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NASH BLOW-UPS OF JET SCHEMES

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Abstract. Given an arbitrary projective birational morphism of varieties, we provide a natural and explicit way of constructing relative compactifications of the maps induced on the main components of the jet schemes. In the case the morphism is the Nash blow-up of a variety, such relative compactifications are shown to be given by the Nash blow-ups of the main components of the jet schemes.

Éclatements de Nash des espaces de jets

Résumé. Étant donné un morphisme birationnel projective de variétés nous fournissons une manière explicite et naturelle de construire des compactifications relatives des applications induites sur les composantes principales des espaces de jets. Dans le cas o le morphisme est l'éclatement de Nash d'une variété, nous montrons que ces compactifications relatives sont données par les éclatements de Nash des composantes principales des espaces de jets.

1. Introduction

The *Nash blow-up* of a variety is defined as the universal projective birational morphism for which the pull-back of the sheaf of differentials admits a locally free quotient of the same rank. The name comes from John Nash, who is generally credited for having promoted the question of whether singularities of algebraic varieties can always be resolved by finitely many iterations of such blow-ups; before him, the question had already been considered by Semple [12]. The property is known to hold for curves of characteristic zero, and to fail in positive characteristics [9]. A variant of this question, where Nash blow-ups are alternated with normalizations, has

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been settled affirmatively for surfaces of characteristic zero by Spivakovsky [13], building on [5]. Higher order Nash blow-ups have been defined and studied by Yasuda [15].

The Nash blow-up can be thought as the universal operation separating multiple limits of tangent spaces, and hence its construction relates to the geometry of the main component of the first jet scheme of the variety. It is however unclear a priori how the Nash blow-up of a variety should relate to the Nash blow-up of such component. Even less obvious is whether there should be a relationship with the Nash blow-ups of the main components of the higher jet schemes of the variety.

The following result shows that these Nash blow-ups are not just related, but in fact they essentially determine each other.

THEOREM 1.1. — *Let X be a variety. For every n, the main component of the n-th jet scheme of the Nash blow-up of X has an open immersion into the Nash blow-up of the main component of the n-th jet scheme of X, and such immersion is compatible with the respective natural map to the n-th jet scheme of X.*

Denoting by $N(X) \to X$ the Nash blow-up of a variety and by $J'_n(X)$ the main component of the n-th jet scheme of X, Theorem 1.1 can be rephrased by saying that the Nash blow-up $N(J'_n(X)) \to J'_n(X)$ gives a relative compactification of the map $J'_n(N(X)) \to J'_n(X)$ induced on n-jets by the Nash blow-up of X. This implies that the Nash blow-up of a variety X can equivalently be characterized as the universal projective birational morphism $Y \to X$ such that, for every n, the pull-back of $\Omega_{J'_n(X)}$ via $J'_n(Y) \to J'_n(X)$ has a locally free quotient of the same rank. The theorem also implies that the Nash blow-up of $J'_n(X)$ induces the Nash blow-up of X under the natural section (the 'zero section') of the projection $J'_n(X) \to X$. It was shown by Ishii [6] that if a variety X is singular then all of its jet schemes are singular, and Theorem 1.1 implies that, if the ground field is algebraically closed of characteristic zero, then in fact the main components of the jet schemes are already singular.

The proof of Theorem 1.1 uses the description of the sheaves of differentials on jet schemes given in [3] in combination with Theorem 1.2 (stated below), which addresses a related question in a more general context.

Suppose that $\mu\colon Y \to X$ is an arbitrary projective birational morphism of varieties. By functoriality, μ induces for every n a morphism on jet schemes $\mu_n\colon J_n(Y) \to J_n(X)$, and hence, by restriction, a birational morphism $\mu'_n\colon J'_n(Y) \to J'_n(X)$ between the main components of the jet schemes. In general, μ'_n is not a projective morphism, and one can ask whether there

are natural ways of constructing relative compactifications of μ'_n. The next theorem provides an answer to this question.

