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Manhattan curves for hyperbolic surfaces with cusps

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Abstract. In this paper, we study an interesting curve, the so-called Manhattan curve, associated with a pair of boundary-preserving Fuchsian representations of a (non-compact) surface; in particular, representations corresponding to Riemann surfaces with cusps. Using thermodynamic formalism (for countable state Markov shifts), we prove the analyticity of the Manhattan curve. Moreover, we derive several dynamical and geometric rigidity results, which generalize results of Burger [Intersection, the Manhattan curve, and Patterson–Sullivan theory in rank 2. *Int. Math. Res. Not.* **1993**(7) (1993), 217–225] and Sharp [The Manhattan curve and the correlation of length spectra on hyperbolic surfaces. *Math. Z.* **228**(4) (1998), 745–750] for convex cocompact Fuchsian representations.

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1. *Introduction*

This paper is devoted to studying relations between Fuchsian representations of a (non-compact) surface through a dynamics tool, namely, thermodynamic formalism (for countable state Markov shifts). Using a symbolic dynamics model associated with these representations, we investigate several closely related and informative geometric and dynamical objects arising from them, such as the critical exponent, the Manhattan curve and Thurston's intersection number. For dynamics, we prove a version of the famous Bowen formula, which characterizes several geometric and dynamical quantities via the (Gurevich) pressure. Moreover, we analyze the phase transition of the pressure function (of weighted geometric potentials) in detail; thus, we have control of the analyticity of the pressure. In geometry, we recover and extend several rigidity results, such as Bishop–Steger entropy rigidity and Thurston's intersection number rigidity, to Riemann surfaces of infinite volume and with cusps.

To put our results in context, we shall start from notation and definitions. Throughout the paper, S denotes a (topological) surface with negative Euler characteristic. Let ρ_1, ρ_2 be two Fuchsian (i.e., discrete and faithful) representations of $G := \pi_1 S$ into $\mathrm{PSL}(2, \mathbb{R})$ where we regard $\mathrm{PSL}(2, \mathbb{R})$ as the space of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$. For short, we denote $\rho_i(G)$ by Γ_i and the Riemann surface of ρ_i for $i = 1, 2$ by $S_i = \Gamma_i \backslash \mathbb{H}$. We write $h_{\mathrm{top}}(S_1)$ and $h_{\mathrm{top}}(S_2)$ for the *topological entropy of the geodesic flow* for S_1 and S_2, respectively. The group G acts diagonally on $\mathbb{H} \times \mathbb{H}$ by $\gamma \cdot (x_1, x_2) = (\rho_1(\gamma)x_1, \rho_2(\gamma)x_2)$, where $(x_1, x_2) \in \mathbb{H} \times \mathbb{H}$ and $\gamma \in G$. We are interested in *weighted Manhattan metrics* $d^{a,b}_{\rho_1,\rho_2}$ associated with S_1 and S_2; more precisely, in fixing $o = (o_1, o_2) \in \mathbb{H} \times \mathbb{H}$, $d^{a,b}_{\rho_1,\rho_2}(o, \gamma o) := a \cdot d(o_1, \rho_1(\gamma)o_1) + b \cdot d(o_2, \rho_2(\gamma)o_2)$. Moreover, we always assume that $a, b \geq 0$ and a, b do not vanish at the same time: i.e., throughout this paper, we assume that $(a, b) \in D := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \backslash (0, 0)$.

Definition 1.1. The Poincaré series of the weighted Manhattan metric $d^{a,b}_{\rho_1,\rho_2}$ is defined as

$$Q^{a,b}_{\rho_1,\rho_2}(s) = \sum_{\gamma \in G} \exp(-s \cdot d^{a,b}_{\rho_1,\rho_2}(o, \gamma o)).$$

Moreover, $\delta^{a,b}_{\rho_1,\rho_2}$ denotes the critical exponent of $Q^{a,b}_{\rho_1,\rho_2}(s)$: i.e., $Q^{a,b}_{\rho_1,\rho_2}(s)$ diverges when $s < \delta^{a,b}_{\rho_1,\rho_2}$ and $Q^{a,b}_{\rho_1,\rho_2}(s)$ converges when $s > \delta^{a,b}_{\rho_1,\rho_2}$. For short, if there is no confusion, we will always drop the subscripts ρ_1, ρ_2.

Notice that the critical exponent $\delta^{a,b}$, by the triangle inequality, is independent on the choice of the reference point $o = (o_1, o_2)$. We remark that when $a = 0$ (or $b = 0$), we are back to the classical critical exponent of $\rho_1(G)$ (or $\rho_2(G)$), and by Sullivan's result we know that $\delta^{1,0} = h_{\mathrm{top}}(S_1)$ and $\delta^{0,1} = h_{\mathrm{top}}(S_2)$.

Definition 1.2. (The Manhattan curve) The Manhattan curve $\mathcal{C} = \mathcal{C}(\rho_1, \rho_2)$ of ρ_1, ρ_2 is the boundary of the set

$$\{(a, b) \in \mathbb{R}^2 : Q^{a,b}_{\rho_1,\rho_2}(1) < \infty\}.$$

Alternatively, $\mathcal{C}$ can be defined as

$$\{(a, b) \in \mathbb{R}^2 : Q^{a,b}_{\rho_1,\rho_2}(s) \text{ has critical exponent } 1\}.$$

In [**Bur93**], using the Patterson–Sullivan argument, Burger proved that for ρ_1 and ρ_2 convex cocompact (i.e., both $\rho_1(G)$ and $\rho_2(G)$ have no parabolic element), one has that $\mathcal{C}$ is C^1. In [**Sha98**], Sharp employed thermodynamic formalism to prove that $\mathcal{C}$ is real analytic. In this work, we are interested in representations that are not convex cocompact.

We mainly work on representations that satisfy the following two geometric conditions, namely, being *boundary-preserving isomorphic* and the *extended Schottky condition* (see Definitions 2.17 and 3.1 for more details). Roughly speaking, an extended Schottky surface is a geometrically finite Riemann surface of infinite volume with cusps, funnels or both ends and whose group of deck transformations is a free group. One example of an extended Schottky surface is the surface with two cusps and two funnels.

From now on, let ρ_1, ρ_2 be two boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition. To simplify the presentation, we leave the precise definition of many dynamical and geometric terms until §2.

Following the work of Dal'bo and Peigné [**DP96**], there exists a symbolic coding of closed geodesics on extended Schottky surfaces. Here we summarize relevant results in [**DP96**].

PROPOSITION. (Proposition 3.6) *There exists a topologically mixing countable state Markov shift* (Σ^+, σ) *and a function* $\tau : \Sigma^+ \to \mathbb{R}^+$ *(respectively,* $\kappa : \Sigma^+ \to \mathbb{R}^+$*) such that all but finitely many closed geodesics on* S_1 *(respectively,* S_2*) are coded by* $\mathrm{Fix}(\Sigma^+)$ *and the fixed points of* σ *and the lengths of these closed geodesics are given by* τ *(respectively,* κ*).*

Because τ and κ are constructed by the geometric potential of the corresponding Bowen-Series map on the boundary of T^1S_1 and T^1S_2, we will continue calling them geometric potentials (see §3 for more details).

The following theorem is our first main result.

THEOREM. (Phase transition and the Bowen formula; Lemma 3.11, Lemma 3.13, Theorem 3.14 and Theorem 4.8). *Let* (Σ^+, σ) *be the countable state Markov shift and let* τ, κ *be the geometric potentials given by the above proposition. We have, for* $a, b \geq 0$,

$$P_\sigma(-t(a\tau + b\kappa)) = \begin{cases} \textit{infinite} & \textit{for } t < \dfrac{1}{2(a+b)}, \\ \textit{real analytic} & \textit{for } t > \dfrac{1}{2(a+b)}. \end{cases}$$

Moreover, the set $\{(a, b) \in D : P_\sigma(-a\tau - b\kappa) = 0\}$ *is a real analytic curve and, for* $(a, b) \in D$*, we have* $P_\sigma(-\delta^{a,b}(a\tau + b\kappa)) = 0$.

Remark.

(1) Recall that for a finite state Markov shift, the (Gurevich) pressure P_σ has no phase transition, that is, the pressure function $t \mapsto P_\sigma(tf)$ is analytic for f a Hölder continuous potential. Whereas, for countable state Markov shifts, Sarig [**Sar99**, **Sar01**] and Mauldin and Urbański [**MU03**] pointed out that, for f a locally Hölder continuous potential, $t \mapsto P_\sigma(tf)$ is not necessarily analytic. Nevertheless, the above theorem gives a precise picture of the pressure function of weighted geometric potentials in the above theorem.

(2) Similar to the Bowen formula for hyperbolic flows over compact metric spaces, we give a geometric interpretation of the solution for the equation $P_\sigma(tf) = 0$ when f is a weighted geometric potential. Namely, the above theorem points out that the critical exponent $\delta^{a,b}$ can be realized by the growth rate of hyperbolic elements (or, equivalently, closed geodesics).

Combining the above results, one concludes that the Manhattan curve $\mathcal{C}(\rho_1, \rho_2)$ possesses the following features.

THEOREM. (Theorem 4.11, Proposition 4.12) *Let* ρ_1, ρ_2 *be two boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition. Then:*

(1) $(h_{\mathrm{top}}(S_1), 0)$ *and* $(0, h_{\mathrm{top}}(S_2))$ *are on* $\mathcal{C}$*;*

(2) $\mathcal{C}(\rho_1, \rho_2)$ *is real analytic;*

(3) *$\mathcal{C}(\rho_1, \rho_2)$ is strictly convex if ρ_1 and ρ_2 are NOT conjugate in* $\mathrm{PSL}(2, \mathbb{R})$*; and*
(4) *$\mathcal{C}(\rho_1, \rho_2)$ is a straight line if and only if ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$.

Furthermore, we have the following rigidity corollaries.

COROLLARY. (Bishop–Steger entropy rigidity; cf. [**BS93**], Corollary 4.14) *We have, for any $o \in \mathbb{H}$,*

$$\delta^{1,1} = \lim_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in G : d(o, \rho_1(\gamma)o) + d(o, \rho_2(\gamma)o) \le T\}.$$

Moreover, $\delta^{1,1} \le (h_{\mathrm{top}}(S_1) \cdot h_{\mathrm{top}}(S_2))/(h_{\mathrm{top}}(S_1) + h_{\mathrm{top}}(S_2))$ and the equality holds if and only if S_1 and S_2 are isometric.

Remark. In Bishop and Steger [**BS93**], their result holds for finite volume Fuchsian representations (i.e., lattices). We extend their result to some infinite volume Fuchsian representations.

Definition 1.3. (Thurston's intersection number) Let S_1 and S_2 be two Riemann surfaces. Thurston's intersection number $\mathrm{I}(S_1, S_2)$ of S_1 and S_2 is given by

$$\mathrm{I}(S_1, S_2) = \lim_{n\to\infty} \frac{l_2[\gamma_n]}{l_1[\gamma_n]},$$

where $\{[\gamma_n]\}_{n=1}^{\infty}$ is a sequence of conjugacy classes for which the associated closed geodesics γ_n become equidistributed on $\Gamma_1\backslash\mathbb{H}$ with respect to area.

COROLLARY. (Thurston rigidity; cf. [**Thu98**], Corollary 4.15) *Let $S_1 = \rho_1(G)\backslash\mathbb{H}$ and $S_2 = \rho_2(G)\backslash\mathbb{H}$. Then $\mathrm{I}(S_1, S_2) \ge h_{\mathrm{top}}(S_1)/h_{\mathrm{top}}(S_2)$ and equality holds if and only if ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$.

The outline of this paper is as follows. In §2, we briefly review the necessary background of thermodynamic formalism (for countable state Markov shifts) and hyperbolic geometry. In §3, we introduce extended Schottky surfaces. Moreover, we study the phase transition of the geodesic flows over them. Section 4 is devoted to the proof of our main results. Using arguments in [**PPS15**], we derive geometric interpretations of the critical exponent $\delta^{a,b}$, and thus we are able to link it with the (symbolic) suspension flow and the Bowen formula.

2. *Preliminaries*

2.1. *Thermodynamic formalism for countable state Markov shifts.*

Let $\mathcal{S}$ be a countable set and let $\mathbb{A} = (t_{ab})_{\mathcal{S}\times\mathcal{S}}$ be a matrix of zeroes and ones indexed by $\mathcal{S} \times \mathcal{S}$.

Definition 2.1. The *one-sided (countable state) Markov shift* $(\Sigma_{\mathbb{A}}^+, \sigma)$ with the set of *alphabet* $\mathcal{S}$ is the set

$$\Sigma_{\mathbb{A}}^+ = \{x = (x_n) \in \mathcal{S}^{\mathbb{N}} : t_{x_n x_{n+1}} = 1 \text{ for every } n \in \mathbb{N}\}$$

coupled with the (left) shift map $\sigma : \Sigma_{\mathbb{A}}^+ \to \Sigma_{\mathbb{A}}^+$, $(\sigma(x))_i = (x)_{i+1}$.

We will always drop the subscript $\mathbb{A}$ of $\Sigma_{\mathbb{A}}^+$ when there is no ambiguity on the adjacency matrix $\mathbb{A}$. Furthermore, we endow Σ^+ with the relative product topology, which is given by the base of *cylinders*

$$[a_0, \ldots, a_{n-1}] := \{x \in \Sigma^+ : a_i = x_i \text{ for } 0 \leq i \leq n-1\}.$$

A *word* on an alphabet $\mathcal{S}$ is an element $(a_0, a_2, \ldots, a_{n-1}) \in \mathcal{S}^n$ ($n \in \mathbb{N}$). The *length* of the word $(a_0, a_2, \ldots, a_{n-1})$ is n. A word is called *admissible* (with respect to an adjacency matrix $\mathbb{A}$) if the cylinder it defines is non-empty.

In the following, we will assume that (Σ^+, σ) is *topologically mixing*: that is, for any $a, b \in \mathcal{S}$, there exists an $N \in \mathbb{N}$ such that $\sigma^{-n}[a] \cap [b]$ is non-empty for all $n > N$. Notice that under the topologically mixing assumption and the big images and preimages (BIP) property below, the thermodynamic formalism for countable state Markov shifts is well studied and very close to the classical thermodynamic formalism for finite state Markov shifts.

