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Unique equilibrium states for geodesic flows over surfaces without focal points

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Abstract

In this paper, we study dynamics of geodesic flows over closed surfaces of genus greater than or equal to 2 without focal points. Especially, we prove that there is a large class of potentials having unique equilibrium states, including scalar multiples of the geometric potential, provided the scalar is less than 1. Moreover, we discuss ergodic properties of these unique equilibrium states, including the Bernoulli property and the fact that weighted regular periodic orbits are equidistributed relative to these unique equilibrium states.

Keywords: surface without focal points, geodesics flow, Bernoulli, uniqueness of equilibrium states
Mathematics Subject Classification numbers: 37D35, 37D40, 37C40, 37D25

(Some figures may appear in colour only in the online journal)

Contents

1. Introduction

This paper is devoted to the study of dynamics of the geodesic flows over closed surfaces without focal points. We focus on the thermodynamic formalism of the geodesic flows, especially, the uniqueness of the equilibrium states and their ergodic properties. For uniformly hyperbolic flows, also known as Anosov flows, thanks to fundamental works of Ornstein, Weiss, Bowen and Ruelle [OW73, Bow75, BR75], we know that every Hölder potential has a unique equilibrium state which enjoys several ergodic features such as Bernoulli and equidistribution properties. It is also well-known that the geodesic flow on a negatively curved manifold is uniformly hyperbolic. However, when the manifold contains subsets with zero or positive curvature, the geodesic flow may no longer be uniformly hyperbolic. The non-uniform hyperbolicity greatly increases the difficulty in understanding the thermodynamics of these flows. Nevertheless, the geometric features of surfaces without focal points allow us to investigate the dynamics of the geodesic flows. Several geometric properties are available in this setting such as the flat strip theorem, C^2-regularity of the horocycles, and more. These properties enable us to extend the existence and the uniqueness result on the measure of maximal entropy by Knieper [Kni98] and on equilibrium states by Burns *et al* [BCFT18] over closed rank 1 nonpositively curved manifolds to closed surfaces without focal points of genus at least 2.

Combining the dynamical and geometric features of surfaces without focal points, in this paper, we are able to prove the uniqueness of equilibrium states for a large class of potentials and Bernoulli and equidistribution properties for such equilibrium states. These results also

generalize Gelfert–Ruggiero's recent work [GR17] on the uniqueness of measure of the maximum entropy for the geodesic flows over surfaces without focal points. We remark that, using a differently approach, Liu *et al* [LWW18] extended the uniqueness of measure of maximum entropy result to manifolds without focal points of arbitrary dimension.

Putting our results in context below, we shall first introduce relevant terminologies briefly (see sections 2 and 3 for more details). Throughout the paper, S denotes a closed (i.e. compact without boundary) C^∞ Riemannian surface of genus greater than or equal to 2 without focal points. The *geodesic flow* $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ on the unit tangent bundle T^1S is the flow given by $f_t(v) = \dot{\gamma}_v(t)$ where γ_v is the (unit speed) geodesic determined by the initial vector $v \in T^1S$.

In this paper, we study topological pressure and equilibrium states of continuous potentials with respect to the geodesic flow $\mathcal{F}$. For a continuous potential (i.e. function) $\varphi : T^1S \to \mathbb{R}$, the *topological pressure* $P(\varphi)$ of φ with respect to $\mathcal{F}$ can be described by the *variational principle*:

$$P(\varphi) = \sup\{h_\mu(\mathcal{F}) + \int \varphi \mathrm{d}\mu : \ \mu \text{ is } a \ \mathcal{F}\text{-invariant Borel probability measure on } T^1S\},$$

where $h_\mu(\mathcal{F})$ is the measure-theoretic entropy of μ with respect to $\mathcal{F}$. An invariant Borel probability measure μ achieving the supremum is called an *equilibrium state*. We notice that when φ is identically equal to 0 then $P(0)$ is equal to the topological entropy $h_{\text{top}}(\mathcal{F})$ of $\mathcal{F}$, and an equilibrium state for $\varphi \equiv 0$ is called a *measure of maximum entropy*.

The non-uniform hyperbolicity of $\mathcal{F}$ comes from the existence of the *singular set* Sing. For surfaces without focal points, we can describe the singular set as $\text{Sing} = \{v \in T^1S : \ K(\pi f_t v) \geqslant 0 \ \forall t \in \mathbb{R}\}$ where $\pi : T^1S \to S$ is the canonical projection and K is the Gaussian curvature (see section 3 for alternative characterizations of the singular set). The complement of Sing is called the *regular set* and denoted by Reg.

Our first result asserts the uniqueness of the equilibrium states for potentials with 'nice' regularity that carry smaller pressure on the singular set. The potentials with 'nice' regularity include Hölder potentials and the *geometric potential* φ^u defined as

$$\varphi^u(v) := -\lim_{t \to 0} \frac{1}{t} \log \det(\mathrm{d}f_t|_{E^u(v)}).$$

Here, $E^u(v)$ is the unstable subspace in $T_v T^1S$ (see section 3 for details).

Theorem A. *Let S be a surface of genus greater than or equal to 2 without focal points and $\mathcal{F}$ be the geodesic flow over S. Let $\varphi : T^1S \to \mathbb{R}$ be a Hölder continuous potential or a scalar multiple of the geometric potential $q\varphi^u$ for some $q \in \mathbb{R}$. Suppose φ verifies the pressure gap property $P(\text{Sing}, \varphi) < P(\varphi)$, then φ has a unique equilibrium state μ_φ.*

The proof of theorem A uses the same idea as the proof of [BCFT18, theorem A]. Both [BCFT18] and this paper follow the general framework introduced by Bowen [Bow75], which was subsequently extended to flows by Franco [Fra77] and recently extended further by Climenhaga and Thompson [CT16]. We have more detailed discussion of this method in section 2. Roughly speaking, the general framework follows the original work of Bowen stating that when the potential has 'nice' regularity (namely, the Bowen property) and the system has 'sufficient hyperbolicity' (namely, the specification property and the expansivity) then this potential has a unique equilibrium state. While we follow the general framework of [BCFT18], our setting of surfaces without focal points does not enjoy properties available in the setting of [BCFT18] coming from the geometry of nonpositively curved manifolds. The most notable such property is the convexity of $\|J(t)\|$ for any Jacobi field J. Due to the absence of such
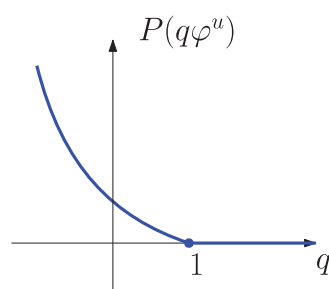
Figure 1. Pressure function.

convexity, we use an alternative way to quantify hyperbolicity on T^1S and to characterize the singular set; see remark 3.10. We discuss more details of this method in sections 2 and 3.

The second result, following theorem A, states several ergodic properties of these unique equilibrium states. We successfully extend several properties known to hold under uniformly hyperbolic cases (see, for example, [PP90]), as well as under nonpositively curved surfaces (see, for example, [Pol96, LLS16] and [BCFT18]). Namely, these unique equilibrium states are Bernoulli and the weak* limit of the weighted regular periodic orbits. Recall that other weaker ergodic properties such as being Kolmogorov and strongly mixing follows once the measure is Bernoulli.

Theorem B. *Suppose φ satisfies the same assumptions in theorem A. Then, the unique equilibrium state μ_φ is fully supported, Bernoulli, and the weak* limit of the weighted regular periodic orbits. Moreover, $\mu_\varphi(\mathrm{Reg}) = 1$.*

In our last main result, we study the geometric potential φ^u and its pressure function $q \mapsto P(q\varphi^u)$. We give the full description of the pressure function, and show that the situation is analogous to the nonpositively curved manifolds (see, for example, [BG14] and [BCFT18]).

Theorem C. *Let S be a surface of genus greater than or equal to 2 without focal points and $\mathcal{F}$ be the geodesic flow over S. Suppose $\varphi = q\varphi^u$ is the scalar multiple the geometric potential with $q < 1$. Then, φ satisfies the pressure gap property.*

Such $q\varphi^q$ has a unique equilibrium state from theorem A, and the unique equilibrium state satisfies the properties listed in theorem B.

Moreover, the map $q \mapsto P(q\varphi^u)$ is C^1 on $q \in (-\infty, 1)$. If $\mathrm{Sing} \neq \emptyset$, then $P(q\varphi^u) = 0$ for $q \geqslant 1$, see figure 1.

This paper is organized as follows. In section 2, we go over the background in thermodynamic formalism; in particular, we describe our primary tool, the Climenhaga–Thompson criteria introduced in [CT16]. In section 3, we recall the definitions and geometric features of surfaces and manifolds without focal points. Sections 4–6 are devoted to setting up the framework for the Climenhaga–Thompson criteria, namely, the orbit decomposition, the specification property, and the Bowen property. We will prove theorem A in section 7 and theorem B in section 8. In section 9, we will show theorem C and provide some examples of potentials satisfying theorem A.

2. Preliminaries of dynamics

In this section, we introduce necessary background in thermodynamics. An excellent reference for terminology introduced in this section is Walters' book [Wal82].

Throughout this section, (X, d) is a compact metric space, $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ is a continuous flow on X, and $\varphi : X \to \mathbb{R}$ is a continuous potential.

2.1. Topological pressure

For convenience, we first define the following terms.

Definition 2.1.　For any $t, \delta > 0$ and $x \in X$,

(1) The *Bowen ball* of radius δ and order t at x is defined as

$$B_t(x, \delta) = \{y \in X : \ d(f_\tau x, f_\tau y) < \delta \text{ for all } 0 \leqslant \tau \leqslant t\}.$$

(2) We say a set E is (t, δ)- *separated* if for all $x, y \in E$ with $x \neq y$, there exists $t_0 \in [0, t]$ such that $d(f_{t_0} x, f_{t_0} y) \geqslant \delta$.

Definition 2.2 (Finite length orbit segments).　Any subset

$$\mathcal{C} \subset X \times [0, \infty)$$

can be identified with a collection of *finite length orbit segments. More precisely, every* $(x, t) \in \mathcal{C}$ *is identified with the orbit segment* $\{f_\tau x : \ 0 \leqslant \tau \leqslant t\}$.

We denote $\Phi(x, t) := \int_0^t \varphi(f_\tau x) d\tau$ the integral of φ along an orbit segment (x, t).

Let $\mathcal{C}_t := \{x \in X : \ (x, t) \in \mathcal{C}\}$ be the set of length t orbit segments in $\mathcal{C}$. We define

$$\Lambda(\mathcal{C}, \varphi, \delta, t) = \sup\Big\{\sum_{x \in E} e^{\Phi(x,t)} : \ E \subset \mathcal{C}_t \text{ is } (t, \delta)\text{-separated}\Big\}.$$

Definition 2.3 (Topological pressure).　The *pressure* of φ on $\mathcal{C}$ is defined as

$$P(\mathcal{C}, \varphi) = \lim_{\delta \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \Lambda(\mathcal{C}, \varphi, \delta, t).$$

When $\mathcal{C} = X \times [0, \infty)$, we denote $P(X \times [0, \infty), \varphi)$ by $P(\varphi)$ and call it the *topological pressure* of φ with respect to $\mathcal{F}$.

As noted in the introduction, the pressure $P(\varphi)$ satisfies the variational principle

$$P(\varphi) = \sup_{\mu \in \mathcal{M}(\mathcal{F})} \Big\{h_\mu(\mathcal{F}) + \int \varphi d\mu\Big\}$$

where $\mathcal{M}(\mathcal{F})$ is the set of $\mathcal{F}$-invariant probability measures on X. Also, a $\mathcal{F}$-invariant probability measure μ realizing the supremum is called an *equilibrium state* for φ.

Remark 2.4.

(1) When the entropy map $\mu \mapsto h_\mu$ is upper semi-continuous, any weak* limit of a sequence of invariant measures approximating the pressure is an equilibrium state. In particular, there exists at least one equilibrium state for every continuous potential.

(2) In our setting, the geodesic flow over surfaces without focal points, the upper semi-continuity of the entropy map is guaranteed by the entropy-expansivity established in [LW16].

2.2. Climenhaga–Thompson's criteria for the uniqueness of equilibrium states

Climenhaga and Thompson have a series of successful results on establishing the uniqueness of the equilibrium states of various non-uniformly hyperbolic systems; see [CT12, CT13, CFT18, CT16, BCFT18]. This work follows the same method, so called, the Climenhaga–Thompson criteria. In this subsection, we introduce the terms used in the Climenhaga–Thompson criteria.

One of the primary ideas in the Climenhaga–Thompson criteria is to relax the original assumptions from the work of Bowen on the uniformly hyperbolic systems [Bow75] by asking that the 'hyperbolic' behavior on the system and the 'good regularity' on the potential hold on a (large) collection of finite orbit segments $\mathcal{C}$ rather than in the whole space. This flexibility is essential for applying this method to non-uniformly hyperbolic systems. To be more precise, the 'hyperbolic' behavior refers to the *specification* property and the property that the pressure of obstructions to expansivity $P_{\exp}^{\perp}(\varphi)$ be strictly smaller than the pressure $P(\varphi)$ of the entire system (see below). The 'good regularity' on φ refers to the potential having the *Bowen property*.

Definition 2.5 (Specification). We say $\mathcal{C} \subset X \times [0, \infty)$ has *specification* at scale $\rho > 0$ if there exists $\tau = \tau(\rho)$ such that for every finite sub-collection of $\mathcal{C}$, i.e. (x_1, t_1), $(x_2, t_2), \ldots, (x_N, t_N) \in \mathcal{C}$, there exists $y \in X$ and transition times $\tau_1, \ldots, \tau_{N-1} \in [0, \tau]$ such that for $s_0 = \tau_0 = 0$ and $s_j = \sum_{i=1}^{j} t_i + \sum_{i=1}^{j-1} \tau_i$, we have

$$f_{s_{j-1}+\tau_{j-1}}(y) \in B_{t_j}(x_j, \rho)$$

for $j \in \{1, 2, \ldots, N\}$. If $\mathcal{C}$ has specification at all scales, then we say $\mathcal{C}$ has *specification*. We say that the flow has specification if the entire orbit space $\mathcal{C} = X \times [0, \infty)$ has specification.

Definition 2.6 (Bowen property). We say $\varphi : X \to \mathbb{R}$ a continuous potential has the *Bowen property* on $\mathcal{C} \subset X \times [0, \infty)$ if there are $\varepsilon, K > 0$ such that for all $(x, t) \in \mathcal{C}$, we have

$$\sup_{y \in B_t(x, \varepsilon)} |\Phi(x, t) - \Phi(y, t)| \leqslant K$$

where $\Phi(x, t) = \int_0^t \varphi(f_\tau x) \mathrm{d}\tau$ as in definition 2.2.

Definition 2.7 (Decomposition of orbit segments). A decomposition of $X \times [0, \infty)$ consists of three collections $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset X \times [0, \infty)$ such that:

(1) There exist $p, g, s : X \times [0, \infty) \to \mathbb{R}$ such that for each $(x, t) \in X \times [0, \infty)$, we have
$t = p(x, t) + g(x, t) + s(x, t)$,
(2) $(x, p(x, t)) \in \mathcal{P}$, $(f_{p(x,t)}x, g(x, t)) \in \mathcal{G}$, and $(f_{p(x,t)+g(x,t)}x, s(x, t)) \in \mathcal{S}$.

In section 4, we will give the precise construction of a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ and prove that such decomposition has required properties in subsequent sections. Due to some technical reasons (see [CT16]), we need to work on with discrete-time versions of $\mathcal{P}$ and $\mathcal{S}$, namely,

$$[\mathcal{P}] := \{(x, n) \in X \times \mathbb{N} : \ (f_{-s}x, n + s + t) \in \mathcal{P} \text{ for some } s, t \in [0, 1]\},$$

and similarly for $[\mathcal{S}]$.

The following three terms are the remaining pieces needed in stating the Climenhaga–Thompson criteria.

Definition 2.8. For $x \in X$, $\varepsilon > 0$ and $\varphi : X \to \mathbb{R}$ a potential

(1) The *bi-infinite Bowen ball* $\Gamma_\varepsilon(x)$ is defined as

$$\Gamma_\varepsilon(x) := \{y \in X : \ d(f_t x, f_t y) \leqslant \varepsilon \text{ for all } t \in \mathbb{R}\}.$$

(2) The set of *non-expansive points at scale* ε is defined as

$$\text{NE}(\varepsilon) := \{x \in X : \ \Gamma_\varepsilon(x) \nsubseteq f_{[-s,s]}(x) \text{ for any } s > 0\}$$

where $f_{[a,b]}(x) = \{f_t x : \ t \in [a, b]\}$.

