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PRESSURE METRICS AND MANHATTAN CURVES FOR TEICHMÜLLER SPACES OF PUNCTURED SURFACES

BY

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ABSTRACT

In this paper, we extend the construction of pressure metrics to Teichmüller spaces of surfaces with punctures. This construction recovers Thurston's Riemannian metric on Teichmüller spaces. Moreover, we prove the real analyticity and convexity of Manhattan curves of finite area type-preserving Fuchsian representations, and thus we obtain several related entropy rigidity results. Lastly, relating the two topics mentioned above, we show that one can derive the pressure metric by varying Manhattan curves.

1. Introduction

Let $S = S_{g,n}$ be an orientable surface of genus g and n punctures with negative Euler characteristic. In this paper, we discuss how one can characterize Fuchsian representations and the geometry of $\mathcal{T}(S)$, the Teichmüller space of S, by studying dynamics objects associated with them. For example, we prove rigidity results via examining the shape of Manhattan curves, and we construct a Riemannian metric on $\mathcal{T}(S)$ by derivatives of pressure.

When S has no punctures, results in this work are not new. Manhattan curves and rigidity results are, for instance, discussed in [Bur93, Sha98], and the pressure metric on $\mathcal{T}(S)$ is discovered in [McM08] and further investigated in [PS16, BCS18]. Nevertheless, when S has punctures, especially when Fuchsian

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representations are not convex co-compact, far fewer results along this line are proved. Indeed, in such cases, their dynamics are much more complicated because of the presence of parabolic elements.

Using similar ideas to those in [LS08, Kao18], we study geodesic flows over hyperbolic surfaces with cusps by countable state Markov shifts and corresponding suspension flows. Notice that for countable state Markov shifts, in contrast to compact cases, for unbounded potentials without sufficient control of their regularity and values around cusps, the pressure of their perturbation might not only lose the analyticity but also information of some thermodynamics data. For example, time changes for suspension flows over a non-compact Markov shift may not take equilibrium states to equilibrium states for some potentials (cf. [CI18]).

To overcome these issues, we carefully study the associated geometric potential (or the roof function of the suspension flow). By doing so, we know exactly where the pressure function (of geometric potentials and their weighted sums) is analytic. Thus, we can mimic the procedure used in compact cases. More precisely, we derive a version of Bowen's formula which relates the topological entropy of the geodesic flow and the corresponding roof function. With Bowen's formula and the analyticity of pressure, we prove the convexity of Manhattan curves, and using the second derivative of pressure we construct a Riemannian metric on $\mathcal{T}(S)$.

To put our results in context, we now introduce necessary notations and definitions. Recall that a representation $\rho \in \mathrm{Hom}(\pi_1 S, \mathrm{PSL}(2,\mathbb{R}))$ is **Fuchsian** if it is discrete and faithful, and ρ has **finite area** if the hyperbolic surface $X_\rho = \rho(\pi_1 S)\backslash\mathbb{H}$ has finite area. We say two finite area Fuchsian representations ρ_1, ρ_2 are **type-preserving** if there exists an isomorphism $\iota : \rho_1(\pi_1 S) \to \rho_2(\pi_1 S)$ sending parabolic elements to parabolic elements and hyperbolic elements to hyperbolic elements. Here $\mathrm{PSL}(2,\mathbb{R})$ refers to the space of orientation preserving isometries of the hyperbolic plane $\mathbb{H}$.

Let ρ_1 and ρ_2 be two Fuchsian representations. Recall that $d^{a,b}_{\rho_1,\rho_2}$, the **weighted Manhattan metric** on $\mathbb{H}\times\mathbb{H}$ with respect to ρ_1, ρ_2, is given by fixing

$$o = (o_1, o_2), \quad d^{a,b}_{\rho_1,\rho_2}(o, \gamma o) := ad(o_1, \rho_1(\gamma)o_1) + bd(o_2, \rho_1(\gamma)o_2) \quad \text{for } \gamma \in \pi_1(S)$$

where d is the hyperbolic distance on $\mathbb{H}$. Notice that we are only interested in non-negative weights, i.e., $a, b \geq 0$ and $a + b \neq 0$. We denote the associated

Poincaré series by

$$Q^{a,b}_{\rho_1,\rho_2}(s) := \sum_{\gamma\in\pi_1(S)} e^{-s\cdot d^{a,b}_{\rho_1,\rho_2}(o,\gamma o)}.$$

Definition 1.1 (Manhattan curve): The **Manhattan curve** $\mathcal{C}(\rho_1,\rho_2)$ of ρ_1, ρ_2 is given by

$$\mathcal{C}(\rho_1,\rho_2) := \{(a,b)\in\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\backslash(0,0) : \delta^{a,b}_{\rho_1,\rho_2}=1\}$$

where $\delta^{a,b}_{\rho_1,\rho_2}$ is the **critical exponent** of $Q^{a,b}_{\rho_1,\rho_2}(s)$, i.e., $Q^{a,b}_{\rho_1,\rho_2}(s)$ is divergent if $s<\delta^{a,b}_{\rho_1,\rho_2}$ and is convergent if $s>\delta^{a,b}_{\rho_1,\rho_2}$.

By definition, one can regard $\mathcal{C}(\rho_1,\rho_2)$ as a generalization of the critical exponents for ρ_1 and ρ_2. Obviously, taking $a=0$ (respectively, $b=0$), $\delta^{a,b}_{\rho_1,\rho_2}$ reduces to δ_{ρ_1}, the classical critical exponent for ρ_1 (respectively, δ_{ρ_2}). By Otal and Peigné [OP04], we know δ_{ρ_1} is also the topological entropy of the geodesic flow over X_{ρ_1}.

As mentioned above, using a symbolic model given in [LS08], for every finite area Fuchsian representation ρ, we can code the geodesic flow over X_ρ. Elaborated discussion of the coding of geodesic flows is in Section 3. We briefly introduce the idea and strategy below. We will associate the geodesic flow on the smaller special section $\Omega_0\subset T^1X_\rho$ with a suspension flow $(\Sigma^+,\sigma,\tau_\rho)$ where (Σ^+,σ) is a countable state Markov shift and $\tau_\rho:\Sigma^+\to\mathbb{R}^+$ is the roof function. Furthermore, by the construction, the roof function τ_ρ is a continuous function prescribing the length of closed geodesics. We sometimes call τ_ρ the **geometric potential** of ρ. Moreover, one important feature of this symbolic model is that if ρ_1,ρ_2 are finite area type-preserving Fuchsian representations, then they correspond to the same Markov shift (Σ^+,σ) but to different roof functions $\tau_{\rho_1},\tau_{\rho_2}$. In other words, we can use roof functions to characterize finite area type-preserving Fuchsian representations.

Using this symbolic model, we can characterize $\mathcal{C}(\rho_1,\rho_2)$ as solutions of a version of Bowen's formula. Furthermore, we derive the first main result of the paper:

Theorem A: *Let ρ_1, ρ_2 be two finite area type-preserving Fuchsian representations. Then $\mathcal{C}(\rho_1,\rho_2)$ is a real analytic curve, and $\mathcal{C}(\rho_1,\rho_2)$ is strictly convex unless ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2,\mathbb{R})$*; in such cases $\mathcal{C}(\rho_1,\rho_2)$ is a straight line.*

Using the shape of the Manhattan curve, we can further prove rigidity results related with following dynamics quantities.

Definition 1.2: Let ρ_1, ρ_2 be a pair of Fuchsian representations.

(1) The **Bishop–Steiger entropy** $h_{BS}(\rho_1, \rho_2)$ of ρ_1 and ρ_2 is defined as

$$h_{\mathrm{BS}}(\rho_1, \rho_2) := \lim_{T\to\infty} \frac{1}{T} \ln(\#\{[\gamma] \in [\pi_1(S)] : d(o, \rho_1(\gamma)o) + d(o, \rho_2(\gamma o)o) \leq T\}).$$

(2) The **intersection number** $\mathrm{I}(\rho_1, \rho_2)$ of ρ_1 and ρ_2 is defined as

$$\mathrm{I}(\rho_1, \rho_2) := \lim_{n\to\infty} \frac{l_2[\gamma_n]}{l_1[\gamma_n]}$$

where $\{[\gamma_n]\}_{n=1}^{\infty}$ is a sequence of conjugacy classes for which the associated closed geodesics γ_n become equidistributed on X_{ρ_1} with respect to area.

Using a dynamics interpretation of $\mathrm{I}(\rho_1, \rho_2)$ and the convexity and analyticity of pressure, we recover the following results of Bishop and Steiger [BS93], and Thurston [Thu98].

Theorem B: *Let ρ_1, ρ_2 be a pair of area type-preserving Fuchsian representations, We have:*

(1) *(Bishop–Steiger Rigidity) $h_{\mathrm{BS}}(\rho_1, \rho_2) \leq \frac{1}{2}$, and the equality holds if and only if ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$.
(2) *(The Intersection Number Rigidity) $\mathrm{I}(\rho_1, \rho_2) \geq 1$, and the equality holds if and only if ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$.

Remark 1.3:

(1) One might prove $\mathcal{C}(\rho_1, \rho_2)$ is C^1 and Theorem B without employing symbolic dynamics. Nevertheless, symbolic dynamics provides a convenient approach to control the analyticity of pressure, and hence to prove the analyticity of $\mathcal{C}(\rho_1, \rho_2)$.
(2) It is not immediately clear why $\mathrm{I}(\rho_1, \rho_2)$ is well-defined. We will justify it in Section 3.
(3) The intersection number rigidity is known, amount the experts, as a work of Thurston. However, due to the limited knowledge of the author, for the non-convex co-compact cases we cannot find a reference to it.

We now change gear from pairs of Fuchsian representations to the space of conjugacy classes of Fuchsian representations, that is, the Teichmüller space of $S = S_{g,n}$. Recall that the **Teichmüller space** of S is defined as

$$\mathcal{T}(S) := \mathrm{Hom}^{\mathrm{F}}_{\mathrm{tp}}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\sim$$

where $\mathrm{Hom}^{\mathrm{F}}_{\mathrm{tp}}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ is the space of finite area type-preserving Fuchsian representations, and $\rho_1 \sim \rho_2$ if they are conjugate in $\mathrm{PSL}(2,\mathbb{R})$.

Through the symbolic model, there is a thermodynamic mapping

$$\Psi : \mathcal{T}(S) \to \mathbf{P}$$

where $\mathbf{P}$ is a special space of continuous functions over Σ^+ containing geometric potentials. Using the pressure and variance we can define a norm $||\cdot||_{\mathrm{P}}$ over $\mathbf{P}$. Using the pullback of $||\cdot||_{\mathrm{P}}$, we can define a Riemannian metric $||\cdot||$ on $\mathcal{T}(S)$. We call this Riemannian metric the *pressure metric.* Moreover, $||\cdot||$ can also be derived by the Hessian of the intersection number:

Theorem C (The Pressure Metric): *Suppose $\rho_t \in \mathcal{T}(S)$ is an analytic path for $t \in (-\varepsilon, \varepsilon)$. Then $\mathrm{I}(\rho_0, \rho_t)$ is real analytic and*

$$||\dot\rho_0||^2 := ||\mathrm{d}\psi(\dot\rho_0)||^2_{\mathrm{P}} = \frac{\mathrm{d}^2\mathrm{I}(\rho_0,\rho_t)}{\mathrm{d}t^2}\Big|_{t=0}$$

defines a Riemannian metric on $\mathcal{T}(S_{g,n})$.

We briefly discuss the history of this Riemannian metric $||\cdot||$ on $\mathcal{T}(S_{g,n})$. When $n = 0$, Thurston first discovered it by using the Hessian of the intersection number. Thus, this Riemannian metric is also known as **Thurston's Riemannian metric**. Moreover, as proved by Wolpert [Wol86], this Riemannian metric is exactly the Weil–Petersson metric on $\mathcal{T}(S_{g,0})$. McMullen [McM08] recovered this Riemannian metric using thermodynamic formalism and called it the **pressure metric**. Carrying over the same spirit, Bridgeman, Canary, Labourie and Sambarino [BCLS15] generalized this dynamics approach and constructed a Riemannian metric on the space of Anosov representations into higher rank Lie groups, i.e., a higher rank generalization of $\mathcal{T}(S_{g,0})$. Using the pressure metric constructed in [BCLS15], Xu [Xu19] showed that the pressure metric on the Teichmüller space of bordered surfaces is incomplete and is not Lipschitz equivalent to the Weil–Petersson metric. We remark that Fuchsian representations considered in Xu's work [Xu19] are convex co-compact (i.e., have no parabolic elements) and with infinite volume. Our Theorem C extends the pressure metric and Thurston's construction to spaces of conjugacy classes of finite area type-preserving Fuchsian representations with parabolic elements, i.e., $\mathcal{T}(S_{g,n})$ for $n > 0$.

