A Geometric View of the Service Rates of Codes Problem and its Application to the Service Rate of the First Order Reed-Muller Codes

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Abstract— **Service rate is an important, recently introduced, performance metric associated with distributed coded storage systems. Among other interpretations, it measures the number of users that can be simultaneously served by the system. We introduce a geometric approach to address this problem. One of the most significant advantages of this approach over the existing ones is that it allows one to derive bounds on the service rate of a code without explicitly knowing the list of all possible recovery sets. To illustrate the power of our geometric approach, we derive upper bounds on the service rates of the first order Reed-Muller codes and the simplex codes. Then, we show how these upper bounds can be achieved. Furthermore, utilizing the proposed geometric technique, we show that given the service rate region of a code, a lower bound on the minimum distance of the code can be obtained.**

I. Introduction

The service rate has been very recently recognized as an important performance metric that measures the number of users that can be simultaneously served by an erasure coded storage system [1]–[6]. Maximizing the service rate reduces the latency experienced by users, particularly in a high traffic regime. See [1] for an easy introduction to the subject.

The service rate problem considers a distributed storage system where k files are encoded into n, and stored across n servers. File i can be recovered by reading data from a single or a set of storage nodes, referred to as a recovery set for file i. Requests to download file i arrive at rate λ_i, and can be split across its recovery sets. Server l can simultaneously serve the requests whose cumulative arrival rate does not exceed μ_l. The service rate problem seeks to determine the service rate region of the coded storage system which is defined as the set of all request arrival rate vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ that can be served by this system.

The service rate problem has been studied only in some special cases: 1) for MDS codes when $n \geq 2k$ and binary simplex codes in [3] and 2) for systems with arbitrary n when $k = 2$ in [3] and $k = 3$ in [4]. The existing techniques for solving the problem require enumeration of all possible recovery sets, which becomes increasingly complex when the number of files k increases. Thus, introducing a technique not depending on the enumeration of recovery sets is of great significance. In this paper, we introduce a novel geometric approach with this goal in mind.

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A. Related Work

The past two decades have seen an ever increasing interest in coding for storage and caching. Special codes that support efficient maintenance of storage under node failures have been proposed in e.g., [7]–[11]. The locality and availability of codes matter in such scenarios. This line of work mostly assumes infinite service rate for servers, and is primarily concerned with reliability of storage rather than with serving a large number of simultaneous users.

Another line of work is focused on caching (see e.g., [12]–[15]). In these work, the limited capacity of the backhaul link is considered as the main bottleneck, and the goal is usually to minimize its traffic by prefetching the popular contents at storage nodes with limited size. Thus, these work do not address the scenarios such as live streaming, where many users wish to get the same content concurrently given the limited capacity of the access part of the network.

The most related to this work are papers concerned with fast content download from coded storage (see e.g., [16], [17], and references therein). These papers strive to compute download latency for increasingly complex queueing systems that appear in coded storage [18]–[20]. The service rate problem is related to the stability region of such queues.

B. Main Contributions

We study the service rates of codes problem by introducing a novel geometric approach. This approach overcomes the main drawback of the previous work which are trying to solve this problem by formulating it as a sequence of linear programs (LP). There, one must exactly know all possible recovery sets to enumerate the constraints in each LP, and must also solve all the LPs.

Leveraging our novel geometric technique, we take initial steps towards deriving bounds on the service rates of some parametric classes of linear codes without explicitly listing the set of all possible recovery sets. In particular, we derive upper bounds on the service rates of first order Reed-Muller codes and simplex codes. Subsequently, we show how the derived upper bounds can be achieved. Moreover, utilizing the proposed geometric technique, we show that given the service rate region of a code, a lower bound on the minimum distance of the code can be derived.

Due to the space constraints, most of the proofs are omitted and can be found in [21].

II. Problem Statement

A. Notation

Throughout this paper, we denote vectors by bold-face small letters and matrices by bold-face capital letters. Let $\mathbb{N}$ denote the set of the non-negative integer numbers. Let $\mathbb{F}_q$ be a finite field for some prime power q, and $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. Let us denote a q-ary linear code $\mathcal{C}$ of length n, dimension k and minimum distance d by $[n,k,d]_q$. We denote the Hamming weight of $\mathbf{x}$ by $\mathrm{w}(\mathbf{x})$. For a positive integer k, let $\mathbf{0}$ and $\mathbf{1}$ denote the all-zero and all-one column vectors of length k, respectively. Let $\mathbf{e}_i$ denote a unit vector of length k, having a one at position i and zeros elsewhere. For a positive integer i, define $[i] \triangleq \{1,\ldots,i\}$. Let us denote the cardinality of a set or multiset $\mathcal{S}$ by $\#\mathcal{S}$.

B. Service Rate of Codes

Consider a storage system in which k files $f_1,\ldots,f_k$ are stored over n servers, labeled $1,\ldots,n$, using a linear $[n,k]_q$ code with generator matrix $\mathbf{G} \in \mathbb{F}_q^{k\times n}$. Let $\mathbf{g}_j$ denote the jth column of $\mathbf{G}$. A recovery set for the file f_i is a set of stored symbols which can be used to recover file f_i. With respect to $\mathbf{G}$, a set $R \subseteq [n]$ is a recovery set for file f_i if there exist α_j's $\in \mathbb{F}_q$ such that $\sum_{j\in R}\alpha_j\mathbf{g}_j = \mathbf{e}_i$, i.e., the unit vector $\mathbf{e}_i$ can be recovered by a linear combination of the columns of $\mathbf{G}$ indexed by the set R. W.l.o.g., we restrict our attention to the reduced recovery sets obtained by considering non-zero coefficients α_j's and linearly independent columns $\mathbf{g}_j$'s.