The morphism μ can be described as the blow-up of an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and a way to approach the question is to look for natural ways of constructing an ideal sheaf $\mathfrak{a}_n \subset \mathcal{O}_{J'_n(X)}$ whose blow-up gives a relative compactification of μ'_n. Doing this directly seems hard: while a posteriori we will provide an explicit formula for computing the local generators of such an ideal $\mathfrak{a}_n$ in terms of the generators of $\mathcal{I}$, the formula will show that the complexity of $\mathfrak{a}_n$ grows fast even in simple examples, an indication that looking at ideals might not be the best approach.

Instead, we view μ as the *Nash transformation* $N(\mathcal{F}) \to X$ of a coherent sheaf $\mathcal{F}$, as defined for instance in [10]. In this language, the blow-up of an ideal $\mathcal{I} \subset \mathcal{O}_X$ is the same as the Nash transformation $N(\mathcal{I}) \to X$ of the ideal, and the Nash blow-up of a variety X is defined to be the Nash transformation $N(\Omega_X) \to X$ of the sheaf of differentials of X. In general, the Nash transformation of a coherent sheaf $\mathcal{F}$ of rank r is defined using the Grassmann bundle of locally free quotients of rank r of $\mathcal{F}$, and is a projective birational morphism. Conversely, every projective birational morphism $\mu\colon Y \to X$ can be realized as a Nash transformation of some coherent sheaf $\mathcal{F}$ on X.

THEOREM 1.2. — *Let X be a variety over a field k, and let $\mu\colon N(\mathcal{F}) \to X$ be the Nash transformation of a coherent sheaf $\mathcal{F}$ on X. For every n, let*

$$\begin{array}{ccc} J'_n(X) \times \Delta_n & \xrightarrow{\gamma'_n} & X \\ \Big\downarrow{\scriptstyle \rho'_n} & & \\ J'_n(X) & & \end{array}$$

be the diagram induced by restriction from the universal n-jet of X; here, we denote $\Delta_n = \operatorname{Spec} k[t]/(t^{n+1})$. Define

$$\mathcal{F}'_n := (\rho'_n)_*(\gamma'_n)^*\mathcal{F}.$$

Then the induced map $\mu'_n\colon J'_n(N(\mathcal{F})) \to J'_n(X)$ factors as

$$J'_n(N(\mathcal{F})) \xhookrightarrow{\iota_n} N(\mathcal{F}'_n) \xrightarrow{\nu_n} J'_n(X)$$

where ι_n is an open immersion and ν_n is the Nash transformation of $\mathcal{F}'_n$.

If in this theorem we take $\mathcal{F} = \mathcal{I} \subset \mathcal{O}_X$, an ideal sheaf on X, then $\mathcal{F}'_n$ is not an ideal sheaf. However, the sheaf $\wedge^{(n+1)}\mathcal{F}'_n$, modulo torsion, is isomorphic to an ideal sheaf $\mathfrak{a}_n$, and $N(\mathcal{F}'_n) = N(\mathfrak{a}_n)$. Our approach

enables us to make explicit computations and hence to provide a formula for the generators of $\mathfrak{a}_n$.

A motivation for Theorem 1.2 comes from the Nash problem on families of arcs through the singularities of a variety [8] and, more specifically, from the problem of lifting wedges [7, 11]. In dimension two, the Nash problem has been settled in characteristic zero in [1] but it remains open in positive characteristics. The algebraic proof given in [4] may be adaptable to positive characteristics, provided one can avoid certain wild ramifications that could occur in the proof. A possible approach is to look for suitable deformations of wedges, and this requires working with relative compactifications of the maps $J_n(Y) \to J_n(X)$ where $Y \to X$ is the minimal resolution of the surface. Theorem 1.2 provides a first step in this direction.

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2. Proofs

We work over an arbitrary field k. For every integer $n \geqslant 0$, the *n-th jet scheme* $J_n(X)$ of a scheme X is the scheme representing the functor of points defined by

$$J_n(X)(Z) = X(Z \times \Delta_n)$$

for any scheme Z, where $\Delta_n = \operatorname{Spec} k[t]/(t^{n+1})$. We denote by

$$\begin{array}{ccc} J_n(X) \times \Delta_n & \xrightarrow{\gamma_n} & X \\ \downarrow{\scriptstyle \rho_n} & & \\ J_n(X) & & \end{array}$$

the universal n-jet of X. For generalities about jet schemes, we refer to [14, 2]. If X is a variety, then there exists a unique irreducible component of $J_n(X)$ dominating X, and this component has dimension $(n+1)\dim X$. We shall denote it by $J'_n(X)$ and call it the *main component* of $J_n(X)$.