(3) *The pressure of obstructions to expansivity* for φ is defined as

$$P_{\exp}^\perp(\varphi) := \lim_{\varepsilon \to 0} P_{\exp}^\perp(\varphi, \varepsilon)$$

where

$$P_{\exp}^\perp(\varphi, \varepsilon) := \sup\{h_\mu(f_1) + \int \varphi \mathrm{d}\mu : \ \mu \in \mathcal{M}^e(\mathcal{F}) \text{ and } \mu(\text{NE}(\varepsilon)) = 1\}$$

and $\mathcal{M}^e(\mathcal{F})$ is the set of $\mathcal{F}$-invariant ergodic probability measures on X.

Remark 2.9.   For uniformly hyperbolic systems, $\text{NE}(\varepsilon) = \emptyset$ for ε sufficiently small; thus $P_{\exp}^\perp(\varphi) = -\infty$. In other words, the condition $P_{\exp}^\perp(\varphi) < P(\varphi)$ always holds in Bowen's setting [Bow75].

Finally, the following theorem is the Climenhaga–Thompson criteria for the uniqueness of equilibrium states. We will use this theorem to prove theorem A in section 7.

Theorem 2.10 ([CT16, theorem A]).   *Let* $(X, \mathcal{F})$ *be a flow on a compact metric space, and* $\varphi : X \to \mathbb{R}$ *be a continuous potential. Suppose that* $P_{\exp}^\perp(\varphi) < P(\varphi)$ *and* $X \times [0, \infty)$ *admits a decomposition* $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ *with the following properties:*

(I) $\mathcal{G}$ has specification;
(II) φ has Bowen property on $\mathcal{G}$;
(III) $P([\mathcal{P}] \cup [\mathcal{S}], \varphi) < P(\varphi)$.

Then $(X, \mathcal{F}, \varphi)$ *has a unique equilibrium state* μ_φ.

Remark 2.11.   From the uniqueness of the equilibrium state μ_φ, it follows that μ_φ is ergodic. See also [CT16, proposition 4.19].

2.3. Gurevich pressure

In this subsection, we introduce another well-studied notion of pressure, the Gurevich pressure, that is, the growth rate of weighted periodic orbits. In the uniformly hyperbolic setting, the Gurevich pressure is equal to the topological pressure. However, it is not always the case for non-uniformly hyperbolic systems (see [GS14] for more details). To make the above discussion more precise, we shall define the following relevant terms.

As before, let M be a Riemannian manifold, $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ be the geodesic flow on $T^1 M$, and $\varphi : T^1 M \to \mathbb{R}$ be a continuous potential. A geodesic γ is *closed* if there exists $L > 0$ such that

γ is periodic with period L, that is, $\gamma(t) = \gamma(t + L)$ for all $t \in \mathbb{R}$. A geodesic γ_v is *regular* if the generating vector v is regular.

We denote the set of closed regular geodesics with length in the interval $(a, b]$ by $\mathrm{Per}_R(a, b]$. For $\gamma \in \mathrm{Per}_R(a, b]$, we define

$$\Phi(\gamma) := \int_\gamma \varphi = \int_0^{|\gamma|} \varphi(f_t v) \mathrm{d}t$$

where $v \in T^1 M$ is tangent to γ and $|\gamma|$ is the length of γ. Given $t, \Delta > 0$, we define

$$\Lambda^*_{\mathrm{Reg},\Delta}(\varphi, t) := \sum_{\gamma \in \mathrm{Per}_R(t-\Delta,t]} \mathrm{e}^{\Phi(\gamma)}.$$

Definition 2.12 (Gurevich pressure).   Given $\Delta > 0$,

(1) The *upper regular Gurevich pressure* $\overline{P}^*_{\mathrm{Reg},\Delta}$ of φ is defined as

$$\overline{P}^*_{\mathrm{Reg},\Delta}(\varphi) := \limsup_{t \to \infty} \frac{1}{t} \log \Lambda^*_{\mathrm{Reg},\Delta}(\varphi, t).$$

(2) The *lower regular Gurevich pressure* $\underline{P}^*_{\mathrm{Reg},\Delta}$ of φ is defined as

$$\underline{P}^*_{\mathrm{Reg},\Delta}(\varphi) := \liminf_{t \to \infty} \frac{1}{t} \log \Lambda^*_{\mathrm{Reg},\Delta}(\varphi, t).$$

When $\overline{P}^*_{\mathrm{Reg},\Delta}(\varphi) = \underline{P}^*_{\mathrm{Reg},\Delta}(\varphi)$, we call this value the *regular Gurevich pressure* and denote it by $P^*_{\mathrm{Reg},\Delta}(\varphi)$.

Remark 2.13.   Our *upper* regular Gurevich pressure $\overline{P}^*_{\mathrm{Reg},\Delta}$ is the regular Gurevich pressure $P_{\mathrm{Gur},\mathcal{R}}$ used in [GS14]. Indeed, using the same argument as in [GS14], one can show that $\overline{P}^*_{\mathrm{Reg},\Delta}$ is independent of $\Delta > 0$. However, to derive the equidistribution property, we need to take the lower regular Gurevich pressure into account (see proposition 2.17).

Definition 2.14.   For a potential $\varphi : T^1 M \to \mathbb{R}$, we say μ is the *weak* limit* of φ-weighted regular periodic orbits, if there exists $\Delta > 0$ such that

$$\mu = \lim_{t \to \infty} \frac{\sum_{\gamma \in \mathrm{Per}_R(t-\Delta,t]} \mathrm{e}^{\Phi(\gamma)} \delta_\gamma}{\Lambda^*_{\mathrm{Reg},\Delta}(\varphi, t)}$$

where δ_γ is the normalized Lebesgue measure along a periodic orbit γ.

In his proof of the variational principle in [Wal82, theorem 9.10], Walters pointed out a way to construct equilibrium states through periodic orbits.

Proposition 2.15 ([Wal82, theorem 9.10).]   *Given $\Delta > 0$, suppose there exists $\{t_k\}$ such that*

$$\lim_{k \to \infty} \frac{1}{t_k} \log \Lambda^*_{\mathrm{Reg},\Delta}(\varphi, t_k) = P(\varphi)$$

and

$$\lim_{k \to \infty} \frac{\sum_{\gamma \in \mathrm{Per}_R(t_k - \Delta, t_k]} \mathrm{e}^{\Phi(\gamma)} \delta_\gamma}{\Lambda^*_{\mathrm{Reg},\Delta}(\varphi, t_k)} = \mu,$$

then μ is an equilibrium state.

Remark 2.16. The proof of the proposition above proceeds by relating the collection of closed regular orbits to a (t, δ)-separated set. This type of argument appears in section 8 as a part of the proof for theorem B. See lemma 8.11 for details.

Since the set of $\mathcal{F}$-invariant probability measures $\mathcal{M}(\mathcal{F})$ is compact with respect to the weak* topology, proposition 2.15 has the following consequence:

Proposition 2.17. *Given $\Delta > 0$, suppose $P^*_{\mathrm{Reg},\Delta}(\varphi) = P(\varphi)$ and φ has a unique equilibrium μ_φ. Then μ_φ is the weak* limit of φ-weighted regular closed geodesics.*

3. Preliminaries of surfaces without focal points

3.1. Geometry of Riemannian manifolds without focal points

In this section, we recall relevant earlier results of manifolds without focal points. These results can be found in [Ebe73, Pes77b, Esc77, Bur83].

Throughout this section M denotes a closed C^∞ Riemannian manifold, and we denote the geodesic flow on its unit tangent bundle T^1M by $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$. Recall that for any Riemannian manifold M, we can naturally equip its tangent bundle T^1M with the *Sasaki metric*. In what follows, without stating specifically, the norm $|| \cdot ||$ on TT^1M always refers to the Sasaki metric (see discussions below remark 3.2 for the definition).

A *Jacobi field* $J(t)$ along a geodesic γ is a vector field along γ satisfying the Jacobi equation:

$$J''(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0, \tag{3.1}$$

where R is the Riemannian curvature tensor, and $\prime$ denotes the covariant derivative along γ. When M is a surface, the Jacobi equation (3.1) simplifies to

$$J''(t) + K(\gamma(t))\dot{\gamma}(t) = 0,$$

where K is the Gaussian curvature.

A Jacobi field J is *orthogonal* if both J and J' are orthogonal to $\dot{\gamma}$ at some $t_0 \in \mathbb{R}$ (and hence for all $t \in \mathbb{R}$).

A Jacobi field J is *parallel* at t_0 if $J'(t_0) = 0$. If $J'(t) = 0$ for all $t \in \mathbb{R}$, then we say J is *parallel*.

Definition 3.1 (No focal points). A Riemannian manifold M has *no focal points* if for any initial vanishing Jacobi field $J(t)$, its length $\|J(t)\|$ is strictly increasing. We say M has *no conjugate points* if any non-zero Jacobi field has at most one zero.

Remark 3.2. There are other equivalent definitions for manifolds without focal points, and many of their geometric features are introduced in [dC13]. The following results are classical and relevant in our setting:

(1) Nonpositively curved $\subsetneq$ no focal points $\subsetneq$ no conjugate points.

(2) One can find examples from each category above from [Gul75], as well as [Ger03], for examples in the above assertion.

It is a classical result that one can identify the tangent space of T^1M with the space of orthogonal Jacobi fields $\mathcal{J}$. Moreover, one can use this relation to define three $\mathcal{F}$-invariant bundles E^u, E^c, and E^s in TT^1M. To be more precise, for each $v \in T^1M$, there exists a direct sum decomposition $T_vT^1M = H_v \oplus V_v$ into the horizontal and vertical subspaces, each equipped with the norm induced from the Riemannian metric on M. The *Sasaki metric* on T^1M is defined by declaring H_v and V_v to be orthogonal. Denoting the space of orthogonal Jacobi fields along a geodesic γ by $\mathcal{J}(\gamma)$, the identification between T_vT^1M and $\mathcal{J}(\gamma_v)$ is given by

$$T_vT^1M \ni \xi = (\xi_h, \xi_v) \mapsto J_\xi \in \mathcal{J}(\gamma_v)$$

where J_ξ is the unique Jacobi field characterized by $J_\xi(0) = \xi_h$ and $J'_\xi(0) = \xi_v$. Moreover, we have

$$\|\mathrm{d}f_t(\xi)\|^2 = \|J_\xi(t)\|^2 + \|J'_\xi(t)\|^2. \tag{3.2}$$

We define $\mathcal{J}^s(\gamma)$ to be the space of *stable (orthogonal) Jacobi fields* as

$$\mathcal{J}^s(\gamma) = \{J(t) \in \mathcal{J}(v): \|J(t)\| \text{ is bounded for } t \geqslant 0\},$$

and $\mathcal{J}^u(\gamma)$ to be the space of *unstable (orthogonal) Jacobi fields* as

$$\mathcal{J}^u(\gamma) = \{J(t) \in \mathcal{J}(v): \|J(t)\| \text{ is bounded for } t \leqslant 0\}.$$

Using these two linear spaces of $\mathcal{J}(\gamma)$ and the identification, we can define two subbundles $E^s(v)$ and $E^u(v)$ of T_vT^1M as the following:

$$E^s(v) := \{\xi \in T_vT^1M: J_\xi \in \mathcal{J}^s(v)\},$$
$$E^u(v) := \{\xi \in T_vT^1M: J_\xi \in \mathcal{J}^u(v)\}.$$

Last, we define $E^c(v)$ given by the flow direction.

Definition 3.3 (Rank). The *rank* of a vector $v \in T^1M$ is the dimension of the space of parallel Jacobi fields. We call M a *rank 1* manifold if it has at least one rank 1 vector.

Definition 3.4 (Singular and regular set). The *singular set* Sing $\subset T^1M$ is the set of vectors with rank greater than or equal to 2. The *regular set* Reg is the complement of Sing.

When M is a surface, the singular set admits a useful alternative characterization (3.3). This fact as well as other facts regarding manifolds with no focal points are summarized in the following proposition.

Proposition 3.5. *Let M be a closed Riemannian manifold without focal points. Then we have:*

(1) *[Hur86, theorem 3.2] The geodesic flow $\mathcal{F}$ is topologically transitive if M is rank 1[3].*
(2) *[Pes77b, propositions 4.7 and 6.2] $\dim E^u(v) = \dim E^s(v) = n - 1$, and $\dim E^c(v) = 1$ where $\dim M = n$.*
(3) *[Pes77b, theorems 4.11 and 6.4] The subbundles $E^u(v)$, $E^s(v)$, $E^{cu}(v)$ and $E^{cs}(v)$ are $\mathcal{F}$—invariant where $E^{cs}(v) = E^c(v) \oplus E^s(v)$ and $E^{cu}(v) = E^c(v) \oplus E^u(v)$.*
(4) *[Pes77b, theorems 6.1 and 6.4] The subbundles $E^u(v)$, $E^s(v)$, $E^{cu}(v)$ and $E^{cs}(v)$ are*

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[3] Ergodicity was claimed in [Hur86] but the argument has an error. Nevertheless the proof for theorem 3.2 is independent of ergodicity, and it remains valid.

integrable to $\mathcal{F}$—invariant foliations $W^u(v)$, $W^s(v)$, $W^{cu}(v)$ and $W^{cs}(v)$, respectively. Moreover, $W^u(v)$ (resp. $W^s(v)$) consists of vectors perpendicular to $H^u(v)$ (resp. $H^s(v)$) and toward to the same side as v (see below for the definition of the horospheres $H^{s/u}(v)$).

(5) *[Esc77, lemma, p 246] $E^u(v) \cap E^s(v) \neq \emptyset$ if and only if $v \in \mathrm{Sing}$.*

(6) *[O'S76, theorem 1, Esc77, theorem 2] The flat strip theorem: suppose M is simply connected and geodesics γ_1, γ_2 are bi-asymptotic in the sense that $d(\gamma_1(t), \gamma_2(t))$ is uniformly bounded for all $t \in \mathbb{R}$. Then γ_1 and γ_2 bound a strip of flat totally geodesically immersed surface.*

(7) *[Ebe73, corollary 3.3, 3.6] Suppose $\dim M = 2$, then*

$$\mathrm{Sing} = \{v \in T^1 M : K(\pi f_t v) = 0 \text{ for all } t \in \mathbb{R}\}, \tag{3.3}$$

where $\pi : T^1 M \to M$ is the canonical projection.

(8) *[Hop48] Suppose $\dim M = 2$, then M is rank 1 if and only if its genus is at least 2.*

(9) *[Esc77, section 5] For any $J \in \mathcal{J}^s(\gamma)$(resp. $\mathcal{J}^u(\gamma)$), $||J(t)||$ is monotonely decreasing (resp. increasing) for all $t \in \mathbb{R}$.*

We shall introduce more metrics on $T^1 M$ and the flow invariant foliations induced in Proposition 3.5 so that we can perform finer analysis. We write d_S for the distance function on $T^1 M$ induced by the Sasaki metric on $TT^1 M$. We will make use of another handy metric d_K on $T^1 M$:

$$d_K(v, w) := \max\{d(\gamma_v(t), \gamma_w(t)) : t \in [0, 1]\}.$$

Such metric d_K also appeared in [Kni98]. It is not hard to see that d_S and d_K are uniformly equivalent. Thus, we will primarily work with the metric d_K throughout the paper. In particular, any Bowen ball $B_t(v, \varepsilon)$ appearing from here onward is with respect to the metric d_K, i.e.

$$B_t(v, \varepsilon) := \{w \in T^1 M : d_K(f_\tau w, f_\tau v) < \varepsilon \text{ for all } 0 \leqslant \tau \leqslant t\}.$$

Furthermore, an *intrinsic metric* d^s on $W^s(v)$ for all $v \in T^1 M$ is given by

$$d^s(u, w) := \inf\{l(\pi\gamma) : \gamma : [0, 1] \to W^s(v), \gamma(0) = u, \gamma(1) = w\}$$

where l is the length of the curve in M, and the infimum is taken over all C^1 curves γ connecting $u, w \in W^s(v)$. Using d^s we can define the *local stable leaf* through v of size ρ as:

$$W^s_\rho(v) := \{w \in W^s(v) : d^s(v, w) \leqslant \rho\}.$$

Moreover, we can locally define a similar intrinsic metric d^{cs} on $W^{cs}(v)$ as:

$$d^{cs}(u, w) = |t| + d^s(f_t u, w)$$

where t is the unique time such that $f_t u \in W^s(w)$. This metric d^{cs} extends to the whole central stable leaf $W^{cs}(v)$. We also define $d^u, W^u_\rho(v), d^{cu}(v)$ analogously. Notice that when ρ is small these intrinsic metrics are uniformly equivalent to d_S and d_K.