Let $\mathcal{R}_i = \{R_{i,1},\ldots,R_{i,t_i}\}$ be the $t_i \in \mathbb{N}$ recovery sets for file f_i. Let $\mu_l \in \mathbb{R}_{\geq 0}$ be the average rate at which the server $l \in [n]$ resolves received file requests. We denote the service rates of servers $1,\ldots,n$ by a vector $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_n)$. We further assume that the requests to download file f_i arrive at rate λ_i, $i \in [k]$. We denote the request rates for files $1,\ldots,k$ by the vector $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_k)$. We consider the class of scheduling strategies that assign a fraction of requests for a file to each of its recovery sets. Let $\lambda_{i,j}$ be the portion of requests for file f_i that are assigned to the recovery set $R_{i,j}$, $j \in [t_i]$. The service rate region $\mathcal{S}(\mathbf{G},\boldsymbol{\mu}) \subseteq \mathbb{R}_{\geq 0}^k$ is defined as the set of all request vectors $\boldsymbol{\lambda}$ that can be served by a coded storage system with generator matrix $\mathbf{G}$ and service rate $\boldsymbol{\mu}$. Alternatively, $\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$ can be defined as the set of all vectors $\boldsymbol{\lambda}$ for which there exist $\lambda_{i,j} \in \mathbb{R}_{\geq 0}$, $i \in [k]$ and $j \in [t_i]$, satisfying the following constraints:

$$\sum_{j=1}^{t_i} \lambda_{i,j} = \lambda_i, \qquad \text{for all} \quad i \in [k], \tag{1a}$$

$$\sum_{i=1}^{k} \sum_{\substack{j\in[t_i]\\ l\in R_{i,j}}} \lambda_{i,j} \leq \mu_l, \quad \text{for all} \quad l \in [n], \tag{1b}$$

$$\lambda_{i,j} \in \mathbb{R}_{\geq 0}, \qquad \text{for all} \quad i \in [k],\ j \in [t_i]. \tag{1c}$$

The constraints (1a) guarantee that the demands for all files are served, and constraints (1b) ensure that no node receives requests at a rate in excess of its service rate.

Lemma 1. *The service rate region $\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$ is a non-empty, convex, closed, and bounded subset of $\mathbb{R}_{\geq 0}^k$.*

Proposition 1. *[22] For any set $\mathcal{A} = \{\mathbf{v}_1,\ldots,\mathbf{v}_p\} \subseteq \mathbb{R}^k$, the convex hull of the set $\mathcal{A}$, denoted by $\mathrm{conv}(\mathcal{A})$, consists of all convex combinations of the elements of $\mathcal{A}$, i.e., all vectors of the form $\sum_{i=1}^p \gamma_i\mathbf{v}_i$, with $\gamma_i \geq 0$, $\sum_{i=1}^p \gamma_i = 1$.*

Corollary 1. *The service rate region $\mathcal{S}(\mathbf{G},\boldsymbol{\mu}) \subseteq \mathbb{R}_{\geq 0}^k$ forms a polytope which can be expressed in two forms: as the intersection of a finite number of half spaces or as the convex hull of a finite set of vectors (vertices of the polytope).*

The service rate problem seeks to determine the service rate region $\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$ of a coded storage system with generator matrix $\mathbf{G}$ and service rate $\boldsymbol{\mu}$. Based on Corollary 1, the first algorithm for computing the service rate region that comes to mind is enumerating all vertices of the polytope $\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$ and then computing the convex hull of the resulting vertices. As we indicate shortly, this problem can be formulated as an optimization problem consisting of a sequence of LPs.

Given that any $k-1$ request arrival rates, $\lambda_{i_1},\ldots,\lambda_{i_{k-1}}$, are zeros, there exists a maximum value of λ_{i_k}, denoted by $\lambda^\star_{i_k}$, where $0 \leq \lambda^\star_{i_k} \leq \sum_{l=1}^n \mu_l$ such that $\lambda^\star_{i_k}.\mathbf{e}_{i_k} \in \mathcal{S}(\mathbf{G},\boldsymbol{\mu})$ and all vectors $\lambda_{i_k}.\mathbf{e}_{i_k}$ with $\lambda_{i_k} > \lambda^\star_{i_k}$ are not in $\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$. Thus, these constrained optimization problems of finding the maximum value $\lambda^\star_{i_k}$ are all LPs. For $i \in [k]$, let $\mathbf{v}_i = \lambda^\star_i\mathbf{e}_i$. Since $\mathcal{J} = \{\mathbf{0},\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\} \subseteq \mathcal{S}(\mathbf{G},\boldsymbol{\mu})$, as an immediate consequence of Lemma 1 and Proposition 1, the set $\mathrm{conv}(\mathcal{J})$ is contained in $\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$. Starting with $\mathcal{J}$, we can iteratively enlarge $\mathcal{J}$ until the subsequent procedure stops. We choose a facet H of $\mathrm{conv}(\mathcal{J})$ described by a vector $\mathbf{h} \in \mathbb{R}_{\geq 0}^k$ and $\eta \in \mathbb{R}_{\geq 0}$, as follows:

$$H = \left\{\mathbf{x} \in \mathbb{R}_{\geq 0}^k \,:\, \mathbf{h}^\top\mathbf{x} = \eta\right\} \cap \mathrm{conv}(\mathcal{J})$$