The *nth variation* of a function $g : \Sigma^+ \to \mathbb{R}$ is defined by

$$V_n(g) = \sup\{|g(x) - g(y)| : x, y \in \Sigma^+, x_i = y_i \text{ for } i = 1, 2, \ldots, n\}.$$

We say that g has *summable variation* if $\sum_{n=1}^{\infty} V_n(g) < \infty$, and g is *locally Hölder* if there exists $c > 0$ and $\theta \in (0, 1)$ such that $V_n(g) \leq c\theta^n$ for all $n \geq 1$.

Definition 2.2. (Gurevich pressure for Markov shifts) Let $g : \Sigma^+ \to \mathbb{R}$ have summable variation. The *Gurevich pressure* of g is defined by

$$P_\sigma(g) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{Fix}^n} e^{S_n g(x)} \chi_{[a]}(x),$$

where $\mathrm{Fix}^n := \{x \in \Sigma^+ : \sigma^n x = x\}$, a is any element of $\mathcal{S}$ and $S_n g(x) := \sum_{i=0}^{n-1} g(\sigma^i x)$.

It was pointed out by Sarig (cf. [**Sar99**, Theorem 1]) that the limit exists, and the limit is independent of the choice of $a \in \mathcal{S}$.

THEOREM 2.3. (Variational principle; [**Sar99**, Theorem 3]) *Let* (Σ^+, σ) *be a topologically mixing countable state Markov shift and let* g *have summable variation. If* $\sup g < \infty$, *then*

$$P_\sigma(g) = \sup\left\{h_\sigma(\mu) + \int_{\Sigma^+} g \, d\mu : \mu \in \mathcal{M}_\sigma \text{ and } -\int_{\Sigma^+} g \, d\mu < \infty\right\},$$

where $\mathcal{M}_\sigma$ *is the set of* σ*−invariant Borel probability measures on* Σ^+.

For $\mu \in \mathcal{M}_\sigma$ such that $P_\sigma(g) = h_\sigma(\mu) + \int_{\Sigma^+} g \, d\mu$, we call such a measure μ an *equilibrium state* for the function g.

Definition 2.4. (BIP) A (countable state) Markov shift $(\Sigma_{\mathbb{A}}^+, \sigma)$ has the BIP property if and only if there exists $\{b_1, b_2, \ldots, b_n\} \subset \mathbb{N}$ such that, for every $a \in \mathbb{N}$, there exists $i, j \in \mathbb{N}$ with $t_{b_i a} t_{a b_j} = 1$.

The following theorem about the analyticity of pressure is found independently by Mauldin and Urbański [**MU03**] and Sarig [**Sar03**]. There are minor differences between their original statements; however, under the topologically mixing and the BIP assumptions their results are the same (see Remark 2.6 for more details).

THEOREM 2.5. (Analyticity of pressure; [**MU03**, Theorems 2.6.12, 2.6.13], [**Sar03**, Corollary 4]) *Let* (Σ^+, σ) *be a topologically mixing countable state Markov shift with the BIP property. If* $\Delta \subset \mathbb{R}$ *is an interval and* $t \to f_t$ *is a real analytic family of locally Hölder continuous functions with* $P_\sigma(f_t) < \infty$*, then* $t \to P_\sigma(f_t) \in \mathbb{R}$*, for* $t \in \Delta$*, is also real analytic. Moreover, the derivative of the pressure function is*

$$\frac{d}{dt} P_\sigma(f_t)\bigg|_{t=0} = \int_{\Sigma^+} f_0 \, \mathrm{d}\mu_{f_0},$$

where μ_{f_0} *is the equilibrium state for* f_0*.*

Remark 2.6.

(1) We combine [**MU03**, Proposition 2.1.9 and Theorem 2.6.12] in the following way to derive Theorem 2.5. By Proposition 2.1.9, we know that $P_\sigma(f_t) < \infty$ implies that f_t are summable Hölder functions (i.e., $f_t \in \mathcal{K}^s_\beta$ in [**MU03**] notation). The rest is a direct consequence of Theorem 2.6.12.

(2) A topologically mixing countable state Markov shift (Σ^+, σ) with the BIP property is indeed a graph directed Markov system with a *finitely irreducible* adjacency matrix defined in [**MU03**]. Hence the definition of (Gurevich) pressure given here (from Sarig [**Sar99**]) matches with the one given in Mauldin and Urbański [**MU03**] (cf. [**MU01**, §7]).

(3) For [**Sar03**, Corollary 4], f_t is required to be *positive recurrent*. However, under the same assumptions as in Theorem 2.5 (i.e., (Σ^+, σ) is topologically mixing with the BIP property and f_t are functions of summable variation with $P_\sigma(f_t) < \infty$), then one can prove that f_t are positive recurrent (cf. [**Sar03**, Corollary 2] or [**Sar09**, Proposition 3.8]).

THEOREM 2.7. (Phase transition; [**Sar99**, **Sar01**, **MU03**]) *Let* (Σ^+, σ) *be a countable state Markov shift with the BIP property and let* $g : \Sigma^+ \to \mathbb{R}$ *be a positive locally Hölder continuous function. Then there exists* $s_\infty > 0$ *such that the pressure function* $t \to P_\sigma(-tg)$ *has the properties*

$$P_\sigma(-tg) = \begin{cases} \infty & \textit{if } t < s_\infty, \\ \textit{real analytic} & \textit{if } t > s_\infty. \end{cases}$$

Moreover, if $t > s_\infty$*, there exists a unique equilibrium state for* $-tg$*.*

Recall that two functions $f, g : \Sigma^+ \to \mathbb{R}$ are said to be *cohomologous*, denoted by $f \sim g$, via a *transfer function* h, if $f = g + h - h \circ \sigma$. A function that is cohomologous to zero is called a *coboundary*.

THEOREM 2.8. (Livšic theorem; [**Sar09**, Theorem 1.1]) *Suppose* (Σ^+, σ) *is topologically mixing and that* $f, g : \Sigma^+ \to \mathbb{R}$ *have summable variation. Then* f *and* g *are cohomologous if and only if, for all* $x \in \Sigma^+$ *and* $n \in \mathbb{N}$ *such that* $\sigma^n(x) = x$*,* $S_n f(x) = S_n g(x)$*.*

2.2. *Thermodynamic formalism for suspension flows.* Let (Σ^+, σ) be a topologically mixing (countable state) Markov shift and let $\tau : \Sigma^+ \to \mathbb{R}^+$ be a positive function of summable variation and bounded away from zero, which we call the *roof function*. We define the *suspension space* (relative to τ) as

$$\Sigma_\tau^+ := \{(x, t) \in \Sigma^+ \times \mathbb{R} : 0 \le t \le \tau(x)\},$$

with the identification $(x, \tau(x)) = (\sigma x, 0)$.

The *suspension flow* ϕ (relative to τ) is defined as the (vertical) translation flow on Σ_τ^+ given by

$$\phi_t(x, s) = (x, s + t) \quad \text{for } 0 \le s + t \le \tau(x).$$

Let $F : \Sigma_\tau^+ \to \mathbb{R}$ be a continuous function. We define $\Delta_F : \Sigma^+ \to \mathbb{R}$ as

$$\Delta_F(x) = \int_0^{\tau(x)} F(x, t)\, \mathrm{d}t.$$

The following version of the Gurevich pressure for suspension flows is given in Kempton [**Kem11**].

Definition 2.9. (Gurevich pressure for suspension flows) Suppose $F : \Sigma_\tau^+ \to \mathbb{R}$ is a function such that $\Delta_F : \Sigma^+ \to \mathbb{R}$ has summable variation. The *Gurevich pressure* of F over the suspension flow (Σ_τ^+, ϕ) is defined as

$$P_\phi(F) := \lim_{T\to\infty} \frac{1}{T} \log\bigg(\sum_{\substack{\phi_s(x,0)=(x,0)\\ 0\le s\le T}} \exp\bigg(\int_0^s F(\phi_t(x, 0))\, \mathrm{d}t \bigg) \chi_{[a]}(x) \bigg),$$

where a is any element of $\mathcal{S}$.

Notice that, as pointed out by Kempton (cf. [**Kem11**, Lemma 3.3]), this definition is independent of the choice of $a \in \mathcal{S}$. Moreover, there are several alternative ways of defining the Gurevich pressure for suspension flows, such as using the variational principle. In the following, we summarize some of these from works of Savchenko [**Sav98**], Barreira and Iommi [**BI06**], Kempton [**Kem11**], and Jaerisch, Kesseböhmer and Lamei [**JKL14**].

THEOREM 2.10. (Characterizations for the Gurevich pressure) *Under the same assumptions as in Definition 2.9,*

$$\begin{aligned} P_\phi(F) &= \inf\{t \in \mathbb{R} : P_\sigma(\Delta_F - t\tau) \le 0\} \\ &= \sup\{t \in \mathbb{R} : P_\sigma(\Delta_F - t\tau) \ge 0\} \\ &= \sup\bigg\{ h_\phi(\nu) + \int_{\Sigma_\tau^+} F\, \mathrm{d}\nu : \nu \in \mathcal{M}_\phi \textit{ and } -\int_{\Sigma_\tau^+} \tau\, \mathrm{d}\nu < \infty \bigg\}, \end{aligned}$$

where $\mathcal{M}_\phi$ is the set of ϕ-invariant Borel probability measures on Σ_τ^+.

As before, we call a measure $\nu \in \mathcal{M}_\phi$ an *equilibrium state* for F if $P_\phi(F) = h_\phi(\nu) + \int F\, \mathrm{d}\nu$.

2.3. *Hyperbolic surfaces.* Let S be a surface with negative Euler characteristic. Recall that a *Fuchsian representation* ρ is a discrete and faithful representation from $G := \pi_1 S$ to $\rho(G) := \Gamma \leq \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}(\mathbb{H})$. It is well known that all hyperbolic surfaces (i.e., surfaces with constant Gaussian curvature -1) can be realized by a Fuchsian representation, and *vice versa*. A Fuchsian representation is called *geometrically finite* if there exists a fundamental domain that is a finite-sided convex polygon. Recall that $\partial_\infty \mathbb{H}$, the boundary of $\mathbb{H}$, is defined as $\mathbb{R} \cup \{\infty\}$, and the *limit set* $\Lambda(\Gamma) \subset \partial_\infty \mathbb{H}$ of Γ is the set of limit points of all Γ-orbits $\Gamma \cdot o$ for $o \in \mathbb{H}$. We call an element $\gamma \in \Gamma$ *hyperbolic* (respectively, *parabolic*), if γ has exactly two (respectively, one) fixed points on $\partial_\infty \mathbb{H}$. For a hyperbolic element γ, we denote the *attracting fixed point* by γ^+ (i.e., $\gamma^+ = \lim_{n\to\infty} \gamma^n o$) and the *repelling fixed point* by γ^- (i.e., $\gamma^- = \lim_{n\to\infty} \gamma^{-n} o$). For each hyperbolic element $\gamma \in \Gamma$, the geodesic on $\mathbb{H}$ connecting γ^- and γ^+ projects to a closed geodesic on $\Gamma\backslash\mathbb{H}$. We denote this closed geodesic on $\Gamma\backslash\mathbb{H}$ by λ_γ. Conversely, each closed geodesic λ on $\Gamma\backslash\mathbb{H}$ corresponds to a unique hyperbolic element (up to conjugation) that is denoted by γ_λ. Moreover, the length $l[\lambda_\gamma]$ of the closed geodesic λ_γ is exactly the translation distance $l[\gamma]$ of γ, where $l[\gamma] := \min\{d(x, \gamma x) : x \in \mathbb{H}\}$.

Definition 2.11. The *Busemann function* $B : \partial_\infty \mathbb{H} \times \mathbb{H} \times \mathbb{H}$ is defined as

$$B_\xi(x, y) := \lim_{z\to\xi} d(x, z) - d(y, z),$$

where $\xi \in \partial_\infty \mathbb{H}$ and $x, y, z \in \mathbb{H}$.

We summarize several well-known properties of the Busemann function.

PROPOSITION 2.12. *Let $B : \partial_\infty \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be the Busemann function. Then, for $\xi \in \partial_\infty \mathbb{H}$ and $x, y, z \in \mathbb{H}$,*

(1) $B_\xi(x, y) + B_\xi(y, z) = B_\xi(x, z)$;
(2) *For* $\gamma \in \mathrm{PSL}(2, \mathbb{R})$, $B_{\gamma(\xi)}(\gamma(x), \gamma(y)) = B_\xi(x, y)$; *and*
(3) $B_\xi(x, y) \leq d(x, y)$.

Remark 2.13.

(1) Equivalently, using the Poincaré disk model, we can replace $\mathbb{H}$ by the unit disk $\mathbb{D}$ (through the map $\Psi : \mathbb{H} \to \mathbb{D}$, where $\Psi(z) = i(z - i)/(z + i)$). We have $\mathrm{Isom}(\mathbb{D}) \cong \mathrm{Isom}(\mathbb{H}) \cong \mathrm{PSL}(2, \mathbb{R})$. In this paper, we will alternate the use of $\mathbb{H}$ and $\mathbb{D}$ depending on the convenience of computation and presentation.
(2) In the Poincaré disk model, $\partial_\infty \mathbb{D}$ is S^1 and the Busemann function $B : \partial_\infty \mathbb{D}^1 \times \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ satisfies the properties stated above.
(3) There is a neat formula for the Busemann function: for $\xi \in \partial_\infty \mathbb{D}$,

$$|\gamma'(\xi)| = e^{B_\xi(o, \gamma^{-1} o)},$$

where $\gamma(z) : \mathbb{D} \to \mathbb{D}$ is the Möbius map associated with $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ and o is the origin.

2.3.1. *Marked length spectrum.* As mentioned in the previous subsection, for a hyperbolic surface $R = \Gamma\backslash\mathbb{H}$, there exists a bijection between free homotopy classes on R and conjugacy classes of Γ. Moreover, we have a bijection between closed geodesics on R and conjugacy classes of hyperbolic elements of Γ.

Definition 2.14. A *marked length spectrum* function $l : [c] \mapsto l[c] \in \mathbb{R}^+$ assigns to a homotopy class $[c]$ the length $l[c]$. In other words, it is also the function $l : [h] \mapsto l[h]$ that assigns to a conjugacy class of a hyperbolic element $[h]$ the length $l[h]$ of the corresponding unique closed geodesic.

