DIFFERENTIALS ON THE ARC SPACE

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Abstract

We provide a description of the sheaves of Kähler differentials of the arc space and jet schemes of an arbitrary scheme where these sheaves are computed directly from the sheaf of differentials of the given scheme. Several applications on the structure of arc spaces are presented.

1. Introduction

The works of Greenberg [19], Nash [37], Kolchin[30], and Denef and Loeser [11] have set the basis for our understanding of the structure of arc spaces and their connections to singularities and birational geometry. In these studies, most of the focus is on the reduced structure of arc spaces and the underlying topological spaces, and little is known about their scheme structure. Notable studies in this direction are those of Reguera [38], [39], which were recently continued in [34], [40].

The present article can be viewed as a continuation of these studies. In the first part we describe the sheaves of Kähler differentials of the arc space and of the jet schemes. The second part of the paper is devoted to applications of the resulting formulas. This approach leads to new results as well as simpler and more direct proofs of some of the theorems in the literature, and it also provides a new way of understanding some of the fundamental properties of the theory.

In the first part, we work over an arbitrary base scheme S. For simplicity, in the Introduction we restrict to the case of a scheme X over a field k; the reader will not loose too much of the spirit of the paper by even assuming that X is a variety. The *arc space* of X over S, denoted X_∞, parameterizes formal arcs on X, and it comes equipped with a universal family known as the *universal arc*:

$$U_\infty \xrightarrow{\ \gamma_\infty\ } X$$
$$\rho_\infty \downarrow$$
$$X_\infty$$

More concretely, if $X \subset \mathbb{A}^n$ is an affine scheme defined by polynomial equations $f_j = 0$, then an arc on X is a vector of power series $(x_1(t), \ldots, x_n(t))$ with coefficients in a field satisfying identically the equations $f_j = 0$. The coefficients of the power series define coordinates on X_∞. The universal arc is a vector of power series $(x_1(t), \ldots, x_n(t))$ whose coefficients are the coordinate functions on X_∞.

As a word of warning, it should be noted that X_∞ and U_∞ are typically not Noetherian even when X is Noetherian. On the other hand, they are affine over X and this gives us concrete tools to work with. We note that all the theorems in the following are local in nature, and thus one can restrict without loss of generality to the case where X is affine.

On U_∞, we construct a sheaf $\mathscr{P}_\infty$ whose $\mathscr{O}_{X_\infty}$-dual is $\mathscr{O}_{U_\infty}$. The sheaf $\mathscr{P}_\infty$ plays the role of "kernel" (as in a sort of Fourier–Mukai transform) in the next formula, which relates the sheaf of differentials of X_∞ directly to the sheaf of differentials of X.

THEOREM A (Kähler differentials on arc spaces)
There is a natural isomorphism

$$\Omega_{X_\infty} \simeq \rho_{\infty *}\big(\gamma_\infty^*(\Omega_X) \otimes \mathscr{P}_\infty\big).$$

We have a similar result for jet schemes. If X_n denotes the nth *jet scheme* of X, then we have a *universal jet* $X_n \xleftarrow{\ \rho_n\ } U_n \xrightarrow{\ \gamma_n\ } X$, and we construct a sheaf $\mathscr{P}_n$ on U_n whose $\mathscr{O}_{X_n}$-dual is $\mathscr{O}_{U_n}$. In this case, it turns out that $\mathscr{P}_n$ is the $\mathscr{O}_{X_n}$-dual of $\mathscr{O}_{U_n}$ and is trivial as an $\mathscr{O}_{U_n}$-module, and the description of the sheaf of differentials gets simplified.

THEOREM B (Kähler differentials on jet schemes)
There are natural isomorphisms

$$\Omega_{X_n} \simeq \rho_{n *}\big(\gamma_n^*(\Omega_X) \otimes \mathscr{P}_n\big) \simeq \rho_{n *}\big(\gamma_n^*(\Omega_X)\big).$$

These theorems provide an efficient way of computing the Jacobian matrices of the jet schemes and the arc space. We illustrate this in some details at the end of Section 5.

We apply Theorems A and B to study the structure of arc spaces. Our starting point is the description of the fibers of the sheaf of differentials of the jet schemes X_n at liftable jets. Suppose that X is a scheme of finite type over k, and let $\alpha \in X_\infty$ be an arc. For $n \gg 1$, let $\alpha_n \in X_n$ be the truncation of α and L_n the residue field of α_n. Using the above theorems, we determine an isomorphism

$$\Omega_{X_n} \otimes L_n \simeq \left(L_n[t]/(t^{n+1})\right)^d \oplus \bigoplus_{i \geq d} L_n[t]/(t^{e_i}).$$

Here the number d and the sequence $\{e_i\}$ are certain Fitting-theoretic invariants of the pullback of Ω_X by α (see Section 6 for the precise definition of these invariants and Theorem 7.2 for the precise statement). If X is a reduced and equidimensional scheme of finite type over a field k, and if α is not fully contained in the singular locus of X, then we have $d = \dim X$ and $\sum_{i \geq d} e_i = \mathrm{ord}_\alpha(\mathrm{Jac}_X)$, where Jac_X is the Jacobian ideal of X, and the above isomorphism recovers in this case one of the main results of [5].

These results have several applications. For simplicity, for the remainder of this Introduction we will assume that X is a variety defined over a perfect field k, although more general results are obtained below.

A natural way of studying the structure of the arc space of X is to analyze its (not necessarily closed) points. Given a point $\alpha \in X_\infty$, we are interested in two invariants: the *embedding dimension*

$$\mathrm{emb.dim}(\mathcal{O}_{X_\infty, \alpha})$$

of the local ring at α, and the *jet codimension* of α in X_∞, which is defined by

$$\mathrm{jet.codim}(\alpha, X_\infty) := \lim_{n \to \infty} \left((n+1)\dim(X) - \dim(\overline{\{\alpha_n\}})\right).$$

These and related invariants have been studied in the literature (e.g., see [8], [9], [14], [34], [40]). Both numbers provide measures of the "size" of the point. Note, however, that while the embedding dimension is computed on the arc space (with its scheme structure), the jet codimension is computed from the truncations of the arc and only depends on the reduced structure of the jet schemes. Our first application is that these two invariants measure the same quantity, as follows.

THEOREM C (Embedding dimension as jet codimension)
Given a variety X over a perfect field, we have

$$\mathrm{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) = \mathrm{jet.codim}(\alpha, X_\infty)$$

for every $\alpha \in X_\infty$.

One of the most important results on arc spaces is the *birational transformation rule* due to Denef and Loeser (see [11] and also [31]). This result implies the change-of-variable formula in motivic integration (see [2], [11], [31], [32]), and it has been applied to study invariants of singularities in birational geometry (see, e.g., [5], [13], [15], [17], [25], [27], [28], [35], [36], [44]). Using our description of the sheaf of differentials, we obtain the following variant.

THEOREM D (Birational transformation rule)
Given a proper birational map $f : Y \to X$ between two varieties over a perfect field, we have

$$\mathrm{emb.dim}(\mathcal{O}_{Y_\infty, \beta}) \leq \mathrm{emb.dim}(\mathcal{O}_{X_\infty, f_\infty(\beta)}) \leq \mathrm{emb.dim}(\mathcal{O}_{Y_\infty, \beta}) + \mathrm{ord}_\beta(\mathrm{Jac}_f)$$

for every $\beta \in Y_\infty$, where Jac_f is the Jacobian of f. Moreover, if Y is smooth at $\beta(0)$, then

$$\mathrm{emb.dim}(\mathcal{O}_{X_\infty, f_\infty(\beta)}) = \mathrm{emb.dim}(\mathcal{O}_{Y_\infty, \beta}) + \mathrm{ord}_\beta(\mathrm{Jac}_f).$$

The connection between this theorem and Denef and Loeser's birational transformation rule becomes evident once we rewrite the second formula of Theorem D using Theorem C, which gives the formula

$$\mathrm{jet.codim}\big(f_\infty(\beta), X_\infty\big) = \mathrm{jet.codim}(\beta, Y_\infty) + \mathrm{ord}_\beta(\mathrm{Jac}_f)$$

for any resolution of singularities $f : Y \to X$. Although it does not retain all the information provided in [11] that is necessary for the change-of-variable formula in motivic integration, this formula suffices for all known applications to the study of singularities in birational geometry.

Our next application regards the *stable points* of the arc space. These are the generic points of the irreducible constructible subsets of X_∞ that are not contained in $(\mathrm{Sing}\,X)_\infty$ (see Section 10). Stable points and their local rings have been extensively studied in [34], [38]–[40]. The following theorem can be viewed as providing a characterization of stable points.

THEOREM E (Characterization of finite embedding dimension)
Given a variety X over a perfect field and a point $\alpha \in X_\infty$, we have

$$\mathrm{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) < \infty$$

if and only if α is a stable point.

It follows from general properties of local rings that the embedding dimension at a point $\alpha \in X_\infty$ is finite if and only if the completion of the local ring is Noetherian.

The fact that the completion of the local ring at a stable point $\alpha \in X_\infty$ is Noetherian is a theorem of Reguera (see [38], [39]), and we get a new proof of this important result. It is the key ingredient in the proof of the *curve selection lemma*, which plays an essential role in the recent progress on the Nash problem (see, e.g., [6], [18]).

There are examples in positive characteristic of varieties X whose arc space X_∞ has irreducible components that are fully contained in $(\mathrm{Sing}\,X)_\infty$, and Theorem E implies that X_∞ has infinite embedding codimension at the generic points of such components (see Remark 10.9). One should contrast this with the main theorem in both [12] and [20], which can be interpreted as saying that the completion of the local ring of X_∞ at any k-valued point that is not contained in $(\mathrm{Sing}\,X)_\infty$ has finite embedding codimension.

A special class of stable points is given by what we call the *maximal divisorial arcs*. By definition, these are the arcs $\alpha \in X_\infty$ whose associated valuation ord_α is a divisorial valuation and that are maximal (with respect to specialization) among all arcs defining the same divisorial valuation. Equivalently, they are the generic points of the maximal divisorial sets defined in [8], [14], [24]. For example, if E is a prime divisor on a resolution of singularities $f : Y \to X$, and $C \subset Y_\infty$ is the set of arcs on Y with positive order of contact along E, then the closure of $f_\infty(C)$ in X_∞ is the maximal divisorial set associated to the valuation ord_E and its generic point is the maximal divisorial arc corresponding to this valuation. Our final application gives the following result.

THEOREM F (Embedding dimension at maximal divisorial arcs)
Let X be a variety over a perfect field, let $f : Y \to X$ be a proper birational morphism from a normal variety Y, let E be a prime divisor on Y, and let q be a positive integer. If $\alpha \in X_\infty$ is the maximal divisorial arc corresponding to the divisorial valuation $q\,\mathrm{ord}_E$, then

$$\mathrm{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = q\big(\mathrm{ord}_E(\mathrm{Jac}_f) + 1\big).$$

The quantity $\mathrm{ord}_E(\mathrm{Jac}_f)$ is also known as the *Mather discrepancy* of E over X, and it is denoted by $\widehat{k}_E(X)$ (see [8]). In view of Theorem C, Theorem F recovers [8, Theorem 3.8]. The theorem is also closely related to a recent result of Mourtada and Reguera in [34], [40], which states that, with the same assumptions as in Theorem F, if the field has characteristic zero, then

$$\mathrm{emb.dim}(\widehat{\mathcal{O}_{X_\infty,\alpha}}) = \mathrm{emb.dim}(\mathcal{O}_{(X_\infty)_{\mathrm{red}},\alpha}) = q\big(\mathrm{ord}_E(\mathrm{Jac}_f) + 1\big).$$

Here, $\widehat{\mathcal{O}_{X_\infty,\alpha}}$ is the completion of $\mathcal{O}_{X_\infty,\alpha}$ with respect to the I-adic topology where $I \subset \mathcal{O}_{X_\infty,\alpha}$ is the maximal ideal. It is regarded with the inverse limit topology, which in general differs from the $\widehat{I}$-topology (a system of the neighborhood of 0 is given by

the closures of the powers of $\widehat{I}$, which can be strictly larger than the powers themselves). It can be shown using results from [39] that, in characteristic zero, Theorem F also follows from the above theorem of Mourtada and Reguera (see Remark 11.8).

In a different direction, the isomorphisms given in Theorem B can be used to study the relationship between the Nash blowup of a variety and the Nash blowup of its jet schemes (see [7]).

Proofs of the results stated in the Introduction are located as follows: both statements in Theorems A and B are contained in Theorem 5.3, Theorem C is proved in Theorem 10.7, Theorem D combines the statements of Theorems 9.2 and 9.3, Theorem E is proved in Theorem 10.8, and Theorem F follows from Theorem 11.4. We should also recognize it worth mentioning that most proofs in this paper rely only on the definition of arc space and on basic facts of commutative algebra.

2. Conventions

Throughout the following, all rings are assumed to be commutative with identity. Unless otherwise specified, rings are regarded with the discrete topology; however, power series rings of the form $R[[t]]$ are considered as complete topological rings. For topological modules M and N over a topological ring R, we define their completed tensor product, denoted by $M \,\hat{\otimes}_R\, N$, as the completion of the ordinary tensor product $M \otimes_R N$. We will mostly encounter completed tensor products of the form $M \,\hat{\otimes}_R\, A[[t]]$, where R is a ring, A is an R-algebra, and M is an R-module, all with the discrete topology.

We fix a base scheme S and we work on the category of schemes over S. Given an object X in this category, we do not impose any condition on the morphism $X \to S$. However, starting with Section 8 we will assume that $S = \operatorname{Spec} k$, where k is a perfect field, and we will mostly focus on schemes of finite type over k.

We also need to consider formal schemes over S. For our purposes, it will be enough to consider the notion of formal scheme introduced in [21]. In fact, we will consider formal schemes only of the form $X \,\hat{\times}_\mathbb{Z}\, \operatorname{Spf} \mathbb{Z}[[t]]$, where X is an ordinary scheme. Here and in the rest of this article, we use the symbol $\hat{\times}$ to denote the product in the category of formal schemes; this emphasizes the fact that it corresponds to the completed tensor product $\hat{\otimes}$ at the level of topological rings. In particular,

$$\operatorname{Spec} R \,\hat{\times}_\mathbb{Z}\, \operatorname{Spf} \mathbb{Z}[[t]] = \operatorname{Spf}\big(R \,\hat{\otimes}_\mathbb{Z}\, \mathbb{Z}[[t]]\big) = \operatorname{Spf} R[[t]].$$

Unless otherwise stated, the letters m and n will be used to denote elements in the set $\mathbb{N} \cup \{\infty\}$ where $\mathbb{N}$ denotes the set of nonnegative integers.

3. Generalities on arcs and jets

Let X be an arbitrary scheme over a base scheme S. The jet schemes and the arc space of X over S are defined, in this generality, in [43] (see that article for more details and proofs; see also [16], [26]).

For every nonnegative integer n, the *n-jet scheme* $(X/S)_n$ of X over S represents the functor from S-schemes to sets given by

$$Z \mapsto \operatorname{Hom}_S\big(Z \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[t]/(t^{n+1}), X\big),$$

while the *arc space* $(X/S)_\infty$ of X over S represents the functor from S-schemes to sets given by

$$Z \mapsto \operatorname{Hom}_S\big(Z \,\hat{\times}_{\mathbb{Z}} \operatorname{Spf} \mathbb{Z}[[t]], X\big).$$

A point of $(X/S)_n$ is called an *n-jet* of X (over S), and a point of $(X/S)_\infty$ is called an *arc* of X (over S).