Remark 3.6. A handy feature of these metrics is that for any $v \in T^1 M$, $\sigma \in \{s, cs\}$ and for any $u, w \in W^\sigma$ the map $t \mapsto d^\sigma(f_t u, f_t w)$ is a non-increasing function. Indeed, let γ be a curve in $W^s(v)$ connecting u and w. Then $f_t \gamma$ lies in $W^s(f_t v)$. $\{f_s(\gamma)\}_{0 \leqslant s \leqslant t}$ is a one-parameter family of geodesics and the associated Jacobi fields are all stable. Since stable Jacobi fields are non-increasing on manifolds without focal points (proposition 3.5 (9)), the length of γ is not less than the length of $f_t(\gamma)$.

Similarly, for $\sigma \in \{u, cu\}$, $t \mapsto d^\sigma(f_t u, f_t w)$ is non-decreasing. These features are used in establishing the specification property in section 5.

Following proposition 3.5, one can define the *stable horosphere* $H^s(v) \subset M$ and the *unstable horosphere* $H^u(v) \subset M$ as the projection of the respective foliations to M:

$$H^s(v) = \pi(W^s(v)) \text{ and } H^u(v) = \pi(W^u(v)).$$

We now summarize some useful properties of them.

Proposition 3.7 ([Esc77, theorem 1i, ii]). *Let M be a Riemannian closed manifold without focal points. Then we have*

(1) $H^u(v)$, $H^s(v)$ are C^2-embedded hypersurfaces when lifted to the universal cover $\widetilde{M}$.
(2) For $\sigma \in \{s, u\}$, thesymmetric linear operator $\mathcal{U}^\sigma(v) : T_{\pi v} H^\sigma(v) \to T_{\pi v} H^\sigma(v)$ given by $v \mapsto \nabla_v N$, i.e. the shape operator on $H^\sigma(v)$, is well-defined, where N is the unit normal vector field on $H^\sigma(v)$ toward the same side as v.
(3) $\mathcal{U}^u$ is positively semidefinite and $\mathcal{U}^s$ is negatively semidefinite.

We are ready to rephrase above two propositions specific to the surface setting. From now on, we denote by S a closed Riemannian surface of genus at least 2 without focal points. Then from propositions 3.5 and 3.7 we have:

- S is rank 1.
- For $v \in T^1 S$, $H^u(v)$ (resp., $H^s(v)$) is one dimensional and called the *unstable* (resp., *stable*) *horocycle*.
- The (one dimensional) linear operator $\mathcal{U}^u(v) : T_{\pi v} H^u(v) \to T_{\pi v} H^u(v)$ is given by the *geodesic curvature $k^u(v)$* of the horocycle of $H^u(v)$ at πv. More precisely, for all $w \in T_{\pi v} H^u(v)$

$$\mathcal{U}^u(v)(w) = k^u(v)w.$$

- Similarly, $\mathcal{U}^s(v)$ is given by $k^s(v)$ the *geodesic curvature $k^s(v)$* of the horocycle of $H^s(v)$ at πv, i.e. $\mathcal{U}^s(v)(w) = -k^s(v)w$ for all $w \in T_{\pi v} H^s(v)$. Moreover, $k^s(-v) = k^u(v)$ which follows from the fact that $H^s(v) = H^u(-v)$.

3.2. Hyperbolicity indices λ and λ_T

In this subsection, using k^s and k^u we introduce several useful functions to quantify the hyperbolicity for any $v \in T^1 S$. These hyperbolicity indices will be used in section 4 to derive the decomposition for orbit segments.

Definition 3.8. For $v \in T^1 S$ and for any $T > 0$, we define:

(1) $\lambda(v) := \min(k^u(v), k^s(v))$.
(2) $\lambda_T(v) := \int_{-T}^{T} \lambda(f_\tau v) \mathrm{d}\tau$.

Remark 3.9.

(1) Since the horocycles are C^2 (by proposition 3.7), we have k^s and k^u are continuous, and so are λ and λ_T.
(2) The λ defined in this paper is exactly the same as the λ introduced in [BCFT18].

Remark 3.10. The main difference between the 'nonpositively curved' setting in [BCFT18] and our 'no focal points' setting is the following: in nonpositively curved manifolds, the norm of Jacobi fields is convex, while it is not necessarily true in manifolds with no focal points. As one can observe in [BCFT18], the convexity on the norm of Jacobi fields can be used to

deduce good estimates on λ (for instance, lemma 3.3 in [BCFT18]), and one can use λ to characterize the singular set.

However, λ does not enjoy such properties in our setting. In order to equip λ with nice properties as in [BCFT18], we introduce a new function λ_T by integrating λ for a longer time T. While $\lambda(v)$ in no focal points setting does not capture the hyperbolicity at v, the integrated function λ_T for large enough T is successful in distinguishing Sing from Reg, and this is the main motivation for introducing the new function λ_T.

The following proposition and lemma establish relations between horocycles and related Jacobi fields. The version we state below is from [BCFT18, lemma 2.9].

Proposition 3.11. *Let $\gamma_v(t)$ be a unit speed geodesic such that $\dot{\gamma}_v(0) = v$, and J^u be the $H^u(v)$-Jacobi field along γ_v, that is, the Jacobi field derived by varying through geodesics perpendicular to $H^u(v)$ and satisfying $\|J^u(0)\| = 1$. Then $J^u \in \mathcal{J}^u$ and*

$$(J^u)'(t) = k^u(f_t v)J^u(t) \text{ for all } t \in \mathbb{R}. \tag{3.4}$$

Similarly, for J^s the $H^s(v)$-Jacobi field along γ_v, we have $J^s \in \mathcal{J}^s$ and $(J^s)'(t) = -k^s(f_t v)J^s(t)$ for all $t \in \mathbb{R}$.

Proof. Let $\alpha(s,t)$ for $(s,t) \in (-\varepsilon, \varepsilon) \times \mathbb{R}$ be the variation of geodesics along $H^u(v)$, i.e. $\alpha(0,t) = \gamma_v(t)$ and $\alpha(s,0) \in H^u(v)$, such that $\frac{\partial}{\partial s}\alpha(s,t)\big|_{s=0} = J^u(t)$. Then, for $t = 0$

$$(J^u)'(0) = \frac{\nabla}{\partial t}\frac{\partial}{\partial s}\alpha(s,t)\bigg|_{s=0,t=0} = \frac{\nabla}{\partial s}\frac{\partial}{\partial t}\alpha(s,t)\bigg|_{s=0,t=0}$$
$$= \nabla_{J^u(0)}N = \mathcal{U}^u(v)(J^u(0)) = k^u(v)J^u(0),$$

where the second equality is by the symmetry of the Levi-Civita connection and the last equality follows from proposition 3.7.

To see this is true for all t, we notice that the flow invariant unstable manifold $W^u(v)$ consists of vectors which are perpendicular to $H^u(v)$ and point toward to the same side as v (see proposition 3.5). That is, when we vary geodesics perpendicularly along $H^u(v)$, these geodesics vary perpendicularly along $H^u(f_t v)$ as well. Thus, $J^u(t)$ is the Jacobi field derived by varying geodesics perpendicular to $H^u(f_t v)$, and we have $(J^u)'(t) = k^u(f_t v)J^u(t)$ by repeating the computation above. For J^s, the same argument applies. $\square$

Let Λ be the maximum value of the function k^u:

$$\Lambda := \max_{v \in T^1 S} k^u(v) = \max_{v \in T^1 S} k^s(v). \tag{3.5}$$

From the proposition above, for $\sigma \in \{s, u\}$ we have $\|(J^\sigma)'(t)\| \leqslant \Lambda \|J^\sigma(t)\|$ for all t. Then by equation (3.2), for any $\xi \in E^u(v)$ or $E^s(v)$ we have

$$\|J_\xi(t)\|^2 \leqslant \|\mathrm{d}f_t \xi\|^2 \leqslant (1 + \Lambda^2)\|J_\xi(t)\|^2.$$

The following lemma is an immediate consequence of proposition 3.11 obtained by integrating (3.4), and it is the analogue of [BCFT18, lemma 2.11]

Lemma 3.12. *Let $v \in T^1 S$ and J^u (resp. J^s) be an unstable (resp. stable) Jacobi field along γ_v. Then*

$$\|J^u(t)\| \geqslant e^{\int_0^t k^u(f_\tau v)\mathrm{d}\tau}\|J^u(0)\| \text{ and } \|J^s(t)\| \leqslant e^{-\int_0^t k^s(f_\tau v)\mathrm{d}\tau}\|J^s(0)\|. \tag{3.6}$$

A handy lemma for computation:

Lemma 3.13. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function and*

$$\psi_T(t) := \int_{-T}^{T} \psi(t+\tau)\mathrm{d}\tau.$$

Then, for every $a \leqslant b$,

$$\int_a^b \psi_T(t)\mathrm{d}t \leqslant 2T \int_{a-T}^{b+T} \psi(t)\mathrm{d}t.$$

Moreover, we have

$$\frac{1}{2T} \int_0^t \lambda_T(f_\tau v)\mathrm{d}\tau - 2T\Lambda \leqslant \int_0^t \lambda(f_\tau v)\mathrm{d}\tau$$

where $\Lambda := \max\limits_{v \in T^1 S} k^u(v) = \max\limits_{v \in T^1 S} k^s(v)$ as in (3.5).

Proof. For $b - a \leqslant 2T$,

$$\int_a^b \psi_T(t)\mathrm{d}t = \int_a^b \int_{-T}^T \psi(t+\tau)\mathrm{d}\tau\mathrm{d}t$$

$$= \int_{a-T}^{b-T} (\tau + T - a)\psi(\tau)\mathrm{d}\tau + \int_{b-T}^{a+T} (b - a)\psi(\tau)\mathrm{d}\tau + \int_{a+T}^{b+T} (b + T - \tau)\psi(\tau)\mathrm{d}\tau$$

$$\leqslant (b - a) \int_{a-T}^{b-T} \psi(\tau)\mathrm{d}\tau + (b - a) \int_{b-T}^{a+T} \psi(\tau)\mathrm{d}\tau + (b - a) \int_{a+T}^{b+T} \psi(\tau)\mathrm{d}\tau$$

$$= (b - a) \int_{a-T}^{b+T} \psi(\tau)\mathrm{d}\tau \leqslant 2T \int_{a-T}^{b+T} \psi(\tau)\mathrm{d}\tau.$$

For $b - a \geqslant 2T$,

$$\int_a^b \psi_T(t)\mathrm{d}t = \int_a^b \int_{-T}^T \psi(t+\tau)\mathrm{d}\tau\mathrm{d}t$$

$$= \int_{a-T}^{a+T} (\tau + T - a)\psi(\tau)\mathrm{d}\tau + \int_{a+T}^{b-T} 2T\psi(\tau)\mathrm{d}\tau + \int_{b-T}^{b+T} (s + T - \tau)\psi(\tau)\mathrm{d}\tau$$

$$\leqslant 2T \int_{a-T}^{a+T} \psi(\tau)\mathrm{d}\tau + 2T \int_{a+T}^{b-T} \psi(\tau)\mathrm{d}\tau + 2T \int_{b-T}^{b+T} \psi(\tau)\mathrm{d}\tau = 2T \int_{a-T}^{b+T} \psi(\tau)\mathrm{d}\tau.$$

Since $\Lambda \geqslant \max\limits_{v \in T^1 S} \lambda(v)$, the last assertion follows from

$$\int_0^t \lambda(f_\tau v)\mathrm{d}\tau = \int_{-T}^{T+t} \lambda(f_\tau v)\mathrm{d}\tau - \int_{-T}^0 \lambda(f_\tau v)\mathrm{d}\tau - \int_t^{T+t} \lambda(f_\tau v)\mathrm{d}\tau$$

$$\geqslant \int_{-T}^{T+t} \lambda(f_\tau v)\mathrm{d}\tau - 2T\Lambda$$

$$\geqslant \frac{1}{2T} \int_0^t \lambda_T(f_\tau v)\mathrm{d}\tau - 2T\Lambda.$$
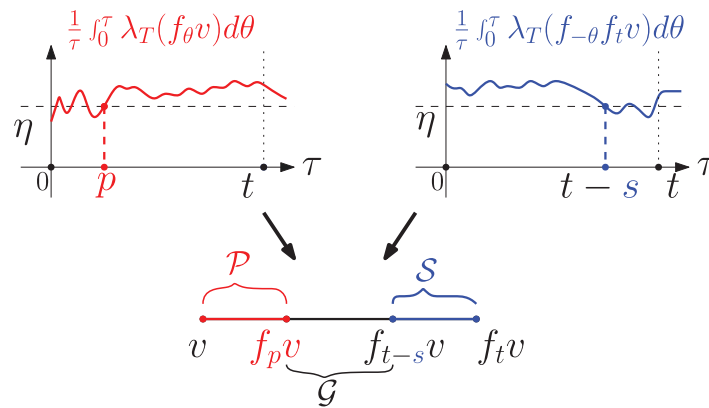
This completes the proof. □

Figure 2. Orbit decomposition.

4. A decompositions of finite orbit segments

4.1. Sing, λ, λ_T, and decompositions

In this subsection, we discuss a decomposition given by λ_T. This decomposition will allow us to apply the Climenhaga–Thompson criteria (i.e. theorem 2.10) to prove the uniqueness of equilibrium states. Throughout the section, we retain the same notations as previous sections.

Definition 4.1 (Good orbits and bad orbits). For any $T, \eta > 0$, we define the two collections of finite orbit segments $\mathcal{G}_T(\eta), \mathcal{B}_T(\eta) \subset T^1 S \times [0, \infty)$ using λ_T:

$$\mathcal{G}_T(\eta) := \{(v,t) \colon \int_0^\tau \lambda_T(f_\theta v) \mathrm{d}\theta \geqslant \tau\eta \text{ and } \int_0^\tau \lambda_T(f_{-\theta} f_t v) \mathrm{d}\theta \geqslant \tau\eta \ \forall \tau \in [0,t]\},$$

$$\mathcal{B}_T(\eta) := \{(v,t) \colon \int_0^t \lambda_T(f_\theta v) \mathrm{d}\theta < t\eta\}.$$

Using $\mathcal{G}_T(\eta)$ and $\mathcal{B}_T(\eta)$, we define the orbit decomposition

$$(\mathcal{P}, \mathcal{G}, \mathcal{S}) = (\mathcal{B}_T(\eta), \mathcal{G}_T(\eta), \mathcal{B}_T(\eta)).$$

More precisely, we define three maps $p, g, s \colon T^1 S \times [0, \infty) \to \mathbb{R}$ as follows. For any given finite orbit segment (v,t), we let $p = p(v,t) \in [0,t]$ be the largest time such that $(v,p) \in \mathcal{B}_T(\eta)$. We then let $s = s(v,t) \in [0, t-p]$ be the largest time such that $(f_{t-s}v, s) \in \mathcal{B}_T(\eta)$, and let $g = g(v,t) = t - s - p$ be the remaining time in the middle. From the choice of p and s, it is not hard to see that $(f_p v, g) \in \mathcal{G}_T(\eta)$. Indeed, if $(f_p v, g)$ did not belong to $\mathcal{G}_T(\eta)$, then one of (or both) p and s can be increased, and this would contradict the choice of p or s as the largest time such that $(v,p) \in \mathcal{B}_T(\eta)$ and $(f_{t-s}v, s) \in \mathcal{B}_T(\eta)$. Please see figure 2 for an example.

Proposition 4.2. *We have:*

(1) Sing *is closed and flow invariant.*
(2) $G_T(\eta) \subset T^1 S \times \mathbb{R}$ *is closed.*
(3) Reg *is dense in* T^1S.

Proof. These assertions are rather straightforward from their definitions (notice that λ_T is continuous). Nevertheless, we elaborate a little more on the last one since it is less obvious

than others. Notice that the geodesic flow is topologically transitive (see proposition 3.5), so there exists a dense orbit $\gamma \subset T^1 S$. Since Reg is an open set, there exists $t \in \mathbb{R}$ such that $\gamma(t) \in$ Reg, and which implies that $\gamma \subset$ Reg because Sing is flow invariant. $\qquad\square$

4.2. Uniform estimates on $\mathcal{G}_T(\eta)$

Let $T, \eta > 0$ be given, and suppose $T > 1$. From the compactness of $T^1 S$, the functions λ and λ_T are uniformly continuous, so there exists $\delta = \delta(T, \eta)$ such that

$$d_K(v, w) < \delta \implies |\Theta(v) - \Theta(w)| < \frac{\eta}{4T}, \tag{4.1}$$

where Θ is one of λ or λ_T.

