The nucleus of an adjunction and the Street monad on monads

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Abstract

An adjunction is a pair of functors related by a pair of natural transformations, and relating a pair of categories. It displays how a structure, or a concept, projects from each category to the other, and back. Adjunctions are the common denominator of Galois connections, representation theories, spectra, and generalized quantifiers. We call an adjunction nuclear when its categories determine each other. We show that every adjunction can be resolved into a nuclear adjunction. The resolution is idempotent in a strict sense. The resulting nucleus displays the concept that was implicit in the original adjunction, just as the singular value decomposition of an adjoint pair of linear operators displays their canonical bases.

The two composites of an adjoint pair of functors induce a monad and a comonad. Monads and comonads generalize the closure and the interior operators from topology, or modalities from logic, while providing a saturated view of algebraic structures and compositions on one side, and of coalgebraic dynamics and decompositions on the other. They are resolved back into adjunctions over the induced categories of algebras and of coalgebras. The nucleus of an adjunction is an adjunction between the induced categories of algebras and coalgebras. It provides new presentations for both, revealing algebras on the side where the coalgebras are normally presented, and vice versa. The new presentations elucidate the central role of idempotents, and of the absolute limits and colimits in monadicity and comonadicity. They suggest interesting extensions of the monad and comonad toolkits, particularly for programming.

In his seminal early work, Ross Street described an adjunction between monads and comonads in 2-categories. Lifting the nucleus construction, we show that the resulting Street monad on monads is strictly idempotent, and extracts the nucleus of a monad. A dual treatment achieves the same for comonads. This uncovers remarkably concrete applications behind a notable fragment of pure 2-category theory. The other way around, driven by the tasks and methods of machine learning and data analysis, the nucleus construction also seems to uncover remarkably pure and general mathematical content lurking behind the daily practices of network computation and data analysis.

*Supported by NSF and AFOSR.

Contents

1 Introduction

We begin with an informal overview of the main result here in Sec. 1, and motivate it through the general examples in Sections 2–5. A categorically minded reader may prefer to start from the categorical definitions in Sec. 5, proceed to the general constructions in Sections 6–9, and come back for examples and explanations. Some general definitions can be found in the Appendix.

1.1 Nuclear adjunctions and the adjunction nuclei

1.1.1 Definition.

We say that an adjunction $F = (F^* \dashv F_* : \mathbb{B} \to \mathbb{A})$ is *nuclear* when the right adjoint F_* is monadic and the left adjoint F^* is comonadic. This means that the categories $\mathbb{A}$ and $\mathbb{B}$ determine one another, and can be reconstructed from each other:

- F_* is monadic when $\mathbb{B}$ is equivalent to the category $\mathbb{A}^{\overleftarrow{F}}$ of algebras for the monad $\overleftarrow{F} = F_* F^* : \mathbb{A} \to \mathbb{A}$, whereas

- F^* is comonadic when $\mathbb{A}$ is equivalent to the category $\mathbb{B}^{\overrightarrow{F}}$ of coalgebras for the comonad $\overrightarrow{F} = F^* F_* : \mathbb{B} \to \mathbb{B}$.

The situation is reminiscent of Maurits Escher's *"Drawing hands"* in Fig.1.

$$\begin{tikzcd}
\mathbb{A} \arrow[loop above, "\overleftarrow{F}"] \arrow[rr, "\simeq"] \arrow[d, bend right, "F^*"'] \arrow[d, phantom, "\dashv"] & & \mathbb{B}^{\overrightarrow{F}} \arrow[d, bend right, "F^\sharp"'] \arrow[d, phantom, "\dashv"] \\
\mathbb{B} \arrow[loop below, "\overrightarrow{F}"] \arrow[rr, "\simeq"] \arrow[u, bend right, "F_*"'] & & \mathbb{A}^{\overleftarrow{F}} \arrow[u, bend right, "F_\sharp"']
\end{tikzcd}$$

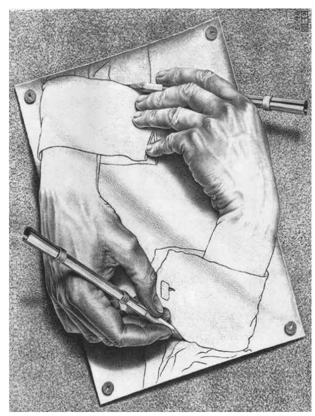

Figure 1: An adjunction $(F^* \dashv F_*)$ is nuclear when $\mathbb{A} \simeq \mathbb{B}^{\overrightarrow{F}}$ and $\mathbb{B} \simeq \mathbb{A}^{\overleftarrow{F}}$.