The following theorem shows that, for each Fuchsian representation, its proportional marked length spectrum determines the surface. We remark that, for convex cocompact cases, the same result was stated (without a proof) in Burger [**Bur93**]. For general Fuchsian representations, we found it in [**Kim01**].

THEOREM 2.15. (Proportional marked length spectrum rigidity [**Kim01**, Theorem A]) *Let $\rho_1, \rho_2 : G \to \mathrm{PSL}(2, \mathbb{R})$ be Zariski dense Fuchsian representations having the proportional marked length spectrum (i.e., there exists a constant $c > 0$ such that $l[\rho_1(\gamma)] = c \cdot l[\rho_2(\gamma)]$ for all $\gamma \in G$). Then ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$.

Remark 2.16.
(1) A representation $\rho : G \to \mathrm{PSL}(2, \mathbb{R})$ is called *Zariski dense* if it is irreducible and non-parabolic, where non-parabolic means that $\rho(G)$ has no global fixed point on the boundary of $\mathbb{H}$. It is clear that Fuchsian representations satisfying the extended Schottky condition (see §3) are Zariski dense.
(2) Kim's result is much more general than the version stated above. However, this version is sufficient for us. We expect that the stated version was known before the work of Kim but we have been unable to find a reference.

2.3.2. *Boundary-preserving isomorphic representations.*

Definition 2.17. Let ρ_1 and ρ_2 be two geometrically finite Fuchsian representations from $G (= \pi_1 S)$ into $\mathrm{PSL}(2, \mathbb{R})$. We say that ρ_1 and ρ_2 are *boundary-preserving isomorphic* if there exists an isomorphism $\iota : \rho_1(G) \to \rho_2(G)$ such that:
(1) ι is *type-preserving*, i.e., ι sends hyperbolic elements to hyperbolic elements and parabolic elements to parabolic elements; and
(2) ι is *peripheral-structure-preserving*, i.e., $\gamma \in \rho_1(G)$ corresponds to a geodesic boundary of S_1 if and only if $\iota(\gamma) \in \rho_2(G)$ corresponds to a geodesic boundary of S_2.

Remark 2.18. For ρ_1 and ρ_2 being two convex cocompact Fuchsian representations, ρ_1 and ρ_2 are always type-preserving isomorphic (because they have no parabolic element). However, it does not guarantee that S_1 and S_2 are homemorphic. For example, a one-holed torus is not homeomorphic to a pair of pants. Therefore, the peripheral-structure-preserving condition is necessary to derive a homeomorphism between S_1 and S_2.

THEOREM 2.19. (Fenchel–Nielsen isomorphism theorem, cf. [**Kap09**, Theorem 5.4], [**Mas88**, Theorem V.H.1]) *Let ρ_1 and ρ_2 be two geometrically finite Fuchsian representations and let $S_1 = \rho_1(G)\backslash\mathbb{H}$ and $S_2 = \rho_2(G)\backslash\mathbb{H}$. Suppose there is a boundary-preserving isomorphism $\iota : \rho_1(G) \to \rho_2(G)$. Then there exists an ι-equivariant bilipschitz homeomorphism $f : S_1 \to S_2$.*

We then lift f, given by the above theorem, to the associated universal coverings, and thus we derive an ι-equivariant bilipschitz homeomorphism between universal coverings (both are $\mathbb{H}$). By abusing the notation, we still denote this homeomorphism by $f : \mathbb{H} \to \mathbb{H}$. More precisely, there exists a constant $C > 0$ such that, for $x, y \in \mathbb{H}$,

$$\frac{1}{C} d(x, y) \leq d(f(x), f(x)) \leq C d(x, y).$$

Remark 2.20.

(1) In [**Kap09**, Theorem 5.4], the ι-equivariant homeomorphism $f : S_1 \to S_2$ is stated to be quasiconformal. Nevertheless, it is well known (cf. Mori's theorem, [**Ahl06**, p. 30]) that quasiconformal homeomorphisms are bilipschitz maps.

(2) Tukia's isomorphism theorem (cf. [**Tuk85**, Theorem 3.3]) points out that the boundaries of these two Fuchsian groups are also strongly related. More precisely, there exists an ι-equivariant Hölder continuous homeomorphism $q : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$.

3. *Extended Schottky surfaces*

In this section, following the notation in Dal'Bo and Peigné [**DP96**], we will mostly use the Poincaré disk model $\mathbb{D}$. Nevertheless, one can easily convert it to the upper-half plane model $\mathbb{H}$. Let us fix two integers N_1 and N_2 such that $N_1 + N_2 \geq 2$ and $N_2 \geq 1$ and consider N_1 hyperbolic isometries $h_1, \ldots, h_{N_1}$ and N_2 parabolic isometries $p_1, \ldots, p_{N_2}$ that satisfy the following conditions.

(C1) For $1 \leq i \leq N_1$, there exists in $\partial_\infty \mathbb{D} = S^1$ a compact neighborhood C_{h_i} of the attracting fixed point h_i^+ of h_i and a compact neighborhood $C_{h_i^{-1}}$ of the repelling fixed point h_i^- of h_i such that

$$h_i(S^1 \backslash C_{h_i^{-1}}) \subset C_{h_i}.$$

(C2) For $1 \leq i \leq N_2$, there exists in S^1 a compact neighborhood C_{p_i} of the unique fixed point $p_i^{\pm}$ of p_i such that, for all $n \in \mathbb{Z}^* := \mathbb{Z} \backslash \{o\}$,

$$p_i^n(S^1 \backslash C_{p_i}) \subset C_{p_i}.$$

(C3) The $2N_1 + N_2$ neighborhoods introduced in (C1) and (C2) are pairwise disjoint.

The group $\Gamma = \langle h_1, \ldots, h_{N_1}, p_1, \ldots, p_{N_2} \rangle \leq \mathrm{Isom}(\mathbb{D}) \cong \mathrm{PSL}(2, \mathbb{R})$ is proved (cf. [**DP96**]) to be a non-elementary free group that acts properly discontinuously and freely on $\mathbb{D}$.

Definition 3.1. We call $\Gamma = \langle h_1, \ldots, h_{N_1}, p_1, \ldots, p_{N_2} \rangle$ an *extended Schottky group* if it satisfies conditions (C1), (C2), (C3) and $N_1 + N_2 \geq 3$. Moreover, if Γ is an extended Schottky group and R is the hyperbolic surface $\Gamma \backslash \mathbb{D}$, then we say that the corresponding Fuchsian representation ρ (i.e., $\rho : \pi_1 R \to \mathrm{PSL}(2, \mathbb{R})$ such that $\rho(\pi_1 R) = \Gamma$) satisfies *the extended Schottky condition*. See Figure 1 for an example.

Remark 3.2.

(1) If $N_2 = 0$, the group Γ is a (classical) Schottky group which is known to be convex cocompact.

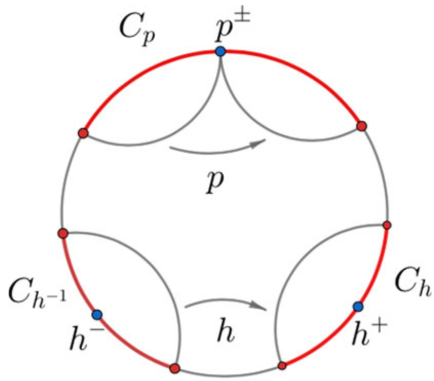

FIGURE 1. An example of extended Schottky groups.

(2) Hyperbolic surfaces satisfying (C1), (C2) and (C3) are geometrically finite with infinite volume.
(3) For a hyperbolic surface satisfying (C1), (C2) and (C3), by the computation in the proof of Lemma 3.10, the elementary parabolic groups $\langle p_i \rangle$ for $1 \leq i \leq N_2$ are of divergent type.
(4) The definition of extended Schottky condition here (for hyperbolic surfaces) is extracted from a more general definition for manifolds with pinched negative curvatures (cf. [**DP96**, **DP98**]).

Let $\mathcal{A}^{\pm} = \{h_1^{\pm 1}, \ldots, h_{N_1}^{\pm 1}, p_1, \ldots, p_{N_2}\}$. For $a \in \mathcal{A}^{\pm}$, denote by U_a the convex hull in $\mathbb{D} \cup \partial_\infty \mathbb{D}$ of the set C_a. For extended Schottky surfaces, we have the following important and very useful lemma.

LEMMA 3.3. *Let Γ be an extended Schottky group. Fix $o \in \mathbb{D}$. Then there exists a universal constant $C > 0$ (depending only on the generators of Γ and the fixed point o) such that, for every $a_1, a_2 \in \mathcal{A}^{\pm}$ satisfying $a_1 \neq a_2^{\pm 1}$, and for every $x \in U_{a_1}$ and $y \in U_{a_2}$, one has*

$$d(x, y) \geq d(x, o) + d(y, o) - C.$$

Remark 3.4. The above lemma is well known. The version that we stated is taken from [**IRV16**, Lemma 4.4].

3.1. *Coding of closed geodesics.* In this subsection, we plan to present a coding of closed geodesics on extended Schottky surfaces. This symbolic coding is given in Dal'Bo and Peigné [**DP96**] (the case of $\mathcal{P} = \emptyset$ in their notation).

Throughout this subsection, let S be a surface with negative Euler characteristic and let ρ_1 and ρ_2 be two boundary-preserving isomorphic Fuchsian representations, from $G = \pi_1 S$ into $\mathrm{PSL}(2, \mathbb{R})$, satisfying the extended Schottky condition. For $i = 1,2$, we write $\Gamma_i = \rho_i(G)$, $S_i = \Gamma_i \backslash \mathbb{D}$, and we let $\Lambda(\Gamma_i)$ denote the limit set of Γ_i.

Since ρ_1 and ρ_2 are boundary-preserving isomorphic and satisfy the extended Schottky condition, we write $G = \langle h_1, h_2, \ldots, h_{N_1}, p_1, p_2, \ldots, p_{N_2} \rangle$, where h_j (respectively, p_k) is called hyperbolic (respectively, parabolic) and corresponds to a hyperbolic (respectively, parabolic) element $\rho_i(h_j)$ (respectively $\rho_i(p_k)$). We denote the set of generators by $\mathcal{A} = \{h_1, h_2, \ldots, h_{N_1}, p_1, p_2, \ldots, p_{N_2}\}$.

We first work on one fixed extended Schottky surface, say, S_1. In the following, we recall definitions and summarize several useful propositions from [**DP96**] about the coding of the geodesics on S_1.

Definition 3.5.

(1) Let $\mathcal{A} = \{h_1, h_2, \ldots, h_{N_1}, p_1, p_2, \ldots, p_{N_2}\}$. The countable state Markov shift (Σ^+, σ) associated with S_1 is defined as

$$\Sigma^+ = \{x = (a_i^{n_i})_{i\geq 1} : a_i \in \mathcal{A},\, n_i \in \mathbb{Z}^*, \text{ and } a_i \neq a_{i+1}^{\pm}\} \quad \text{where } \mathbb{Z}^* = \mathbb{Z}\backslash\{0\},$$

and the shift map $\sigma(a_1^{n_1}a_2^{n_2}a_3^{n_3}\ldots) = a_2^{n_2}a_3^{n_3}\ldots$.

(2) Λ_1^0 is a subset of $\Lambda(\Gamma_1)$ defined as

$$\Lambda_1^0 = \Lambda(\Gamma_1)\backslash\{\Gamma_1\xi : \xi \text{ is a fixed point of } \rho_1(\alpha) \text{ for } \alpha \in \mathcal{A}\}.$$

(3) $\mathcal{G}_{S_1}$ is the set of all closed geodesics on S_1 except those corresponding to hyperbolic elements in $\mathcal{A}$.

PROPOSITION 3.6. (Coding property and the geometric potential)

(1) [**DP96**, p. 759] *There exists a bijection* $\omega_1 : \Lambda_1^0 \to \Sigma^+$.

(2) [**DP96**, p. 760] *The Bowen-Series map* $T : \Lambda_1^0 \to \Lambda_1^0$ *is given by* $T(\xi) = \omega_1^{-1}(\sigma(\omega_1(\xi)))$ *for* $\xi \in \Lambda_1^0$.

(3) [**DP96**, Lemma II.1] *There exists a bijection (up to cyclic permutations)* $\mathcal{H} : \mathcal{G}_{S_1} \to \mathrm{Fix}(\Sigma^+)$*, where* $\mathrm{Fix}(\Sigma^+) = \cup_n \mathrm{Fix}^n(\Sigma^+)$ *is the set of fixed points of* σ.

(4) [**DP96**, p. 759] *Let* $\tau : \Sigma^+ \to \mathbb{R}$ *be the geometric potential (relative to* T*), that is,*

$$\tau(x) := -\log|T'(\omega_1^{-1}(x))| = B_{\omega_1^{-1}(x)}(o, \rho_1(a_1^{n_1})o), \text{ where } x = a_1^{n_1}a_2^{n_2}\ldots \in \Sigma^+.$$

Suppose $\gamma \in \Gamma_1$ *is a hyperbolic element and* $\omega_1(\gamma^+) = \overline{a_1^{n_1}\ldots a_k^{n_k}} \in \mathrm{Fix}^k(\Sigma^+)$. *Then*

$$l_1[\gamma] = S_k(\tau(\omega_1(\gamma^+))).$$

(5) [**DP96**, Lemma II.4] *There exist* $K, C > 0$ *such that* $S_n\tau(x) \geq C$ *for all* $n > K$ *and* $x \in \Sigma^+$.

(6) [**DP96**, Lemma V.2,V.5] τ *is locally Hölder continuous.*

Furthermore, the countable state Markov shift (Σ^+, σ) derived above satisfies the following two favorable conditions.

PROPOSITION 3.7. (Properties of the Markov shift) *Let* (Σ^+, σ) *be the countable state Markov shift associated to* S_1. *Then:*

(1) *the Markov shift* (Σ^+, σ) *satisfies the BIP property; and*

(2) *if* $N_1 + N_2 \geq 3$, *then* (Σ^+, σ) *is topologically mixing.*

Proof. Taking the finite set to be $\mathcal{A} = \{h_1, h_2, \ldots, h_{N_1}, p_1, p_2, \ldots, p_{N_2}\}$, it is clear that (Σ^+, σ) satisfies the BIP property (see Definition 2.4). The topologically mixing property for Markov shifts is a combinatorics condition.