Also, define

$$\widetilde{\lambda}(v) = \max\left\{0, \lambda(v) - \frac{\eta}{4T}\right\}.$$

Then, for $w \in B_t(v, \delta)$, we have

$$\int_0^t \lambda(f_\tau w)\mathrm{d}\tau \geqslant \int_0^t \widetilde{\lambda}(f_\tau v)\mathrm{d}\tau \geqslant \int_0^t \lambda(f_\tau v)\mathrm{d}\tau - \frac{\eta t}{4T}. \tag{4.2}$$

It follows from (4.2) and lemma 3.13 that

$$\begin{aligned}
\int_0^t \widetilde{\lambda}(f_\tau v)\mathrm{d}\tau &\geqslant \int_0^t \lambda(f_\tau v)\mathrm{d}\tau - \frac{\eta t}{4T}, \\
&\geqslant \frac{1}{2T}\int_0^t \lambda_T(f_\tau v)\mathrm{d}\tau - 2T\Lambda - \frac{\eta t}{4T},
\end{aligned} \tag{4.3}$$

where $\Lambda = \max\limits_{v \in T^1 S} k^u(v)$ as in (3.5).

Lastly, using the notations above, we have the following control of the expansion and contraction along stable and unstable leaves.

Lemma 4.3 ([BCFT18, lemma 3.10]).   *For any $T, \eta > 0$, pick $\delta = \delta(T, \eta)$ as in (4.1). Then for any $v \in T^1 S$ and $w, w' \in W_\delta^s(v)$, we have the following for every $t \geqslant 0$:*

$$d^s(f_t w, f_t w') \leqslant d^s(w, w')\mathrm{e}^{-\int_0^t \widetilde{\lambda}(f_\tau v)\mathrm{d}\tau}.$$

Similarly, if $w, w' \in W_\delta^u(v)$, then for any $t \geqslant 0$,

$$d^u(f_{-t} w, f_{-t} w') \leqslant d^u(w, w')\mathrm{e}^{-\int_0^t \widetilde{\lambda}(f_{-\tau} v)\mathrm{d}\tau}.$$

Remark 4.4.   Lemma 4.3 can be proved in the exact same way as [BCFT18, lemma 3.10]. Although the setting of [BCFT18] is nonpositively curved manifolds and $\tilde{\lambda}$ in [BCFT18] is slightly different from our $\tilde{\lambda}$, the proof of [BCFT18, lemma 3.10] still applies to lemma 4.3 without any modification. Indeed, the proof of [BCFT18, lemma 3.10] is based on [BCFT18, lemma 2.11], and we have the corresponding lemma 3.12 available in our setting as well. The difference in the definitions of $\tilde{\lambda}$ also does not cause any problem because the only inequality used in proving [BCFT18, lemma 3.10] is $\lambda \geqslant \tilde{\lambda}$, and this inequality remains true from the definition of $\tilde{\lambda}$.

The following lemma refines lemma 4.3. In other words, it provides us with a nice control on the expansion and contraction for orbit segments in G_T.

Lemma 4.5. *For any* $T > 1$ *and* $\eta > 0$, *pick* $\delta = \delta(T, \eta)$ *as in (4.1), and suppose* $(v, t) \in G_T(\eta)$. *Then every* $v' \in B_t(v, \delta)$ *satisfies* $(v', t) \in G_T(\frac{\eta}{2})$.

Moreover, there exists $C = C(T, \eta) > 0$ *such that for any* $(v, t) \in \mathcal{G}_T(\eta)$, *any* $w, w' \in W_\delta^s(v)$ *and any* $0 \leqslant \tau \leqslant t$,

$$d^s(f_\tau w, f_\tau w') \leqslant C d^s(w, w') e^{-\frac{\eta}{4T}\tau}.$$

Similarly, for $w, w' \in f_{-t} W_\delta^u(f_t v)$ *and* $0 \leqslant \tau \leqslant t$, *we have*

$$d^u(f_\tau w, f_\tau w') \leqslant C d^u(f_t w, f_t w') e^{-\frac{\eta}{4T}(t-\tau)}.$$

Proof. The first statement follows from the choice of $\delta = \delta(T, \eta)$ in (4.1): for any $v' \in B_t(v, \delta)$ where $(v, t) \in G_T(\eta)$ and any $0 \leqslant \tau \leqslant t$, we have

$$\int_0^\tau \lambda_T(f_\theta v') \mathrm{d}\theta \geqslant \int_0^\tau \lambda_T(f_\theta v) \mathrm{d}\theta - \tau \cdot \frac{\eta}{4T} > \tau\eta - \frac{\tau\eta}{4T} > \frac{\tau\eta}{2}.$$

The last inequality used the assumption that $T > 1$. Similarly, $\int_0^\tau \lambda_T(f_{-\theta+t} v') \mathrm{d}\theta > \tau\eta/2$. Hence, $(v', t) \in G_T(\eta/2)$.

By lemma 4.3 and inequality (4.3), since $(v, t) \in G_T(\eta)$, we have

$$d^s(f_\tau w, f_\tau w') \leqslant d^s(w, w') e^{-\int_0^\tau \widetilde{\lambda}(f_x v) \mathrm{d}x},$$

$$\leqslant d^s(w, w') \exp\left(\frac{-1}{2T} \underbrace{\int_0^\tau \lambda_T(f_x v) \mathrm{d}x}_{\geqslant \tau\eta} + 2T\Lambda + \frac{\eta\tau}{4T} \right),$$

$$\leqslant d^s(w, w') \exp\left(\frac{-\eta\tau}{2T} + 2T\Lambda + \frac{\eta\tau}{4T} \right) = C \cdot d^s(w, w') e^{-\frac{\eta\tau}{4T}},$$

where $C = e^{2T\Lambda}$. Similarly, we have the other inequality. $\qquad\square$

Definition 4.6. We define the uniformly regular set as

$$\mathrm{Reg}_T(\eta) := \{v \in T^1 S : \lambda_T(v) \geqslant \eta\}.$$

Lemma 4.7. *Given* $\eta, T > 0$, *there exists* $\theta > 0$ *so that for any* $v \in \mathrm{Reg}_T(\eta)$, *we have for any* $-T \leqslant t \leqslant T$

$$\angle(E^u(f_t v), E^s(f_t v)) \geqslant \theta.$$

Proof. Assume the contrary. Then there exists $\{(v_i, t_i)\}_{i \in \mathbb{N}} \subset \mathrm{Reg}_T(\eta) \times [-T, T]$ such that

$$\angle(E^s(f_{t_i} v_i), E^u(f_{t_i} v_i)) \to 0.$$

Since $\mathrm{Reg}_T(\eta) \times [-T, T]$ is compact, there exist subsequences $t_{i_j} \to t_0$, and $v_{i_j} \to v_0$ such that $\angle(E^s(f_{t_0} v_0), E^u(f_{t_0} v_0)) = 0$. Then, $f_{t_0} v_0 \in \mathrm{Sing}$ from proposition 3.5 (5). On the other hand, $\mathrm{Reg}_T(\eta)$ is closed so $v_0 \in \mathrm{Reg}_T(\eta)$. However, this is a contradiction because Sing is flow invariant. $\qquad\square$

4.3. Relations between $k^s, k^u, \lambda, \lambda_T$, *and* Sing

The aim of this subsection is to show how one can use these hyperbolicity indices λ and λ_T to characterize the singular set Sing.

Lemma 4.8. *The following are equivalent for* $v \in T^1 S$.

(1) $v \in$ Sing.
(2) $k^u(f_t v) = 0$ *for all* $t \in \mathbb{R}$.
(3) $k^s(f_t v) = 0$ *for all* $t \in \mathbb{R}$.

Proof. It is clear that $(1) \Longrightarrow (2)$ and (3). We will prove $(2) \Longrightarrow (1)$ which then $(3) \Longrightarrow (1)$ similarly follows.

To see $(2) \Longrightarrow (1)$, it is enough to show that J^u the unstable Jacobi field along γ_v is parallel. By proposition 3.11, we have for all $t \in \mathbb{R}$

$$(J^u)'(t) = k^u(f_t v) J^u(t) = 0.$$

Thus J^u is a parallel Jacobi field. $\square$

Lemma 4.9. $\lambda_T(v) = 0$ *for all* T *if and only if* $v \in$ Sing.

Proof. The if direction is clear. In the following we prove the only if direction.

First we notice that since λ is non-negative, continuous, we have that $\lambda_T(v) = 0$ for all $T \in \mathbb{R}$ implies $\lambda(f_t v) = 0$ for all $t \in \mathbb{R}$.

Claim: There are only three possible cases such that $\lambda(f_t v) = 0$ for all $t \in \mathbb{R}$:

  (i) $k^s(f_t v) = 0$ for all $t \in \mathbb{R}$.
 (ii) $k^u(f_t v) = 0$ for all $t \in \mathbb{R}$.
(iii) There exists $t_0 \in \mathbb{R}$ such that $k^s(f_{t_0} v) = k^u(f_{t_0} v) = 0$.

It is clear from lemma 4.8 that both (i) and (ii) give $v \in$ Sing. To see (iii) also implies $v \in$ Sing, we recall that, for $\sigma \in \{s, u\}$, $k^\sigma(f_{t_0} v) = 0$ implies that there exists $0 \neq w^\sigma \in T_{\pi(f_{t_0} v)} H^\sigma(f_{t_0} v)$ such that $k^\sigma(w^\sigma) = 0$. Since both w^u, w^s are orthogonal to $f_{t_0} v$ and S is a surface, we know $w^u = w^s$ (by taking the same length, and reversing the sign if necessary). It is not hard to see that the $H^u(f_{t_0} v)$-Jacobi field J^u matches the $H^s(f_{t_0} v)$-Jacobi field J^s, that implies, $E^u(f_{t_0} v) \cap E^s(f_{t_0} v) \neq 0$. Thus we have $f_{t_0} v \in$ Sing, and because Sing is flow invariant we have $v \in$ Sing.

To see the claim, let $U := \{t \in \mathbb{R} : k^u(f_t v) = 0\}$ and $W := \{t \in \mathbb{R} : k^s(f_t v) = 0\}$. Since both k^u, k^s are continuous, U and W are closed sets in $\mathbb{R}$. Notice that if $U \cap W = \emptyset$ then $U = \mathbb{R} \backslash W$; thus U, W are clopen sets. Since $\mathbb{R}$ is connected, if $U \cap W = \emptyset$, then $U = \mathbb{R}$ or $W = \mathbb{R}$. $\square$

Remark 4.10.

(1) We remark that we are using the fact that S is a surface in the proof of lemma 4.9. Indeed, in the process of showing that $v \in$ Sing from $\lambda_T(v) = 0$ for all T, we obtained a parallel Jacobi field along v by showing that the stable Jacobi field J^s is equal the unstable Jacobi field J^u, and this step required that S is a surface.

(2) There are a few other places in this paper where we make use of the fact that S is a surface. For instance, when we establish the Bowen property for the scalar multiples of the geometric potential in section 6, our analysis heavily rely on the fact that S is a 2-dimensional manifold. Moreover, the Bernoulli property on μ_φ from theorem B as well as the differentiability of the map $q \mapsto P(q\varphi^u)$ on the interval $(-\infty, 1)$ from theorem C rely on the fact that S is a surface.

Lemma 4.11. *Let μ be a $\mathcal{F}$-invariant probability measure on T^1S. Suppose $\lambda(v) = 0$ for μ-a.e. $v \in T^1S$, then $\mathrm{supp}(\mu) \subset \mathrm{Sing}$.*

Proof. Suppose $\mathrm{supp}(\mu) \not\subseteq \mathrm{Sing}$. Since μ is Borel, there exists $v \in \mathrm{Reg} \cap \mathrm{supp}(\mu)$ such that for any $r > 0$ we have $\mu(B(v, r)) > 0$. We also notice that since $v \in \mathrm{Reg}$ there exists t_0 such that $\lambda(f_{t_0}v) > 0$ (otherwise $v \in \mathrm{Sing}$ by lemma 4.9). By the continuity of λ, there exists a neighborhood $B(f_{t_0}v, r_0)$ of $f_{t_0}v$ such that $\lambda|_{B(f_{t_0}v, r_0)} > 0$. Then there exists $r > 0$ such that $B(v, r) \subset f_{-t_0}(B(f_{t_0}v, r_0))$ and we have

$$\mu(B(f_{t_0}v, r_0)) = \mu(f_{-t_0}(B(f_{t_0}v, r_0))) \geqslant \mu(B(v, r)) > 0.$$

Hence, λ cannot vanish μ-almost everywhere. $\qquad\square$

5. The specification property

Let X be a compact metric space with metric d and $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ be a flow on X. For any $t \in \mathbb{R}^+$, we set $d_t(v, w) = \sup_{s \in [0,t]} d(f_s v, f_s w)$ for any $v, w \in X$.

In what follows, X will be T^1S and d the metric d_K. With respect to the intrinsic metric d^{cs} and d^u on W^{cs} and W^u, these metrics relate to each other by (from the fact that the stable manifold is non-increasing in forward time; see remark 3.6)

$$d_K(v, w) \leqslant d^{cs}(v, w) \text{ and } d_K(v, w) \leqslant \mathrm{e}^\Lambda d^u(v, w)$$

where $\Lambda = \max_{v \in T^1S} k^u(v) = \max_{v \in T^1S} k^s(v)$ as defined in (3.5). This then implies

$$\begin{aligned} d_t(v, w) &\leqslant d^{cs}(v, w), \\ d_t(v, w) &\leqslant d^u(f_{t+1}v, f_{t+1}w) \leqslant \mathrm{e}^\Lambda d^u(f_t v, f_t w). \end{aligned} \tag{5.1}$$

Definition 5.1. The foliations W^{cs} and W^u have *local product structure* at scale $\delta > 0$ with constant $\kappa \geqslant 1$ at v if for any $w_1, w_2 \in B(v, \delta)$, the intersection $[w_1, w_2] := W^u_{\kappa\delta}(w_1) \cap W^{cs}_{\kappa\delta}(w_2)$ is a unique point and satisfies

$$\begin{aligned} d^u(w_1, [w_1, w_2]) &\leqslant \kappa d_K(w_1, w_2), \\ d^{cs}(w_2, [w_1, w_2]) &\leqslant \kappa d_K(w_1, w_2). \end{aligned}$$

For any $T, \eta > 0$, we define $\mathcal{C}_T(\eta) := \{(v, t) : v, f_t v \in \mathrm{Reg}_T(\eta)\}$. The uniform lower bound of λ_T on the endpoints of the orbits in $\mathcal{C}_T(\eta)$ guarantees the uniform local product structure on $\mathcal{C}_T(\eta)$:

Lemma 5.2. *For any $T, \eta > 0$, there exist $\delta > 0$ and $\kappa \geqslant 1$ such that $\mathcal{C}_T(\eta)$ has local product structure at scale δ with constant κ.*

Proof. The lemma follows from the uniform angle gap from lemma 4.7 together with the continuity of the distribution E^s and E^u. $\qquad\qquad\square$

The following proposition is due to the transitivity of the geodesic flow.

Proposition 5.3. *Let* $T, \eta > 0$ *be given. Then there exists* $\delta > 0$ *such that for any* $\rho \in (0, \delta]$, *there exists* $a = a(\rho)$ *such that the following holds: for any* $v, w \in T^1 M$ *with* $d_K(v, \text{Reg}_T(\eta)) < \delta$ *and* $d_K(w, \text{Reg}_T(\eta)) < \delta$, *there exists* $\tau \in [0, a]$ *and* $[v, w]_\tau \in T^1 S$ *such that*

$$[v, w]_\tau \in W^u_\rho(v) \text{ and } f_\tau [v, w]_\tau \in W^{cs}_\rho(w). \tag{5.2}$$

Proof. Let ε and κ be the constants from the local product structure on $\text{Reg}_T(\eta)$. By taking $\delta \in (0, \varepsilon/2)$ sufficiently small, we can ensure that the δ-neighborhood of $\text{Reg}_T(\eta)$ has local product structure at scale $\varepsilon/2$ with constant 2κ. Now using the transitivity of the flow $\mathcal{F}$, for any $\rho \in (0, \delta)$, we can find $a = a(\rho)$ such that the following holds: for any v, w, there exists $x = x(v, w) \in B(v, \rho/4\kappa^2)$ and $\tau \in (0, a)$ with $f^\tau x \in B(w, \rho/4\kappa^2)$.

If v, w happen to be δ-close to $\text{Reg}_T(\eta)$, then the uniform local product structure on δ-neighborhood of $\text{Reg}_T(\eta)$ gives $[v, w]_\tau$ as follows: take $z = [v, x]$ and set $[v, w]_\tau := f_{-\tau}[f_\tau z, w]$. Then, $[v, w]_\tau := f_{-\tau}[f_\tau z, w]$ satisfies (5.2). $\qquad\square$

Remark 5.4. It is worth noting that the choices of τ and $[v, w]_\tau$ are not unique; we simply choose any one of $[v, w]_\tau$'s that satisfy (5.2).

Proposition 5.5. *For any* $\eta, T > 0$, $\mathcal{C}_T(\eta)$ *has specification as in definition* 2.5. *Hence, so does* $G_T(\eta)$.