With this, we solve $\max \mathbf{h}^\top\boldsymbol{\lambda}$, where $\boldsymbol{\lambda} \in \mathbb{R}_{\geq 0}^k$ satisfies the demand constraints (1a) and the capacity constraints (1b). If the optimal target value is strictly larger than η, then we add the solution vector $\boldsymbol{\lambda}^\star$ to $\mathcal{J}$ and continue. Note that for any $\mathbf{h} = (h_1,\ldots,h_k)$, the primal LP is given by

$$\max \sum_{i=1}^{k} h_i\lambda_i \quad \text{s.t.} \quad (1a),(1b),(1c) \text{ hold.} \tag{2}$$

The corresponding dual LP is given by

$$\begin{aligned} \min \quad & \sum_{l=1}^{n} \gamma_l\mu_l & & \\ \text{s.t.} \quad & h_i \leq \beta_i & & \forall i \in [k] \\ & \beta_i \leq \sum_{l\in R_{i,j}} \gamma_l & & \forall i \in [k], \forall j \in [t_i] \\ & \beta_i \in \mathbb{R},\ \gamma_l \in \mathbb{R}_{\geq 0} & & \forall i \in [k], \forall l \in [n] \end{aligned} \tag{3}$$

According to the Duality Theorem, if both the primal LP and the corresponding dual LP have feasible solutions, then their optimal target values coincide. A feasible solution for the primal LP (2) can be given by $\lambda_{i,j} = 0$ and $\lambda_i = 0$, and a feasible solution for the dual LP (3) can be given by $\beta_i = h_i$ and $\gamma_l = \sum_{i=1}^k h_i$.

Given a generator matrix $\mathbf{G}$ of a linear code and a service rate $\boldsymbol{\mu}$, the LP (2) can be utilized to compute the maximum value of $\eta = \sum_{i=1}^{k} h_i\lambda_i$, denoted by $\eta^\star$, for every $\mathbf{h} \in \mathbb{R}^k_{\geq 0}$. Having $\eta^\star$ at hand, we know that all $\boldsymbol{\lambda} \in \mathcal{S}(\mathbf{G}, \boldsymbol{\mu})$ satisfy $\sum_{i=1}^{k} h_i\lambda_i \leq \eta^\star$, which is a valid inequality for $\mathcal{S}(\mathbf{G}, \boldsymbol{\mu})$. The downside of this approach is that we have to exactly know the set of all possible recovery sets for each file and also have to be able to optimally solve all the LP (2). Using the dual LP (3), we run into a similar problem since in order to formulate all the inequalities in (3), again we require to know the elements of all the recovery sets for each file.

Therefore, determining the service rate region of a code is a challenging problem, and in general we have to be pleased with lower and upper bounds. Thus, characterizing the exact service rate region of some parametric classes of linear codes or deriving some bounds on the service rate of a code without knowing explicitly all recovery sets is of great significance, which we aim to address in this paper. Towards this goal, we introduce a novel geometric approach. Leveraging our geometric approach, we derive upper bounds on the service rates of the first order Reed-Muller codes and simplex codes. Note that our approach can be applied to any linear code.

C. Geometric View on Linear Codes [23]–[25]

Definition 1. *For a vector space $\mathcal{V}$ of dimension v over $\mathbb{F}_q$, ordered by inclusion, the set of all $\mathbb{F}_q$-subspaces of $\mathcal{V}$ forms a finite modular geometric lattice with meet $X \wedge Y = X \cap Y$, join $X \vee Y = X + Y$, and rank function $X \mapsto \dim(X)$. This subspace lattice of $\mathcal{V}$ is known as the projective geometry of $\mathcal{V}$, denoted by $\mathrm{PG}(\mathcal{V})$.*

Note that for a vector space $\mathcal{V}$ of dimension v over $\mathbb{F}_q$, the 1-dimensional subspaces of $\mathcal{V}$ are the points of $\mathrm{PG}(\mathcal{V})$, the 2-dimensional subspaces of $\mathcal{V}$ are the lines of $\mathrm{PG}(\mathcal{V})$, and the $v-1$ dimensional subspaces of $\mathcal{V}$ are called the hyperplanes of $\mathrm{PG}(\mathcal{V})$. The projective geometry $\mathrm{PG}(\mathcal{V})$ is also denoted by $\mathrm{PG}(v{-}1, q)$, which is referred to as the $v-1$ dimensional projective space over $\mathbb{F}_q$. This notion makes sense because of the fact that, up to isomorphism, the projective geometry $\mathrm{PG}(\mathcal{V})$ only depends on the order q of the base field and the (*algebraic*) dimension v, justifying the notion $\mathrm{PG}(v-1, q)$ of (*geometric*) dimension $v-1$ over $\mathbb{F}_q$.