Given a coherent sheaf $\mathcal{F}$ on a scheme X, and a positive integer r, we denote by $\mathrm{Gr}(\mathcal{F}, r)$ the *Grassmann bundle* over X parameterizing locally

free quotients of $\mathcal{F}$ of rank r, where by the term *quotient* we mean an equivalence class of surjective maps from the same source where two surjections are identified whenever they have the same kernel. This scheme represents the functor of points given by

$$\mathrm{Gr}(\mathcal{F},r)(Z) = \big\{\big(Z \xrightarrow{p} X,\, p^*\mathcal{F} \twoheadrightarrow \mathcal{Q}\big) \mid \mathcal{Q} \text{ locally free sheaf on } Z \text{ of rank } r\big\}$$

for any scheme Z.

Suppose now that X is a variety, and let $\mathcal{F}$ be a coherent sheaf on X of rank r. The *Nash transformation* of $\mathcal{F}$ is defined to be the irreducible component of $\mathrm{Gr}(\mathcal{F},r)$ dominating X, and is denoted by $N(\mathcal{F})$. The natural projection $\mathrm{Gr}(\mathcal{F},r) \to X$ induces the blow-up map $N(\mathcal{F}) \to X$. The *Nash blow-up* $N(X) \to X$ is, by definition, the Nash transformation of the sheaf of Kähler differentials Ω_X.

Proof of Theorem 1.2. — The sheaf $\mathcal{F}'_n$ is the restriction, under the inclusion $J'_n(X) \subset J_n(X)$, of the sheaf

$$\mathcal{F}_n := (\rho_n)_*\gamma_n^*\mathcal{F}.$$

By construction, $J'_n(N(\mathcal{F}))$ is an irreducible component of the jet scheme $J_n(\mathrm{Gr}(\mathcal{F},r))$. Similarly, observing that $\mathcal{F}'_n$ is a sheaf of rank $(n+1)r$ and keeping in mind that $J'_n(X)$ is an irreducible component of $J_n(X)$, we see that $N(\mathcal{F}'_n)$ is an irreducible component of $\mathrm{Gr}(\mathcal{F}_n,(n+1)r)$. We claim that there is a universally injective map

$$i\colon J_n(\mathrm{Gr}(\mathcal{F},r)) \hookrightarrow \mathrm{Gr}(\mathcal{F}_n,(n+1)r),$$

defined over X, which agrees with the natural identification of these schemes over the open set where X is smooth and $\mathcal{F}$ is locally free, and restricts to an open immersion from $J'_n(N(X))$ to $N(J'_n(X))$. Note that the existence of such a map implies the statement of the theorem.

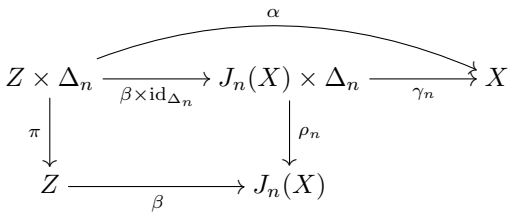
In order to prove this claim, we compare the functors of points of the schemes $J_n(\mathrm{Gr}(\mathcal{F},r))$ and $\mathrm{Gr}(\mathcal{F}_n,(n+1)r)$. For every scheme Z, we have

$$\begin{aligned} J_n(\mathrm{Gr}(\mathcal{F},r))(Z) &= \mathrm{Gr}(\mathcal{F},r)(Z\times\Delta_n) \\ &= \big\{\big(Z\times\Delta_n \xrightarrow{\alpha} X,\, \alpha^*\mathcal{F} \twoheadrightarrow \mathcal{Q}\big) \mid \\ &\qquad \mathcal{Q} \text{ locally free sheaf on } Z\times\Delta_n \text{ of rank } r\big\} \end{aligned}$$

and

$$\mathrm{Gr}(\mathcal{F}_n,(n+1)r)(Z) = \big\{\big(Z \xrightarrow{\beta} J_n(X),\, \beta^*\mathcal{F}_n \twoheadrightarrow \mathcal{R}\big) \mid \mathcal{R} \text{ locally free sheaf on } Z \text{ of rank } (n+1)r\big\}.$$

By the description of $J_n(X)$ via the functor of points, every $\beta\colon Z \to J_n(X)$ corresponds to a unique $\alpha\colon Z \times \Delta_n \to X$, and for any such pair of maps there is a commutative diagram



where π is the projection onto the first factor. Note that taking push-forward along π of a sheaf on $Z \times \Delta_n$ simply means that we are restricting scalars to $\mathcal{O}_Z$ and forgetting the given $\mathcal{O}_{Z\times\Delta_n}$-module structure of the sheaf.

