geqslant g_n(s)$ *for all* $n \in \mathbf{N}^*$ *and* $s \in \mathbb{S}$, *and*

$$-\infty < \int_{\mathbb{S}} \limsup_{n\to\infty} g_n(s)\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} g_n(s)\,\mu_n(ds). \tag{2.7}$$

Under the condition that $\{\mu_n\}_{n\in\mathbf{N}^*} \subset \mathcal{M}(\mathbb{S})$ converges in total variation to $\mu \in \mathcal{M}(\mathbb{S})$, Theorem 2.1 in [9] establishes the uniform FL, which is a stronger result than the classical FL.

THEOREM 2.4 (uniform FL for measures converging in total variation [9, Theorem 2.1]). *Let* $(\mathbb{S}, \Sigma)$ *be a measurable space, let a sequence of measures* $\{\mu_n\}_{n\in\mathbf{N}^*}$ *from* $\mathcal{M}(\mathbb{S})$ *converge in total variation to a measure* $\mu \in \mathcal{M}(\mathbb{S})$, *let* $\{f_n\}_{n\in\mathbf{N}^*}$ *be a sequence of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$, *and let* f *be a measurable* $\overline{\mathbf{R}}$*-valued function. Assume that* $f \in L^1(\mathbb{S};\mu)$ *and* $f_n \in L^1(\mathbb{S};\mu_n)$ *for each* $n \in \mathbf{N}^*$. *Then the inequality*

$$\liminf_{n\to\infty} \inf_{C\in\Sigma} \left(\int_C f_n(s)\,\mu_n(ds) - \int_C f(s)\,\mu(ds) \right) \geqslant 0 \tag{2.8}$$

holds if and only if the following two assertions hold:

(i) *For each* $\varepsilon > 0$, $\mu(\{s \in \mathbb{S}\colon f_n(s) \leqslant f(s) - \varepsilon\}) \to 0$, *and therefore there exists a subsequence* $\{f_{n_k}\}_{k\in\mathbf{N}^*} \subset \{f_n\}_{n\in\mathbf{N}^*}$ *such that* $f(s) \leqslant \liminf_{k\to\infty} f_{n_k}(s)$ *for* μ*-a.e.* $s \in \mathbb{S}$;

(ii) $\{f_n^-\}_{n\in\mathbf{N}^*}$ *is a.u.i. w.r.t.* $\{\mu_n\}_{n\in\mathbf{N}^*}$.

3. Semiconvergence conditions for sequences of functions. Let $(\mathbb{S},\Sigma)$ be a measurable space, μ a measure on $(\mathbb{S},\Sigma)$, $\{f_n\}_{n\in\mathbf{N}^*}$ a sequence of measurable $\overline{\mathbf{R}}$-valued functions, and f a measurable $\overline{\mathbf{R}}$-valued function. In this section, we introduce the notions of lower and upper semiconvergence in measure μ (see Definition 3.2) for a sequences of functions $\{f_n\}_{n\in\mathbf{N}^*}$ defined on a measurable space $\mathbb{S}$. Next, under the assumption that $\mathbb{S}$ is a metric space, we examine necessary and sufficient conditions for the following equalities (see Theorem 3.1, Corollary 3.1, and Example 3.1):

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') = \liminf_{n\to\infty} f_n(s), \tag{3.1}$$

$$\lim_{n\to\infty,\, s'\to s} f_n(s') = \lim_{n\to\infty} f_n(s), \tag{3.2}$$

which improve the statements of FL and Lebesgue's convergence theorem for weakly converging measures; see Theorem 4.1 and Corollary 5.1. For example, these equalities are important for approximating average-cost relative value functions for MDPs with weakly continuous transition probabilities by discounted relative value functions; see section 7. For this purpose we introduce the notions of lower and upper semiequicontinuous families of functions; see Definition 3.3. Finally, we provide sufficient conditions for lower semiequicontinuity; see Definition 3.1 and Corollary 3.2.

Remark 3.1. Since

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') \leqslant \liminf_{n\to\infty} f_n(s), \tag{3.3}$$

(3.1) is equivalent to the inequality

$$\liminf_{n\to\infty} f_n(s) \leqslant \liminf_{n\to\infty,\, s'\to s} f_n(s'). \tag{3.4}$$

To formulate sufficient conditions for (3.1) to hold we introduce the definitions of uniform semiconvergences from below and from above.

DEFINITION 3.1 (uniform semiconvergence). *A sequence of real-valued functions $\{f_n\}_{n\in\mathbf{N}^*}$ on $\mathbb{S}$ semiconverges uniformly from below to a real-valued function f on $\mathbb{S}$ if, for each $\varepsilon>0$, there exists $N\in\mathbf{N}^*$ such that*

$$f_n(s) > f(s)-\varepsilon \tag{3.5}$$

for each $s\in\mathbb{S}$ and $n=N,N+1,\dots$. A sequence of real-valued functions $\{f_n\}_{n\in\mathbf{N}^*}$ on $\mathbb{S}$ semiconverges uniformly from above to a real-valued function f on $\mathbb{S}$ if $\{-f_n\}_{n\in\mathbf{N}^*}$ semiconverges uniformly from below to $-f$ on $\mathbb{S}$.

Remark 3.2. A sequence $\{f_n\}_{n\in\mathbf{N}^*}$ *converges uniformly to* f on $\mathbb{S}$ if and only if it uniformly semiconverges from below and from above.

Let us consider the following definitions of semiconvergence in measure.

DEFINITION 3.2 (semiconvergence in measure). *A sequence of measurable $\overline{\mathbf{R}}$-valued functions $\{f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges to a measurable real-valued function f in measure μ if, for each $\varepsilon>0$,*

$$\mu\big(\{s\in\mathbb{S}\colon f_n(s)\leqslant f(s)-\varepsilon\}\big)\to 0 \quad \text{as} \quad n\to\infty.$$

A sequence of measurable $\overline{\mathbf{R}}$-valued functions $\{f_n\}_{n\in\mathbf{N}^*}$ upper semiconverges to a measurable real-valued function f in measure μ if $\{-f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges to $-f$ in measure μ, that is, for each $\varepsilon>0$,

$$\mu\big(\{s\in\mathbb{S}\colon f_n(s)\geqslant f(s)+\varepsilon\}\big)\to 0 \quad \text{as} \quad n\to\infty.$$

Remark 3.3. A sequence of measurable $\overline{\mathbf{R}}$-valued functions $\{f_n\}_{n\in\mathbf{N}^*}$ converges to a measurable real-valued function f *in measure* μ, that is, for each $\varepsilon > 0$,

$$\mu\big(\{s \in \mathbb{S}\colon |f_n(s) - f(s)| \geqslant \varepsilon\}\big) \to 0 \quad \text{as} \quad n \to \infty$$

if and only if this sequence of functions both lower and upper semiconverges to f in measure μ.

Remark 3.4. If $f(s) \leqslant \liminf_{n\to\infty} f_n(s)$, $f(s) \geqslant \limsup_{n\to\infty} f_n(s)$, or $f(s) = \lim_{n\to\infty} f_n(s)$ for μ-a.e. $s \in \mathbb{S}$, then $\{f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges, upper semiconverges, or converges, respectively, to f in measure μ. Conversely, [9, Lemma 3.1] implies that if $\{f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges, upper semiconverges, or converges to f in measure μ, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbf{N}^*} \subset \{f_n\}_{n\in\mathbf{N}^*}$ such that $f(s) \leqslant \liminf_{k\to\infty} f_{n_k}(s)$, $f(s) \geqslant \limsup_{k\to\infty} f_{n_k}(s)$, or $f(s) = \lim_{k\to\infty} f_{n_k}(s)$, respectively, for μ-a.e. $s \in \mathbb{S}$.

Now let $\mathbb{S}$ be a metric space, and let $B_\delta(s)$ be the open ball in $\mathbb{S}$ of radius $\delta > 0$ centered at $s \in \mathbb{S}$. We consider the notions of lower and upper semiequicontinuity for a sequence of functions.

DEFINITION 3.3 (semiequicontinuity). *A sequence $\{f_n\}_{n\in\mathbf{N}^*}$ of real-valued functions on a metric space $\mathbb{S}$ is called lower semiequicontinuous at a point $s \in \mathbb{S}$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$f_n(s') > f_n(s) - \varepsilon \quad \text{for all } s' \in B_\delta(s) \text{ and for all } n \in \mathbf{N}^*.$$

A sequence $\{f_n\}_{n\in\mathbf{N}^*}$ is called lower semiequicontinuous (on $\mathbb{S}$) if it is lower semiequicontinuous at all $s \in \mathbb{S}$. A sequence $\{f_n\}_{n\in\mathbf{N}^*}$ of real-valued functions on a metric space $\mathbb{S}$ is called upper semiequicontinuous at a point $s \in \mathbb{S}$ (on $\mathbb{S}$) if the sequence $\{-f_n\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous at the point $s \in \mathbb{S}$ (on $\mathbb{S}$).

Recall the definition of equicontinuity of a sequence of functions; see, e.g., [18, p. 177].

DEFINITION 3.4 (equicontinuity). *A sequence $\{f_n\}_{n\in\mathbf{N}^*}$ of real-valued functions on a metric space $\mathbb{S}$ is called equicontinuous at the point $s \in \mathbb{S}$ (on $\mathbb{S}$) if this sequence is both lower and upper semiequicontinuous at the point $s \in \mathbb{S}$ (on $\mathbb{S}$).*

Theorem 3.1 states necessary and sufficient conditions for equality (3.1). This theorem and Corollary 3.1 generalize [12, Lemma 3.3], where the equicontinuity was considered.