CLAIM. *For every* $x, y \in \{a_i^m : a_i \in \mathcal{A}, m \in \mathbb{Z}\}$, *there exists* $N = N(x, y) \in \mathbb{N}$ *such that, for all* $k > N$, *there is an admissible word of length* k *of the form* $x a_2^{n_2} a_3^{n_3} \ldots a_{k-1}^{n_{k-1}} y$ *for some* $n_i \in \mathbb{Z}^*$ *and* $i = 2, \ldots, k-1$.

Proof. Recall that $\Sigma^+ = \{x = (a_i^{n_i})_{i \geq 1} : a_i \in \mathcal{A},\ n_i \in \mathbb{Z}^*,\ \text{and } a_i \neq a_{i+1}^{\pm}\}$. Since $N_1 + N_2 \geq 3$, we have at least three distinct elements in $\mathcal{A}$, say, a_1, a_2, a_3. Pick two elements x, y in $\{a_i^m : a_i \in \mathcal{A}, m \in \mathbb{Z}\}$ without loss of generality, say, $x = a_1^{m_1}$ and $y = a_2^{m_2}$.

For $k = 2t + 2$ for any $t \in \mathbb{N}$, the following word is admissible: i.e.,

$$a_1^{m_1} \underbrace{(a_2 a_3) \ldots (a_2 a_3)}_{t \text{ pairs}} a_2^{m_2}.$$

For $k = 2t + 3$ for any $t \in \mathbb{N}$, the following word is admissible: i.e.,

$$a_1^{m_1} \underbrace{(a_2 a_3) \ldots (a_2 a_3)}_{t \text{ pairs}} a_1 a_2^{m_2}.$$

We have completed the proof of the claim. □

Using a standard argument in symbolic dynamics, we observe the following handy lemma for the geometric potential τ.

LEMMA 3.8. *There exists a locally Hölder continuous function* τ' *such that* $\tau \sim \tau'$ *and* τ' *is bounded away from zero.*

Proof. By the above proposition, we know that there exist $K, C > 0$ such that $\tau + \tau \circ \sigma + \cdots + \tau \circ \sigma^m \geq C$ for all $m > K$. Let $\lambda = 1/K$ and consider $h'(x) = \sum_{n=0}^{K-1} a_n \cdot \tau \circ \sigma^n(x)$, where $a_n = 1 - n\lambda$. Notice that $a_0 = 1$, $a_{K-1} = \lambda$ and $a_K = 0$. Moreover, we have $a_n - a_{n-1} = -\lambda$ for $n = 1, 2, \ldots, K$. Therefore,

$$\begin{aligned} h'(x) - h(\sigma x) &= \sum_{n=0}^{K-1} a_n \cdot \tau \circ \sigma^n(x) - \sum_{n=0}^{K-1} a_n \cdot \tau \circ \sigma^{n+1}(x) \\ &= a_0 \cdot \tau(x) - \lambda \cdot (\tau \circ \sigma x + \tau \circ \sigma^2 x + \cdots + \tau \circ \sigma^{K-1} x) - a_{K-1} \tau \circ \sigma^K(x) \\ &= \tau(x) - \lambda \sum_{n=1}^{K} \tau \circ \sigma^n x. \end{aligned}$$

Let $\tau'(x) := \lambda \sum_{n=1}^{K} \tau \circ \sigma^n x$. It is clear that $\tau'(x)$ is locally Hölder; moreover, we have

$$\tau'(x) = \lambda \sum_{n=1}^{K} \tau \circ \sigma^n x \geq \frac{C}{K} > 0.$$ □

Notice that the coding above is completely determined by the type of generators (i.e., hyperbolic or parabolic) in Γ_1. Because Γ_1 and Γ_2 are boundary-preserving isomorphic, repeating the same construction as above for Γ_2, we derive for S_2 the same countable state Markov shift (Σ^+, σ) as for S_1. In other words, the same Proposition 3.6 holds for S_2. More precisely, there exists a bijection $\omega_2 : \Lambda_2^0 \to \Sigma^+$ and the geometric potential $\kappa : \Sigma^+ \to \mathbb{R}$ given by $\kappa(x) := B_{\omega_2^{-1}(x)}(o, \rho_2(a_1^{n_1})o)$ for $x = a_1^{n_1} a_2^{n_2} \ldots \in \Sigma^+$. Furthermore, κ is cohomologus to a locally Hölder continuous function κ' that is bounded away from zero (i.e., Lemma 3.8).

Remark 3.9.

(1) Suppose $\iota : \Gamma_1 \to \Gamma_2$ is a type-preserving isomorphism. Then, by Tukia's isomorphism theorem (cf. Remark 2.20.2), there exists an ι-equivariant homeomorphism $q : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$. One can also prove that, for $\xi \in \Lambda_1^0$, we have $\omega_2(\xi) = \omega_1(q(\xi))$. Moreover, we can write $\kappa(x) = B_{(\omega_1 \circ q)^{-1}(x)}(o, (\iota \circ \rho_1)(a_1^{n_1}) \cdot o)$, where $a_1^{n_1}$ is the first element of $\omega_1^{-1}(x)$.

(2) Notice that since τ and τ' (constructed in Corollary 3.8) are cohomologous, the thermodynamics for τ (respectively, κ) and τ' (respectively, κ') are the same. Therefore, for brevity, we will abuse our notation and continue to denote the function τ' by τ and, similarly, κ' by κ.

3.2. *Phase transition of the geodesic flow.* We continue this subsection with the same notation and assumptions as in the previous subsection. Recall that $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \backslash (0, 0)$. Throughout, let ρ_1 and ρ_2 be two boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition.

LEMMA 3.10. *Suppose $(a, b) \in D$. For any parabolic element $p \in G$ (i.e., $\rho_1(p)$ and $\rho_2(p)$ are parabolic), we have $\delta^{a,b}_{\langle p \rangle} = \inf\{t \in \mathbb{R} : Q^{a,b}_{\langle p \rangle}(t) < \infty\} = 1/2(a + b)$, where $Q^{a,b}_{\langle p \rangle}(t) = \sum_{n \in \mathbb{Z}} e^{-t(d^{a,b}(o, p^n o))}$. For $h \in \Gamma$ hyperbolic (i.e., $\rho_1(h)$ and $\rho_2(h)$ are hyperbolic), then $\delta^{a,b}_{\langle h \rangle} = 0$.*

Proof. Let $p \in G$ be a parabolic element. Without loss of generality, we can assume $\rho_i(p) : \mathbb{H} \to \mathbb{H}$ to be the Möbius transformation $\rho_i(p)(z) = z + c_i$ for $i = 1, 2$, where $c_i \in \mathbb{R}$. Then direct computation shows that

$$d(i, \rho_i(p^n)(i)) = d(i, i + nc_i) = \log \frac{\sqrt{(nc_i)^2 + 4} + |nc_i|}{\sqrt{(nc_i)^2 + 4} - |nc_i|}.$$

Notice that

$$\frac{\sqrt{(nc_i)^2 + 4} + |nc_i|}{\sqrt{(nc_i)^2 + 4} - |nc_i|} = \frac{2n^2 c_i^2 + 4 + 2|nc_i|\sqrt{(nc_i)^2 + 4}}{4},$$

so when $|n|$ is big enough (say, $|n| > M_p$), there exist m_i and M_i such that

$$2\log|n| + m_i \leq d(i, i + nc_i) \leq 2\log|n| + M_i.$$

Converting the above inequalities to the disk model gives

$$2\log|n| + m_i \leq d(o, p^n o) \leq 2\log|n| + M_i.$$

Therefore,

$$\begin{aligned} Q^{a,b}_{\langle p \rangle}(t) &= \sum_{n \in \mathbb{Z}} e^{-td^{a,b}(o, p^n o)} \\ &= \sum_{|n| \leq M_p} e^{-td^{a,b}(o, p^n o)} + \sum_{|n| > M_p} e^{-td^{a,b}(o, p^n o)}, \end{aligned}$$

where $\sum_{|n|\leq M_p} e^{-t\cdot d^{a,b}(o,p^n o)} < \infty$ is a finite sum. Furthermore, for $|n| > M$,

$$\begin{aligned}-tad(o,\rho_1(p^n)(o)) - tbd(o,\rho_1(p^n)(o)) &\geq -ta(2\log|n| + M_1) - tb(2\log|n| + M_2)\\ &= -t\underbrace{(aM_1 + bM_2)}_{C_1^{a,b}(p)} - 2t(a+b)\log|n|\end{aligned}$$

and

$$\begin{aligned}-tad(o,\rho_1(p^n)(o)) - tbd(o,\rho_1(p^n)(o)) &\leq -ta(2\log|n| + m_1) - tb(2\log|n| + m_2)\\ &= -t\underbrace{(am_1 + bm_2)}_{C_2^{a,b}(p)} - 2t(a+b)\log|n|.\end{aligned}$$

Hence,

$$\begin{aligned}\bigg(\frac{1}{C_1^{a,b}(p)}\bigg)^t \sum_{|n|>M_p}\bigg(\frac{1}{|n|}\bigg)^{2t(a+b)} &\leq \sum_{|n|>M_p} e^{-td^{a,b}(o,p^n o)}\\ &\leq \bigg(\frac{1}{C_2^{a,b}(p)}\bigg)^t \sum_{|n|>M_p}\bigg(\frac{1}{|n|}\bigg)^{2t(a+b)},\end{aligned}$$

and thus $\delta_{\langle p\rangle}^{a,b} = 1/2(a+b)$.

For any hyperbolic element $h \in G$,

$$\begin{aligned}Q_{\langle h\rangle}^{a,b}(t) &= \sum_{n\in\mathbb{Z}} e^{-td^{a,b}(o,h^n o)}\\ &= \sum_{n\in\mathbb{Z}} e^{-tad(o,\rho_1(h^n)o)-tbd(o,\rho_2(h^n)o)}\\ &= 2\sum_{n\in\mathbb{N}} e^{-tanB_{\rho_1(h)^+}(o,\rho_1(h)o)-tnbB_{\rho_2(h)^+}(o,\rho_2(h)o)}\\ &= 2\sum_{n\in\mathbb{N}} e^{-tn(aB_{\rho_1(h)^+}(o,\rho_1(h)o)+bB_{\rho_2(h)^+}(o,\rho_2(h)o))}.\end{aligned}$$

Since $B_{\rho_i(h)^+}(o,\rho_i(h)o) > 0$ for $i = 1, 2$, we get $\delta_{\langle h\rangle}^{a,b} = 0$. □

Recall that the Markov shift (Σ^+, σ) defined above (see Definition 3.5) for ρ_1 and ρ_2 is topologically mixing and satisfies the BIP property. Also, the geometric potentials τ and κ defined above (see Proposition 3.6) are locally Hölder and bounded away from zero. Therefore, we are in the scenario that was introduced in §2. The following result is inspired by Iommi, Riquelme and Velozo [**IRV16**].

LEMMA 3.11. *Let ρ_1 and ρ_2 be two boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition. Let (Σ^+, σ) be the Markov shift and let τ and κ be the geometric potentials defined in the above subsection.*

Then, for $a, b \geq 0$,

$$P_\sigma(-t(a\tau + b\kappa)) = \begin{cases}\textit{infinite} & \textit{for } t < \delta_{\langle p\rangle}^{a,b},\\ \textit{analytic} & \textit{for } t > \delta_{\langle p\rangle}^{a,b}.\end{cases}$$

Proof. By definition,

$$P_\sigma(-t(a\tau+b\kappa)) = \lim_{n\to\infty}\frac{1}{n+1}\log\bigg(\sum_{x\in\mathrm{Fix}^n}\exp(-t(aS_n\tau+bS_n\kappa))\cdot\chi_{[h_1]}\bigg)$$

$$= \lim_{n\to\infty}\frac{1}{n+1}\log\bigg(\sum_{x=\overline{h_1x_2\ldots x_{n+1}}}\exp(-t(aS_n\tau+bS_n\kappa))\bigg).$$

Notice that

$$\mathrm{Fix}^{n+1}(\Sigma^+)$$
$$= \{\overline{a_1^{m_1}a_2^{m_2}\ldots a_{n+1}^{m_1}} : a_i\in\mathcal{A},\ a_i\neq a_{i+1}^{\pm 1} \text{ and } m_i\in\mathbb{Z}^* \text{ for } i=1,2,\ldots,n+1\}.$$

For each $k\in\mathbb{N}$ and set $n+1=k(N_1+N_2-1)$, we consider a subset $B^k\subset\mathrm{Fix}^{n+1}$ defined as

$$B^k = \bigg\{\overline{h_1a_1^{m_1}\ldots a_n^{m_n}}\in\mathrm{Fix}^{n+1} : a_{i+j(N_1+N_2-1)} = \begin{cases} h_{i+1} & \text{for } 1\le i\le N_1-1 \\ p_{i+1-N_1} & \text{for } N_1\le i\le N_1+N_2-1\end{cases}\bigg\}.$$

In other words, elements $b\in B^k$ are in the form

$$b = \overline{h_1\,\underbrace{h_2^{m_1}\ldots h_{N_1}^{m_{N_1-1}}p_1^{m_{N_1}}\ldots p_{N_2}^{m_{N_1+N_2-1}}}\ldots\underbrace{h_2^{m_{(k-1)(N_1+N_2-1)}}\ldots p_{N_2}^{m_{k(N_1+N_2-1)}}}}.$$

For brevity, we denote N_1+N_2-1 by N_3. Then, for $\xi_0\in\Lambda_1^0$,

$$P_\sigma(-t(a\tau+b\kappa)) \ge \lim_{k\to\infty}\frac{1}{kN_3}\log\bigg(\sum_{\substack{\xi=\rho_1(x)\xi_0\\ x\in B^k}}\exp(-t(aS_{kN_3}\tau+bS_{kN_3}\kappa))\bigg)$$

$$= \lim_{k\to\infty}\frac{1}{kN_3}\log\bigg(\sum_{\substack{\xi=\rho_1(x)\xi_0\\ x\in B^k}}\exp(f(a,b,t,kN_3))\bigg),$$

where

$$f(a,b,t,n) = -t\bigg(\sum_{i=1}^n aB_{\omega_1^{-1}(\sigma^i x)}(o,\rho_1(x_{i+1})o) + bB_{\omega_2^{-1}(\sigma^i x)}(o,\rho_2(x_{i+1})o)\bigg).$$