The last result of the paper is to link the two main topics in this work: Manhattan curves and the pressure metric. We prove that when we look at a path in $\mathcal{T}(S)$, the variation of corresponding Manhattan curves contains information on the pressure metric. As similar result has been proved by Pollicott and Sharp [PS16] when S is a closed surface. We generalize it to surfaces with punctures.

Theorem D: *Let $(s, \chi_t(s))$ be the coordinates of points on the Manhattan curve $\mathcal{C}(\rho_0, \rho_t)$. Then we have*

$$\frac{\mathrm{d}^2\chi_t(s)}{\mathrm{d}t^2}\Big|_{t=0} = s(s-1)\cdot ||\dot{\rho}_0||^2 \quad \textit{for } s \in (0,1).$$

The paper is organized as follows. In Section 2, we introduce some background knowledge of geometry and thermodynamic formalism of countable state Markov shifts. In Section 3 we discuss the coding of geodesic flows and important properties of the corresponding roof functions. We study the analyticity of the pressure function in Section 4. Section 5 is devoted to investigating the shape of the Manhattan curve and rigidity. In Section 6, we construct the pressure metric. In the last section, we focus on the relation between Manhattan curves and the pressure metric.

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2. Preliminaries

2.1. Geometry. Throughout this paper, $S = S_{g,n}$ is an orientable surface of genus g and n punctures and with negative Euler characteristic. In this work, we are interested in finite area hyperbolic surfaces homemorphic to S, that is, S pairs with a Riemannian metric g of Gaussian curvature -1. Notice that every such surface (S, g) can be obtained by a Fuchsian representation. More precisely, (S, g) is isomorphic to the hyperbolic surface $X_\rho = \rho(\pi_1(S))\backslash\mathbb{H}$.

For short, let us denote $\rho(\pi_1 S)$ by Γ. Recall that $\partial_\infty\mathbb{H}$, the **boundary** of $\mathbb{H}$, is defined as $\mathbb{R}\cup\{0\}$, and $\Lambda(\Gamma) := \overline{\{\gamma \cdot o : \gamma \in \Gamma\}}$ denotes the **limit set** of Γ. An element $\gamma\in\Gamma$ is called **hyperbolic** if γ has two fixed points on $\Lambda(\Gamma)$, namely, the **attracting fixed point** γ_+ (i.e., $\lim_{n\to\infty}\gamma^n o=\gamma_+$) and the **repelling fixed point** γ_- (i.e., $\lim_{n\to-\infty}\gamma^n o = \gamma_-$); γ is called **parabolic** if it has one fixed point. Because X_ρ is negatively curved, we know that every closed geodesic λ on X_ρ corresponds to a unique hyperbolic element γ (up to conjugation), and vice versa. Moreover, the length of λ equals $l[\gamma]$, the **translation distance** of γ, that is,

$$l[\gamma] := \min\{d(x, \gamma x) : x \in \mathbb{H}\}.$$

A natural dynamical system associated to X_ρ is the geodesic flow

$$g_t : T^1X_\rho \to T^1X_\rho$$

on the unit tangent bundle T^1X_ρ, which translates many geometric problems to dynamics problems. We recall that the **Busemann function** $B : \partial_\infty\mathbb{H}\times\mathbb{H}\times\mathbb{H}$ is defined as

$$B_\xi(x, y) := \lim_{z\to\xi} d(x, z) - d(y, z)$$

for $x, y, z \in \mathbb{H}$ and $\xi \in \partial_\infty\mathbb{H}$. Lift the geodesic flow $g_t : T^1X_\rho \to T^1X_\rho$ to its universal covering $T^1\mathbb{H}$; by abusing notation, we have the geodesic flow $g_t : T^1\mathbb{H} \to T^1\mathbb{H}$.

Recall that two Fuchsian representations ρ_1, ρ_2 are **type-preserving** if there exists an isomorphism $\iota : \rho_1(\pi_1 S) \to \rho_2(\pi_1 S)$ such that ι sends hyperbolic elements to hyperbolic elements and parabolic elements to parabolic elements. The following theorem indicates that if ρ_1, ρ_2 are type-preserving finite area Fuchsian representations, then we can link X_{ρ_1} and X_{ρ_2} in a controlled manner.

Theorem 2.1 (Fenchel–Nielsen Isomorphism Theorem; [Kap09, Theorem 5.5, 8.16, 8.29]): *Suppose ρ_1, ρ_2 are two finite area type-preserving Fuchsian representations of $\pi_1 S$. Then there exists a bilipschitz homeomorphism* $\mathrm{b} : X_{\rho_1} \to X_{\rho_2}$. *Moreover, one can extend* b *to an equivarient bilipschitz map, abusing the notation,* $\mathrm{b} : \partial_\infty\mathbb{H}\cup\mathbb{H} \to \partial_\infty\mathbb{H}\cup\mathbb{H}$.

Remark 2.2: In [Kap09], the homeomorphism $\mathrm{b} : X_{\rho_1} \to X_{\rho_2}$ is stated to be quasiconformal. Nevertheless, using Mori's Theorem (cf. p. 30 [Ahl06]) it is not hard to see that quasiconformal homeomorphisms are indeed bilipschitz maps.

In the following, we state a special case of [Kim01, Theorem A].

Theorem 2.3 (Marked Length Spectrum Rigidity): *Let $\rho_1,\rho_2 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ be Zariski dense Fuchsian representations. There exists a finite collection of $\gamma \in \pi_1(S)$ such that if there exists $k > 0$ such that $l[\rho_1(\gamma)] = k \cdot l[\rho_2(\gamma)]$ for all these γ, then ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2,\mathbb{R})$.

Remark 2.4:

(1) A representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is called **Zariski dense** if it is irreducible and $\rho(\pi_1(S))$ has no global fixed point on $\partial_\infty\mathbb{H}$. It is clear that finite area Fuchsian representations are Zariski dense.

(2) Theorem A in [Kim01] is much more general than the special case that we stated in Theorem 2.3, and this special case should be known before [Kim01]. Nevertheless, for convenience, we quote [Kim01, Theorem A].

2.2. Countable state Markov shifts.

In this subsection we aim to introduce terminologies of thermodynamic formalism for countable state (topological) Markov shifts. The reader can find more details in Mauldin's and Urbański's book [MU03] and Sarig's notes [Sar09].

Let $\mathcal{A}$ a countable set and $\mathbb{A} = (t_{ab})_{\mathcal{A}\times\mathcal{A}}$ be a matrix of zeros and ones with no columns or rows consisting entirely of zeros.

Definition 2.5 (Countable State Markov Shift): The (one-sided) **countable state Markov shift** with **alphabet** (or states) $\mathcal{A}$ and **transition matrix** $\mathbb{A}$ is defined by

$$\Sigma_{\mathbb{A}}^{+} := \{x = (x_i) \in \mathcal{A}^{\mathbb{N}} : t_{x_n x_{n+1}} = 1 \ \forall n \in \mathbb{N}\}$$

equipped with the topology generated by the collection of **cylinders**

$$[a_0, \ldots, a_n] := \{x \in \Sigma_{\mathbb{A}}^{+} : x_i = a_i, 0 \le i \le n\} \quad (n \in \mathbb{N}, a_0, \ldots, a_n \in \mathcal{A})$$

and coupled to the (left) shift map $\sigma : (x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, \ldots)$.

A **word** of **length** n on an alphabet $\mathcal{A}$ is a finite sequence

$$(a_0, a_1, \ldots, a_{n-1}) \in \mathcal{A}^{n-1}$$

for all $n \in \mathbb{N}\backslash\{0\}$, and a word $(a_0, a_1, \ldots, a_{n-1})$ is **admissible** with respect to $\mathbb{A} = (t_{ab})_{\mathcal{A}\times\mathcal{A}}$ if $t_{a_i a_j} = 1$.

From now on we will omit the subscript $\mathbb{A}$ from $\Sigma_{\mathbb{A}}^+$ and simply use Σ^+ for one-sided Markov shifts because our discussion here only focuses on a fixed transition matrix.

Recall that a Markov shift (Σ^+, σ) is **topologically transitive** if for all $a, b \in \mathcal{A}$ there exists an admissible word $(a, \dots, b)$, and is **topological mixing** if for all $a, b \in \mathcal{A}$ there exists a number N_{ab} such that for all $n \geq N_{ab}$ there exists an admissible word $(a, \dots, b)$ of length n.

Let $g : \Sigma^+ \to \mathbb{R}$ be a function. For $n \geq 1$, the n-th **variation** of g is defined by

$$\mathrm{V}_n(g) := \sup\{|g(x) - g(y)| : x, y \in \Sigma^+, x_i = y_i \text{ for } 0 \leq i \leq n-1\}.$$

When $\sum_{n=0}^{\infty} \mathrm{V}_n(g) < \infty$ we say that g has **summable variations,** and in particular, we call g a **locally Hölder continuous function** if there exist $C > 0$ and $\theta \in (0,1)$ such that $\mathrm{V}_n(g) \leq C \cdot \theta^n$ for $n \geq 1$.

We remark that when the alphabet $\mathcal{A}$ is finite the Markov shift is called a **subshift of finite type**, and in that case Σ^+ is a compact set. When $\mathcal{A}$ is infinite, Σ^+ is no longer compact. Nevertheless, countable state Markov shifts with the following property can be studied similarly as in the compact cases.

Definition 2.6 (BIP): We say $(\Sigma_{\mathbb{A}}^+, \sigma)$ has the **big image and preimages (BIP) property** if there exists a finite collection of states $s_1, s_2, \dots, s_n \in \mathcal{A}$ such that for every state $s \in \mathcal{A}$ there are some $i, j \in \{1, 2, \dots, n\}$ such that (s_i, s), (s, s_j) are admissible.

Definition 2.7 (Topological Pressure for Countable State Markov Shifts): Let (Σ^+, σ) be a topologically mixing Markov shifts and $g : \Sigma^+ \to \mathbb{R}$ has summable variations. The **topological pressure** (or the **Gurevich pressure**) of g is defined by

$$P_\sigma(g) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{Fix}^n} e^{S_n g(x)} \mathbb{1}_{[a]}(x),$$

where $\mathrm{Fix}^n := \{x \in \Sigma^+ : \sigma^n(x) = x\}$, $a \in \mathcal{A}$ is any state, and

$$S_n g(x) = g(x) + \cdots + g(\sigma^{n-1}(x))$$

is the n-th ergodic sum of g.

Notice that the topological pressure is independent of the state $a \in \mathcal{A}$ (cf. [Sar09]).

Theorem 2.8 (Variational Principle; [Sar99] Theorem 3): *Let* (Σ^+, σ) *be a topologically mixing Markov shift and* $g : \Sigma^+ \to \mathbb{R}$ *has summable variations. If* $\sup g < \infty$ *then*

$$P_\sigma(g) = \sup \left\{ h_\sigma(m) + \int_{\Sigma^+} g \, \mathrm{d}m : m \in \mathcal{M}_\sigma \text{ and } -\int_{\Sigma^+} g \, \mathrm{d}m < \infty \right\}$$

where $h_\sigma(m)$ *is the measure theoretic entropy of* m *and* $\mathcal{M}_\sigma$ *is the set of* σ*-invariant Borel probability measures on* Σ^+.

We want to remark that although Mauldin and Urbański and also Sarig defined countable state Markov shifts and the topological pressure differently, when the Markov shift is topologically mixing and has the BIP property, their definitions are the same (cf. [MU01, Section 7]). Since in this paper we only focus on topologically mixing Markov shifts with the BIP property, we will use results from both Mauldin and Urbański, and Sarig.