By the definition of $\mathcal{F}_n$ and base-change, which holds in this setting because ρ_n and π are affine, we have

$$\beta^*\mathcal{F}_n = \pi_*\alpha^*\mathcal{F}.$$

Using the identification $J_n(X)(Z) = X(Z \times \Delta_n)$, the above formula yields the following alternative description of the functor of points:

$$\mathrm{Gr}(\mathcal{F}_n, (n+1)r)(Z) = \big\{\big(Z \times \Delta_n \xrightarrow{\alpha} X,\, \pi_*\alpha^*\mathcal{F} \twoheadrightarrow \mathcal{R}\big) \mid \mathcal{R} \text{ locally free sheaf on } Z \text{ of rank } (n+1)r\big\}.$$

For every locally free sheaf $\mathcal{Q}$ on $Z \times \Delta_n$ of rank r, the push-forward $\pi_*\mathcal{Q}$ is a locally free sheaf on Z of rank $(n+1)r$. Taking push-forwards via π is exact, and any two quotients of $\alpha^*\mathcal{F}$ are identified (i.e., they define the same kernel in $\alpha^*\mathcal{F}$) if and only if their push-forwards are identified as quotients of $\pi_*\alpha^*\mathcal{F}$ (i.e., they define the same kernel in $\pi_*\alpha^*\mathcal{F}$). This means that taking push-forwards via π defines a natural injection

$$J_n(\mathrm{Gr}(\mathcal{F}, r))(Z) \hookrightarrow \mathrm{Gr}(\mathcal{F}_n, (n+1)r)(Z).$$

As this holds for every scheme Z, we deduce that there is a naturally defined universally injective morphism

$$i\colon J_n(\mathrm{Gr}(\mathcal{F}, r)) \hookrightarrow \mathrm{Gr}(\mathcal{F}_n, (n+1)r).$$

It is immediate to see that i is defined over X and therefore it agrees with the natural identification of these schemes over the open set where X is smooth and $\mathcal{F}$ is locally free. Furthermore, the restriction of i to $J'_n(N(\mathcal{F}))$ gives a universally injective map $\iota_n\colon J'_n(N(\mathcal{F})) \to N(\mathcal{F}'_n)$. To finish the proof, we need to show that ι_n is a local isomorphism, that is, it

induces isomorphisms on all local rings. To this end, we prove the following property.

LEMMA 2.1. — *Let $(A, \mathfrak{m})$ be a Noetherian local domain over k, and set $U = \operatorname{Spec} A$ and $P = \operatorname{Spec} A/\mathfrak{m}$. Assume that f_0 and g are morphisms as in the diagram*

$$\begin{array}{ccc} P & \xrightarrow{f_0} & J_n(\mathrm{Gr}(\mathcal{F}, r)) \\ \downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle i} \\ U & \xrightarrow[g]{} & \mathrm{Gr}(\mathcal{F}_n, (n+1)r) \end{array}$$

such that the square sub-diagram commutes and the image of g is a dense subset of $N(\mathcal{F}'_n)$. Then there exists a unique morphism f (marked by the dotted arrow in the diagram) making the whole diagram commute.