Note that an arc $\alpha \in (X/S)_\infty$ can be equivalently thought of as a map $\operatorname{Spf} L[[t]] \to X$ or as a map $\operatorname{Spec} L[[t]] \to X$, where L is the residue field of α and the composition of the map with the structure map $X \to S$ factors through $\operatorname{Spec} L$ (see Lemma 3.1 below). In particular, if $S = \operatorname{Spec} R$, then we can view α as an arc with coefficients in R via the map $R \to L$. The advantage of considering α as a map $\operatorname{Spec} L[[t]] \to X$ is that it allows us to talk about the *generic point of the arc*, by which we mean the image $\alpha(\eta) \in X$ of the generic point η of $\operatorname{Spec} L[[t]]$. We will denote by $\alpha(0)$ the image of the closed point of $\operatorname{Spec} L[[t]]$. If $S = \operatorname{Spec} R$ where R is a ring, then one can replace $\mathbb{Z}$ with R in the above formulas. If $S = \operatorname{Spec} k$, where k is a field, then we will simply denote $(X/S)_n$ by X_n and $(X/S)_\infty$ by X_∞.

For any $n \in \mathbb{N} \cup \{\infty\}$, the scheme $(X/S)_n$ is equipped with a universal family

$$
\begin{array}{ccc}
U_n & \xrightarrow{\ \gamma_n\ } & X \\
{\scriptstyle \rho_n}\big\downarrow & & \big\downarrow \\
(X/S)_n & \longrightarrow & S
\end{array}
\tag{3.1}
$$

For n finite, the family is given by

$$U_n = (X/S)_n \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[t]/(t^{n+1}),$$

and is called the *universal n-jet*. For $n = \infty$, it is given by

$$U_\infty = (X/S)_\infty \,\hat{\times}_{\mathbb{Z}} \operatorname{Spf} \mathbb{Z}[[t]]$$

and is called the *universal arc*. Notice that U_∞ is a formal scheme. The completed fiber of $\rho_\infty \colon U_\infty \to (X/S)_\infty$ over a point $\alpha \in (X/S)_\infty(L)$ is $\operatorname{Spf} L[[t]]$, and the

restriction of γ_∞ to it agrees with the induced map $\alpha\colon \operatorname{Spf} L[[t]] \to X$. While the definition of U_∞ may be hard to grasp at first sight, we will show in a moment that, in the affine case, its structure can be described very explicitly.

We will use the following notation for the natural truncation maps:

$$\pi_n\colon (X/S)_\infty \to (X/S)_n, \qquad \pi_{m,n}\colon (X/S)_m \to (X/S)_n, \qquad \psi_n\colon (X/S)_n \to X.$$

Note that γ_n is different from the composition map $\psi_n \circ \rho_n$. Furthermore, we observe that there is no natural map between U_m and U_n when $m > n$. The natural map is

$$\mu_{m,n}\colon U_n \times_{(X/S)_n} (X/S)_m \longrightarrow U_m,$$

where the fiber product is taken with respect to the maps ρ_n and $\pi_{m,n}$.

It will be useful to have notation in place for the affine case. The following basic properties are proved in [43].

LEMMA 3.1 ([43, Corollary 1.8, Theorem 4.5])
Assume that $S = \operatorname{Spec} R$ for a ring R and that $X = \operatorname{Spec} A$ for an R-algebra A. Then $(X/S)_n$ is affine for all $n \in \mathbb{N} \cup \{\infty\}$. If we write $(X/S)_n = \operatorname{Spec} A_n$, then for every R-algebra C, we have

$$(X/S)_n(C) = \operatorname{Hom}_{R\text{-alg}}(A_n, C) = \operatorname{Hom}_{R\text{-alg}}\big(A, C[t]/(t^{n+1})\big),$$

if n is finite, and

$$(X/S)_\infty(C) = \operatorname{Hom}_{R\text{-alg}}(A_\infty, C) = \operatorname{Hom}_{R\text{-alg}}\big(A, C[[t]]\big).$$

Moreover, A_n is characterized by the above property.

More explicitly, jet schemes and arc spaces can be defined using Hasse–Schmidt derivations. With the notation of Lemma 3.1, the A-algebra A_n can be constructed as the algebra of Hasse–Schmidt differentials $\operatorname{HS}^n_{A/R}$ (see [43, Definition 1.3 and Theorem 4.5]). It comes equipped with the universal Hasse–Schmidt derivation, which is a sequence $(D_0, D_1, \ldots, D_n)$ (by which we mean $(D_0, D_1, \ldots)$ if $n = \infty$), where $D_0\colon A \to \operatorname{HS}^n_{A/R}$ is the natural inclusion and $D_i\colon A \to \operatorname{HS}^n_{A/R}$, for $i \geq 1$, are group homomorphisms such that $D_i(r) = 0$ for $r \in R$ and

$$D_i(xy) = \sum_{j+k=i} D_j(x) D_k(y)$$

for all $x, y \in A$.

Remark 3.2
The definition of Hasse–Schmidt derivations is tailored to work in arbitrary characteristic. In characteristic zero, Hasse–Schmidt derivations can be computed from usual

derivations (see [43, Section 1, Example]) but this is no longer true if the characteristic is positive.

Defining $B_n := A_n[t]/(t^{n+1})$ when n is finite and $B_\infty := A_\infty[[t]]$ when $n = \infty$, we have

$$(X/S)_n = \operatorname{Spec} A_n \qquad \text{and} \qquad U_n = \operatorname{Spf} B_n.$$

The universal jet (or arc) is given by

$$
\begin{array}{ccc}
B_n & \xleftarrow{\gamma_n^\sharp} & A \\
{\scriptstyle \rho_n^\sharp}\uparrow & & \uparrow \\
A_n & \longleftarrow & R
\end{array}
\tag{3.2}
$$

where the map $\rho_n^\sharp$ is the natural inclusion $A_n \subset B_n$, and the map $\gamma_n^\sharp$ is defined by

$$\gamma_n^\sharp(f) = \sum_{p=0}^{n} D_p(f) t^p,$$

where $(D_p)_{p=0}^n$ is the universal Hasse–Schmidt derivation.

We will consider in B_n the A-module structure given by $\gamma_n^\sharp$ and the A_n-module structure given by $\rho_n^\sharp$. Notice that B_n has a second A-module structure (induced from the inclusion $A \subset A_n \subset B_n$), but we will have no use for it.

For $m > n$, the map $\mu_{m,n} \colon U_n \times_{(X/S)_n} (X/S)_m \to U_m$ is defined by the natural projection

$$\mu_{m,n}^\sharp \colon B_m = A_m[t]/(t^{m+1}) \longrightarrow B_n \otimes_{A_n} A_m = A_m[t]/(t^{n+1})$$

when m is finite and by

$$\mu_{\infty,n}^\sharp \colon B_\infty = A_\infty[[t]] \longrightarrow B_n \otimes_{A_n} A_\infty = A_\infty[t]/(t^{n+1})$$

when $m = \infty$.

Remark 3.3

If X is a quasicompact and quasiseparated scheme over a field k, then it follows from the results in [1] that the functor of points of the arc space X_∞ can also be described as

$$X_\infty(C) = \operatorname{Hom}_k\big(\operatorname{Spec} C[[t]], X\big)$$

for any k-algebra C. Notice that this description avoids the use of formal schemes, and, in particular, it gives a universal arc which is defined as an ordinary scheme. The category of quasicompact and quasiseparated schemes is probably large enough to contain all arc spaces of geometric interest, but the theory developed in [1] in very delicate, and we preferred to avoid relying on it. Our results are local in nature, so it is enough for us to have an analogue of the above formula in the affine case. This is precisely the content of Lemma 3.1, which is an elementary fact from commutative algebra. Our results do not require the quasicompact and quasiseparated conditions.

4. The sheaves $\mathcal{P}_n$

The goal of this section is to define the sheaves $\mathcal{P}_n$ appearing in Theorems A and B. We start by looking at the affine case. We continue with the notation introduced in Section 3, so that, given an R-algebra A, we have $A_n = \mathrm{HS}^n_{A/R}$, $B_n = A_n[t]/(t^{n+1})$ when n is finite, and $B_\infty = A_\infty[[t]]$.

Definition 4.1
For any $n \in \mathbb{N} \cup \{\infty\}$, we define P_n to be the B_n-module given by

$$P_n := t^{-n} A_n[t] / t A_n[t]$$

when n is finite and

$$P_\infty := A_\infty((t)) / t A_\infty[[t]]$$

when $n = \infty$.

As A_n-modules, we have $B_n = \prod_{i=0}^n A_n t^i$ and $P_n = \bigoplus_{j=0}^n A_n t^{-j}$. It is convenient to view an element $b \in B_n$ as a power series $b = \sum_{i=0}^n a_i t^i$ (a polynomial if n is finite), and an element $p \in P_n$ as a polynomial $p = \sum_{j=0}^n a'_{-j} t^{-j}$. With this in mind, we can view the B_n-module structure as follows: the action of an element $b \in B_n$ on an element $p \in P_n$ is simply given by the product $b \cdot p$ of the two series, modulo $t A_n[[t]]$.

Note that the A_n-module $\mathrm{Hom}_{A_n}(P_n, A_n)$ has a natural B_n-module structure given by precomposition. That is, given $b \in B_n$ and $\phi \colon P_n \to A_n$, we define $b \cdot \phi$ to be the homomorphism $P_n \to A_n$ defined by $(b \cdot \phi)(p) := \phi(b \cdot p)$.

LEMMA 4.2
For every $n \in \mathbb{N} \cup \{\infty\}$, there exists a canonical isomorphism $B_n \simeq \mathrm{Hom}_{A_n}(P_n, A_n)$ as B_n-modules.

Proof
Since $B_n = \prod_{i=0}^n A_n t^i$ and $P_n = \bigoplus_{j=0}^n A_n t^{-j}$, there is a canonical isomorphism of

A_n-modules $B_n \simeq \mathrm{Hom}_{A_n}(P_n, A_n)$ given by

$$b = \sum_{i=0}^{n} a_i t^i \mapsto \Big(\phi_b \colon p = \sum_{j=0}^{n} a'_{-j} t^{-j} \mapsto \sum_{i=0}^{n} a_i a'_{-i} \Big),$$

and it is immediate to check that this isomorphism is compatible with the respective B_n-module structures. □

Remark 4.3
Lemma 4.2 generalizes to all A_n-modules in the following way. For every A_n-module M, the space $\mathrm{Hom}_{A_n}(P_n, M)$ has a natural B_n-module structure given by precomposition, and there is a canonical isomorphism

$$M \mathbin{\hat{\otimes}}_{A_n} B_n \simeq \mathrm{Hom}_{A_n}(P_n, M)$$

as B_n-modules. The proof follows the same arguments of the proof of Lemma 4.2 once we observe that $M \mathbin{\hat{\otimes}}_{A_n} B_n = \prod_{i=0}^{n} M t^i$.

Remark 4.4
When n is finite, we can view $\{t^{-j}\}_{j=0}^{n}$ as the dual basis of $\{t^i\}_{i=0}^{n}$, and we have $P_n \simeq \mathrm{Hom}_{A_n}(B_n, A_n)$. Note, however, that P_∞ is not the A_∞-dual of B_∞.

LEMMA 4.5
For $n \in \mathbb{N}$, the morphism that sends t^{-j} to t^{-j+n} gives an isomorphism of B_n-modules between P_n and B_n. By contrast, P_∞ and B_∞ are not isomorphic, not even as A_∞-modules.

Proof
Multiplication by t^n clearly gives an isomorphism of A_n-modules $P_n \simeq B_n$, and we can check that this is compatible with the B_n-module structures. The last assertion is also clear since $P_\infty \simeq A_\infty[t]$ (as A_∞-module), whereas $B_\infty = A_\infty[[t]]$. □

Remark 4.6
For $m > n$, the homomorphism $\mu_{m,n}^{\sharp} \colon B_m \longrightarrow B_n \otimes_{A_n} A_m$ defining the morphism $\mu_{m,n} \colon U_n \times_{(X/S)_n} (X/S)_m \to U_m$ corresponds, via the duality given in Lemma 4.2, to the inclusion

$$P_n \otimes_{A_n} A_m \longrightarrow P_m$$

that sends t^{-j} in $P_n \otimes_{A_n} A_m$ to t^{-j} in P_m. When m is finite, this inclusion corresponds via the natural isomorphisms $P_n \simeq B_n$ and $P_m \simeq B_m$ to the homomorphism $B_n \otimes_{A_n} A_m \to B_m$ given by multiplication by t^{m-n}.

The definition of P_n globalizes as follows. Given an arbitrary morphism of schemes $X \to S$, source and target can be covered by affine charts $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} R \subset S$ so that the morphism is determined by gluing affine morphisms $\operatorname{Spec} A \to \operatorname{Spec} R$. For every n, let U_n be the universal family given in (3.1). Then the sheaves P_n constructed above for the corresponding charts $\operatorname{Spec} B_n \subset U_n$ glue together to give a sheaf $\mathscr{P}_n$ on U_n.

For every $n \in \mathbb{N} \cup \{\infty\}$, there is a natural isomorphism

$$\rho_{n*}(\mathcal{O}_{U_n}) \simeq \operatorname{Hom}_{\mathcal{O}_{(X/S)_n}}\big(\rho_{n*}(\mathscr{P}_n), \mathcal{O}_{(X/S)_n}\big).$$

Moreover, the right-hand side has a natural $\mathcal{O}_{U_n}$-module structure given by precomposing with the $\mathcal{O}_{U_n}$-module action on $\mathscr{P}_n$, and with this structure is isomorphic to $\mathcal{O}_{U_n}$. Furthermore, if n is finite, then we have $\mathscr{P}_n \simeq \mathcal{O}_{U_n}$. All these statements can be checked locally on X, and therefore it suffices to consider the case where $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$, where they reduce to Lemmas 4.2 and 4.5.

The analysis done in the affine case can be carried out in an identical way in this more general setting. In particular, for each $m > n$, we get a natural injective morphism

$$\pi_{m,n}^*\big(\rho_{n*}(\mathscr{P}_n)\big) \longrightarrow \rho_{m*}(\mathscr{P}_m).$$

If m and n are finite, then $\mathscr{P}_m \simeq \mathcal{O}_{U_m}$ and $\mathscr{P}_n \simeq \mathcal{O}_{U_n}$, and the above injection is conjugate to the morphism $\pi_{m,n}^*(\rho_{n*}(\mathcal{O}_{U_n})) \to \rho_{m*}(\mathcal{O}_{U_m})$ given by multiplication by t^{m-n}.

5. Derivations and differentials

In this section, we prove the description of $\Omega_{(X/S)_n}$ stated in Theorems A and B. We continue with the notation introduced in the preceding section, so that, given an R-algebra A, we have $A_n = \operatorname{HS}_{A/R}^n$, $B_n = A_n[t]/(t^{n+1})$ when n is finite, and $B_\infty = A_\infty[[t]]$. As before, we regard B_n an A_n-module via $\rho_n^\sharp$ and as an A-module via $\gamma_n^\sharp$, where these maps are defined in (3.2).