Recall that a measure $m \in \mathcal{M}_\sigma$ is called an **equilibrium state** for g if

$$P(g) = h_\sigma(m) + \int g \, \mathrm{d}m.$$

A measure $\nu \in \mathcal{M}_\sigma$ is called a **Gibbs measure** for g if there exist constants $G > 1$ and P such that for all cylinders $[a_0, \ldots, a_{n-1}]$ and for every $x \in [a_0, \ldots, a_{n-1}]$ we have

$$\frac{1}{G} \leq \frac{\nu[a_0, a_1, \ldots, a_{n-1}]}{\exp[S_n g(x) - nP]} \leq G.$$

Remark 2.9: We would like to point out that there are subtle differences between Gibbs states and equilibrium states. Every equilibrium state is a Gibbs state but not vice versa. More precisely, if g is locally Hölder with finite pressure and $\sup g < \infty$, then g has a unique Gibbs measure ν_g, and g has at most one equilibrium state. Furthermore, with the additional condition $-\int g \, \mathrm{d}\nu_g < \infty$, we know the unique Gibbs state ν_g is the equilibrium state for g (cf. [Sar09, Theorem 4.5, 4.6, 4.9] and [MU03, Theorem 2.2.4, 2.2.9]).

Two functions $f, g : \Sigma^+ \to \mathbb{R}$ are **cohomologous**, denoted by $f \sim g$, if there exists a function $h : \Sigma^+ \to \mathbb{R}$ such that $f = g + h - h \circ \sigma$ where h is called a **transition function**. The following theorem shows that the thermodynamic data are invariant in each cohomologous class of locally Hölder continuous functions.

Theorem 2.10 ([MU03, Theorem 2.2.7]): *Suppose* (Σ^+, σ) *is topologically mixing, and* $f, g : \Sigma^+ \to \mathbb{R}$ *are locally Hölder continuous function with Gibbs measures* ν_f *and* ν_g*, respectively. Then the following are equivalent:*

(1) $\nu_f = \nu_g$.
(2) *(Livšic Theorem) There exists a constant* $R > 0$ *such that* $\forall\, n \geq 1$ *and* $x \in \mathrm{Fix}^n$ *we have* $S_n f(x) - S_n g(x) = nR$.
(3) $f - g$ *is cohomologous to a constant* R *via a bounded Hölder continuous transition function.*

Moreover, when the above assertions are true, then $R = P_\sigma(f) - P_\sigma(g)$.

We remark that we can define a two-sided countable state Markov shift $\Sigma_{\mathbb{A}}$ as

$$\Sigma_{\mathbb{A}} := \{x = (x_i) \in \mathcal{A}^{\mathbb{Z}} : t_{x_n x_{n+1}} = 1\ \forall n \in \mathbb{Z}\}$$

and define similarly all the thermodynamic data. Notice that if a potential on a two-sided shift space (Σ, σ) only depends on its future coordinate, then to understand the associated thermodynamic data, it is sufficient to study its behavior on the one-sided shift (Σ^+, σ). For a two-sided sequence $(, a, \dot{b}, c,)$, $\dot{b}$ means b is at the zero-th coordinate, i.e., $a = x_{-1}, b = x_0, c = x_1$.

Let (Σ^+, σ) be a topologically mixing countable state Markov shift with the BIP property. In the following, we list a few theorems about the analyticity of pressure and phase transition phenomena.

Theorem 2.11 (Analyticity of Pressure; [MU03, Theorem 2.6.12 and 2.6.13], [Sar03, Corollary 4]): *Suppose* $t \mapsto f_t$ *is a real analytic family of locally Hölder continuous functions for* $t \in \Delta$*, where* Δ *is an interval of* $\mathbb{R}$ *and* $P_\sigma(f_t) < \infty$ *for* $t \in \Delta$*. Then the pressure function* $t \mapsto P_\sigma(f_t)$*, for* $t \in \Delta$*, is also real analytic. Moreover, the derivative of the pressure is*

$$\frac{\mathrm{d}}{\mathrm{d}t} P_\sigma(f_t)\Big|_{t=0} = \int_{\Sigma^+} \dot{f}_0 \,\mathrm{d}\nu_{f_0},$$

where ν_{f_0} *is the unique Gibbs state for* f_0.

Theorem 2.12 (Phase Transition; [Sar99, Sar01], [MU03]): *Let* $g : \Sigma^+ \to \mathbb{R}$ *be a locally Hölder continuous function with* $g > 0$*. Then there exists* $s_\infty > 0$ *such that*

$$P_\sigma(-tg) = \begin{cases} \infty & \text{if } t < s_\infty, \\ \text{real analytic} & \text{if } t > s_\infty. \end{cases}$$

Moreover, $-tg$ *has a unique Gibbs state* ν_{-tg} *for* $t > s_\infty$.

Let $f : \Sigma^+ \to \mathbb{R}$ be a locally Hölder continuous function and let $m \in \mathcal{M}_\sigma$ be an invariant measure. Recall that the **variance** $\mathrm{Var}(f, m)$ of f with respect to m is defined by

$$\mathrm{Var}(f,m) := \lim_{n\to\infty} \left(\frac{1}{n} \int_{\Sigma^+} (S_n f - \int_{\Sigma^+} f \,\mathrm{d}m)^2 \right)^{\frac{1}{2}}.$$

Using Theorem 2.11 and [Sar09, Theorem 5.10, 5.12] (or [MU03, Theorem 2.6.14, Lemma 4.8.8]), we have the following corollary.

Corollary 2.13 (Derivatives of Pressure): *Suppose $f + tg$ is a family of locally Hölder continuous functions with finite pressure for $t \in (-\varepsilon, \varepsilon)$. If g is bounded, then*

$$P_\sigma(f+tg) = P_\sigma(f) + t \cdot \int_{\Sigma^+} g \,\mathrm{d}\nu_f + \frac{t^2}{2} \cdot \mathrm{Var}(g, \nu_f) + o(t^2)$$

where ν_f is the Gibbs measure for f. Moreover,

$$\mathrm{Var}(g, \nu_f) = 0$$

if and only if g is cohomologous to zero.

2.3. Suspension flows over countable state Markov shifts. Let (Σ^+, σ) be a topologically mixing countable state Markov shift with the BIP property and $\tau : \Sigma^+ \to \mathbb{R}^+$ be bounded away from zero and locally Hölder continuous. The **suspension space** (relatively to τ) is the set

$$\Sigma^+_\tau := \{(x,t) \in \Sigma^+ \times \mathbb{R} : 0 \le t \le \tau(x)\}/\sim,$$

where $(x, \tau(x)) \sim (\sigma x, 0)$ for every $x \in \Sigma^+$. The **suspension flow** ϕ_t with **roof function** τ is the (vertical) translation flow on Σ^+_τ given by

$$\phi_t(x,s) = (x, s+t) \quad \text{for } x \in \Sigma^+ \text{ and } 0 \le s+t \le \tau(x).$$

Similarly, we can define suspension flows over a two-sided shift.

In the following, we list several equivalent definitions of the topological pressure for suspension flows. These definitions are from Savchenko [Sav98]; Barreira and Iommi [BI06]; Kempton [Kem11]; and Jaerisch, Kesseböhmer and Lamei [JKL14].

Given a continuous function $F : \Sigma_\tau^+ \to \mathbb{R}$, we define the function $\Delta_F : \Sigma^+ \to \mathbb{R}$ by

$$\Delta_F(x) := \int_0^{\tau(x)} F(x,t)\,\mathrm{d}t.$$

Definition/Theorem 2.14 (Topological Pressure for Suspension Flows): *Suppose $F : \Sigma_\tau^+ \to \mathbb{R}$ is a function such that $\Delta_F : \Sigma^+ \to \mathbb{R}$ is locally Hölder continuous. The following descriptions of $P_\phi(F)$, the* ***topological pressure*** *of F over the suspension flow (Σ_τ^+, ϕ), are equivalent:*

$$\begin{aligned} P_\phi(F) &= \lim_{T\to\infty} \frac{1}{T} \log \Bigg(\sum_{\substack{\phi_s(x,0)=(x,0) \\ 0\le s\le T}} \exp\left(\int_0^s F(\phi_t(x,0))\,\mathrm{d}t \right) \mathbb{1}_{[a]}(x) \Bigg) \\ &= \sup\left\{ h_\phi(\mu) + \int_{\Sigma_\tau^+} F\,\mathrm{d}\mu : \mu \in \mathcal{M}_\phi \text{ and } -\int_{\Sigma_\tau^+} \tau\,\mathrm{d}\mu < \infty \right\}, \end{aligned}$$

where a is any state in $\mathcal{A}$ and $\mathcal{M}_\phi$ is the set of ϕ-invariant Borel probability measures on Σ_τ^+. Moreover, if $\mu \in \mathcal{M}_\phi$ such that $P_\phi(F) = h_\phi(\mu) + \int_{\Sigma_\tau^+} F\,\mathrm{d}\mu$, then we call μ an ***equilibrium state*** *for F.*

We finish this subsection by recalling an important observation of relations between invariant measures on Σ^+ and on Σ_τ^+.

Theorem 2.15 ([AK42]): *Let $\mathcal{M}_\sigma(\tau) := \{m \in \mathcal{M}_\sigma : \int_{\Sigma^+} \tau\,\mathrm{d}m < \infty\}$. Then there exists a bijection*

$$\begin{aligned} R : \mathcal{M}_\sigma(\tau) &\to \mathcal{M}_\phi \\ m &\mapsto \frac{m \times \mathrm{Leb}}{m \times \mathrm{Leb}(\Sigma_\tau^+)} \end{aligned}$$

where Leb *is the Lebesgue measure for the flow direction.*

In other words, for any continuous function $F : \Sigma_\tau^+ \to \mathbb{R}$, we have

$$\int_{\Sigma_\tau^+} F\,\mathrm{d}R(m) = \frac{\int_{\Sigma^+} \Delta_F\,\mathrm{d}m}{\int_{\Sigma^+} \tau\,\mathrm{d}m}.$$

Theorem 2.16 (Equilibrium States for Flows; [IJT15] Theorem 3.4, 3.5): *Let $F : \Sigma_\tau^+ \to \mathbb{R}$ be a continuous function such that Δ_F is locally Hölder. Suppose Δ_F has an equilibrium state m_{Δ_F} such that $\int \tau\,\mathrm{d}m_{\Delta_F} < \infty$. Then F has a unique equilibrium state $\mu = R(m_{-P_\phi(F)\tau+\Delta_F})$.*

3. Geodesic flows for finite area hyperbolic surfaces

3.1. A symbolic model for geodesics flows. In this section, we survey a symbolic model for the geodesic flow. More precisely, we will construct a geodesic flow invariant subset Ω_0 of the unit tangent bundle, and study it through a symbolic model. This construction is given by Ledrappier and Sarig in [LS08]. We will mostly follow their notations and use the Poincaré disk model $\mathbb{D}$ in this section.

Let $S = S_{g,n}$ be a surface with genus g and n punctures, $X = X_\rho$ be the finite area hyperbolic surface given by the Fuchsian representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, and $g_t : T^1X \to T^1X$ be the geodesic flow for X. In this paper, we are only interested in non-compact surfaces, because the compact case has been studied before. In other words, in our discussion n is at least 1.

Theorem 3.1 ([Tuk72, Tuk73]): *Suppose X is a non-compact finite area hyperbolic surface with negative Euler characteristic. Then there exists a closed ideal hyperbolic polygon $D_0 \subset \mathbb{D}$ such that the following hold:*

(1) *The origin is in D_0.*
(2) *D_0 has $2k$ vertices, and all vertices are on $\partial_\infty \mathbb{D}$, where*

$$k = 2g + n - 1 = -\chi(X) + 1 \geq 2.$$

(3) *These vertices partition $\partial_\infty \mathbb{D}$ to $2k$ intervals I_i, $i \in \mathcal{S}$ where*

$$\mathcal{S} := \{1, 1', 2, 2', \ldots, k, k'\}.$$

Moreover, each I_i can be paired with the other interval $I_{i'}$ such that there exists a pair of Möbius transformations $g_i, g_{i'} = g_i^{-1}$ with g_i maps I_i onto $\partial_\infty \mathbb{D} \backslash I_{i'}$ and $g_{i'}$ maps $I_{s'}$ onto $\partial_\infty \mathbb{D} \backslash I_i$.
(4) *X is isomorphic to the space obtained by identifying all pairs of $(I_i, I_{i'})$ through g_i for all $i \in \mathcal{S}$.*
(5) *Take i (or i') from each side pair $(I_i, I_{i'})$ and consider the corresponding Möbius transformation g_i. Then*

$$\Gamma = \rho(\pi_1(X)) = \langle g_1, \ldots, g_k \rangle$$

where ρ is the Fuchsian representation such that $X = \Gamma \backslash \mathbb{D}$.

