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Evaluation of Abramowitz functions in the right half of the complex plane

Zydrunas Gimbutas [a], Shidong Jiang [b], Li-Shi Luo [c,d,*]

[a] Information Technology Laboratory, National Institute of Standards and Technology, Boulder, CO 80305, USA
[b] Department of Mathematics Sciences, New Jersey Institute of Technology, Newark, NJ 07102, USA
[c] Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA 23529, USA
[d] Beijing Computational Science Research Center, Beijing 100193, China

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ABSTRACT

A numerical scheme is developed for the evaluation of Abramowitz functions J_n in the right half of the complex plane. For $n = -1, \ldots, 2$, the scheme utilizes series expansions for $|z| < 1$, asymptotic expansions for $|z| > R$ with R determined by the required precision, and least squares Laurent polynomial approximations on each sub-region in the intermediate region $1 \leq |z| \leq R$. For $n > 2$, J_n is evaluated via a forward recurrence relation. The scheme achieves nearly machine precision for $n = -1, \ldots, 2$ at a cost that is competitive as compared with software packages for the evaluation of other special functions in the complex domain.

1. Introduction

The Abramowitz functions J_n of order n, defined by

$$J_n(z) := \int_0^\infty t^n e^{-t^2 - z/t} \mathrm{d}t, \quad n \in \mathbb{Z}, \tag{1}$$

are frequently encountered in kinetic theory (cf., *e.g.*, [8,17]), where the integral equations resulting from linearization of the Boltzmann equation have these functions (cf., *e.g.*, [8,17,26,21]) as the kernels. The n-th order Abramowitz function J_n satisfies the third order ordinary differential equation (ODE) [1,2]

$$zJ_n''' - (n-1)J_n'' + 2J_n = 0 \tag{2}$$

and the recurrence relations

$$J_n'(z) = -J_{n-1}(z), \tag{3}$$

$$2J_n(z) = (n-1)J_{n-2}(z) + zJ_{n-3}(z). \tag{4}$$

The integral representation (1) also leads to

* Corresponding author at: Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA 23529, USA.
E-mail addresses: zydrunas.gimbutas@nist.gov (Z. Gimbutas), shidong.jiang@njit.edu (S. Jiang), lluo@odu.edu (L.-S. Luo).

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$$J_n(\bar{z}) = \overline{J_n(z)}. \tag{5}$$

Research on Abramowitz functions is rather limited. In [2], about two pages of Section 27.5 are devoted to Abramowitz functions, which contain series and asymptotic expansions, originally developed in [1,25,38]. In [10], numerical computation of Abramowitz functions is discussed when z is a positive real number, and, in particular, it is shown that the recurrence relation for J_n is stable in both directions. In [27], a more efficient and reliable numerical algorithm using Chebyshev expansions has been developed for the evaluation of J_n $(n = 0, 1, 2)$ when z is a positive real number.

For time-dependent or time-harmonic problems in kinetic theory, evaluation of Abramowitz functions with complex arguments is often required. However, we are not aware of any work on the evaluation of Abramowitz functions in complex domains.

In this paper, we develop an efficient and accurate numerical scheme for the evaluation of Abramowitz functions when its argument z is in the right half of the complex plane (denoted as $\overline{\mathbb{C}^+} = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq 0\}$) for $n \geq -1$. We first note that Chebyshev expansions are not good representations in the complex domain since Chebyshev polynomials are orthogonal polynomials only when the argument is real. Second, when $|z|$ is small, say, less than r for some $r > 0$, a series expansion can be used to evaluate $J_n(z)$ accurately with small number of terms. Third, when $|z|$ is large, say, greater than R for some $R > 0$, the truncated asymptotic expansion can be used to evaluate $J_n(z)$ accurately.

We now consider the intermediate region $D = \{z \in \overline{\mathbb{C}^+} \,|\, r \leq |z| \leq R\}$, where neither the series expansion nor the asymptotic expansion can be used to achieve the required precision. Since 0 and ∞ are the only singularities of the ODE (2) satisfied by J_n, standard ODE theory [20, Chapter 16] together with the series expansion (7) shows that $J_n(z) = f_n(z) + g_n(z) \ln z$ where both f and g are entire functions. Thus, J_n admits an infinite Laurent series representation in D by theory of complex variables [5]. One may naturally ask whether $J_n(z)$ can be well approximated by a Laurent polynomial in D. It turns out that such an approximation requires excessively large number of terms to achieve high accuracy. Furthermore, this global approximation is extremely ill-conditioned due to the fact that J_n behaves like an exponential function asymptotically, making its dynamic range too wide to be resolved numerically with high accuracy and rendering the scheme useless.

We propose two techniques to deal with the extreme ill-conditioning associated with the global approximation of J_n in D. First, we extract out the leading factor in the asymptotic expansion (18) of $J_n(z)$ and make a change of variable as follows:

$$J_n(z) = \sqrt{\frac{\pi}{3}} \left(\frac{\nu}{3}\right)^{n/2} e^{-\nu} U_n(\nu), \quad \nu := 3\left(\frac{z}{2}\right)^{2/3}. \tag{6}$$

It has been shown in [1,25] that $U_n(\nu)$ also satisfies a third order ODE with 0 a regular singularity and ∞ an irregular one. Thus, $U_n(\nu)$ is analytic for $z \in D$ and therefore can be represented by an infinite Laurent series in ν in the transformed domain. The main advantage of working with $U_n(\nu)$ instead of $J_n(z)$ is that $U_n(\nu)$ has a much narrower dynamic range and thus admits more accurate and efficient approximation.

Next, we divide the intermediate region D into several sub-regions $D_i = \{z \in \overline{\mathbb{C}^+} \,|\, r_i \leq |z| \leq r_{i+1}\}$ $(i = 0, \ldots, M-1,\ r_0 = r,\ r_M = R)$. By symmetry, we may further restrict ourselves to consider the quarter-annulus domain $Q_i = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, r_i \leq |z| \leq r_{i+1}\}$ $(i = 0, \ldots, M-1,\ r_0 = r,\ r_M = R)$. On each sub-region Q_i, we approximate $U_n(\nu)$ via a Laurent polynomial [24] in ν where the coefficients are obtained by solving a least squares problem. Here the linear system is set up by matching the function values with the values of the Laurent polynomial approximation on a set of N points on the boundary of Q_i. The least squares problem is still ill-conditioned and the conditioning becomes worse as N increases, but its solution can be used to produce very accurate approximation to the function being approximated.

Here, we would like to remark that recently least squares method has been applied to construct accurate and stable approximation for many classes of functions. In [7], it is used together with method of fundamental solutions to solve boundary value problems for the Helmholtz equation. In [15], it is used to construct rational approximation for functions on the unit circle. In [4,3], it is shown that a wide class of functions can be approximated in an accurate and well-conditioned manner using frames and the least squares method. The least squares method is used in [16] to construct efficient and accurate sum-of-Gaussians approximations for a class of kernels in mathematical physics and in [6,35] to construct sum-of-poles approximations for certain functions. Needless to say, the least squares problem itself has to be solved using suitable algorithms. Many such algorithms exist (cf., *e.g.*, [11,14,18,28,32]).

For $n \geq 3$, we apply the recurrence relation (4) to compute $J_n(z)$. We note that the recurrence relation only needs the values of J_n for $n = 0, 1, 2$. Since many applications in kinetic theory require the evaluation of J_{-1}, we provide the direct evaluation of J_{-1} as well via our scheme since it is more efficient than using the recurrence relation.

Clearly, the scheme presented in this paper may be applied to the accurate evaluation of a very broad class of special functions in complex domains. Very often these special functions satisfy an ODE with a finite number of singularities. Therefore, they are analytic in complex domains excluding singular points and branch cuts. Complex analysis then ensures that Laurent series is a suitable representation to such functions in the domain. With a careful choice of the domain and suitable transformation, the least squares method becomes a reliable tool for constructing efficient, accurate and stable approximation for these functions.

The remainder of this paper is organized as follows. Section 2 collects analytic results used in the construction of the algorithm. Section 3 discusses numerical algorithms for the evaluation of Abramowitz functions. Section 4 illustrates the performance and accuracy of the algorithm. The paper is concluded with a short discussion on possible extensions and applications of the work.

2. Analytic apparatus

The series expansion of J_n takes the form

$$2J_n(z) = \sum_{k=0}^{\infty} (a_k^{(n)} \ln z + b_k^{(n)}) z^k. \tag{7}$$

For $n = 1$, the coefficients can be found in [2, §27.5.4] with $a_0^{(1)} = a_1^{(1)} = 0$, $a_2^{(1)} = -1$, $b_0^{(1)} = 1$, $b_1^{(1)} = -\sqrt{\pi}$, $b_2^{(1)} = 3(1-\gamma)/2$, and

$$a_k^{(1)} = -\frac{2a_{k-2}^{(1)}}{k(k-1)(k-2)}, \qquad b_k^{(1)} = -\frac{2b_{k-2}^{(1)} + (3k^2 - 6k + 2)a_k^{(1)}}{k(k-1)(k-2)}, \quad k \geq 3, \tag{8}$$

where $\gamma \approx 0.5772156649015328606065 12$ is Euler's constant. For $n = -1, 0$, the coefficients can be obtained from term-by-term differentiation of (7), together with (3):

$$a_k^{(n)} = -(k+1)a_{k+1}^{(n+1)}, \qquad b_k^{(n)} = -(k+1)b_{k+1}^{(n+1)} - a_{k+1}^{(n+1)}, \quad k \geq 0. \tag{9}$$

For $n = 2$, the coefficients can be obtained from term-by-term integration of (7) together with $J_2(0) = \sqrt{\pi}/4$, *i.e.*, $a_0^{(2)} = 0$, $b_0^{(2)} = \sqrt{\pi}/2$, and

$$a_k^{(2)} = -\frac{a_{k-1}^{(1)}}{k}, \qquad b_k^{(2)} = -\frac{b_{k-1}^{(1)}}{k} + \frac{a_{k-1}^{(1)}}{k^2}, \quad k \geq 1. \tag{10}$$

We have the following lemma regarding the convergence of the power series $\sum_{k=0}^{\infty} a_k^{(n)} z^k$ and $\sum_{k=0}^{\infty} b_k^{(n)} z^k$ in the series expansion (7).