Let $\mathcal{V}$ be a vector space of dimension v over $\mathbb{F}_q$. The set of all k-dimensional subspaces of $\mathcal{V}$, referred to as *k-subspaces*, will be denoted by $\begin{bmatrix}\mathcal{V}\\k\end{bmatrix}_q$. The cardinality of this set is given by the Gaussian binomial coefficient as

$$\begin{bmatrix}v\\k\end{bmatrix}_q = \begin{cases} \frac{(q^v-1)(q^{v-1}-1)\cdots(q^{v-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} & \text{if } 0 \leq k \leq v; \\ 0 & \text{otherwise.}\end{cases}$$

A multiset is a modification of the concept of a set that, unlike a set, allows for multiple instances for each of its elements. The positive integer number of instances, given for each element is called the multiplicity of this element in the multiset. More formally, a multiset $\mathcal{S}$ on a base set $\mathcal{X}$ can be identified with its characteristic function $\chi_{\mathcal{S}} : \mathcal{X} \to \mathbb{N}$, mapping $x \in \mathcal{X}$ to the multiplicity of x in $\mathcal{S}$. The *cardinality* of $\mathcal{S}$ is $\#\mathcal{S} = \sum_{x \in X} \chi_{\mathcal{S}}(x)$. $\mathcal{S}$ is also called *$\#\mathcal{S}$-multiset*.

Definition 2. *Let $\mathcal{V}$ be a vector space of dimension v over $\mathbb{F}_q$, $\mathcal{P}$ be a multiset of points p in $\mathrm{PG}(\mathcal{V})$ with characteristic function $\chi_{\mathcal{P}} : \mathrm{PG}(\mathcal{V}) \to \mathbb{N}$, and $\mathcal{H}$ denotes a hyperplane in $\mathrm{PG}(\mathcal{V})$. The restricted multiset $\mathcal{P} \cap \mathcal{H}$ is defined via its characteristic function as*

$$\chi_{\mathcal{P}\cap\mathcal{H}}(p) = \begin{cases} \chi_{\mathcal{P}}(p) & \text{if } p \in \begin{bmatrix}\mathcal{H}\\1\end{bmatrix}_q; \\ 0 & \text{otherwise.}\end{cases}$$

Then $\#(\mathcal{P} \cap \mathcal{H}) = \sum_{p \in \begin{bmatrix}\mathcal{H}\\1\end{bmatrix}_q} \chi_{\mathcal{P}}(p)$.

Let $\mathbf{G} \in \mathbb{F}_q^{k \times n}$ be the generator matrix of a linear $[n, k]_q$ code $\mathcal{C}$, a k-subspace of the n-dimensional vector space $\mathbb{F}_q^n$. Let $\mathbf{g}_i \in \mathbb{F}_q^k$, $i \in [n]$ denote the ith column of $\mathbf{G}$. Suppose that none of the $\mathbf{g}_i$'s is $\mathbf{0}$. (The code $\mathcal{C}$ is said to be of full length.) Then each $\mathbf{g}_i$ determines a point in the projective space $\mathrm{PG}(k-1, q)$, and $\mathcal{G} := \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n\}$ is a set of n points in $\mathrm{PG}(k-1, q)$ if the $\mathbf{g}_i$ happen to be pair-wise independent. When dependence occurs, $\mathcal{G}$ is interpreted as a multiset and each point is counted with the appropriate multiplicity. In general, $\mathcal{G}$ is called n-multiset induced by $\mathcal{C}$.

Proposition 2. *Different generator matrices of a code yield projectively equivalent codes. In other words, there exist a bijective correspondence between the equivalence classes of full-length q-ary linear codes and the projective equivalence classes of multisets in finite projective spaces.*

Note that the importance of this correspondence lies in the fact that it relates the coding-theoretic properties of $\mathcal{C}$ to the geometric or the combinatorial properties of $\mathcal{G}$.

Proposition 3. *Let $\mathbf{G} \in \mathbb{F}_q^{k \times n}$ be the generator matrix of a linear $[n, k, d]_q$ code $\mathcal{C}$, and $\mathcal{G}$ be the n-multiset induced by code $\mathcal{C}$. Then, the minimum distance $d = n - \max \#(\mathcal{G} \cap \mathcal{H})$ where $\mathcal{H}$ runs through all the hyperplanes of $\mathrm{PG}(k-1, q)$.*

D. First Order Reed-Muller Codes [26]–[29]

In this paper, we consider binary first order Reed-Muller codes $\mathrm{RM}_2(1, k-1)$ with the integer parameter $k \geq 2$. It is known that $\mathrm{RM}_2(1, k-1)$ is a linear $[2^{k-1}, k, 2^{k-2}]_2$ code. For a given k, one way of obtaining this code is to evaluate all multilinear polynomials with the binary coefficients, $k-1$ variables and the total degree of one on the elements of $\mathbb{F}_2^{k-1}$. The encoding polynomial for $\mathrm{RM}_2(1, k{-}1)$ can be written as $c_1 + c_2 \cdot Z_1 + c_3 \cdot Z_2 + \cdots + c_k \cdot Z_{k-1}$ where $Z_1, \ldots, Z_{k-1}$ are the $k-1$ variables, and $c_1, \ldots, c_k$ are the binary coefficients of this polynomial. Indeed, the data symbols $f_1, \ldots, f_k$ are used as the coefficients of the encoding polynomial, and the codeword symbols are obtained by evaluating the encoding polynomial on all vectors $(Z_1, \ldots, Z_{k-1}) \in \mathbb{F}_2^{k-1}$.