From now on, for the finite area hyperbolic surface X, we use the generator given in the above theorem, and denote $\Gamma = \langle g_1, \dots, g_k \rangle$. Roughly speaking, there are two steps to construct the Ledrappier–Sarig coding. One first uses the generators $\{g_1, \dots, g_k\}$ to derive a Markov shift (Σ_1, σ_1) (i.e., cutting sequences), then modify (Σ_1, σ_1) to get another Markov shift (Σ_A, σ_A) on which the first returning map has better regularity. We will discuss their construction in detail below.

The shape of the fundamental D_0 plays a crucial role in the Ledrappier and Sarig's coding. We start by looking at vertices of D_0. Notice that for every vertex v of D_0, there exists a (shortest) cycle, say l elements, of edge-pairing isometries g_{s_i} for $1 \le i \le l$ such that v is the unique fixed point of $g_{s_l} g_{s_{l-1}} g_{s_2} g_{s_1}$ provided $g_{s_i} g_{s_2} g_{s_1}(D_0)$ and $(g_{s_1}^{-1} g_{s_2}^{-1} g_{s_i}^{-1})(D_0)$ touch $\partial_\infty \mathbb{D}$ at v for all $1 \le i \le l$. We call

$$\underline{w} = (s_1, \dots, s_l) \quad \text{and} \quad \underline{w'} = (s_l', \dots, s_1')$$

the **cycles** of v. We denote the set of all vertex cycles by $\mathfrak{C}$, and $N(\mathfrak{C})$ is the least common multiplier of length of cycles of all vertices (see Figure 3.1).

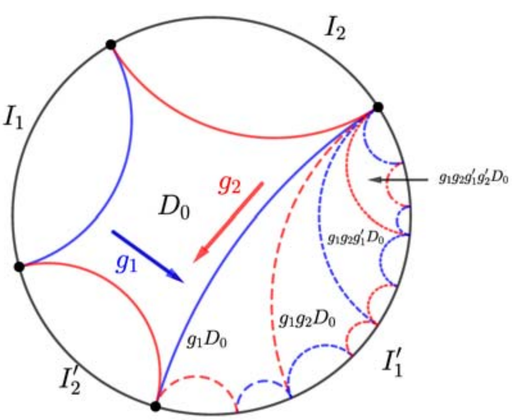

Figure 3.1. Finite area surfaces with cusps.

3.1.1. *The classical coding.* Recall that a vector $v \in T^1X$ **escapes to infinity** if $g_t(v)$ leaves, eventually, all compact sets $K \subset T^1M$ as $t \to \infty$ or $-\infty$. Let $\Omega_0 \subset T^1X$ be the set of non-escaping vectors. It is clear that Ω_0 is a flow invariant set and contains most of the interesting dynamics.

A unit vector $v \in T^1\mathbb{D}$ based at $\mathbb{D} \cap \partial D_0$ is called **inward pointing** if $g_t(v) \in \mathrm{int}(D_0)$ for sufficiently small t. We denote by $(\partial D_0)_{\mathrm{in}}$ the set of all inward pointing vectors. It is not hard to see that $(\partial D_0)_{\mathrm{in}}$ projects to a Poincaré section of $g_t : \Omega_0 \to \Omega_0$; by abusing notation, we also denote this section by $(\partial D_0)_{\mathrm{in}}$.

In the following, we recall two equivalent methods of coding of geodesic flows on Ω_0: cutting sequences and boundary expansion. To derive the coding, we first label edges of D_0 in the following manner. For each edge e of D_0, it determines a boundary interval $I_{s(e)}$ for some $s(e) \in \mathcal{S}$ such that $I_{s(e)}$ has the same vertices as e and is on the side of e which does not contain D_0. We call $s = s(e) \in \mathcal{S}$ the **external label** of e, and $s' = s'(e)$ the **internal label** of e. See Figure 3.2 for an illustration.

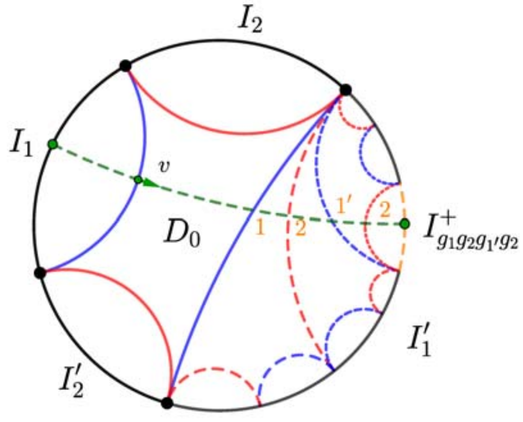

Figure 3.2. Classical coding.

Now we are ready to state two canonical codings or Markov partitions associated to $(\partial D_0)_{\mathrm{in}}$. For every $v \in (\partial D_0)_{\mathrm{in}}$ it is determined by:

(1) **Cutting sequence** $(x_k) \in \mathcal{S}^{\mathbb{Z}}$: x_k are the internal labels of the edges of D_0 cut by $g_t(v)$ where $k = 1$ is the first cut in positive time and $k = 0$ is the first cut in non-negative time.

(2) **Boundary expansion** $(y_k) \in \mathcal{S}^{\mathbb{Z}}$: the lift $\widetilde{(g_t v)} \subset T^1\mathbb{D}$ is a geodesic on $T^1\mathbb{D}$ with an **attracting limit point** (or ending point) in $\bigcap_{k\geq 1} I^+_{y_1,\ldots,y_k}$, and a **repelling limit point** (or beginning point)

in $\bigcap_{k\le 0} I^-_{y_0,\ldots,y_k}$, where

$$I^+_{s_1,s_2,\ldots,s_k} = g_{s_1}g_{s_2}\cdots g_{s_{k-1}}(I_{s'_k})$$

and

$$I^-_{s_1,s_2,\ldots,s_k} = g^{-1}_{s_1}g^{-1}_{s_2}\cdots g^{-1}_{s_{k-1}}(I_{s_k}).$$

It is not hard to see that $(x_k)_{k\in\mathbb{Z}} = (y_k)_{k\in\mathbb{Z}}$ because all vertices of D_0 are on $\partial_\infty\mathbb{D}$. Thus we can and will interchange between these two perspectives. In summary, the classical coding means that for every $v \in (\partial D_0)_{\text{in}}$, the geodesic $g_t(v)$ corresponds to an element in

$$\Sigma_1 := \{(x_k) \in \mathcal{S}^{\mathbb{Z}} : x_{k+1} \neq (x_k)'\}$$

and σ_1 is the left shift on Σ_1.

3.1.2. *The modified coding.* As pointed out in [LS08], (Σ_1, σ_1) is not "good" enough for our purpose. For example, the classical coding is not necessarily one to one, and the first return map is not regular enough to push the machinery. Thus we need to modify (Σ_1, σ_1) by looking at a smaller section of the flow $g_t : \Omega_0 \to \Omega_0$.

Fix a number n^* large, set $N^* = 4n^*N(\mathfrak{C})$, and the set of length N^* repeating vertex cycles defined as

$$\mathfrak{C}^* := \{\underbrace{(\underline{w}, \underline{w}, \ldots, \underline{w})}_{N^*/|\underline{w}| \text{ copies}} : \underline{w} \in \mathfrak{C}\}.$$

We write $N^\# := \frac{1}{2}N^* - 1$. Now consider the following set:

$$A := \{y \in \Sigma_1 : \underbrace{(y_{-N^\#}, \ldots, y_{\frac{N^*}{2}})}_{N^*} \notin \mathfrak{C}^*\} \subset \Sigma_1.$$

The smaller section $S_A \subset (\partial D_0)_{\text{in}}$ is given by

$$S_A := \{v \in (\partial D_0)_{\text{in}} : \text{ the cutting sequence of } g_t(v) \text{is in } A\}$$

(see Figure 3.3).

It is not hard to see that S_A is a Poincaré section of $g_t : \Omega_0 \to \Omega_0$. Moreover, by the combinatorial property of $\mathfrak{C}$ pointed out in [LS08, Section 2.1], we know that for a geodesic $g_t(v)$ with the cutting sequence $(x_n)_{n\in\mathbb{Z}}$ which stops returning to A at some point, (x_n) will eventually repeat an element $\underline{w} \in \mathfrak{C}^*$, i.e., $(x_n)_{n\in\mathbb{Z}} = (\ldots, x_n, \ldots, \underline{w}, \underline{w}, \underline{w}, \ldots)$.

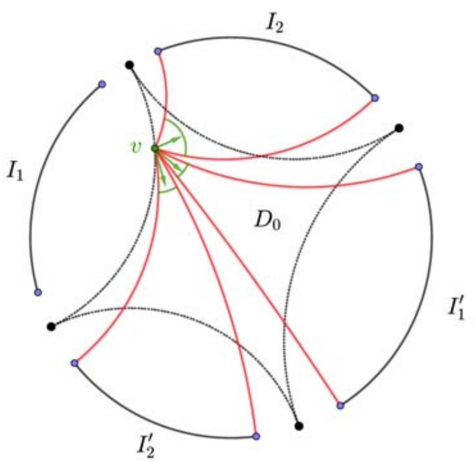

Figure 3.3. The smaller section.

In other words, if v does not escape to infinity, then the cutting sequence of $g_t(v)$ always returns to A. More precisely, $\forall x \in A$, there exists $N = N(x) \in \mathbb{R}$ such that $\sigma_1^N(x) \in A$. We define the induced shift map on A by

$$\sigma_A(x) := \sigma_1^{N_A(x)}$$

where

$$N_A(x) = \min\{N \in \mathbb{N} : \sigma_1^N(x) \in A\}.$$

Now, we are ready to describe a Markov partition of this modified Markov shift $\sigma_A : A \to A$:

(1) **Type I**, denoted by $\Sigma_A(\mathrm{I})$: cylinders of length $N^* + 1$, namely

$$[x_{-N^\#}, \ldots, \dot{x}_0, \ldots, x_{\frac{N}{2}^*-1}, x_{\frac{N}{2}^*}],$$

such that

$$\underbrace{[x_{-N^\#}, \ldots, \dot{x}_0, \ldots, x_{\frac{N}{2}^*-1}]}_{\text{length}=N^*} \subset A$$

and

$$[x_{-N^\#+1}, \ldots, \dot{x}_1, \ldots, x_{\frac{N}{2}^*}] = \sigma_1([x_{-N^\#}, \ldots, \dot{x}_0, \ldots, x_{\frac{N}{2}^*-1}]) \subset A.$$

The **shape** of $[\underline{e}] \in \Sigma_A(\mathrm{I})$ is defined as $\mathfrak{s}(\underline{e}) = (\underline{e})$.

(2) **Type II**, denoted by $\Sigma_A(\mathrm{II})$: cylinders of length bigger than N^*+1, namely

$$B_{l,k}(a,\underline{w},c) := [x_{-N^\#}, w_1, \ldots, \dot{w}_{N\#}, \ldots, w_{N^*}, (\underline{w})^{l-1}, w_1, w_2, \ldots, w_k, \underline{b}]$$

where $a := x_{-N\#}, c \in \mathcal{S}$, $\underline{w} \in \mathfrak{C}^*$, $l \geq 0$, $0 \leq k < N^*$ are not both zero, and

$$\underline{b} := \begin{cases} (w_{k+1}, \ldots, w_{N^*}, w_1, \ldots, w_{k-1}, c), & l = 0, k \neq 0 \\ (w_1, \ldots, w_{N^*-1}, c), & l \neq 0, k = 0 \\ (w_{k+1}, \ldots, w_{N^*}, w_1, \ldots, w_{k-1}, c), & l, k \neq 0 \end{cases}$$

provided $B_{l,k}(a,\underline{w},c) \subset A$, $[\underline{b}] \subset A$. The *shape* of $[\underline{e}] \in \Sigma_A(\mathrm{II})$ and of the form $B_{l,k}(a,\underline{w},c)$ is defined as

$$\mathfrak{s}(\underline{e}) := (k, a, \underline{w}, c) \in \{0, \ldots, N^*-1\} \times \mathcal{S} \times \mathfrak{C}^* \times \mathcal{S}.$$

Proposition 3.2 ([LS08, Lemma 2.1]): *$\sigma_A : A \to A$ is topologically mixing, and the Markov partition given by $\Sigma_A(\mathrm{I})$ and $\Sigma_A(\mathrm{II})$ has the BIP property.*

Let (Σ_A, σ_A) be the countable state Markov shift derived by the Markov partition $\Sigma_A(\mathrm{I})$ and $\Sigma_A(\mathrm{II})$. We write the alphabet set of Σ_A^+ by

$$\mathcal{S}_A := \left\{ \underline{e} \in \bigcup_{n \geq N^*+1} \mathcal{S}^n : \sigma_1^{\#}[\underline{e}] \in \Sigma_A(\mathrm{I}) \cup \Sigma_A(\mathrm{II}) \right\}.$$

Let $\pi_A : \Sigma_A \to A \subset \Sigma_1$ denote the natural coding map. For $x \in \Sigma_A$, we use x_0 to denote the letter in the zero-th coordinate. Notice that we can always write $x_0 = (s_{-N\#}, \ldots, s_{n-N\#-1})$ in terms of $\mathcal{S}$ letters, and in this representation $n - N^*$ is the σ_1-return time of $[x_0]$.