Lemma 1. *For $n = -1, \ldots, 2$, the power series $\sum_{k=0}^{\infty} a_k^{(n)} z^k$ and $\sum_{k=0}^{\infty} b_k^{(n)} z^k$ in (7) converge in $\mathbb{C}$.*

Proof. For $n = 1$, direct calculation shows that

$$a_{2k-1} = 0, \quad a_{2k}^{(1)} = \frac{(-1)^k 2}{(2k)!(k-1)!}, \quad k > 0. \tag{11}$$

Thus, the radius of convergence for $\sum_{k=0}^{\infty} a_k^{(n)} z^k$ is ∞ by the ratio test and the series converges for all complex numbers. We now split $\sum_{k=0}^{\infty} b_k^{(n)} z^k$ into the odd and even parts:

$$\sum_{k=0}^{\infty} b_k^{(1)} z^k = z \sum_{k=0}^{\infty} b_{2k+1}^{(1)} (z^2)^k + \sum_{k=1}^{\infty} b_{2k}^{(1)} (z^2)^k. \tag{12}$$

For the odd part, direct calculation shows

$$b_{2k+1}^{(1)} = \frac{(-2)^k b_1^{(1)}}{(2k+1)!(2k-1)!!}, \tag{13}$$

where $(2k-1)!! := (2k-1)(2k-3)\cdots 3 \cdot 1$. Using the root test and Stirling's formula for factorials [5, p. 201], we observe that the odd part converges for all complex numbers. For the even part, we claim that

$$|b_{2k}^{(1)}| < \frac{2}{[(k-1)!]^3}, \quad k \geq 1. \tag{14}$$

We prove (14) by induction. First, (14) holds for $k = 1$ by direct calculation. Now, assume (14) holds for $2k - 2$, *i.e.*,

$$|b_{2k-2}^{(1)}| < \frac{2}{[(k-2)!]^3}. \tag{15}$$

By (11), it is easy to see that

$$|a_{2k}^{(1)}| < \frac{1}{2^k [(k-1)!]^3}, \quad k > 1. \tag{16}$$

Using the second equation in (8), we have

$$
\begin{aligned}
|b_{2k}^{(1)}| &\le \frac{2|b_{2k-2}^{(1)}|}{2k(2k-1)(2k-2)} + \frac{3|a_{2k}^{(1)}|}{k-1} + \frac{2|a_{2k}^{(1)}|}{2k(2k-1)(2k-2)} \\
&< \frac{2|b_{2k-2}^{(1)}|}{2k(2k-1)(2k-2)} + \frac{1}{[(k-1)!]^3} \\
&< \frac{4}{2k(2k-1)(2k-2)[(k-2)!]^3} + \frac{1}{[(k-1)!]^3} \\
&< \frac{2}{[(k-1)!]^3},
\end{aligned}
\tag{17}
$$

where the first inequality follows from the triangle inequality, the second one follows from (16), the third one follows from the induction assumption. Thus, the even part also converges for all complex numbers by the comparison and root tests, and Stirling's formula. Finally, the convergence of the power series for $n = -1, 0, 2$ follows from (9), (10), (11), (13), and (14), the comparison and root tests, and Stirling's formula. □

Even though (7) was originally derived under the assumption that z is positive real, it indeed makes sense for any $z \neq 0$. Furthermore, it provides a natural analytic continuation [5, p. 283] of J_n to $\mathbb{C}$ with the branch cut along negative real axis and the principal branch for $\ln z$ chosen to be, say, $\mathrm{Im}(\ln z) \in (-\pi, \pi]$.

The asymptotic expansion of J_n via the expansion of U_n is given by [2, §27.5.8]:

$$
J_n(z) \sim \sqrt{\frac{\pi}{3}} \left(\frac{\nu}{3}\right)^{n/2} e^{-\nu} \left(c_0^{(n)} + \frac{c_1^{(n)}}{\nu} + \frac{c_2^{(n)}}{\nu^2} + \cdots \right), \quad z \to \infty, \tag{18}
$$

where $\nu := 3(z/2)^{2/3}$, $c_0^{(n)} = 1$, $c_1^{(n)} = (3n^2 + 3n - 1)/12$, and

$$
\begin{aligned}
12(k+2)c_{k+2}^{(n)} = &-(12k^2 + 36k - 3n^2 - 3n + 25)c_{k+1}^{(n)} \\
&+ \frac{1}{2}(n-2k)(2k+3-n)(2k+3+2n)c_k^{(n)}, \quad k \ge 0.
\end{aligned}
\tag{19}
$$

Once again, (18) was originally derived under the assumption that z is real and positive [1,25]. One may, however, verify that the expansion inside the parentheses on the right hand side of (18) is a formal solution to the third order ODE satisfied by U_n in (6). Furthermore, the exponential factor decays when $\arg z \in (-\frac{3\pi}{4}, \frac{3\pi}{4})$. Hence, (18) is valid for any $z \in \overline{\mathbb{C}^+}$ as $z \to \infty$.

The following lemma is the theoretical foundation of our algorithm.

Lemma 2. *Suppose that $D \subset \mathbb{C}$ is a closed bounded domain that does not contain the origin and the function f is analytic in D. Let $L(z) = \sum_{k=-N_1}^{N_2} c_k z^k$. Then*

(i) if $|f(z) - L(z)| \le \epsilon$ for $z \in \partial D$, then $|f(z) - L(z)| \le \epsilon$ for $z \in D$;
(ii) if $|f(z) - L(z)|/|f(z)| \le \epsilon$ for $z \in \partial D$ and f has no zeros in D, then $|f(z) - L(z)|/|f(z)| \le \epsilon$ for $z \in D$.

Proof. This follows from the analyticity of $L(z)$ on D and the maximum principle [5, p. 133]. □

3. Numerical algorithms

3.1. Series and asymptotic expansions

As we have shown in Lemma 1, the coefficients $a_k^{(n)}$ and $b_k^{(n)}$ in (8)–(10) decay very rapidly and the corresponding series expansions converge for any $z \neq 0$. However, they cannot be used for numerical calculation for large $|z|$ due to cancellation errors and increasing number of terms for achieving the desired precision. Thus, we will use the series expansions only for $|z| < 1$ (*i.e.*, $r = 1$). In this region, both power series $\sum_{k=0}^{\infty} a_k^{(n)} z^k$ and $\sum_{k=0}^{\infty} b_k^{(n)} z^k$ converge exponentially fast and very few terms are needed to reach the desired precision.

The coefficients $c_k^{(n)}$ in (19) diverge rapidly and the asymptotic expansion (18) has to be truncated in order to be of any use. For any truncated asymptotic expansion, it is well-known that its accuracy increases as $|z|$ increases. For a prescribed precision ϵ_{mach}, one needs to determine N_a — the number of terms in the truncated series, and R with $|z| > R$ the applicable region of the truncated series. This is straightforward to determine numerically. We have found that $N_a = 18$ and $R = 120$ are sufficient to achieve 10^{-19} precision for J_n ($n = -1, \ldots, 2$).

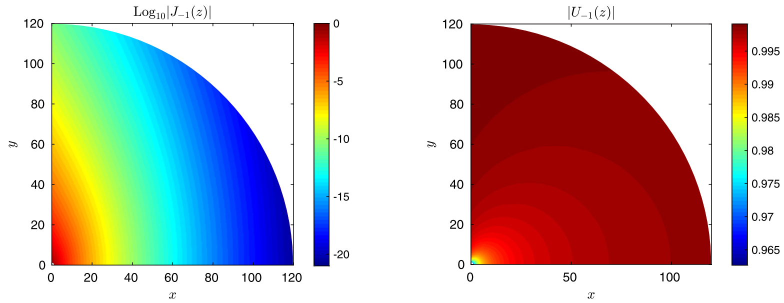

Fig. 1. Dynamic ranges of $J_2(z)$ and $U_2(z)$ in Q. For comparison purposes, both figures are plotted in the variable z. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

3.2. Construction of the Laurent polynomial approximation for the intermediate region

We now discuss the evaluation of J_n in the intermediate region $D = \{z \in \overline{\mathbb{C}^+} \,|\, r \le |z| \le R\}$. First, by the conjugate property (5), we only need to discuss the evaluation of J_n in the first quadrant $Q = \{z \in \mathbb{C} \,|\, r \le |z| \le R,\ 0 \le \arg z \le \frac{\pi}{2}\}$. As discussed in the introduction, it is very difficult to directly approximate $J_n(z)$ in Q due to its large dynamic range. We use the transformation (6) and consider the approximation of $U_n(\nu)$ instead, U_n has a very small dynamic range. Fig. 1 shows $\log_{10}|J_2(z)|$ in Q on the left and $|U_2(z)|$ in Q on the right, where the left panel shows that the magnitude of $J_2(z)$ ranges from 10^{-19} to 10^0, and the right panel shows that the magnitude of $U_2(z)$ ranges from 1.0 to 1.7. Other $J_n(z)$ and $U_n(z)$ exhibit similar pattern with much narrower ranges for $|U_n(z)|$ ($n = -1,\ 0,\ 1$). Thus, we will consider the evaluation of $U_n(\nu)$ in Q.

To this end, we divide Q into several quarter-annulus domains:

$$Q_i := \{z \in \mathbb{C} | r_i \le |z| \le r_{i+1}, 0 \le \arg z \le \frac{\pi}{2}\},\ i = 0, \dots, M-1,\ r_0 = 1,\ r_M = R. \tag{20}$$

We will try to approximate $U_n(\nu)$ in each Q_i via a Laurent polynomial

$$U_n(\nu) \simeq L_n^{(i)}(\nu) = \sum_{k=-N_1}^{N_2} d_k^{(i)} \nu^k, \qquad z \in Q_i. \tag{21}$$

As noted before, $U_n(\nu)$ satisfies a third order ODE with 0 and ∞ as the only singular points [1,25]. Thus, $U_n(\nu)$ is analytic in Q_i. By Lemma 2, in order to guarantee the accuracy of the approximation in the whole domain Q_i, it is sufficient to ensure the same accuracy is achieved on the boundary of Q_i, *i.e.*,

$$\left| U_n(\nu) - \sum_{k=-N_1}^{N_2} d_k^{(i)} \nu^k \right| \le \epsilon, \qquad z \in \partial Q_i. \tag{22}$$

The error-bound in (22) is achieved by solving the least squares problem:

$$\mathbf{Ad}^{(i)} = \mathbf{f}, \quad A_{jk} := \nu_j^k,\ f_j := U_n(\nu_j),\ j = 1, \dots, 4N_b, \tag{23}$$

where $\nu_j := 3(z_j/2)^{2/3}$, and z_j are chosen to be the images of Gauss-Legendre nodes on each segment of ∂Q_i, N_b is chosen to ensure that the error of approximation of $U_n(\nu)$ by the corresponding Legendre polynomial interpolation on each segment of ∂Q_i is bounded by ϵ. The right hand side $\mathbf{f}$ in (23) is computed *via* symbolic software system MATHEMATICA to at least 50 digits. In other words, we do not use the actual analytic Laurent series to approximate U_n on each quarter-annulus Q_i. Instead, a numerical procedure is applied to find much more efficient "modified" Laurent series for approximating U_n on each Q_i.

The linear system (23) is ill-conditioned. However, since we always use $\mathbf{d}^{(i)}$ in the Laurent polynomial approximation to evaluate U_n, we obtain (by the maximum principle) high accuracy in function evaluation in the entire sub-region as long as the residual of the least squares problem (23) is small.

The least squares solver also reveals the numerical rank of $\mathbf{A}$, which is used to obtain the optimal value of $N_{\mathrm{T}} = N_2 - N_1 + 1$, the total number of terms in the Laurent polynomial approximation. It is then straightforward to use a simple search to find the value for N_1, which completes the algorithm for finding a nearly optimal and highly accurate Laurent polynomial approximation for U_n in Q_i.