Proof. — Suppose f_0 and g are given. Let $\pi_0 \colon P \times \Delta_n \to P$ and $\pi \colon U \times \Delta_n \to U$ denote the respective projections to the first components. By the descriptions of the functors of points, we can write

$$f_0 = \big(P \times \Delta_n \xrightarrow{\alpha_0} X,\, \alpha_0^* \mathcal{F} \twoheadrightarrow \mathcal{Q}\big),$$

where $\mathcal{Q}$ is a locally free $A[t]/(t^{n+1})$-module of rank r, and

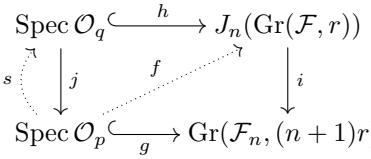
$$g = \big(U \times \Delta_n \xrightarrow{\alpha} X,\, \pi_* \alpha^* \mathcal{F} \twoheadrightarrow \mathcal{R}\big)$$

where $\mathcal{R}$ is a locally free A-module of rank $(n+1)r$. The commutativity of the square sub-diagram in the statement means that α_0 is the restriction of α and $\mathcal{R} \otimes_A A/\mathfrak{m} = (\pi_0)_* \mathcal{Q}$. The fact that the image of g is dense in $N(\mathcal{F}'_n)$ implies that α is dominant, and hence $\pi_* \alpha^* \mathcal{F}$ is a sheaf of rank $(n+1)r$. Since $\mathcal{R}$ is a locally free quotient of the same rank of $\pi_* \alpha^* \mathcal{F}$, the kernel $\mathcal{K}$ of $\pi_* \alpha^* \mathcal{F} \to \mathcal{R}$ is the torsion A-submodule of $\pi_* \alpha^* \mathcal{F}$. Every element of $\sum_{i \geqslant 0} t^i \mathcal{K}$, viewed as an A-submodule of $\pi_* \alpha^* \mathcal{F}$, is torsion, and therefore we have $\sum_{i \geqslant 0} t^i \mathcal{K} = \mathcal{K}$. This shows that $\mathcal{K}$ is an $A[t]/(t^{n+1})$-submodule of $\pi_* \alpha^* \mathcal{F}$ and hence $\mathcal{R}$ is an $A[t]/(t^{n+1})$-module quotient of $\pi_* \alpha^* \mathcal{F}$. This gives the lift f of g as in the diagram, which is clearly unique and makes the diagram commute. □

We apply Lemma 2.1 to the local rings of $N(\mathcal{F}'_n)$ at the points in the image of ι_n. Using the fact that i is injective on the functors of points, we deduce that i induces isomorphisms on the local rings.

To see this last implication, let $\mathcal{O}_q$ denote the local ring of $J'_n(N(\mathcal{F}))$ at a point q, and let $\mathcal{O}_p$ denote the local ring of $N(\mathcal{F}'_n)$ at $p = \iota_n(q)$. Let $g \colon \operatorname{Spec} \mathcal{O}_p \to \mathrm{Gr}(\mathcal{F}_n, (n+1)r)$ and $h \colon \operatorname{Spec} \mathcal{O}_q \to J_n(\mathrm{Gr}(\mathcal{F}, r))$ be the natural maps, and let $j \colon \operatorname{Spec} \mathcal{O}_q \to \operatorname{Spec} \mathcal{O}_p$ be the map induced by ι_n.

We have the following diagram:



Here, the square sub-diagram is commutative, f exists by Lemma 2.1 and hence satisfies

$$i \circ f = g, \tag{2.1}$$

and the universal property of local rings implies that f factors through h, so that we have a morphism s, as in the diagram, satisfying

$$h \circ s = f. \tag{2.2}$$

Using the commutativity of the square sub-diagram and Eq. (2.1), we get

$$i \circ h = g \circ j = i \circ f \circ j.$$

Then, using the fact that i is injective at the level of functors of points and hence is a monomorphism, we deduce that

$$h = f \circ j. \tag{2.3}$$

Now, using Eqs. (2.2) and (2.3), we get

$$h = f \circ j = h \circ s \circ j,$$

and since h is a monomorphism, this implies that $s \circ j$ is the identity of $\operatorname{Spec} \mathcal{O}_q$. Using Eqs. (2.2) and (2.3) in a different order, we get

$$f = h \circ s = f \circ j \circ s.$$

Since g is a monomorphism, it follows by Eq. (2.1) that f is a monomorphism, and this implies that $j \circ s$ is the identity of $\operatorname{Spec} \mathcal{O}_p$. This proves that j is an isomorphism, which completes the proof of the theorem. □