Proof. Let $T, \eta > 0$ be given. We begin by fixing any regular periodic orbit (v'_0, t'_0) as our reference orbit. From Lemma 4.9, there exists $T', \eta' > 0$ such that the entire orbit segment (v'_0, t'_0) is contained in $\text{Reg}_{T'}(\eta')$. By comparing T' and the given T, we re-define T as the larger of the two. Similarly, we re-define η as the smaller of η' and the given η. It then follows that $(v'_0, t'_0) \in G_T(\eta)$. We set $v_0 := f_{-T} v'_0$ and $t_0 := 2T + t'_0$. Then (v_0, t_0) is just an extended orbit segment obtained from (v'_0, t'_0), and its endpoints $v_0, f_{t_0} v_0$ belong to $\text{Reg}_T(\eta)$.

Using the uniform continuity of λ, we can choose $\delta_1 > 0$ such that $|\lambda(v) - \lambda(w)| < \frac{\eta t'_0}{4 T t_0}$ whenever $d_K(v, w) < \delta_1$. For such choice of δ_1, for any $w \in B_{t_0}(v_0, \delta_1)$ we have

$$\begin{aligned} 2T \int_0^{t_0} \lambda(f_s w)\mathrm{d}s &\geqslant 2T \int_0^{t_0} \lambda(f_s v_0)\mathrm{d}s - (2T) t_0 \cdot \frac{\eta t'_0}{4 T t_0}, \\ &\geqslant \int_0^{t'_0} \lambda_T(f_s v'_0)\mathrm{d}s - \frac{\eta t'_0}{2}, \\ &\geqslant \eta t'_0 - \frac{\eta t'_0}{2} = \frac{\eta t'_0}{2}. \end{aligned}$$

The second and third inequalities are due to lemma 3.13 and the assumption that $(v'_0, t'_0) \in G_T(\eta)$, respectively. In particular, setting $\alpha := \exp(\frac{\eta t'_0}{4T}) > 1$, for any $w, w' \in B_{t_0}(v_0, \delta_1)$ with $w' \in f_{-t_0} W^u_{\delta_1}(f_{t_0} w)$, we have

$$\alpha d^u(w, w') \leqslant d^u(f_{t_0} w, f_{t_0} w'). \tag{5.3}$$

Let $\delta_2 > 0$ be from proposition 5.3, and set $\delta := \min\{\delta_1, \delta_2\}$. Given an arbitrary small scale $0 < \rho < \delta$, we will show that by setting $\rho' := \rho/\big(6\mathrm{e}^\Lambda \sum_{i=1}^\infty \alpha^{-i}\big)$, $\mathcal{C}_T(\eta)$ has specification at scale ρ with corresponding $\tau(\rho) := t_0 + 2a$ where $a := a(\rho')$ is from proposition 5.3.

Let $(v_1, t_1), \ldots, (v_n, t_n) \in \mathcal{C}_T(\eta)$ be given. We will inductively define orbit segments (w_j, s_j) such that for each $1 \leqslant j \leqslant n$, we have

$$f_{s_j} w_j \in W^{cs}_{\rho'}(f_{t_j} v_j). \tag{5.4}$$

We begin by setting $(w_1, s_1) := (v_1, t_1)$. Supposing that (w_j, s_j) satisfies (5.4), we want to define (w_{j+1}, s_{j+1}) in a way that the orbit of w_{j+1} closely shadows that of w_j for time s_j, then jumps (via proposition 5.3 with transition time $\leqslant a$) to v_0 and shadows v_0 for time t_0, then jumps to (again via proposition 5.3) and shadows v_{j+1} for time t_{j+1}.

Since proposition 5.3 only allows one jump at a time, we define an auxiliary orbit segment

$$(u_j, l_j) := \big(f_{-s_j}[f_{s_j} w_j, v_0]_{\tau_j}, s_j + \tau_j + t_0\big)$$

by applying proposition 5.3 to $f_{s_j} w_j$ and v_0. Note that proposition 5.3 can be successfully applied because $f_{s_j} w_j \in W^{cs}_{\rho'}(f_{t_j} v_j)$ from (5.4) and $f_{t_j} v_j \in \mathrm{Reg}_T(\eta)$ from $(v_j, t_j) \in \mathcal{C}_T(\eta)$. Moreover, $f_{l_j} u_j \in W^{cs}_{\rho'}(v_0)$ because $f_{s_j + \tau_j} u_j \in W^{cs}_{\rho'}(v_0)$ and d^{cs} does not increase in forward time; see remark 3.6.

We then apply proposition 5.3 again to $f_{l_j} u_j$ and v_{j+1} to obtain

$$(w_{j+1}, s_{j+1}) := \big(f_{-l_j}[f_{l_j} u_j, v_{j+1}]_{\tau_j'}, l_j + \tau_j' + t_{j+1}\big).$$

From the same reasoning as in the construction of (u_j, l_j), the new orbit segment (w_{j+1}, s_{j+1}) is well-defined and $f_{s_{j+1}} w_{j+1} \in W^{cs}_{\rho'}(f_{t_{j+1}} v_{j+1})$.

Now we show that (w_j, s_j) constructed as above shadows each (v_i, t_i) up to $i = j$ with scale ρ'; that is, $d_{t_i}(f_{s_i - t_i} w_j, v_i) < \rho$. Notice that for any $i \leqslant m \leqslant j$, we have

$$d^u(f_{s_i} w_m, f_{s_i} u_m) \leqslant \rho' \alpha^{-(m-i)}.$$

This is because $d^u(f_{s_m} w_m, f_{s_m} u_m) \leqslant \rho'$ from the construction of u_m and each time $f_{s_m} u_m$ and $f_{s_m} w_m$ pass through the reference orbit (v_0, t_0) in backward time, their d^u distance decrease by a factor of at least α from (5.3). Similarly, we have

$$d^u(f_{s_i} u_m, f_{s_i} w_{m+1}) \leqslant \rho' \alpha^{-(1+m-i)}.$$

Hence, for any $i \leqslant j$, we can uniformly bound the d^u distance $d^u(f_{s_i} w_j, f_{s_i} w_i)$ by $\frac{\rho}{3\mathrm{e}^\Lambda}$:

$$\begin{aligned}
d^u(f_{s_i} w_j, f_{s_i} w_i) &\leqslant \sum_{m=i}^{j-1} d^u(f_{s_i} w_m, f_{s_i} w_{m+1}), \\
&\leqslant \sum_{m=i}^{j-1} d^u(f_{s_i} w_m, f_{s_i} u_m) + d^u(f_{s_i} u_m, f_{s_i} w_{m+1}), \\
&\leqslant \rho' \sum_{m=i}^{j-1} \alpha^{-(m-i)} + \rho' \sum_{m=i}^{j-1} \alpha^{-(1+m-i)}, \\
&\leqslant \frac{\rho}{3\mathrm{e}^\Lambda},
\end{aligned}$$

where the last inequality is due to the definition of ρ'. From the relations among various metrics (5.1), we obtain that

$$d_{t_i}(f_{s_i-t_i}w_j, v_i) \leqslant d_{t_i}(f_{s_i-t_i}w_j, f_{s_i-t_i}w_i) + d_{t_i}(f_{s_i-t_i}w_i, v_i),$$
$$\leqslant \frac{\rho}{3e^\Lambda} \cdot e^\Lambda + \rho' \leqslant \rho,$$

where we have used that $d^s(f_{s_i-t_i}w_i, v_i) \leqslant \rho'$ from the construction of w_i. Since ρ was arbitrary, this finishes the proof. $\qquad\square$

One useful corollary of the specification property is the closing lemma which creates lots of periodic orbits, and later allows $\mathcal{C}_T(\eta)$ to be approximated by regular periodic orbits. The proof of the closing lemma below follows the same idea as [BCFT18, lemma 4.7].

Lemma 5.6 (The closing lemma). *For any given $T, \eta, \varepsilon > 0$, there exists $s = s(\varepsilon) > 0$ such that for any $(v,t) \in \mathcal{C}_T(\eta)$ there exists $w \in B_t(v, \varepsilon)$ and $\tau \in [0, s(\varepsilon)]$ with $f_{t+\tau}w = w$.*

Proof.   The proof is based on Brouwer's fixed point theorem. We begin by fixing a regular periodic orbit $(v_0', t_0') \in G_T$ and set $(v_0, t_0) := (f_{-T}v_0', 2T + t_0')$ as in proposition 5.5, after possibly re-defining T and η. Reasoning as in proposition 5.5, there exists $\delta > 0$ such that the distance between any $w, w' \in W_\delta^s(v_0)$ contract (and likewise expand for any $w, w' \in f_{-t_0}W_\delta^u(f_{t_0}v_0)$) under f_{t_0} by factor $\alpha := \exp(\frac{\eta t_0'}{4T})$.

Let $\varepsilon = \varepsilon_0/4$. We may suppose ε is small enough that $\mathcal{C}_T(\eta)$ has local product structure at scale ε and constant κ. Let $n \in \mathbb{N}$ such that $\alpha^n > 2\kappa$. Also, we may assume $nt_0 \geqslant 1 + \varepsilon$ without loss of generality (otherwise, simply increase n).

Now, for any $(v,t) \in \mathcal{C}_T(\eta)$, we use proposition 5.5 to find $w_0 \in B(v, \varepsilon/4\kappa)$ whose orbits shadows (v,t) once, then (v_0, t_0) n-times, and then (v,t) once again at scale $\varepsilon/4\kappa$ with each transition time bounded above by $\widetilde{\tau}$. Since w_0 has to eventually shadow (v,t) again, there exists $\tau_0 \in [nt_0, n(t_0 + \widetilde{\tau}) + \widetilde{\tau}]$ such that $f_{t+\tau_0}w_0 \in B(v, \varepsilon/4\kappa)$. From the triangle inequality pivoted at v, we have $d_K(w_0, f_{t+\tau_0}w_0) < 2 \cdot \varepsilon/4\kappa = \varepsilon/2\kappa$. Also, using the forward contraction of the stable manifold near the reference orbit (v_0, t_0), for any $u \in W_\varepsilon^s(w_0)$, we have

$$d_K(f_{t+\tau_0}u, w_0) \leqslant d_K(f_{t+\tau_0}u, f_{t+\tau_0}w_0) + d_K(f_{t+\tau_0}w_0, w_0),$$
$$\leqslant \alpha^{-n}d_K(u, w_0) + \varepsilon/2\kappa \leqslant \varepsilon/\kappa.$$

Since v has local product structure at scale ε with constant κ and ω_0 is $\varepsilon/4\kappa$-close to v, the point $W_\varepsilon^s(w_0) \cap W_\varepsilon^{cu}(f_{t+\tau_0}u)$ is well-defined and belongs to $W_\varepsilon^s(w_0)$. In particular, the continuous map from $W_\varepsilon^s(w_0)$ to itself given by

$$u \mapsto W_\varepsilon^s(w_0) \cap W_\varepsilon^{cu}(f_{t+\tau_0}u)$$

is well-defined. Hence, by Brouwer fixed point theorem, we can find a fixed point $w_1 \in W_\varepsilon^s(w_0)$ under this map. Since the map is not given by f_s for some s, the fixed point w_1 is not quite $\mathcal{F}$ invariant yet. Instead, its characterizing property is that $w_1 \in W_\varepsilon^{cu}(f_{t+\tau_0}w_1)$.

By adjusting τ_0 by a unique small constant less than ε, we have $w_1 \in W_\varepsilon^u(f_{t+\tau}w_1)$ where τ is adjusted constant from τ_0. Since the unstable manifold shrinks in backward time near (v_0, t_0) by factor α, this time we obtain a continuous map defined by the flow $f_{-t-\tau}$:

$$f_{-t-\tau} \colon W_{2\varepsilon}^u(f_{t+\tau}w_1) \to W_{2\varepsilon}^u(f_{t+\tau}w_1).$$

Hence, the Brouwer fixed point theorem applies again and we obtain $w \in W_{2\varepsilon}^u(f_{t+\tau}w_1)$ with $f_{t+\tau}w = w$. We are left to show that $d_t(v, w) \leqslant \varepsilon_0$. This follows because

$$d_t(v, w) \leqslant d_t(v, w_0) + d_t(w_0, w_1) + d_t(w_1, w),$$
$$\leqslant \varepsilon/4\kappa + d^s(w_0, w_1) + d^u(w_1, w),$$
$$\leqslant \varepsilon/4\kappa + \varepsilon + 2\varepsilon \leqslant \varepsilon_0.$$

Here, we have used (5.1) and the fact that $d_t(w_1, w) \leqslant d^u(f_{t+1}w_1, f_{t+1}w) \leqslant d^u(f_{t+\tau}w_1, f_{t+\tau}w)$ because $\tau \geqslant nt_0 - \varepsilon \geqslant 1$. Lastly, setting $s(\varepsilon_0) := n(t_0 + \widetilde{\tau}) + \widetilde{\tau} + \varepsilon$, we are done. $\qquad\square$

Using the same argument as [BCFT18, corollary 4.8], we have the following corollary of the closing lemma.

Corollary 5.7. *For any given $T, \eta > 0$, there exist $\varepsilon = \varepsilon(T, \eta) > 0$ such that for any $\varepsilon_0 < \varepsilon$ there exists $s = s(\varepsilon_0) > 0$ satisfying the following: for any $(v, t) \in \mathcal{C}_T(\eta)$ there exists*

(1) a regular vector w with $w \in B_t(v, \varepsilon_0)$, and
(2) $\tau \in [0, s]$ with $f_{t+\tau}w = w$.

Proof. From the uniform continuity of λ, there exists $\varepsilon = \varepsilon(\eta) > 0$ such that for all $w \in B(v, \varepsilon)$, we have $\lambda(w) > 0$.

Since $v \in \mathcal{C}_T(\eta)$, there exists $v' = f_\sigma v$ for some $\sigma \in [-T, T]$ such that $\lambda(v') > \eta$. Also, we must have $(v', t + \sigma) \in \mathcal{C}_{2T}(\eta)$ from the definition of $\mathcal{C}_T(\eta)$. By lemma 5.6, for any $2T, \eta, \varepsilon_0 > 0$, there exists $s = s(\varepsilon_0) > 0$ such that $w \in B_{t+\sigma}(v', \varepsilon_0)$ and $\tau \in [0, s(\varepsilon_0)]$ such that $f_{t+\sigma+\tau}(w) = w$.

Also, it follows that w is a regular vector because $\lambda(w) > 0$ from $d_K(v', w) < \varepsilon_0 < \varepsilon$. $\qquad\square$

6. The Bowen property

In this section, we prove the Bowen property for Hölder potentials and the geometric potential φ^u. Lemmas in this section have their corresponding versions in [BCFT18] and the proofs follow the same ideas. Nevertheless, in contrast to [BCFT18], we have an extra time parameter T for accumulating hyperbolicity, thus we have to modify proofs in [BCFT18] accordingly. In particular, we take a slightly different approach from [BCFT18] to derive the Bowen property for geometric potentials, because several crucial estimates in [BCFT18] do not extend to the no focal point setting.

6.1. The Bowen property for Hölder potentials

Definition 6.1. A function $\varphi : T^1S \to \mathbb{R}$ is called *Hölder along stable leaves* if there exist $C, \theta, \delta > 0$ such that for $v \in T^1S$ and $w \in W_\delta^s(v)$, one has $|\varphi(v) - \varphi(w)| \leqslant Cd^s(v, w)^\theta$. Similarly, φ is called *Hölder along unstable leaves* if there exist $C, \theta, \delta > 0$ such that for $v \in T^1S$ and $w \in W_\delta^u(v)$, one has $|\varphi(v) - \varphi(w)| \leqslant Cd^u(v, w)^\theta$.

Since d_K is equivalent to d^u and d^s along unstable and stable leaves when δ is small, we know φ is Hölder implies that φ is Hölder along stable and unstable leaves.

Definition 6.2. A function φ is said to have the *Bowen property along stable leaves* with respect to $\mathcal{C} \subset T^1S \times [0, \infty)$ if there exist $\delta, K > 0$ such that

$$\sup\{|\Phi(v,t) - \Phi(w,t)| : (v,t) \in \mathcal{C}, \, w \in W^s_\delta(v)\} \leqslant K.$$

Similarly, a function φ is said to have the *Bowen property along unstable leaves* with respect to $\mathcal{C} \subset T^1 S \times [0, \infty)$ if there exist $\delta, K > 0$ such that

$$\sup\{|\Phi(v,t) - \Phi(w,t)| : (v,t) \in \mathcal{C}, \, w \in f_{-t} W^u_\delta(f_t v)\} \leqslant K.$$

Lemma 6.3. *For any $T > 1$ and $\eta > 0$, if φ is Hölder along stable leaves (resp. unstable leaves), then φ has the Bowen property along stable leaves (resp. unstable leaves) with respect to $G_T(\eta)$.*

Proof. It is a direct consequence of lemma 4.5. We prove the stable leaves case, and for unstable leaves one uses the same argument.