LEMMA 5.1
Let $m, n \in \mathbb{N} \cup \{\infty\}$. Let R be a ring, and let A be an R-algebra. Let M be an A_n-module, and consider $M \hat{\otimes}_{A_n} B_n$ with the A-module structure induced from the A-module structure on B_n (notice that $\hat{\otimes}_{A_n} = \otimes_{A_n}$ when n is finite). Then there is a natural isomorphism

$$\operatorname{Der}_R(A_n, M) \simeq \operatorname{Der}_R(A, M \hat{\otimes}_{A_n} B_n).$$

If $m > n$ and M is an A_m-module, then the natural map $\mathrm{Der}_R(A_m, M) \to \mathrm{Der}_R(A_n, M)$ corresponds via the above isomorphism to the map induced by $\mu_{m,n}^{\sharp} \colon B_m \to B_n \otimes_{A_n} A_m$.

Proof

To treat the cases of arcs and jets at the same time, we will identify $R[[t]]$ with $R[t]/(t^{n+1})$ when $n = \infty$.

Fix an A_n-module M as in the statement of the lemma, and consider the A_n-module $A_n \oplus \varepsilon M$ with the A_n-algebra structure defined by $(r \oplus \varepsilon m) \cdot (r' \oplus \varepsilon m') = (rr' \oplus \varepsilon(rm' + r'm))$. The symbol ε should be thought as a variable with $\varepsilon^2 = 0$. Since $A_n \hat{\otimes}_R R[t]/(t^{n+1}) = B_n$, Lemma 3.1 gives a natural isomorphism

$$\mathrm{Hom}_{R\text{-alg}}(A_n, A_n \oplus \varepsilon M) \simeq \mathrm{Hom}_{R\text{-alg}}\big(A, B_n \oplus \varepsilon(M \hat{\otimes}_{A_n} B_n)\big).$$

The two modules of derivations that we are interested in are mapped into each other via this isomorphism. More precisely, we have

$$\begin{aligned}
\mathrm{Der}_R(A_n, M) &\simeq \big\{\phi \in \mathrm{Hom}_{R\text{-alg}}(A_n, A_n \oplus \varepsilon M) \,\big|\, \phi = \mathrm{id}_{A_n} \bmod \varepsilon\big\} \\
&\simeq \big\{\phi \in \mathrm{Hom}_{R\text{-alg}}\big(A, B_n \oplus \varepsilon(M \hat{\otimes}_{A_n} B_n)\big) \,\big|\, \phi = \gamma_n^{\sharp} \bmod \varepsilon\big\} \\
&\simeq \mathrm{Der}_R(A, M \hat{\otimes}_{A_n} B_n).
\end{aligned}$$

For the second statement of the lemma, it suffices to note that the map $\mathrm{Der}_R(A_m, M) \to \mathrm{Der}_R(A_n, M)$ corresponds, via the above isomorphisms, to the map

$$\mathrm{Hom}_{R\text{-alg}}\big(A, B_m \oplus \varepsilon(M \hat{\otimes}_{A_m} B_m)\big) \to \mathrm{Hom}_{R\text{-alg}}\big(A, B_n \oplus \varepsilon(M \hat{\otimes}_{A_n} B_n)\big)$$

induced by the projection $R[t]/(t^{m+1}) \to R[t]/(t^{n+1})$, and the latter is exactly the projection that induces $\mu_{m,n}^{\sharp}$. All the isomorphisms in the proof are functorial with respect to all the data involved, and therefore the resulting isomorphisms are natural. □

Remark 5.2

Lemma 5.1 is the algebraic incarnation of an intuitive geometric fact about tangent vectors on arc spaces and jet schemes. For concreteness, we look at the case of arcs when $R = k$ is a field. Consider a point α in X_∞ with residue field L, here regarded as an A_∞-module. Then an element of $\mathrm{Der}_k(A_\infty, L)$ corresponds to a tangent vector to X_∞ at α. Using the isomorphism of Lemma 5.1, this tangent vector gets identified with an element of $\mathrm{Der}_k(A, L[[t]])$, which corresponds to a vector field on X along the image of the arc α. These types of identifications are expected for moduli spaces

of maps. For example, given two smooth projective varieties X and Y, we can consider the space $\mathcal{M} = \mathrm{Mor}(Y, X)$ parameterizing morphisms from Y to X. Then, for a morphism $f : Y \to X$, we have the well-known formula

$$T_{\mathcal{M}, f} \simeq H^0(Y, f^* T_X),$$

which is analogous to Lemma 5.1.

THEOREM 5.3
Let $X \to S$ be a morphism of schemes. For every $n \in \mathbb{N} \cup \{\infty\}$ we have a natural isomorphism

$$\Omega_{(X/S)_n/S} \simeq \rho_{n*}\big(\gamma_n^*(\Omega_{X/S}) \otimes \mathcal{P}_n\big),$$

where $\rho_n \colon U_n \to (X/S)_n$ and $\gamma_n \colon U_n \to X$ are defined in (3.1), and these sheaves are isomorphic to $\rho_{n*}(\gamma_n^*(\Omega_{X/S}))$ whenever n is finite. Moreover, for every $m, n \in \mathbb{N} \cup \{\infty\}$ with $m > n$, the morphisms

$$\pi_{m,n}^*(\Omega_{(X/S)_n/S}) \to \Omega_{(X/S)_m/S}$$

induced by the truncation maps are obtained from the natural inclusion $\pi_{m,n}^*(\rho_{n*}(\mathcal{P}_n)) \to \rho_{m*}(\mathcal{P}_m)$ by tensoring with $\rho_{n*}(\gamma_n^*(\Omega_{X/S}))$, and correspond to the maps $\pi_{m,n}^*(\rho_{n*}(\mathcal{O}_{U_n})) \to \rho_{m*}(\mathcal{O}_{U_m})$ given by multiplication by t^{m-n} whenever m is finite.

Proof
Since these properties are local in X, we can assume that $X = \mathrm{Spec}\, A$ and $S = \mathrm{Spec}\, R$. Recall that all B_n-modules are regarded as A_n-modules via $\rho_n^{\sharp}$ and as A-modules via $\gamma_n^{\sharp}$. With the same notation as in Section 4, let M be an arbitrary A_n-module. By Remark 4.3, the natural morphism

$$M \,\hat{\otimes}_{A_n} B_n \longrightarrow \mathrm{Hom}_{A_n}(P_n, M)$$

is an isomorphism of B_n-modules. Then, by Lemma 5.1, we have a chain of natural isomorphisms

$$
\begin{aligned}
\mathrm{Hom}_{A_n}(\Omega_{A_n/R}, M) &\simeq \mathrm{Der}_R(A_n, M) \\
&\simeq \mathrm{Der}_R(A, M \,\hat{\otimes}_{A_n} B_n) \\
&\simeq \mathrm{Hom}_A(\Omega_{A/R}, M \,\hat{\otimes}_{A_n} B_n) \\
&\simeq \mathrm{Hom}_A\big(\Omega_{A/R}, \mathrm{Hom}_{A_n}(P_n, M)\big) \\
&\simeq \mathrm{Hom}_{A_n}(\Omega_{A/R} \otimes_A P_n, M).
\end{aligned}
$$

It follows that there is a natural isomorphism of A_n-modules $\Omega_{A_n/R} \simeq \Omega_{A/R} \otimes_A P_n$. Since $\Omega_{A/R} \otimes_A P_n = (\Omega_{A/R} \otimes_A B_n) \otimes_{B_n} P_n$, this and the fact that, by Lemma 4.5, P_n is isomorphic to B_n as a B_n-module if n is finite give the first statement. The other statements follow immediately from the second part of Lemma 5.1 and Remark 4.6.

$\square$

Remark 5.4

Suppose for simplicity that $X = \operatorname{Spec} A$ is affine over $S = \operatorname{Spec} R$ so that the formula in Theorem 5.3 becomes, for $n = \infty$,

$$\Omega_{A_\infty/R} \simeq \Omega_{A/R} \otimes_A P_\infty.$$

For every A_∞-algebra L (e.g., a field corresponding to a point of $(X/S)_\infty$), we have

$$\Omega_{A_\infty/R} \otimes_{A_\infty} L \simeq \Omega_{A/R} \otimes_A P_\infty \otimes_{A_\infty} L$$

$$\simeq \Omega_{A/R} \otimes_A L[[t]] \otimes_{L[[t]]} (P_\infty \otimes_{A_\infty} L).$$

Note that $L[[t]] = B_\infty \hat{\otimes}_{A_\infty} L$. In the second step above, we have used the fact that $P_\infty \otimes_{A_\infty} L = L((t))/tL[[t]]$, and hence it is not just a module over $B_\infty \otimes_{A_\infty} L$ but also over $L[[t]]$. The above formula will be useful in order to compute fibers of $\Omega_{X_\infty/S}$.

It is worthwhile to work out explicitly what Theorem 5.3 is telling us in the affine case, when both X and S are affine and X is of finite type over S. We start with the case when n is finite. If $S = \operatorname{Spec} R$ and $X = \operatorname{Spec} A$, then we can write

$$A = \frac{R[x_1, \ldots, x_r]}{(f_1, \ldots, f_s)}.$$

Then

$$A_n = \frac{R[x_i^{(p)} \mid i = 1, \ldots, r, p = 0, \ldots, n]}{(f_j^{(q)} \mid j = 1, \ldots, s, q = 0, \ldots, n)},$$

where, for every g and k, we set $g^{(k)} := D_k(g)$. The presentation of A yields the following presentation for $\Omega_{A/R}$:

$$\bigoplus_j A\, df_j \xrightarrow{J} \bigoplus_i A\, dx_i \to \Omega_{A/R} \to 0.$$

Here $J = [\partial f_j/\partial x_i]$ is the Jacobian matrix. Similarly, the presentation of A_n gives

$$\bigoplus_{j,q} A_n\, df_j^{(q)} \xrightarrow{\widetilde{J}_n} \bigoplus_{i,p} A_n\, dx_i^{(p)} \to \Omega_{A_n/R} \to 0,$$

where $\widetilde{J_n} = [\partial f_j^{(q)}/\partial x_i^{(p)}]$. Theorem 5.3 provides an efficient way of writing down this matrix. Explicitly, the theorem gives the presentation

$$\bigoplus_j P_n\, df_j \xrightarrow{J(t)} \bigoplus_i P_n\, dx_i \to \Omega_{A_n/R} \to 0,$$

where $J(t)$ is, entry by entry, the pullback of J via the universal jet. In particular, we can write

$$J(t) = J + J't + J''t^2 + \cdots + J^{(n)}t^n,$$

where $J^{(k)} = [D_k(\partial f_j/\partial x_i)]$. Using the decomposition $P_n = \bigoplus_{k=0}^n A_n t^{-k}$, we get

$$\bigoplus_{j,q} A_n t^{-q}\, df_j \xrightarrow{J_n} \bigoplus_{i,p} A_n t^{-p}\, dx_i \to \Omega_{A_n/R} \to 0,$$

where

$$J_n = \begin{bmatrix} J & J' & \cdots & J^{(n)} \\ 0 & J & \cdots & J^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{bmatrix}.$$

The isomorphism in Lemma 5.1 maps $\partial/\partial x_i^{(p)} \mapsto t^p \partial/\partial x_i$. By duality, we see that the isomorphism in Theorem 5.3 maps $dx_i^{(p)} \mapsto t^{-p}\, dx_i$ and $df_j^{(q)} \mapsto t^{-q}\, df_j$. From this, we see that

$$\widetilde{J_n} = J_n.$$

In particular, we can immediately recover from this that

$$\frac{\partial f_j^{(q)}}{\partial x_i^{(p)}} = \frac{\partial f_j^{(q+1)}}{\partial x_i^{(p+1)}} \quad \text{for all } p, q < n$$

and

$$\frac{\partial f_j^{(q)}}{\partial x_i^{(p)}} = D_{q-p}\left(\frac{\partial f_j}{\partial x_i}\right) \quad \text{for all } p < q.$$

These relations are well known and can be viewed as an elementary consequence of the chain rule. It is a fun exercise to verify them starting from an explicit polynomial. The above presentation of $\Omega_{A_n/R}$ is compatible with the differentials $\Omega_{A_n/R} \otimes_{A_n} A_m \to \Omega_{A_m/R}$ of the truncation maps, and it can be used to compute them. Letting $n \to \infty$, one gets a similar description of the presentation matrix for $\Omega_{A_\infty/R}$, where now the matrix is infinite-dimensional.

6. Invariant factors and fitting invariants

In the next section, we will be interested in studying fibers of the sheaves of differentials on jet schemes $(X/S)_n$. Using Theorem 5.3, this will involve understanding the pullback of $\Omega_{X/S}$ along a jet. As a preparation, in the present section we include some remarks on these types of pullbacks.

Let X be an arbitrary scheme over base scheme S. For a given $n \in \mathbb{N} \cup \{\infty\}$, consider a point $\alpha_n \in (X/S)_n$ and let L_n denote the residue field of α_n. We do not assume that α_n is a closed point of $(X/S)_n$. By shrinking X around $\psi_n(\alpha_n)$, we may assume without loss of generality that $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$ are affine. Let A_n and B_n be the algebras defined in Section 4, so that $(X/S)_n = \operatorname{Spec} A_n$ and $U_n = \operatorname{Spec} B_n$.

Algebraically, α_n is given by a morphism

$$\alpha_n^\sharp \colon A \longrightarrow B_n \,\hat{\otimes}_{A_n} L_n,$$

where $B_n \,\hat{\otimes}_{A_n} L_n = L_n[t]/(t^{n+1})$ if n is finite and $B_\infty \,\hat{\otimes}_{A_\infty} L_\infty = L_\infty[[t]]$. For simplicity, in what follows we will identify $L_\infty[[t]]$ with $L_n[t]/(t^{n+1})$ when $n = \infty$, and we will always use the more suggestive notation $L_n[t]/(t^{n+1})$ instead of $B_n \,\hat{\otimes}_{A_n} L_n$.

Consider a finitely generated A-module M. We are interested in understanding the structure of its pullback along α_n, which is given by

$$M \otimes_A L_n[t]/(t^{n+1}).$$

Notice that $L_n[t]/(t^{n+1})$ is a principal ideal ring (and a domain when $n = \infty$). Since M is finitely generated, the pullback is also finitely generated, and the structure theory for finitely generated modules over principal ideal rings gives a unique decomposition

$$M \otimes_A L_n[t]/(t^{n+1}) \simeq \left(L_n[t]/(t^{n+1})\right)^d \oplus \bigoplus_{i \geq d} L_n[t]/(t^{e_i}),$$

where the direct sum in the right-hand side has finitely many nonzero terms and $n + 1 > e_d \geq e_{d+1} \geq \cdots \geq e_{d+r}$. When $0 \leq i < d$, we set $e_i = n + 1$ (so $e_i = \infty$ if $n = \infty$). If the dependency on α_n and M needs to be emphasized, we will write $d = d(\alpha_n) = d(\alpha_n, M)$ and $e_i = e_i(\alpha_n) = e_i(\alpha_n, M)$.