Another way of describing a Reed-Muller $\mathrm{RM}_2(1, k-1)$ is based on the generator matrix which can be constructed as follows. Let us write the set of all $(k{-}1)$-dimensional binary vectors as $\mathcal{X} = \mathbb{F}_2^{k-1} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ where $n = 2^{k-1}$ and for $i \in [n]$, $\mathbf{x}_i = (x_{i_{k-1}}, \ldots, x_{i_1})$ with $x_{i_j} \in \mathbb{F}_2$, $j \in [k-1]$. For any $\mathcal{A} \subseteq \mathcal{X}$, define the indicator vector $\mathbb{I}_{\mathcal{A}} \in \mathbb{F}_2^{k-1}$ as,

$$(\mathbb{I}_{\mathcal{A}})_i = \begin{cases} 1 & \text{if } \mathbf{x}_i \in \mathcal{A}; \\ 0 & \text{otherwise.}\end{cases}$$

For the k rows of the generator matrix of $\mathrm{RM}_2(1,k-1)$, define k row vectors of length 2^{k-1} as $\mathbf{r}_0=(1,\ldots,1)$ and $\mathbf{r}_j=\mathbb{I}_{\mathcal{H}_j},\ j\in[k-1]$, where $\mathcal{H}_j=\{\mathbf{x}_i\in\mathcal{X}\mid x_{i_j}=0\}$. It should be noted that the set $\{\mathbf{r}_{k-1},\ldots,\mathbf{r}_1,\mathbf{r}_0\}$ gives the rows of a non-systematic generator matrix of the $\mathrm{RM}_2(1,k-1)$. For a systematic generator matrix of the $\mathrm{RM}_2(1,k-1)$, the set of rows $\{\mathbf{r}_{k-1},\ldots,\mathbf{r}_1,\sum_{i=0}^{k-1}\mathbf{r}_i\}$ can be considered.

Example 1. Consider $\mathrm{RM}_2(1,3)$ which is a linear $[8,4,4]_2$ code. Define $\mathcal{X}=\mathbb{F}_2^3=\{(0,0,0),(0,0,1),\ldots,(1,1,1)\}$. According to the definition, $\mathcal{H}_3=\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\}$ that gives $\mathbf{r}_3=(1,1,1,1,0,0,0,0)$, and $\mathcal{H}_2=\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_5,\mathbf{x}_6\}$ which gives $\mathbf{r}_2=(1,1,0,0,1,1,0,0)$, and $\mathcal{H}_1=\{\mathbf{x}_1,\mathbf{x}_3,\mathbf{x}_5,\mathbf{x}_7\}$ which results $\mathbf{r}_1=(1,0,1,0,1,0,1,0)$. Let $\mathbf{r}_0$ be all-one row vector of dimension eight. The set $\{\mathbf{r}_3,\mathbf{r}_2,\mathbf{r}_1,\mathbf{r}_0\}$ defines the rows of a non-systematic generator matrix of the $\mathrm{RM}_2(1,3)$.

$$\mathbf{G}=\begin{bmatrix}1&1&1&1&0&0&0&0\\1&1&0&0&1&1&0&0\\1&0&1&0&1&0&1&0\\1&1&1&1&1&1&1&1\end{bmatrix}$$

Also, $\sum_{i=0}^{3}\mathbf{r}_i=(0,1,1,0,1,0,0,1)$. Hence, a systematic generator matrix of the $\mathrm{RM}_2(1,3)$ is given by:

$$\mathbf{G}=\begin{bmatrix}1&1&1&1&0&0&0&0\\1&1&0&0&1&1&0&0\\1&0&1&0&1&0&1&0\\0&1&1&0&1&0&0&1\end{bmatrix}$$

III. Geometric View on Service Rate of Codes

In this section, we use the geometric description of linear codes. For a linear code $\mathcal{C}$ with generator matrix $\mathbf{G}\in\mathbb{F}_q^{k\times n}$, we consider the n-multiset $\mathcal{G}$ induced by $\mathcal{C}$ in $\mathrm{PG}(k-1,q)$ with the characteristic function $\chi_{\mathcal{G}}$ as defined in section II-C. Thus, each point $p\in\mathrm{PG}(k-1,q)$ has a certain multiplicity $\chi_{\mathcal{G}}(p)\in\mathbb{N}$. In this language, the reduced recovery sets are subsets of $\mathcal{G}$, where each point can be taken once in a reduced recovery set. Also, the service rate of each point p, denoted by $\mu(p)$, can be defined as the sum of the service rates of the nodes (columns of $\mathbf{G}$) corresponding to the point p. Based on this definition, $\mu(p)=\sum_{l\in\mathcal{L}_p}\mu_l$ where $\mathcal{L}_p$ is the set of nodes that correspond to the same point $p\in\mathrm{PG}(k-1,q)$. Since $\#\mathcal{L}_p=\chi_{\mathcal{G}}(p)$, if all nodes in the set $\mathcal{L}_p$ have the same service rate, say μ_p, then we have $\mu(p)=\chi_{\mathcal{G}}(p)\cdot\mu_p$.