Remark 1. We would like to emphasize that the Laurent polynomial approximation may not be unique, but this non-uniqueness has no effect on the accuracy of the approximation.

Remark 2. We have computed the integrals

$$I_n = \int_{\partial Q} \frac{J_n'(z)}{J_n(z)} \mathrm{d}z = -\int_{\partial Q} \frac{J_{n-1}(z)}{J_n(z)} \mathrm{d}z \tag{24}$$

for $n = -1, \ldots, 2$ and found numerically that they are all close to zero. By the argument principle [5, p. 152], we have

$$I_n = 2\pi i (Z_n - P_n), \tag{25}$$

where Z_n and P_n denote respectively the number of zeros and poles of $J_n(z)$ inside ∂Q. Since $J_n(z)$ is analytic in Q, it has no poles in Q, *i.e.*, $P_n = 0$. Thus, the fact that I_n is very close to zero shows that $Z_n = 0$, that is, J_n has no zeros in Q. Further numerical investigation shows that functions $|U_n(\nu)|$ $(n = -1, \ldots, 2)$ range from 0.95 to 1.7 on ∂Q. Combining these two facts, we conclude that the absolute error bound on the approximation of U_n gives roughly the same relative error bound.

3.3. Evaluation of J_n for $n = -1, \ldots, 2$

Once the coefficients of Laurent polynomial approximation for each sub-region are obtained and stored, the evaluation of $J_n(z)$ is straightforward. That is, we first compute $|z|$ to decide on which region the point lies, then use the proper representation to evaluate $J_n(z)$ accordingly. We summarize the algorithm for calculating $J_n(z)$ for $z \in \overline{\mathbb{C}^+}$, $n = -1, \ldots, 2$ in Algorithm 1.

Algorithm 1 Evaluation of $J_n(z)$ for $z \in \overline{\mathbb{C}^+}$.

```
procedure ABRAM(z, f)
    ▷ Input parameter: z – the complex number for which the Abramowitz function J_n is to be evaluated.
    ▷ Output parameter: f – the value of Abramowitz function J_n(z).
  assert Re(z) ≥ 0.
  if |z| ≤ 1 then                                              ▷ z is in the series expansion region.
      Use the series expansion (7) to evaluate f = J_n(z).
  else if |z| ≥ 120 then                                       ▷ z is in the asymptotic region.
      Set ν = 3(z/2)^(2/3).
      Use the asymptotic expansion (18) to compute U_n(ν).
      Set f = sqrt(π/3) (ν/3)^(n/2) e^(−ν) U_n(ν).
  else                                                         ▷ z is in the intermediate region.
      Set ν = 3(z/2)^(2/3).
      Use a precomputed Laurent polynomial approximation (21) to compute U_n(ν).
      Set f = sqrt(π/3) (ν/3)^(n/2) e^(−ν) U_n(ν).
  end if
end procedure
```

Remark 3. All these expansions can be converted into a polynomial of a certain transformed variable. We use Horner's method [23, §4.6.4] to evaluate the polynomial in the optimal number of arithmetic operations.

Remark 4. The accuracy of $J_n(z)$ deteriorates as $|z|$ increases since the condition number of evaluating the exponential function $e^{-\nu}$ is $|\nu|$ and ν has to be evaluated numerically via $\nu = 3(z/2)^{2/3}$.

3.4. Evaluation of J_n for $n > 2$

In [10], it is shown that (4) is stable in both directions when z is a positive real number. We have implemented the forward recurrence to evaluate $J_n(z)$ for $n > 2$. We have not observed any numerical instability during our numerical tests for $z \in \overline{\mathbb{C}^+}$.

4. Numerical results

We have implemented the algorithms in Section 3 and the code is available at https://github.com/zgimbutas/abramowitz. Numerical experiments were performed on a desktop computer with a 3.10 GHz Intel(R) Xeon(R) CPU.

For the series expansion (7), a straightforward calculation shows that 18 terms in $\sum b_k^{(n)} z^k$ and 9 nonzero terms in $\sum a_k^{(n)} z^k$ are needed to reach 10^{-19} precision for J_n $(n = -1, \ldots, 2)$. For the asymptotic expansion (18), we find that it is sufficient to choose $N_a = 18$, $R = 120$ for 10^{-19} precision. All coefficients are precomputed with 50 digit precision.

For the intermediate region, we divide $|z|$ on [1, 120] into three subintervals [1, 3], [3, 15], [15, 120] and Q into Q_1, Q_2, Q_3, respectively. We use IEEE binary128 precision to carry out the precomputation step and solve the least squares

Table 1
The relative L^∞ error of Algorithm 1 over 100,000 uniformly distributed random points in $\overline{\mathbb{C}^+}$. The reference value is computed via MATHEMATICA to at least 50 digit accuracy. S denotes the series expansion region and A denotes the asymptotic expansion region.

	S	Q_1	Q_2	Q_3	A
J_{-1}	1.5×10^{-15}	2.1×10^{-15}	4.4×10^{-16}	6.4×10^{-16}	8.6×10^{-16}
J_0	1.3×10^{-15}	2.4×10^{-15}	2.2×10^{-16}	2.2×10^{-16}	2.2×10^{-16}
J_1	1.1×10^{-15}	2.4×10^{-15}	4.7×10^{-16}	6.0×10^{-16}	8.0×10^{-16}
J_2	1.2×10^{-15}	2.9×10^{-15}	5.6×10^{-16}	8.4×10^{-16}	1.2×10^{-15}

Table 2
The maximum relative error for evaluating J_{100} using the forward recurrence relation (4) over 100,000 uniformly distributed random points in the domain $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq 0,\ 0 < |z| < 1000\}$. The reference values are calculated using MATHEMATICA with 240-digit precision arithmetic.

S	Q_1	Q_2	Q_3	A
1.3×10^{-15}	2.9×10^{-15}	1.3×10^{-15}	2.0×10^{-15}	3.7×10^{-15}

problem with 10^{-20} threshold for the residual. We have found that for Q_1 we need $N_2 = 11$, $N_T = 30$ for J_0 and J_1, $N_2 = 10$, $N_T = 32$ for J_{-1}, and $N_2 = 11$, $N_T = 32$ for J_2. For all four functions J_n $(n = -1, 0, 1, 2)$, we need $N_2 = 0$, $N_T = 30$ for Q_2 and $N_2 = 0$, $N_T = 20$ for Q_3. The coefficients of Laurent polynomial approximations for J_n $(n = -1, 0, 1, 2)$ on Q_i $(i = 1, 2, 3)$ are listed in Tables B.4–B.15 in Appendix B.

Remark 5. The coefficients in Tables B.4–B.15 for Q_2 and Q_3 do not have small norms. However, for Q_2, $\left|\frac{1}{\nu}\right| \leq \frac{1}{3(3/2)^{(2/3)}} = 0.254\ldots$; and for Q_3, $\left|\frac{1}{\nu}\right| \leq \frac{1}{3(15/2)^{(2/3)}} \approx 0.087$. It is easy to see that terms $c_j \left(\frac{1}{\nu}\right)^j$ decrease as j increases. Alternatively, we could consider the Laurent series of the form $\sum \tilde{c}_j \left(\frac{\nu_i}{\nu}\right)^j$ with $\nu_i = 3(r_i/2)^{(2/3)}$ (r_i is the lower bound for $|z|$ in Q_i). Then the coefficient vector $\tilde{\mathbf{c}}$ will have small norm, as required in [7,4]. However, this corresponds to the column scaling in the least squares matrix and almost all methods for solving the least squares problems do column normalization. Thus, it has no effect on the accuracy of the solution and stability of the algorithm.

Remark 6. The partition of the sub-regions is by no means optimal or unique. There is an obvious trade-off between the number of sub-regions and the number of terms in the Laurent polynomial approximation. For example, one may use a finer partition for the regions closer to the origin. We have tried to divide the intermediate region into 14 regions with $Q_i := \{z \in \overline{\mathbb{C}^+} | (\sqrt{2})^{i-1} \leq |z| \leq (\sqrt{2})^i\}$ $(i = 1, \ldots, 14)$, and we observe that only 20 terms are needed for all regions. However, our numerical experiments indicate that the partition has very mild effect on the overall performance (*i.e.*, speed and accuracy) of the algorithm.

4.1. Accuracy check

We first check the accuracy of Algorithm 1. The reference function values are calculated via MATHEMATICA to at least 50 digit accuracy. The error is measured in terms of maximum relative error, *i.e.*,

$$E := \max_i \frac{|\hat{J}_n(z_i) - \tilde{J}_n(z_i)|}{|\tilde{J}_n(z_i)|},$$

where $\tilde{J}_n(z_i) := e^{\nu_i} J_n(z_i)$ $(\nu_i := 3(z_i/2)^{2/3})$ is the reference value of the scaled Abramowitz function computed via MATHEMATICA, and $\hat{J}_n(z_i)$ is the value computed via our algorithm. The points z_i are sampled randomly with uniform distribution in both its magnitude and angle in $\overline{\mathbb{C}^+}$. Table 1 lists the errors for evaluating $\tilde{J}_n$ $(n = -1, 0, 1, 2)$ in various regions, where we observe that the errors are within $10\epsilon_{\text{mach}}$ with the machine epsilon $\epsilon_{\text{mach}} \approx 2.22 \times 10^{-16}$ for IEEE double precision. In general, the errors in the first intermediate region Q_1 are slightly bigger due to mild cancellation errors.

For $n > 2$, extensive numerical experiments indicate that the forward recurrence relation (4) is stable for evaluating J_n in $\overline{\mathbb{C}^+}$. The relative errors are shown in Table 2 for a typical run.

4.2. Timing results

Since all three representations (*i.e.*, Laurent polynomials, series and asymptotic expansions) mainly involve polynomials of degree less than 30, the algorithm takes about constant time per function evaluation in $\overline{\mathbb{C}^+}$. We have tested the CPU time of Algorithm 1 for evaluating $\tilde{J}_n(z)$ and compared it with that of evaluating the complex error function $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$. The complex error function is a well studied special function that has received much attention in the community of scientific computing (cf., *e.g.*, [9,12,13,19,29–31,33,34,36,37]). Here we use the well-regarded Faddeeva package [22] to evaluate erf(z).

Table 3
The total CPU time T in seconds for evaluating $J_n(z)$ using Algorithm 1 and the error function erf(z) over 1, 000, 000 uniformly distributed random points in $0 \le \mathrm{Re}(z) \le 10$, $0 \le \mathrm{Im}(z) \le 10$.

	$J_{-1}(z)$	$J_0(z)$	$J_1(z)$	$J_2(z)$	erf(z)
T	0.44	0.41	0.44	0.41	0.34

The results are shown in Table 3. First, we note that erf(z) is an entire function which is somewhat simpler than the Abramowitz functions and the Faddeeva package guarantees about 10^{-13} accuracy. Second, the numbers of terms in all three representations in our algorithm are chosen so that 10^{-19} precision may be achieved if the calculation were carried out in 80-bit floating-point arithmetic (it achieves about 10^{-15} accuracy in double precision arithmetic as shown in Table 1).