1.1.2 Result

The nucleus construction $\overleftarrow{\mathfrak{N}}$ extracts from any adjunction F its nucleus $\overleftarrow{\mathfrak{N}}F$

$$\begin{aligned} F &= (F^* \dashv F_* : \mathbb{B} \to \mathbb{A}) \\ \overleftarrow{\mathfrak{N}}F &= \left(F^\sharp \dashv F_\sharp : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{B}^{\overrightarrow{F}}\right) \end{aligned} \tag{1}$$

The functor $F_\sharp$ is formed by composing the forgetful functor $\mathbb{A}^{\overleftarrow{F}} \to \mathbb{A}$ with the comparison functor $\mathbb{A} \to \mathbb{B}^{\overrightarrow{F}}$, whereas $F^\sharp$ is the composite of the forgetful functor $\mathbb{B}^{\overrightarrow{F}} \to \mathbb{B}$ with the comparison $\mathbb{B} \to \mathbb{A}^{\overleftarrow{F}}$. Hence the left-hand square in Fig. 2. We show that the functors $F^\sharp$ and $F_\sharp$ are adjoint,

$$
\begin{tikzcd}
\mathbb{A} \arrow[loop above, "\overleftarrow{F}"] \arrow[rr] \arrow[dd, bend right, "F^*"'] \arrow[dd, phantom, "\dashv"] & & \mathbb{B}^{\overrightarrow{F}} \arrow[loop above, "\overset{\Leftarrow}{F}"] \arrow[rr, "\simeq"] \arrow[dd, bend right, "F^\sharp"'] \arrow[dd, phantom, "\dashv"] & & \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Rightarrow}{F}} \arrow[dd, bend right, "F^{\sharp\sharp}"'] \arrow[dd, phantom, "\dashv"] \\
& & & & \\
\mathbb{B} \arrow[loop below, "\overrightarrow{F}"] \arrow[rr] \arrow[uu, bend right, "F_*"'] & & \mathbb{A}^{\overleftarrow{F}} \arrow[loop below, "\overset{\Rightarrow}{F}"] \arrow[rr, "\simeq"] \arrow[uu, bend right, "F_\sharp"'] & & \left(\mathbb{B}^{\overrightarrow{F}}\right)^{\overset{\Leftarrow}{F}} \arrow[uu, bend right, "F_{\sharp\sharp}"']
\end{tikzcd}
$$

Figure 2: The nucleus construction induces an idempotent monad on adjunctions.

which means that we can iterate the nucleus construction $\overleftarrow{\mathfrak{N}}$ in (1) and induce a tower of adjunctions

$$F \to \overleftarrow{\mathfrak{N}}F \to \overleftarrow{\mathfrak{N}}\overleftarrow{\mathfrak{N}}F \to \overleftarrow{\mathfrak{N}}\overleftarrow{\mathfrak{N}}\overleftarrow{\mathfrak{N}}F \to \cdots \tag{2}$$

We show that $\overleftarrow{\mathfrak{N}}F = \left(F^\sharp \dashv F_\sharp\right)$ is a *nuclear* adjunction, which means that the right-hand square in Fig. 2 is an equivalence of adjunctions. The tower in (2) thus settles at the second step. The $\overleftarrow{\mathfrak{N}}$-construction is an *idempotent monad* on adjunctions. Since the adjunctions form a 2-category, $\overleftarrow{\mathfrak{N}}$ is a 2-monad. We emphasize that its idempotence is *strict*, i.e. (up to a natural family of equivalences), and not *lax* (up to a natural family of adjunctions). While lax idempotence is frequently encountered and well-studied in categorical algebra [48, 50, 81, 85][1], strictly idempotent categorical constructions are relatively rare, and occur mostly in the context of absolute completions. The nucleus construction suggests a reason [74].