Proof of Theorem 1.1. — By [3, Theorem B], there is an isomorphism

$$\Omega_{J_n(X)} \cong (\rho_n)_* \gamma_n^* \Omega_X,$$

and this implies that

$$N(J_n'(X)) = N\big((\rho_n)_* \gamma_n^* \Omega_X \otimes_{\mathcal{O}_{J_n(X)}} \mathcal{O}_{J_n'(X)}\big) = N\big((\rho_n')_* (\gamma_n')^* \Omega_X\big),$$

where ρ_n' and γ_n' are the restrictions of ρ_n and γ_n to $J_n'(X) \times \Delta$. Therefore Theorem 1.1 reduces to Theorem 1.2 with $\mathcal{F} = \Omega_X$. □

Corollary 2.2. — *For any variety X, the following properties are equivalent:*

(1) *the Nash blow-up* $N(J'_n(X)) \to J'_n(X)$ *is an isomorphism for some* $n \geqslant 0$*;*
(2) *the Nash blow-up* $N(J'_n(X)) \to J'_n(X)$ *is an isomorphism for every* $n \geqslant 0$.

Proof. — By Theorem 1.1, both properties are equivalent to the fact that the Nash blow-up $N(X) \to X$ is an isomorphism. □

In positive characteristics, there are examples of singular varieties whose Nash blow-up is an isomorphism (see [9, Example 1]), and Corollary 2.2 implies that this property, whenever it holds, propagates through all the jet schemes, and conversely.

By contrast, when the ground field is algebraically closed of characteristic zero the Nash blow-up is an isomorphism if and only if the variety is smooth (see [9, Theorem 2]). It is elementary to show that the jet schemes of a smooth variety are smooth, and conversely it was proved in [6] that if X is a singular variety then all its jet schemes $J_n(X)$ are singular. With the above assumptions on the ground field, we deduce the following stronger statement from Corollary 2.2.

COROLLARY 2.3. — *If X is a singular variety defined over an algebraically closed field of characteristic zero, then the main component of $J'_n(X)$ of $J_n(X)$ is singular for every n.*

3. Computational aspects

After viewing a projective birational morphism $\mu\colon Y \to X$ as the Nash transformation of a coherent sheaf $\mathcal{F}$ on a variety X, Theorem 1.2 provides a construction of a relative compactification of the induced map $\mu'_n\colon J'_n(Y) \to J'_n(X)$ by taking the Nash transformation of an explicitly described sheaf $\mathcal{F}'_n$ on $J'_n(X)$. Such transformation is a projective birational morphism, and therefore can also be described as the blow-up of an ideal sheaf $\mathfrak{a}_n$ on $J'_n(X)$. In this section we explain how to compute such ideal.

For simplicity, we assume that $X = \operatorname{Spec} R$ is affine. The following diagram provides the algebraic counterpart of the restriction to $J'_n(X)$ of the universal n-jet:

$$\begin{array}{ccc} R'_n[t]/(t^{n+1}) & \xleftarrow{(\gamma'_n)^\sharp} & R \\ \uparrow{\scriptstyle (\rho'_n)^\sharp} & & \\ R'_n & & \end{array}$$

Here R'_n is a quotient of R_n, the algebra of Hasse–Schmidt differentials of order at most n, $(\rho'_n)^\sharp$ is the natural inclusion map, and $(\gamma'_n)^\sharp$ is induced by the homomorphism

$$\gamma_n^\sharp \colon R \to R_n[t]/(t^{n+1}), \quad f \mapsto \sum_{i=0}^{n} D_i(f)\, t^i,$$

where $(D_0, D_1, \dots, D_n)$ is the universal Hasse–Schmidt derivation of order n. With this notation, we have $J_n(X) = \operatorname{Spec} R_n$ and $J'_n(X) = \operatorname{Spec} R'_n$.

If $\tau(\mathcal{F})$ denotes the torsion of $\mathcal{F}$, then the two sheaves $\mathcal{F}'_n$ and $(\mathcal{F}/\tau(\mathcal{F}))'_n$ have the same torsion free quotient. We can therefore assume without loss of generality that $\mathcal{F}$ is torsion free. Let F denote the R-module associated to $\mathcal{F}$. If r is the rank of F, we can then realize F as a submodule of R^r. Picking a set of generators for F of cardinality s, we obtain a matrix $M \in \operatorname{Mat}_{r\times s}(R)$ such that $F = \operatorname{Im} M$. Notice that, to produce an ideal whose blow-up gives $Y \to X$, one can take the ideal generated by the $r \times r$ minors of M.