Remark 3.3:

(1) $\Sigma_A(\mathrm{I})$ is composed of return time 1 (i.e., $N_A = 1$) cylinders, and

$$\underbrace{[x_{-N\#,\ldots,}x_{n-N\#}]}_{n+1 \text{ terms}} \in \Sigma_A(\mathrm{II})$$

has return time $n - N^*$.

(2) There are only finitely many different shapes $\mathfrak{s}(\underline{a})$ for all $\underline{a} \in \mathcal{S}_A$.

(3) The length $|\underline{a}|$ for $\underline{a} \in \mathcal{S}_A$ is unbounded.

Recall that every $x = (x_k) \in \Sigma_A$ determines a point $\pi_A(x) = (s_i) \in \Sigma_1$, and $\pi_A(x)$ corresponds to a unit tangent vector $v(x) \in S_A \subset (\partial D_0)_{\text{in}}$. We denote by $\xi(x)$ the attracting limit point of $v(x)$ and $\eta(x)$ the repelling fixed point of $v(x)$. Since $\xi(x) = \bigcap_{k\geq 1} I^+_{s_1,\ldots,s_k}$ and $\eta(x) = \bigcup_{k\leq 0} I^-_{s_0,\ldots,s_k}$ where $\pi_A(x) = (s_i)_{i\in\mathbb{Z}}$, we know that $\xi(x)$ only depends on $x^+ = (x_k)_{k\geq 0}$ and $\eta(x)$ only depends on $x^- = (x_k)_{k\leq 0}$.

Definition 3.4: The **geometric potential** $\tau : \Sigma_A \to \mathbb{R}$ is defined as

$$\tau(x) := B_{\xi(x)}(o, (g_{x_0})o)$$

where o is the origin, $x_0 = (s_{-N^\#}, \ldots, s_{n-N^\#-1}) \in \mathcal{S}_A$, and

$$g_{x_0} = g_{s_1} \circ \cdots \circ g_{s_{n-N^*}}.$$

Proposition 3.5 (Geometric Potential (I), [LS08, Lemma 2.2]): *Let (Σ_A, σ_A) be the Markov shift constructed above. Then:*

1. *Suppose v generates a closed geodesics, namely $g_{l(v)}v = v$. Then there exists a unique (up to permutations) $x = (\overline{x_1 x_2 \cdots x_m}) \in \mathrm{Fix}^m(\Sigma_A)$ such that $l(v) = S_m\tau(x)$, and vice versa.*
2. *τ is locally Hölder continuous.*
3. *τ only depends on the future coordinates, that is, if $x_0^\infty = y_0^\infty$ then $\tau(x) = \tau(y)$.*
4. *$\exists C, K > 0$ such that $\tau(x) + \tau(\sigma(x)) + \cdots + \tau(\sigma^n(x)) \geq C$ for all $n \geq K$.*

Since the geometric potential τ only depends on the future coordinates, we can focus on (Σ_A^+, σ_A), the one-sided countable Markov shift induced from (Σ_A, σ_A), by forgetting the past coordinate.

Proposition 3.6 (Geometric Potential (II), [LS08, Lemma 3.1]): *On the one-sided countable Markov shift (Σ_A^+, σ_A), we have the following:*

1. *τ has a unique equilibrium state $m_{-\tau}$ and $\int_{\Sigma_A^+} \tau \, \mathrm{d}m_{-\tau} < \infty$.*
2. *The Liouville measure m_L on T^1X is given by $m_L = R(m_{-\tau}) \circ \widetilde{\pi}_A^{-1}$ where $R : \mathcal{M}_\sigma \to \mathcal{M}_\tau$ given in Theorem 2.15.*
3. *$P(-\tau) = 0$.*
4. *τ is bounded on $\Sigma_A(\mathrm{I})$, and there exists $C_1 > 0$ such that*

$$2\ln|x_0| - C_1 \leq \tau(x) \leq 2\ln|x_0| + C_1$$

for all $x \in \Sigma_A(\mathrm{II})$.

Proof. Everything is in [LS08, Lemma 3.1], and only the first assertion of (4) needs more exploration. Let $[x_0] \in \Sigma_A(\mathrm{I})$ and $x \in [x_0]$. We can write

$$x_0 = (s_{-N^\#}, \dots, s_{\frac{N^*}{2}-1}, s_{\frac{N^*}{2}}), \quad g_{x_0}(x) = g_{s_1}, \quad \text{and} \quad \tau(x) = B_{\xi(x)}(o, g_{s_1} o)$$

where $s_i \in \mathcal{S}$ for $i = -N^\#, \dots, \frac{N^*}{2}$. Recall that in the disc model,

$$B_\xi(o, y) = \ln \frac{1 - |y|^2}{|\xi - y|^2}.$$

Since $\xi(x) \in I^+_{s_1, s_2, \dots, s_{\frac{N^*}{2}}}$, it is not hard to see $B_{\xi(x)}(o, g_{s_1} o) = \ln(\frac{1-|g_{s_1} o|^2}{|\xi(x) - g_{s_1} o|})$ is (uniformly) bounded for all $x \in [x_0]$. Notice that this bound depends on $[x_0] \in \Sigma_A(\mathrm{I})$. We can find a universal bound $\tau(x)$ on $\Sigma_A(\mathrm{I})$ because

$$|\Sigma_A(\mathrm{I})| < \infty. \qquad \blacksquare$$

Remark 3.7:

(1) By standard techniques in symbolic dynamics, we know that τ is cohomologous to τ' which is locally Hölder and $\tau'(x) > c > 0$ for some constant c' (cf. [Kao18, Lemma 3.8]). From now on, we will use τ' to replace τ whenever τ needs to be bounded away from zero. Abusing the notation, we will continue to denote τ' by τ.
(2) In [LS08], the constant C_1 in Proposition 3.6 (4) depends on the shape of x_0. Because there are only finitely many shapes, we can replace it by a universal constant.

3.2. Type-preserving finite area Fuchsian representations. In this subsection, we consider ρ_1, ρ_2, two type-preserving finite area Fuchsian representations. The Fenchel–Neilsen Isomorphism Theorem (cf. Theorem 2.1) shows that there exists a bilipschitz map taking the limit set $\Lambda(\rho_1(\pi_1 S))$ and fundamental domain of X_{ρ_1} to $\Lambda(\rho_2(\pi_1 S))$ and the fundamental domain of X_{ρ_2}, and hence $\Lambda_0(\rho_1)$, to $\Lambda_0(\rho_2)$. Hence, the suspension flows corresponding to the geodesic flows on $\Omega_0(\rho_1)$ and $\Omega_0(\rho_2)$ correspond to the same Markov shift (Σ_A^+, σ_A) but different roof functions τ_{ρ_1}, τ_{ρ_2}, respectively. The following result shows that we have good control of these roof functions.

Corollary 3.8: *There exists $C > 0$ such that $|\tau_{\rho_1}(x) - \tau_{\rho_2}(x)| < C$ for all Σ_A^+. In particular, $|\frac{\tau_{\rho_2}(x)}{\tau_{\rho_1}(x)}| < C'$ for some constant C'.*

Proof. It follows immediately from Proposition 3.6 (4) and Remark 3.7. $\blacksquare$

In the second part of this subsection, we discuss the intersection number $\mathrm{I}(\rho_1,\rho_2)$ of ρ_1 and ρ_2 proposed by Thurston. Recall that $\mathrm{I}(\rho_1,\rho_2)$ of ρ_1 and ρ_2 is defined as

$$\mathrm{I}(\rho_1,\rho_2) := \lim_{n\to\infty} \frac{l_2[\gamma_n]}{l_1[\gamma_n]}$$

where $\{[\gamma_n]\}_{n=1}^{\infty}$ is a sequence of conjugacy classes for which the associated closed geodesics γ_n become equidistributed on $\rho_1(\pi_1 S)\backslash\mathbb{H}$ with respect to the Liouville measure. However, it is unclear why $\mathrm{I}(\rho_1,\rho_2)$ is well-defined, especially when S has punctures. We will discuss this issue in Proposition 3.10 where we give $\mathrm{I}(\rho_1,\rho_2)$ a dynamical characterization.

To link the suspension flows on $\Sigma^+_{\tau_{\rho_1}}$ and $\Sigma^+_{\tau_{\rho_2}}$, we consider the following reparametrization function $\psi : \Sigma^+_{\tau_{\rho_1}} \to \mathbb{R}$.

Definition 3.9 (Symbolic reparametrization function): Let $\tau_{\rho_i}(x)$ be the roof function of ρ_i for $i = 1,2$. We define the **reparametrization function** $\psi : \Sigma^+_{\tau_{\rho_1}} \to \mathbb{R}$ as

$$\psi(x,t) := \frac{\tau_{\rho_2}(x)}{\tau_{\rho_1}(x)} f\Big(\frac{t}{\tau_{\rho_1}(x)}\Big)$$

where $f : [0,1] \to \mathbb{R}$ is a smooth function such that $f(0) = f(1) = 0$, $f(t) > 0$ for $0 < t < 1$ and $\int_0^1 f(t)\,\mathrm{d}t = 1$.

We first notice that ψ is well-defined since $\psi(x,\tau(x)) = \psi(\sigma(x),0)$ for all $x \in \Sigma^+$. By Corollary 3.8, we know that ψ is bounded and locally Hölder continuous. Recall that any periodic orbit λ of the suspension flow $\phi_t : \Sigma^+_{\tau_{\rho_1}} \to \Sigma^+_{\tau_{\rho_1}}$ corresponds to a unique hyperbolic element $\gamma_\lambda \in \pi_1 S$. It is not hard to verify that $\int_0^{l_1[\gamma_\lambda]} \psi(\phi_t)\,\mathrm{d}t = l_2[\gamma_\lambda]$, which is the reason why we call ψ a reparametrization function.

We now can state and prove the main result of this subsection: characterizing $\mathrm{I}(\rho_1,\rho_2)$ by the symbolic model.

Proposition 3.10: *Suppose ρ_1,ρ_2 are two type-preserving finite area Fuchsian representations. Then the intersection $\mathrm{I}(\rho_1,\rho_2)$ is well-defined. Moreover, if τ, κ are the geometric potentials for ρ_1, ρ_2, respectively, then*

$$\mathrm{I}(\rho_1,\rho_2) = \frac{\int \kappa\,\mathrm{d}m_{-\tau}}{\int \tau\,\mathrm{d}m_{-\tau}}$$

where $m_{-\tau}$ is the equilibrium state of τ.

Proof. Since the Bowen–Margulis measure m_L, or the Liouville measure, of $g_t : T^1S \to T^1S$ is supported on the recurrent part, we have $m_L = m_L|_{\Omega_0}$. Thus, it is sufficient to focus on the geodesic flow $g_t : T^1\Omega_0 \to T^1\Omega_0$. Notice that, by the construction, we know that $g_t : T^1\Omega_0 \to T^1\Omega_0$ conjugates to the suspension flow $\phi_t : \Sigma_\tau \to \Sigma_\tau$ by the map $\varpi : T^1\Omega_0 \to \Sigma_\tau$. Moreover, it is not hard to verify that given a bounded and continuous function F on Σ_τ, $\overline{F} := F \circ \varpi$ is bounded and continuous on $T^1\Omega_0$.