Because $B_\xi(x,y)\le d(x,y)$, we have

$$P_\sigma(-t(a\tau+b\kappa))$$

$$\ge \lim_{k\to\infty}\frac{1}{kN_3}\log\sum_{\substack{\xi=\rho_1(x)\xi_0\\ x\in B^k}}\exp\bigg(-t\bigg(\sum_{i=1}^{kN_3} ad(o,\rho_1(x_{i+1})o)+bd(o,\rho_2(x_{i+1})o\bigg)\bigg)$$

$$= \lim_{k\to\infty}\frac{1}{kN_3}\log\bigg(\sum_{\substack{\xi=\rho_1(x)\xi_0\\ x\in B^k}}\exp\bigg(-t\sum_{i=1}^{kN_3} d^{a,b}(o,x_{i+1}o)\bigg)\bigg).$$

Moreover, by the definition of B^k,

$$\sum_{\substack{\xi=\rho_1(x)\xi_0\\ x\in B^k}} \exp\bigg(-t\sum_{i=1}^{kN_3} d^{a,b}(o, x_{i+1}o)\bigg)$$
$$= e^{-td^{a.b}(o,h_1o)}\cdot \sum_{(m_1,\ldots,m_{kN_3})\in(\mathbb{Z}^*)^{kN_3}} \exp\bigg(-t\sum_{i=1}^{kN_3} d^{a,b}(o, a_i^{m_i}o)\bigg).$$

Also, notice that

$$\sum_{(m_1,\ldots,m_{kN_3})\in(\mathbb{Z}^*)^{kN_3}} \exp\bigg(-t\sum_{i=1}^{kN_3} d^{a,b}(o, a_i^{m_i}o)\bigg)$$
$$=\prod_{i=1}^{kN_3}\sum_{m_i\in\mathbb{Z}^*} \exp\bigg(-t\sum_{i=1}^{kN_3} d^{a,b}(o, a_i^{m_i}o)\bigg)$$
$$=\bigg(\prod_{i=2}^{N_1}\sum_{m\in\mathbb{Z}^*} e^{-td^{ab}(o,h_i^m o)}\bigg)^k\bigg(\prod_{i=1}^{N_2}\sum_{m\in\mathbb{Z}^*} e^{-td^{ab}(o,p_i^m o)}\bigg)^k.$$

Hence,

$$P_\sigma(-t(a\tau+b\kappa))$$
$$\geq \lim_{k\to\infty}\frac{1}{kN_3}\log\bigg(e^{-td^{a.b}(o,h_1o)}\bigg(\prod_{i=2}^{N_1}\sum_{m\in\mathbb{Z}^*} e^{-td^{ab}(o,h_i^m o)}\bigg)^k\bigg(\prod_{i=1}^{N_2}\sum_{m\in\mathbb{Z}^*} e^{-td^{ab}(o,p_i^m o)}\bigg)^k\bigg)$$
$$=\frac{1}{N_3}\bigg(\log\bigg(\prod_{i=2}^{N_1}\sum_{m\in\mathbb{Z}^*} e^{-td^{ab}(o,h_i^m o)}\bigg)\bigg(\prod_{i=1}^{N_2}\sum_{m\in\mathbb{Z}^*} e^{-td^{ab}(o,p_i^m o)}\bigg)\bigg)$$
$$=\frac{1}{N_3}\log\bigg(\prod_{g\in\mathcal{A}\setminus h_1}(Q^{a,b}_{\langle g\rangle}(t)-1)\bigg),$$

where $Q^{a,b}_{\langle g\rangle}(t)=\sum_{m\in\mathbb{Z}} e^{-td^{ab}(o,g^m o)} = 1+\sum_{m\in\mathbb{Z}^*} e^{-td^{ab}(o,g^m o)}$.

In the following, we derive an upper bound for $P_\sigma(-t(a\tau+b\kappa))$. Let (ξ_t^i) be the end of the geodesic ray $[o, \omega_1^{-1}(\sigma^{i+1}x))$. Then, by Lemma 3.3,

$$\begin{aligned}\tau(\sigma^i x) &= B_{\omega_1^{-1}(\sigma^i x)}(o, \rho_1(x_i)o)\\ &= B_{\omega_1^{-1}(\sigma^{i+1}x)}(\rho_1^{-1}(x_i)o, o)\\ &= \lim_{t\to\infty} d(\xi_t^i, \rho_1(x_i)o) - d(\xi_t^i, o)\\ &\geq (d(\xi_t^i, o) - d(o, \rho_1(x_i)o) - C_1) - d(\xi_t^i, o)\\ &= d(o, \rho_1(x_i)o) - C_1.\end{aligned}$$

Similarly, we have $\kappa(\sigma^i x)\geq d(o, \rho_2(x_i)o) - C_2$ for some constant C_2. Thus,

$$e^{-t(a\tau(\sigma^i x)+b\kappa(\sigma^i x))}\leq e^{t(aC_1+bC_2)}e^{-t(d^{a,b}(o,x_io))}.$$

Hence,

$$P_\sigma(-ta\tau - tb\kappa) \le \lim_{n\to\infty} \frac{1}{n} \log\bigg(\sum_{a_1,\ldots,a_n} \sum_{m_1,\ldots,m_n\in\mathbb{Z}^*} \prod_{i=1}^{n} e^{t(aC_1+bC_2)} e^{-t(d^{a,b}(o,a_i^{m_i} o))} \bigg)$$
$$= t(aC_1 + bC_2) + \log\bigg(\prod_{g\in\mathcal{A}} (Q^{a,b}_{\langle g\rangle}(t) - 1) \bigg).$$

Then, by Lemma 3.10,

$$P_\sigma(-t(a\tau + b\kappa)) = \begin{cases} \text{infinite} & \text{for } t < \delta^{a,b}_{\langle p\rangle}, \\ \text{finite} & \text{for } t > \delta^{a,b}_{\langle p\rangle}. \end{cases}$$

Finally, by Theorem 2.5, we know that the finiteness of the pressure function implies the analyticity. □

Remark 3.12. When a (or b) is zero, we recover the well known result

$$P_\sigma(-t\tau) = \begin{cases} \infty & \text{for } t \ge \frac{1}{2}, \\ \text{finite} & \text{for } t < \frac{1}{2}. \end{cases}$$

LEMMA 3.13. *For each $(a, b) \in D$, there exists a unique $t_{a,b} \in (1/2(a + b), \infty)$ such that*

$$P_\sigma(-t_{a,b}(a\tau + b\kappa)) = 0.$$

Proof. Let (a, b) be a point in D and let $f(t) = P_\sigma(-t(a\tau + b\kappa))$. It is obvious that $-t(a\tau + b\kappa)$ is a locally Hölder continuous function. By Theorem 2.5, $f(t)$ is real analytic on t when $P_\sigma(-t(a\tau + b\kappa)) < \infty$. Let $K = \{t \in \mathbb{R} : f(t) < \infty\}$. Then, for $t_0 \in K$,

$$\frac{d}{dt} f(t)\bigg|_{t=t_0} = -\int (a\tau + b\kappa)\, d\mu_{-t_0(a\tau+b\kappa)} < -(ac + bc) < 0,$$

where $\tau, \kappa > c > 0$ and $\mu_{-t_0(a\tau+b\kappa)}$ is the equilibrium state of $-t_0(a\tau + b\kappa)$.

Hence, $f(t) = P_\sigma(-t(a\tau + b\kappa))$ is real analytic and strictly decreasing on K. Moreover, we know that $P_\sigma(-t(a\tau + b\kappa)) < 0$ when t is positive and big enough. More precisely, because $\kappa > c > 0$, we know that $P_\sigma(-t(a\tau + b\kappa)) < P_\sigma(-ta\tau) - tbc$. Furthermore, we know that $P_\sigma(-h_{\text{top}}(S_1)\tau) = 0$, so when $ta > h_{\text{top}}(S_1)$, we have $P_\sigma(-ta\tau) < 0$. Therefore, it remains to say that there exists $t'_{a,b} \in (1/2(a + b), \infty)$ such that $0 < P_\sigma(-t'_{a,b}(a\tau + b\kappa)) < \infty$.

Notice that, by the computation made in the proof of Lemma 3.10, for a parabolic element $p \in G$ and for $t > 1/2(a + b)$,

$$Q^{a,b}_{\langle p\rangle}(t) - 1 = -1 + \sum_{|n|\le M_p} e^{-td^{a,b}(o,p^n o)} + \sum_{|n|>M_p} e^{-td^{a,b}(o,p^n o)}$$
$$> \bigg(\frac{1}{C^{a,b}_1(p)}\bigg)^t \sum_{|n|>M_p} \bigg(\frac{1}{|n|}\bigg)^{2t(a+b)}$$
$$> \bigg(\frac{1}{C^{a,b}_1(p)}\bigg)^t \cdot 2\int_{M_p+1}^{\infty} x^{-2t(a+b)}\, dx$$
$$= \bigg(\frac{1}{C^{a,b}_1(p)}\bigg)^t \bigg(\frac{2}{2t(a + b) - 1}\bigg)\bigg(\frac{1}{M_p + 1}\bigg)^{2t(a+b)-1} > 0.$$

Moreover,

$$\log(Q^{a,b}_{\langle p\rangle}(t)-1) > -t\log(C^{a,b}_1(p)) + \log 2 + \log\left(\frac{1}{2t(a+b)-1}\right) + (2t(a+b)-1)\log\left(\frac{1}{M_p+1}\right) > 0, \text{ when } t \text{ is big enough.}$$

Indeed, $\log(1/2t(a+b)-1)\to\infty$ as $t\to(1/2(a+b))^+$ and other terms remain bounded when $t\to(1/2(a+b))^+$.

For any hyperbolic element $h\in G$,

$$Q^{a,b}_{\langle h\rangle}(t)-1 = 2\sum_{n\in\mathbb{N}} e^{-tn\cdot c_{a,b}(h)} = \frac{2}{e^{t\cdot c_{a,b}(h)}-1},$$

where $c_{a,b}(h) = (aB_{\rho_1(h)^+}(o,\rho_1(h)o) + bB_{\rho_2(h)^+}(o,\rho_2(h)o))$, one has

$$\log(Q^{a,b}_{\langle h\rangle}(t)-1) = \log 2 + \log(e^{t\cdot c_{a,b}(h)}-1),$$

which remains bounded when $t\to(1/2(a+b))^+$.

By repeating the argument above for $g\in\mathcal{A}\backslash h_1$ and using the computation in the proof of Lemma 3.11, we can choose $t'_{a,b}\in(1/2(a+b),0)$ such that

$$\infty > P_\sigma(t'_{a,b}(a\tau+b\kappa)) > \frac{1}{N_3}\log\left(\prod_{g\in\mathcal{A}\backslash h_1}(Q^{a,b}_{\langle g\rangle}(t)-1)\right) > 0. \qquad \square$$

THEOREM 3.14. *The set* $\{(a,b)\in D : P_\sigma(-a\tau-b\kappa)=0\}$ *is a real analytic curve.*

Proof. By Lemma 3.13, it makes sense to discuss solutions to $P_\sigma(-a\tau-b\kappa)=0$. Moreover, for $(a,b)\in D$ such that $f(a,b)=P_\sigma(-a\tau-b\kappa)<\infty$, we have that $f(a,b)$ is real analytic on both variables, and

$$\partial_b f(a,b)|_{(a,b)=(a_0,b_0)} = -\int \kappa \,\mathrm{d}\mu_{-a_0\tau-b_0\kappa} < -c,$$

where $\tau,\kappa>c>0$ and $\mu_{-a_0\tau-b_0\kappa}$ is the equilibrium state of $-a_0\tau-b_0\kappa$.

Therefore, by the implicit function theorem, the solutions to $P_\sigma(-a\tau-b\kappa)=0$ in D are real analytic, i.e., $b=b(a)$ is real analytic on a. $\square$

4. *The Manhattan curve*

4.1. *The Manhattan curve, critical exponent and Gurevich pressure.* For any pair of Fuchsian representations ρ_1 and ρ_2, we recall that the Manhattan curve $\mathcal{C}(\rho_1,\rho_2)$ of ρ_1 and ρ_2 is the boundary of the convex set

$$\{(a,b)\in\mathbb{R}^2 : Q^{a,b}_{\rho_1,\rho_2}(s) \text{ has critical exponent } 1\},$$

where $Q^{a,b}_{\rho_1,\rho_2}(s)=\sum_{\gamma\in G}\exp(-s\cdot d^{a,b}_{\rho_1,\rho_2}(o,\gamma o))$ is the Poincaré series of the weighted Manhattan metric $d^{a,b}_{\rho_1,\rho_2}$.

We have a rough picture of the corresponding Manhattan curve $\mathcal{C}(\rho_1,\rho_2)$ for all Fuchsian representations.

THEOREM 4.1. *Let S be a surface with negative Euler characteristic, and let ρ_1 and ρ_2 be two Fuchsian representations of $G = \pi_1 S$ into* $\mathrm{PSL}(2, \mathbb{R})$. *We denote $S_1 = \rho_1(G)\backslash\mathbb{H}$ and $S_2 = \rho_2(G)\backslash\mathbb{H}$. Then:*

(1) $(h_{\mathrm{top}}(S_1), 0)$ *and* $(0, h_{\mathrm{top}}(S_2))$ *are on* $\mathcal{C}(\rho_1, \rho_2)$*;*
(2) $\mathcal{C}(\rho_1, \rho_2)$ *is convex; and*
(3) $\mathcal{C}(\rho_1, \rho_2)$ *is a continuous curve.*

Proof. The first assertion is obvious. The second assertion is because that the domain

$$\{(a, b) : Q^{a,b}_{\rho_1,\rho_2}(1) < \infty\}$$

is convex. To see that it is convex, by the Hölder inequality, for $(a_1, b_1), (a_2, b_2) \in D$,

$$Q^{ta_1+(1-t)b_1, ta_2+(1-t)b_2}(1) \le (Q^{a_1,b_1}(1))^t \cdot (Q^{a_2,b_2}(1))^{1-t}.$$

To see that $\mathcal{C}$ is continuous, we notice that because $\mathcal{C}$ is convex, we know that $\mathcal{C}$ is homeomorphic to the straight line connecting $(h_{\mathrm{top}}(S_1), 0)$ and $(0, h_{\mathrm{top}}(S_2))$. □

In the rest of this subsection, we focus on ρ_1 and ρ_2 being boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition. Considering these representations will give us a much better understanding of the Manhattan curve $\mathcal{C}(\rho_1, \rho_2)$. As pointed out in[**OP04**], the critical exponent for a geometrically finite negatively curved manifold is the (exponential) growth rate of closed geodesics. Similarly, we show that the critical exponent $\delta^{a,b}_{\rho_1,\rho_2}$ is the growth rate of hyperbolic elements (or, equivalently, closed orbits). To reach that, inspired by Paulin, Pollicott and Schapira [**PPS15**], we introduce several related geometric growth rates. Through analyzing these growth rates, we are able to link the dynamical critical exponent $t_{a,b}$ (i.e., the solution to the Bowen formula) with the geometric critical exponent $\delta^{a,b}_{\rho_1,\rho_2}$. As a result, these geometric growth rates give us the full picture of the Manhattan curve $\mathcal{C}(\rho_1, \rho_2)$.