LEMMA 3.1. *Let $\{f_n\}_{n\in\mathbf{N}^*}$ be a pointwise nondecreasing sequence of lower semicontinuous $\overline{\mathbf{R}}$-valued functions on a metric space $\mathbb{S}$. Then*

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') = \lim_{n\to\infty} f_n(s), \qquad s \in \mathbb{S}. \tag{3.6}$$

Proof. For each $s \in \mathbb{S}$,

$$\begin{aligned}\liminf_{n\to\infty,\, s'\to s} f_n(s') &= \sup_{n\geqslant 1} \liminf_{s'\to s} \inf_{k\geqslant n} f_k(s') = \sup_{n\geqslant 1} \liminf_{s'\to s} f_n(s') \\ &= \sup_{n\geqslant 1} f_n(s) = \lim_{n\to\infty} f_n(s),\end{aligned}$$

where the first equality follows from the definition of lim inf, the third follows from the lower semicontinuity of the function f_n, and the second and last equalities hold because the sequence $\{f_n\}_{n\in\mathbf{N}^*}$ is pointwise nondecreasing. Hence (3.6) holds. Lemma 3.1 is proved.

THEOREM 3.1 (necessary and sufficient conditions for (3.1)). *Let $\{f_n\}_{n\in\mathbf{N}^*}$ be a sequence of real-valued functions on a metric space $\mathbb{S}$, and let $s \in \mathbb{S}$. Then the following assertions hold*:

(i) *If the sequence of functions $\{f_n\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous at s, then each function f_n, $n \in \mathbf{N}^*$, is lower semicontinuous at s and* (3.1) *holds*;

(ii) *if $\{f_n\}_{n\in\mathbf{N}^*}$ is the sequence of lower semicontinuous functions satisfying* (3.1) *and if $\{f_n(s)\}_{n\in\mathbf{N}^*}$ is converging, that is,*

$$\liminf_{n\to\infty} f_n(s) = \limsup_{n\to\infty} f_n(s), \tag{3.7}$$

then the sequence $\{f_n\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous at s.

Example 3.1 demonstrates that assumption (3.7) is essential in Theorem 3.1(ii). Without this assumption, the remaining conditions of Theorem 3.1(ii) imply only the existence of a subsequence $\{f_{n_k}\}_{k\in\mathbf{N}^*} \subset \{f_n\}_{n\in\mathbf{N}^*}$ such that $\{f_{n_k}\}_{k\in\mathbf{N}^*}$ is lower semiequicontinuous at s. This is true because every subsequence $\{f_{n_k}\}_{k\in\mathbf{N}^*}$ satisfying $\lim_{k\to\infty} f_{n_k}(s) = \liminf_{n\to\infty} f_n(s)$ is lower semiequicontinuous at s in view of Theorem 3.1(ii) since (3.7) holds for such subsequences.

Example 3.1. Consider $\mathbb{S} := [-1, 1]$ endowed with the standard Euclidean metric and put

$$f_n(t) := \begin{cases} 0 & \text{if } n = 2k-1, \\ \max\{1 - n|t|, 0\} & \text{if } n = 2k, \end{cases} \qquad k \in \mathbf{N}^*, \quad t \in \mathbb{S}.$$

Each function f_n, $n \in \mathbf{N}^*$, is nonnegative and continuous on $\mathbb{S}$. Equality (3.1) holds because

$$0 \leqslant \liminf_{n\to\infty, s'\to 0} f_n(s') \leqslant \liminf_{n\to\infty} f_n(0) = f_{2k-1}(0) = 0, \qquad k \in \mathbf{N}^*.$$

Equality (3.7) is not satisfied, because

$$\limsup_{n\to\infty} f_n(0) = 1 > 0 = \liminf_{n\to\infty} f_n(0),$$

where the first equality holds because $f_{2k}(0) = 1$ for each $k \in \mathbf{N}^*$, and the second equality holds because $f_{2k-1}(0) = 0$ for each $k \in \mathbf{N}^*$. The sequence of functions $\{f_n\}_{n\in\mathbf{N}^*}$ is not lower semiequicontinuous at $s = 0$ because $f_{2k}(1/(2k)) = 0 < 1/2 = f_{2k}(0) - 1/2$ for each $k \in \mathbf{N}^*$. Therefore, the conclusion of Theorem 3.1(ii) does not hold, which shows that assumption (3.7) is essential.

Proof of Theorem 3.1. (i) We observe that the lower semicontinuity at s of each function f_n, $n \in \mathbf{N}^*$, follows from lower semiequicontinuity of $\{f_n\}_{n\in\mathbf{N}^*}$ at s. Thus, to prove assertion (i) it is sufficient to verify (3.1), which is equivalent to (3.4) because of Remark 3.1.

Let us prove (3.4). Fix an arbitrary $\varepsilon > 0$. According to Definition 3.3, there exists $\delta(\varepsilon) > 0$ such that, for each $n \in \mathbf{N}^*$ and $s' \in B_{\delta(\varepsilon)}(s)$,

$$f_n(s') \geqslant f_n(s) - \varepsilon. \tag{3.8}$$

Since

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') = \sup_{n\geqslant 1,\, \delta>0} \inf_{k\geqslant n,\, s'\in B_\delta(s)} f_k(s') \geqslant \sup_{n\geqslant 1} \inf_{k\geqslant n,\, s'\in B_{\delta(\varepsilon)}(s)} f_k(s'), \tag{3.9}$$

(3.8) implies

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') \geqslant \sup_{n\geqslant 1} \inf_{k\geqslant n} f_k(s) - \varepsilon = \liminf_{n\to\infty} f_n(s) - \varepsilon, \tag{3.10}$$

where the equalities in (3.9) and (3.10) follow from the definition of lim inf, the inequality in (3.9) holds because $\{\delta(\varepsilon)\} \subset \{\delta\colon \delta > 0\}$, and the inequality in (3.10) is secured by (3.8) and (3.9). Now inequality (3.4) follows from (3.10) since $\varepsilon > 0$ is arbitrary. Assertion (i) is proved.

(ii) We prove assertion (ii) by contradiction. Assume that the sequence of functions $\{f_n\}_{n\in\mathbf{N}^*}$ is not lower semiequicontinuous at s. Then there exist $\varepsilon^* > 0$, a sequence $\{s_n\}_{n\in\mathbf{N}^*}$ converging to s, and a sequence $\{n_k\}_{k\in\mathbf{N}^*} \subset \mathbf{N}^*$ such that

$$f_{n_k}(s_k) \leqslant f_{n_k}(s) - \varepsilon^*, \qquad k \in \mathbf{N}^*. \tag{3.11}$$

If a sequence $\{n_k\}_{k\in\mathbf{N}^*}$ is bounded, then (3.11) contradicts the lower semicontinuity of each function f_n, $n \in \mathbf{N}^*$. Otherwise, without loss of generality, we may assume that the sequence $\{n_k\}_{k\in\mathbf{N}^*}$ is strictly increasing. Therefore, from (3.11) and (3.7) we have

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') \leqslant \lim_{n\to\infty} f_n(s) - \varepsilon^*,$$

which is a contradiction to (3.1). Hence the sequence of functions $\{f_n\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous at s. Theorem 3.1 is proved.

Let us investigate necessary and sufficient conditions for equality (3.2).

COROLLARY 3.1. *Let $\{f_n\}_{n\in\mathbf{N}^*}$ be a sequence of real-valued functions on a metric space $\mathbb{S}$, and let $s \in \mathbb{S}$. If $\{f_n(s)\}_{n\in\mathbf{N}^*}$ is a convergent sequence, that is, if* (3.7) *holds, then the sequence of functions $\{f_n\}_{n\in\mathbf{N}^*}$ is equicontinuous at s if and only if each function f_n, $n \in \mathbf{N}^*$, is continuous at s and* (3.2) *holds.*

Proof. Corollary 3.1 follows directly from Theorem 3.1 applied twice to the families $\{f_n\}_{n\in\mathbf{N}^*}$ and $\{-f_n\}_{n\in\mathbf{N}^*}$.

In the following corollary we establish sufficient conditions for lower semiequicontinuity.

COROLLARY 3.2 (sufficient conditions for lower semiequicontinuity). *Let $\mathbb{S}$ be a metric space, and let $\{f_n\}_{n\in\mathbf{N}^*}$ be a sequence of real-valued lower semicontinuous functions on $\mathbb{S}$ semiconverging uniformly from below to a real-valued lower semicontinuous function f on $\mathbb{S}$. If the sequence $\{f_n\}_{n\in\mathbf{N}^*}$ converges pointwise to f on $\mathbb{S}$, then $\{f_n\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous on $\mathbb{S}$.*

Proof. If inequality (3.4) holds for all $s \in \mathbb{S}$, then Remark 3.1 and Theorem 3.1(ii) imply that $\{f_n\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous on $\mathbb{S}$ because the sequence of functions $\{f_n\}_{n\in\mathbf{N}^*}$ converges pointwise to f on $\mathbb{S}$. Therefore, to complete the proof, let us prove that (3.4) holds for each $s \in \mathbb{S}$. Indeed, the uniform semiconvergence from below of $\{f_n\}_{n\in\mathbf{N}^*}$ to f on $\mathbb{S}$ implies that, for an arbitrary $\varepsilon > 0$,

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') \geqslant f(s) - \varepsilon \tag{3.12}$$

for each $s \in \mathbb{S}$. Now (3.1) follows from (3.4), since $\varepsilon > 0$ is arbitrary and $f(s) = \lim_{n\to\infty} f_n(s)$, $s \in \mathbb{S}$. Corollary 3.2 is proved.

Let $\mathbb{S}$ be a compact metric space. The Ascoli theorem (see [14, p. 96] or [18, p. 179]) implies that a sequence of real-valued continuous functions $\{f_n\}_{n\in\mathbf{N}^*}$ on $\mathbb{S}$ converges uniformly on $\mathbb{S}$ to a continuous real-valued function f on $\mathbb{S}$ if and only if $\{f_n\}_{n\in\mathbf{N}^*}$ is equicontinuous and this sequence converges pointwise to f on $\mathbb{S}$. According to Corollary 3.2, a sequence of real-valued lower semicontinuous functions $\{f_n\}_{n\in\mathbf{N}^*}$ on $\mathbb{S}$, converging pointwise to a real-valued lower semicontinuous function f on $\mathbb{S}$, is lower semiequicontinuous on $\mathbb{S}$ if $\{f_n\}_{n\in\mathbf{N}^*}$ semiconverges uniformly from below to f on $\mathbb{S}$. Example 3.2 illustrates that the converse of Corollary 3.2 does not hold in the general case; that is, there is a lower semiequicontinuous sequence $\{f_n\}_{n\in\mathbf{N}^*}$ of continuous functions on $\mathbb{S}$ converging pointwise to a lower semicontinuous function f such that $\{f_n\}_{n\in\mathbf{N}^*}$ does not semiconverge uniformly from below to f on $\mathbb{S}$.