Definition 6.1
We call $\{e_i(\alpha_n, M)\}_{i=0}^\infty$ the sequence of *invariant factors* of M with respect to α_n. The number $d(\alpha_n, M)$ is called the *Betti number* of M with respect to α_n.

The invariant factors of M with respect to α_n determine the pullback $M \otimes_A L[t]/(t^{n+1})$ up to isomorphism. If α_n is the truncation of another jet α_m (so

$\pi_{m,n}(\alpha_m) = \alpha_n$), the invariant factors with respect to α_n and α_m are related. We have:

$$e_i(\alpha_n) = \min\{n+1, e_i(\alpha_m)\}.$$

Notice that the Betti numbers with respect to α_n and α_m could be different. We always have $d(\alpha_n) \geq d(\alpha_m)$.

The invariant factors are related to the Fitting ideals of M, whose definition we recall briefly. Since M is finitely generated, we can find a presentation

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \to 0, \tag{6.1}$$

where F_0 and F_1 are free A-modules and F_0 is finitely generated. Then $\mathrm{Fitt}^i(M) \subset A$ is the ideal generated by the minors of size $\mathrm{rank}(F_0) - i$ of the matrix representing φ. Geometrically, a point $x \in X$ belongs to the zero locus of $\mathrm{Fitt}^i(M)$ if and only if the dimension of the fiber $M \otimes_A k(x)$ is greater than i.

Recall that if $J \subset A$ is an ideal, then $c = \mathrm{ord}_{\alpha_n}(J)$ is defined as the number for which $\alpha_n^{\sharp}(J) = (t^c)$. When n is finite, we pick $c \in \{0, 1, \ldots, n+1\}$, and when $n = \infty$, we have $c \in \{0, 1, \ldots, \infty\}$ (we use the convention $t^{\infty} = 0$).

Definition 6.2
Consider the numbers

$$c_i(\alpha_n, M) := \mathrm{ord}_{\alpha_n}\big(\mathrm{Fitt}^i(M)\big).$$

We call $\{c_i(\alpha_n, M)\}_{i=0}^{\infty}$ the sequence of *Fitting invariants* of M with respect to α_n.

The Fitting invariants are determined by the invariant factors. To see this, consider the pullback via α_n of the presentation in (6.1):

$$\widetilde{F_1} \xrightarrow{\widetilde{\varphi}} \widetilde{F_0} \longrightarrow M \otimes_A L_n[t]/(t^{n+1}) \to 0. \tag{6.2}$$

The structure theory says that, after picking appropriate bases, $\widetilde{\varphi}$ is represented by a matrix with entries $t^{e_0}, t^{e_1}, t^{e_2}, \ldots$ along the main diagonal and zeros elsewhere. The minors of $\widetilde{\varphi}$ are the pullbacks of the minors of φ. This property is expressed by saying that "the formation of Fitting ideals commutes with base change," and it gives

$$\mathrm{Fitt}^i\big(M \otimes_R L_n[t]/(t^{n+1})\big) = \alpha_n^{\sharp}\big(\mathrm{Fitt}^i(M)\big) \cdot L_n[t]/(t^{n+1}) = (t^{c_i}) \subset L_n[t]/(t^{n+1}),$$

where $c_i = \mathrm{ord}_{\alpha_n}(\mathrm{Fitt}^i(M))$. Here the Fitting ideals of $M \otimes_A L_n[t]/(t^{n+1})$ are computed with respect to its structure as a module over $L_n[t]/(t^{n+1})$. On the other hand, we can compute these Fitting ideals directly using the presentation in (6.2). We get:

$$c_i = \min\{n + 1, e_i + e_{i+1} + e_{i+2} + \cdots\}.$$

If $n = \infty$, we get that $c_i = e_i + e_{i+1} + e_{i+2} + \cdots$, and we see that in this case the invariant factors are determined by the Fitting invariants. The Betti number counts the number of infinite Fitting invariants and can be interpreted geometrically as the dimension of the fiber $M \otimes_A k(\xi)$, where $\xi = \alpha_\infty(\eta)$ is the generic point of the arc α_∞. In particular, we have that $\mathrm{ord}_\alpha(\mathrm{Fitt}^d(M)) < \infty$.

7. The fiber over a jet

In this section, we assume that X is a scheme of finite type over an arbitrary base scheme S. This condition on X guarantees that the sheaf of differentials $\Omega_{X/S}$ is a finitely generated $\mathcal{O}_X$-module. In particular, we can consider its Fitting ideals and can apply the results of Section 6.

Remark 7.1

To compute $\mathrm{Fitt}^i(\Omega_{X/S})$, we can work locally on X and S, and we use a relatively closed embedding of X in some affine space $\mathbb{A}_S^N$ over S to get a presentation as in (6.1), where φ is the Jacobian matrix of the embedding. As for any module, a point $x \in X$ belongs to the zero locus of $\mathrm{Fitt}^i(\Omega_{X/S})$ if and only if the dimension of the fiber $\Omega_{X/S} \otimes_{\mathcal{O}_X} k(x)$ is $> i$. In particular, if X is a reduced and equidimensional scheme of finite type over a field k, then $\mathrm{Fitt}^i(\Omega_{X/k})$ is zero when $i < \dim X$ and equals the Jacobian ideal Jac_X when $i = \dim X$.

We are interested in studying the fibers of $\Omega_{(X/S)_n/S}$, and relating them to the Fitting invariants of $\Omega_{X/S}$.

THEOREM 7.2

Let X be a scheme of finite type over a base scheme S, let $n \in \mathbb{N}$, and consider a jet $\alpha_n \in (X/S)_n$. We do not assume that α_n is a closed point of $(X/S)_n$, and we let L_n be its residue field. Let d_n and $\{e_i\}$ be the Betti number and invariant factors of $\Omega_{X/S}$ with respect to α_n. Then the isomorphism $\Omega_{(X/S)_n/S} \simeq \rho_{n*}(\gamma_n^*(\Omega_{X/S}))$ given by Theorem 5.3 induces an isomorphism

$$\Omega_{(X/S)_n/S} \otimes_{\mathcal{O}_{(X/S)_n}} L_n \simeq \left(L_n[t]/(t^{n+1})\right)^{d_n} \oplus \bigoplus_{i \geq d_n} L_n[t]/(t^{e_i}).$$

Proof

After restricting to a suitable open set of X, we can assume that $X = \mathrm{Spec}\, A$ and $S = \mathrm{Spec}\, R$. As before, we use the notation from Section 4. From Theorem 5.3, since n is finite, we see that $\Omega_{A_n/S} \simeq \Omega_{A/S} \otimes_A B_n$, where B_n is considered as an A-module via $\gamma_n^\sharp$. This implies that

$$\Omega_{A_n/S} \otimes_{A_n} L_n \simeq \Omega_{A/S} \otimes_A L_n[t]/(t^{n+1}),$$

where $L_n[t]/(t^{n+1})$ is considered as an A-module via $\alpha_n^{\sharp}$. The theorem now follows from the definition of the invariant factors with respect to the jet α_n.   □

We now restrict ourselves to *liftable jets*. By definition, these are points in a jet scheme $(X/S)_n$ that lie in the image of the truncation map $\pi_n \colon (X/S)_\infty \to (X/S)_n$. From the above theorem it is immediate to compute the dimensions of the fibers of $\Omega_{(X/S)_n/S}$ over liftable jets.

COROLLARY 7.3
In addition to the assumptions of Theorem 7.2, assume that $\alpha_n = \pi_n(\alpha)$ for some arc $\alpha \in (X/S)_\infty$. Consider the ideal sheaf $\mathcal{J}_{d_n} := \mathrm{Fitt}^{d_n}(\Omega_{X/S})$. Then

$$\dim_{L_n}(\Omega_{(X/S)_n/S} \otimes_{\mathcal{O}_{(X/S)_n}} L_n) = (n+1)d_n + \mathrm{ord}_\alpha(\mathcal{J}_{d_n}).$$

Proof
Starting at the position $i = d_n$, we have an equality of invariant factors $e_i(\alpha_n) = e_i(\alpha)$. Since $\mathrm{ord}_\alpha(\mathcal{J}_{d_n}) = c_{d_n} = e_{d_n} + e_{d_n+1} + \cdots$, the result follows immediately from Theorem 7.2.   □

Remark 7.4
The Betti number $d_n = d(\alpha_n, \Omega_{X/S})$ appearing in the two immediately preceding results is hard to interpret in geometric terms. On the other hand, the Betti number $d = d(\alpha, \Omega_{X/S})$ has a clear meaning: it is the dimension of the fiber $\Omega_{X/S} \otimes_{\mathcal{O}_X} k(\xi)$, where $\xi = \alpha(\eta) \in X$ is the generic point of α. Recall that when n is large enough (bigger than all the invariant factors of α), the Betti numbers of $\Omega_{X/S}$ with respect to α and $\alpha_n = \pi_n(\alpha)$ coincide.

The next corollary recovers [5, Proposition 5.1].

COROLLARY 7.5
In addition to the assumptions of Corollary 7.3, assume that $S = \mathrm{Spec}\, k$ for a field k, assume that X is reduced and equidimensional over k, and assume that the arc α is not completely contained in the singular locus of X. Then, for finite $n \geq \mathrm{ord}_\alpha(\mathrm{Jac}_X)$, we have

$$\dim_{L_n}(\Omega_{X_n/k} \otimes_{\mathcal{O}_{X_n}} L_n) = (n+1)\dim X + \mathrm{ord}_\alpha(\mathrm{Jac}_X).$$

Proof
With the additional assumptions, we see that the Betti number of $\Omega_{X/S}$ with respect

to α is $d = \dim X$, and therefore $\mathcal{J}_d = \mathrm{Fitt}^d(\Omega_{A/k}) = \mathrm{Jac}_X$. The condition $n \geq \mathrm{ord}_\alpha(\mathrm{Jac}_X)$ guarantees that the Betti numbers of $\Omega_{A/k}$ with respect to α and α_n coincide. The result is just a restatement of Corollary 7.3 in this case. $\square$

8. Embedding dimension

We now study embedding dimensions of arcs and jets. Starting with this section and for the reminder of this article, we assume that X is a scheme of finite type over a perfect field k. In the following, let $\alpha \in X_\infty$ be a point, and denote by $L = L_\infty$ the residue field of α. We do not assume that α is a closed point of X_∞. For every $n \in \mathbb{N}$, we let $\alpha_n = \pi_n(\alpha)$ be the truncations, and we denote by L_n their residue fields. It will be convenient to also allow the notation α_∞ for α.

For $n \in \mathbb{N}$, we denote $\dim(\alpha_n) := \mathrm{tr.deg}(L_n/k)$. Since the ground field k is assumed to be perfect, we have

$$\dim(\alpha_n) = \dim_{L_n}(\Omega_{L_n/k}).$$

We start with some preliminary lemmas. For ease of notation, in our discussion of these preliminary properties we will restrict ourselves to the affine setting and will assume that $X = \mathrm{Spec}\, A$, where A is a finitely generated k-algebra. We apply the notation from Section 4 with $R = k$.

For each $n \in \mathbb{N} \cup \{\infty\}$, we let $I_n \subset A_n$ be the prime ideal defining α_n. When $m > n$, we have inclusions $I_n \subset I_m$. The Zariski tangent space of X_n at α_n is the dual of the L_n-vector space I_n/I_n^2, and hence the embedding dimension of X_n at α_n is given by

$$\mathrm{emb.dim}(\mathcal{O}_{X_n,\alpha_n}) = \dim_{L_n}(I_n/I_n^2).$$

Note that there are natural maps $I_n/I_n^2 \to I_m/I_m^2$ whenever $m > n$ and also that

$$I_\infty/I_\infty^2 = \mathrm{inj}\lim_{n \to \infty}(I_n/I_n^2).$$

LEMMA 8.1
As above, let $X = \mathrm{Spec}\, A$, where A is a finitely generated k-algebra, and let $\alpha \in X_\infty$. For every $n \in \mathbb{N}$, let d_n be the Betti number of $\Omega_{A/k}$ with respect to the truncation $\alpha_n = \pi_n(\alpha)$, and consider the ideal $J_{d_n} := \mathrm{Fitt}^{d_n}(\Omega_{A/k})$. Then

$$\mathrm{emb.dim}(\mathcal{O}_{X_n,\alpha_n}) = (n+1)d_n - \dim(\alpha_n) + \mathrm{ord}_\alpha(J_{d_n}).$$

Proof
Applying [33, Theorem 25.2] to the sequence $k \to (A_n)_{I_n} \to L_n$, we get an exact sequence

$$0 \to I_n/I_n^2 \longrightarrow \Omega_{A_n/k} \otimes_{A_n} L_n \longrightarrow \Omega_{L_n/k} \to 0.$$

Here we used the assumption that k is perfect. The lemma now follows from Corollary 7.3 and the equality $\dim(\alpha_n) = \dim_{L_n}(\Omega_{L_n/k})$.   □

LEMMA 8.2
With the same assumptions as Lemma 8.1, let d and e_i be the Betti number and invariant factors of $\Omega_{A/k}$ with respect to the arc α, and let $\Omega_{A_n/k} \otimes_{A_n} A_m \to \Omega_{A_m/k}$ be the map induced by the truncation morphism $\pi_{m,n} \colon X_m \to X_n$. Then, for finite $m \geq n + \mathrm{ord}_\alpha(J_d)$, we have

$$K := \ker(\Omega_{A_n/k} \otimes_{A_n} L \to \Omega_{A_m/k} \otimes_{A_m} L) \simeq \left(\frac{L[t]}{(t^{n+1})}\right)^{d_n-d} \oplus \left(\bigoplus_{i \geq d_n} \frac{L[t]}{(t^{e_i})}\right),$$

and hence $\dim_L(K) = (n+1)(d_n - d) + \mathrm{ord}_\alpha(J_{d_n})$. In particular, for $n \geq \mathrm{ord}_\alpha(J_d)$ and $m \geq n + \mathrm{ord}_\alpha(J_d)$, we have

$$K = \ker(\Omega_{A_n/k} \otimes_{A_n} L \to \Omega_{A_m/k} \otimes_{A_m} L) \simeq \bigoplus_{i \geq d} \frac{L[t]}{(t^{e_i})}$$

and $\dim_L(K) = \mathrm{ord}_\alpha(J_d)$.