Let $\{\gamma_n\}$ be any sequence of equidistributed geodesics with respect to m_L and let $\lambda_n = \varpi \circ \gamma_n$ be the corresponding closed orbits of γ_n on Σ_τ. Let us denote δ_{γ_n} (resp. δ_{λ_n}) the 1-dimensional Lebesgue measure supported on γ_n (resp. λ_n). Moreover, by definition, we know that $l(\lambda_n)$ the length of λ_n, is exactly $l_1[\gamma_n]$.

Let ψ be the symbolic reparametrization given in Definition 3.9. Notice that ψ can be defined over the two-sided suspension flow Σ_τ in the same manner. For convenience, we abuse the notation and continue calling it ψ. As discussed above, we know ψ is bounded and continuous on Σ_τ, and thus $\overline{\psi} := \psi \circ \varpi$ is bounded and continuous on $T^1\Omega_0$. We get

$$\frac{l_2[\gamma_n]}{l_1[\gamma_n]} = \int \psi \, \mathrm{d}\Big(\frac{\delta_{\lambda_n}}{l(\lambda_n)}\Big) := \frac{\int_0^{l[\lambda_n]} \psi(\phi_t)\, \mathrm{d}t}{l(\lambda_n)} = \frac{\int_0^{l_1[\gamma_n]} \overline{\psi}(g_t)\, \mathrm{d}t}{l_1[\gamma_n]} = \int \overline{\psi} \, \mathrm{d}\Big(\frac{\delta_{\gamma_n}}{l_1[\gamma_n]}\Big)$$

$$\to \int_{T^1\Omega_0} \overline{\psi} \, \mathrm{d}m_L = \int_{\Sigma_\tau^+} \psi \, \mathrm{d}R(m_{-\tau}) = \frac{\int_{\Sigma^+} \kappa \, \mathrm{d}m_{-\tau}}{\int_{\Sigma^+} \tau \, \mathrm{d}m_{-\tau}}$$

where the convergence is because $\{\gamma_n\}$ is equidistributed with respect to m_L (i.e., $\frac{\delta_{\gamma_n}}{l_1[\gamma_n]} \overset{weak^*}{\to} m_L$), the second last equality comes from the conjugation map ϖ taking the measure of maximal entropy of g_t to the measure of maximal entropy of ϕ_t, and the last equality follows Theorem 2.15. ∎

4. Phase transitions for geodesic flows

Throughout this section, let ρ_1 and ρ_2 be two type-preserving finite volume Fuchsian representations, and we write $X_1 = \Gamma_1 \backslash \mathbb{D}$ and $X_2 = \Gamma_2 \backslash \mathbb{D}$ where $\Gamma_1 = \rho_1(\pi_1(S))$ for $i = 1, 2$. Following the above section, let $(\Sigma^+, \sigma) = (\Sigma_A^+, \sigma_A)$ be the Markov shift associated with X_1 and X_2, and we denote their geometric potentials by τ and κ, respectively.

To derive the analyticity of pressure, we need to locate the place where phase transition happens. As in [Kao18], we have the following observation.

Theorem 4.1 (Phase Transition): *Suppose $a, b \geq 0$, $a+b \neq 0$, and τ, κ are given above. Then we have*

$$P_\sigma(-t(a\tau + b\kappa)) = \begin{cases} \textit{analytic}, & t > \frac{1}{2(a+b)}, \\ \infty, & t < \frac{1}{2(a+b)}. \end{cases}$$

Moreover, there exists a unique $t_{ab} \in (\frac{1}{2(a+b)}, \infty)$ such that

$$P_\sigma(-t_{a,b}(a\tau + b\kappa)) = 0.$$

Proof. By Theorem 2.11, we know it is sufficient to show that

$$P_\sigma(-t(a\tau + b\kappa)) = \begin{cases} \text{finite}, & t > \frac{1}{2(a+b)}, \\ \infty, & t < \frac{1}{2(a+b)}. \end{cases}$$

Recall [MU03, Theorem 2.19]; we know that for any locally Hölder continuous function f, $P_\sigma(f) < \infty$ if and only if

$$Z_1(f) := \sum_{x_0 \in \mathcal{S}_A} e^{\sup\{f(x): x \in [x_0]\}} < \infty.$$

By Proposition 3.6, there exist constants C_1, $C_2 > 0$ such that

$$\begin{aligned} Z_1(-t(a\tau + b\kappa)) = & \sum_{x_0 \in \Sigma_A(\mathrm{I})} e^{\sup\{-t(a\tau+b\kappa): x \in [x_0]\}} + \sum_{x_0 \in \Sigma_A(\mathrm{II})} e^{\sup\{-t(a\tau+b\kappa): x \in [x_0]\}} \\ \leq & C_1 + \sum_{r=N^*+1}^{\infty} \sum_{\substack{x_0 \in \Sigma_A(\mathrm{II}) \\ |x_0| = r}} e^{-2t(a\tau+b\kappa)) \log|x_0| + C} \\ = & C_1 + C_2 \sum_{r=N^*+1}^{\infty} \sum_{\substack{x_0 \in \Sigma_A(\mathrm{II}) \\ |x_0| = r}} e^{-2t(a\tau+b\kappa)) \log|x_0|}. \end{aligned}$$

Similarly, there exist constants C_3, $C_4 > 0$ such that

$$Z_1(-t(a\tau + b\kappa)) \geq C_3 + C_4 \sum_{r=N^*+1}^{\infty} \sum_{\substack{x_0 \in \Sigma_A(\mathrm{II}) \\ |x_0| = r}} e^{-2t(a\tau+b\kappa)) \log|x_0|}.$$

Thus, it is clear that $Z_1(-t(a\tau + b\kappa)) < \infty$ if and only if $t > \frac{1}{2(a+b)}$.

Lastly, fix a, b with $a, b \geq 0, a+b \neq 0$. Then the computation in [MU03, Theorem 2.19] showed that, in our case, $Z_1(-t(a\tau + b\kappa)) \to \infty$ as $t \to \frac{1}{2(a+b)}$ implies

$$P_\sigma(-t(a\tau + b\kappa)) \to \infty \quad \text{as } t \to \frac{1}{2(a+b)}.$$

In particular, taking t close to $\frac{1}{2(a+b)}$, we have $P_\sigma(-t(a\tau+b\kappa))>0$. Moreover, it is obvious that $P_\sigma(-t(a\tau+b\kappa))<0$ when t is big enough. Hence, by the analyticity and the monotonicity of the pressure, we know there exists a unique $t_{a,b}$ such that $P_\sigma(-t_{a,b}(a\tau+b\kappa))=0$. ∎

Corollary 4.2: *The set*

$$\{(a,b): a,b\geq 0, a+b\neq 0, \text{ and } P_\sigma(-a\tau-b\kappa)=0\}$$

is a real analytic curve.

Proof. The proof of [Kao18, Theorem 3.14] applies here. In short, by Theorem 4.1, it makes sense to discuss solutions to $P_\sigma(-a\tau-b\kappa)=0$. To see the solution set as a real analytic curve one only needs to apply the Implicit Function Theorem, because we know that

$$\partial_b P_\sigma(-a\tau-b\kappa)|_{(a_0,b_0)} = -\int \kappa \, \mathrm{d}\nu_{-a_0\tau-b_0\kappa} < -c$$

where $c>0$ is a lower bound for κ and $\nu_{-a_0\tau-b_0\kappa}$ is the Gibbs measure for $-a_0\tau-b_0\kappa$. ∎

5. Manhattan curves, critical exponents, and rigidity

In this section, we will prove Theorem A and Theorem B. The ideas mostly follow [Kao18]. In [Kao18], the author used results of Paulin, Pollicott and Schapira [PPS15] to analyze the geometric Gurevich pressure over the geodesics flow. The general framework in [PPS15] includes finite area hyperbolic surfaces. Nevertheless, for completeness, we will give outlines of the proofs, and the reader can find all the details in [Kao18].

Following the notations in Section 4, let ρ_1,ρ_2 be two type-preserving finite area Fuchsian representations, $X_1=X_{\rho_1}$ and $X_2=X_{\rho_2}$ the corresponding hyperbolic surfaces, and τ, κ the corresponding geometric potentials over the Markov shift $(\Sigma^+,\sigma)=(\Sigma_A^+,\sigma_A)$. Recall that the Poincaré series $Q^{a,b}_{\rho_1,\rho_2}(s)$ of the weighted Manhattan metric $d^{a,b}_{\rho_1,\rho_2}$ is defined by

$$Q^{a,b}_{\rho_1,\rho_2}(s) := \sum_{\gamma\in\pi_1(S)} \exp(-s\cdot d^{a,b}_{\rho_1,\rho_2}(o,\gamma o)),$$

$\delta^{a,b}_{\rho_1,\rho_2}$ is the critical exponent of $Q^{a,b}_{\rho_1,\rho_2}$, and the Manhattan curve $\mathcal{C}(\rho_1,\rho_2)$ of ρ_1 and ρ_2 is the set $\{(a,b)\in\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\backslash(0,0) : \delta^{a,b}_{\rho_1,\rho_2}=1\}$. For brevity, we will drop the subscript ρ_1,ρ_2 in the rest of this section.

The goal of this subsection is to prove the following theorem:

THEOREM 5.1 ([Kao18, Section 4]): *Suppose $a, b \geq 0$ and $a+b \neq 0$. Then*

$$P_\sigma(-\delta^{a,b}_{\rho_1,\rho_2}(a\tau + b\kappa)) = 0.$$

In particular, $(a,b) \in \mathcal{C}(\rho_1,\rho_2)$ if and only if the pair (a,b) satisfies

$$P_\sigma(-a\tau - b\kappa) = 0.$$

Proof. As we mentioned before, the results in [Kao18, Section 4] are applicable here. We give here a brief outline of the proof. We consider the following growth rates and their relations:

- The geometric Gurevich pressure $P^{a,b}_{\mathrm{Geo}}$ given by growth rates of closed orbits on T^1X_1:

  $$P^{a,b}_{\mathrm{Geo}} := \limsup_{s\to\infty} \frac{1}{s} \log Z_W(s)$$

  where

  $$Z_W(s) := \sum_{\substack{\lambda\cap W\neq\phi \\ \lambda\in \mathrm{Per}_1(s)}} e^{-al_1[\lambda]-bl_2[\lambda]};$$

  here $W \subset T^1X_1$ is a relatively compact open set and

  $$\mathrm{Per}_1(s) := \{\lambda : \lambda \text{ is a closed orbit on } T^1X_1 \text{ and } l_1[\lambda] \leq s\}.$$

- The critical exponent $\overline{\delta}^{a,b}$ proposed in [PPS15]: $\overline{\delta}^{a,b}$ is the critical exponent of

  $$Q^{a,b}_{\mathrm{PPS},x,y}(s) := \sum_{\gamma\in\pi_1(S)} e^{-d^{a,b}(x,\gamma y)-sd(x,\rho_1(\gamma)y)}$$

  the **Paulin–Pollicott–Schapira** (**PPS**) Poincaré series.
- Let $\psi(x,t) := \frac{\kappa(x)}{\tau(x)} : \Sigma^+_\tau \to \mathbb{R}$ for $t \in [0,\tau(x)]$. Then [Kao18, Lemma 4.7] showed that $P_\phi(-a-b\psi) = 0 \iff P_\sigma(-a\tau - b\kappa) = 0$.
- [Kao18, Lemma 4.3, 4.4] showed that $P^{a,b}_{\mathrm{Geo}} = \overline{\delta}^{a,b} = P_\phi(-a-b\psi)$.
- [Kao18, Lemma 4.5] pointed out that $\overline{\delta}^{a,b} = 0 \iff \delta^{a,b} = 1$.