Let $(v,t) \in G_T(\eta)$, $\delta_1 > 0$ be as in lemma 4.5 and $\delta_2 > 0$ be given by the Hölder continuity along stable leaves. Then for $\delta = \min\{\delta_1, \delta_2\}$ and $w \in W^s_\delta(v)$, we have

$$
\begin{aligned}
|\Phi(v,t) - \Phi(w,t)| &\leqslant \int_0^t |\varphi(f_\tau v) - \varphi(f_\tau w)| \, d\tau \leqslant \int_0^t C_1 \cdot d^s(f_\tau v, f_\tau w)^\theta d\tau \\
&\leqslant \int_0^t C_1 \cdot \left(C d^s(v,w) \cdot e^{-\frac{\eta}{4T}\tau} \right)^\theta d\tau \leqslant C_1 \cdot C^\theta \cdot d^s(v,w)^\theta \int_0^t e^{\frac{-\eta\theta}{4T}\tau} d\tau \\
&\leqslant C_1 \cdot C^\theta \delta^\theta \frac{4T}{\eta\theta}.
\end{aligned}
$$

This completes the proof. $\qquad\square$

It was proved in [BCFT18, lemma 7.4] that the Bowen property along invariant leaves implies the Bowen property on the entire phase space. With minor modification on the proof of [BCFT18, lemma 7.4], we have the following similar result for geodesic flows over manifolds without focal points. More precisely, the lemma below follows after replacing $\mathcal{G}(\eta)$ and [BCFT18, corollary 3.11] used in the proof of [BCFT18, lemma 7.4] by $\mathcal{G}_T(\eta)$ and lemma 4.5, respectively.

Lemma 6.4. *For any $T > 1$ and $\eta > 0$, suppose φ has the Bowen property along stable leaves and unstable leaves with respect to $G_T(\frac{\eta}{2})$. Then $\varphi : T^1 S \to \mathbb{R}$ has the Bowen property on $G_T(\eta)$.*

Summing up two lemmas above, we have the desired result for Hölder potentials:

Theorem 6.5. *If φ is Hölder continuous, then it has the Bowen property with respect to $G_T(\eta)$ for any $T > 1$ and $\eta > 0$.*

6.2. The Bowen property for the geometric potential

Definition 6.6. The *geometric potential* $\varphi^u : T^1 S \to \mathbb{R}$ is defined as: for $v \in T^1 S$

$$\varphi^u(v) := -\lim_{t \to 0} \frac{1}{t} \log \det(\, df_t|_{E^u(v)}) = -\left. \frac{d}{dt} \right|_{t=0} \log \det(\, df_t|_{E^u(v)}).$$

In general, we do not know if φ^u is Hölder continuous. There are some partial results under the nonpositively curved assumption; however, not much is known in the no focal points setting. Nevertheless, in this subsection we prove φ^u has the Bowen property on $\mathcal{G}_T(\eta)$.

We denote by J_v^u the unstable Jacobi field along γ_v with $J_v^u(0) = 1$. Let $U_v^u := (J_v^u)'/J_v^u$. Since J_v^u satisfies the Jacobi equation (3.1), U_v^u is a solution to the Riccati equation

$$U' + U^2 + K(f_t v) = 0.$$

Notice that we also have $U_v^u(t) = k^u(f_t v)$ by proposition 3.11. Notice the following lemma relates $\varphi^u(t)$ and $-U_v^u(t)$.

Lemma 6.7 ([BCFT18, lemma 7.6]).   *There exists a constant C such that for all $v \in T^1 S$ and $t > 0$ we have*

$$\left| \int_0^t \varphi^u(f_\tau v) d\tau - \int_0^t -U_v^u(\tau) d\tau \right| \leqslant C.$$

Proof.   The proof follows exactly as that in [BCFT18]. ψ^u in [BCFT18] is exactly $-U_v^u$ when $n = 2$. $\square$

Hence, in order to prove the Bowen property of φ^u on $\mathcal{G}_T(\eta)$, we only have to prove lemma 6.8 below which follows from lemma 6.9. Lemma 6.8 is similar to proposition 7.7 in [BCFT18]. However, their proof relies heavily on the convexity of Jacobi fields, hence we cannot translate it directly. Nevertheless, in the surface case, comparison of Ricatti solutions is nothing but comparison of real functions, thus we manage to apply different techniques to overcome the absence of convexity.

Lemma 6.8.   *For any $T > 1$ and $\eta > 0$, there are $\delta, Q, \xi > 0$ such that given any $(v,t) \in \mathcal{G}_T(\eta), w_1 \in W_\delta^s(v)$ and $w_2 \in f_{-t} W_\delta^u(f_t v)$, for every $0 \leqslant \tau \leqslant t$ we have*

$$|U_v^u(\tau) - U_{w_1}^u(\tau)| \leqslant Q e^{-\xi\tau},$$

$$|U_v^u(\tau) - U_{w_2}^u(\tau)| \leqslant Q(e^{-\xi\tau} + e^{-\xi(t-\tau)}).$$

Lemma 6.9.   *For any $T > 1$ and $\eta > 0$, there are δ, Q such that given any $(v,t) \in \mathcal{G}_T(\eta), w \in B_t(v, \delta)$, for every $0 \leqslant \tau \leqslant t$ we have*

$$|U_v^u(\tau) - U_w^u(\tau)| \leqslant Q \exp\left(-\frac{\eta\tau}{T}\right) + \int_0^\tau \exp\left(-\int_s^\tau 2\tilde{\lambda}(f_a v) da\right) |K(f_s v) - K(f_s w)| ds.$$

We will show how lemma 6.8 follows from lemma 6.9 first, and then prove lemma 6.9.

Proof of lemma 6.8.   Let $\delta > 0$ be given from lemma 6.9. We will use Q to denote a uniform constant that is updated as necessary when the context is clear.

Since $w_1 \in W_\delta^s(v)$, the smoothness of K together with lemma 4.3 implies

$$|K(f_s v) - K(f_s w_1)| \leqslant Q d_K(f_s v, f_s w_1) \leqslant Q d^s(f_s v, f_s w_1) \leqslant Q\delta \exp\left(-\int_0^s \tilde{\lambda}(f_a v) da\right)$$

for any $s \in [0, t]$. Thus by lemma 6.9, there exists $Q > 0$ such that

$$|U_v^u(\tau) - U_{w_1}^u(\tau)| \leqslant Q \exp\left(-\frac{\eta\tau}{T}\right) + Q \int_0^\tau \exp\left(-\int_s^\tau 2\tilde{\lambda}(f_a v)\mathrm{d}a\right) \exp\left(-\int_0^s \tilde{\lambda}(f_a v)\mathrm{d}a\right) \mathrm{d}s,$$

$$\leqslant Q \exp\left(-\frac{\eta\tau}{T}\right) + Q \int_0^\tau \exp\left(-\int_0^\tau \tilde{\lambda}(f_a v)\mathrm{d}a\right) \mathrm{d}s,$$

$$\leqslant Q \exp\left(-\frac{\eta\tau}{T}\right) + Q\tau \exp\left(-\frac{\eta\tau}{4T}\right),$$

$$\leqslant Q \mathrm{e}^{-\xi\tau},$$

once we fix $\xi < \eta/4T$. Hence $|U_v^u(\tau) - U_{w_1}^u(\tau)| \leqslant Q\mathrm{e}^{-\xi\tau}$.

For $w_2 \in f_{-t} W_\delta^u(f_t v)$, we similarly have the following estimate:

$$|K(f_s v) - K(f_s w_2)| \leqslant Q d_K(f_s v, f_s w_2) \leqslant Q d^u(f_{s-} f_t v, f_{s-} f_t w_2) \leqslant Q\delta \exp\left(-\int_s^t \tilde{\lambda}(f_a v)\mathrm{d}a\right)$$

for any $s \in [0, t]$. We use lemma 6.9 again and get:

$$|U_v^u(\tau) - U_{w_2}^u(\tau)| \leqslant Q \exp\left(-\frac{\eta\tau}{T}\right) + Q \int_0^\tau \exp\left(-\int_s^\tau 2\tilde{\lambda}(f_a v)\mathrm{d}a\right) \exp\left(-\int_s^t \tilde{\lambda}(f_a v)\mathrm{d}a\right) \mathrm{d}s,$$

$$\leqslant Q \exp\left(-\frac{\eta\tau}{T}\right) + Q \int_0^\tau \exp\left(-\int_s^t \tilde{\lambda}(f_a v)\mathrm{d}a\right) \mathrm{d}s,$$

$$\leqslant Q \exp\left(-\frac{\eta\tau}{T}\right) + Q \int_0^\tau \exp\left(-\frac{\eta(t-s)}{4T}\right) \mathrm{d}s,$$

$$\leqslant Q \exp\left(-\frac{\eta\tau}{T}\right) + Q \exp\left(-\frac{\eta(t-\tau)}{4T}\right).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of lemma 6.9. We set $\delta > 0$ from (4.1). Without loss of generality, we may assume $U_w^u(0) \geqslant U_v^u(0)$ and let U_1 be the solution of the Riccati equation along γ_v with $U_1(0) = U_w^u(0)$. We have

$$|U_v^u(\tau) - U_w^u(\tau)| \leqslant |U_v^u(\tau) - U_1(\tau)| + |U_1(\tau) - U_w^u(\tau)|.$$

Since $U_w^u(0) \geqslant U_v^u(0)$ and both U_1 and U_v^u satisfy the same first order ODE, their graphs do not intersect. Thus we have $U_1(\tau) \geqslant U_v^u(\tau) = k^u(f_\tau v)$ for all τ. Hence

$$(U_1 - U_v^u)' = -(U_1 - U_v^u)(U_1 + U_v^u) \leqslant -2k^u(f_\tau v)(U_1 - U_v^u) \leqslant -2\lambda(f_\tau v)(U_1 - U_v^u).$$

Thus $(U_1(\tau) - U_v^u(\tau)) \exp\left(\int_0^\tau 2\lambda(f_s v)\mathrm{d}s\right)$ is non-increasing. From lemma 3.13 and the assumption that $(v, t) \in \mathcal{G}_T(\eta)$, we have

$$0 \leqslant U_1(\tau) - U_v^u(\tau) \leqslant (U_w^u(0) - U_v^u(0)) \exp\left(-\int_0^\tau 2\lambda(f_s v)\mathrm{d}s\right)$$

$$\leqslant Q \exp\left(-\frac{1}{T}\int_0^\tau \lambda_T(f_s v)\mathrm{d}s\right) \leqslant Q \exp\left(-\frac{\eta\tau}{T}\right).$$

Now we estimate $|U_1(\tau) - U_w^u(\tau)|$. We may assume $U_1(\tau) > U_w^u(\tau)$ (the other case is similar). Let $s_0 \in [0, t]$ such that $U_1(s_0) = U_w^u(s_0)$ and $U_1(s) > U_w^u(s)$ for any $s \in (s_0, t)$. By taking

difference of the corresponding Riccati equations, for any $s \in (s_0, t)$, we have

$$(U_1 - U_w^u)'(s) = -(U_1(s) - U_w^u(s))(U_1(s) + U_w^u(s)) + K(f_s v) - K(f_s w)$$
$$\leqslant -2k^u(f_s w)(U_1 - U_w^u)(s) + |K(f_s v) - K(f_s w)|.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\left((U_1(s) - U_v^u(s)) \exp\left(\int_{s_0}^s 2k^u(f_a w)\mathrm{d}a \right) \right)$$
$$= \exp\left(\int_{s_0}^s 2k^u(f_a w)\mathrm{d}a \right) ((U_1 - U_w^u)'(s) + 2k^u(f_s w)(U_1 - U_w^u)(s)),$$
$$\leqslant \exp\left(\int_{s_0}^s 2k^u(f_a w)\mathrm{d}a \right) |K(f_s v) - K(f_s w)|.$$

Integrating from s_0 to τ, we have

$$U_1(\tau) - U_v^u(\tau) \leqslant \exp\left(-\int_{s_0}^\tau 2k^u(f_a w)\mathrm{d}a \right) \int_{s_0}^\tau \exp\left(\int_{s_0}^s 2k^u(f_a w)\mathrm{d}a \right) |K(f_s v) - K(f_s w)|\mathrm{d}s,$$
$$= \int_{s_0}^\tau \exp\left(-\int_s^\tau 2k^u(f_a w)\mathrm{d}a \right) |K(f_s v) - K(f_s w)|\mathrm{d}s,$$
$$\leqslant \int_{s_0}^\tau \exp\left(-\int_s^\tau 2\lambda(f_a w)\mathrm{d}a \right) |K(f_s v) - K(f_s w)|\mathrm{d}s,$$
$$\leqslant \int_{s_0}^\tau \exp\left(-\int_s^\tau 2\tilde{\lambda}(f_a v)\mathrm{d}a \right) |K(f_s v) - K(f_s w)|\mathrm{d}s,$$
$$\leqslant \int_0^\tau \exp\left(-\int_s^\tau 2\tilde{\lambda}(f_a v)\mathrm{d}a \right) |K(f_s v) - K(f_s w)|\mathrm{d}s,$$

where the last inequality follows because $s_0 \geqslant 0$ and the integrand is non-negative. $\qquad\square$

Putting together lemmas 6.4 and 6.8, we have the following result:

Theorem 6.10. *The geometric potential φ^u has the Bowen property with respect to $G_T(\eta)$ for any $T > 1$ and $\eta > 0$.*

7. Pressure gap and the proof of theorem A

The aim of this section is to prove theorem A. In order to do that, we spend most part of this section on related estimates on pressures, such as $P(\cdot)$, $P(\mathrm{Sing}, \cdot)$, $P_{\exp}^\perp(\cdot)$, and relations between them.

We know when the collection $\mathcal{C} = X \times [0, \infty)$ we can use the variational principle to understand the topological pressure $P(\cdot)$. However, when the collection $\mathcal{C}$ is not the set of all finite orbits, the variational principle does not hold any more. Nevertheless, one can still use empirical measures along orbit segments in $\mathcal{C}$ to 'understand' $P(\mathcal{C}, \cdot)$. To be more precise, we start from recalling related terms and estimates given in [BCFT18].

Let X be a compact metric space, $\mathcal{F}$ be a continuous flow, and $\varphi : X \to \mathbb{R}$ be a continuous potential. Given a collection of finite orbit segments $\mathcal{C} \subset X \times [0, \infty)$, for $(x, t) \in \mathcal{C}$ the *empirical measure* $\delta_{(x,t)}$ is defined as, for any $\psi \in C(X)$,

$$\int \psi \mathrm{d}\delta_{(x,t)} = \frac{1}{t} \int_0^t \psi(f_\tau x)\mathrm{d}\tau.$$

We further write $\mathcal{M}_t(\mathcal{C})$ for the convex linear combinations of empirical measures of length t, that is,

$$\mathcal{M}_t(\mathcal{C}) := \{\sum_{i=1}^k a_i \delta_{(x_i,t)} : a_i \geqslant 0, \sum a_i = 1, (x_i, t) \in \mathcal{C}\}.$$

Finally, let $\mathcal{M}(\mathcal{C})$ denote the set of $\mathcal{F}$-invariant Borel probability measures which are limits of measures in $\mathcal{M}_t$, i.e.

$$\mathcal{M}(\mathcal{C}) := \{\lim_{k \to \infty} \mu_{t_k} : t_k \to \infty, \mu_{t_k} \in \mathcal{M}_{t_k}(\mathcal{C})\}.$$

Notice that when $\mathcal{C}$ contains arbitrary long orbit segments, $\mathcal{M}(\mathcal{C})$ is a non-empty set.

We recall a useful general result from [BCFT18]:

Proposition 7.1 ([BCFT18, proposition 5.1]). *Suppose φ is a continuous function, then*

$$P(\mathcal{C}, \varphi) \leqslant \sup_{\mu \in \mathcal{M}(\mathcal{C})} P_\mu(\varphi)$$

where $P_\mu(\varphi) := h_\mu + \int \varphi \mathrm{d}\mu$.

Let us apply above results to our specific setting: S a closed surface of genus greater than or equal to 2 without focal points, $\mathcal{F}$ the geodesic flow for S, and $\varphi : T^1 S \to \mathbb{R}$ a continuous potential.

The following lemma establishes that the pressure of the obstruction to expansivity is strictly less than the entire pressure. It is a direct consequence of the flat strip theorem.

Proposition 7.2 ([BCFT18, proposition 5.4]). *For a continuous potential φ, $P_{\exp}^\perp(\varphi) \leqslant P(\mathrm{Sing}, \varphi)$.*

Proof. It is a straightforward consequence of the flat strip theorem. Since the flat strip theorem holds for manifolds without focal points (see proposition 3.5), the proof goes verbatim as in [BCFT18, proposition 5.4]. □

The following proposition shows that, using the pressure gap condition, one can control the size of bad orbit segments in the sense of pressure.