Proof
Since $m \geq \mathrm{ord}_\alpha(J_d)$, we see that the Betti numbers of $\Omega_{A/k}$ with respect to α and α_m coincide. By Theorems 7.2 and 5.3 (see also Remark 4.6), we see that K is isomorphic to the kernel of the map

$$\left(\frac{L[t]}{(t^{n+1})}\right)^d \oplus \left(\frac{L[t]}{(t^{n+1})}\right)^{d_n-d} \oplus \left(\bigoplus_{i \geq d_n} \frac{L[t]}{(t^{e_i})}\right)$$

$$\xrightarrow{\cdot t^{m-n}} \left(\frac{L[t]}{(t^{m+1})}\right)^d \oplus \left(\bigoplus_{i=d}^{d_n-1} \frac{L[t]}{(t^{e_i})}\right) \oplus \left(\bigoplus_{i \geq d_n} \frac{L[t]}{(t^{e_i})}\right)$$

given by multiplication by t^{m-n}. Since we have $m - n \geq \mathrm{ord}_\alpha(J_d) \geq e_i$ for all $i \geq d$, the first assertion follows. Note that $\sum_{i \geq d_n} e_i = \mathrm{ord}_\alpha(J_{d_n})$. The last assertion follows from this and from the fact that, if $n \geq \mathrm{ord}_\alpha(J_d)$, then $d_n = d$.   □

LEMMA 8.3
With the same assumptions as Lemma 8.2, consider the natural morphism

$$\lambda_n \colon I_n/I_n^2 \otimes_{A_n} L \longrightarrow I_\infty/I_\infty^2$$

induced by the truncation map. Then

$$\dim_L\big(\mathrm{Im}(\lambda_n)\big) \geq (n+1)d - \dim(\alpha_n).$$

Proof

Consider $m \geq n + \mathrm{ord}_\alpha(J_d)$. We have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_n/I_n^2 \otimes L & \longrightarrow & \Omega_{A_n/k} \otimes L & \longrightarrow & \Omega_{L_n/k} \otimes L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_{n,m}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_m/I_m^2 \otimes L & \longrightarrow & \Omega_{A_m/k} \otimes L & \longrightarrow & \Omega_{L_m/k} \otimes L & \longrightarrow & 0
\end{array}
$$

Recall that, by Lemmas 8.1 and 8.2, we have

$$\dim_L(I_n/I_n^2 \otimes L) = (n+1)d_n - \dim(\alpha_n) + \mathrm{ord}_\alpha(J_{d_n}),$$

$$\dim_L(K) = (n+1)(d_n - d) + \mathrm{ord}_\alpha(J_{d_n}).$$

Then from the first column of the diagram we see that

$$
\begin{aligned}
\dim_L\big(\mathrm{Im}(\lambda_{n,m})\big) &= \dim_L(I_n/I_n^2 \otimes L) - \dim_L(K') \\
&\geq \dim_L(I_n/I_n^2 \otimes L) - \dim_L(K) \\
&= (n+1)d - \dim(\alpha_n).
\end{aligned}
$$

Since $\mathrm{Im}(\lambda_n) = \mathrm{inj}\lim_m \mathrm{Im}(\lambda_{n,m})$, the result follows. $\qquad\square$

LEMMA 8.4

With the same assumptions as Lemma 8.2, we have

$$\mathrm{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) \geq \limsup_{n\to\infty}\big((n+1)d - \dim(\alpha_n)\big).$$

Proof

Consider the maps λ_n of Lemma 8.3. Since $I_\infty/I_\infty^2 = \mathrm{inj}\lim(I_n/I_n^2)$, it also holds that $I_\infty/I_\infty^2 = \mathrm{inj}\lim(\mathrm{Im}(\lambda_n))$. Therefore the assertion follows from Lemma 8.3. $\quad\square$

We now return to the global case of schemes of finite type over k. Recall that the dimension of a scheme X of finite type over k at a point ξ, denoted $\dim_\xi(X)$, is

defined as the infimum of the dimensions (over k) of all open neighborhoods of ξ in X.

THEOREM 8.5
Assume that X is a scheme of finite type over k. Then for any arc $\alpha \in X_\infty$, letting $\xi = \alpha(\eta) \in X$ denote the generic point of the arc, we have

$$\mathrm{emb.dim}(\mathscr{O}_{X_\infty, \alpha}) \geq \dim_\xi(X) - \dim(\alpha_0).$$

Proof
The proof follows from Lemma 8.4 and the fact that the Betti number of $\Omega_{X/k}$ with respect to α is bounded below by $\dim_\xi(X)$. $\qquad\square$

Since the ground field is perfect, a point x on a scheme of finite type X is singular if and only if $\dim(\Omega_X \otimes k(x)) > \dim_x(X)$. In particular, nonreduced points of X are singular. We denote by $\mathrm{Sing}\, X$ the singular locus of X. Without further mention, we will use the fact that, as the ground field is perfect, if X is a variety, then $\mathrm{Sing}\, X$ is a proper closed subset of X and hence, in particular, has smaller dimension.

LEMMA 8.6
Let X be a a scheme of finite type over k, let $\alpha \in X_\infty$, and let $Z \subset X$ be the closure of the generic point $\alpha(\eta)$ of α. Then for every $n \in \mathbb{N}$ we have

$$\dim(\alpha_n) \leq (n+1)\dim(Z).$$

Proof
In characteristic zero, the statement follows easily from *Kolchin's irreducibility theorem* [30, Chapter 4, Proposition 10]. Over an arbitrary perfect field, the argument can be adjusted as follows. Note that $\alpha \in Z_\infty \setminus (\mathrm{Sing}\, Z)_\infty$. Then, by [39, Theorem 2.9], α belongs to the unique irreducible component C of Z_∞ that dominates Z. It follows that the closure of the projection of C to Z_n has dimension $(n+1)\dim(Z)$, and the assertion follows. $\qquad\square$

PROPOSITION 8.7
Let X be scheme of finite type, let $\alpha \in X_\infty$, and denote by $\xi = \alpha(\eta) \in X$ the generic point of α. Assume that one of the following two conditions holds:
(1) $\alpha \in Y_\infty$ *where $Y \subset X$ is a closed subscheme with $\dim_\xi(Y) < \dim_\xi(X)$,*
(2) $\alpha \in (\mathrm{Sing}\, X)_\infty$.
Then

$$\mathrm{emb.dim}(\mathscr{O}_{X_\infty, \alpha}) = \infty.$$

Proof

Suppose first that (1) holds. By the geometric interpretation of Betti numbers, we have $d(\alpha, \Omega_X) \geq \dim_\xi(X)$. Since $\dim(\alpha_n) \leq (n+1)\dim_\xi(Y)$ by Lemma 8.6, and $\dim_\xi(Y) < \dim_\xi(X)$, Lemma 8.4 implies that

$$\text{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) \geq \limsup_{n \to \infty} (n+1)\big(d(\alpha, \Omega_X) - \dim(Y)\big) = \infty.$$

Suppose then that (2) holds. If $\dim_\xi(\text{Sing }X)_\infty < \dim_\xi(X)$, then the assertion follows from case (1). Otherwise, X is nonreduced at ξ and hence $d(\alpha, \Omega_X) > \dim_\xi(X)$, and we conclude that $\text{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) = \infty$ by Lemmas 8.4 and 8.6. $\qquad\square$

Remark 8.8

The inequalities stated in Lemmas 8.3 and 8.4 are both equalities whenever the map $K' \to K$ in the proof of Lemma 8.3 is an isomorphism. This is clearly the case if L_m is separable over L_n, as in this case $K'' = 0$, and therefore we get equalities in both formulas if k has characteristic zero or α is a k-rational point of X_∞. More interestingly, we will see later in Section 10 that the condition that L_m is separable over L_n is always guaranteed if X is a variety and α is a stable point of X_∞, and hence we will deduce that equalities hold in this case, too. It will turn out in the end (see Lemma 10.6) that in fact the equality holds in general, and the limsup is a limit, in the formula stated in Lemma 8.4.

9. The birational transformation rule

Here we study how birational morphisms affect the embedding dimension of arcs. Given two schemes X and Y of finite type over a perfect field k, we say that a morphism $f : Y \to X$ is *birational over a union of components* if there exist a dense open set $V \subset Y$ and a (not necessarily dense) open set $U \subset X$ such that $f(V) \subset U$ and the restriction $f|_V : V \to U$ is an isomorphism. If U is dense in X, or equivalently if f is dominant, then we say that f is *birational*.

If $f : Y \to X$ is birational over a union of components, then the sheaf of relative differentials $\Omega_{Y/X}$ is torsion. We consider the *Jacobian ideal* $\text{Jac}_f := \text{Fitt}^0(\Omega_{Y/X})$ of f.

LEMMA 9.1

Let X and Y be schemes of finite type over a perfect field, and consider a proper map $f : Y \to X$ that is birational over a union of components. Let $\beta \in Y_\infty$ and consider $\alpha = f_\infty(\beta) \in X_\infty$. If f is locally an isomorphism at the generic point $\beta(\eta)$ of the arc (that is, one can pick $V \subset Y$ as above such that $\beta(\eta) \in V$), then the residue fields of α and β are equal.

Proof

Let L and K be the residue fields of α and β, respectively. Since $\alpha = f_\infty(\beta)$, we have $L \subset K$. Consider α as map $\alpha \colon \operatorname{Spec} L[[t]] \to X$. The hypothesis on β guarantees that the generic point $\alpha(\eta)$ of α lies in the locus over which f is an isomorphism, and therefore it can be lifted. The valuative criterion of properness gives a unique lift of α to an arc $\widetilde{\alpha} \colon \operatorname{Spec} L[[t]] \to Y$. This corresponds to a morphism $\operatorname{Spec} L \to Y_\infty$ whose image is β by construction. This implies that $K \subset L$, as required. $\qquad\square$

THEOREM 9.2
Let X and Y be schemes of finite type over a perfect field. Consider a proper map $f \colon Y \to X$ that is birational over a union of components. Let $\beta \in Y_\infty$ and consider $\alpha = f_\infty(\beta) \in X_\infty$. Assume that Y is smooth at $\beta(0)$. Then

$$\operatorname{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \operatorname{emb.dim}(\mathcal{O}_{Y_\infty,\beta}) + \operatorname{ord}_\beta(\operatorname{Jac}_f).$$

Note that several of these numbers could be infinite. For example, if β has infinite embedding dimension, the theorem implies that α also has infinite embedding dimension. Conversely, if α has infinite embedding dimension and β is not completely contained in the vanishing locus of Jac_f, then β has infinite embedding dimension.

Proof of Theorem 9.2
Let $V \subset Y$ be a dense open subset as above, so that f restricts to an isomorphism from V to an open set $U \subset X$. If β is contained in $Z := Y \setminus V$, then α is contained in the image of Z, and hence both embedding dimensions are infinite by Proposition 8.7. Thus we assume that β is not contained in Z, which means that $\beta(\eta) \in V$. By Lemma 9.1 both α and β have the same residue field, which we call L. Let I and J be the ideals defining α and β. Since the ground field is perfect, we have the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_{X_\infty/k} \otimes_{\mathcal{O}_{X_\infty}} L & \longrightarrow & \Omega_{L/k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \| & & \\
0 & \longrightarrow & J/J^2 & \longrightarrow & \Omega_{Y_\infty/k} \otimes_{\mathcal{O}_{Y_\infty}} L & \longrightarrow & \Omega_{L/k} & \longrightarrow & 0
\end{array}
\tag{9.1}
$$

Therefore, the theorem will follow if we show that

$$\dim_L(\ker\varphi) - \dim_L(\operatorname{coker}\varphi) = \operatorname{ord}_\beta(\operatorname{Jac}_f).$$

Recall the universal arc $\rho_\infty \colon U_\infty \to X_\infty$ and the sheaf $\mathcal{P}_\infty$ on U_∞ defined in Section 4. We rely on Remark 5.4 for the computation of the fibers appearing in the middle column of (9.1). For ease of notation, we denote

$$B_L := L[[t]] = \rho_{\infty *}(\mathcal{O}_{U_\infty}) \, \hat{\otimes}_{\mathcal{O}_{X_\infty}} L$$

and

$$P_L := L((t))/tL[[t]] = \rho_{\infty *}(\mathcal{P}_\infty) \otimes_{\mathcal{O}_{X_\infty}} L.$$

We can regard B_L both as an $\mathcal{O}_X$-algebra via the arc α and as an $\mathcal{O}_Y$-algebra via β. Then P_L, which is naturally a B_L-module, becomes both an $\mathcal{O}_X$-module and an $\mathcal{O}_Y$-module.

The map f induces a natural sequence of sheaves of differentials:

$$\Omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \xrightarrow{f^*} \Omega_{Y/k} \longrightarrow \Omega_{Y/X} \longrightarrow 0.$$

After pulling back to the arcs α and β, we get:

$$\Omega_{X/k} \otimes_{\mathcal{O}_X} B_L \xrightarrow{\psi} \Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L \longrightarrow \Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L \longrightarrow 0.$$

All of the terms in this sequence are finitely generated modules over $B_L = L[[t]]$, and therefore they are direct sums of cyclic modules. Since Y is smooth at $\beta(0)$, the middle term $F_Y := \Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L$ is free. Write $\Omega_{X/k} \otimes_{\mathcal{O}_X} B_L = F_X \oplus T_X$, where F_X is free and T_X is torsion. Since f is an isomorphism at the generic point of β, the restriction $\overline{\psi} = \psi|_{F_X}$ is injective. Consider $Q_{Y/X} = \operatorname{coker}(\overline{\psi})$. We have an exact sequence:

$$0 \longrightarrow F_X \xrightarrow{\overline{\psi}} F_Y \longrightarrow Q_{Y/X} \longrightarrow 0. \tag{9.2}$$

Note that $\psi(T_X) = 0$, and therefore that $Q_{Y/X} = \Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L$.

Theorem 5.3 says that φ is obtained from ψ by tensoring with P_L:

$$0 \to K \longrightarrow \Omega_{X/k} \otimes_{\mathcal{O}_X} P_L \xrightarrow{\varphi} \Omega_{Y/k} \otimes_{\mathcal{O}_Y} P_L \longrightarrow \Omega_{Y/X} \otimes_{\mathcal{O}_Y} P_L \to 0$$

$$\Big\| \qquad\qquad\qquad \Big\|$$

$$\Omega_{X_\infty/k} \otimes_{\mathcal{O}_{X_\infty}} L \qquad \Omega_{Y_\infty/k} \otimes_{\mathcal{O}_{Y_\infty}} L$$

$$\tag{9.3}$$

Note that P_L is a divisible B_L-module, and hence tensoring with P_L kills torsion. We get the following diagram:

$$0 \rightarrow K \longrightarrow F_X \otimes_{B_L} P_L \xrightarrow{\overline{\varphi}} F_Y \otimes_{B_L} P_L \longrightarrow Q_{Y/X} \otimes_{B_L} P_L \rightarrow 0$$

$$0 \rightarrow K \rightarrow \Omega_{X/k} \otimes_{\mathcal{O}_X} P_L \xrightarrow{\varphi} \Omega_{Y/k} \otimes_{\mathcal{O}_Y} P_L \rightarrow \Omega_{Y/X} \otimes_{\mathcal{O}_Y} P_L \rightarrow 0$$

$$(9.4)$$

where $\overline{\varphi}$ is induced by $\overline{\psi}$.

Since $Q_{Y/X} = \Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L$ is torsion (because β is not contained in the exceptional locus), we see that $Q_{Y/X} \otimes_{B_L} P_L = 0$. Moreover, $K = \mathrm{Tor}_1^{B_L}(Q_{Y/X}, P_L) = Q_{Y/X}$. Therefore $\dim_L(K) = \dim_L(Q_{Y/X}) = \mathrm{ord}_\beta(\mathrm{Jac}_f)$, and the result follows.