Example 3.2. Let $\mathbb{S} := [0,1]$ be endowed with the standard Euclidean metric, let $f(s) := \mathbf{I}\{s \neq 0\}$, and let, for $s \in \mathbb{S}$,

$$f_n(s) := \begin{cases} ns & \text{if } s \in \left[0, \dfrac{1}{n}\right], \\ 1 & \text{otherwise.} \end{cases}$$

Then the functions f_n, $n \in \mathbf{N}^*$, are *continuous* on $\mathbb{S}$, the function f is *lower semicontinuous* on $\mathbb{S}$, and the sequence $\{f_n\}_{n\in\mathbf{N}^*}$ *converges pointwise* to f on $\mathbb{S}$. In addition, the sequence of functions $\{f_n\}_{n\in\mathbf{N}^*}$ is *lower semiequicontinuous*, because, for each $\varepsilon > 0$ and $s \in \mathbb{S}$, (i) if $s > 0$, then there exists $\delta(s,\varepsilon) = \min\{s - 1/(\lfloor 1/s\rfloor + 1), \varepsilon/\lfloor 1/s\rfloor\}$ such that $f_n(s') \geqslant f_n(s) - \varepsilon$ for all $n \in \mathbf{N}^*$ and $s' \in B_{\delta(s,\varepsilon)}(s)$; and (ii) if $s = 0$, then $f_n(s') \geqslant 0 = f_n(0)$ for all $n \in \mathbf{N}^*$ and $s' \in \mathbb{S}$. The uniform semiconvergence from below of $\{f_n\}_{n\in\mathbf{N}^*}$ to f *does not hold* because

$$f_n\left(\frac{1}{n(n+1)}\right) = \frac{1}{n+1} \leqslant 1 - \frac{1}{2} = f\left(\frac{1}{n(n+1)}\right) - \frac{1}{2}$$

for each $n \in \mathbf{N}^*$; that is, the converse to Corollary 3.2 does not hold.

4. Fatou's lemmas in the classical form for varying measures. In this section, we establish Fatou's lemmas in their classical form for varying measures. This section consists of two subsections dealing with weakly and setwise converging measures, respectively.

4.1. Fatou's lemmas in the classical form for weakly converging measures. The following theorem is the main result of this subsection.

THEOREM 4.1 (FL for weakly converging measures). *Let $\mathbb{S}$ be a metric space, let the sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ converge weakly to $\mu \in \mathcal{M}(\mathbb{S})$, let $\{f_n\}_{n\in\mathbf{N}^*}$ be a lower semiequicontinuous sequence of real-valued functions on $\mathbb{S}$, and let f be a measurable real-valued function on $\mathbb{S}$. Assume that the following conditions hold*:

(i) *The sequence $\{f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges to f in measure μ*;

(ii) *either $\{f_n^-\}_{n\in\mathbf{N}^*}$ is a.u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$ or assumption* (ii) *of Theorem* 2.2 *holds. Then*

$$\int_{\mathbb{S}} f(s)\,\mu(ds) \leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds). \tag{4.1}$$

We recall that the asymptotic uniform integrability of $\{f_n^-\}_{n\in\mathbf{N}^*}$ w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$ neither implies nor is implied by assumption (ii) of Theorem 2.2 [4, Examples 3.1 and 3.2].

Proof of Theorem 4.1. Consider a subsequence $\{f_{n_k}\}_{k\in\mathbf{N}^*} \subset \{f_n\}_{n\in\mathbf{N}^*}$ such that

$$\lim_{k\to\infty} \int_{\mathbb{S}} f_{n_k}(s)\,\mu_{n_k}(ds) = \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds). \tag{4.2}$$

Assumption (i) implies that $\mu(\{s \in \mathbb{S}\colon f_{n_k}(s) \leqslant f(s) - \varepsilon\}) \to 0$ as $k \to \infty$ for each $\varepsilon > 0$. Therefore, according to Remark 3.4, there exists a subsequence $\{f_{k_j}\}_{j\in\mathbf{N}^*} \subset \{f_{n_k}\}_{k\in\mathbf{N}^*}$ such that $f(s) \leqslant \liminf_{j\to\infty} f_{k_j}(s)$ for μ-a.e. $s \in \mathbb{S}$. Thus, Theorem 3.1(i) implies that

$$f(s) \leqslant \liminf_{j\to\infty, s'\to s} f_{k_j}(s')$$

for μ-a.e. $s \in \mathbb{S}$, and therefore,

$$\int_{\mathbb{S}} f(s)\,\mu(ds) \leqslant \int_{\mathbb{S}} \liminf_{j\to\infty,\, s'\to s} f_{k_j}(s')\,\mu(ds). \tag{4.3}$$

Theorem 2.2, as applied to $\{f_{k_j}\}_{j\in\mathbf{N}^*}$, implies

$$\int_{\mathbb{S}} \liminf_{j\to\infty, s'\to s} f_{k_j}(s')\,\mu(ds) \leqslant \liminf_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mu_{k_j}(ds). \tag{4.4}$$

Now (4.1) follows directly from (4.3), (4.4), and (4.2). Theorem 4.1 is proved.

The following corollary states that the setwise convergence in Theorem 2.3 can be substituted by the weak convergence if the integrands form a lower semiequicontinuous sequence of functions.

COROLLARY 4.1 (FL for weakly converging measures). *Let $\mathbb{S}$ be a metric space, let a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ converge weakly to $\mu \in \mathcal{M}(\mathbb{S})$, and let $\{f_n\}_{n\in\mathbf{N}^*}$ be a lower semiequicontinuous sequence of real-valued functions on $\mathbb{S}$. If assumption* (ii) *of Theorem* 4.1 *is satisfied, then inequality* (2.6) *holds.*

Proof. Inequality (2.6) follows directly from Theorem 4.1 and Remark 3.4.

The following example illustrates that Theorem 4.1 can provide a more exact lower bound for the lower limit of the integral than Theorem 2.2.

Example 4.1. Let $\mathbb{S} := [0, 2]$. We endow $\mathbb{S}$ with the metric

$$\begin{aligned}\rho(s_1, s_2) = {}& \mathbf{I}\{s_1 \in [0,1)\}\,\mathbf{I}\{s_2 \in [0,1)\}|s_1 - s_2| \\ &+ \big(1 - \mathbf{I}\{s_1 \in [0,1)\}\mathbf{I}\{s_2 \in [0,1)\}\big)\,\mathbf{I}\{s_1 \neq s_2\}.\end{aligned}$$

To see that ρ is a metric, note that for $s_1, s_2 \in \mathbb{S}$,

(i) $\rho(s_1, s_2) \in [0, 1]$;

(ii) $\rho(s_1, s_2) = 0$ if and only if $s_1 = s_2$;

(iii) $\rho(s_1, s_2)$ is symmetric in s_1 and s_2; and

(iv) for $s_1 \neq s_2$ and $s_3 \in \mathbb{S}$, the triangle inequality holds because $\rho(s_1, s_2) = |s_1 - s_2| \leqslant |s_1 - s_3| + |s_3 - s_2| = \rho(s_1, s_3) + \rho(s_3, s_2)$ if $s_1,\ s_2,\ s_3 \in [0, 1)$, and $\rho(s_1, s_2) \leqslant 1 \leqslant \rho(s_1, s_3) + \rho(s_3, s_2)$ otherwise.

Let μ be the Lebesgue measure on $\mathbb{S}$, and let $\{\mu_n\}_{n\in\mathbf{N}^*} \subset \mathcal{M}(\mathbb{S})$ be defined as

$$\mu_n(C) := \sum_{k=0}^{n-1} \frac{1}{n}\mathbf{I}\left\{\frac{k}{n} \in C\right\} + \mu(C \cap [1, 2]), \qquad C \in \Sigma, \quad n \in \mathbf{N}^*.$$

Then the sequence $\{\mu_n\}_{n\in\mathbf{N}^*}$ converges weakly to μ (see [2, Example 2.2]), and $\{\mu_n\}_{n\in\mathbf{N}^*}$ does not converge setwise to μ because $\mu_n([0, 1]\setminus\mathbb{Q}) = 0 \nrightarrow 1 = \mu([0, 1]\setminus\mathbb{Q})$,

where $\mathbb{Q}$ is the set of all rational numbers in $[0,1]$. Define $f \equiv 1$ and $f_n(s) = 1 - \mathbf{I}\{s \in (1 + j/2^k, 1 + (j+1)/2^k]\}$, where $k = \lfloor \log_2 n \rfloor$, $j = n - 2^k$, $s \in \mathbb{S}$, and $n \in \mathbf{N}^*$.