Recall that there is a natural one-to-one correspondence between closed geodesics on S_1 and on S_2 (indexed by the non-trivial conjugacy classes in the fundamental group). If λ is a closed geodesic on S_1, then, by abusing notation, we will also use λ to denote the corresponding closed geodesic. Moreover, we will write $l_i[\gamma]$ for the length of the closed geodesic λ on S_i, $i = 1, 2$.

Definition 4.2. (Geometric growth rates counted from S_1) Let S be a surface with negative Euler characteristic, and let $G := \pi_1 S$. Suppose $\rho_1, \rho_2 : G \to \mathrm{PSL}(2, \mathbb{R})$ are boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition. For $x, y \in \mathbb{H}$ and $\gamma \in G$:

(1) $\overline{Q}^{a,b}_{x,y}(s) := \sum_{\gamma\in G} e^{-d^{a,b}(x,\gamma y)-sd(x,\rho_1(\gamma)y)}$ is called the *Paulin–Pollicott–Schapira (PPS) Poincaré series*, where $d^{a,b}(x, \gamma y) = ad(x, \rho_1(\gamma)y) + bd(x, \rho_2(\gamma)y)$;
(2) $\overline{\delta}^{a,b}$ is the *critical exponent* of $\overline{Q}^{a,b}_{x,y}(s)$, i.e., $\overline{Q}^{a,b}_{x,y}(s)$ converges when $s > \overline{\delta}^{a,b}$ and $\overline{Q}^{a,b}_{x,y}(s)$ diverges when $s < \overline{\delta}^{a,b}$, and is called the *PPS critical exponent;*
(3) $G^{a,b}_{x,y}(s) := \sum_{\gamma\in G; d(x,\rho_1(\gamma)y)\le s} e^{-d^{a,b}(x,\gamma y)}$;
(4) $Z_W(s) := \sum_{\substack{\lambda\cap W\neq\phi \\ \lambda\in \mathrm{Per}_1(s)}} e^{-al_1[\lambda]-bl_2[\lambda]}$, where $W \subset T^1S_1$ is a relatively compact open set and $\mathrm{Per}_1(s) := \{\lambda : \lambda \text{ is a closed orbit on } T^1S_1 \text{ and } l_1[\lambda] \le s\}$; and

(5) $P^{ab}_{\mathrm{Gur}} := \limsup_{s\to\infty} (1/s) \log Z_W(s)$ is the *geometric Gurevich pressure*.

LEMMA 4.3. $\overline{\delta}^{a,b} = P^{ab}_{\mathrm{Gur}} = \lim_{s\to\infty} (1/s) \log G^{a,b}_{x,y}(s) = \lim_{s\to\infty} (1/s) \log Z_W(s)$ *for any relative compact* $W \subset T^1 S_1$.

Proof. This proof follows the (short) proof of [**PPS15**, Corollary 4.2, Corollary 4.5 and Theorem 4.7] (also the proof of [**Pei13**, Theorem 2.4]). The strategy is standard but tedious. We leave the proof to the appendix. □

Furthermore, we show below that the geometric Gurevich pressure P^{ab}_{Gur} matches the symbolic Gurevich pressure (for the suspension flow).

In what follows, (Σ^+, σ) stands for the countable state Markov shift associated with ρ_1 and ρ_2 defined in §3, and $\tau, \kappa : \Sigma^+ \to \mathbb{R}^+$ stand for the corresponding geometric potentials. Recall that (Σ^+, σ) is topologically mixing and satisfies the BIP property, and that τ and κ are locally Hölder continuous functions and bounded away from zero. Let Σ^+_τ be the suspension space relative to τ and let $\phi : \Sigma^+_\tau \to \Sigma^+_\tau$ be the suspension flow.

We consider a function $\psi : \Sigma^+_\tau \to \mathbb{R}^+$ given by $\psi(x, t) := \kappa(x)/\tau(x)$ for $x \in \Sigma^+$, $0 \le t \le \tau(x)$ and $\psi(x, \tau(x)) = \psi(\sigma(x), 0)$. Using this function ψ, we can reparametrize the suspension flow $\phi : \Sigma^+_\tau \to \Sigma^+_\tau$ and derive information about orbits of the geodesic flow over $T^1 S_2$. Roughly speaking, ψ is a reparametrization function, in the symbolic sense, of the geodesic flow over $T^1 S_1$ such that the reparametrized flow is conjugated to the geodesic flow over $T^1 S_2$.

LEMMA 4.4. *Suppose* $\psi : \Sigma^+_\tau \to \mathbb{R}^+$ *is defined as* $\psi(x, t) := \kappa(x)/\tau(x)$ *for* $x \in \Sigma^+$, $0 \le t \le \tau(x)$ *and* $\psi(x, \tau(x)) = \psi(\sigma(x), 0)$. *Then* $P_\phi(-a - b\psi) = P^{ab}_{\mathrm{Gur}}$.

Proof. Notice that since S_1 is geometrically finite, there exists a relatively compact open set W such that W meets every closed orbit on $T^1 S_1$. Therefore, for any $g_0 \in \mathcal{A} = \{h_1, \ldots, h_{N_1}, p_1, \ldots, p_{N_2}\}$,

$$\frac{1}{s} Z_{g_0}(s) \le Z^{a,b}_W(s) \le \sum_{g\in\mathcal{A}} Z_g(s) + C,$$

where $Z_g(T) = \sum_{\substack{\phi_s(x,0)=(x,0),\\ 0\le s\le T}} e^{\int_0^s (-a-b\psi)\circ\phi_t(x,t)\,\mathrm{d}t} \chi_{[g]}(x)$ for $g \in \mathcal{A}$.

The first inequality is because a closed orbit $\phi_t(x, 0) = (x, 0)$, $x = g_0 x_2 x_3 \ldots$, $0 \le t \le s$, of the suspension flow corresponds to at most s closed orbits on $T^1 S_1$. The constant C in the second inequality is from closed geodesics corresponding to the hyperbolic generators h_i (because these closed geodesics are not in our coding).

Recall that, by definition, we have $P_\phi(-a - b\psi) = \lim_{s\to\infty} (1/s) \log Z_{g_0}(s)$, and by Definition 2.9,

$$P_\phi(-a - b\psi) = \lim_{s\to\infty} \frac{1}{s} \log Z_{g_0}(s) \quad \text{for any } g_0 \in \mathcal{A}.$$

Hence $P_\phi(-a - b\psi) = P^{ab}_{\mathrm{Gur}}$. □

LEMMA 4.5. $\overline{\delta}^{a,b} = 0$ *if and only if* $\delta^{a,b} = 1$.

Proof. We first notice that the critical exponents are irrelevant with base point. Therefore we can choose

$$d^{a,b}(o, \gamma o) = ad(o, \rho_1(\gamma)o) + bd(fo, \rho_2(\gamma)fo),$$

where $f : \mathbb{H} \to \mathbb{H}$ is the ι-equivalent bilipschitz given in Theorem 2.19. Since $f : \mathbb{H} \to \mathbb{H}$ is bilipschitz, there exists $C > 1$ such that, for $\gamma \in G$ and a fixed $o \in \mathbb{H}$,

$$\frac{1}{C} d(fo, \rho_2(\gamma)fo) \leq d(o, \rho_1(\gamma)o) \leq Cd(fo, \rho_2(\gamma)fo).$$

With the inequalities above, the desired results are straightforward. To simplify the notation, in this proof $d(o, \rho_1(\gamma)o)$ is denoted by $d_1(\gamma)$ and $d(fo, \rho_2(\gamma)fo)$ is denoted by $d_2(\gamma)$.

($\Longrightarrow$) Suppose $\delta^{a,b}_{PPS} = 0$.

CLAIM.

$$\sum_{\gamma \in G} e^{s(-ad_1(\gamma) - bd_2(\gamma))} < \infty \quad \textit{for } s > 1.$$

Proof. Let $s = 1 + t_0$ for some $t_0 > 0$.

$$\begin{aligned}\sum_{\gamma \in G} e^{s(-ad_1(\gamma) - bd_2(\gamma))} &= \sum_{\gamma \in G} e^{-ad_1(\gamma) - bd_2(\gamma) + t_0(-ad_1(\gamma) - bd_2(\gamma))} \\ &\leq \sum_{\gamma \in G} e^{-ad_1(\gamma) - bd_2(\gamma) + t_0(-ad_1(\gamma) - b((1/C)d_1(\gamma)))} \\ &= \sum_{\gamma \in G} e^{-ad_1(\gamma) - bd_2(\gamma) - t_0(a + b/C)d_1(\gamma)} \\ &< \infty.\end{aligned}$$

We have completed the proof of the claim. □

Similarly,

$$\sum_{\gamma \in G} e^{s(-ad_1(\gamma) - bd_2(\gamma))} = \infty \quad \text{for } s < 1.$$

Hence $\delta^{a,b} = 1$.

($\Longleftarrow$) Suppose $\delta^{a,b} = 1$.

CLAIM.

$$\sum_{\gamma \in G} e^{-ad_1(\gamma) - bd_2(\gamma) - td_1(\gamma)} < \infty \quad \textit{for } t > 0.$$

Proof. Recall that there exists $C > 1$ such that $(1/C)d_1(\gamma) < d_2(\gamma) < Cd_1(\gamma)$. For any $t > 0$, we pick $s_0 = (a + bC + t)/(a + bC) > 1$, and we have

$$s_0 = \frac{a + bC + t}{a + bC} \iff \frac{-as_0 + a + t}{s_0 b - b} = C > \frac{d_2}{d_1},$$

which implies that

$$ad_1(\gamma) + bd_2(\gamma) + td_1(\gamma) > s_0(ad_1(\gamma) + bd_2(\gamma)),$$

so that, since $s_0 > 1 = \delta^{a,b}$,

$$\sum_{\gamma \in G} e^{-ad_1(\gamma)-bd_2(\gamma)-td_1(\gamma)} \le \sum_{\gamma \in G} e^{-s_0(ad_1(\gamma)+bd_2(\gamma))} < \infty.$$

We have completed the proof the claim. □

Similarly, one can show that

$$\sum_{\gamma \in G} e^{-ad_1(\gamma)-bd_2(\gamma)-td_1(\gamma)} = \infty \quad \text{for } t < 0.$$

Therefore $\overline{\delta}^{a,b} = 1$. □

We have an immediate corollary.

COROLLARY 4.6. *$P_\phi(-a - b\psi) = P^{a,b}_{\mathrm{Gur}} = 0$ if and only if $\delta^{a,b} = 1$.*

4.2. *Proof of main results.* Throughout this subsection, ρ_1 and ρ_2 are boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition, and $S_1 = \rho_1(G)\backslash\mathbb{H}$ and $S_2 = \rho_2(G)\backslash\mathbb{H}$. Let (Σ^+, σ) be the topologically mixing countable state Markov shift associated with ρ_1 and ρ_2 defined in §3, and let $\tau, \kappa : \Sigma^+ \to \mathbb{R}^+$ be the corresponding geometric potentials. Recall that Σ^+_τ is the suspension space relative to τ, $\phi : \Sigma^+_\tau \to \Sigma^+_\tau$ is the suspension flow and the reparametrization function $\psi : \Sigma^+_\tau \to \mathbb{R}^+$ is given by $\psi(x, t) := \kappa(x)/\tau(x)$ for $x \in \Sigma^+$, $0 \le t \le \tau(x)$ and $\psi(x, \tau(x)) = \psi(\sigma(x), 0)$.

LEMMA 4.7. *Suppose $\psi : \Sigma^+_\tau \to \mathbb{R}^+$ is defined by $\psi(x, t) := \kappa(x)/\tau(x)$ for $x \in \Sigma^+$, $0 \le t \le \tau(x)$ and $\psi(x, \tau(x)) = \psi(\sigma(x), 0)$. Then $P_\sigma(-a\tau - b\kappa) = 0$ if and only if $P_\phi(-a - b\psi) = 0$.*

Proof. ($\Longrightarrow$) Suppose $P_\sigma(-a\tau - b\kappa) = 0 < \infty$. Then, when $t \in (-\varepsilon, \varepsilon)$, $P_\sigma(-a\tau - b\kappa - t\tau)$ is real analytic and is strictly decreasing, i.e.,

$$P_\sigma(-a\tau - b\kappa - t\tau) \begin{cases} < 0 & \text{for } t > 0, \\ = 0 & \text{for } t = 0, \\ > 0 & \text{for } t < 0. \end{cases}$$

Therefore, by Theorem 2.10 and $\Delta_{-a-b\psi} = -a\tau - b\kappa$, we have $P_\phi(-a - b\psi) = 0$.

($\Longleftarrow$) To see that $P_\phi(-a - b\psi) = 0$ implies $P_\sigma(-a\tau - b\kappa) = 0$. Notice that because $\tau > c > 0$ implies $\sum_{i=0}^{\infty} \tau \circ \sigma^i = \infty$, by Lemma 4.1 and Remark 4.1 in Jaerisch–Kesseböhmer–Lamei [**JKL14**],

$$\begin{aligned} 0 &= P_\phi(-a - b\psi) \\ &= \sup\left\{ \frac{h_\sigma(\mu)}{\int \tau \, d\mu} + \frac{\int (-a\tau - b\kappa) \, d\mu}{\int \tau \, d\mu} : \mu \in \mathcal{M}_\sigma(\tau) \text{ with } -a\tau - b\kappa \in L^1(\mu) \right\}, \end{aligned}$$

where $\mathcal{M}_\sigma(\tau) := \{\mu : \mu \in \mathcal{M}_\sigma \text{ and } \int \tau \, d\mu < \infty\}$.