In the asymptotic region, our algorithm is slightly faster than the numbers shown in Table 3, while the Faddeeva package is faster by a factor of about 3. However, the efficiency in the asymptotic region (*i.e.*, the asymptotic expansion) heavily depends on the properties of the given special functions and is thus independent of the algorithm for other regions. Combining all these factors, we may conclude that our algorithm is competitive with the highly optimized Faddeeva package.

5. Conclusions and further discussions

We have designed an efficient and accurate algorithm for the evaluation of Abramowitz functions J_n in the right half of the complex plane. Some useful observations in the design of the algorithm are applicable for evaluating many other special functions in the complex domain. First, it is better to pull out the leading asymptotic factor from the given function when $|z|$ is large. Second, the maximum principle reduces the dimensionality of the approximation problem by one. Third, the least squares scheme is generally a reliable and accurate method to find an approximation of a prescribed form. That is, analytic representations should be used with caution even if they are available, as they often lead to large cancellation error or very inefficient approximations or both.

Finally, though we have used Laurent polynomials to approximate Abramowitz functions in the intermediate region, there are many other representations for function approximations. This includes truncated series expansion, rational functions (cf., *e.g.*, [15]), etc. We have actually tested the truncated series expansion in the sub-region (*i.e.*, Q_1) closest to the origin for J_n. Our numerical experiments indicate that the performance is about the same as the one presented in this paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Zeros of $J_n(z)$

We have used NIntegrate in Mathematica to evaluate I_n defined in (24). When WorkingPrecision is set to 100, $|I_n|$ are about 10^{-59} for $n = -1, 0, 1, 2$. When it is set to 200, the values of $|I_n|$ decrease to 10^{-160}. By the argument principle, I_n can only take integral multiples of $2\pi i$. Thus, the numerical calculation clearly shows that J_n $(n = -1, 0, 1, 2)$ have no zeros in the intermediate region Q. Analytically, we can only show that J_n has no zeros in the sector $|\arg(z)| \le \frac{\pi}{4}$. The proof is presented below.

Lemma 3. *If $z_0 \in \mathbb{C}$ is a zero of $J_n(z)$, then so is $\bar{z}_0$.*

Proof. This simply follows from the conjugate property (5). $\square$

Lemma 4. *Suppose that $n \ge 0$. Then $J_n(z)$ has no zero in the sector $|\arg z| \le \frac{\pi}{4}$.*

Proof. Let $z_0 = x_0 + iy_0 \in \overline{\mathbb{C}^+}$ be a zero of $J_n(z)$. Then by Lemma 3, $\bar{z}_0$ is also a zero of $J_n(z)$. Consider functions $f(t) = J_n(z_0 t)$ and $g(t) = J_n(\bar{z}_0 t)$. Then $f(1) = g(1) = 0$, and f, g and their derivatives decay exponentially fast to 0 as $t \to \infty$ by the asymptotic expansion (18).

The differential equation (2) implies that

$$tf'''(t) - (n-1)f''(t) + 2z_0^2 f(t) = 0, \tag{A.1}$$

$$tg'''(t) - (n-1)g''(t) + 2\bar{z}_0^2 g(t) = 0. \tag{A.2}$$

Multiplying both sides of (A.1) by g, integrating both sides from 1 to ∞, and performing integration by parts, we obtain

$$\begin{aligned} 0 &= \int_1^\infty [tf'''g - (n-1)f''g + 2z_0^2 fg]\mathrm{d}t \\ &= tgf''\Big|_1^\infty - \int_1^\infty f''(g + g't)\mathrm{d}t + \int_1^\infty [-(n-1)f''g + 2z_0^2 fg]\mathrm{d}t \\ &= \int_1^\infty [-tf''g' - nf''g + 2z_0^2 fg]\mathrm{d}t \\ &= \int_1^\infty [-tf''g' + nf'g' + 2z_0^2 fg]\mathrm{d}t. \end{aligned} \tag{A.3}$$

Similarly,

$$0 = \int_1^\infty [-tf'g'' + nf'g' + 2\bar{z}_0^2 fg]\mathrm{d}t. \tag{A.4}$$

Moreover,

$$\begin{aligned} \int_1^\infty [-tf'g'' - tf'g'']\mathrm{d}t &= -\int_1^\infty t\mathrm{d}(f'g') \\ &= -tf'g'\Big|_1^\infty + \int_1^\infty f'g'\mathrm{d}t \\ &= f'(1)g'(1) + \int_1^\infty f'g'\mathrm{d}t. \end{aligned} \tag{A.5}$$

Adding (A.3), (A.4) and using (A.5) to simplify the result, we obtain

$$0 = f'(1)g'(1) + (2n+1)\int_1^\infty f'g'\mathrm{d}t + 2(z_0^2 + \bar{z}_0^2)\int_1^\infty fg\mathrm{d}t. \tag{A.6}$$

Rearranging (A.6), we have

$$4(y_0^2 - x_0^2)\int_1^\infty |J_n(z_0 t)|^2\mathrm{d}t = |z_0|^2\left|J_n'(z_0)\right|^2 + (2n+1)|z_0|^2\int_1^\infty \left|J_n'(z_0 t)\right|^2\mathrm{d}t. \tag{A.7}$$

Since the right side of (A.7) and the integral on its left side are both positive, we must have $y_0^2 - x_0^2 > 0$ and the lemma follows. $\square$

Lemma 5. *$J_n(z)$ has no zero in $D = \{z \in \overline{\mathbb{C}^+} \,|\, |z| > R\}$, where R is sufficiently large.*

Proof. Subtracting (A.4) from (A.3), we have

$$0=\int_1^\infty t(f'g''-f''g')\mathrm{d}t+2(z_0^2-\bar{z}_0^2)\int_1^\infty fg\mathrm{d}t. \tag{A.8}$$

That is,

$$4x_0y_0\int_1^\infty |J_n(z_0t)|^2\,\mathrm{d}t=|z_0|^2\int_1^\infty \mathrm{Im}\,(\bar{z}_0tJ_{n-1}(z_0t)J_{n-2}(\bar{z}_0t))\,\mathrm{d}t. \tag{A.9}$$

In the domain D, $J_n(z)$ is well approximated by the leading term of its asymptotic expansion. Let $z_0=r_0e^{i\theta_0}$ with $r_0>0$ and $\theta_0\in[-\pi/2,\pi/2]$. Substituting the leading terms of the asymptotic expansions into both sides of (A.9) and simplifying the resulting expressions, we obtain

$$\sin(2\theta_0)\sim-\sin(2\theta_0/3). \tag{A.10}$$

In other words, two sides of (A.9) have opposite sign unless they are both equal to zero, *i.e.*, unless $\theta_0=0$ or z_0 is a positive real number. However, $J_n(x)>0$ when $x>0$, as seen from its integral representation (1). And the lemma follows. □

Appendix B. The coefficients of Laurent polynomial approximations for J_n

We list the coefficients c_j of Laurent polynomial approximations for evaluating J_n ($n=-1$, 0, 1, and 2) on each quarter-annulus domain Q_i ($i=1$, 2, and 3) in Tables B.4–B.15. That is,

$$J_n(z)\approx\sqrt{\frac{\pi}{3}}\left(\frac{\nu}{3}\right)^{n/2}e^{-\nu}\nu^{N_2}\sum_{j=0}^{N_T-1}c_j\left(\frac{1}{\nu}\right)^j, \tag{B.1}$$

Table B.4
The coefficients c_j ($j=0,\ldots,31$) of the Laurent polynomial approximation given by (B.1) to evaluate $J_{-1}(z)$ to 19-digit precision in $Q_1:=\{z\in\mathbb{C}\mid \mathrm{Re}(z)\geq 0,\mathrm{Im}(z)\geq 0, 1\leq|z|\leq 3\}$. $N_2=10$.

Real part	Imaginary part
$0.508\,404\,632\,082\,606\,781\,52\times10^{-17}$	$-0.174\,608\,152\,994\,637\,499\,48\times10^{-15}$
$-0.745\,912\,235\,026\,426\,206\,60\times10^{-14}$	$0.124\,626\,002\,002\,964\,530\,12\times10^{-13}$
$0.510\,342\,448\,563\,248\,242\,07\times10^{-12}$	$-0.294\,298\,471\,469\,682\,176\,69\times10^{-12}$
$-0.155\,278\,534\,850\,271\,007\,09\times10^{-10}$	$0.263\,158\,514\,306\,763\,567\,96\times10^{-12}$
$0.264\,414\,045\,122\,879\,630\,95\times10^{-9}$	$0.139\,834\,751\,397\,682\,449\,07\times10^{-9}$
$-0.247\,487\,638\,713\,530\,933\,63\times10^{-8}$	$-0.374\,213\,198\,230\,171\,159\,33\times10^{-8}$
$0.482\,268\,582\,740\,909\,041\,08\times10^{-8}$	$0.543\,401\,289\,326\,601\,410\,72\times10^{-7}$
$0.216\,253\,553\,725\,866\,075\,08\times10^{-6}$	$-0.508\,720\,998\,708\,511\,613\,98\times10^{-6}$
$-0.368\,717\,051\,178\,481\,237\,97\times10^{-5}$	$0.301\,340\,166\,557\,595\,939\,20\times10^{-5}$
$0.346\,284\,048\,895\,070\,301\,60\times10^{-4}$	$-0.739\,108\,230\,704\,056\,172\,19\times10^{-5}$
$0.999\,777\,374\,590\,696\,606\,94$	$-0.576\,030\,836\,245\,300\,251\,51\times10^{-4}$
$-0.823\,278\,581\,628\,196\,930\,45\times10^{-1}$	$0.849\,768\,580\,372\,614\,021\,53\times10^{-3}$
$0.619\,747\,893\,545\,737\,665\,66\times10^{-3}$	$-0.605\,139\,381\,903\,151\,159\,82\times10^{-2}$
$0.566\,151\,822\,947\,680\,796\,37\times10^{-1}$	$0.302\,907\,889\,349\,127\,277\,55\times10^{-1}$
$-0.135\,136\,779\,991\,090\,296\,79$	$-0.115\,958\,015\,111\,906\,821\,78$
$0.209\,718\,152\,961\,885\,801\,67$	$0.349\,173\,944\,608\,271\,288\,28$
$-0.135\,593\,023\,999\,587\,351\,43$	$-0.830\,316\,528\,148\,841\,088\,13$
$-0.399\,898\,988\,541\,072\,716\,42$	$0.153\,226\,178\,650\,705\,161\,48\times10^{1}$
$0.173\,982\,716\,383\,338\,408\,50\times10^{1}$	$-0.207\,207\,422\,408\,323\,786\,54\times10^{1}$
$-0.382\,180\,642\,772\,971\,751\,42\times10^{1}$	$0.166\,556\,450\,780\,677\,180\,66\times10^{1}$
$0.574\,046\,443\,633\,433\,309\,31\times10^{1}$	$0.364\,413\,737\,004\,387\,528\,95$
$-0.602\,185\,574\,538\,225\,680\,30\times10^{1}$	$-0.366\,998\,894\,320\,715\,544\,24\times10^{1}$
$0.386\,640\,982\,933\,784\,632\,50\times10^{1}$	$0.656\,323\,653\,065\,047\,142\,02\times10^{1}$
$-0.256\,979\,491\,396\,712\,909\,42$	$-0.717\,303\,344\,007\,278\,311\,01\times10^{1}$
$-0.261\,383\,686\,683\,662\,857\,52\times10^{1}$	$0.521\,017\,298\,717\,892\,892\,59\times10^{1}$
$0.331\,280\,620\,490\,481\,945\,83\times10^{1}$	$-0.227\,447\,202\,390\,353\,554\,39\times10^{1}$
$-0.229\,475\,501\,388\,204\,966\,70\times10^{1}$	$0.218\,248\,685\,401\,302\,644\,77$
$0.979\,988\,419\,462\,684\,815\,18$	$0.430\,819\,146\,717\,141\,565\,57$
$-0.232\,732\,788\,689\,187\,012\,41$	$-0.308\,816\,943\,645\,394\,016\,56$
$0.135\,738\,167\,246\,491\,846\,59\times10^{-1}$	$0.101\,035\,914\,706\,240\,915\,10$
$0.647\,176\,653\,107\,878\,954\,82\times10^{-2}$	$-0.162\,049\,762\,735\,533\,721\,87\times10^{-1}$
$-0.113\,530\,822\,404\,964\,078\,13\times10^{-2}$	$0.911\,116\,455\,095\,110\,768\,69\times10^{-3}$

Table B.5
Similar to Table B.4, c_j ($j = 0, \ldots, 29$) for $J_{-1}(z)$ in $Q_2 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 3 \leq |z| \leq 15\}$. $N_2 = 0$.