Since the subspace $(1,2] \subset \mathbb{S}$ is endowed with the discrete metric, every sequence of functions on $(1,2]$ is equicontinuous. Since $f_n(s) = 1$ for $n \in \mathbf{N}^*$ and $s \in [0,1]$, the sequence $\{f_n\}_{n\in\mathbf{N}^*}$ is equicontinuous on $[0,1]$. Therefore, $\{f_n\}_{n\in\mathbf{N}^*}$ is equicontinuous and, thus, lower semiequicontinuous on $\mathbb{S}$. In addition, (2.5) holds, and $\{f_n^-\}_{n\in\mathbf{N}^*}$ is a.u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$ because f_n is nonnegative for $n \in \mathbf{N}^*$. Since $\mu(\{s \in \mathbb{S}\colon f_n(s) < f(s)\}) = 1/2^{\lfloor \log_2 n \rfloor} \to 0$ as $n \to \infty$, condition (i) from Theorem 4.1 holds. In view of Theorem 4.1,

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds) &= \liminf_{n\to\infty} \left(\int_0^1 f_n(s)\,\mu_n(ds) + \int_1^2 f_n(s)\,\mu_n(ds) \right) \\ &= \liminf_{n\to\infty} \left(1 + 1 - \frac{1}{2^{\lfloor \log_2 n \rfloor}} \right) = 2 \geqslant \int_{\mathbb{S}} f(s)\,\mu(ds) = 2.\end{aligned}$$

By assertion (i) of Theorem 3.1,

$$\liminf_{n\to\infty,\, s'\to s} f_n(s') = \liminf_{n\to\infty} f_n(s) = 1 - \mathbf{I}\{s \in [1,2]\}, \qquad s \in \mathbb{S}.$$

In view of Theorem 2.2, (1.2) and (2.6) imply

$$\begin{aligned}2 = \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds) &\geqslant \int_{\mathbb{S}} \liminf_{n\to\infty,\, s'\to s} f_n(s')\,\mu(ds) \\ &= \int_{\mathbb{S}} \liminf_{n\to\infty} f_n(s)\,\mu(ds) = 1.\end{aligned}$$

Therefore, Theorem 4.1 provides a more exact lower bound (4.1) for the lower limit of integrals than (1.2) and (2.6) for weakly converging measures and lower semiequicontinuous sequences of functions.

4.2. Fatou's lemmas for setwise converging measures. The main results of this subsection, Theorem 4.2 and Corollary 4.2, are counterparts to Theorem 4.1 for setwise converging measures.

THEOREM 4.2 (FL for setwise converging measures). *Let $(\mathbb{S},\Sigma)$ be a measurable space, let a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ converge setwise to a measure $\mu \in \mathcal{M}(\mathbb{S})$, and let $\{f_n\}_{n\in\mathbf{N}^*}$ be a sequence of $\overline{\mathbf{R}}$-valued measurable functions on $\mathbb{S}$. If $\{f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges to a real-valued function f in measure μ and $\{f_n^-\}_{n\in\mathbf{N}^*}$ is a.u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$, then inequality* (4.1) *holds.*

Proof. The proof repeats several lines of the proofs of Theorems 4.1 and 2.2. Consider a subsequence $\{f_{n_k}\}_{k\in\mathbf{N}^*} \subset \{f_n\}_{n\in\mathbf{N}^*}$ such that

$$\lim_{k\to\infty} \int_{\mathbb{S}} f_{n_k}(s)\,\mu_{n_k}(ds) = \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds). \tag{4.5}$$

Since the sequence $\{f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges to f in measure μ, we have $\mu(\{s \in \mathbb{S}\colon f_{n_k}(s) \leqslant f(s) - \varepsilon\}) \to 0$ as $k \to \infty$ for each $\varepsilon > 0$. Therefore, Remark 3.4 implies that there exists a subsequence $\{f_{k_j}\}_{j\in\mathbf{N}^*} \subset \{f_{n_k}\}_{k\in\mathbf{N}^*}$ such that $f(s) \leqslant \liminf_{j\to\infty} f_{k_j}(s)$ for μ-a.e. $s \in \mathbb{S}$. Thus,

$$\int_{\mathbb{S}} f(s)\,\mu(ds) \leqslant \int_{\mathbb{S}} \liminf_{j\to\infty} f_{k_j}(s)\,\mu(ds). \tag{4.6}$$

Now we prove that

$$\int_{\mathbb{S}} \liminf_{j\to\infty} f_{k_j}(s)\,\mu(ds) \leqslant \lim_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mu_{k_j}(ds). \tag{4.7}$$

For this purpose, given a fixed arbitrary $K > 0$, we have

$$\begin{aligned}\liminf_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mu_{k_j}(ds) &\geqslant \liminf_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_{k_j}(s) > -K\}\,\mu_{k_j}(ds)\\ &+ \liminf_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_{k_j}(s) \leqslant -K\}\,\mu_{k_j}(ds).\end{aligned} \tag{4.8}$$

Next, we claim that

$$\liminf_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_{k_j}(s) > -K\}\,\mu_{k_j}(ds) \geqslant \int_{\mathbb{S}} \liminf_{j\to\infty} f_{k_j}(s)\,\mu(ds). \tag{4.9}$$

Indeed, applying Lemma 2.2 of [20] to the nonnegative sequence $\bigl\{f_{k_j}(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_{k_j}(s) > -K\} + K\bigr\}_{j\in\mathbf{N}^*}$, we get

$$\begin{aligned}\liminf_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_{k_j}(s) > -K\}\,\mu_{k_j}(ds)\\ \geqslant \int_{\mathbb{S}} \liminf_{j\to\infty} f_{k_j}(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_{k_j}(s) > -K\}\,\mu(ds).\end{aligned} \tag{4.10}$$

Here we note that

$$f_{k_j}(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_{k_j}(s) > -K\} \geqslant f_{k_j}(s) \tag{4.11}$$

for each $s\in\mathbb{S}$ because $K > 0$. Now (4.9) follows from (4.10) and (4.11).

Inequalities (4.8) and (4.9) imply

$$\begin{aligned}\liminf_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mu_{k_j}(ds) &\geqslant \int_{\mathbb{S}} \liminf_{j\to\infty,\,s'\to s} f_{k_j}(s')\,\mu(ds)\\ &+ \lim_{K\to+\infty} \liminf_{j\to\infty} \int_{\mathbb{S}} f_{k_j}(s)\,\mathbf{I}\{s\in\mathbb{S}\colon f_{k_j}(s) \leqslant -K\}\,\mu_{k_j}(ds),\end{aligned}$$

which is equivalent to (4.7) because $\{f_{k_j}^-\}_{j\in\mathbf{N}^*}$ is a.u.i. w.r.t. $\{\mu_{k_j}\}_{j\in\mathbf{N}^*}$. Hence (4.1) follows directly from (4.6), (4.7), and (4.5). Theorem 4.2 is proved.

The following corollary to Theorem 4.2 generalizes Theorem 2.3.

COROLLARY 4.2. *Let $(\mathbb{S},\Sigma)$ be a measurable space, let a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ converge setwise to a measure $\mu\in\mathcal{M}(\mathbb{S})$, and let $\{f_n\}_{n\in\mathbf{N}^*}$ be a sequence of $\overline{\mathbf{R}}$-valued measurable functions on $\mathbb{S}$ lower semiconverging to a real-valued function f in measure μ. If there exists a sequence of measurable real-valued functions $\{g_n\}_{n\in\mathbf{N}^*}$ on $\mathbb{S}$ such that $f_n(s) \geqslant g_n(s)$ for all $n\in\mathbf{N}^*$ and $s\in\mathbb{S}$, and if* (2.7) *holds, then inequality* (4.1) *holds.*

Proof. Consider an increasing sequence $\{n_k\}_{k\in\mathbf{N}^*}$ of natural numbers such that

$$\lim_{k\to\infty} \int_{\mathbb{S}} f_{n_k}(s)\,\mu_{n_k}(ds) = \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\,\mu_n(ds). \tag{4.12}$$

Since the sequence $\{f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges to f in measure μ, we have $\mu(\{s\in\mathbb{S}\colon f_{n_k}(s)\leqslant f(s)-\varepsilon\})\to 0$ as $k\to\infty$ for each $\varepsilon>0$. Therefore, Remark 3.4 implies that there exists a subsequence $\{f_{k_j}\}_{j\in\mathbf{N}^*}\subset\{f_{n_k}\}_{k\in\mathbf{N}^*}$ such that $f(s)\leqslant\liminf_{j\to\infty} f_{k_j}(s)$ for μ-a.e. $s\in\mathbb{S}$. Thus,

$$\int_{\mathbb{S}} f(s)\,\mu(ds)\leqslant\int_{\mathbb{S}}\liminf_{j\to\infty} f_{k_j}(s)\,\mu(ds). \tag{4.13}$$

Now we prove that

$$\int_{\mathbb{S}}\liminf_{j\to\infty} f_{k_j}(s)\,\mu(ds)\leqslant\lim_{j\to\infty}\int_{\mathbb{S}} f_{k_j}(s)\,\mu_{k_j}(ds). \tag{4.14}$$

Applying Theorem 4.2 to the sequence $\{f_{k_j}-g_{k_j}\}_{j\in\mathbf{N}^*}$, we get

$$\begin{aligned}
&\int_{\mathbb{S}}\liminf_{j\to\infty} f_{k_j}(s)\,\mu(ds)-\int_{\mathbb{S}}\limsup_{j\to\infty} g_{k_j}(s)\,\mu(ds)\\
&\quad\leqslant\int_{\mathbb{S}}\liminf_{j\to\infty}(f_{k_j}(s)-g_{k_j}(s))\,\mu(ds)\\
&\quad\leqslant\liminf_{j\to\infty}\int_{\mathbb{S}} f_{k_j}(s)\,\mu_{k_j}(ds)-\limsup_{j\to\infty}\int_{\mathbb{S}} g_{k_j}(s)\,\mu_{k_j}(ds).
\end{aligned}$$

Now (2.7) gives (4.14). Hence (4.1) follows directly from (4.13), (4.14), and (4.12). Corollary 4.2 is proved.

Theorem 4.2 provides a more exact lower bound for the lower limit of the integral than Theorem 2.3. This fact is illustrated in Example 4.2.

Example 4.2 (cf. [10, Example 4.1]). Let $\mathbb{S}=[0,1]$ and $\Sigma=\mathcal{B}([0,1])$, let μ be the Lebesgue measure on $\mathbb{S}$, and let, for $C\in\mathcal{B}(\mathbb{S})$ and $n\in\mathbf{N}^*$,

$$\mu_n(C):=\int_C 2\,\mathbf{I}\left\{s\in\mathbb{S}\colon\frac{2k}{2^n}<s<\frac{2k+1}{2^n},\ k=0,1,\dots,2^{n-1}-1\right\}\mu(ds).$$

Next, let $f\equiv 1$ and $f_n(s)=1-\mathbf{I}\{s\in[j/2^k,(j+1)/2^k]\}$, where $k=\lfloor\log_2 n\rfloor$, $j=n-2^k$, $s\in\mathbb{S}$, and $n\in\mathbf{N}^*$. Then the sequence $\{\mu_n\}_{n\in\mathbf{N}^*}$ converges setwise to μ, (2.7) holds, $\{f_n^-\}_{n\in\mathbf{N}^*}$ is a.u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$, and the sequence $\{f_n\}_{n\in\mathbf{N}^*}$ lower semiconverges to f in measure μ. In view of Theorem 4.2 and (2.6),

$$1=\liminf_{n\to\infty}\int_{\mathbb{S}} f_n(s)\,\mu_n(ds)\geqslant\int_{\mathbb{S}} f(s)\,\mu(ds)=1>0=\int_{\mathbb{S}}\liminf_{n\to\infty} f_n(s)\,\mu(ds).$$

Therefore, Theorem 4.2 provides a more exact lower bound for the lower limit of the integral than inequality (2.6).