1.1.3 Upshot

The fact that the the adjunction $F^\sharp \dashv F_\sharp$ in nuclear means that, for any adjunction $F = (F^* \dashv F_*)$, the category of algebras $\mathbb{A}^{\overleftarrow{F}}$ and the category of coalgebras $\mathbb{B}^{\overrightarrow{F}}$ can be reconstructed from one another: $\mathbb{A}^{\overleftarrow{F}}$ as a category of algebras over $\mathbb{B}^{\overrightarrow{F}}$, and $\mathbb{B}^{\overrightarrow{F}}$ as a category of coalgebras over $\mathbb{A}^{\overleftarrow{F}}$. They are always an instance of the Escher situation in Fig. 1. Simplifying these mutual reconstructions provides a new view of the final resolutions of monads and comonads, complementing the original

[1]Monads over 2-categories and bicategories have been called *doctrines* [58], and the lax idempotent ones are often called the *Kock-Zöberlein doctrines* [81].

Eilenberg-Moore construction [27]. It was described in [75] as a programming tool, and was used as a mathematical tool in [74]. Presenting algebras and coalgebras as idempotents provides a rational reconstruction of monadicity (and comonadicity) in terms of idempotent splitting, echoing Paré's explanations in terms of absolute colimits [65, 66], and contrasting with Beck's fascinating but somewhat mysterious proof of his fundamental theorem in terms of split coequalizers [14, 15]. Concrete applications of the nuclei spread in many directions, some of which are indicated in the examples, which had to be trimmed, in some cases radically.

1.1.4 Background

Nuclear adjunctions have been studied since the early days of category theory, albeit without a name. The problem of characterizing situations when the left adjoint of a monadic functor is comonadic is the topic of Michael Barr's paper in the proceedings of the legendary Battelle conference [8]. From a different direction, in his seminal work on the formal theory of monads, Ross Street identified the 2-adjunction between the 2-categories of monads and of comonads [80, Sec. 4]. This adjunction leads to a formal view of the nucleus construction on either side, as a 2-monad. We show that this construction is idempotent in the strong sense. On the side of applications, the quest for comonadic adjoints of monadic functors continued in descent theory, and an important step towards characterizing them was made by Mesablishvili in [63]. Coalgebras over algebras, and algebras over coalgebras, have also been regularly used for a variety of modeling purposes in semantics of computation (see e.g. [7, 41, 43], and the references therein).

As the vanishing point of monadic descent, nuclear adjunctions arise in many branches of geometry, tacitly or explicitly. In abstract homotopy theory, they are tacitly in [44, 78], and explicitly in [1]. There are, however, different ways in which monad-comonad couplings may arise. In [1], Applegate and Tierney formed such couplings on the two sides of comparison functors and their adjoints, and they found that such monad-comonad couplings generally induce further monad-comonad couplings along the further comparison functors, and may form towers of transfinite length. We describe this in more detail in Sec. 10. Confusingly, the Applegate-Tierney towers of monad-comonad couplings *formed by comparison functor adjunctions* left a false impression that the monad-comonad couplings *formed by the adjunctions between categories of algebras over coalgebras, of coalgebras over algebras, etc.* also lead to towers of transfinite length. This impression blended into folklore, and the towers of alternating monads over coalgebras and comonads over algebras, extending out of sight, persist in categorical literature.[2]

[2]There is an interesting exception outside the categorical literature. In a fax message sent to Paul Taylor on 9/9/99 [53], a copy of which was kindly provided after the present paper appeared on arxiv, Steve Lack set out to determine the conditions under which the tower of coalgebras over algebras, which "a priori continues indefinitely", settles to equivalence at a finite stage. Within 7 pages of diagrams, the question was reduced to splitting a certain idempotent. While the argument is succinct, it does seem to prove a claim which, together with its dual, implies our Prop. 7.4. The claim was, however, not pursued in further work. This amusing episode from the early life of the nucleus underscores its message: that a concept is technically within reach whenever there is an adjunction, but it does need to be spelled out and applied to be recognized.