Real part	Imaginary part
$0.999\,999\,999\,999\,961\,653\,01$	$0.141\,806\,832\,347\,584\,925\,36 \times 10^{-12}$
$-0.833\,333\,333\,158\,883\,431\,56 \times 10^{-1}$	$-0.183\,554\,755\,025\,424\,015\,39 \times 10^{-10}$
$0.347\,222\,020\,992\,143\,062\,18 \times 10^{-2}$	$0.664\,295\,120\,902\,317\,816\,28 \times 10^{-9}$
$0.554\,592\,179\,369\,355\,251\,95 \times 10^{-1}$	$0.192\,090\,703\,259\,659\,827\,96 \times 10^{-7}$
$-0.174\,770\,093\,094\,885\,488\,35$	$-0.272\,358\,724\,156\,552\,434\,93 \times 10^{-5}$
$0.475\,579\,850\,792\,853\,198\,78$	$0.123\,293\,398\,001\,490\,185\,87 \times 10^{-3}$
$-0.120\,447\,196\,014\,882\,443\,81 \times 10^{1}$	$-0.333\,799\,891\,312\,548\,583\,84 \times 10^{-2}$
$0.241\,605\,349\,770\,769\,985\,85 \times 10^{1}$	$0.599\,204\,046\,472\,687\,860\,33 \times 10^{-1}$
$0.714\,019\,341\,240\,202\,213\,24$	$-0.697\,646\,996\,515\,841\,414\,17$
$-0.603\,675\,406\,822\,103\,741\,45 \times 10^{2}$	$0.386\,077\,962\,390\,076\,721\,58 \times 10^{1}$
$0.605\,451\,350\,482\,099\,861\,87 \times 10^{3}$	$0.324\,292\,794\,756\,158\,453\,98 \times 10^{2}$
$-0.459\,463\,671\,083\,445\,667\,27 \times 10^{4}$	$-0.108\,419\,490\,933\,568\,204\,60 \times 10^{4}$
$0.283\,585\,737\,521\,554\,577\,24 \times 10^{5}$	$0.147\,667\,142\,274\,551\,196\,33 \times 10^{5}$
$-0.135\,548\,409\,522\,738\,422\,75 \times 10^{6}$	$-0.136\,296\,993\,207\,088\,386\,68 \times 10^{6}$
$0.427\,223\,358\,854\,162\,769\,83 \times 10^{6}$	$0.936\,405\,385\,625\,510\,558\,57 \times 10^{6}$
$-0.187\,177\,344\,190\,171\,379\,32 \times 10^{6}$	$-0.490\,146\,368\,535\,523\,402\,06 \times 10^{7}$
$-0.766\,746\,981\,331\,305\,086\,47 \times 10^{7}$	$0.192\,939\,969\,196\,334\,868\,42 \times 10^{8}$
$0.566\,216\,201\,198\,774\,900\,02 \times 10^{8}$	$-0.533\,796\,877\,619\,324\,138\,68 \times 10^{8}$
$-0.245\,446\,260\,540\,985\,694\,13 \times 10^{9}$	$0.769\,699\,843\,063\,185\,308\,11 \times 10^{8}$
$0.736\,524\,060\,830\,223\,396\,55 \times 10^{9}$	$0.118\,988\,708\,450\,170\,908\,57 \times 10^{9}$
$-0.152\,002\,939\,636\,990\,115\,85 \times 10^{10}$	$-0.111\,677\,447\,076\,659\,508\,37 \times 10^{10}$
$0.181\,576\,363\,392\,016\,524\,60 \times 10^{10}$	$0.370\,018\,124\,502\,864\,503\,98 \times 10^{10}$
$0.236\,970\,051\,052\,140\,740\,56 \times 10^{9}$	$-0.769\,732\,352\,121\,328\,283\,29 \times 10^{10}$
$-0.605\,418\,652\,742\,096\,914\,12 \times 10^{10}$	$0.105\,094\,370\,501\,909\,818\,95 \times 10^{11}$
$0.134\,473\,475\,911\,834\,175\,29 \times 10^{11}$	$-0.828\,696\,601\,026\,570\,423\,17 \times 10^{10}$
$-0.165\,386\,003\,269\,058\,328\,99 \times 10^{11}$	$0.878\,893\,331\,337\,860\,555\,48 \times 10^{9}$
$0.121\,080\,386\,549\,490\,128\,13 \times 10^{11}$	$0.583\,201\,605\,488\,309\,253\,71 \times 10^{10}$
$-0.458\,813\,237\,705\,320\,160\,82 \times 10^{10}$	$-0.645\,912\,107\,582\,825\,318\,47 \times 10^{10}$
$0.335\,597\,695\,613\,487\,923\,57 \times 10^{9}$	$0.300\,129\,749\,468\,952\,920\,83 \times 10^{10}$
$0.215\,904\,420\,673\,766\,075\,26 \times 10^{9}$	$-0.515\,536\,276\,388\,964\,358\,29 \times 10^{9}$

Table B.6
Similar to Table B.4, c_j ($j = 0, \ldots, 19$) for $J_{-1}(z)$ in $Q_3 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 15 \leq |z| \leq 120\}$. $N_2 = 0$.

Real part	Imaginary part
$0.100\,000\,000\,000\,000\,002\,11 \times 10^{1}$	$0.178\,673\,059\,693\,174\,710\,10 \times 10^{-16}$
$-0.833\,333\,333\,333\,370\,624\,47 \times 10^{-1}$	$-0.977\,231\,664\,374\,838\,609\,03 \times 10^{-14}$
$0.347\,222\,222\,196\,623\,078\,73 \times 10^{-2}$	$0.185\,750\,995\,314\,155\,635\,50 \times 10^{-11}$
$0.554\,591\,050\,637\,792\,915\,69 \times 10^{-1}$	$-0.170\,367\,606\,540\,725\,010\,33 \times 10^{-9}$
$-0.174\,766\,524\,353\,726\,066\,09$	$0.720\,973\,568\,196\,242\,083\,86 \times 10^{-8}$
$0.475\,521\,803\,699\,479\,611\,03$	$0.458\,867\,227\,971\,042\,147\,45 \times 10^{-7}$
$-0.120\,457\,482\,849\,866\,306\,05 \times 10^{1}$	$-0.234\,325\,765\,233\,684\,458\,86 \times 10^{-4}$
$0.244\,764\,600\,694\,641\,417\,08 \times 10^{1}$	$0.144\,884\,043\,026\,931\,818\,55 \times 10^{-2}$
$-0.194\,435\,702\,473\,795\,297\,07$	$-0.512\,841\,062\,799\,477\,565\,51 \times 10^{-1}$
$-0.447\,750\,705\,123\,945\,998\,08 \times 10^{2}$	$0.119\,733\,980\,838\,481\,655\,62 \times 10^{1}$
$0.421\,634\,597\,094\,092\,230\,79 \times 10^{3}$	$-0.187\,325\,842\,170\,831\,902\,17 \times 10^{2}$
$-0.309\,908\,462\,261\,138\,328\,47 \times 10^{4}$	$0.180\,645\,013\,045\,163\,968\,11 \times 10^{3}$
$0.209\,138\,849\,163\,905\,853\,68 \times 10^{5}$	$-0.592\,962\,111\,536\,245\,581\,48 \times 10^{3}$
$-0.128\,959\,963\,550\,548\,747\,85 \times 10^{6}$	$-0.108\,214\,826\,602\,820\,261\,69 \times 10^{5}$
$0.670\,852\,955\,256\,816\,110\,41 \times 10^{6}$	$0.198\,088\,627\,508\,652\,376\,78 \times 10^{6}$
$-0.263\,378\,771\,825\,866\,161\,50 \times 10^{7}$	$-0.168\,133\,321\,132\,881\,930\,78 \times 10^{7}$
$0.670\,963\,418\,948\,175\,610\,42 \times 10^{7}$	$0.859\,319\,504\,013\,814\,145\,32 \times 10^{7}$
$-0.765\,712\,819\,081\,208\,414\,43 \times 10^{7}$	$-0.265\,726\,278\,587\,171\,822\,34 \times 10^{8}$
$-0.598\,034\,480\,268\,757\,485\,09 \times 10^{7}$	$0.448\,016\,842\,841\,870\,047\,03 \times 10^{8}$
$0.192\,093\,223\,477\,658\,710\,37 \times 10^{8}$	$-0.300\,660\,136\,102\,592\,770\,74 \times 10^{8}$

Table B.7
The coefficients c_j ($j = 0, \ldots, 29$) of the Laurent polynomial approximation given by (B.1) to evaluate $J_0(z)$ to 19-digit precision in $Q_1 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 1 \leq |z| \leq 3\}$. $N_2 = 11$.