For all $\mu \in \mathcal{M}_\sigma$ such that $-a\tau - b\kappa \in L^1(\mu)$, we have $\int \tau \, \mathrm{d}\mu > c > 0$; hence,

$$0 = \sup\left\{h_\sigma(\mu) + \int (-a\tau - b\kappa) \, \mathrm{d}\mu : \mu \in \mathcal{M}_\sigma(\tau) \text{ and } -a\tau - b\kappa \in L^1(\mu)\right\}.$$

Recall that

$$P_\sigma(-a\tau - b\kappa) = \sup\left\{h_\sigma(\mu) + \int (-a\tau - b\kappa) \, \mathrm{d}\mu : \mu \in \mathcal{M}_\sigma \text{ and } -a\tau - b\kappa \in L^1(\mu)\right\}.$$

Notice that, for $\mu \in \mathcal{M}_\sigma$, if $-a\tau - b\kappa \in L^1(\mu)$, then $\int \tau \, \mathrm{d}\mu < \infty$ (i.e., $\mu \in \mathcal{M}_\sigma(\tau)$). Moreover, it is obvious that $\mathcal{M}_\sigma(\tau) \subset \mathcal{M}_\sigma$. Thus,

$$\begin{aligned} P_\sigma(-a\tau - b\kappa) &= \sup\left\{h_\sigma(\mu) + \int (-a\tau - b\kappa) \, \mathrm{d}\mu : \mu \in \mathcal{M}_\sigma \text{ and } -a\tau - b\kappa \in L^1(\mu)\right\} \\ &= \sup\left\{h_\sigma(\mu) + \int (-a\tau - b\kappa) \, \mathrm{d}\mu : \mu \in \mathcal{M}_\sigma(\tau) \text{ and } -a\tau - b\kappa \in L^1(\mu)\right\} \\ &= 0. \end{aligned}$$ □

The following theorem gives more geometric characterizations to $t_{a,b}$ (i.e., the solution to the equation $P_\sigma(-t_{a,b}(a\tau + b\kappa)) = 0$). This is unsurprising, as the famous Bowen formula, $t_{a,b}$ is indeed the critical exponent $\delta^{a,b}$ and the growth rate of hyperbolic elements.

THEOREM 4.8. (The Bowen formula) *For $(a, b) \in D$, suppose that $t_{a,b}$ is the solution to $P_\sigma(-t_{a,b}(a\tau + b\kappa)) = 0$. Then*

$$t_{a,b} = \delta^{a,b} = \lim_{s\to\infty} \frac{1}{s} \log \underline{G}^{a,b}_{x,y}(s),$$

where $\underline{G}^{a,b}_{x,y}(s) := \#\{\gamma \in G : d^{a,b}(x, \gamma y) \le s\}$.

Proof. We first notice that

$$\begin{aligned} \delta^{a,b} = 1 &\iff \overline{\delta}^{a,b} = 0 && \text{Lemma 4.5} \\ &\iff P^{a,b}_{\text{Gur}} = 0 && \text{Lemma 4.3} \\ &\iff P_\phi(-a - b\psi) = 0 && \text{Lemma 4.4} \\ &\iff P_\sigma(-a\tau - b\kappa) = 0 && \text{Lemma 4.7.} \end{aligned}$$

Thus $P_\sigma(-t_{a,b}(a\tau + b\kappa)) = 0$ if and only $\delta^{t_{a,b}a,t_{a,b}b} = 1$, i.e., $Q^{t_{a,b}a,t_{a,b}b}(s) = \sum_{\gamma\in G} e^{-t_{ab}d^{a,b}(o,\gamma o)}$ has critical exponent one. Hence $Q^{a,b}(s) = \sum_{\gamma\in G} e^{-sd^{a,b}(o,\gamma o)}$ has critical exponent $t_{a,b}$, i.e., $\delta^{a,b} = t_{a,b}$.

For the second inequality, the proof is the same as the proof of Lemma 4.3 with some simplification (in other words, the proof is a modification of [**PPS15**, Lemma 3.3, Corollary 4.5, Theorem 4.7] or [**Pei13**, §2.2]). □

Remark 4.9. Using the same argument as in the proof of Lemma 4.3, one can also prove that the critical exponent $\delta^{a,b}$ is the growth rate of closed geodesics on S_1 and S_2. One notices that each closed geodesic on S_1 (and S_2) corresponds to a hyperbolic element in Γ_1 (and Γ_2). In other words,

$$\delta^{a,b} = h^{a,b} := \lim_{s\to\infty} \frac{1}{s} \#\{\gamma \in G : \gamma \text{ is hyperblic and } al_1[\gamma] + bl_2[\gamma] \le s\}.$$

LEMMA 4.10. *The Manhattan curve $\mathcal{C}(\rho_1, \rho_2)$ is the set of solutions to $P_\sigma(-a\tau - b\kappa) = 0$ in D.*

Proof. This follows from the same argument as in the above theorem.

$$\begin{aligned} (a, b) \in \mathcal{C}(\rho_1, \rho_2) &\iff \delta^{a,b} = 1 && \text{by definition} \\ &\iff \overline{\delta}^{a,b} = 0 && \text{Lemma 4.5} \\ &\iff P_{\text{Gur}}^{a,b} = 0 && \text{Lemma 4.3} \\ &\iff P_\phi(-a - b\psi) = 0 && \text{Lemma 4.4} \\ &\iff P_\sigma(-a\tau - b\kappa) = 0 && \text{Lemma 4.7.} \end{aligned}$$

□

THEOREM 4.11. *The Manhattan curve $\mathcal{C}(\rho_1, \rho_2)$ is real analytic.*

Proof. This is a direct consequence of Theorem 3.14 and Lemma 4.10. □

PROPOSITION 4.12. *Let ρ_1 and ρ_2 be two boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition, and let $S_1 = \rho_1(G)\backslash\mathbb{H}$ and $S_2 = \rho_2(G)\backslash\mathbb{H}$. Then:*

(1) *$\mathcal{C}$ is strictly convex if S_1 and S_2 are NOT conjugate in* $\mathrm{PSL}(2, \mathbb{R})$*; and*

(2) *$\mathcal{C}$ is a straight line if and only if S_1 and S_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$.

Proof. This result is a direct consequence of Theorem 4.1 and Theorem 4.11. Indeed, the strict convexity comes from the analyticity and the convexity of $\mathcal{C}$.

It is clear that if S_1 and S_2 are isometric, then $\mathcal{C}$ is a straight line. Conversely, suppose that $\mathcal{C}$ is a straight line. Then the slope of the tangent line of the Manhattan curve $\mathcal{C}$ is a constant, i.e.,

$$b' = -\frac{h_{\text{top}}(S_2)}{h_{\text{top}}(S_1)} = \frac{-\int \tau \, \mathrm{d}m_{-a\tau - b(a)\kappa}}{\int \kappa \, \mathrm{d}m_{-a\tau - b(a)\kappa}},$$

where $m_{-a\tau - b(a)\kappa}$ is the equilibrium state for $-a\tau - b(a)\kappa$ for all $a \in [0, h_{\text{top}}(S_1)]$. In particular,

$$b' = -\frac{\int \tau \, \mathrm{d}m_{-h_{\text{top}}(S_1)\tau}}{\int \kappa \, \mathrm{d}m_{-h_{\text{top}}(S_1)\tau}} = -\frac{\int \tau \, \mathrm{d}m_{-h_{\text{top}}(S_2)\kappa}}{\int \kappa \, \mathrm{d}m_{-h_{\text{top}}(S_2)\kappa}}.$$

CLAIM. *$h_{\text{top}}(S_1)\tau$ and $h_{\text{top}}(S_2)\kappa$ are cohomologous.*

It is clear that we have the desired result after we prove the claim. Because $h_{\text{top}}(S_1)\tau \sim h_{\text{top}}(S_2)\kappa$ means that S_1 and S_2 have proportional marked length spectra. Then by proportional marked length spectrum rigidity (i.e., Theorem 2.15) the proof is complete.

Proof. For short, we denote $m_1 = m_{-h_{\text{top}}(S_1)\tau}$ and $m_2 = m_{-h_{\text{top}}(S_2)\kappa}$. We prove this claim by the uniqueness of the equilibrium states. In other words, we want to show that m_2 is the equilibrium state for $-h_{\text{top}}(S_1)\tau$, i.e.,

$$0 = P_\sigma(-h_{\text{top}}(S_1)\tau) = h(m_2) - h_{\text{top}}(S_1) \int \tau \, \mathrm{d}m_2.$$

Notice that, by definition,

$$0 = P_\sigma(-h_{\text{top}}(S_2)\kappa) = h(m_2) - h_{\text{top}}(S_2) \int \kappa \, \mathrm{d}m_2,$$

and, by the above observation,

$$\frac{h_{\text{top}}(S_1)}{h_{\text{top}}(S_2)} = \frac{\int \kappa \, \mathrm{d}m_2}{\int \tau \, \mathrm{d}m_2}.$$

Thus,

$$\begin{aligned} h(m_2) - h_{\text{top}}(S_1) \int \tau \, \mathrm{d}m_2 &= h_{\text{top}}(S_2) \int \kappa \, \mathrm{d}m_2 - h_{\text{top}}(S_1) \int \tau \, \mathrm{d}m_2 \\ &= 0 \\ &= P_\sigma(-h_{\text{top}}(S_1)\tau). \end{aligned}$$

By the uniqueness of the equilibrium states (cf. Theorem 2.7), we know that $m_1 = m_2$. Moreover, [**Sar09**, Theorem 4.8] showed that this only happens when $-h_{\text{top}}(S_1)\tau$ and $-h_{\text{top}}(S_2)\kappa$ are cohomologous. □

Remark 4.13. Using arguments in Paulin, Pollicott and Schapira [**PPS15**], as well as the Patterson–Sullivan theory approach in [**DK08**], it is possible to recover some of the above results without using symbolic dynamics. However, due to the author's limited knowledge, without using symbolic dynamics, there seems no clear path to proving the analyticity of the Manhattan curve $\mathcal{C}(\rho_1, \rho_2)$.

COROLLARY 4.14. (Bishop–Steger entropy rigidity [**BS93**]) *Let ρ_1 and ρ_2 be two boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition, and let $S_1 = \rho_1(G)\backslash\mathbb{H}$ and $S_2 = \rho_2(G)\backslash\mathbb{H}$. Then, for any fixed $o \in \mathbb{H}$,*

$$\delta^{1,1} = \lim_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in G : d(o, \rho_1(\gamma)o) + d(o, \rho_2(\gamma)o) \le T\}.$$

Moreover, $\delta^{1,1} \le (h_{\text{top}}(S_1) \cdot h_{\text{top}}(S_2))/(h_{\text{top}}(S_1) + h_{\text{top}}(S_2))$ and equality holds if and only if S_1 and S_2 are isometric.

Proof. By Theorem 4.8, we know that $\delta^{1,1}(1, 1) \in \mathcal{C}$ is the intersection of $\mathcal{C}$ and the line $a = b$. By the convexity of $\mathcal{C}$, we know that the intersection of the line $a = b$ and $b = (-h_{\text{top}}(S_2)/h_{\text{top}}(S_1))a + h_{\text{top}}(S_2)$ lies above $\delta^{1,1}(1, 1)$. See Figure 2.

Therefore $\delta^{1,1} \le (h_{\text{top}}(S_1) \cdot h_{\text{top}}(S_2))/(h_{\text{top}}(S_1) + h_{\text{top}}(S_2))$. Moreover, when the equality holds, $\mathcal{C}$ is a straight line. By Proposition 4.12, the proof is complete. □

Definition. (Thurston's intersection number, Definition 1.3) Let S_1 and S_2 be two Riemann surfaces. Thurston's intersection number $\mathrm{I}(S_1, S_2)$ of S_1 and S_2 is given by

$$\mathrm{I}(S_1, S_2) = \lim_{n\to\infty} \frac{l_2[\gamma_n]}{l_1[\gamma_n]},$$

where $\{[\gamma_n]\}_{n=1}^{\infty}$ is a sequence of conjugacy classes for which the associated closed geodesics γ_n become equidistributed on $\Gamma_1\backslash\mathbb{H}$ with respect to area.

COROLLARY 4.15. (Thurston rigidity) *Let ρ_1 and ρ_2 be two boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition, and let $S_1 = \rho_1(G)\backslash\mathbb{H}$ and $S_2 = \rho_2(G)\backslash\mathbb{H}$. Then $\mathrm{I}(S_1, S_2) \ge (h_{\text{top}}(S_1))/(h_{\text{top}}(S_2))$ and equality holds if and only if ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$.

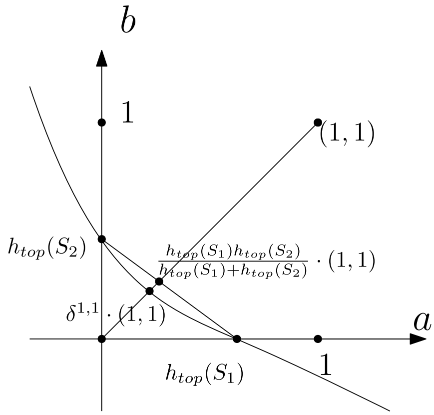

FIGURE 2. The Manhattan curve and the Bishop–Steger entropy rigidity.

Proof. It is enough to show that the normal of the tangent of $\mathcal{C}(S_1, S_2)$ at $(h_{\mathrm{top}}(S_1), 0)$ is $I(S_1, S_2)$.

Recall that

$$b'(a) = \frac{-\int \tau \, dm}{\int \kappa \, dm},$$

where $m = m_{-a\tau - b\kappa}$ is the equilibrium state of $-a\tau - b\kappa$. So, for $a = h_{\mathrm{top}}(S_1)$, $b = 0$,

$$b'(-h_{\mathrm{top}}(S_1)) = -\frac{\int \tau \, dm_{-h_{\mathrm{top}}(S_1)\tau}}{\int \kappa \, dm_{-h_{\mathrm{top}}(S_1)\tau}}.$$

Thus, it is sufficient to show that

$$\mathrm{I}(S_1, S_2) := \lim_{T\to\infty} \frac{\sum_{\lambda\in \mathrm{Per}_1(T)} l_2[\lambda]}{\sum_{\lambda\in \mathrm{Per}_1(T)} l_1[\lambda]} = \frac{\int \kappa \, dm_{-h_{\mathrm{top}}(S_1)\tau}}{\int \tau \, dm_{-h_{\mathrm{top}}(S_1)\tau}}.$$

Because $m_{-h_{\mathrm{top}}(S_1)\tau}$ is the Bowen–Margulis measure for the geodesic flow on $T^1 S_1$, and S_1 is geometrically finite, we know that the Bowen–Margulis measure is equidistributed with respect to closed orbits (see, for example, [**Rob03**, Theorem 4.1.1]). Therefore, the above equation is true. □

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A. *Appendix.*

Recall our notation that ρ_1 and ρ_2 are two boundary-preserving isomorphic Fuchsian representations satisfying the extended Schottky condition and that $S_1 = \rho_1(G)\backslash\mathbb{H}$ and

$S_2 = \rho_2(G)\backslash\mathbb{H}$. Let $d^{a,b}_{\rho_1,\rho_2}$ be the weighted Manhattan metric. Recall that $\delta^{a,b}$ is the critical exponent of the Poincaré series associated with $d^{a,b}_{\rho_1,\rho_2}$.