5. Lebesgue's convergence theorem for varying measures. In this section, we present Lebesgue's convergence theorem for varying measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ and functions that are a.u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$. The following corollary follows from Theorem 2.2. It also follows from Theorem 3.5 in [20] adapted to general metric spaces. We provide it here for completeness.

Corollary 5.1 (Lebesgue's convergence theorem for weakly converging measures [4, Corollary 2.8]). *Let $\mathbb{S}$ be a metric space, let $\{\mu_n\}_{n\in\mathbf{N}^*}$ be a sequence of*

measures on $\mathbb{S}$ *converging weakly to* $\mu \in \mathcal{M}(\mathbb{S})$, *and let* $\{f_n\}_{n\in\mathbf{N}^*}$ *be an a.u.i.* (*see* (2.4)) *w.r.t.* $\{\mu_n\}_{n\in\mathbf{N}^*}$ *sequence of measurable* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$ *such that* $\lim_{n\to\infty,\, s'\to s} f_n(s')$ *exists for* μ*-a.e.* $s \in \mathbb{S}$. *Then*

$$\lim_{n\to\infty} \int_{\mathbb{S}} f_n(s)\, \mu_n(ds) = \int_{\mathbb{S}} \lim_{n\to\infty,\, s'\to s} f_n(s')\, \mu(ds) = \int_{\mathbb{S}} \lim_{n\to\infty} f_n(s)\, \mu(ds).$$

The following corollary states the convergence theorem for weakly converging measures μ_n and for an equicontinuous sequence of functions $\{f_n\}_{n\in\mathbf{N}^*}$.

COROLLARY 5.2 (Lebesgue's convergence theorem for weakly converging measures). *Let* $\mathbb{S}$ *be a metric space, let a sequence of measures* $\{\mu_n\}_{n\in\mathbf{N}^*}$ *converge weakly to* $\mu \in \mathcal{M}(\mathbb{S})$, *let* $\{f_n\}_{n\in\mathbf{N}^*}$ *be a sequence of real-valued equicontinuous functions on* $\mathbb{S}$, *and let* f *be a measurable real-valued function on* $\mathbb{S}$. *If the sequence* $\{f_n\}_{n\in\mathbf{N}^*}$ *converges to* f *in measure* μ *and is a.u.i.* (*see* (2.4)) *w.r.t.* $\{\mu_n\}_{n\in\mathbf{N}^*}$, *then*

$$\lim_{n\to\infty} \int_{\mathbb{S}} f_n(s)\, \mu_n(ds) = \int_{\mathbb{S}} f(s)\, \mu(ds). \tag{5.1}$$

Proof. Corollary 5.2 follows from Theorem 4.1 applied to $\{f_n\}_{n\in\mathbf{N}^*}$ and $\{-f_n\}_{n\in\mathbf{N}^*}$.

The following corollary follows directly from Theorem 4.2.

COROLLARY 5.3 (Lebesgue's convergence theorem for setwise converging measures). *Let* $(\mathbb{S}, \Sigma)$ *be a measurable space, let a sequence of measures* $\{\mu_n\}_{n\in\mathbf{N}^*}$ *converge setwise to a measure* $\mu \in \mathcal{M}(\mathbb{S})$, *and let* $\{f_n\}_{n\in\mathbf{N}^*}$ *be a sequence of* $\overline{\mathbf{R}}$*-valued measurable functions on* $\mathbb{S}$. *If the sequence* $\{f_n\}_{n\in\mathbf{N}^*}$ *converges to a measurable real-valued function* f *in measure* μ *and this sequence is a.u.i.* (*see* (2.4)) *w.r.t.* $\{\mu_n\}_{n\in\mathbf{N}^*}$, *then* (5.1) *holds.*

Proof. Corollary 5.3 follows from Theorem 4.2 applied to $\{f_n\}_{n\in\mathbf{N}^*}$ and $\{-f_n\}_{n\in\mathbf{N}^*}$.

6. Monotone convergence theorem for varying measures. In this section, we present monotone convergence theorems for varying measures.

THEOREM 6.1 (monotone convergence theorem for weakly converging measures). *Let* $\mathbb{S}$ *be a metric space, let* $\{\mu_n\}_{n\in\mathbf{N}^*}$ *be a sequence of measures on* $\mathbb{S}$ *that converges weakly to* $\mu \in \mathcal{M}(\mathbb{S})$, *let* $\{f_n\}_{n\in\mathbf{N}^*}$ *be a sequence of lower semicontinuous* $\overline{\mathbf{R}}$*-valued functions on* $\mathbb{S}$ *such that* $f_n(s) \leqslant f_{n+1}(s)$ *for each* $n \in \mathbf{N}^*$ *and* $s \in \mathbb{S}$, *and let* $f(s) := \lim_{n\to\infty} f_n(s)$, $s \in \mathbb{S}$. *Assume that the following conditions are satisfied*:

(i) *The function* f *is upper semicontinuous*,

(ii) *the functions* f_1^- *and* f^+ *are a.u.i. w.r.t.* $\{\mu_n\}_{n\in\mathbf{N}^*}$.

Then (5.1) *holds.*

Remark 6.1. The lower semicontinuity of f_n and nondecreasing pointwise convergence of f_n to f imply the lower semicontinuity of f. Therefore, under the assumptions in Theorem 6.1 the function f is continuous.

The following example demonstrates the necessity of condition (i) in Theorem 6.1.

Example 6.1. Consider $\mathbb{S} = [0, 1]$ endowed with the standard Euclidean metric, $f(s) = \mathbf{I}\{s \in (0, 1]\}$, $s \in \mathbb{S}$, $f_n(s) = \min\{ns, 1\}$, $n \in \mathbf{N}^*$, and $s \in \mathbb{S}$, and consider the

probability measures

$$\mu_n(C) := \int_C n\,\mathbf{I}\left\{s \in \left[0, \frac{1}{n}\right]\right\}\nu(ds), \qquad \mu(C) := \mathbf{I}\{0 \in C\}, \qquad C \in \mathcal{B}(\mathbb{S}), \quad n \in \mathbf{N}^*, \tag{6.1}$$

where ν is the Lebesgue measure on $\mathbb{S}$.

Hence $f_n(s) \uparrow f(s)$ for each $s \in \mathbb{S}$ as $n \to \infty$, and the sequence of probability measures μ_n converges weakly to μ. Since the functions f_1 and f are bounded, condition (ii) from Theorem 6.1 holds. The function f_n is continuous, and the function f is lower semicontinuous, but f is not upper semicontinuous. Since $\int_{\mathbb{S}} f_n(s)\,\mu_n(ds) = 1/2$, $n \in \mathbf{N}^*$, and $\int_{\mathbb{S}} f(s)\,\mu(ds) = 0$, formula (5.1) does not hold.

Proof of Theorem 6.1. Since $f_n(s) \leqslant f(s)$,

$$f(s) = \liminf_{n\to\infty,\, s'\to s} f_n(s') \leqslant \limsup_{n\to\infty,\, s'\to s} f_n(s') \leqslant \limsup_{s'\to s} f(s') \leqslant f(s), \qquad s \in \mathbb{S},$$

where the first equality follows from Lemma 3.1, and the last inequality holds because f is upper semicontinuous. Hence $\lim_{n\to\infty,\, s'\to s} f_n(s') = f(s)$, $s \in \mathbb{S}$. In addition, condition (ii) implies that the sequence $\{f_n\}_n$ is a.u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$. Now (5.1) follows from Corollary 5.1. Theorem 6.1 is proved.

COROLLARY 6.1. *Let $\mathbb{S}$ be a metric space, let $\{\mu_n\}_{n\in\mathbf{N}^*}$ be a sequence of measures on $\mathbb{S}$ that converges weakly to $\mu \in \mathcal{M}(\mathbb{S})$, and let $\{f_n\}_{n\in\mathbf{N}^*}$ be a pointwise nondecreasing sequence of measurable $\overline{\mathbf{R}}$-valued functions on $\mathbb{S}$. Let $f(s) := \lim_{n\to\infty} f_n(s)$ and $\underline{f}_n(s) := \liminf_{s'\to s} f_n(s')$, $s \in \mathbb{S}$. If*

(i) *the function f is real-valued and upper semicontinuous,*

(ii) *the sequence $\{\underline{f}_n\}_{n\in\mathbf{N}^*}$ lower semiconverges to f in measure μ, and*

(iii) *the functions $\underline{f}_1^-$ and f^+ are a.u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$,*

then (5.1) *holds.*

The following example demonstrates the necessity of condition (ii) in Corollary 6.1.

Example 6.2. Consider $\mathbb{S} = [0,1]$ endowed with the standard Euclidean metric, $f(s) = 1$,

$$f_n(s) = \begin{cases} 1 & \text{if } s = 0, \\ \min\{ns, 1\} & \text{if } s \in (0,1], \end{cases} \qquad n \in \mathbf{N}^*, \quad s \in \mathbb{S},$$

and the probability measures μ_n, $n \in \mathbf{N}^*$, and μ defined in (6.1). Then $\underline{f}_n(s) = \min\{ns, 1\}$, $f_n(s) \uparrow f(s)$ for each $s \in \mathbb{S}$ as $n \to \infty$, and the sequence of probability measures μ_n converges weakly to μ. Since the functions $\underline{f}_1$ and f are bounded, condition (iii) from Corollary 6.1 holds. Condition (ii) from Corollary 6.1 does not hold because $f(0) = f_n(0) = 1$ and $\underline{f}_n(0) = 0$ for each $n \in \mathbf{N}^*$. Since $\int_{\mathbb{S}} f_n(s)\,\mu_n(ds) = 1/2$, $n \in \mathbf{N}^*$, and $\int_{\mathbb{S}} f(s)\,\mu(ds) = 1$, formula (5.1) does not hold.