Alternatively, we can check directly that $\dim_L(K) = \mathrm{ord}_\beta(\mathrm{Jac}_f)$. To do this, note that, after appropriate choices of bases, the map $\overline{\psi}$ can be represented by a matrix with entries $t^{e_0}, t^{e_1}, \dots$ along the main diagonal and zeros elsewhere. In the language of Section 6, the e_i can be chosen to be the invariant factors of the module $\Omega_{Y/X}$ with respect to the arc β. In particular, we have $\mathrm{ord}_\beta(\mathrm{Jac}_f) = \sum_{i \geq 0} e_i$. Since β is not contained in the exceptional locus, we have $e_i < \infty$ for all i. The map $\overline{\varphi}$ is represented by the same matrix as $\overline{\psi}$. We get $K = \bigoplus_{i \geq 0} K_i$, where K_i is the kernel of the map $P_L \rightarrow P_L$ given by multiplication by t^{e_i}. An easy computation shows that $\dim_L(K_i) = e_i$, and therefore $\dim_L(K) = \sum_{i \geq 0} e_i = \mathrm{ord}_\beta(\mathrm{Jac}_f)$. $\qquad\square$

THEOREM 9.3
Let X and Y be schemes of finite type over a perfect field. Consider a proper map $f : Y \rightarrow X$ that is birational over a union of components. Let $\beta \in Y_\infty$, and consider $\alpha = f_\infty(\beta) \in X_\infty$. Then

$$\mathrm{emb.dim}(\mathcal{O}_{Y_\infty, \beta}) \leq \mathrm{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) \leq \mathrm{emb.dim}(\mathcal{O}_{Y_\infty, \beta}) + \mathrm{ord}_\beta(\mathrm{Jac}_f).$$

Proof
The proof is almost identical to that of Theorem 9.2. Using the notation of that theorem, the main difference is that $\Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L$ is no longer a free B_L-module. Write $\Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L = F_Y \oplus T_Y$, where F_Y is a free B_L-module and T_Y is torsion, and let $\overline{\psi}$ be the composition of $\psi|_{F_X}$ with the projection to F_Y. Then $\overline{\psi}$ is still injective, and we can consider the module $Q_{Y/X}$ given by the sequence in (9.2). The diagrams in (9.3) and (9.4) remain valid. (Observe that $\Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L$ is a torsion B_L-module, so $\mathrm{coker}\,\varphi = 0$, and the diagram of (9.1) shows that $\mathrm{emb.dim}(\mathcal{O}_{Y_\infty, \beta}) \leq \mathrm{emb.dim}(\mathcal{O}_{X_\infty, \alpha})$.)

The module $Q_{Y/X}$ is a quotient of $\Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L$, and therefore it is also torsion. As in the proof of Theorem 9.2, this implies that:

$$\ker(\varphi) = K = \operatorname{Tor}_1^{B_L}(Q_{Y/X}, P_L) = Q_{Y/X}.$$

Using again that $Q_{Y/X}$ is a quotient of $\Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L$, we see that

$$\dim_L(K) = \dim_L(Q_{Y/X}) \leq \dim_L(\Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L) = \operatorname{ord}_\beta(\operatorname{Jac}_f),$$

and the result follows.   □

COROLLARY 9.4

Let $f : Y \to X$ be a proper birational morphism between schemes of finite type over a perfect field. Then the induced map $f_\infty : Y_\infty \to X_\infty$ induces a bijection

$$\{\beta \in Y_\infty \mid \text{emb.dim}(\mathcal{O}_{Y_\infty,\beta}) < \infty\} \xrightarrow{\ 1-1\ } \{\alpha \in X_\infty \mid \text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) < \infty\}.$$

Proof

Theorem 9.3 implies that Y_∞ has finite embedding dimension at a point β if and only if X_∞ has finite embedding dimension at $f_\infty(\beta)$. To conclude, it suffices to observe that if $\alpha \in X_\infty$ is not in the image of f_∞, then X_∞ has infinite embedding dimension at α. Indeed, by the valuative criterion of properness, α must be fully contained in the indeterminacy locus of $f^{-1} : X \dashrightarrow Y$. Since f is a birational map, the indeterminacy locus of f^{-1} has dimension strictly smaller than the dimension of X at any of its points, and therefore we have $\text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \infty$ by Proposition 8.7.   □

Theorem 9.3 also implies the following useful property.

COROLLARY 9.5

Let X be a scheme of finite type over a perfect field k, and let $\alpha \in X_\infty$ be an arc such that $\alpha \notin (\operatorname{Sing} X)_\infty$. Let $Y \subset X$ be the irreducible component of X such that $\alpha \in Y_\infty$. Then

$$\text{emb.dim}(\mathcal{O}_{Y_\infty,\alpha}) = \text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}).$$

Proof

The first remark, which is implicit in the statement, is that for every $\alpha \in X_\infty$ there is always an irreducible component Y of X such that $\alpha \in Y_\infty$, and if $\alpha \notin (\operatorname{Sing} X)_\infty$, then such component is unique.

If $f : Y \to X$ is the natural inclusion, then $\operatorname{Jac}_f = \mathcal{O}_Y$ and hence the assertion follows directly from Theorem 9.3.   □

Corollary 9.5 would follow immediately if we knew that the local rings $\mathcal{O}_{X_\infty,\alpha}$ and $\mathcal{O}_{Y_\infty,\alpha}$ are isomorphic. This is, however, not true in general. While the reductions

of these rings can be shown to be isomorphic, it is not always the case that the rings themselves are isomorphic. An example where this fails is given next.

Example 9.6

Let $X = \{xy = 0\} \subset \mathbb{A}^2$ be the union of two lines and let $Y = \{y = 0\} \subset \mathbb{A}^2$ be one of the components, and consider the arc $\alpha = (-t, 0) \in X_\infty$. Let $\Gamma = \mathbb{Z} \oplus \epsilon \mathbb{Z}$ with the lexicographic order, so that, for instance,

$$\cdots < 0 < \epsilon < 2\epsilon < \cdots < 1 - 2\epsilon < 1 - \epsilon < 1 < 1 + \epsilon < \cdots,$$

and let $R \subset k(r, s)$ be the rank-2 valuation ring with value group Γ associated to the monomial valuation v defined by $v(r) = \epsilon$ and $v(s) = 1$. The map $k[x, y]/(xy) \to (R/(rs))[[t]]$ defined by

$$x \mapsto r - t, \qquad y \mapsto s + \frac{s}{r}t + \frac{s}{r^2}t^2 + \frac{s}{r^3}t^3 + \cdots$$

does not factor through the quotient $k[x, y]/(xy) \to k[x]$, and this shows that the corresponding map $\mathcal{O}_{X_\infty, \alpha} \to R/(rs)$ does not factor through the quotient $\mathcal{O}_{X_\infty, \alpha} \to \mathcal{O}_{Y_\infty, \alpha}$.

Remark 9.7

If Y is a union of components of a nonreduced scheme X, the inclusion $f: Y \to X$ is not necessarily a birational map over a union of components. In particular, Theorems 9.2 and 9.3 may not apply. Consider for instance the case where X is nowhere reduced and $Y = X_{\mathrm{red}}$. If f is the inclusion, then $\mathrm{Jac}_f = \mathcal{O}_Y$, and hence $\mathrm{ord}_\alpha(\mathrm{Jac}_f) = 0$ for every $\alpha \in Y_\infty$. As we will see in the next section, we can always find an arc $\alpha \in Y_\infty$ with $\mathrm{emb.dim}(\mathcal{O}_{Y_\infty, \alpha}) < \infty$. However, since $\mathrm{Sing}\, X = Y$, for any α we have $\mathrm{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) = \infty$ by Proposition 8.7.

10. Constructible and stable points

Throughout this section, let k be a perfect field and let X be a scheme of finite type over k. The goal of this section is to characterize local rings $\mathcal{O}_{X_\infty, \alpha}$ of finite embedding dimension. We start by recalling the notion of constructibility.

Let Z be an arbitrary scheme. The definition of constructible subset of Z is here intended in the sense of [22] (see also [42, Tag 005G]). That is, a subset of Z is *constructible* if and only if it is a finite union of finite intersections of retrocompact open sets and their complements, where a subset $W \subset Z$ is said to be *retrocompact* if, for every quasicompact open set $U \subset Z$, the intersection $W \cap U$ is quasicompact. A constructible set is *irreducible* if it contains a unique *generic point*, namely, a point whose closure in Z contains the set. With small abuse of terminology, we say that a point $z \in Z$ is *constructible* if z is the generic point of an irreducible constructible

subset of Z. Note that the fact that a point $z \in Z$ is constructible in this sense does not mean necessarily that the 1-point subset $\{z\}$, or its closure, are constructible subsets of Z.

The above definition gives a notion of what it means for a point $\alpha \in X_\infty$ to be a constructible point. It is a general fact that a subset of X_∞ is constructible if and only if it is the (reduced) inverse image of a constructible subset of X_n for some finite n (see [23, Théorème 8.3.11]).

LEMMA 10.1
Let X be a scheme of finite type and let $\alpha \in X_\infty$ be an arc such that $\alpha \notin (\operatorname{Sing} X)_\infty$. Let $Y \subset X$ be the irreducible component of X such that $\alpha \in Y_\infty$. Then α is a constructible point of X_∞ if and only if it is a constructible point of Y_∞.

Proof
Suppose first that α is a constructible point of X_∞. Then α is the generic point of an irreducible constructible subset $C \subset X_\infty$. Note that $C \cap Y_\infty$ is a constructible subset of Y_∞, and since $\alpha \in Y_\infty$ and Y_∞ is closed in X_∞, we actually have $C \subset Y_\infty$. It follows that α is a constructible point of Y_∞.

Conversely, suppose that α is a constructible point of Y_∞, and let $D \subset Y_\infty$ be an irreducible constructible subset with generic point α. We can find an integer n such that $D = (\pi_n|_{Y_\infty})^{-1}(D_n)$, where D_n is a constructible subset of Y_n. Since X_n is a scheme of finite type and Y_n is a closed subscheme of X_n, the set D_n is also constructible in X_n. Since $\alpha \notin (\operatorname{Sing} X)_\infty$, we can find a closed subscheme $Z \subset X$ such that $X = Y \cup Z$ and $\alpha \notin Z_\infty$. This means that α has finite order of contact with Z. If m is an integer larger than this order, then $\alpha \notin \pi_m^{-1}(Z_m)$. Then $\pi_n^{-1}(D_n) \setminus \pi_m^{-1}(Z_m)$ is a constructible subset of X_∞ with generic point α, and hence α is a constructible point of X_∞. $\square$

When X is a variety, constructible subsets of X_∞ are called *weakly stable semialgebraic sets* in [11] and *cylinders* in [8], [14], [16], [24], among other places. Related notions are those of *generically stable set* introduced in [38] and *quasicylinder* introduced in [8]. The more restrictive notion of *stable semialgebraic set* was introduced in [11]. We recall this notion next.

Following the terminology of [11] (see also [16]), a morphism $g \colon V' \to V$ of schemes of finite type is said to induce a *piecewise trivial fibration* $W' \to W$ with fiber F, where $W' \subset V'$ and $W \subset V$ are constructible subsets and F is a reduced scheme, if there is a decomposition $W = T_1 \sqcup \cdots \sqcup T_r$, with all T_i locally closed subsets of W such that each $W' \cap g^{-1}(T_i)$ is locally closed in V' and, with the reduced scheme structure, it is isomorphic to $T_i \times F$.

Let X be a variety. A subset $W \subset X_\infty$ is said to be a *stable semialgebraic subset* if, for all $n \gg 1$, the truncation $\pi_{n+1}(X_\infty) \to \pi_n(X_\infty)$ induces a piecewise trivial fibration over $\pi_n(W)$ with fiber $\mathbb{A}^{\dim(X)}$. The term *stable point* was coined in [39] to denote the generic point of an irreducible stable semialgebraic subset of X_∞. As explained in [11, (2.7)], a simple consequence of [11, Lemma 4.1] is that a point $\alpha \in X_\infty$ is stable if and only if it is constructible and is not contained in $(\operatorname{Sing} X)_\infty$. Note that [11, Lemma 4.1] is stated in characteristic zero, but the proof works over an arbitrary perfect field (see [16, Proposition 4.1] where the property is proved over algebraically closed fields of arbitrary characteristic).

Remark 10.2

The formulation of [39, 3.1(i)(c)] is equivalent to asking that the point is the generic point of a generically stable subset of X_∞ as defined in [38]. It seems that the condition that the generic point of a generically stable subset be not contained in $(\operatorname{Sing} X)_\infty$ was overlooked in [38], [39]. An easy and well-known computation of the fiber over the 1-jet $(t, 0, 0)$ in the arc space of the Whitney umbrella $X = \{xy^2 = z^2\} \subset \mathbb{A}^3$ shows that such condition is necessary.

Remark 10.3

Let X be a variety and let $\alpha \in X_\infty$ be a stable point. For every $n \in \mathbb{N}$, let $\alpha_n = \pi_n(\alpha)$ be the truncation of α and L_n its residue field. Then it follows immediately from the definition that, for all $n \gg 1$, the field extension $L_{n+1} \subset L_n$ is purely transcendental of degree $\dim(X)$ (cf. [39, Section 3.1(i)(b)]).

Remark 10.4

The reason for introducing the term *constructible point* in the context of arc spaces is that it is not completely clear what the definition of stable point of X_∞ should be if X is a nonreduced scheme. The point is that the difference between weakly stable and stable when X is a variety can be characterized in two equivalent ways: either by imposing the condition that α is not contained in $(\operatorname{Sing} X)_\infty$ or by requiring the piecewise trivial fibration condition with fiber $\mathbb{A}^{\dim(X)}$. If X is not generically reduced, these two conditions are no longer equivalent.

Implicit in the works on motivic integration is the definition of codimension of a constructible subset of the arc space of a smooth variety. This was formalized and extended to singular varieties in [8], [9], [14]. Let X be a variety and let $\alpha \in X_\infty$, and for any $n \in \mathbb{N}$ let $\alpha_n = \pi_n(\alpha)$. The *jet codimension* of α in X_∞ is defined to be

$$\operatorname{jet.codim}(\alpha, X_\infty) := \lim_{n \to \infty} \big((n+1)\dim(X) - \dim(\alpha_n) \big).$$

The fact that the limit exists is an easy application of [11, Lemma 4.1] (e.g., see [9, Lemma 4.13]). Note also that jet.codim$(\alpha, X_\infty) \geq 0$ for every $\alpha \in X_\infty$ (cf. Lemma 8.6).

The next property, which is well-known to experts, formalizes the relationship between stable points and jet codimension.

PROPOSITION 10.5
Let X be a variety and let $\alpha \in X_\infty$. Then α is a stable point if and only if jet.codim$(\alpha, X_\infty) < \infty$.

Proof
If $\alpha \notin (\operatorname{Sing} X)_\infty$ then this follows easily from [11, Lemma 4.1]. If $\alpha \in (\operatorname{Sing} X)_\infty$, then, since the ground field is perfect, the closure $Z \subset X$ of the generic point $\alpha(\eta)$ of α has dimension $\dim(Z) < \dim(X)$ and the same argument applied to Z implies that jet.codim$(\alpha, X_\infty) = \infty$.   □

The following more precise version of Lemma 8.4 holds on varieties.