Lemma 2. *Let $\mathbf{G}\in\mathbb{F}_q^{k\times n}$ be the generator matrix of a linear $[n,k]_q$ code $\mathcal{C}$, and $\mathcal{G}$ be the n-multiset induced by code $\mathcal{C}$ with service rate $\mu(p)$ of each point $p\in\mathrm{PG}(k-1,q)$. If for some $i\in[k]$, $s\cdot\mathbf{e}_i\in\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$ and a hyperplane $\mathcal{H}$ of $\mathrm{PG}(k-1,q)$ is not containing $\mathbf{e}_i$, then we have*

$$s\leq\sum_{p\in\mathrm{PG}(k-1,q)\setminus\mathcal{H}}\mu(p).$$

Corollary 2. *Let $\mathbf{G}\in\mathbb{F}_q^{k\times n}$ be the generator matrix of a linear $[n,k,d]_q$ code $\mathcal{C}$ with service rate $\mu_l=1$ of all nodes $l\in[n]$, and $\mathcal{G}$ be the n-multiset induced by code $\mathcal{C}$. If for all $i\in[k]$, $s\cdot\mathbf{e}_i\in\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$, then the minimum distance d of code $\mathcal{C}$ is at least $\lceil s\rceil$.*

Corollary 3. *Let $\mathbf{G}\in\mathbb{F}_q^{k\times n}$ be the generator matrix of a linear $[n,k]_q$ code $\mathcal{C}$, and $\mathcal{G}$ be the n-multiset induced by code $\mathcal{C}$ with service rate $\mu(p)$ of each point $p\in\mathrm{PG}(k-1,q)$. Let $\mathcal{I}\subseteq[k]$. If for all $i\in\mathcal{I}$, there exist $s_i\in\mathbb{R}_{\geq 0}$ such that $\sum_{i\in\mathcal{I}}s_i\cdot\mathbf{e}_i\in\mathcal{S}(\mathbf{G},\boldsymbol{\mu})$ and a hyperplane $\mathcal{H}$ of $\mathrm{PG}(k-1,q)$ is not containing $\mathbf{e}_i$ for all $i\in\mathcal{I}$, then*

$$s\leq\sum_{p\in\mathrm{PG}(k-1,q)\setminus\mathcal{H}}\mu(p).$$

where $s=\sum_{i\in\mathcal{I}}s_i$.

Note that Corollary 3 enables us to derive upper bounds on the service rate of the first order Reed-Muller and simplex codes. In what follows, without loss of generality, we assume that the service rate of all servers in the coded storage system is 1, i.e., $\mu_l=1$ for all $l\in[n]$. Thus, by this assumption, the service rate region of a code only depends on the generator matrix $\mathbf{G}$ of the code and can be denoted by $\mathcal{S}(\mathbf{G})$.

IV. Service Rate Region of Simplex Codes

In this section, by leveraging a novel geometric approach, we characterize the service rate region of the binary simplex codes which are special rate-optimal subclass of availability codes that are known as an important family of distributed storage codes. As we will show, the determined service rate region coincides with the region derived in [3, Theorem 1].

Theorem 1. *For each integer $k\geq 1$, the service rate region of the k-dimensional binary simplex code $\mathcal{C}$, which is a linear $[2^k-1,k,2^{k-1}]_2$ code with generator matrix $\mathbf{G}$ is given by*

$$\mathcal{S}(\mathbf{G})=\left\{\boldsymbol{\lambda}\in\mathbb{R}_{\geq 0}^k\ :\ \sum_{i=1}^{k}\lambda_i\leq 2^{k-1}\right\}.$$

Proof: Note that the simplex code is projective. Since the projective space $\mathrm{PG}(k-1,2)$ contains exactly 2^k-1 points, the generator matrix $\mathbf{G}$ consists of all non-zero vectors of $\mathbb{F}_2^k$. (Up to column permutations the generator matrix is unique.) Given an arbitrary $i\in[k]$, we partition the columns of $\mathbf{G}$ into $\mathbf{e}_i$ and $\{\mathbf{x},\mathbf{x}+\mathbf{e}_i\}$ for all $2^{k-1}-1$ non-zero vectors $\mathbf{x}\in\mathbb{F}_2^k$ with ith coordinate being equal to zero. Thus, for all $i\in[k]$, $2^{k-1}\cdot\mathbf{e}_i\in\mathcal{S}(\mathbf{G})$. Let $\mathbf{v}_i=2^{k-1}\cdot\mathbf{e}_i$ for $i\in[k]$. Since $\mathcal{J}=\{\mathbf{0},\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}\subseteq\mathcal{S}(\mathbf{G})$, based on Lemma 1 and Proposition 1, the $\mathrm{conv}(\mathcal{J})$ is contained in $\mathcal{S}(\mathbf{G})$, i.e.,

$$\mathcal{S}(\mathbf{G})\supseteq\left\{\boldsymbol{\lambda}\in\mathbb{R}_{\geq 0}^k\ :\ \sum_{i=1}^{k}\lambda_i\leq 2^{k-1}\right\}$$

For the other direction, we consider the hyperplane $\mathcal{H}$ given by $\sum_{i=1}^{k}x_i=0$, which does not contain any unit vector $\mathbf{e}_i$. Thus, for any demand vector $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_k)$ in the service rate region, the Corollary 3 results in $\sum_{i=1}^{k}\lambda_i\leq 2^{k-1}$. The reason is that half of the vectors in $\mathbb{F}_2^k$ which are the columns of $\mathbf{G}$ and so the elements of $\mathcal{G}$, are not contained in $\mathcal{H}$. □

V. Service Rate Region of Reed-Muller Codes

This section seeks to characterize the service rate region of the $\mathrm{RM}_2(1,k-1)$ code with a non-systematic and systematic generator matrix $\mathbf{G}$ constructed as described in section II-D.