The relative compactification of $\mu'_n \colon J'_n(Y) \to J'_n(X)$ constructed in Theorem 1.2 is given by the Nash transformation of the R'_n-module

$$F'_n := (\gamma'_n)^\sharp(F) \cdot (R'_n[t]/(t^{n+1}))^r,$$

where the R'_n-module structure is defined via $(\rho'_n)^\sharp$. A straightforward computation shows that $F_n = \operatorname{Im} M_n$ where $M_n \in \operatorname{Mat}_{(n+1)r\times(n+1)s}(R'_n)$ is the matrix given in block form by

$$M_n = \begin{bmatrix} D_0(M) & 0 & \cdots & 0 \\ D_1(M) & D_0(M) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_n(M) & D_{n-1}(M) & \cdots & D_0(M) \end{bmatrix}.$$

Here $D_i(M)$ is the matrix obtained from M by applying D_i to each entry. By construction, we have the following property.

PROPOSITION 3.1. — *With the above notation, the morphism $N(F'_n) \to J'_n(X)$ is the blow-up of the ideal $\mathfrak{a}_n \subset R'_n$ generated by the $(n+1)r \times (n+1)r$ minors of M_n.*

The next example shows the computation of the first few ideals $\mathfrak{a}_n$ in a simple case.

Example 3.2. — Consider $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$, and let $Y \to X$ be the blow-up of the maximal ideal (x, y). Taking F to be the maximal ideal,

we have

$$M_0 = \begin{bmatrix} x & y \end{bmatrix},$$

$$M_1 = \begin{bmatrix} x & y & 0 & 0 \\ x_1 & y_1 & x & y \end{bmatrix},$$

$$M_2 = \begin{bmatrix} x & y & 0 & 0 & 0 & 0 \\ x_1 & y_1 & x & y & 0 & 0 \\ x_2 & y_2 & x_1 & y_1 & x & y \end{bmatrix},$$

$$M_3 = \begin{bmatrix} x & y & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & y_1 & x & y & 0 & 0 & 0 & 0 \\ x_2 & y_2 & x_1 & y_1 & x & y & 0 & 0 \\ x_3 & y_3 & x_2 & y_2 & x_1 & y_1 & x & y \end{bmatrix},$$

where $x_i = D_i(x)$ and $y_i = D_i(y)$. Letting $\mathfrak{a}_n$ be the ideal generated by the $(n+1) \times (n+1)$ minors of M_n, we have

$$\begin{aligned}
\mathfrak{a}_0 &= (x, y), \\
\mathfrak{a}_1 &= \mathfrak{a}_0^2 + (xy_1 - yx_1), \\
\mathfrak{a}_2 &= \mathfrak{a}_0\mathfrak{a}_1 + (y_2x^2 - yx_2x - x_1y_1x + yx_1^2,\ x_2y^2 - x_1y_1y - xy_2y + xy_1^2), \\
\mathfrak{a}_3 &= \mathfrak{a}_0\mathfrak{a}_2 + \mathfrak{a}_1^2 + \\
&\quad + (y_3x^3 - yx_3x^2 - x_2y_1x^2 - x_1y_2x^2 + 2yx_1x_2x + x_1^2y_1x - yx_1^3, \\
&\qquad y_1y_2x^2 - yy_3x^2 - x_1y_1^2x + y^2x_3x + y^2x_1x_2 + yx_1^2y_1, \\
&\qquad x_2y_1y^2 + x_1y_2y^2 + xy_3y^2 - x_1y_1^2y - 2xy_1y_2y + xy_1^3 - y^3x_3, \\
&\qquad y_2^2x^2 - y_1y_3x^2 + x_2y_1^2x + yx_3y_1x - 2yx_2y_2x - x_1y_1y_2x + \\
&\qquad\qquad + yx_1y_3x + y^2x_2^2 - y^2x_1x_3 - yx_1x_2y_1 + yx_1^2y_2).
\end{aligned}$$

Notice that while the matrices M_n remain simple and have an easily recognizable structure, the corresponding ideals $\mathfrak{a}_n$ grow in complexity quite fast.

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