Real part	Imaginary part
$-0.90832607641433626723 \times 10^{-16}$	$-0.12971716857438253177 \times 10^{-15}$
$0.12389804620230878343 \times 10^{-14}$	$0.12374086345769475560 \times 10^{-13}$
$0.18925665936446973863 \times 10^{-12}$	$-0.43594042425001909808 \times 10^{-12}$
$-0.93436124699082728782 \times 10^{-11}$	$0.71610240171455947215 \times 10^{-11}$
$0.21005471373426356192 \times 10^{-9}$	$-0.31570720100508497322 \times 10^{-10}$
$-0.27921469604412283831 \times 10^{-8}$	$-0.10267754766776108185 \times 10^{-8}$
$0.22210697286892781643 \times 10^{-7}$	$0.25281961277933484578 \times 10^{-7}$
$-0.71494613675879873372 \times 10^{-7}$	$-0.30771058781715872427 \times 10^{-6}$
$-0.63954539217286436591 \times 10^{-6}$	$0.24264961215641857661 \times 10^{-5}$
$0.11477599934330755236 \times 10^{-4}$	$-0.12692783111400141826 \times 10^{-4}$
$-0.94447601385518738118 \times 10^{-4}$	$0.36582122360442313063 \times 10^{-4}$
$0.10005227131416389419 \times 10^{1}$	$0.42527679632933790645 \times 10^{-4}$
$-0.85416339695541973974 \times 10^{-1}$	$-0.11668274275597700974 \times 10^{-2}$
$0.92680088758957499003 \times 10^{-1}$	$0.75525130431164108091 \times 10^{-2}$
-0.12830962183399732914	$-0.31999279672322587569 \times 10^{-1}$
0.18262801902460105636	0.10112601789451786253
-0.22086524963618477505	-0.24794786230219646677
0.16700979504519707012	0.47574461283438653640
$0.62230841342939847177 \times 10^{-1}$	-0.70546898276037268906
-0.46160590435660301333	0.77405336431438665012
0.86282905145667261861	-0.54883285003677036633
$-0.10157725409050061178 \times 10^{1}$	$0.89653554833115686588 \times 10^{-1}$
0.81127042933258073193	0.34223426976661392785
-0.40588870780744941328	-0.50414950527612358637
$0.70646295770216518095 \times 10^{-1}$	0.39020053902964327900
$0.62576278207207491689 \times 10^{-1}$	-0.18665248819340615755
$-0.56069463746156990540 \times 10^{-1}$	$0.51369345024189480453 \times 10^{-1}$
$0.20854565059593331201 \times 10^{-1}$	$-0.49622460298807761080 \times 10^{-2}$
$-0.37736411371630856848 \times 10^{-2}$	$-0.10706620427594972634 \times 10^{-2}$
$0.24658062816300990462 \times 10^{-3}$	$0.24793548656986345025 \times 10^{-3}$

Table B.8
Similar to Table B.7, c_j ($j = 0, \ldots, 29$) for $J_0(z)$ in $Q_2 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 3 \leq |z| \leq 15\}$. $N_2 = 0$.

Real part	Imaginary part
0.99999999999988637217	$-0.86635400939375232846 \times 10^{-13}$
$-0.83333333323131499972 \times 10^{-1}$	$0.22519321671024025250 \times 10^{-10}$
$0.86805555689429826898 \times 10^{-1}$	$-0.20734497580441697938 \times 10^{-8}$
-0.11815206514175737947	$0.96339270441630372077 \times 10^{-7}$
0.17969333057187777654	$-0.23057043651727157918 \times 10^{-5}$
-0.24337342169790375842	$0.98098354524518453957 \times 10^{-5}$
$0.14764429473872620763 \times 10^{-1}$	$0.12782321450918857049 \times 10^{-2}$
$0.23309627937902507555 \times 10^{1}$	$-0.51073707172260220779 \times 10^{-1}$
$-0.16480981490288764106 \times 10^{2}$	$0.11291585827876199238 \times 10^{1}$
$0.92305117544447684520 \times 10^{2}$	$-0.17262096251706180600 \times 10^{2}$
$-0.51600417922753718015 \times 10^{3}$	$0.19328752764190907274 \times 10^{3}$
$0.32464931691139105961 \times 10^{4}$	$-0.15901053336317843819 \times 10^{4}$
$-0.22314060512920598345 \times 10^{5}$	$0.90599663123843335441 \times 10^{4}$
$0.14604708231321899614 \times 10^{6}$	$-0.26233323039837437629 \times 10^{5}$
$-0.81145513740976803311 \times 10^{6}$	$-0.98104807756320344084 \times 10^{5}$
$0.35457971304583219348 \times 10^{7}$	$0.18294948704040264718 \times 10^{7}$
$-0.11084057351725538629 \times 10^{8}$	$-0.13170160809055557556 \times 10^{8}$
$0.17946742587727355394 \times 10^{8}$	$0.62986605485232028006 \times 10^{8}$
$0.35800094666248698896 \times 10^{8}$	$-0.21614451081432945159 \times 10^{9}$
$-0.37304955427569800721 \times 10^{9}$	$0.52194699072674118229 \times 10^{9}$
$0.14479420488451058753 \times 10^{10}$	$-0.75867692917492941354 \times 10^{9}$
$-0.35951788816564166863 \times 10^{10}$	$0.22618398696580184017 \times 10^{8}$
$0.60001115021506662033 \times 10^{10}$	$0.31006219604826621451 \times 10^{10}$
$-0.60780548332669974319 \times 10^{10}$	$-0.87891548726761143921 \times 10^{10}$
$0.15576242961336566425 \times 10^{10}$	$0.13885136549901956758 \times 10^{11}$
$0.55250083598538675266 \times 10^{10}$	$-0.13622496533575828228 \times 10^{11}$
$-0.92625760934489411655 \times 10^{10}$	$0.75732780185372212772 \times 10^{10}$
$0.69543035636142439747 \times 10^{10}$	$-0.12850366647773799533 \times 10^{10}$
$-0.25630570764916110006 \times 10^{10}$	$-0.85541751450691424086 \times 10^{9}$
$0.33863352297553708594 \times 10^{9}$	$0.36939690751181780209 \times 10^{9}$

Table B.9
Similar to Table B.7, c_j $(j = 0, \ldots, 19)$ for $J_0(z)$ in $Q_3 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 15 \leq |z| \leq 120\}$. $N_2 = 0$.

Real part	Imaginary part
0.999 999 999 999 999 969 30	$0.175\,938\,640\,339\,119\,357\,46 \times 10^{-16}$
$-0.833\,333\,333\,333\,199\,009\,02 \times 10^{-1}$	$-0.168\,461\,478\,982\,929\,779\,50 \times 10^{-15}$
$0.868\,055\,555\,534\,238\,161\,81 \times 10^{-1}$	$-0.114\,533\,947\,041\,601\,480\,63 \times 10^{-11}$
−0.118 152 006 023 476 722 98	$0.230\,483\,859\,692\,902\,846\,78 \times 10^{-9}$
0.179 689 507 723 700 402 85	$-0.219\,453\,859\,989\,587\,483\,45 \times 10^{-7}$
−0.243 237 776 508 143 247 72	$0.120\,879\,546\,268\,738\,763\,83 \times 10^{-5}$
$0.117\,005\,703\,951\,529\,384\,11 \times 10^{-1}$	$-0.381\,038\,547\,987\,292\,836\,35 \times 10^{-4}$
$0.237\,452\,068\,486\,745\,862\,41 \times 10^{1}$	$0.413\,464\,001\,734\,930\,254\,58 \times 10^{-3}$
$-0.167\,588\,370\,665\,203\,899\,57 \times 10^{2}$	$0.204\,376\,305\,776\,969\,609\,42 \times 10^{-1}$
$0.889\,665\,457\,654\,216\,509\,42 \times 10^{2}$	$-0.120\,643\,172\,387\,860\,813\,73 \times 10^{1}$
$-0.394\,943\,740\,653\,525\,533\,20 \times 10^{3}$	$0.339\,120\,804\,377\,190\,498\,06 \times 10^{2}$
$0.140\,598\,972\,643\,933\,533\,50 \times 10^{4}$	$-0.624\,748\,204\,647\,099\,272\,86 \times 10^{3}$
$-0.404\,513\,553\,220\,860\,199\,35 \times 10^{4}$	$0.808\,763\,586\,506\,789\,969\,96 \times 10^{4}$
$0.193\,845\,431\,043\,598\,119\,33 \times 10^{5}$	$-0.745\,595\,997\,903\,222\,393\,86 \times 10^{5}$
$-0.205\,176\,883\,526\,801\,360\,30 \times 10^{6}$	$0.480\,862\,899\,008\,954\,123\,20 \times 10^{6}$
$0.170\,435\,218\,408\,308\,883\,84 \times 10^{7}$	$-0.205\,186\,473\,609\,185\,214\,09 \times 10^{7}$
$-0.877\,050\,770\,331\,537\,793\,72 \times 10^{7}$	$0.499\,039\,829\,050\,756\,294\,08 \times 10^{7}$
$0.267\,971\,952\,909\,986\,438\,93 \times 10^{8}$	$-0.308\,748\,073\,455\,144\,363\,82 \times 10^{7}$
$-0.434\,487\,954\,511\,809\,855\,46 \times 10^{8}$	$-0.141\,981\,907\,205\,855\,775\,90 \times 10^{8}$
$0.266\,930\,744\,198\,889\,886\,36 \times 10^{8}$	$0.256\,745\,115\,830\,298\,117\,22 \times 10^{8}$

Table B.10
The coefficients c_j $(j = 0, \ldots, 29)$ of the Laurent polynomial approximation given by (B.1) for evaluation of $J_1(z)$ to 19-digit precision in $Q_1 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 1 \leq |z| \leq 3\}$. $N_2 = 11$.