In summary, we have

$$\begin{aligned}\delta^{a,b}=1 &\iff \overline{\delta}^{a,b}=0\\ &\iff P^{a,b}_{\text{Geo}}=0\\ &\iff P_{\phi}(-a-b\psi)=0\\ &\iff P_{\sigma}(-a\tau-b\kappa)=0.\end{aligned}$$

Thus, $P_\sigma(-t_{a,b}(a\tau+b\kappa))=0 \iff \delta^{t_{a,b}a,t_{a,b}b}=1$, i.e.,

$$Q^{t_{a,b}a,t_{a,b}b}_{\text{PPS},o,o}(s)=\sum_{\gamma\in\pi_1(S)} e^{-t_{ab}d^{a,b}(o,\gamma o)}$$

has critical exponent 1. Hence,

$$Q^{t_{a,b}a,t_{a,b}b}_{\text{PPS},o,o}(s)=\sum_{\gamma\in\pi_1(S)} e^{-sd^{a,b}(o,\gamma o)}$$

has critical exponent $t_{a,b}$, and thus $\delta^{a,b}=t_{a,b}$. ∎

Remark 5.2: We wish to point out that the reparametrization $\psi:\Sigma^+_\tau\to\Sigma^+_\tau$ given in [Kao18] is not well-defined. One needs to replace the definition of ψ in [Kao18] by Definition 3.9. Since the reparametrization function defined in [Kao18] and in the current paper have the same regularity and periodic orbit information (such as lengths and weights), all arguments in [Kao18] stay valid and unchanged using the ψ defined in Definition 3.9.

By Corollary 4.2 and the above theorem, we have:

Corollary 5.3: *The Manhattan curve $\mathcal{C}(\rho_1,\rho_2)$ is a real analytic curve given, for $a,b\geq 0$ and $a+b\neq 0$, by*

$$\mathcal{C}(\rho_1,\rho_2)=\{(a,b):P_\sigma(-a\tau-b\kappa)=0\}.$$

The following theorem is Bowen's formula which characterizes the topological entropy of the geodesic flow in terms of the pressure and the geometric potential.

Corollary 5.4: *Suppose ρ_1 is a finite volume Fuchsian representation ρ_1. Then*

$$P_\sigma(-1\cdot\tau)=0$$

where 1 is the critical exponent of $\rho_1(\pi_1(S))$.

Proof. It follows from Proposition 3.6 and the fact that when ρ_1 is a finite area Fuchsian representation then the critical exponent of $\rho_1(\pi_1(S))$ is 1 (cf. [OP04]). ∎

Notice that by Bowen's formula and the Implicit Function Theorem, we can prove that the pressure varies analytically when τ varies analytically with

$$P_\sigma(-\tau) = 0.$$

Now we are ready to prove Theorem A.

THEOREM 5.5 (Theorem A): *$\mathcal{C}(\rho_1, \rho_2)$ is a convex real analytic curve. Moreover, $\mathcal{C}(\rho_1, \rho_2)$ is strictly convex unless ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$*. In such cases $\mathcal{C}(\rho_1, \rho_2)$ is a straight line.*

Proof. The analyticity of $\mathcal{C}$ is proved in Corollary 5.3. To show the remaining parts, we first notice that by Hölder's inequality the Manhattan curve $\mathcal{C}$ is always convex, and because $\mathcal{C}$ is real analytic we know $\mathcal{C}$ is either a straight line or strictly convex. It is clear that if ρ_1 and ρ_2 are conjugate then $\mathcal{C}$ is a straight line. We claim that if $\mathcal{C}$ is a straight line then ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}(2, \mathbb{R})$.

To see this, suppose $\mathcal{C}$ is a straight line. Then the slope of this line is -1 because $(1, 0)$,$(0, 1) \in \mathcal{C}$. In other words, we have

$$-1 = -\frac{\int \tau \, \mathrm{d}m_{-\tau}}{\int \kappa \, \mathrm{d}m_{-\tau}} = -\frac{\int \tau \, \mathrm{d}m_{-\kappa}}{\int \kappa \, \mathrm{d}m_{-\kappa}} \tag{5.1}$$

where $m_{-\tau}$, $m_{-\kappa}$ are the equilibrium states for $-\tau$ and $-\kappa$, respectively.

It is sufficient to show that τ and κ are cohomologous, because $\tau \sim \kappa$ implies that X_1 and X_2 have the same marked length spectrum, and which implies that ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}(2, \mathbb{R})$ (cf. Theorem 2.3).

To see that τ and κ are cohomologous, it is enough to show that $m_{-\tau} = m_{-\kappa}$. Indeed, by Theorem 2.10, we know $m_{-\tau} = m_{-\kappa}$ implies $\tau - \kappa \sim c_0$ where c_0 is a constant, and $c_0 = 0$ follows from

$$\int \kappa \, \mathrm{d}m_{-\tau} = \int \tau \, \mathrm{d}m_{-\tau}.$$

Notice that $m_{-\tau}$, $m_{-\kappa}$ are the equilibrium states of $-\tau$, $-\kappa$, respectively; we have

$$h_\sigma(m_{-\tau}) - \int \tau \, \mathrm{d}m_{-\tau} = P_\sigma(-\tau) = 0 = P_\sigma(-\kappa) = h_\sigma(m_{-\kappa}) - \int \kappa \, \mathrm{d}m_{-\kappa}.$$

Moreover, by equation (5.1), we know $\int \kappa \, \mathrm{d}m_{-\tau} = \int \tau \, \mathrm{d}m_{-\tau}$. Thus, we get

$$h_\sigma(m_{-\tau}) + \int (-\kappa) \, \mathrm{d}m_{-\tau} = 0 = P_\sigma(-\kappa).$$

In other words, $m_{-\tau}$ is an equilibrium state for $-\kappa$, and by the uniqueness of equilibrium states we get $m_{-\kappa} = m_{-\tau}$. ∎

Using the strictly convexity of the Manhattan curve, we have the following rigidity results.

THEOREM 5.6 (Bishop–Steiger Rigidity; Theorem B): *Suppose ρ_1, ρ_2 are two type-preserving finite volume Fuchsian representations. Then*

$$h_{\mathrm{BS}}(\rho_1, \rho_2) \leq \frac{1}{2}.$$

Moreover, equality holds if and only if ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}(2, \mathbb{R})$.

Proof. We first notice that it is a standard and well-known procedure (cf. [Kao18, Theorem 4.8]) to show

$$\delta^{1,1} = h_{\mathrm{BS}} = \lim_{T \to \infty} \frac{1}{T} \ln(\#\{[\gamma] \in [\pi_1(S)] : d(o, \rho_1(\gamma)o + d(o, \rho_2(\gamma)o \leq T\}).$$

Moreover, since $(\frac{1}{2}, \frac{1}{2})$ is the middle point of $(0,1), (1,0) \in \mathcal{C}(\rho_1, \rho_2)$, by Theorem 5.5, we know $(\frac{1}{2}, \frac{1}{2})$ is above $\delta^{1,1} \cdot (1,1) \in \mathcal{C}(\rho_1, \rho_2)$ and $\delta^{1,1} = \frac{1}{2}$ if and only if $\mathcal{C}$ is a straight line. ∎

THEOREM 5.7 (Thurston's Rigidity; Theorem B): *Suppose ρ_1, ρ_2 are two type-preserving finite volume Fuchsian representations. Then*

$$\mathrm{I}(\rho_1, \rho_2) \geq 1$$

and equals 1 *if and only if ρ_1 and ρ_2 are conjugate in* $\mathrm{PSL}(2, \mathbb{R})$.

Proof. By Proposition 3.10, we know that

$$\mathrm{I}(\rho_1, \rho_2) = \frac{\int \kappa \, \mathrm{d}m_{-\tau}}{\int \tau \, \mathrm{d}m_{-\tau}}.$$

Recall that $m_{-\kappa}$ is the equilibrium state of $-\kappa$ and $m_{-\tau} \in \mathcal{M}_\sigma$; we have

$$0 = P_\sigma(-\kappa) = h_\sigma(m_{-\kappa}) - \int \kappa \, \mathrm{d}m_{-\kappa} \geq h_\sigma(m_{-\tau}) - \int \kappa \, \mathrm{d}m_{-\tau}.$$

Notice that $m_{-\tau}$ is the equilibrium state of $-\tau$, i.e.,

$$0 = P_\sigma(-\tau) = h_\sigma(m_{-\tau}) - \int \tau \, \mathrm{d}m_{-\tau};$$

we have

$$\int \tau \, \mathrm{d}m_{-\tau} = h_\sigma(m_{-\tau}) \leq \int \kappa \, \mathrm{d}m_{-\tau}.$$

The rigidity part was proved in Theorem 5.5. More precisely, we proved in Theorem 5.5 that if $1 = \frac{\int \kappa \, \mathrm{d}m_{-\tau}}{\int \tau \, \mathrm{d}m_{-\tau}}$ then $\rho_1 \sim \rho_2$ in $\mathrm{PSL}(2, \mathbb{R})$. ∎

6. The pressure metric

6.1. The pressure metric and Thurston's Riemannian metric. The aim of this subsection is to construct a Riemannian metric for the Teichmüller space of surfaces with punctures. Using the symbolic model of geodesics flows discussed in Section 3, we can relate the Teichmüller space with the space of geometric potentials.

Recall that $S = S_{g,n}$ is an orientable surface of genus g and n punctures and with negative Euler characteristic. The Teichmüller space $\mathcal{T}(S)$ is the space of conjugacy classes of finite area type-preserving Fuchsian representations. By Section 3, we know that for every $\rho \in \mathcal{T}(S)$, the geodesics flow on a smaller section $\Omega_0 \subset T^1 X_\rho$ conjugates the suspension flow over a Markov shift $(\Sigma^+, \sigma) = (\Sigma_A^+, \sigma_A)$ with a unique (up to cohomology) locally Hölder continuous roof function τ. We point out again that the Markov shift (Σ^+, σ) is constructed through the shape of the fundamental domain. Since type-preserving Fuchsian representations have the same shape as the fundamental domain, we know that the suspension flow models for all $\rho \in \mathcal{T}(S)$ have the same base space (Σ^+, σ) yet with different roof functions.

Let $\mathbf{P}$ be the set of pressure zero locally Hölder continuous functions on Σ^+, that is,

$$\mathbf{P} := \{\tau \in C(\Sigma^+) : \tau \text{ is locally Hölder}, \ P_\sigma(-\tau) = 0\}.$$

In the following, we will discuss the relations between $\mathcal{T}(S)$ and $\mathbf{P}$. Notice that since $\mathcal{T}(S)$ is composed by representations in $\mathrm{PSL}(2, \mathbb{R})$, it inherits a natural analytic structure from $\mathrm{PSL}(2,\mathbb{R})$ (see [Ham03] for more details). The following proposition indicates that there exists an analytic thermodynamic mapping $\Phi : \mathrm{Hom}^{\mathrm{F}}_{\mathrm{tp}}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R})) \to \mathbf{P}$.

Proposition 6.1 (Thermodynamic Mapping): *Let $0 < \varepsilon \ll 1$ and $\{\rho_t\}_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{T}(S)$ be an analytic one-parameter family in $\mathcal{T}(S)$. Then $\Phi(\{\rho_t\}) = \{\tau_t\} \subset \mathbf{P}$ is an analytic one-parameter family in $\mathbf{P}$.*

Proof. We first notice that if $\{\rho_t\} \subset \mathcal{T}(S)$ is analytic, then the boundary map (derived in Theorem 2.1) $\mathrm{b}_t : \partial_\infty\mathbb{H} = \Lambda(\rho_0(\pi_1 S)) \to \Lambda(\rho_t(\pi_1 S)) = \partial_\infty\mathbb{H}$ is real analytic (see [McM08, Section 2] or [BCS18, Proposition 4.1]). For completeness, we summarize the proof of this fact. The idea of [McM08, Section 2], as well as [BCS18, Proposition 4.1], is a complex analytic approach, namely, using holomorphic motions and the λ-lemma.

Let us denote $QF(S)$ the space conjugacy classes of quasi-Fuchsian (i.e., the limit set is a Jordan curve) representations of $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$. Recall that $QF(S)$ is an open neighborhood of $\mathcal{T}(S)$ in the $\mathrm{PSL}(2,\mathbb{C})$-character variety of $\pi_1(S)$. Let ρ_t vary in $QF(S)$. Then there exist embeddings $\overline{\mathrm{b}}_t : \partial_\infty\mathbb{H} \to \Lambda(\rho_t(\pi_1 S)) \subset \hat{\mathbb{C}}$. Notice that $\rho_0 \in \mathcal{T}(S)$ is fixed. It is clear that if $\xi \in \Lambda(\rho_0(\pi_1 S))$ is fixed by a nontrivial element $\rho_0(\gamma)$, then $b_t(\xi)$ varies holomorphically. Thus by Slodkowski's generalized λ-lemma (cf. [Slo91]), we know that $\overline{\mathrm{b}}_t$ varies complex analytically when ρ_t varies in $QF(S)$; hence, $\mathrm{b}_t (= \overline{\mathrm{b}}_t)$ varies real analytically when ρ_t varies in $\mathcal{T}(S)$.