1.1.5 Terminology

Despite all of their roles and avatars, adjunctions where the right adjoint is monadic and the left adjoint is comonadic were not given a name. We call them nuclear because of the link with nuclear operators on Banach spaces, which generalize the spectral decomposition of hermitians and the singular value decomposition of matrices and lift them all the way to linear operators on topological vector spaces. This was the subject of Grothendieck's thesis, where the terminology was introduced [34]. We describe this conceptual link in Sec. 3, for the very special case of finite-dimensional Hilbert spaces.

1.1.6 Schema

Fig. 3 maps the paths that lead to the nucleus. We will follow it as an itinerary, first through familiar

matrices ***extensions*** ***localizations*** ***nuclei***

$$\begin{tikzcd}
& & \mathsf{Mnd} \arrow[dl, bend right, "\mathsf{EM}"'] \arrow[dr, bend left, "\mathsf{MN}"] & \\
\mathsf{Mat} \arrow[r, "\mathsf{MA}"] & \mathsf{Adj} \arrow[ur, bend right, "\mathsf{AM}"'] \arrow[ur, phantom, "\top"] \arrow[dr, bend left, "\mathsf{AC}"] \arrow[dr, phantom, "\top"] & & \mathsf{Nuc} \arrow[ul, bend left, "\mathsf{NM}"] \arrow[ul, phantom, "\top"] \arrow[dl, bend right, "\mathsf{NC}"'] \arrow[dl, phantom, "\top"] \\
& & \mathsf{Cmn} \arrow[ul, bend left, "\mathsf{KC}"] \arrow[ur, bend right, "\mathsf{CN}"'] &
\end{tikzcd}$$

Figure 3: The nucleus setting

examples and special cases in Sections 2–4, and then as a general pattern. Most definitions are in Sec. 5. Some readers may wish to skip the rest of the present section, have a look at the examples, and come back as needed. For others we provide here an informal overview of the terminology, mostly just naming names.

Who is who. While the production line of mathematical tools is normally directed from theory to applications, ideas often flow in the opposite direction. The idea of the nucleus is familiar, in fact central, in data mining and concept analysis, albeit without a name, but has remained elusive in general [47]. Data analysis usually begins from data *matrices*, which we view as objects of an abstract category Mat. To be analyzed, data matrices are usually completed or *extended* into some sort of *adjunctions*, which we view as objects of an abstract category Adj. The functor $\mathsf{MA} : \mathsf{Mat} \to \mathsf{Adj}$ represents this extension. The adjunctions are then *localized* along the functors $\mathsf{AM} : \mathsf{Adj} \to \mathsf{Mnd}$ and $\mathsf{AC} : \mathsf{Adj} \to \mathsf{Cmn}$ at *monads* and *comonads*, which form categories Mnd and Cmn. In some areas and periods of category theory, a functor is called a localization when it has a full and faithful adjoint. The functors AM and AC in Fig. 3 have both left and right adjoints, both full and faithful. We display only the right adjoint $\mathsf{EM} : \mathsf{Mnd} \to \mathsf{Adj}$ of AM, which maps a monad to the adjunction induced by its (Eilenberg-Moore) category of algebras, and the left adjoint $\mathsf{KC} : \mathsf{Cmn} \to \mathsf{Adj}$, which maps a comonad to the adjunction induced by its (Kleisli) category of cofree coalgebras. The nucleus construction is composed of such couplings. Alternatively, it can be composed of the left adjoint $\mathsf{KM} : \mathsf{Mnd} \to \mathsf{Adj}$ of AM and the right adjoint $\mathsf{EC} : \mathsf{Cmn} \to \mathsf{Adj}$ of AC. There is, in general, an entire gamut of different adjunctions localized along $\mathsf{AM} : \mathsf{Adj} \to \mathsf{Mnd}$ at the same monad. We call them the *resolutions*[3] of the monad. Dually, the adjunctions localized along $\mathsf{AC} : \mathsf{Adj} \to \mathsf{Cmn}$ at the same comonad are the resolutions of that comonad. For readers unfamiliar with monads and comonads, we note that monads over posets are called closure operators, whereas comonads over posets are the interior operators. In general, the (Kleisli) cofree coalgebra construction $\mathsf{KC} : \mathsf{Cmn} \to \mathsf{Adj}$ in Fig. 3 (and the free algebra construction $\mathsf{KM} : \mathsf{Mnd} \to \mathsf{Adj}$ that is not displayed) captures the *initial resolutions* of comonads (resp. monads); whereas the (Eilenberg-Moore) algebra construction $\mathsf{EM} : \mathsf{Mnd} \to \mathsf{Adj}$ (and the coalgebra construction $\mathsf{EC} : \mathsf{Cmn} \to \mathsf{Adj}$ that is not displayed) captures the *final resolutions* of monads (resp. comonads). For closure operators and interior operators over posets, and more generally for idempotent monads and comonads over categories, the initial and the final resolutions coincide. In any case, the categories Mnd and Cmn are embedded in Adj fully and faithfully; idempotent monads and comonads are mapped to their unique resolutions, whereas monads, in general, are embedded in two extremal ways, with a gamut of resolutions in-between. The composites of these extremal resolution functors from Mnd and Cmn to Adj with the localizations from Adj to Mnd and Cmn induce the idempotent monad $\overleftarrow{\mathsf{EM}} = \mathsf{EM} \circ \mathsf{AM}$ over Adj which maps any adjunction to the Eilenberg-Moore resolution of the induced monad, and the idempotent comonad $\overrightarrow{\mathsf{KC}} = \mathsf{KC} \circ \mathsf{AC}$, still over Adj, which maps any adjunction to the Kleisli resolution of the induced comonad. Just as there is a category of categories, there is thus a monad of monads, and a comonad of comonads; and both happen to be idempotent. Since the subcategories fixed by idempotent monads or comonads, in general, are usually viewed as localizations, we view monads and comonads as localizations of adjunctions; and we call all the adjunctions that induce a given monad (or comonad) its resolutions. The resolution functors not displayed in Fig. 3 induce a comonad $\overrightarrow{\mathsf{KM}} = \mathsf{KM} \circ \mathsf{AM}$, mapping adjunctions to the Kleisli resolutions of the induced monads, and a monad $\overleftarrow{\mathsf{EC}} = \mathsf{EC} \circ \mathsf{AC}$, mapping adjunctions to the Eilenberg-Moore resolutions of the induced comonads. They are all spelled out in Sec. 5.