LEMMA 10.6
Let X be a variety, let $\alpha \in X_\infty$ be any point, and let d be the Betti number of $\Omega_{X/k}$ with respect to α. Then

$$\operatorname{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \lim_{n \to \infty} \big((n+1)d - \dim(\alpha_n)\big).$$

Proof
Since the right-hand side of the equation is bounded below by the jet codimension, it follows by Lemmas 8.4, 8.6 and Proposition 10.5 that both terms of the equation are infinite unless α is a stable point. If α is a stable point, then Remark 10.3 implies that, for $m \geq n \gg 1$, the field extensions $L_n \subset L_m$ are separable, and therefore equality holds by Remark 8.8.   □

THEOREM 10.7
Let X be a variety over a perfect field k. Then for every $\alpha \in X_\infty$ we have

$$\operatorname{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \operatorname{jet.codim}(\alpha, X_\infty).$$

Proof
If $\alpha \in (\operatorname{Sing} X)_\infty$, then both sides of the equation are infinite by Propositions 8.7 and 10.5. Assume then that $\alpha \notin (\operatorname{Sing} X)_\infty$. By Lemma 8.4 and the fact that $d(\alpha, \Omega_{X/k}) = \dim(X)$, we have

$$\text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) \geq \lim_{n\to\infty} \big((n+1)\dim(X) - \dim(\alpha_n)\big)$$

$$= \text{jet.codim}(\alpha, X_\infty).$$

If α is stable then the inequality in the first step of this formula is an equality by Lemma 10.6, and hence, combining the formula with Proposition 10.5, we get

$$\text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \text{jet.codim}(\alpha, X_\infty) < \infty.$$

If α is not stable, then we have $\text{jet.codim}(\alpha, X_\infty) = \infty$ by Proposition 10.5, and we conclude that $\text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \text{jet.codim}(\alpha, X_\infty) = \infty$. $\qquad\square$

We obtain the following characterization of local rings of finite embedding dimension.

THEOREM 10.8
Let X be a scheme of finite type over a perfect field. For every $\alpha \in X_\infty$, we have

$$\text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) < \infty$$

if and only if α is a constructible point and is not contained in $(\text{Sing } X)_\infty$.

Proof
If $\alpha \in (\text{Sing } X)_\infty$, then $\text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \infty$ by Proposition 8.7.

Assume then that $\alpha \notin (\text{Sing } X)_\infty$. This implies that X is reduced and irreducible at the generic point $\xi = \alpha(\eta)$. By Corollary 9.5 and Lemma 10.1, we can replace X with its irreducible component containing ξ and thus assume that it is a variety. Then Theorem 10.7 gives us $\text{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \text{jet.codim}(\alpha)$, and we can conclude from the fact that $\text{jet.codim}(\alpha) < \infty$ if and only if α is stable by Proposition 10.5. $\qquad\square$

Remark 10.9
By [39, Theorem 2.9], if X is a variety defined over a perfect field of positive characteristic, then X_∞ has finitely many irreducible components only one of which is not contained in $(\text{Sing } X)_\infty$. For an example where X_∞ has more than one component given by the p-fold Whitney umbrella $X = \{xy^p = z^p\} \subset \mathbb{A}^3$ in characteristic p, see [4, Example 8.1]. Theorem 10.8 implies that, if α is the generic point of an irreducible component of X_∞ that is contained in $(\text{Sing } X)_\infty$, then $\mathcal{O}_{X_\infty,\alpha}$, which is a 0-dimensional ring, has infinite embedding dimension.

COROLLARY 10.10
Let X be a variety over a perfect field. For any $\alpha \in X_\infty$ and $n \in \mathbb{N}$, let $\alpha_n = \pi_n(\alpha)$

be the truncation of α. If $\mathrm{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \infty$, *then* $\mathrm{emb.dim}(\mathcal{O}_{X_n,\alpha_n})$ *becomes arbitrarily large as n increases. Otherwise, we have*

$$\mathrm{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \mathrm{emb.dim}(\mathcal{O}_{X_n,\alpha_n}) - \mathrm{ord}_\alpha(\mathrm{Jac}_X)$$

for all sufficiently large integers n.

Proof
For short, let $d_n := d(\alpha_n, \Omega_{X/k})$ be the Betti number. If $\mathrm{emb.dim}(\mathcal{O}_{X_\infty,\alpha}) = \infty$, then $\mathrm{jet.codim}(\alpha, X_\infty) = \infty$ by Theorem 10.7, and hence, since $d_n \geq \dim(X)$ for all n and the sequence of numbers $\mathrm{ord}_\alpha(J_{d_n})$ stabilizes for n large enough, $\mathrm{emb.dim}(\mathcal{O}_{X_n,\alpha_n})$ goes to ∞ as $n \to \infty$ by Lemma 8.1. Otherwise, α is a stable point and $d_n = \dim(X)$ for $n \gg 1$ by Theorem 10.8, and hence the corollary follows by Lemmas 8.1 and 10.6. □

COROLLARY 10.11
Let X be a variety over a perfect field and let $f : Y \to X$ be a resolution of singularities. Then for every $\beta \in Y_\infty$, letting $\alpha = f_\infty(\beta)$, we have

$$\mathrm{jet.codim}(\alpha, X_\infty) = \mathrm{jet.codim}(\beta, Y_\infty) + \mathrm{ord}_\beta(\mathrm{Jac}_f).$$

Proof
The proof follows from Theorem 9.2 and 10.7. □

One of the nice features of local rings of finite embedding dimension comes from the following elementary fact.

LEMMA 10.12
For any scheme Z over a field, the completion $\widehat{\mathcal{O}_{Z,z}}$ of the local ring of Z at a point z is Noetherian if and only if $\mathrm{emb.dim}(\mathcal{O}_{Z,z}) < \infty$.

Proof
This is in fact a general result about completions of local rings. Let $(\widehat{R}, \widehat{\mathfrak{m}})$ be the $\mathfrak{m}$-adic completion of a local ring $(R, \mathfrak{m})$. If $\mathfrak{m}/\mathfrak{m}^2$ is finite-dimensional, then $\widehat{\mathfrak{m}}$ is finitely generated by [42, Tag 0315], and this implies that $\widehat{R}$ is Noetherian. The converse follows by the fact that since $\widehat{\mathfrak{m}}^2 \subset \widehat{\mathfrak{m}^2}$, there is always a surjection $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \to \mathfrak{m}/\mathfrak{m}^2$. □

The following property is an immediate consequence of Theorem 10.8.

COROLLARY 10.13

Let X be a reduced scheme of finite type over a perfect field. The completion $\widehat{\mathscr{O}_{X_\infty,\alpha}}$ of the local ring at a point $\alpha \in X_\infty$ is Noetherian if and only if α is a constructible point and is not contained in $(\operatorname{Sing} X)_\infty$.

Proof

The proof follows from Theorem 10.8 and Lemma 10.12.   □

When X is a variety, the fact that the completion of the local ring at a stable point $\alpha \in X_\infty$ is Noetherian is a result of Reguera. It follows from [38, Corollary 4.6], which proves that $\mathscr{O}_{\widehat{(X_\infty)_{\mathrm{red}}},\alpha}$ is Noetherian (cf. [34, Section 2.3, (vii)]), and [39, Theorem 3.13], which proves that there is an isomorphism $\widehat{\mathscr{O}_{X_\infty,\alpha}} \simeq \mathscr{O}_{\widehat{(X_\infty)_{\mathrm{red}}},\alpha}$. Notice that this last result of Reguera is stated in characteristic zero, but the proof extends to all perfect fields using Hasse–Schmidt derivations.

The *curve selection lemma* [38, Corollary 4.8] easily follows from *Cohen's structure theorem* once one knows that these rings are Noetherian. It is a powerful statement that allows the study of certain containments among sets in the arc space via the use of arcs in the arc space. All the current proofs solving the Nash problem in dimension 2 (see [6], [18]) use the curve selection lemma in an essential way.

We conclude with the following property which was obtained by different methods in [39, Proposition 4.1]. A more general statement in characteristic zero which applies to maps that are not necessarily birational is given in [39, Proposition 4.5].

COROLLARY 10.14

Let $f : Y \to X$ be a proper birational morphism between schemes of finite type over a perfect field. Then the induced map $f_\infty : Y_\infty \to X_\infty$ induces a bijection from the set of constructible points of Y_∞ that are not contained in $(\operatorname{Sing} Y)_\infty$ and the set of constructible points of X_∞ that are not contained in $(\operatorname{Sing} X)_\infty$. In particular, if X and Y are varieties, then f_∞ induces a bijection

$$\{\text{stable points } \beta \in Y_\infty\} \xrightarrow{\ 1-1\ } \{\text{stable points } \alpha \in X_\infty\}.$$

Proof

The proof follows from Corollary 9.4 and 10.8.   □

Notice, by contrast, that the image $f_\infty(C)$ of a constructible set $C \subset Y_\infty$ needs not be constructible in X_∞. This is shown in the next example.

Example 10.15

Let $f : Y \to X$ be the blowup of a smooth closed point $x \in X$ of a variety of dimen-

sion at least 2, let $E \subset Y$ be the exceptional divisor, and let $y \in E$ be a closed point. The set $C \subset Y_\infty$ of arcs with positive order of contact with E at points in $E \setminus \{y\}$ is constructible, but its image $f_\infty(C) \subset X_\infty$ is not constructible, since it is equal to $W \setminus \bigcup_{i \geq 1} Z_i$, where W is the set of arcs through x and Z_i is the set of arcs with order i at x and where the principal tangent direction is equal to y.

11. Maximal divisorial arcs

In this section, we study arcs that are naturally associated with divisorial valuations. Let X be a reduced scheme of finite type over a perfect field k.

Definition 11.1

A *valuation* on X is intended to be a k-trivial valuation of the function field of one of the irreducible components of X with center in X. A *divisorial valuation* on X is a valuation v of the form $v = q \operatorname{ord}_E$ where q is a positive integer and E is a prime divisor on a normal scheme Y with a morphism $f : Y \to X$ that is birational over a union of irreducible components of X. For a divisorial valuation v, the number $\widehat{k}_v(X) := v(\operatorname{Jac}_f)$ depends only on v (not on the particular map f), and is called the *Mather discrepancy* of v over X. When $v = \operatorname{ord}_E$ (so $q = 1$), we write $\widehat{k}_E(X)$.

If we denote by $\mathcal{K}_X$ the sheaf of rational functions of X in the sense of [29] (i.e., $\mathcal{K}_X = \bigoplus_\eta \mathcal{O}_{X,\eta}$, where η ranges among the generic points of the irreducible components of X), then a valuation of X can be thought of as a function $v : \mathcal{K}_X \to (-\infty, \infty]$ which restricts to a Krull valuation on one of the summands $\mathcal{O}_{X,\eta}$ and is constant equal to ∞ on the other summands. If $v = q \operatorname{ord}_E$ is a divisorial valuation on X, then E is a divisor on one of the irreducible components of Y, and the component of X dominated by it corresponds to the summand of $\mathcal{K}_X$ where the valuation is nontrivial.

Definition 11.2

A point $\alpha \in X_\infty$ is a *maximal divisorial arc* if $\operatorname{ord}_\alpha$ extends to a divisorial valuation on X and α is maximal among all points $\gamma \in X_\infty$ with $\operatorname{ord}_\gamma = \operatorname{ord}_\alpha$ (that is, α is not the specialization of any other such point γ).

In general, for an arc $\alpha \in X_\infty$, the function $\operatorname{ord}_\alpha$ is only defined on $\mathcal{O}_{X,\alpha(0)}$. If α is a maximal divisorial arc, then we write $\operatorname{ord}_\alpha = q \operatorname{ord}_E$ and we think of it as a function on $\mathcal{K}_X$. Note that for other arcs β (for instance, if β is contained in Y_∞ for a smaller dimensional scheme $Y \subset X$) there may not be a natural way to extend ord_β to $\mathcal{K}_X$.

Let $f : Y \to X$ be a proper morphism from a normal scheme Y that is birational over a union of components. Let $E \subset Y$ be a prime divisor, and let $E^\circ \subset E$ be the open set where both Y and E are smooth and none of the other components of the exceptional locus of f intersect E. For any positive integer q, consider the *contact locus*

$$\mathrm{Cont}^{\geq q}(E^\circ, Y) \subset Y_\infty,$$

which is defined to be the set of arcs in Y with order of contact at least q with E at a point in E°. Since Y is smooth along E°, the truncations $Y_m \to Y_n$ are affine bundles over an open set containing E°, and this implies that $\mathrm{Cont}^{\geq q}(E^\circ, Y)$ is irreducible.

The following property is well known to experts; we include a proof for the convenience of the reader.

LEMMA 11.3
With the above notation, the image under $f_\infty : Y_\infty \to X_\infty$ of the generic point of $\mathrm{Cont}^{\geq q}(E^\circ, Y)$ is a maximal divisorial arc on X, and any such arc arises in this way.

Proof
Let β be the generic point of $\mathrm{Cont}^{\geq q}(E^\circ, Y)$ and $\alpha = f_\infty(\beta)$. It is elementary to see that $\mathrm{ord}_\beta = q \, \mathrm{ord}_E$. By the definition of f_∞, we have $\mathrm{ord}_\alpha = \mathrm{ord}_\beta$, and therefore $\mathrm{ord}_\alpha = q \, \mathrm{ord}_E$.

We may assume without loss of generality that f is dominant (this is not essential, but it makes the wording of the proof more clear). If $\gamma \in X$ is any arc with $\mathrm{ord}_\gamma = \mathrm{ord}_\alpha$, then γ cannot be fully contained in the indeterminacy locus of f^{-1}, and therefore it lifts to an arc $\widetilde{\gamma}$ on Y by the valuative criterion of properness. Since $\mathrm{ord}_{\widetilde{\gamma}} = q \, \mathrm{ord}_E$, we see that $\widetilde{\gamma}$ must dominate the generic point of E and hence lie in $\mathrm{Cont}^{\geq q}(E^\circ, Y)$. It follows that γ is a specialization of α, and therefore α is a maximal divisorial arc. This argument also shows that any maximal divisorial arc arises in this way. $\square$

THEOREM 11.4
Let X be a reduced scheme of finite type over a perfect field. For every divisorial valuation $q \, \mathrm{ord}_E$ on X there exists a unique maximal divisorial arc $\alpha \in X_\infty$ with $\mathrm{ord}_\alpha = q \, \mathrm{ord}_E$. Moreover,

$$\mathrm{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) = q\big(\widehat{k}_E(X) + 1\big).$$

Proof
The first assertion is well known and can be viewed as a direct consequence of Theorem 11.3. The formula for the embedding dimension follows from Theorem 9.2 after

we notice that, if β is the generic point of $\mathrm{Cont}^{\geq q}(E^{\circ}, Y)$, then $\mathrm{emb.dim}(\mathcal{O}_{Y_{\infty}, \beta}) = q$, which is an easy computation, given that Y is smooth. $\square$

From Theorem 10.8 we recover the following fact about maximal divisorial arcs (due to [14] when X is a smooth variety, and due to [8], [39] for arbitrary varieties).

COROLLARY 11.5
Let X be a reduced scheme of finite type. Then every maximal divisorial arc $\alpha \in X_{\infty}$ is a constructible point and is not contained in $(\mathrm{Sing}\, X)_{\infty}$.

Proof
By Theorem 11.4, the local ring $\mathcal{O}_{X_{\infty}, \alpha}$ has finite embedding dimension, and hence α is a constructible point not contained in $(\mathrm{Sing}\, X)_{\infty}$ by Theorem 10.8. $\square$

Since by Theorem 10.7, if X is a variety then we have $\mathrm{jet.codim}(\alpha) = \mathrm{emb.dim}(\mathcal{O}_{X_{\infty}, \alpha})$, we obtain the next corollary, which recovers the formula in [8, Theorem 3.8].