Proof of Corollary 6.1. Since the function $\underline{f}_n$ is lower semicontinuous, Theorem 6.1 implies

$$\lim_{n\to\infty} \int_{\mathbb{S}} \underline{f}_n(s)\,\mu_n(ds) = \int_{\mathbb{S}} \lim_{n\to\infty} \underline{f}_n(s)\,\mu(ds). \tag{6.2}$$

Condition (i) implies that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbf{N}^*} \subset \{f_n\}_{n\in\mathbf{N}^*}$ such that

$$\liminf_{k\to\infty} \underline{f}_{n_k}(s) \geqslant f(s) \quad \text{for } \mu\text{-a.e. } s \in \mathbb{S}. \tag{6.3}$$

Since $\underline{f}_n(s) \leqslant f_n(s) \leqslant f(s)$, $n \in \mathbf{N}^*$ and $s \in \mathbb{S}$, and since the sequence $\{\underline{f}_n\}_{n\in\mathbf{N}^*}$ is pointwise nondecreasing, (6.3) gives that

$$f(s) = \lim_{n\to\infty} \underline{f}_n(s) \quad \text{for } \mu\text{-a.e. } s \in \mathbb{S}. \tag{6.4}$$

Hence (6.2) and (6.4) imply

$$\lim_{n\to\infty} \int_{\mathbb{S}} \underline{f}_n(s)\, \mu_n(ds) = \int_{\mathbb{S}} f(s)\, \mu(ds). \tag{6.5}$$

Since $\underline{f}_n(s) \leqslant f_n(s) \leqslant f(s)$, $n \in \mathbf{N}^*$, and $s \in \mathbb{S}$,

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{S}} \underline{f}_n(s)\, \mu_n(ds) &\leqslant \liminf_{n\to\infty} \int_{\mathbb{S}} f_n(s)\, \mu_n(ds) \\ &\leqslant \limsup_{n\to\infty} \int_{\mathbb{S}} f_n(s)\, \mu_n(ds) \leqslant \limsup_{n\to\infty} \int_{\mathbb{S}} f(s)\, \mu_n(ds). \end{aligned} \tag{6.6}$$

Applying Theorem 2.2 to the sequence $\{-f\}$, we have, since f is upper semicontinuous,

$$\limsup_{n\to\infty} \int_{\mathbb{S}} f(s)\, \mu_n(ds) \leqslant \int_{\mathbb{S}} f(s)\, \mu(ds). \tag{6.7}$$

Now (5.1) follows from (6.5), (6.6), and (6.7).

The following corollary from Theorem 4.2 is the counterpart to Theorem 6.1 for setwise converging measures.

COROLLARY 6.2 (monotone convergence theorem for setwise converging measures). *Let $(\mathbb{S}, \Sigma)$ be a measurable space, let a sequence of measures $\{\mu_n\}_{n\in\mathbf{N}^*}$ converge setwise to a measure $\mu \in \mathcal{M}(\mathbb{S})$, and let $\{f_n\}_{n\in\mathbf{N}^*}$ be a pointwise nondecreasing sequence of measurable $\overline{\mathbf{R}}$-valued functions on $\mathbb{S}$. Let $f(s) := \lim_{n\to\infty} f_n(s)$, $s \in \mathbb{S}$. If the functions f_1^- and f^+ are a.u.i. w.r.t. $\{\mu_n\}_{n\in\mathbf{N}^*}$, then (5.1) holds.*

Proof. Since $f_n \uparrow f$, (5.1) follows directly from Theorem 4.2 applied to the sequences $\{f_n\}_{n\in\mathbf{N}^*}$ and $\{-f_n\}_{n\in\mathbf{N}^*}$. Corollary 6.2 is proved.

7. Applications to Markov decision processes. Consider a discrete-time MDP with a state space $\mathbb{X}$, an action space $\mathbb{A}$, one-step costs c, and transition probabilities q. Assume that $\mathbb{X}$ and $\mathbb{A}$ are Borel subsets of Polish (complete separable metric) spaces. Let $c(x,a)\colon \mathbb{X} \times \mathbb{A} \mapsto \overline{\mathbf{R}}$ be the one-step cost and $q(B|x,a)$ be the transition kernel representing the probability that the next state is in $B \in \mathcal{B}(\mathbb{X})$, given that the action a is chosen at the state x. The cost function c is assumed to be measurable and bounded below.

The decision process proceeds as follows: at each time epoch $t = 0, 1, \ldots$, the current state of the system, x, is observed. A decisionmaker chooses an action a, the cost $c(x,a)$ is accrued, and the system moves to the next state according to $q(\,\cdot \mid x, a)$. Let $H_t = (\mathbb{X}\times\mathbb{A})^t \times \mathbb{X}$ be the set of histories for $t = 0, 1, \ldots$. A (randomized) decision

rule at period $t = 0, 1, \dots$ is a regular transition probability π_t from H_t to $\mathbb{A}$; that is, (i) $\pi_t(\cdot \mid h_t)$ is a probability distribution on $\mathbb{A}$, where $h_t = (x_0, a_0, x_1, \dots, a_{t-1}, x_t)$; and (ii) for any measurable subset $B \subset \mathbb{A}$, the function $\pi_t(B \mid \cdot)$ is measurable on H_t. A policy π is a sequence $(\pi_0, \pi_1, \dots)$ of decision rules. Let Π be the set of all policies. A policy π is called nonrandomized if each probability measure $\pi_t(\cdot \mid h_t)$ is concentrated at one point. A nonrandomized policy is called stationary if all decisions depend only on the current state.

Ionescu Tulcea's theorem implies that an initial state x and a policy π define a unique probability $\mathbf{P}_x^\pi$ on the set of all trajectories $\mathbb{H}_\infty = (\mathbb{X} \times \mathbb{A})^\infty$ endowed with the product of σ-fields defined by Borel σ-fields of $\mathbb{X}$ and $\mathbb{A}$; see [1, pp. 140–141] or [14, p. 178]. Let $\mathbb{E}_x^\pi$ be an expectation w.r.t. $\mathbf{P}_x^\pi$.

For a finite-horizon $N \in \mathbf{N}^*$, let us define the expected total discounted costs,

$$v_{N,\alpha}^\pi(x) := \mathbb{E}_x^\pi \sum_{t=0}^{N-1} \alpha^t c(x_t, a_t), \qquad x \in \mathbb{X}, \tag{7.1}$$

where $\alpha \in [0, 1]$ is the discount factor. When $N = \infty$ and $\alpha \in [0, 1)$, (7.1) defines an infinite-horizon expected total discounted cost denoted by $v_\alpha^\pi(x)$. Let $v_\alpha(x) := \inf_{\pi \in \Pi} v_\alpha^\pi(x)$, $x \in \mathbb{X}$. A policy π is called optimal for the discount factor α if $v_\alpha^\pi(x) = v_\alpha(x)$ for all $x \in \mathbb{X}$.

The *average cost per unit time* is defined as

$$w_1^\pi(x) := \limsup_{N \to \infty} \frac{1}{N} v_{N,1}^\pi(x), \qquad x \in \mathbb{X}.$$

Define the optimal value function $w_1(x) := \inf_{\pi \in \Pi} w_1^\pi(x)$, $x \in \mathbb{X}$. A policy π is called average-cost optimal if $w_1^\pi(x) = w_1(x)$ for all $x \in \mathbb{X}$.

We note that, in general, action sets may depend on current states, and usually the state-dependent sets $A(x)$ are considered for all $x \in \mathbb{X}$. In our problem formulation $A(x) = \mathbb{A}$ for all $x \in \mathbb{X}$. This problem formulation is simpler than a formulation with the sets $A(x)$, and these two problem formulations are equivalent because we allow that $c(x, a) = +\infty$ for some $(x, a) \in \mathbb{X} \times \mathbb{A}$. For example, we may set $A(x) = \{a \in \mathbb{A} \colon c(x, a) < +\infty\}$. For a formulation with the sets $A(x)$, one may define $c(x, a) = +\infty$ when $a \in \mathbb{A} \setminus A(x)$ and use the action sets $\mathbb{A}$ instead of $A(x)$.

To establish the existence of the average-cost optimal policies via an optimality inequality for problems with compact action sets, Schäl [19] considered two continuity conditions **W** and **S** for problems with weakly and setwise continuous transition probabilities, respectively. For setwise continuous transition probabilities, Hernández-Lerma [13] generalized Assumption **S** to Assumption S* to cover MDPs with possibly noncompact action sets. For a similar purpose, when transition probabilities are weakly continuous, Feinberg et al. [5] generalized Assumption **W** to Assumption W*.

We recall that a function $f \colon \mathbb{U} \mapsto \overline{\mathbf{R}}$ defined on a metric space $\mathbb{U}$ is called inf-compact (on $\mathbb{U}$) if, for every $\lambda \in \mathbf{R}$, the level set $\{u \in \mathbb{U} \colon f(u) \leqslant \lambda\}$ is compact. A subset of a metric space is also a metric space with respect to the same metric. For $U \subset \mathbb{U}$, if the domain of f is narrowed to U, then this function is called the restriction of f to U.

Definition 7.1 (see [7, Definition 1.1], [3, Definition 2.1]). *A function $f \colon \mathbb{X} \times \mathbb{A} \mapsto \overline{\mathbf{R}}$ is called $\mathbb{K}$-inf-compact if, for every nonempty compact subset $\mathcal{K}$ of $\mathbb{X}$, the restriction of f to $\mathcal{K} \times \mathbb{A}$ is an inf-compact function.*

Assumption W* ([5], [10], [11], [3]). (i) The function c is $\mathbb{K}$-inf-compact.
(ii) The transition probability $q(\cdot \mid x,a)$ is weakly continuous in $(x,a) \in \mathbb{X} \times \mathbb{A}$.