The category Nuc of nuclei is the intersection of Mnd and Cmn, as embedded into Adj along their resolutions in Fig. 3. However, we will see in Sec. 7 that any other resolutions will do, as long as the last one is final. The nucleus of an adjunction can thus be viewed as the joint resolution of the induced monad and comonad.

1.2 The Street monad

The composites $\mathfrak{E}^* = \mathsf{AM} \circ \mathsf{KC}$ and $\mathfrak{E}_* = \mathsf{AC} \circ \mathsf{EM}$ in Fig. 3 are adjoint to one another, and thus form a monad $\overleftarrow{\mathfrak{E}} = \mathfrak{E}_* \circ \mathfrak{E}^*$ on the category Cmn of comonads, and a comonad $\overrightarrow{\mathfrak{E}} = \mathfrak{E}^* \circ \mathfrak{E}_*$

[3] This terminology was proposed by Jim Lambek. Although it does not seem to have caught on, it is convenient in the present context, and naturally extends from its roots in algebra.

on the category Mnd of monads. The initial (Kleisli) resolution KM of the monads and the final (Eilenberg-Moore) resolution EC of comonads give the adjoints $\mathfrak{M}^* = \mathsf{AC} \circ \mathsf{KM}$ and $\mathfrak{M}_* = \mathsf{AM} \circ \mathsf{EC}$, which form a monad $\overleftarrow{\mathfrak{M}} = \mathfrak{M}_* \circ \mathfrak{M}^*$ on the category Mnd of monads, and a comonad $\overrightarrow{\mathfrak{M}} = \mathfrak{M}^* \circ \mathfrak{M}_*$ on the category Cmn of comonads. All is summarized in Figures 14 and 15. In Ross Street's paper on the *Formal theory of monads*, the latter adjunction between monads and comonads is spelled out directly [80, Thm. 11]. This was the main result of that seminal analysis, and remains the central theorem of the theory. We prove that Street's monad is strictly idempotent. This added wrinkle steers the theory into the practice: the adjunctions, and their monads and comonads, are not just the foundation of the categorical analysis, but also a convenient tool for concept mining from it. The nucleus of a monad, or of a comonad, displays its conceptual content.