Real part	Imaginary part
$0.110\,051\,983\,428\,464\,857\,55 \times 10^{-15}$	$-0.694\,974\,798\,976\,949\,017\,98 \times 10^{-16}$
$-0.102\,537\,173\,909\,528\,367\,50 \times 10^{-13}$	$0.573\,447\,330\,612\,984\,007\,54 \times 10^{-15}$
$0.355\,419\,807\,461\,472\,502\,13 \times 10^{-12}$	$0.171\,380\,359\,477\,394\,369\,87 \times 10^{-12}$
$-0.570\,656\,634\,267\,733\,169\,25 \times 10^{-11}$	$-0.800\,977\,526\,803\,848\,613\,53 \times 10^{-11}$
$0.213\,779\,424\,023\,220\,328\,01 \times 10^{-10}$	$0.177\,457\,611\,434\,888\,666\,34 \times 10^{-9}$
$0.925\,041\,777\,735\,636\,816\,59 \times 10^{-9}$	$-0.234\,916\,902\,824\,469\,120\,66 \times 10^{-8}$
$-0.219\,581\,322\,067\,737\,848\,69 \times 10^{-7}$	$0.187\,152\,555\,411\,620\,792\,36 \times 10^{-7}$
$0.267\,441\,468\,134\,350\,880\,20 \times 10^{-6}$	$-0.605\,381\,927\,275\,615\,665\,71 \times 10^{-7}$
$-0.214\,179\,643\,388\,840\,263\,93 \times 10^{-5}$	$-0.551\,667\,774\,710\,601\,484\,67 \times 10^{-6}$
$0.116\,052\,298\,034\,036\,980\,59 \times 10^{-4}$	$0.100\,892\,068\,271\,577\,766\,27 \times 10^{-4}$
$-0.369\,022\,862\,459\,559\,263\,31 \times 10^{-4}$	$-0.857\,345\,236\,304\,648\,588\,44 \times 10^{-4}$
0.999 988 860 265 205 370 47	$0.499\,847\,603\,957\,965\,000\,59 \times 10^{-3}$
0.417 648 343 367 488 728 67	$-0.217\,591\,528\,352\,878\,862\,50 \times 10^{-2}$
−0.128 804 358 762 548 012 79	$0.723\,846\,234\,663\,040\,534\,06 \times 10^{-2}$
0.100 017 970 753 934 941 04	$-0.182\,046\,813\,784\,092\,073\,36 \times 10^{-1}$
−0.123 269 945 334 165 195 99	$0.324\,943\,247\,688\,919\,815\,32 \times 10^{-1}$
0.193 908 157 240 139 101 77	$-0.310\,480\,560\,080\,575\,564\,12 \times 10^{-1}$
−0.304 462 209 484 766 030 76	$-0.280\,943\,713\,750\,866\,947\,29 \times 10^{-1}$
0.402 171 783 043 948 309 19	0.187 014 328 290 689 364 87
−0.390 454 113 780 130 146 41	−0.429 651 957 392 473 659 92
0.202 994 860 670 515 644 92	0.642 709 594 134 958 603 86
0.100 869 854 237 938 765 17	−0.677 127 070 547 480 866 83
−0.347 063 116 270 971 730 40	0.489 430 856 309 116 678 32
0.397 171 277 728 308 542 81	−0.203 886 519 877 986 375 36
−0.276 279 701 906 586 144 95	$-0.369\,206\,393\,782\,648\,768\,81 \times 10^{-2}$
0.120 939 113 708 687 558 32	$0.671\,683\,984\,139\,715\,248\,04 \times 10^{-1}$
$-0.288\,457\,249\,888\,844\,212\,84 \times 10^{-1}$	$-0.454\,423\,099\,541\,989\,405\,39 \times 10^{-1}$
$0.972\,537\,936\,714\,344\,693\,28 \times 10^{-3}$	$0.152\,367\,452\,768\,735\,573\,99 \times 10^{-1}$
$0.119\,899\,114\,841\,701\,766\,70 \times 10^{-2}$	$-0.254\,009\,262\,500\,862\,960\,20 \times 10^{-2}$
$-0.204\,365\,653\,486\,630\,713\,65 \times 10^{-3}$	$0.146\,720\,578\,798\,766\,712\,50 \times 10^{-3}$

Table B.11
Similar to Table B.10, c_j $(j = 0, \ldots, 29)$ for $J_1(z)$ in $Q_2 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 3 \leq |z| \leq 15\}$. $N_2 = 0$.

Real part	Imaginary part
$0.100000000000015598 22 \times 10^{1}$	$-0.644325809756137710 82 \times 10^{-13}$
$0.416666666637650457 92$	$-0.309292903803165853 08 \times 10^{-11}$
$-0.121527775756020318 81$	$0.138478874954507795 12 \times 10^{-8}$
$0.641395990107306846 01 \times 10^{-1}$	$-0.118041438905059638 78 \times 10^{-6}$
$0.193403338768682509 14 \times 10^{-1}$	$0.525251291036592228 82 \times 10^{-5}$
$-0.310853962881174583 25$	$-0.142099426151803529 58 \times 10^{-3}$
$0.140761123934977405 95 \times 10^{1}$	$0.228432708182397970 21 \times 10^{-2}$
$-0.530346035828585911 37 \times 10^{1}$	$-0.125084269021042670 53 \times 10^{-1}$
$0.168439698370425810 97 \times 10^{2}$	$-0.396338591511731227 45$
$-0.308961039620087435 37 \times 10^{2}$	$0.133668672429219894 70 \times 10^{2}$
$-0.123540786588963380 72 \times 10^{3}$	$-0.231002965578393101 61 \times 10^{3}$
$0.164860287291898027 72 \times 10^{4}$	$0.277980113167897259 15 \times 10^{4}$
$-0.793656002819625726 17 \times 10^{4}$	$-0.250853794383456416 90 \times 10^{5}$
$-0.178763381620329697 92 \times 10^{4}$	$0.173377113022011027 56 \times 10^{6}$
$0.360421953744320980 49 \times 10^{6}$	$-0.909323038984623505 15 \times 10^{6}$
$-0.338183293087006495 02 \times 10^{7}$	$0.342839554438727027 45 \times 10^{7}$
$0.194492028977494948 34 \times 10^{8}$	$-0.746254995983527439 42 \times 10^{7}$
$-0.790470833722513988 93 \times 10^{8}$	$-0.617040364456367436 42 \times 10^{7}$
$0.227878894195504202 17 \times 10^{9}$	$0.135424549695310366 97 \times 10^{9}$
$-0.415506385950006041 77 \times 10^{9}$	$-0.654833732803006626 71 \times 10^{9}$
$0.172907123696176881 12 \times 10^{9}$	$0.196648952333792101 38 \times 10^{10}$
$0.167637779713995208 71 \times 10^{10}$	$-0.400050752276151932 94 \times 10^{10}$
$-0.630971170189396346 89 \times 10^{10}$	$0.514426712304306953 93 \times 10^{10}$
$0.126396237086149185 40 \times 10^{11}$	$-0.242505339192332000 87 \times 10^{10}$
$-0.160214248381859378 66 \times 10^{11}$	$-0.510632232601052458 52 \times 10^{10}$
$0.121944700391779730 74 \times 10^{11}$	$0.128012956581150484 67 \times 10^{11}$
$-0.366175697409280990 53 \times 10^{10}$	$-0.139069953249516583 35 \times 10^{11}$
$-0.208824308492656407 04 \times 10^{10}$	$0.823514139103719163 18 \times 10^{10}$
$0.222563247596899852 06 \times 10^{10}$	$-0.236069992598179064 85 \times 10^{10}$
$-0.573572754664664525 87 \times 10^{9}$	$0.181001679632682649 95 \times 10^{9}$

Table B.12
Similar to Table B.10, c_j $(j = 0, \ldots, 19)$ for $J_1(z)$ in $Q_3 := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 15 \leq |z| \leq 120\}$. $N_2 = 0$.

Real part	Imaginary part
$0.100000000000000000 88 \times 10^{1}$	$-0.371046820944367410 73 \times 10^{-16}$
$0.416666666666656932 68$	$0.106331057865606799 43 \times 10^{-13}$
$-0.121527777775325301 35$	$-0.817834833097519209 64 \times 10^{-12}$
$0.641396602068443950 55 \times 10^{-1}$	$-0.532067463095992156 52 \times 10^{-10}$
$0.193403761465063495 34 \times 10^{-1}$	$0.147143090852591082 66 \times 10^{-7}$
$-0.310929014736007604 33$	$-0.128476337578447411 59 \times 10^{-5}$
$0.141082302044215261 48 \times 10^{1}$	$0.640298043092671354 69 \times 10^{-4}$
$-0.538142870351929351 74 \times 10^{1}$	$-0.197291639374589200 69 \times 10^{-2}$
$0.180990405065459285 10 \times 10^{2}$	$0.339592660467105515 28 \times 10^{-1}$
$-0.442225211017710052 59 \times 10^{2}$	$-0.464431291493640173 64 \times 10^{-1}$
$-0.492817125056098922 21 \times 10^{2}$	$-0.146733386982745397 90 \times 10^{2}$
$0.191207310397992978 66 \times 10^{4}$	$0.446546557664915277 24 \times 10^{3}$
$-0.184802964071395561 13 \times 10^{5}$	$-0.765292571761824189 14 \times 10^{4}$
$0.119494512833017511 33 \times 10^{6}$	$0.880375481155049965 27 \times 10^{5}$
$-0.514113640570026638 24 \times 10^{6}$	$-0.705634002812068307 70 \times 10^{6}$
$0.109735353928198287 12 \times 10^{7}$	$0.390994526691565767 03 \times 10^{7}$
$0.173314355651991350 31 \times 10^{7}$	$-0.143550076611335737 35 \times 10^{8}$
$-0.191271957480326461 77 \times 10^{8}$	$0.317572532452279463 71 \times 10^{8}$
$0.508015744433299636 57 \times 10^{8}$	$-0.337006579463013313 33 \times 10^{8}$
$-0.479102880592342539 94 \times 10^{8}$	$0.611719070376090119 58 \times 10^{7}$

Table B.13
The coefficients c_j ($j = 0, \ldots, 31$) of the Laurent polynomial approximation given by (B.1) to evaluate $J_2(z)$ to 19-digit precision in $Q_1 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 1 \leq |z| \leq 3\}$. $N_2 = 11$.

Real part	Imaginary part
$0.318\,666\,326\,858\,196\,122\,21 \times 10^{-16}$	$0.622\,787\,389\,698\,301\,379\,87 \times 10^{-16}$
$0.233\,742\,024\,881\,147\,144\,31 \times 10^{-15}$	$-0.583\,681\,868\,290\,920\,313\,91 \times 10^{-14}$
$-0.124\,647\,632\,629\,216\,011\,44 \times 10^{-12}$	$0.202\,591\,995\,287\,227\,709\,99 \times 10^{-12}$
$0.549\,389\,148\,673\,893\,951\,95 \times 10^{-11}$	$-0.306\,881\,697\,703\,554\,016\,91 \times 10^{-11}$
$-0.121\,585\,758\,421\,062\,813\,73 \times 10^{-9}$	$0.207\,061\,195\,679\,195\,922\,98 \times 10^{-12}$
$0.160\,180\,032\,739\,285\,601\,65 \times 10^{-8}$	$0.882\,168\,424\,734\,363\,412\,38 \times 10^{-9}$
$-0.119\,414\,758\,814\,497\,678\,65 \times 10^{-7}$	$-0.188\,155\,398\,002\,929\,243\,03 \times 10^{-7}$
$0.153\,068\,687\,903\,453\,394\,46 \times 10^{-7}$	$0.225\,670\,345\,515\,736\,497\,10 \times 10^{-6}$
$0.804\,072\,544\,812\,413\,845\,44 \times 10^{-6}$	$-0.177\,728\,604\,980\,304\,237\,20 \times 10^{-5}$
$-0.114\,650\,402\,869\,713\,448\,49 \times 10^{-4}$	$0.889\,930\,973\,639\,227\,927\,29 \times 10^{-5}$
$0.926\,984\,174\,506\,952\,816\,24 \times 10^{-4}$	$-0.171\,530\,442\,583\,568\,098\,74 \times 10^{-4}$
$0.999\,481\,719\,915\,493\,824\,51$	$-0.150\,461\,944\,586\,964\,532\,45 \times 10^{-3}$
$0.141\,870\,622\,524\,129\,318\,02 \times 10^{1}$	$0.183\,491\,416\,917\,293\,879\,94 \times 10^{-2}$
$-0.126\,478\,358\,733\,725\,358\,21$	$-0.114\,282\,767\,059\,959\,086\,59 \times 10^{-1}$
$0.186\,035\,981\,114\,746\,633\,25$	$0.506\,468\,994\,335\,975\,245\,05 \times 10^{-1}$
$-0.287\,649\,487\,485\,808\,553\,21$	$-0.172\,682\,434\,765\,531\,714\,63$
$0.376\,719\,333\,723\,802\,524\,36$	$0.464\,960\,764\,809\,206\,824\,74$
$-0.286\,735\,627\,303\,066\,851\,62$	$-0.992\,159\,994\,955\,746\,183\,74$
$-0.252\,608\,025\,036\,984\,377\,99$	$0.165\,119\,884\,035\,805\,682\,51 \times 10^{1}$
$0.141\,712\,614\,729\,715\,384\,18 \times 10^{1}$	$-0.203\,812\,445\,017\,954\,268\,09 \times 10^{1}$
$-0.294\,579\,672\,497\,088\,580\,73 \times 10^{1}$	$0.157\,820\,772\,036\,354\,306\,07 \times 10^{1}$
$0.402\,844\,065\,189\,415\,904\,62 \times 10^{1}$	$-0.377\,233\,543\,355\,642\,928\,19 \times 10^{-1}$
$-0.381\,289\,500\,139\,206\,926\,43 \times 10^{1}$	$-0.199\,070\,648\,765\,300\,931\,67 \times 10^{1}$
$0.224\,752\,491\,800\,986\,001\,79 \times 10^{1}$	$0.333\,439\,877\,698\,488\,751\,42 \times 10^{1}$
$-0.298\,455\,079\,954\,699\,949\,80$	$-0.323\,517\,065\,419\,137\,574\,06 \times 10^{1}$
$-0.895\,442\,874\,444\,819\,929\,84$	$0.204\,544\,678\,700\,893\,659\,42 \times 10^{1}$
$0.101\,756\,395\,675\,792\,407\,08 \times 10^{1}$	$-0.774\,743\,518\,505\,746\,398\,68$
$-0.587\,887\,788\,658\,000\,930\,21$	$0.835\,031\,192\,819\,333\,227\,86 \times 10^{-1}$
$0.200\,093\,190\,200\,811\,827\,63$	$0.776\,897\,300\,954\,775\,936\,36 \times 10^{-1}$
$-0.359\,654\,320\,908\,390\,960\,74 \times 10^{-1}$	$-0.437\,976\,033\,677\,438\,831\,19 \times 10^{-1}$
$0.167\,694\,205\,585\,345\,301\,17 \times 10^{-2}$	$0.957\,563\,654\,301\,825\,227\,46 \times 10^{-2}$
$0.272\,567\,105\,531\,954\,481\,21 \times 10^{-3}$	$-0.766\,126\,822\,662\,540\,928\,89 \times 10^{-3}$