Assumption S* ([13, Assumption 2.1]). (i) The function $c(x,a)$ is inf-compact in $a \in \mathbb{A}$ for each $x \in \mathbb{X}$.
(ii) The transition probability $q(\cdot \mid x,a)$ is setwise continuous in $a \in \mathbb{A}$ for each $x \in \mathbb{X}$.

Let

$$\begin{aligned} m_\alpha &:= \inf_{x\in\mathbb{X}} v_\alpha(x), \qquad u_\alpha(x) := v_\alpha(x) - m_\alpha, \\ \underline{w} &:= \liminf_{\alpha\uparrow 1}(1-\alpha)m_\alpha, \qquad \overline{w} := \limsup_{\alpha\uparrow 1}(1-\alpha)m_\alpha. \end{aligned} \tag{7.2}$$

The function u_α is called the *discounted relative value function.* If either Assumption W* or Assumption S* holds, we consider the following assumption.

Assumption B. (i) $w^* := \inf_{x\in\mathbb{X}} w_1(x) < +\infty$;
(ii) $\sup_{\alpha\in[0,1)} u_\alpha(x) < +\infty$, $x \in \mathbb{X}$.

According to [19, Lemma 1.2(a)], Assumption B(i) implies that $m_\alpha < +\infty$ for all $\alpha \in [0,1)$. Thus, all of the quantities in (7.2) are defined.

In [5], [19] it was proved that, if a stationary policy ϕ satisfies the average-cost optimality inequality (ACOI)

$$\underline{w} + u(x) \geqslant c(x,\phi(x)) + \int_{\mathbb{X}} u(y)\, q(dy \mid x, \phi(x)), \qquad x \in \mathbb{X}, \tag{7.3}$$

for some nonnegative measurable function $u\colon \mathbb{X} \to \mathbf{R}$, then the stationary policy ϕ is average-cost optimal. A nonnegative measurable function $u(x)$ satisfying inequality (7.3) with some stationary policy ϕ is called an average-cost relative value function. The following two theorems state the validity of the ACOI under Assumption W* (or Assumption S*) and Assumption B.

THEOREM 7.1 (see [5, Corollary 2 and p. 603]). *Let Assumptions* W* *and* B *hold. For an arbitrary sequence* $\{\alpha_n \uparrow 1\}_{n\in\mathbf{N}^*}$, *let*

$$u(x) := \liminf_{n\to\infty,\, y\to x} u_{\alpha_n}(y), \qquad x \in \mathbb{X}. \tag{7.4}$$

Then there exists a stationary policy ϕ *satisfying ACOI* (7.3) *with the function* u *defined in* (7.4). *Therefore,* ϕ *is a stationary average-cost optimal policy. In addition, the function* u *is lower semicontinuous, and*

$$w_1^\phi(x) = \underline{w} = \lim_{\alpha\uparrow 1}(1-\alpha)v_\alpha(x) = \lim_{\alpha\uparrow 1}(1-\alpha)m_\alpha = \overline{w} = w^*, \qquad x \in \mathbb{X}. \tag{7.5}$$

THEOREM 7.2 (see [13, section 4]). *Let Assumptions* S* *and* B *hold. For an arbitrary sequence* $\{\alpha_n \uparrow 1\}_{n\in\mathbf{N}^*}$, *let*

$$u(x) := \liminf_{n\to\infty} u_{\alpha_n}(x), \qquad x \in \mathbb{X}. \tag{7.6}$$

Then there exists a stationary policy ϕ *satisfying ACOI* (7.3) *with the function* u *defined in* (7.6). *Therefore,* ϕ *is a stationary average-cost optimal policy. In addition,* (7.5) *holds.*

The following corollary to Theorem 7.1 provides a sufficient condition for the validity of ACOI (7.3) with a relative value function u defined in (7.6).

COROLLARY 7.1. *Let Assumptions* W* *and* B *hold, and let there exist a sequence* $\{\alpha_n \uparrow 1\}_{n\in\mathbf{N}^*}$ *of nonnegative discount factors such that the sequence of functions* $\{u_{\alpha_n}\}_{n\in\mathbf{N}^*}$ *is lower semiequicontinuous. Then the conclusions of Theorem* 7.1 *hold for the function* u *defined in* (7.6) *for this sequence* $\{\alpha_n\}_{n\in\mathbf{N}^*}$.

Proof. Since the sequence of functions $\{u_{\alpha_n}\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous, the functions u, as defined in (7.4) and (7.6), coincide in view of Theorem 3.1(i). Corollary 7.1 is proved.

Consider the following equicontinuity condition (EC) on the discounted relative value functions.

Assumption EC. There exists a sequence $\{\alpha_n\}_{n\in\mathbf{N}^*}$ of nonnegative discount factors such that $\alpha_n \uparrow 1$ as $n\to\infty$, and the following two conditions hold:

(i) The sequence of functions $\{u_{\alpha_n}\}_{n\in\mathbf{N}^*}$ is equicontinuous;

(ii) there exists a nonnegative measurable function $U(x)$, $x\in\mathbb{X}$, such that $U(x)\geqslant u_{\alpha_n}(x)$, $n\in\mathbf{N}^*$, and $\int_{\mathbb{X}} U(y)\,q(dy\mid x,a)<+\infty$ for all $x\in\mathbb{X}$ and $a\in\mathbb{A}$.

It is known that, if either Assumption W* or [14, Assumption 4.2.1] holds (the latter one is stronger than Assumption S*), then under Assumptions B and EC there exist a sequence $\{\alpha_n\uparrow 1\}_{n\in\mathbf{N}^*}$ of nonnegative discount factors and a stationary policy ϕ satisfying the average-cost optimality equations (ACOEs)

$$\begin{aligned} w^*+u(x) &= c(x,\phi(x))+\int_{\mathbb{X}} u(y)\,q(dy\mid x,\phi(x)) \\ &= \min_{a\in\mathbb{A}}\left[c(x,a)+\int_{\mathbb{X}} u(y)\,q(dy\mid x,a)\right] \end{aligned} \tag{7.7}$$

with u defined in (7.4) for the sequence $\{\alpha_n\uparrow 1\}_{n\in\mathbf{N}^*}$, and the function u is continuous; see [12, Theorem 3.2] for W* and [14, Theorem 5.5.4]. We note that the quantity w^* in (7.7) can be replaced with any other quantity in (7.5).

In addition, since the first equation in (7.7) implies inequality (7.3), every stationary policy ϕ satisfying (7.7) is average-cost optimal. Observe that in these cases the function u is continuous (see [12, Theorem 3.2] for W* and [14, Theorem 5.5.4]), while under conditions of Theorems 7.1 and 7.2 the corresponding functions u may not be continuous; see Examples 7.1 and 7.2. Below we provide more general conditions for the validity of the ACOEs. In particular, under these conditions the relative value functions u may not be continuous.

Now, we introduce Assumption LEC, which is weaker than Assumption EC. Indeed, Assumption EC(i) is obviously stronger than LEC(i). In view of the Ascoli theorem (see [14, p. 96] or [18, p. 179]), EC(i) and the first claim in EC(ii) imply LEC(ii). The second claim in EC(ii) implies LEC(iii). It is shown in Theorem 7.3 that the ACOEs hold under Assumptions W*, B, and LEC.

Assumption LEC. There exists a sequence $\{\alpha_n\}_{n\in\mathbf{N}^*}$ of nonnegative discount factors such that $\alpha_n\uparrow 1$ as $n\to\infty$ and the following three conditions hold:

(i) The sequence of functions $\{u_{\alpha_n}\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous,

(ii) $\lim_{n\to\infty} u_{\alpha_n}(x)$ exists for each $x\in\mathbb{X}$,

(iii) for each $x\in\mathbb{X}$ and $a\in\mathbb{A}$ the sequence $\{u_{\alpha_n}\}_{n\in\mathbf{N}^*}$ is a.u.i. w.r.t. $q(\cdot\mid x,a)$.

THEOREM 7.3. *Let Assumptions* W* *and* B *hold. Consider a sequence* $\{\alpha_n\uparrow 1\}_{n\in\mathbf{N}^*}$ *of nonnegative discount factors. If Assumption* LEC *is satisfied for the sequence*

$\{\alpha_n\}_{n\in\mathbf{N}^*}$, then there exists a stationary policy ϕ such that the ACOEs (7.7) *hold with the function $u(x)$ defined in* (7.6).

Proof. Since Assumptions W* and B hold, and $\{u_{\alpha_n}\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous, Corollary 7.1 implies that there exists a stationary policy ϕ satisfying (7.3) with u defined in (7.6),

$$w^* + u(x) \geqslant c(x,\phi(x)) + \int_{\mathbb{X}} u(y)\, q(dy \mid x, \phi(x)). \tag{7.8}$$

To prove the ACOEs, it remains to establish the opposite inequality to (7.8). According to [5, Theorem 2(iv)], for each $n \in \mathbf{N}^*$ and $x \in \mathbb{X}$ the discounted-cost optimality equation is

$$v_{\alpha_n}(x) = \min_{a\in\mathbb{A}}\left[c(x,a) + \alpha_n \int_{\mathbb{X}} v_{\alpha_n}(y)\, q(dy \mid x, a)\right],$$

which, by subtracting m_α from both sides and by replacing α_n with 1, implies that for all $a \in \mathbb{A}$,

$$(1-\alpha_n)m_{\alpha_n} + u_{\alpha_n}(x) \leqslant c(x,a) + \int_{\mathbb{X}} u_{\alpha_n}(y)\, q(dy \mid x, a), \qquad x \in \mathbb{X}. \tag{7.9}$$

Let $n \to \infty$. In view of (7.5), Assumptions LEC(ii), (iii), and Fatou's lemma [21, p. 211], (7.9) implies that, for all $a \in \mathbb{A}$,

$$w^* + u(x) \leqslant c(x,a) + \int_{\mathbb{X}} u(y)\, q(dy \mid x, a), \qquad x \in \mathbb{X}. \tag{7.10}$$

We note that the integral in (7.9) converges to the integral in (7.10) since the sequence $\{u_{\alpha_n}\}_{n\in\mathbf{N}^*}$ converges pointwise to u and is u.i.; see Theorem 2.1. Now by (7.10),

$$\begin{aligned} w^* + u(x) &\leqslant \min_{a\in\mathbb{A}}\left[c(x,a) + \int_{\mathbb{X}} u(y)\, q(dy \mid x, a)\right] \\ &\leqslant c(x,\phi(x)) + \int_{\mathbb{X}} u(y)\, q(dy \mid x, \phi(x)), \qquad x \in \mathbb{X}. \end{aligned} \tag{7.11}$$

Thus, (7.8) and (7.11) imply (7.7). Theorem 7.3 is proved.