Proposition 7.3. *Let $\mathcal{B}_T(\eta)$ be the collection of bad orbit segments defined as in definition 4.1. Then there exist $T_0 > 1$ and $\eta_0 > 0$ such that*

$$P([\mathcal{B}_{T_0}(\eta_0)], \varphi) < P(\varphi).$$

Proof. Let D be the metric compatible with the weak* topology on the space of $\mathcal{F}$-invariant probability measures $\mathcal{M}(\mathcal{F})$. Abusing the notation, we will also use D to denote the Hausdorff distance induced by D. Fix $\delta < P(\varphi) - P(\mathrm{Sing}, \varphi)$ and choose $\varepsilon > 0$ such that

$$\mu \in \mathcal{M}(\mathcal{F}) \text{ with } D(\mu, \mathcal{M}(\mathrm{Sing})) < \varepsilon \implies P_\mu(\varphi) - P(\mathrm{Sing}) < \delta.$$

The existence of such ε is guaranteed by the upper semi-continuity of the entropy map $\mathcal{M}(\mathcal{F}) \ni \mu \mapsto h_\mu(f)$ which follows from the geodesic flow $\mathcal{F} : T^1 S \to T^1 S$ being entropy-

expansive (see Liu–Wang [LW16]). From lemmas 4.9 and 4.11, we have

$$\mathcal{M}(\mathrm{Sing}) = \bigcap_{\eta>0, T>0} \mathcal{M}_{\lambda_T}(\eta),$$

where $\mathcal{M}_{\lambda_T}(\eta) = \{\mu \in \mathcal{M}(\mathcal{F}): \int \lambda_T \mathrm{d}\mu \leqslant \eta\}$. Hence, we can find $T_0, \eta_0 > 0$ such that

$$D(\mathcal{M}(\mathrm{Sing}), \mathcal{M}_{\lambda_{T_0}}(\eta_0)) < \varepsilon.$$

Since $\mathcal{M}_{\lambda_T}(\eta)$ is nested, we can increase T_0 if necessary to be bigger than 1.

In particular, for any $\mu \in \mathcal{M}_{\lambda_{T_0}}(\eta_0)$, we have

$$P_\mu(\varphi) < P(\mathrm{Sing}, \varphi) + \delta.$$

Since it follows from the definition that $\mathcal{M}([\mathcal{B}_T(\eta)]) \subset \mathcal{M}_{\lambda_T}(\eta)$, we can verify that the pressure gap $P([\mathcal{B}_{T_0}(\eta_0)], \varphi) < P(\varphi)$ holds for such choice of η_0 and T_0:

$$P([\mathcal{B}_{T_0}(\eta_0)], \varphi) \leqslant \sup_{\mu \in \mathcal{M}([\mathcal{B}_{T_0}(\eta_0)])} P_\mu(\varphi) \leqslant \sup_{\mu \in \mathcal{M}_{\lambda_{T_0}}(\eta_0)} P_\mu(\varphi) \leqslant \delta + P(\mathrm{Sing}, \varphi) < P(\varphi).$$

This proves the proposition. $\qquad\square$

Remark 7.4. We remark that the conclusion of proposition 7.3 remains to hold if we take (T_0, η_1) for any $\eta_1 \in (0, \eta_0)$.

Now, we are ready to prove our first main theorem.

Theorem (Theorem A). *Let S be a surface of genus greater than or equal to 2 without focal points and $\mathcal{F}$ be the geodesic flow over S. Let $\varphi: T^1 S \to \mathbb{R}$ be a Hölder continuous potential or $\varphi = q \cdot \varphi^u$ for some $q \in \mathbb{R}$. Suppose φ verifies the pressure gap property $P(\mathrm{Sing}, \varphi) < P(\varphi)$, then φ has a unique equilibrium state μ_φ.*

Proof. This follows from theorem 2.10 (Climenhaga–Thompson's criteria for the uniqueness of equilibrium states).

We first notice that by proposition 7.2, φ satisfies the first assumption in theorem 2.10. For any $T > 1$ and $\eta > 0$, we can take the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S}) = (\mathcal{B}_T(\eta), \mathcal{G}_T(\eta), \mathcal{B}_T(\eta))$ given in definition 4.1, then by proposition 5.5, theorems 6.5 and 6.10, the conditions (I) and (II) of theorem 2.10 are verified.

Lastly, by proposition 7.3, we know there exists $(T, \eta) = (T_0, \eta_0)$ with $T_0 > 1$ such that the set of bad orbit segments has strictly less pressure than that of φ, that is, $P([\mathcal{B}_{T_0}(\eta_0)], \varphi) < P(\varphi)$, which verifies the condition (III) of theorem 2.10. $\qquad\square$

We conclude this section by remarking on the possibility of further extending theorem A in various directions.

Remark 7.5. A natural question would be whether theorem A can be further extended to more general settings such as manifolds without conjugate points or manifolds without focal points of arbitrary dimension.

For manifolds without conjugate points, the geometric information is much coarser than manifolds without focal points. This causes many difficulties in applying Climenhaga–Thompson criteria [CT16] to prove similar results for manifolds without conjugate points; such difficul-

ties include the unavailability of the flat strip theorem, C^2-regularity of horoshperes as well as the positive semi-definiteness of second fundamental form. Recently, however, Climenhaga *et al* [CKW19] established the uniqueness of the measure of maximal entropy (namely, the special case when $\varphi \equiv 0$) for geodesic flows over surfaces without conjugates points; their approach is genuinely new and differs from that of [CT16] and [BCFT18].

In order to establish analogous results of theorem A for manifolds without focal points of arbitrary dimension, we would need to re-establish via other approaches the corresponding lemmas and estimates that depended on the fact that S is a surface; see remark 4.10.

8. Properties of the equilibrium states and the proof of theorem B

In this section, we prove theorem B.

Theorem (Theorem B). *Let $\varphi : T^1S \to \mathbb{R}$ be a Hölder continuous function or $\varphi = q \cdot \varphi^u$ satisfying $P(\mathrm{Sing}, \varphi) < P(\varphi)$. Then the equilibrium state μ_φ is fully supported, $\mu_\varphi(\mathrm{Reg}) = 1$, Bernoulli, and is the weak* limit of the weighted regular periodic orbits.*

Proof. The proof is separated into following propositions, namely, propositions 8.1, 8.13, 8.10 and 8.6. $\qquad\square$

8.1. $\mu_\varphi(\mathrm{Reg}) = 1$ and μ_φ is Bernoulli

Proposition 8.1. $\mu_\varphi(\mathrm{Reg}) = 1$.

Proof. Since μ_φ is the unique equilibrium state for φ, we have that μ_φ is ergodic (see [CT16] proposition 4.19). Because Sing is $\mathcal{F}$-invariant we have either $\mu_\varphi(\mathrm{Sing}) = 1$ or $\mu_\varphi(\mathrm{Sing}) = 0$. Suppose $\mu_\varphi(\mathrm{Sing}) = 1$, then

$$P(\mathrm{Sing}, \varphi) \geqslant h_{\mu_\varphi}(\mathcal{F}) + \int \varphi|_{\mathrm{Sing}} \, \mathrm{d}\mu_\varphi = P(\varphi),$$

which contradicts the pressure gap condition. Thus $\mu_\varphi(\mathrm{Reg}) = 1$. $\qquad\square$

Definition 8.2 (Bernoulli). Let X be a compact metric space and $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ be a continuous flow on X. We call a $\mathcal{F}$-invariant measure μ *Bernoulli* if the system (X, f_1, μ) is measurably isomorphic to a Bernoulli shift, where f_1 is the time-1 map of the flow $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$.

To prove μ_φ is Bernoulli, we use a result in Ledrappier–Lima–Sarig [LLS16]. In order to apply their result, we recall that for $v \in T^1S$, $\chi(v)$, the *Lyapunov exponent* at v associated to the unstable bundle $E^u(v)$ is given by

$$\chi(v) = \lim_{t \to \pm\infty} \frac{1}{t} \log \left\| \mathrm{d}f_t|_{E^u(v)} \right\|$$

whenever both limits exist and are equal. Such $v \in T^1S$ whose Lyapunov exponent $\chi(v)$ exists are called *Lyapunov regular* vectors. Notice that since the Liouville measure is invariant under the geodesic flow $\mathcal{F}$, the Lyapunov exponent is zero along the flow direction, and is $-\chi(v)$ on the stable bundle $E^s(v)$. Moreover, it is well-known (by Oseledec multiplicative ergodic

theorem) that the set of Lyapunov regular vectors has full measure for any $\mathcal{F}$-invariant probability measure.

Remark 8.3. For $v \in \mathrm{Sing}$, notice f_t does not expand along the unstable bundle $E^u(v)$; indeed, the unstable Jacobi field J_v^u has constant length for $v \in \mathrm{Sing}$. Thus we have $\chi|_{\mathrm{Sing}} = 0$.

Using following lemmas, we can show that the unique equilibrium state for μ_φ is a hyperbolic measure (i.e. $\chi(v) \neq 0$ for μ_φ-a.e. $v \in T^1 S$, which is equivalent to $\chi(\mu_\varphi) := \int \chi(v) \mathrm{d}\mu_\varphi \neq 0$ from the ergodicity of μ_φ) which allows us to use Ledrappier–Lima–Sarig [LLS16] to conclude μ_φ is Bernoulli.

Lemma 8.4. *Let μ be a $\mathcal{F}$-invariant probability measure. Suppose $\chi(v) = 0$ for μ-a.e. $v \in T^1 S$, then $\mathrm{supp}(\mu) \subset \mathrm{Sing}$.*

Proof. We first recall that for $\xi \in T_v T^1 S$ we have $||J_\xi(t)||^2 \leqslant ||\mathrm{d}f_t \xi||^2$. Let $\mu \in \mathcal{M}(\mathcal{F})$ and, without loss of generality, we may assume v is a Lyapunov regular vector for $\xi \in E^u(v)$. Then, by lemma 3.12

$$
\begin{aligned}
\chi(v) &= \lim_{t \to \infty} \frac{1}{t} \log ||\mathrm{d}f_t|_{E^u(v)}|| \\
&\geqslant \lim_{t \to \infty} \frac{1}{t} \log ||J_\xi^u(t)|| \\
&\geqslant \lim_{t \to \infty} \frac{1}{t} \log \left(e^{\int_0^t k^u(f_\tau v) \mathrm{d}\tau} ||J_\xi^u(0)|| \right) \\
&= \lim_{t \to \infty} \frac{1}{t} \int_0^t k^u(f_\tau v) \mathrm{d}\tau \geqslant 0.
\end{aligned}
$$

Integrating with respect to μ, the Birkhoff ergodic theorem gives $\int \chi(v) \mathrm{d}\mu \geqslant \int k^u(v) \mathrm{d}\mu \geqslant 0$. Therefore, if $\chi(v) = 0$ for μ-a.e. $v \in T^1 S$, then $k^u(v) = 0$ for μ-a.e. $v \in T^1 S$; hence, $\lambda(v) = 0$ for μ-a.e. $v \in T^1 S$. By lemma 4.11, we are done. ☐

Remark 8.5.

(1) The computation in the above lemma also points out that if μ is a $\mathcal{F}$-invariant probability measure and v is a Lyapunov regular vector with respect to μ, then $\chi(v) \geqslant 0$. In other words, we know that $\chi(v)$ is indeed the non-negative Lyapunov exponent at v.

(2) If, in addition, μ is ergodic and $\mu(\mathrm{Reg}) > 0$, we have μ is hyperbolic. Indeed, otherwise, there exists $A \subset T^1 S$ such that $\mu(A) > 0$ and $\chi|_A = 0$. Then by the ergodicity of μ we have that $\mu(A) = 1$. Hence, by lemma 8.4, we get $\mathrm{supp}\mu \subset \mathrm{Sing}$ which contradicts $\mu(\mathrm{Reg}) > 0$.

Proposition 8.6. *The unique equilibrium state μ_φ is Bernoulli.*

Proof. [CT16, proposition 4.19] Shows that the unique equilibrium state μ_φ is ergodic, thus by proposition 8.1 and remark 8.5 (2) we get that μ_φ is hyperbolic. Therefore, applying results in [LLS16], we have that μ_φ is Bernoulli. ☐

Remark 8.7. Originally Ledrappier–Lima–Sarig [LLS16] required that $h_\mu(\mathcal{F}) > 0$; nevertheless, it has been clarified in Lima–Sarig [LS19, theorem 1.3] that one only needs to check μ is hyperbolic.

8.2. μ_φ is fully supported

In this subsection, unless stated otherwise, we fix the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ to be $(\mathcal{B}_{T_0}(\eta_0), \mathcal{G}_{T_0}(\eta_0), \mathcal{B}_{T_0}(\eta_0))$ where T_0 and η_0 are given in proposition 7.3. We notice that this decomposition $(\mathcal{B}_{T_0}(\eta_0), \mathcal{G}_{T_0}(\eta_0), \mathcal{B}_{T_0}(\eta_0))$ satisfies the Climenhaga–Thompson criteria for the uniqueness of equilibrium states (i.e. theorem 2.10).

For any decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ and $M > 0$, the collection $\mathcal{G}^M$ is defined as

$$\mathcal{G}^M := \{(x,t) : \, s(x,t), p(x,t) \leqslant M\}.$$

The following lemma shows that if the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ satisfies theorem 2.10, then $\mathcal{G}^M$ captures much thermodynamic information whenever M is large enough.

Lemma 8.8 ([BCFT18, lemma 6.1]). *There exists $M, C, \delta > 0$ such that for all $t > 0$,*

$$\Lambda(\mathcal{G}^M, \varphi, \delta, t) > C\mathrm{e}^{tP(\varphi)}. \tag{8.1}$$

Hence, for large enough M, we have $P(\mathcal{G}^M, \varphi) = P(\varphi)$. Moreover, the equilibrium state μ_ϕ has the lower Gibbs property on $\mathcal{G}^M$. More precisely, for any $\rho > 0$, there exist $Q, \tau, M > 0$ such that for every $(v,t) \in \mathcal{G}^M$ with $t \geqslant \tau$,

$$\mu_\varphi(B_t(v,\rho)) \geqslant Q\mathrm{e}^{-tP(\varphi) + \int_0^t \varphi(f_s v)\mathrm{d}s}.$$

In particular, if v such that $(v,t) \in \mathcal{G}$ for some $t \geqslant \tau$, then $\mu_\varphi(B(v,\rho)) > 0$ for all $\rho > 0$.

Lemma 8.9 ([BCFT18, lemma 6.2]). *Given $\rho, \eta, T > 0$, there exists $\eta_1 > 0$ so that for any $v \in \mathrm{Reg}_T(\eta)$, $t > 0$, there are $s \geqslant t$ and $w \in B(v,\rho)$ such that $(w,s) \in \mathcal{G}_T(\eta_1)$. In particular, we can choose $\eta_1 \leqslant \eta_0$ where η_0 is given in proposition 7.3*

Proof. The proof follows, *mutatis mutandis*, the proof of [BCFT18, lemma 6.2]. One only needs to replace the [BCFT18, corollary 3.11] in their proof by lemma 4.5, and the last assertion follows because for $0 < \eta' \leqslant \eta''$, we have $\mathrm{Reg}_T(\eta'') \subset \mathrm{Reg}_T(\eta')$. $\square$

Proposition 8.10. *The unique equilibrium state μ_φ is fully supported.*

Proof. Since Reg dense in $T^1 M$, it is enough to show that for any $v \in \mathrm{Reg}$ and $r > 0$ we have $\mu_\varphi(B(v,r)) > 0$.

Since $v \in \mathrm{Reg}$, there exists $t_0 \in \mathbb{R}$ such that $\lambda(f_{t_0} v) > 0$. For convenience, let us denote $v' = f_{t_0} v$. By the continuity of λ, there exists $\rho > 0$ such that $\lambda|_{B(v', 2\rho)} > \eta$ for some $\eta > 0$, and we have $v' \in \mathrm{Reg}_T(2\rho\eta)$. We make sure to pick ρ small enough so that $f_{-t_0} B(v', 2\rho) \subset B(v, r)$. By lemma 8.9, there exists $\eta_1 > 0$ such that there is $w \in B(v', \rho)$ satisfying $(w,t) \in \mathcal{G}_T(\eta_1)$ for arbitrary large t (depending on ρ, η).