Table B.14
Similar to Table B.13, c_j ($j = 0, \ldots, 29$) for $J_2(z)$ in $Q_2 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 3 \leq |z| \leq 15\}$. $N_2 = 0$.

Real part	Imaginary part
$0.999\,999\,999\,999\,980\,618\,08$	$0.162\,482\,521\,133\,093\,156\,78 \times 10^{-12}$
$0.141\,666\,666\,668\,292\,754\,03 \times 10^{1}$	$-0.230\,492\,784\,284\,204\,436\,93 \times 10^{-10}$
$-0.121\,527\,779\,888\,094\,063\,80$	$0.105\,786\,214\,112\,157\,548\,90 \times 10^{-8}$
$0.185\,667\,565\,930\,831\,821\,16$	$0.284\,242\,784\,011\,271\,800\,14 \times 10^{-8}$
$-0.351\,998\,911\,746\,989\,632\,56$	$-0.241\,909\,745\,765\,441\,133\,33 \times 10^{-5}$
$0.745\,140\,284\,771\,892\,955\,14$	$0.126\,275\,058\,337\,217\,683\,89 \times 10^{-3}$
$-0.156\,986\,886\,654\,073\,007\,95 \times 10^{1}$	$-0.369\,069\,066\,997\,730\,890\,97 \times 10^{-2}$
$0.244\,026\,634\,964\,360\,786\,03 \times 10^{1}$	$0.711\,508\,801\,323\,843\,873\,16 \times 10^{-1}$
$0.425\,647\,783\,840\,440\,145\,01 \times 10^{1}$	$-0.920\,280\,282\,347\,422\,815\,69$
$-0.866\,287\,814\,850\,868\,846\,13 \times 10^{2}$	$0.696\,596\,700\,416\,120\,258\,01 \times 10^{1}$
$0.767\,804\,318\,971\,574\,775\,13 \times 10^{3}$	$0.123\,360\,228\,557\,911\,306\,38 \times 10^{1}$
$-0.560\,542\,914\,138\,584\,485\,09 \times 10^{4}$	$-0.868\,862\,439\,778\,485\,142\,46 \times 10^{3}$
$0.347\,097\,600\,115\,435\,632\,56 \times 10^{5}$	$0.139\,974\,327\,917\,219\,547\,13 \times 10^{5}$
$-0.172\,687\,375\,146\,298\,482\,98 \times 10^{6}$	$-0.139\,229\,094\,314\,962\,945\,16 \times 10^{6}$
$0.611\,878\,401\,649\,523\,159\,07 \times 10^{6}$	$0.100\,633\,027\,909\,147\,705\,06 \times 10^{7}$
$-0.902\,292\,802\,539\,091\,436\,53 \times 10^{6}$	$-0.550\,900\,707\,205\,202\,075\,41 \times 10^{7}$
$-0.582\,396\,935\,100\,244\,393\,15 \times 10^{7}$	$0.228\,100\,677\,011\,447\,712\,36 \times 10^{8}$
$0.555\,853\,053\,034\,783\,477\,57 \times 10^{8}$	$-0.682\,265\,209\,957\,832\,557\,32 \times 10^{8}$
$-0.263\,019\,290\,324\,975\,273\,08 \times 10^{9}$	$0.122\,936\,418\,112\,390\,372\,82 \times 10^{9}$
$0.840\,529\,096\,164\,811\,422\,18 \times 10^{9}$	$0.211\,968\,273\,244\,948\,845\,33 \times 10^{8}$
$-0.186\,577\,921\,546\,062\,951\,88 \times 10^{10}$	$-0.101\,312\,678\,036\,070\,864\,55 \times 10^{10}$
$0.259\,153\,889\,041\,130\,760\,82 \times 10^{10}$	$0.384\,271\,569\,921\,601\,960\,76 \times 10^{10}$
$-0.925\,179\,224\,709\,660\,621\,67 \times 10^{9}$	$-0.860\,343\,020\,334\,517\,826\,37 \times 10^{10}$
$-0.510\,369\,269\,351\,709\,789\,03 \times 10^{10}$	$0.126\,265\,243\,143\,448\,689\,26 \times 10^{11}$
$0.136\,571\,708\,191\,722\,730\,67 \times 10^{11}$	$-0.113\,016\,421\,524\,042\,590\,95 \times 10^{11}$
$-0.182\,348\,154\,094\,510\,081\,01 \times 10^{11}$	$0.355\,467\,235\,144\,743\,314\,42 \times 10^{10}$
$0.143\,551\,216\,572\,205\,965\,89 \times 10^{11}$	$0.458\,394\,235\,179\,540\,658\,93 \times 10^{10}$
$-0.610\,320\,528\,375\,417\,573\,28 \times 10^{10}$	$-0.645\,093\,443\,224\,018\,395\,65 \times 10^{10}$
$0.842\,751\,754\,996\,283\,692\,28 \times 10^{9}$	$0.327\,738\,449\,951\,987\,873\,42 \times 10^{10}$
$0.158\,570\,795\,668\,019\,918\,82 \times 10^{9}$	$-0.605\,703\,525\,454\,168\,580\,98 \times 10^{9}$

Table B.15
Similar to Table B.13, c_j ($j = 0, \ldots, 19$) for $J_2(z)$ in $Q_3 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, 15 \leq |z| \leq 120\}$. $N_2 = 0$.

Real part	Imaginary part
$0.100\,000\,000\,000\,000\,002\,68 \times 10^{1}$	$0.162\,743\,868\,019\,552\,950\,04 \times 10^{-16}$
$0.141\,666\,666\,666\,666\,076\,32 \times 10^{1}$	$-0.102\,878\,834\,626\,948\,767\,61 \times 10^{-13}$
$-0.121\,527\,777\,777\,735\,963\,34$	$0.212\,780\,643\,404\,207\,813\,15 \times 10^{-11}$
$0.185\,667\,438\,382\,225\,585\,48$	$-0.213\,467\,395\,888\,552\,059\,15 \times 10^{-9}$
$-0.351\,994\,534\,212\,129\,746\,48$	$0.108\,100\,209\,498\,546\,703\,51 \times 10^{-7}$
$0.745\,056\,201\,785\,816\,040\,77$	$-0.127\,583\,354\,158\,386\,725\,36 \times 10^{-6}$
$-0.156\,944\,002\,222\,837\,240\,37 \times 10^{1}$	$-0.191\,947\,509\,138\,627\,125\,85 \times 10^{-4}$
$0.246\,550\,589\,993\,223\,561\,59 \times 10^{1}$	$0.146\,930\,730\,940\,053\,255\,64 \times 10^{-2}$
$0.336\,070\,891\,425\,158\,927\,31 \times 10^{1}$	$-0.570\,883\,175\,245\,060\,610\,35 \times 10^{-1}$
$-0.698\,533\,324\,578\,313\,000\,99 \times 10^{2}$	$0.143\,780\,143\,817\,580\,756\,90 \times 10^{1}$
$0.556\,108\,184\,991\,142\,516\,23 \times 10^{3}$	$-0.246\,434\,105\,357\,333\,767\,07 \times 10^{2}$
$-0.373\,824\,599\,719\,869\,186\,36 \times 10^{4}$	$0.279\,932\,734\,513\,352\,828\,71 \times 10^{3}$
$0.238\,685\,728\,413\,736\,435\,77 \times 10^{5}$	$-0.177\,305\,292\,583\,538\,875\,68 \times 10^{4}$
$-0.144\,152\,750\,317\,028\,372\,09 \times 10^{6}$	$-0.956\,973\,793\,342\,533\,989\,76 \times 10^{3}$
$0.758\,535\,918\,118\,737\,569\,81 \times 10^{6}$	$0.142\,715\,671\,673\,830\,354\,12 \times 10^{6}$
$-0.309\,728\,616\,687\,691\,455\,38 \times 10^{7}$	$-0.150\,030\,990\,512\,279\,299\,84 \times 10^{7}$
$0.853\,766\,582\,377\,944\,168\,73 \times 10^{7}$	$0.845\,725\,298\,441\,645\,188\,14 \times 10^{7}$
$-0.122\,886\,685\,657\,618\,339\,34 \times 10^{8}$	$-0.279\,453\,101\,422\,051\,076\,64 \times 10^{8}$
$0.268\,699\,632\,595\,193\,750\,12 \times 10^{6}$	$0.499\,737\,837\,887\,756\,884\,47 \times 10^{8}$
$0.163\,768\,535\,791\,747\,020\,84 \times 10^{8}$	$-0.359\,498\,303\,220\,644\,799\,62 \times 10^{8}$

where $\nu := 3\left(\frac{z}{2}\right)^{2/3}$. (B.1) is obtained by combining (6) and (21), and rewriting the Laurent polynomial as a power series in $\frac{1}{\nu}$ by pulling out the factor ν^{N_2}.

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