In the following example, Assumptions W*, B, and LEC hold. Hence the ACOEs hold. However, Assumption EC does not hold. Therefore, Assumption LEC is more general than Assumption EC.

Example 7.1. Consider $\mathbb{X} = [0,1]$ equipped with the Euclidean metric and consider $\mathbb{A} = \{a^{(1)}\}$. The transition probabilities are $q(0 \mid x, a^{(1)}) = 1$ for all $x \in \mathbb{X}$. The cost function is $c(x, a^{(1)}) = \mathbf{I}\{x \neq 0\}$, $x \in \mathbb{X}$. Then the discounted-cost value is $v_\alpha(x) = u_\alpha(x) = \mathbf{I}\{x \neq 0\}$, $\alpha \in [0,1)$, and $x \in \mathbb{X}$, and the average-cost value is $w^* = w_1(x) = 0$, $x \in \mathbb{X}$. It is straightforward that Assumptions W* and B hold. In addition, since the function $u(x) = \mathbf{I}\{x \neq 0\}$ is lower semicontinuous but is not continuous, the sequence of functions $\{u_{\alpha_n}\}_{n\in\mathbf{N}^*}$ is lower semiequicontinuous but is not equicontinuous for each sequence $\{\alpha_n \uparrow 1\}_{n\in\mathbf{N}^*}$. Therefore, Assumption LEC holds since $0 \leqslant u_{\alpha_n}(x) \leqslant 1$, $x \in \mathbb{X}$, and Assumption EC does not hold. Now (7.7) holds with $w^* = 0$, $u(x) = \mathbf{I}\{x \neq 0\}$, and $\phi(x) = a^{(1)}$, $x \in \mathbb{X}$.

The following theorem states the validity of ACOEs under Assumptions S*, B, and LEC(ii), (iii).

THEOREM 7.4. *Let Assumptions* S* *and* B *hold. Consider a sequence* $\{\alpha_n \uparrow 1\}_{n\in\mathbf{N}^*}$ *of nonnegative discount factors. If Assumptions* LEC(ii), (iii) *are satisfied for the sequence* $\{\alpha_n\}_{n\in\mathbf{N}^*}$, *then there exists a stationary policy* ϕ *such that* (7.7) *holds with the function* $u(x)$ *defined in* (7.6).

Proof. According to Theorem 7.2, if Assumptions S* and B hold, then we have that

(i) equalities in (7.5) hold,

(ii) there exists a stationary policy ϕ satisfying ACOI (7.8) with the function u defined in (7.6), and

(iii) for each $n \in \mathbf{N}^*$ and $x \in \mathbb{X}$ the discounted-cost optimality equation reads as

$$v_{\alpha_n}(x) = \min_{a\in\mathbb{A}}\left[c(x,a) + \alpha_n \int_{\mathbb{X}} v_{\alpha_n}(y)\, q(dy \mid x,a)\right].$$

Therefore, the same arguments as in the proof of Theorem 7.3 starting from (7.9) imply the validity of (7.7) with u defined in (7.6). Theorem 7.4 is proved.

Observe that the MDP described in Example 7.1 also satisfies Assumptions S*, B, and LEC(ii), (iii). We provide Example 7.2, where Assumptions S*, B, and LEC(ii), (iii) hold. Hence the ACOEs also hold. However, Assumptions W*, LEC(i), and EC do not hold.

Example 7.2. Let $\mathbb{X} = [0,1]$ and $\mathbb{A} = \{a^{(1)}\}$. The transition probabilities are $q(0 \mid x, a^{(1)}) = 1$ for all $x \in \mathbb{X}$. The cost function is $c(x, a^{(1)}) = D(x)$, where D is the Dirichlet function defined as

$$D(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational,} \end{cases} \qquad x \in \mathbb{X}.$$

Since there is only one available action, Assumption S* holds. The discounted-cost value is $v_\alpha(x) = u_\alpha(x) = D(x) = u(x)$, $\alpha \in [0,1)$, and $x \in \mathbb{X}$, and the average-cost value is $w^* = w_1(x) = 0$, $x \in \mathbb{X}$. Hence Assumptions B and LEC(ii), (iii) hold. Therefore the ACOEs (7.7) hold with $w^* = 0$, $u(x) = D(x)$, and $\phi(x) = a^{(1)}$, $x \in \mathbb{X}$. Thus, the average-cost relative function u is not lower semicontinuous. However, since the function $c(x, a^{(1)}) = D(x)$ is not lower semicontinuous, Assumption W* does not hold. Since the function $u(x) = u_\alpha(x) = D(x)$ is not lower semicontinuous, Assumptions LEC(i) and EC do not hold either.

Acknowledgment. The authors thank Huizhen (Janey) Yu for valuable remarks.

REFERENCES

[1] D. P. BERTSEKAS AND S. E. SHREVE, *Stochastic Optimal Control. The Discrete Time Case*, reprint of the 1978 original, Athena Scientific, Belmont, MA, 1996.

[2] P. BILLINGSLEY, *Convergence of Probability Measures*, 2nd ed., Wiley Ser. Probab. Statist. Probab. Statist., John Wiley & Sons, New York, 1999, https://doi.org/10.1002/9780470316962.

[3] E. A. FEINBERG, *Optimality conditions for inventory control*, in Optimization Challenges in Complex, Networked, and Risky Systems, Tutor. Oper. Res., INFORMS, Cantonsville, MD, 2016, pp. 14–44, https://doi.org/10.1287/educ.2016.0145.

[4] E. A. Feinberg, P. O. Kasyanov, and Y. Liang, *Fatou's lemma for weakly converging measures under the uniform integrability condition*, Theory Probab. Appl., 64 (2020), pp. 615–630. https://doi.org/10.1137/S0040585X97T989738; preprint version available at https://arxiv.org/abs/1807.07931.
[5] E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk, *Average cost Markov decision processes with weakly continuous transition probabilities*, Math. Oper. Res., 37 (2012), pp. 591–607, https://doi.org/10.1287/moor.1120.0555.
[6] E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk, *Fatou's lemma for weakly converging probabilities*, Theory Probab. Appl., 58 (2014), pp. 683–689, https://doi.org/10.1137/S0040585X97986850.
[7] E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk, *Berge's theorem for noncompact image sets*, J. Math. Anal. Appl., 397 (2013), pp. 255–259, https://doi.org/10.1016/j.jmaa.2012.07.051.
[8] E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky, *Convergence of probability measures and Markov decision models with incomplete information*, Proc. Steklov Inst. Math., 287 (2014), pp. 96–117, https://doi.org/10.1134/S0081543814080069.
[9] E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky, *Uniform Fatou's lemma*, J. Math. Anal. Appl., 444 (2016), pp. 550–567, https://doi.org/10.1016/j.jmaa.2016.06.044.
[10] E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky, *Partially observable total-cost Markov decision processes with weakly continuous transition probabilities*, Math. Oper. Res., 41 (2016), pp. 656–681, https://doi.org/10.1287/moor.2015.0746.
[11] E. A. Feinberg and M. E. Lewis, *On the convergence of optimal actions for Markov decision processes and the optimality of* (s, S) *inventory policies*, Naval Res. Logist., 65 (2018), pp. 619–637, https://doi.org/10.1002/nav.21750.
[12] E. A. Feinberg and Y. Liang, *On the optimality equation for average cost Markov decision processes and its validity for inventory control*, Ann. Oper. Res., 2017, https://doi.org/10.1007/s10479-017-2561-9.
[13] O. Hernández-Lerma, *Average optimality in dynamic programming on Borel spaces—unbounded costs and controls*, Systems Control Lett., 17 (1991), pp. 237–242, https://doi.org/10.1016/0167-6911(91)90069-Q.
[14] O. Hernández-Lerma and J. B. Lasserre, *Discrete-time Markov Control Processes. Basic Optimality Criteria*, Appl. Math. (N.Y.) 30, Springer-Verlag, New York, 1996, https://doi.org/10.1007/978-1-4612-0729-0.
[15] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, 2nd ed., Grundlehren Math. Wiss. 288, Springer-Verlag, Berlin, 2003, https://doi.org/10.1007/978-3-662-05265-5.
[16] Yu. M. Kabanov and R. Sh. Liptser, *On convergence in variation of the distributions of multivariate point processes*, Z. Wahrsch. Verw. Gebiete, 63 (1983), pp. 475–485, https://doi.org/10.1007/BF00533721.
[17] M. V. Kartashov, *Probability, Processes, Statistics*, Kyiv Univ. Press, Kyiv, 2008 (in Ukrainian).
[18] H. L. Royden, *Real Analysis*, 2nd ed., The Macmillan Co., New York, 1968.
[19] M. Schäl, *Average optimality in dynamic programming with general state space*, Math. Oper. Res., 18 (1993), pp. 163–172, https://doi.org/10.1287/moor.18.1.163.
[20] R. Serfozo, *Convergence of Lebesgue integrals with varying measures*, Sankhyā Ser. A, 44 (1982), pp. 380–402.
[21] A. N. Shiryaev, *Probability*, 2nd ed., Grad. Texts in Math. 95, Springer-Verlag, New York, 1996, https://doi.org/10.1007/978-1-4757-2539-1.
[22] A. W. van der Vaart, *Asymptotic Statistics*, Camb. Ser. Stat. Probab. Math. 3, Cambridge Univ. Press, Cambridge, 1998, https://doi.org/10.1017/CBO9780511802256.