Furthermore, the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S}) = (\mathcal{B}_{T_0}(\eta_1), \mathcal{G}_{T_0}(\eta_1), \mathcal{B}_{T_0}(\eta_1))$ verifies theorem 2.10, assuming that we take η_1 smaller than η_0. Thus by lemma 8.8 we know μ_φ satisfies the lower Gibbs property, i.e.

$$\mu_\varphi(B(w,\rho)) > 0.$$

Now, because μ_φ is flow invariant, it follows that

$$\mu_\varphi(B(v,r)) \geqslant \mu_\varphi(B(v', 2\rho)) \geqslant \mu_\varphi(B(w,\rho)) > 0.$$

$\square$

8.3. Periodic regular orbits are equidistributed relative to μ_φ

Let us continue the discussion on ergodic properties of the equilibrium state. Recall that S is a closed surface without focal point with genus $\geqslant 2$, and $\varphi : T^1S \to \mathbb{R}$ is a potential satisfying theorem A, and μ_φ the equilibrium state. In what follows, the good orbit segment collection $\mathcal{G}$ always refers to $\mathcal{G}_{T_0}(\eta_0)$ where T_0, η_0 are given in proposition 7.3.

Lemma 8.11. *Suppose $\varphi : T^1S \to \mathbb{R}$ is a potential satisfying theorem A. For any $\Delta > 0$, there exists $C > 0$ such that*

$$\Lambda^*_{\text{Reg},\Delta}(\varphi, t) \leqslant C e^{tP(\varphi)}$$

for all $t > \Delta$.

Proof. Claim: for all $\Delta > 0$ and $\delta < \text{inj}(S)$, $\text{Per}_R(t - \Delta, t]$ is a (t, δ)-separated set.

To prove this claim, assume the contrary; suppose γ_1, γ_2 are two closed geodesics in $\text{Per}_R(t - \Delta, t]$ such that $d(\gamma_1(s), \gamma_2(s)) \leqslant \delta$ for all $s \in [0, t]$, and thus γ_2 is covered by $B_\delta(\gamma_1(t_i))$ for finitely many i. Because $\delta < \text{inj}(S)$, each $B_\delta(\gamma_1(t_i))$ is diffeomorphic to the δ-ball on $T_{\gamma_1(t_i)}S$ centerd at $\gamma_1(t_i)$. One can easily construct a homopy between γ_1 and γ_2 by choosing and connecting points from $B_\delta(\gamma_1(t_i))$. Since γ_1, γ_2 are in the same free homotopy class, their lifts $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are bi-asymptotic. Thus by the flat strip theorem (proposition 3.5) $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ bound a flat strip, and hence they are singular. This contradicts to $\gamma_1, \gamma_2 \in \text{Per}_R(t - \Delta, t]$, and we have completed the proof of the claim.

Notice that for every $\gamma \in \text{Per}_R(t - \Delta, t]$, let v_γ be a vector tangent to γ, we have

$$|\Phi(\gamma) - \Phi(v_\gamma, t)| \leqslant \Delta ||\varphi||, \text{ and thus } \Lambda^*_{\text{Reg},\Delta}(\varphi, t) \leqslant e^{\Delta||\varphi||}\Lambda(\varphi, \delta, t).$$

Lastly, by [CT16, lemma 4.11], there exists $C > 0$ such that for $t > \Delta$ we have

$$\Lambda^*_{\text{Reg},\Delta}(\varphi, t) \leqslant e^{\Delta||\varphi||}\Lambda(\varphi, \delta, t) < C e^{tP(\varphi)}.$$

$\square$

Lemma 8.12. *Suppose $\varphi : T^1S \to \mathbb{R}$ is a potential satisfying theorem A. There exists $\Delta, C > 0$ such that*

$$\frac{C}{t} e^{tP(\varphi)} \leqslant \Lambda^*_{\text{Reg},\Delta}(\varphi, t)$$

for all large t .

Proof. By lemma 8.8, we know when M is big, there exists $C_1, \delta_1 > 0$ such that for all $t > 0$

$$C_1 e^{tP(\varphi)} \leqslant \Lambda(\mathcal{G}^M, \delta_1, t).$$

Hence, it suffices to find $\delta, C_2, \Delta, s > 0$ with $\delta < \delta_1$ such that for any $t > \max\{s, \Delta, 2M\}$, we have

$$\Lambda(\mathcal{G}^M, \delta, t) \leqslant C_2(t + s)\Lambda^*_{\text{Reg},\Delta}(\varphi, t + s).$$

Indeed, the lemmas follows from these inequalities because

$$\Lambda^*_{\text{Reg},\Delta}(\varphi, t + s) \geqslant \frac{C_1 C_2^{-1}}{t + s} e^{tP(\varphi)} = \frac{C_1 C_2^{-1} e^{-sP(\varphi)}}{t + s} e^{(t+s)P(\varphi)}.$$

We start from labeling sizes of Bowen balls relative to different propositions. In what follows, we fix $T_0, \eta_0 > 0$ and M large so that theorem A and lemma 8.8 hold. Let $\varepsilon_1 = \varepsilon_1(T_0, \eta_0)$ be given in corollary 5.7. Since φ verifies the Bowen property on $\mathcal{G}^M$, let $\varepsilon_2 = \varepsilon_2(T_0, \eta_0)$ denote the radius of Bowen balls for the Bowen property. Lastly, because S is compact and f_t is uniformly continuous, for any $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon)$ such that when $d_K(u, w) < \delta_1$ we have $d_K(f_\sigma u, f_\sigma w) < \varepsilon$ for any $\sigma \in [-M, M]$, without loss of generality, we may choose $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$.

The first step is to associate each $(v, t) \in \mathcal{G}^M$ with a regular closed orbit whose length is in the interval $[t - t_1, t + t_2]$ for some t_1 and t_2 as follows. Recall that for each $(v, t) \in \mathcal{G}^M$ there exists $0 < s_0, p_0 < M$ such that $(f_{p_0} v, t - s_0 - p_0) = (v', t') \in \mathcal{G}$.

We claim that given $\varepsilon > 0$ as above and $\delta_2 = \min\{\varepsilon, \delta_1(\varepsilon)\}$, there exists $s = s(\delta_2)$ such that for any $(v', t') \in \mathcal{G}^M$ defined as above, there exists a regular vector $w \in B_{t'}(v', \delta_2)$ with $f_{t'+\tau}(w) = w$ for some $\tau \in [0, s]$.

Indeed, the claim is a direct consequence of corollary 5.7, because $(v', t') \in \mathcal{G} \subset \mathcal{C}_{T_0}(\eta_0)$. Moreover, we also have $f_{-p} w \in B_t(v, \varepsilon)$ because $w \in B(v', \delta_2) \subset B(v', \delta_1)$ and the choice of δ_1. Thus, we have the claim.

Moreover, since $\varepsilon < \varepsilon_2$ we have

$$|\Phi(v, t) - \Phi(w, t' + \tau)| = \left| \int_0^t \varphi(f_\sigma v) d\sigma - \int_0^{t'+\tau} \varphi(f_\sigma w) d\sigma \right|$$

$$\leqslant \left| \int_0^{p_0} \varphi(f_\sigma v) d\sigma + \int_0^{t'} \varphi(f_\sigma v') d\sigma + \int_0^{s_0} \varphi(f_{\sigma+t'} v) d\sigma - \int_0^{t'+\tau} \varphi(f_\sigma w) d\sigma \right|$$

$$\leqslant (2M + \tau)||\varphi|| + \left| \int_0^{t'} (\varphi(f_\sigma v') - \varphi(f_\sigma w)) d\sigma \right|,$$

$$\leqslant (2M + \tau)||\varphi|| + K$$

where K is the constant given by the Bowen property.

In sum, given $\varepsilon > 0$ as above, we can define a map $\Psi : \mathcal{G}^M \ni (v, t) \mapsto (w, t' + \tau)$ where w is tangent to a regular closed orbit $\gamma_w \in \mathrm{Per}_R[t', t' + \tau] \subset \mathrm{Per}_R[t - 2M, t + s]$ and $|\Phi(v, t) - \Phi(\gamma_w)| \leqslant (2M + s)||\varphi|| + K$.

We notice that $\Psi|_{E_t}$ is an injection for every (t, δ)-separated set $E_t \subset \mathcal{G}^M$ provided $\delta > 3\varepsilon$ (because for every $(v, t) \in E_t$, its image $\Psi(v, t) = (w, t' + \tau)$ satisfies $w \in B_t(v, \varepsilon)$). Moreover, because $\Psi(E_t)$ is (t, ε)-separated, each $\gamma \in \mathrm{Per}_R[t - 2M, t + s]$ has at most $\frac{t+s}{\varepsilon}$ elements of $\Psi(E_t)$ tangent to it.

Hence, for $\delta > 3\varepsilon$ and for all (t, δ)-separated set $E_t \subset \mathcal{G}^M$ we have

$$\sum_{(v,t) \in E_t} e^{\Phi(v,t)} \leqslant \frac{t+s}{\varepsilon} \cdot e^{(2M+s)||\varphi||+K} \cdot \sum_{\gamma \in \mathrm{Per}_R[t-2M, t+s]} e^{\Phi(\gamma)}.$$

The lemma now follows with by setting $C_2 = e^{(2M+s)||\varphi||+K}/\varepsilon$ and $\Delta = 2M + s$. $\qquad\square$

From the above two lemmas, we can conclude:

Proposition 8.13. *The unique equilibrium state μ_φ obtained in theorem A is the weak* limit of the weighted regular periodic orbits. More precisely, there exists $\Delta > 0$ such that*

$$\mu_\varphi = \lim_{T \to \infty} \frac{\sum_{\gamma \in \mathrm{Per}_R(T-\Delta, T]} e^{\Phi(\gamma)} \delta_\gamma}{\Lambda^*_{\mathrm{Reg}, \Delta}(\varphi, T)}.$$

Proof. It follows immediately from lemmas 8.11, 8.12 and proposition 2.17. $\qquad\square$

9. The proof of theorem C and examples

In this section, we present the proof of theorem C and also provide examples satisfying the pressure gap property. The following lemmas show that the scalar multiple $q\varphi^u$ of the geometric potential possesses the pressure gap property provided $q < 1$.

Lemma 9.1. *If S is a closed surface of genus greater than or equal to 2 without focal points, then $P(q\varphi^u) > 0 = P(\mathrm{Sing}, q\varphi^u)$ for each $q \in (-\infty, 1)$; in particular, $h_{\mathrm{top}}(\mathrm{Sing}) = 0$.*

Proof. It is a classical result proved by Burns [Bur83, theorem, p 6] that $\mu_L(\mathrm{Reg}) > 0$ where μ_L is the Liouville measure. Thus by lemma 8.4 and remark 8.5 we get

$$0 < \chi(\mu_L) := \int_{T^1 S} \chi(v) \mathrm{d}\mu_L.$$

This follows because if $\chi(\mu_L) = 0$, then $\chi(v) = 0$ for μ_L-a.e. $v \in T^1 S$, and hence, by lemma 8.4, we would have $\mathrm{supp}(\mu_L) \subset \mathrm{Sing}$ contradicting $\mu_L(\mathrm{Reg}) > 0$

Therefore, we know

$$0 < \chi(\mu_L) = \int_{T^1 S} \chi(v) \mathrm{d}\mu_L = -\int_{T^1 S} \varphi^u \mathrm{d}\mu_L,$$

where the last equality follows from the Birkhoff ergodic theorem.

Moreover, by Pesin's entropy formula [Pes77a], we have

$$h_{\mu_L}(\mathcal{F}) = \int_{T^1 S} \chi(v) \mathrm{d}\mu_L.$$

Thus for $q \in (-\infty, 1)$

$$P(q\varphi^u) \geqslant h_{\mu_L}(\mathcal{F}) + \int q\varphi^u \mathrm{d}\mu_L = (q-1) \int \varphi^u \mathrm{d}\mu_L > 0.$$

We claim that $P(\mathrm{Sing}, q\varphi^u) = 0$. Indeed, for any $\mu \in \mathcal{M}(\mathrm{Sing})$, $P_\mu(q\varphi^u) := h_\mu(\mathcal{F}) + P_\mu(q\varphi^u) := h_\mu(\mathcal{F}) + q \int_{T^1 S} \varphi^u \mathrm{d}\mu = h_\mu(\mathcal{F}) + q \int_{\mathrm{Sing}} \varphi^u \mathrm{d}\mu = h_\mu(\mathcal{F})$.

By Ruelle's inequality [Rue78] we have $h_\mu(\mathcal{F}) \leqslant \int_{T^1 S} \chi(v) \mathrm{d}\mu_L = 0$ (because $\chi|_{\mathrm{Sing}} = 0$, see remark 8.3). Therefore, $P(\mathrm{Sing}, q\varphi^u) = \sup_{\mu \in \mathcal{M}(\mathrm{Sing})} P_\mu(q\varphi^u) = 0$. $\square$

Now, we are ready to prove theorem C.

Proof of theorem C. From the above lemma, it remains to show that the map $q \mapsto P(q\varphi^u)$ is C^1 for $q < 1$ and $P(q\varphi^u) = 0$ for $q \geqslant 1$ when $\mathrm{Sing} \neq \emptyset$.

We first notice that when $\mathrm{Sing} \neq \emptyset$, we have $P(q\varphi^u) \geqslant 0$. It is because for any invariant measure μ such that with $\mathrm{supp}(\mu) \subset \mathrm{Sing}$, we have

$$h_\mu(\mathcal{F}) + \int_{T^1 S} \varphi^u \mathrm{d}\mu = h_\mu(\mathcal{F}) + \int_{\mathrm{Sing}} \varphi^u \mathrm{d}\mu \geqslant 0.$$

Moreover, the non-negative Lyapunov exponent χ is the Birkhoff average of $-\varphi^u$; thus together with Ruelle's inequality we have for any invariant measure $\nu \in \mathcal{M}(\mathcal{F})$:

$$h_\nu(\mathcal{F}) \leqslant \int_{T^1 S} \chi(v) \mathrm{d}\nu$$

and for $q \geqslant 1$

$$
h_\nu(\mathcal{F}) + \int \varphi^u \mathrm{d}\nu = \underbrace{h_\nu(\mathcal{F}) - \int_{T^1S} \chi(v)\mathrm{d}\nu}_{\leqslant 0}
$$

$$
\geqslant h_\nu(\mathcal{F}) - q \int_{T^1S} \chi(v)\mathrm{d}\nu
$$

$$
= h_\nu(\mathcal{F}) + q \int_{T^1S} \varphi^u \mathrm{d}\nu
$$

Therefore, we have for $q \geqslant 1$

$$
P(q\varphi^u) = \sup\{h_\nu(\mathcal{F}) + q \int_{T^1S} \varphi^u \mathrm{d}\nu : \nu \in \mathcal{M}(\mathcal{F})\} \leqslant 0;
$$

hence we have $P(q\varphi^u) = 0$ for $q \geqslant 1$.

Lastly, Liu–Wang [LW16] proved that the geodesic flow is entropy expansive for manifolds without conjugate points. So by Walters [Wal92], we know that $q \mapsto P(q\varphi^u)$ is C^1 on the domain where $q\varphi^u$ has a unique equilibrium state. In particular, $q \mapsto P(q\varphi^u)$ is C^1 for $q < 1$. $\square$

The proposition below gives us an easy criteria for the pressure gap property.

Proposition 9.2 ([BCFT18, lemma 9.1]). *Let S be a closed surface of genus greater than or equal to 2 without focal points and $\varphi : T^1S \to \mathbb{R}$ continuous. If*

$$
\sup_{v\in\mathrm{Sing}} \varphi(v) - \inf_{v\in T^1S} \varphi(v) < h_{\mathrm{top}}(\mathcal{F}),
$$

then $P(\mathrm{Sing}, \varphi) < P(\varphi)$. In particular, constant functions have the pressure gap property.

Proof. The proof follows from the variational principle. More precisely,

$$
\sup_{v\in\mathrm{Sing}} \varphi(v) - \inf_{v\in T^1S} \varphi(v) < h_{\mathrm{top}}(\mathcal{F}) - \underbrace{h_{\mathrm{top}}(\mathrm{Sing})}_{=0}
$$

$$
\iff \sup_{v\in\mathrm{Sing}} \varphi(v) + h_{\mathrm{top}}(\mathrm{Sing}) < h_{\mathrm{top}}(\mathcal{F}) + \inf_{v\in T^1S} \varphi(v)
$$

and

$$
P(\mathrm{Sing}, \varphi) \leqslant h_{\mathrm{top}}(\mathrm{Sing}) + \sup_{v\in\mathrm{Sing}} \varphi(v) < h_{\mathrm{top}}(\mathcal{F}) + \inf_{v\in T^1S} \varphi(v) \leqslant P(\varphi).
$$

$\square$

By the above proposition, the following class of potentials also possesses the pressure gap property.

Corollary 9.3. *Let S be a closed surface of genus greater than or equal to 2 without focal points and $\varphi : T^1S \to \mathbb{R}$ continuous. If $\varphi|_{\mathrm{Sing}} = 0$ and $\varphi \geqslant 0$, then $P(\mathrm{Sing}, \varphi) < P(\varphi)$.*

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