A. Non-Systematic First Order Reed-Muller Codes

Theorem 2. *For each integer $k \geq 2$, the service rate region of the first order Reed-Muller code $RM_2(1, k-1)$ (or binary affine k-dimensional simplex code) with a non-systematic generator matrix $\mathbf{G}$ constructed as described in section* II-D, *if $k \in \{2,3\}$ is given by*

$$S(\mathbf{G}) = \left\{ \lambda \in \mathbb{R}^k_{\geq 0} \,:\, \sum_{i=1}^{k} \lambda_i \leq 2^{k-2} \right\}$$
$$= \operatorname{conv}\left(\{\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_k\}\right)$$

and if $k \geq 4$, $S(\mathbf{G})$ is given by

$$\left\{ \lambda \in \mathbb{R}^k_{\geq 0} \,:\, \sum_{i=1}^{k} \lambda_i \leq 2^{k-2}, \sum_{i=1}^{k-1} \lambda_i + \frac{3}{2}\lambda_k - 1 \leq 2^{k-2} \right\}$$
$$= \operatorname{conv}\left(\{\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{u}_k, \mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\}\right),$$

where $\mathbf{v}_i = 2^{k-2} \cdot \mathbf{e}_i$ and $\mathbf{w}_j = (2^{k-2} - 2) \cdot \mathbf{e}_j + 2 \cdot \mathbf{e}_k$ for $i \in [k]$ and $j \in [k-1]$, respectively. Also, $\mathbf{u}_k = \frac{2^{k-1}+2}{3} \cdot \mathbf{e}_k$.

Proof: The proof consists of a converse and an achievability. Converse: The unit vector $\mathbf{e}_i$ for all $i \in [k-1]$ is not a column of $\mathbf{G}$ which means that file f_i does not have any systematic recovery set. Therefore, for file f_i, $i \in [k-1]$, all recovery sets have cardinality at least two, and the minimum system capacity utilized by λ_i, $i \in [k-1]$, is $2\lambda_i$. For file f_k, the cardinality of every reduced recovery set is odd since all columns of generator matrix $\mathbf{G}$ has one in the last row. Hence, for file f_k, the unit vector $\mathbf{e}_k$ that is a column of $\mathbf{G}$, forms a systematic recovery set of cardinality one, while all other recovery sets have cardinality at least three. Hence, the minimum capacity used by $\lambda_k \geq 1$ is $1 + 3(\lambda_k - 1)$. Since the system has 2^{k-1} servers, each of service rate (capacity) 1, based on the capacity constraints, the total capacity utilized by the requests for download must be less than or equal to 2^{k-1}. Thus, any vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ in the service rate region must satisfy the following valid constraint,

$$\sum_{i=1}^{k-1} \lambda_i + \frac{3}{2}\lambda_k - 1 \leq 2^{k-2} \tag{4}$$

Consider the hyperplane $\mathcal{H}$ given by $\sum_{i=1}^{k} x_i = 0$, which does not contain any unit vector $\mathbf{e}_i$. The columns of generator matrix $\mathbf{G}$ and so the elements of $\mathcal{G}$ which are not contained in $\mathcal{H}$, are the vectors in $\mathbb{F}_2^k$ with one in the last coordinate that satisfy $\sum_{i=1}^{k-1} x_i = 0$. It is easy to see that there are 2^{k-2} such vectors. Thus, applying Corollary 3 for hyperplane $\mathcal{H}$ impose another valid constraint as follows that any demand vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ in the service rate region must satisfy,