1.3 A simplifying assumption

The claimed results have been verified for the general 2-categories of adjunctions, monads and comonads, and the earlier drafts of this paper attempted to present the claims in full generality. The present version presents them under the simplifying assumption that ***the 2-cell components of the morphisms of adjunctions, monads, and comonads are invertible***. This restriction cuts the length of the paper by half. While suppressing the general 2-cells simplifies some of the verifications, it does not eliminate or modify any of the presented structures, since all 2-categorical equipment of the nucleus construction already comes with invertible 2-cells. The general 2-categorical theory of nuclei is thus a *conservative* extension of the simplified theory presented here: it does not introduce any additional structure or side-conditions, but only a more general domain of validity, and verification. The suppressed 2-cell chasing is, of course, interesting and important on its own; yet it does not seem to provide any information specific to the nucleus construction itself. Our efforts to present the result in its full 2-categorical generality therefore seemed to be at the expense of the main message. We hope that suitable presentation tools under development[4] will soon make the results of this kind communicable with a more rational communication overhead.

1.4 Overview of the paper

We begin with simple and familiar examples of the nucleus, and progress towards the general construction. In the posetal case, the nucleus construction boils down to the fixed points of a Galois connection. It is familiar and intuitive as the posetal method of Formal Concept Analysis, which is presented in Sec. 2. The spectral methods of concept analysis, based on Singular Value Decomposition of linear operators, are perhaps even more widely known from their broad applications on the web. They also subsume under the nucleus construction, this time in linear algebra. This is the content of Sec. 3. Sec. 4 pops up to the level of an abstract categorical version of the nucleus, that emerged in the framework of $*$-autonomous categories and semantics of linear logic, as the separated-extensional core of the Chu construction. We discuss a modification that combines

[4] Our hopes have been vested in the framework of string diagrams for 2-categories, where the 2-cells are the vertices, the 1-cells are the edges, and the 0-cells are the faces of the underlying graphs. The project of drawing a sufficient supply of diagrams for the present paper remained beyond our reach, but it might soon come within reach [39].

the separated-extensional core with the spectral decomposition of matrices and refers back to the conceptual roots in early studies of topological vector spaces. In Sec. 5, we introduce the general categorical framework for the nucleus of adjoint functors, and we state the main theorem in Sec. 6. The proof of the main theorem is built in Sec. 7, through a series of lemmas, propositions, and corollaries. As the main corollary, Sec. 8 presents a simplified version of the nucleus, which provides alternative presentations of categories of algebras for a monad as algebras for a corresponding comonad; and analogously of coalgebras for a comonad as arising from a corresponding monad. These presentations are used in Sec. 9 to present a weaker version of the nucleus construction, obtained by applying the Kleisli construction at the last step, where the Eilenberg-Moore construction is applied in the stronger version. Although the resulting weak nuclei are equivalent to strong nuclei only in degenerate cases, the categories of strong nuclei and of weak nuclei turn out to be equivalent. In Sec. 10 we discuss how the nucleus approach compares and contrasts with the standard localization-based approaches to homotopy theory, from which the entire conceptual apparatus of adjunctions, extensions, and localizations originally emerged. In the final section of the paper, we discuss the problems that it leaves open.

2 Example 1: Concept lattices and poset bicompletions

2.1 From context matrices to concept lattices, intuitively

Consider a market with A sellers and B buyers. Their interactions are recorded in an adjacency matrix $A \times B \xrightarrow{\Phi} 2$, where 2 is the set $\{0, 1\}$, and the entry Φ_{ab} is 1 if the seller $a \in A$ at some point sold goods to the buyer $b \in B$; otherwise it is 0. Equivalently, a matrix $A \times B \xrightarrow{\Phi} 2$ can be viewed as the binary relation $\widehat{\Phi} = \{\langle a, b\rangle \in A \times B \mid \Phi_{ab} = 1\}$, in which case we write $a\widehat{\Phi}b$ instead of $\Phi_{ab} = 1$. In Formal Concept Analysis [17, 31, 30], such matrices or relations are called *contexts*, and used to extract some relevant *concepts*.