COROLLARY 11.6
With the same assumptions as in Theorem 11.4, if X is a variety, then we have

$$\mathrm{jet.codim}(\alpha, X_{\infty}) = q\big(\widehat{k}_E(X) + 1\big).$$

The following related result has been recently proved, by different methods, by Mourtada and by Reguera.

THEOREM 11.7 ([34, Theorem 3.4], see also [40])
Let X be a variety defined over a field of characteristic zero and let $\alpha \in X_{\infty}$ be a maximal divisorial point corresponding to a valuation $q\,\mathrm{ord}_E$. Then

$$\mathrm{emb.dim}(\widehat{\mathcal{O}_{X_{\infty}, \alpha}}) = \mathrm{emb.dim}(\mathcal{O}_{(X_{\infty})_{\mathrm{red}}, \alpha}) = q\big(\widehat{k}_E(X) + 1\big).$$

Remark 11.8
The proof of Theorem 11.4 does not use the fact that maximal divisorial arcs are stable points, which is here deduced as a corollary (see Corollary 11.5). Granting this well-known fact from the start, if X is a variety over a field of characteristic zero, then one can also deduce Theorem 11.4 from Theorem 11.7. Indeed, under these assumptions, if $\alpha \in X_{\infty}$ is a stable point and we denote by $I \subset \mathcal{O}_{X_{\infty}, \alpha}$ and $\bar{I} \subset \mathcal{O}_{(X_{\infty})_{\mathrm{red}}, \alpha}$ the respective maximal ideals, then using the isomorphism $\widehat{\mathcal{O}_{X_{\infty}, \alpha}} \simeq \widehat{\mathcal{O}_{(X_{\infty})_{\mathrm{red}}, \alpha}}$ proved in [39, Theorem 3.13] and the fact that the $\bar{I}$-adic topology of $\mathcal{O}_{(X_{\infty})_{\mathrm{red}}, \alpha}$ is separated

by [39, Corollary 4.3], one deduces that the natural surjection $I/I^2 \to \bar{I}/\bar{I}^2$ is an isomorphism and hence

$$\text{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) = \text{emb.dim}(\mathcal{O}_{(X_\infty)_{\text{red}}, \alpha}).$$

Theorem 11.4 can be used to control Mather discrepancies. For example, the following result is an immediate consequence of Theorem 8.5.

COROLLARY 11.9
Let X be a reduced and equidimensional scheme of finite type over a perfect field, and consider a prime divisor E over X whose center in X is a closed point. Then

$$\widehat{k}_E(X) + 1 \geq \dim(X).$$

In fact, using Lemma 8.3 (with $n = 0$ and $n = 1$) and Remark 8.8, it is not hard to see that, if equality holds in this formula, then the valuation ord_E has center of codimension 1 in the normalized blowup of the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{X,x}$ at x, and $\text{ord}_E(\mathfrak{m}) = 1$.

These facts should be compared with the following result of Ishii. In the statement of the theorem, $\widehat{\text{mld}}_x(X)$ denotes the *minimal Mather log discrepancy* of X at the point x, which is defined as the infimum of the Mather log discrepancies $\widehat{k}_E(X) + 1$ as E ranges among all divisors E over X with center x.

THEOREM 11.10 ([25, Theorem 1.1])
Let X be a variety over a perfect field, and let $x \in X$ be a closed point. Then

$$\widehat{\text{mld}}_x(X) \geq \dim(X)$$

and equality holds if and only if the normalized blowup of the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{X,x}$ at x extracts a divisor E over X such that $\text{ord}_E(\mathfrak{m}) = 1$.

Mather log discrepancies are closely related to the usual log discrepancies, which are defined on $\mathbb{Q}$-Gorenstein varieties. Minimal log discrepancies are conjectured to be bounded above by the dimension of the variety and to characterize smooth points (see [41]). The preceding result of Ishii shows the different behavior of minimal Mather log discrepancies, and it has useful applications in connection to Shokurov's conjecture and the study of isolated singularities with simple links (see [10]).

Corollary 11.9 immediately implies the first statement of Theorem 11.10. Alternatively, the full result can also be obtained by analyzing the behavior of Mather discrepancies under general linear projections, in the spirit of [9, Proposition 2.4]; the argument is essentially contained in the proof of [3, Proposition 4.6].

References

[1]   B. BHATT, *Algebraization and Tannaka duality*, Camb. J. Math. **4** (2016), no. 4, 403–461. MR 3572635. DOI 10.4310/CJM.2016.v4.n4.a1. *(361, 362)*

[2]   V. BATYREV, *Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs*, J. Eur. Math. Soc. (JEMS) **1** (1999), no 1, 5–33. MR 1677693. DOI 10.1007/PL00011158. *(356)*

[3]   T. DE FERNEX, *Birational rigidity of singular Fano hypersurfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **17** (2017), no. 3, 911–929. MR 3726830. *(392)*

[4]   ———, "The space of arcs of an algebraic variety" in *Algebraic Geometry (Salt Lake City, 2015)*, Proc. Sympos. Pure Math. **97**, Amer. Math. Soc., Providence, 2018, 169–197. MR 3821149. *(386)*

[5]   T. DE FERNEX and R. DOCAMPO, *Jacobian discrepancies and rational singularities*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 1, 165–199. MR 3141731. DOI 10.4171/JEMS/430. *(355, 356, 372)*

[6]   ———, *Terminal valuations and the Nash problem*, Invent. Math. **203** (2016), no. 1, 303–331. MR 3437873. DOI 10.1007/s00222-015-0597-5. *(357, 388)*

[7]   ———, *Nash blow-ups of jet schemes*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 6, 2577–2588. DOI 10.5802/aif.3302. *(358)*

[8]   T. DE FERNEX, L. EIN, and S. ISHII, *Divisorial valuations via arcs*, Publ. Res. Inst. Math. Sci. **44** (2008), no.2, 425–448. MR 2426354. DOI 10.2977/prims/1210167333. *(355, 357, 383, 384, 391)*

[9]   T. DE FERNEX and M. MUSTAŢĂ, *The volume of a set of arcs on a variety*, Rev. Roumaine Math. Pures Appl. **60** (2015), no. 3, 375–401. MR 3436273. *(355, 384, 385, 392)*

[10]   T. DE FERNEX and Y.-C. TU, *Towards a link theoretic characterization of smoothness*,
         Math. Res. Lett. **24** (2017), no. 5, 1239–1262. MR 3747166.
         DOI 10.4310/MRL.2017.v24.n5.a1. *(392)*

[11]   J. DENEF and F. LOESER, *Germs of arcs on singular algebraic varieties and motivic
         integration*, Invent. Math. **135** (1999), no. 1, 201–232. MR 1664700.
         DOI 10.1007/s002220050284. *(353, 356, 383, 384, 385)*

[12]   V. DRINFELD, *On the Grinberg–Kazhdan formal arc theorem*, preprint,
         arXiv:math/0203263v1 [math.AG]. *(357)*

[13]   L. EIN and S. ISHII, "Singularities with respect to Mather–Jacobian discrepancies" in
         *Commutative Algebra and Noncommutative Algebraic Geometry, Vol. II
         (Berkeley, 2012/2013)*, Math. Sci. Res. Inst. Publ. **68**, Cambridge Univ. Press,
         New York, 2015, 125–168. MR 3496863. *(356)*

[14]   L. EIN, R. LAZARSFELD, and M. MUSTAŢĂ, *Contact loci in arc spaces*, Compos. Math.
         140 (2004), no. 5, 1229–1244. MR 2081163.
         DOI 10.1112/S0010437X04000429. *(355, 357, 383, 384, 391)*

[15]   L. EIN and M. MUSTAŢĂ, *Inversion of adjunction for local complete intersection
         varieties*, Amer. J. Math. **126** (2004), no. 6, 1355–1365. MR 2102399. *(356)*

[16]   ———, "Jet schemes and singularities" in *Algebraic Geometry, Part 2 (Seattle 2005)*,
         Proc. Sympos. Pure Math. **80**, Amer. Math. Soc., Providence, 2009, 505–546.
         MR 2483946. DOI 10.1090/pspum/080.2/2483946. *(359, 383, 384)*

[17]   L. EIN, M. MUSTAŢĂ, and T. YASUDA, *Jet schemes, log discrepancies and inversion of
         adjunction*, Invent. Math. **153** (2003), no. 3, 519–535. MR 2000468.
         DOI 10.1007/s00222-003-0298-3. *(356)*

[18]   J. FERNÁNDEZ DE BOBADILLA and M. PEREIRA, *The Nash problem for surfaces*, Ann.
         of Math. (2) **176** (2012), no. 3, 2003–2029. MR 2979864.
         DOI 10.4007/annals.2012.176.3.11. *(357, 388)*

[19]   M. GREENBERG, *Rational points in Henselian discrete valuation rings*, Publ. Math.
         Inst. Hautes Études Sci. **31** (1966), 59–64. MR 0207700. *(353)*

[20]   M. GRINBERG and D. KAZHDAN, *Versal deformations of formal arcs*, Geom. Funct.
         Anal. **10** (2000), no. 3, 543–555. MR 1779611. DOI 10.1007/PL00001628.
         (357)

[21]   A. GROTHENDIECK and J. A. DIEUDONNÉ, *Eléments de géométrie algébrique, I*,
         Grundlehren Math. Wiss. **166**, Springer, Berlin, 1971. MR 3075000. *(358)*

[22]   A. GROTHENDIECK, *Éléments de géométrie algébrique, III: Étude cohomologique des
         faisceaux cohérents, I*, Publ. Math. Inst. Hautes Études Sci. **11** (1961).
         MR 0217085. *(382)*

[23]   ———, *Éléments de géométrie algébrique, IV: Étude locale des schémas et des
         morphismes de schémas, III*, Publ. Math. Inst. Hautes Études Sci. **28** (1966).
         MR 0217086. *(383)*

[24]   S. ISHII, "Maximal divisorial sets in arc spaces" in *Algebraic Geometry in East Asia
         (Hanoi, 2005)*, Adv. Stud. Pure Math. **50**, Math. Soc. Japan, Tokyo, 2008,
         237–249. MR 2409559. DOI 10.2969/aspm/05010237. *(357, 383)*

[25]   ———, *Mather discrepancy and the arc spaces*, Ann. Inst. Fourier (Grenoble) **63**
         (2013), no. 1, 89–111. MR 3089196. DOI 10.5802/aif.2756. *(356, 392)*

[26]   S. ISHII and J. KOLLÁR, *The Nash problem on arc families of singularities*, Duke Math.
       J. **120** (2003), no. 3, 601–620. MR 2030097.
       DOI 10.1215/S0012-7094-03-12034-7. *(359)*

[27]   S. ISHII and A. J. REGUERA, *Singularities with the highest Mather minimal log
       discrepancy*, Math. Z. **275** (2013), no. 3–4, 1255–1274. MR 3127058.
       DOI 10.1007/s00209-013-1181-6. *(356)*

[28]   S. ISHII, A. SANNAI, and K. WATANABE, "Jet schemes of homogeneous hypersurfaces"
       in *Singularities in Geometry and Topology*, IRMA Lect. Math. Theor. Phys. **20**,
       Eur. Math. Soc., Zürich, 2012, 39–49. MR 3074928. DOI 10.4171/118-1/3.
       (356)

[29]   S. L. KLEIMAN, *Misconceptions about K_X*, Enseign. Math. **25** (1979), no. 3–4,
       203–206. MR 0570309. *(389)*

[30]   E. KOLCHIN, *Differential Algebra and Algebraic Groups*, Pure Appl. Math. **54**,
       Academic Press, New York, 1973. MR 0568864. *(353, 376)*

[31]   M. KONTSEVICH, *String cohomology*, lecture at the Université Paris-Sud, Orsay,
       France, December 7, 1995. *(356)*

[32]   E. LOOIJENGA, "Motivic measures" in *Séminaire Bourbaki, Vol. 1999/2000, Exposés
       865-879*, Astérisque **276**, Soc. Math. France, Paris, 2002, 267–297. MR 1886763.
       (356)

[33]   H. MATSUMURA, *Commutative Ring Theory*, 2nd ed., Cambridge Stud. Adv. Math. **8**,
       Cambridge Univ. Press, Cambridge, 1989. MR 1011461. *(373)*

[34]   H. MOURTADA and A. J. REGUERA, *Mather discrepancy as an embedding dimension in
       the space of arcs*, Publ. Res. Inst. Math. Sci. **54** (2018), no. 1, 105–139.
       MR 3749346. DOI 10.4171/PRIMS/54-1-4. *(353, 355, 356, 357, 388, 391, 393)*

[35]   M. MUSTAȚĂ, *Jet schemes of locally complete intersection canonical singularities*,
       with an appendix by D. Eisenbud and E. Frenkel, Invent. Math. **145** (2001), no. 3,
       397–424. MR 1856396. DOI 10.1007/s002220100152. *(356)*

[36]   ———, *Singularities of pairs via jet schemes*, J. Amer. Math. Soc. **15** (2002), no. 3,
       599–615. MR 1896234. DOI 10.1090/S0894-0347-02-00391-0. *(356)*

[37]   J. F. NASH, JR., *Arc structure of singularities*, Duke Math. J. **81** (1995), no. 1, 31–38.
       MR 1381967. DOI 10.1215/S0012-7094-95-08103-4. *(353)*

[38]   A. J. REGUERA, *A curve selection lemma in spaces of arcs and the image of the Nash
       map*, Compos. Math. **142** (2006), no. 1, 119–130. MR 2197405.
       DOI 10.1112/S0010437X05001582. *(353, 356, 357, 383, 384, 388)*

[39]   ———, *Towards the singular locus of the space of arcs*, Amer. J. Math. **131** (2009),
       no. 2, 313–350. MR 2503985. DOI 10.1353/ajm.0.0046. *(353, 356, 357, 358,
       376, 384, 386, 388, 391, 392)*

[40]   ———, *Coordinates at stable points of the space of arcs*, J. Algebra **494** (2018),
       40–76. MR 3723170. DOI 10.1016/j.jalgebra.2017.09.031. *(353, 355, 356, 357,
       391)*

[41]   V. V. SHOKUROV, "Letters of a bi-rationalist, IV: Geometry of log flips" in *Algebraic
       Geometry: A Volume in Memory of Paolo Francia*, de Gruyter, Berlin, 2002,
       313–328. MR 1954071. *(392)*

[42]   THE STACKS PROJECT AUTHORS, *Stacks Project*, http://stacks.math.columbia.edu
          (accessed 14 November 2019). *(382, 387)*

[43]   P. VOJTA, "Jets via Hasse–Schmidt derivations" in *Diophantine Geometry (Pisa,
          2005)*, CRM Series 4, Edizioni della Normale, Pisa, 2007, 335–361.
          MR 2349665. *(359, 360, 361)*

[44]   Z. ZHU, *Jet schemes and singularities of $W_d^r(C)$ loci*, Comm. Algebra **43** (2015), no.
          8, 3134–3159. MR 3354083. DOI 10.1080/00927872.2014.910794. *(356)*

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