$$\sum_{i=1}^{k} \lambda_i \leq 2^{k-2} \tag{5}$$

It should be noted that for $\lambda_k < 2$, the Inequality (5) is tighter than (4), while for $\lambda_k > 2$ Inequality (4) is tighter than (5). This means that for $k \in \{2,3\}$ Inequality (4) is redundant. Achievability: For the other direction, we provide solutions (constructions) for the vertices of the corresponding polytope as follows. Let $\mathcal{R}' \subseteq \mathbb{F}_2^k$, $|\mathcal{R}'| = 2^{k-1}$ be the set of columns of $\mathbf{G}$ with one in the last coordinate. For all $i \in [k-1]$, consider all the 2^{k-2} vectors $\mathbf{x} \in \mathcal{R}'$ with zero in the ith coordinate, then $\mathbf{x} + \mathbf{e}_i \in \mathcal{R}'$, and so $\{\mathbf{x}, \mathbf{x} + \mathbf{e}_i\}$ constitutes a recovery set of cardinality two for file f_i. Thus, for each file f_i, $i \in [k-1]$, the columns of $\mathbf{G}$ can be partitioned into 2^{k-2} pairs $\{\mathbf{x}, \mathbf{x} + \mathbf{e}_i\}$ which determines 2^{k-2} disjoint recovery sets for file f_i, $i \in [k-1]$. Therefore, the demand vectors $2^{k-2} \cdot \mathbf{e}_i$ for all $i \in [k-1]$ can be satisfied, i.e., $2^{k-2} \cdot \mathbf{e}_i \in S(\mathbf{G})$. For file f_k, there are exactly one systematic recovery set of cardinality one which is the column $\mathbf{e}_k$ of $\mathbf{G}$, and $(2^{k-1} - 1).(2^{k-1} - 2)/6$ recovery sets of cardinality three which are the sets $\{\mathbf{x}, \mathbf{x}', \mathbf{x} + \mathbf{x}' + \mathbf{e}_k\}$ for all pairs $\mathbf{x}, \mathbf{x}' \in \mathcal{R}' \setminus \mathbf{e}_k$. Note that for $k = 2$, according to Inequality (5), one can readily confirm that $\lambda_k \leq 1$. Thus, for $k = 2$ the systematic recovery set of file f_k can be utilized for satisfying the demand vector $1 \cdot \mathbf{e}_k$. For $k \geq 3$, it should be noted that that each column $\mathbf{x} \in \mathcal{R}' \setminus \mathbf{e}_k$ is contained in exactly $(2^{k-1} - 2)/2$ recovery sets of file f_k of cardinality three. Since the capacity of each node is one, from each recovery set the request rate of $1/(2^{k-2} - 1)$ can be satisfied without violating the capacity constraints. Thus, the demand vector $\frac{2^{k-1}+2}{3} \cdot \mathbf{e}_k$ can be satisfied. For the remaining part, we consider $k \geq 4$. Let $i, j \in [k-1]$ with $i \neq j$ be arbitrary. With this $\{\mathbf{e}_k, \mathbf{e}_i + \mathbf{e}_k\}$ and $\{\mathbf{e}_j + \mathbf{e}_k, \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k\}$ are two of 2^{k-2} recovery sets of cardinality two for file f_i. Thus, the elements in $\mathcal{R}' \setminus \{\mathbf{e}_k, \mathbf{e}_i + \mathbf{e}_k, \mathbf{e}_j + \mathbf{e}_k, \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k\}$ can be partitioned into $2^{k-2} - 2$ recovery sets for file $f_i, i \in [k-1]$. Also, the sets $\{\mathbf{e}_k\}$ and $\{\mathbf{e}_i + \mathbf{e}_k, \mathbf{e}_j + \mathbf{e}_k, \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k\}$ can be utilized as two disjoint recovery sets for file f_k. Thus, the demand vector $\left(2^{k-2} - 2\right) \cdot \mathbf{e}_i + 2 \cdot \mathbf{e}_k$ can be satisfied. $\square$

B. Systematic First Order Reed-Muller Codes

Theorem 3. *For each integer $k \geq 2$, the service rate region $\mathcal{S}(\mathbf{G})$ of the first order Reed-Muller code $RM_2(1, k-1)$ (or binary affine k-dimensional simplex code) with a systematic generator matrix $\mathbf{G}$ constructed as described in section* II-D, *if $k = 2$ is given by*

$$\mathcal{S}(\mathbf{G}) = \left\{ \lambda \in \mathbb{R}^k_{\geq 0} \,:\, \lambda_1 \leq 1, \lambda_2 \leq 1 \right\} = \operatorname{conv}\left(\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2\right)$$

if $k = 3$, is given by

$$\mathcal{S}(\mathbf{G}) = \left\{ \lambda \in \mathbb{R}^k_{\geq 0} \,:\, -\lambda_i + \sum_{j=1}^{3} \lambda_j \leq 2, \forall i \in [k] \right\}$$
$$= \operatorname{conv}\left(\mathbf{0}, 2 \cdot \mathbf{e}_1, 2 \cdot \mathbf{e}_2, 2 \cdot \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right)$$

if $k = 4$, $\mathcal{S}(\mathbf{G})$ is given by

$$\left\{ \lambda \in \mathbb{R}^k_{\geq 0} \,: -\lambda_i + \sum_{j=1}^{k} \lambda_j \leq 4, 2\lambda_i + \sum_{j=1}^{k} \lambda_j \leq 10 \, \forall i \in [k] \right\}$$
$$= \operatorname{conv}\left(\mathbf{0}, \mathbf{p}_i \, \forall i \in [k], \mathbf{q}_{i,j} \, \forall i, j \in [k] \text{ with } i \neq j, \tfrac{4}{3} \cdot \mathbf{1}\right)$$

and if $k \geq 5$, $\mathcal{S}(\mathbf{G})$ lies inside the region given by

$$\left\{ \lambda \in \mathbb{R}^k_{\geq 0} \,:\, \sum_{i \in [k] \setminus \mathcal{S}} \lambda_i + \sum_{j \in \mathcal{S}} (3\lambda_j - 2) \,\leq\, 2^{k-1} \, \forall \mathcal{S} \subseteq [k] \right\}.$$

where $\mathbf{p}_i = \frac{10}{3} \cdot \mathbf{e}_i$ and $\mathbf{q}_{i,j} = 3 \cdot \mathbf{e}_i + 1 \cdot \mathbf{e}_j$ for $i, j \in [k]$.

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