The proof of Lemma 4.3. We first recall two useful lemmas.

LEMMA A.1. [**Sch04**, Lemma 2.2] *Suppose $a, b, c \in \mathbb{H}$ and $d(a, b) + d(a, c) - d(b, c) \le C$ for some $C > 0$. Then a is in a D−neighborhood of the geodesic segment $[b, c]$, where D is a constant depending only on C.*

LEMMA A.2. [**PPS15**, Lemma 4.4] *Let $b_n \ge 0$ such that there exist $C > 0$ and $N \in \mathbb{N}$ such that, for all $n, m \in \mathbb{N}$,*

$$b_n b_m \le C \sum_{i=-N}^{i=N} b_{n+m+i}.$$

Then with $a_n = \sum_{k=0}^{n-1} b_n$, the limit of $a_n^{1/n}$ as $n \to \infty$ exists (and hence is equal to its limit-sup).

Recall that $\overline{\delta}^{a,b}$ is the critical exponent of the PPS Poincaré series $\overline{Q}^{a,b}_{x,y}(s)$.

Without loss of generality, we can write $d^{a,b}(x, \gamma y) = ad(x, \gamma y) + bd(fx, \iota(\gamma) fy)$ for $x, y \in \mathbb{H}$ and $\gamma \in \Gamma_1$, where $\iota : \Gamma_1 \to \Gamma_2$ is a boundary-preserving isomorphism and $f : \mathbb{H} \to \mathbb{H}$ is the bilipschitz map given by Theorem 2.19. To simply our notation, we denote $d_1(x, \gamma y) := d(x, \gamma y)$ and $d_2(x, \gamma y) := d(fx, \iota(\gamma) fy)$. Therefore $G^{a,b}_{x,y}(s)$ can be equivalently defined as

$$G^{a,b}_{x,y}(s) := \sum_{\gamma\in\Gamma_1; d_1(x,\gamma y)\le s} e^{-d^{a,b}(x,\gamma y)}.$$

Similarly, the PPS Poincaré series $\overline{Q}^{a,b}_{x,y}(s)$ can be rewritten as

$$\overline{Q}^{a,b}_{x,y}(s) = \sum_{\gamma\in\Gamma_1} e^{-d^{a,b}(x,\gamma y) - s d_1(x,\gamma y)}.$$

Let us now define several useful growth rates.

- $G^{a,b}_{x,y,1}(s) := \sum_{\gamma\in\Gamma_1; s-1<d_1(x,\gamma y)\le s} e^{-d^{a,b}(x,\gamma y)}$.
- $A_{x,y,U'}(s) := \{\gamma \in \Gamma_1 : d_1(x, \gamma y) \le s \text{ and } \gamma y \in U'\}$ where U' is an open set in $\partial_\infty\mathbb{H} \times \mathbb{H}$.
- $a_{x,y,U'}(s) := \sum_{\gamma\in A_{x,y,U'}(s)} e^{-d^{a,b}(x,\gamma y)}$.
- $B_{x,y,U',V'}(s) := \{\gamma \in \Gamma_1 : d_1(x, \gamma y) \le s,\ \gamma y \in U' \text{ and } \gamma^{-1}x \in V'\}$ where U', V' are open sets in $\partial_\infty\mathbb{H} \times \mathbb{H}$.
- $b_{x,y,U',V'}(s) := \sum_{\gamma\in B_{x,y,U',V'}(s)} e^{-d^{a,b}(x,\gamma y)}$.

By the triangle inequality, we know that

$$\limsup_{s\to\infty} \frac{1}{s} \log a^{a,b}_{x,y,U'}(s), \quad \limsup_{s\to\infty} \frac{1}{s} \log b^{a,b}_{x,y,U',V'}(s) \quad \text{and} \quad \limsup_{s\to\infty} \frac{1}{s} \log G^{a,b}_{x,y}(s)$$

are independent of the choice of bases points x and y, and it is obvious that $b^{a,b}_{x,y,U',V'}(s) \le a^{a,b}_{x,y,U'}(s) \le G^{a,b}_{x,y}$.

LEMMA. (Lemma 4.3)

$$\overline{\delta}^{a,b} = P^{ab}_{\text{Gur}} = \lim_{s\to\infty} (1/s) \log G^{a,b}_{x,y}(s) = \lim_{s\to\infty} (1/s) \log Z_W(s)$$

for any relative compact $W \subset T^1 S_1$.

The proof of the above lemma will be separated into several lemmas. Their proofs use the same argument as [**PPS15**, Lemma 4.2], [**PPS15**, Corollary 4.5] and [**PPS15**, Theorem 4.7] with minor modifications. Therefore, except for Lemma A.3, instead of proving everything in detail again, we will only point out places that require modification to adapt the proofs in [**PPS15**].

LEMMA A.3. *We have*

$$\overline{\delta}^{a,b} = \lim_{s\to\infty} \frac{1}{s} \log G^{a,b}_{x,y}(s).$$

Proof. The proof of this Lemma follows the idea of the (short) proof of [**PPS15**, Lemma 4.2] (see also the proof of [**Pei13**, Theorem 4.2]). Here we give a complete proof because the (short) proof of [**PPS15**, Lemma 4.2] is only an outline.

We notice that, by the triangle inequality, it is obvious that the $\limsup_{s\to\infty} (1/s) \log G^{a,b}_{x,y}(s)$ does not depend on the reference points x and y. Without loss of generality, we pick $x = y = o$. Recall that the generating set of the extended Schottky group $G = \pi_1 S$ is $\mathcal{A}^{\pm} = \{h_1^{\pm}, \ldots, h_{N_1}^{\pm}, p_1, \ldots, p_{N_2}\}$ with $N_1 + N_2 \geq 3$.

Let:

- $E_n := \{\gamma \in \Gamma_1 : n - 1 < d_1(o, \gamma o) \leq n\}$; and
- $b_n := G^{a,b}_{x,y,1}(n) = \sum_{\gamma\in E_n} e^{-d^{a,b}(o,\gamma o)}$.

By Lemma A.2, it is enough prove that there exist $M > 0$ and $N \in \mathbb{N}$ such that, for all $n, m \in \mathbb{N}$,

$$b_n b_m \leq M \sum_{i=-N}^{i=N} b_{n+m+i}.$$

CLAIM. *There exist* $N \in \mathbb{N}$ *and* $M > 0$ *such that* $\#E_n \times \#E_m \leq M \cdot \sum_{i=-N}^{i=N} \#E_{n+m+i}$.

Proof. Let $\gamma_n \in E_n$ and $\gamma_m \in E_m$. By Lemma 3.3, there exists $\alpha \in \mathcal{A}^{\pm}$ (more precisely, if $\gamma_n = g_i \ldots$ and $\gamma_m = g_j \ldots$ for $g_i, g_j \in \mathcal{A}$, then we take $\alpha = g_k$ for $g_k \in \mathcal{A}^{\pm}\backslash\{g_i^{\pm}, g_j^{\pm}\}$) such that

$$|d(o, \gamma_n\rho_1(\alpha)\gamma_m o) - d(o, \gamma_n o) - d(o, \gamma_m o)| < C_1$$

and

$$|d(o, (\iota\circ\gamma_n)\rho_2(\alpha)(\iota\circ\gamma_m)o) - d(o, (\iota\circ\gamma_n)o) - d(o, (\iota\circ\gamma_m)o)| < C_2,$$

where C_1 only depends on ρ_1 and C_2 only depends on ρ_2.

Thus,

$$n + m - C_1 - 2 < d(o, \gamma_n\rho_1(\alpha)\gamma_m o) \leq n + m + C_1 + 2.$$

Let us consider the map

$$\Psi : E_n \times E_m \to \sum_{i=-C_1-2}^{i=C_1+2} \#E_{n+m+i}$$

$$(\gamma_n, \gamma_m) \mapsto \gamma_n\rho_1(\alpha)\gamma_m.$$

This map is obviously not one-to-one. Nevertheless, we claim that $\#\Psi^{-1}(\gamma_n\rho_1(\alpha)\gamma_m)$ is finite. By Lemma A.1, we know that $d(\gamma_n o, [o, \gamma_n\rho_1(\alpha)\gamma_m o]) \le D$ (where D only depends on C_1), which implies that if there exist $\gamma_n' \in E_n$ and $\gamma_m' \in E_m$ such that $\gamma_n'\rho_1(\alpha)\gamma_m' = \gamma_n\rho_1(\alpha)\gamma_m = \gamma$, then $d(\gamma_n o, \gamma_n' o) \le 2(D+1)$ (because $n-1 < d(\gamma_n o, o), d(\gamma_n' o, o) \le n$ and $\gamma_n o, \gamma_n' o$ are in a D-neighborhood of $[o, \gamma o]$). Moreover, by the discreteness of Γ_1, the set $\{\gamma \in \Gamma_1 : d(\gamma o, o) \le 2(D+1)\}$ is finite (say, smaller than or equal to M_1). Hence $\#\Psi^{-1}(\gamma_n\rho_1(\alpha)\gamma_m) \le M_1^2$.

Therefore,

$$\#E_n \times \#E_m \le (2N_1 + N_2)M_1^2 \cdot \sum_{i=-C_1-2}^{i=C_1+2} \#E_{n+m+i},$$

where $2N_1 + N_2$ is the cardinality of $\mathcal{A}^\pm$. We have completed the proof of the claim. □

Moreover, we know that

$$|d^{a,b}(o, \gamma_n\rho_1(\alpha)\gamma_m o) - d^{a,b}(o, \gamma_n o) - d^{a,b}(o, \gamma_m o)| \le aC_1 + bC_2.$$

Thus we have proved the lemma. More precisely,

$$b_n b_m \le (N_1 + N_2)M_1^2 \cdot e^{aC_1 + bC_2} \sum_{i=-(C_1+2)}^{i=C_1+2} b_{n+m+i}. \qquad \square$$

As mentioned above, the proof of Lemma 4.3 follows closely the proof of [**PPS15**, Corollary 4.5] and [**PPS15**, Theorem 4.7]. Notice that [**PPS15**] focuses on the critical exponent $\delta_{\Gamma_M, F}$ associated with a Hölder continuous function $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$, where $\widetilde{M}$ is the universal covering of a complete negatively curved manifold M with pinched curvature and Γ_M is the fundamental group of M. Recall that the critical exponent $\delta_{\Gamma_M, F}$ is defined as

$$\delta_{\Gamma_M, F} := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma_M; n-1 < d_{\widetilde{M}}(x, \gamma y) \le n} e^{-\int_x^{\gamma y} \widetilde{F}},$$

where $d_{\widetilde{M}}$ is the distance function on $\widetilde{M}$.

In our context, we shall take $M = S_1$, $\widetilde{M} = \mathbb{H}$ and $\int_x^{\gamma y} \widetilde{F} = d^{a,b}(x, \gamma y)$ for all $x, y \in \mathbb{H}$ and $\gamma \in \Gamma_1$. However, in our case, the existence of such a Hölder continuous function $\widetilde{F}$ is unclear. Nevertheless, in the proof of [**PPS15**, Corollary 4.5] and [**PPS15**, Theorem 4.7], the Hölder continuity of $\widetilde{F}$ is only used to guarantee [**PPS15**, Lemma 3.2], i.e.,

$$\left| \int_x^z \widetilde{F} - \int_y^z \widetilde{F} \right| \le c_1 e^{d_{\widetilde{M}}(x,y)} + d_{\widetilde{M}}(x, y) \cdot \max_{\pi^{-1}(B(x, d_{\widetilde{M}}(x,y)))} |\widetilde{F}|, \tag{A.1}$$

where c_1 is a (universal) constant and $\pi : T^1\widetilde{M} \to \widetilde{M}$ is the canonical projection. It is not hard to verify that $d^{a,b}(x, \gamma y)$ satisfies (A.1). Indeed, for all $x, y \in \mathbb{H}$ and $\gamma \in \Gamma_1$, without loss generality, we can define $d^{a,b}(\gamma x, y) := d^{a,b}(x, \gamma^{-1} y)$ and $d^{a,b}(x, y) := ad_1(x, y) + bd_2(x, y)$. Hence, by the triangle inequality,

$$\begin{aligned} |d^{a,b}(x, z) - d^{a,b}(y, z)| &= |ad_1(x, z) - bd_2(x, z) - ad_1(y, z) + bd_2(y, z)| \\ &= |a(d_1(x, z) - d_1(y, z)) + b(-d_2(x, z) + d_2(y, z))| \\ &\le d^{a,b}(x, y). \end{aligned}$$

In summary, by taking $M = S_1$, $\widetilde{M} = \mathbb{H}$ and replacing $\int_x^y \widetilde{F}$ by $d^{a,b}(x, y)$ in the proof of [**PPS15**, Corollary 4.5] and [**PPS15**, Theorem 4.7], we have the following lemma, and hence Lemma 4.3.

LEMMA A.4.

$$\lim_{s\to\infty} \frac{1}{s} \log a^{a,b}_{x,y,U'}(s) = \lim_{s\to\infty} \frac{1}{s} \log b^{a,b}_{x,y,U',V'}(s) = \lim_{s\to\infty} \frac{1}{s} \log Z^{a,b}_{W}(s) = \overline{\delta}^{a,b}.$$

We remark that [**PPS15**, Theorem 4.2] is used in the proof of [**PPS15**, Corollary 4.5] and [**PPS15**, Theorem 4.7]. In other words, Lemma A.3 was used implicitly in the proof of Lemma A.4 and Lemma 4.3.

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