To see $\{\tau_t\}$ is real analytic, by definition

$$\tau_t(x) = B_{\xi_t(x)}(o, \rho_t(g_{x_0})o)$$

where $x = x_0 \cdots$ and $\xi_t = \mathrm{b}_t \circ \xi_0 : \Sigma^+ \to \Lambda(\rho_t(\pi_1 S))$. Recall that in the disk model, we know that

$$B_\xi(x,y) = \ln\Big(\frac{1-|y|^2}{|\xi-y|^2}\frac{|\xi-x|^2}{1-|x|^2}\Big).$$

Thus we have, without loss of generality, taking o to be the origin,

$$\tau_t(x) = B_{\mathrm{b}_t\circ\xi_0}(o, \rho_t(g_{x_0})o) = \ln\frac{1-|\rho_t(g_{x_0})o|^2}{|\mathrm{b}_t\circ\xi_0(x) - \rho_t(g_{x_0})o|^2}.$$

Since both ρ_t and b_t vary real analytically, from the above expression we know that τ_t also varies real analytically. ∎

By Corollary 3.8 we know that τ_{ρ_1} is locally Hölder continuous and

$$|\Phi(\rho_1) - \Phi(\rho_0)| = |\tau_{\rho_1} - \tau_{\rho_0}|$$

is bounded for all $\rho_0, \rho_1 \in \mathcal{T}(S)$. Thus, consider an analytic path $\rho_t \subset \mathcal{T}(S)$, and we write out the analytic path $\tau_t = \Psi(\rho_t)$ in terms of a Taylor expansion, $\tau_t = \tau_0 + t \cdot \dot{\tau}_0 + \cdots$. We know that the perturbation $\dot{\tau}_0$ is a bounded locally Hölder continuous function. Therefore, it is sufficient to consider $T_{\tau_0}\mathbf{P}$, the corresponding tangent space of $T_{\rho_0}\mathcal{T}(S)$, as

$$\begin{aligned} T_{\tau_0}\mathbf{P} :=& \left\{ f \in C(\Sigma^+) : \int_{\Sigma^+} f \, \mathrm{d}m_{-\tau_0} = 0, f \text{ is locally Hölder and bounded} \right\} \\ &\subset \mathrm{Ker} D_{-\tau_0} P_\sigma . \end{aligned}$$

Moreover, we are interested in the pressure norm $||\cdot||_P$ on $\mathbf{P}$ given by

$$||f||_{\mathrm{P}} := \frac{\mathrm{Var}(f, m_{-\tau_0})}{\int \tau_0 \, \mathrm{d}m_{-\tau_0}}.$$

Notice that this norm degenerates precisely when $f \sim 0$. In the theorem below, we prove that one can define the **pressure metric** $||\cdot||$ on $\mathcal{T}(S_{g,n})$ through $||\cdot||_P$:

Theorem 6.2 (Theorem C): *Suppose* $0 < \varepsilon \ll 1$ *and* $\rho_t \in \mathcal{T}(S_{g,n})$ *is an analytic path for* $t \in (-\varepsilon, \varepsilon)$. *Then* $\mathrm{I}(\rho_0, \rho_t)$ *is real analytic and*

$$||\dot{\rho}_0||^2 := ||d\Psi(\dot{\rho}_0)||^2_{\mathrm{P}} = \frac{\mathrm{d}^2 \mathrm{I}(\rho_0, \rho_t)}{\mathrm{d}t^2}\Big|_{t=0}$$

defines a Riemannian metric on $\mathcal{T}(S_{g,n})$.

Proof. Following Proposition 3.10 and Proposition 6.1, we know that $\mathrm{I}(\rho_0, \rho_t)$ is real analytic. Thus, it is sufficient to show that $\frac{\mathrm{d}^2 \mathrm{I}(\rho_0,\rho_t)}{\mathrm{d}t^2}|_{t=0} = ||d\Psi(\dot{\rho}_0)||^2_{\mathrm{P}}$ and $||d\Psi(\dot{\rho}_0)||^2_{\mathrm{P}} > 0$ when $\dot{\rho}_0 \neq 0$.

By Proposition 3.10 and Proposition 6.1, we know that

$$\frac{\mathrm{d}^2 \mathrm{I}(\rho_0, \rho_t)}{\mathrm{d}t^2}\Big|_{t=0} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\int \tau_t \, \mathrm{d}m_{-\tau_0}}{\int \tau_0 \, \mathrm{d}m_{-\tau_0}}\right)\Big|_{t=0} = \frac{\int \ddot{\tau}_0 \, \mathrm{d}m_{-\tau_0}}{\int \tau_0 \, \mathrm{d}m_{-\tau_0}}$$

where $\tau_t = \Psi(\rho_t)$. Moreover, by Corollary 2.13, we know that

$$\begin{aligned} 0 =& \frac{\mathrm{d}^2 P_\sigma(-\tau_t)}{\mathrm{d}t^2}\Big|_{t=0} \\ =&(D_{-\tau_0} P_\sigma)(-\ddot{\tau}_0) + (D^2_{-\tau_0} P_\sigma)(-\dot{\tau}_0) \\ =& - \int \ddot{\tau}_0 \, \mathrm{d}m_{-\tau_0} + \mathrm{Var}(-\dot{\tau}_0, m_{-\tau_0}) \end{aligned}$$

and $\mathrm{Var}(-\dot{\tau}_0, m_{-\tau_0}) = 0$ if and only if $\dot{\tau}_0 \sim 0$.

To see the non-degeneracy, suppose $\dot{\tau}_0 \sim 0$ and let h be any hyperbolic element. Then $l(\rho_t[h]) = S_m \tau_t(x)$ for some $x \in \mathrm{Fix}^m$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} l(\rho_t[h]) = S_m \dot{\tau}_0(x) = 0.$$

Moreover, since $\mathcal{T}(S)$ can be parametrized by finitely many (simple) closed geodesics (cf., for example, [Ham03]), $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} l(\rho_t[h]) = 0$ for all h is hyperbolic implies $\dot{\rho}_0 = 0$. Hence, we have $||\dot{\rho}_0||^2 := ||d\Psi(\dot{\rho}_0)||^2_{\mathrm{P}} = \frac{\mathrm{d}^2 \mathrm{I}(\rho_0, \rho_t)}{\mathrm{d}t^2}|_{t=0}$ and $||\dot{\rho}_0||^2 = 0$ if and only if $\dot{\rho}_0 = 0$ in $T_{\rho_0}\mathcal{T}(S_{g,n})$. ∎

6.2. The pressure metric and Manhattan curves. In this subsection, we will prove Theorem D, which points out that one can recover Thurston's Riemannian metric through varying the Manhattan curves. Let $\{\rho_t\} \in \mathcal{T}(S_{g,n})$ be an analytic path, and $\mathcal{C}(\rho_0, \rho_t)$ be the Manhattan curve of ρ_0, ρ_t. By Theorem A, we know $\mathcal{C}(\rho_0, \rho_t)$ is a real analytic curve. Thus we can parametrize $\mathcal{C}(\rho_0, \rho_t)$ by writing $\mathcal{C}(\rho_0, \rho_t) = \{(s, \chi_t(s)) : s \in [0,1]\}$ where $\chi_t(s)$ is a real analytic function. See Figure 6.1.

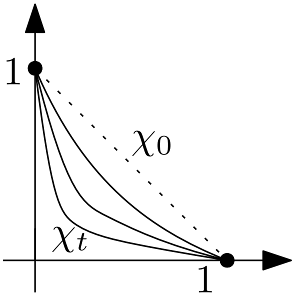

Figure 6.1. Manhattan curves.

THEOREM 6.3 (Theorem D): *Following the above notations, we have for all* $s \in (0,1)$

$$\frac{\mathrm{d}^2\chi_t(s)}{\mathrm{d}t^2}\Big|_{t=0} = s(s-1)\cdot ||\dot{\rho}_0||^2.$$

Proof. For a fixed t, $\mathcal{C}(\rho_0,\rho_t)$ can be identified as

$$\begin{aligned}&\{(a,b) : a,b \geq 0, a+b \neq 0, \text{ and } P_\sigma(-a\tau_0 - b\tau_t) = 0\}\\ &\qquad =\{(s,\chi_t(s)) : P_\sigma(-s\tau_0 - \chi_t(s)\tau_t) = 0, s \in [0,1]\}.\end{aligned}$$

For convenience, let us denote

$$\varphi_t(s) := -s\tau_0 - \chi_t(s)\tau_t.$$

Since when $t=0$, χ_0 is a straight line satisfying $s+\chi_0(s) = 1$, we have $\varphi_0 = -\tau_0$. Thus, we know that $\dot{\varphi}_0 = -\dot{\chi}_0\tau_0 - \chi_0\dot{\tau}_0$ and $\ddot{\varphi}_0 = -\ddot{\chi}_t\tau_0 - 2\dot{\chi}_0\dot{\tau}_0 - \chi_0\ddot{\tau}_0$. By Corollary 2.13, we get

$$\begin{aligned}0 = \frac{\mathrm{d}}{\mathrm{d}t}P_\sigma(\varphi_t)\Big|_{t=0} &= \int \dot{\varphi}_0 \,\mathrm{d}m_{\varphi_0}\\ &= \int -\dot{\chi}_0\tau_0 - \chi_0\dot{\tau}_0 \,\mathrm{d}m_{-\tau_0}\\ &= -\dot{\chi}_0(s)\int \tau_0 \,\mathrm{d}m_{-\tau_0} - \chi_0(s)\int \dot{\tau}_0 \,\mathrm{d}m_{-\tau_0}.\end{aligned}$$

Since $\int \dot{\tau}_0 \,\mathrm{d}m_{-\tau_0} = 0$ (because $P_\sigma(-\tau_t) = 0$) and $\int -\tau_0 \,\mathrm{d}m_{-\tau_0} < 0$, we have $\dot{\chi}_0(s) = 0$, $\forall s \in [0,1]$. Furthermore, by taking the second derivative of pressure (as in the proof of Theorem 6.2) we get

$$\begin{aligned}0 =&\frac{\mathrm{d}^2}{\mathrm{d}t^2}P_\sigma(\varphi_t)\Big|_{t=0}\\ =&\mathrm{Var}(\dot{\varphi}_0, m_{\varphi_0}) + \int \ddot{\varphi}_0 \,\mathrm{d}m_{\varphi_0}\\ =&\mathrm{Var}(\underbrace{-\dot{\chi}_0}_{\overset{\shortparallel}{0}}\tau_0 - \chi_0\dot{\tau}_0, m_{-\tau_0}) - \int (\ddot{\chi}_0\tau_0 + 2\underbrace{\dot{\chi}_0}_{\overset{\shortparallel}{0}}\dot{\tau}_0 + \chi_0\ddot{\tau}_0)\,\mathrm{d}m_{-\tau_0}\\ =&(\chi_0(s))^2\cdot \mathrm{Var}(\dot{\tau}_0, m_{-\tau_0}) - \ddot{\chi}_0\int \tau_0 \,\mathrm{d}m_{-\tau_0} - \chi_0\int \ddot{\tau}_0 \,\mathrm{d}m_{-\tau_0}.\end{aligned}$$

Notice that $P_\sigma(-\tau_t) = 0$; similarly we have

$$0 = \frac{\mathrm{d}^2 P_\sigma(-\tau_t)}{\mathrm{d}t^2}\Big|_{t=0} = -\int \ddot{\tau}_0 \,\mathrm{d}m_{-\tau_0} + \mathrm{Var}(-\dot{\tau}_0, m_{-\tau_0}).$$

Therefore, we have

$$\begin{aligned}\ddot{\chi}_0(s) &= (\chi_0(s)^2 - \chi_0(s))\frac{\mathrm{Var}(\dot{\tau}_0, m_{-\tau_0})}{\int \tau_0 \,\mathrm{d}m_{-\tau_0}} \\ &= ((1-s)^2 - (1-s)))\frac{\mathrm{Var}(\dot{\tau}_0, m_{-\tau_0})}{\int \tau_0 \,\mathrm{d}m_{-\tau_0}} = (s^2 - s)||\dot{\rho}_0||^2. \qquad \blacksquare\end{aligned}$$

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