The idea is illustrated in Fig. 4. The binary relation $\widehat{\Phi} \subseteq A \times B$ is displayed as a bipartite graph. If buyers a_0 and a_4 have farms, and sellers b_1, b_2 and b_3 sell farming equipment, but seller b_0 does not, then the sets $X = \{a_0, a_4\}$ and $Y = \{b_1, b_2, b_3\}$ form a complete subgraph $\langle X, Y\rangle$ of the bipartite graph Φ, which corresponds to the concept *"farming"*. If the buyers from the set $X' = \{a_0, a_1, a_2, a_3\}$ have cars, but the buyer a_4 does not, and the sellers $Y' = \{b_0, b_1, b_2\}$ sell car accessories, but the seller b_3 does not then $\langle X', Y'\rangle$ is another complete subgraph, corresponding to the concept *"car"*. The idea is thus that a context is viewed as a bipartite graph, and the concepts are then extracted as its complete bipartite subgraphs.

2.2 Formalizing concept analysis

A pair $\langle U, V\rangle \in \wp A \times \wp B$ forms a complete subgraph of a bipartite graph $\widehat{\Phi} \subseteq A \times B$ if

$$U = \bigcap_{v \in V} \{x \in A \mid x\widehat{\Phi}v\} \qquad\qquad V = \bigcap_{u \in U} \{y \in B \mid u\widehat{\Phi}y\}$$

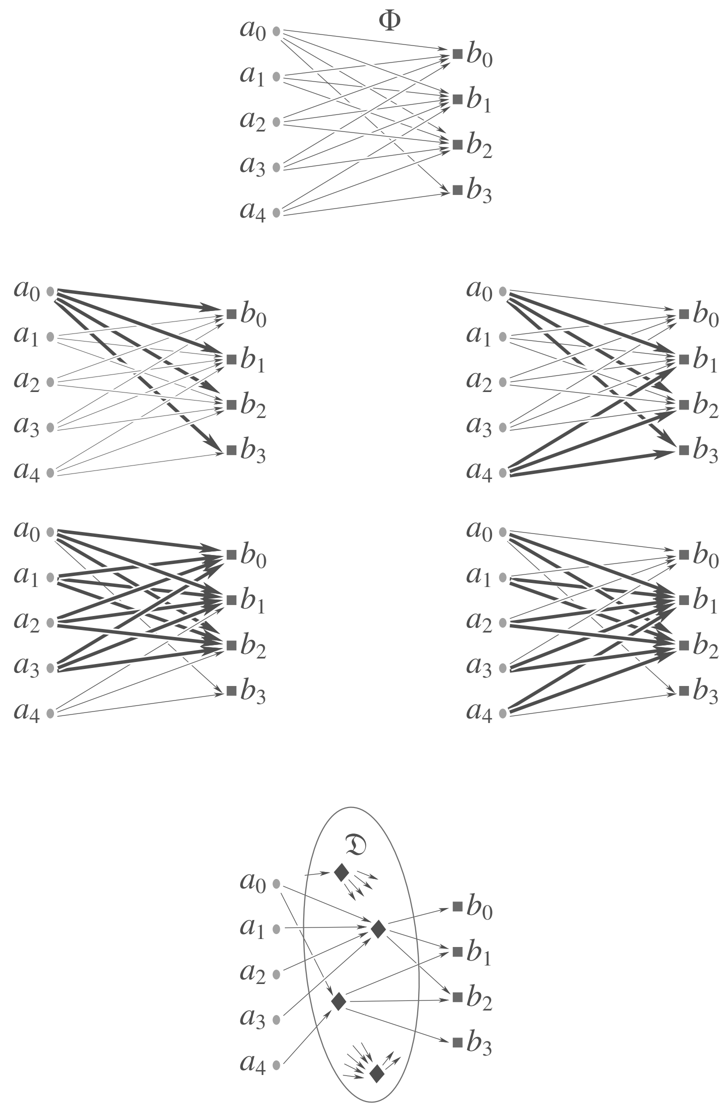

Figure 4: A context Φ, its four concepts, and their concept lattice

It is easy to see that such pairs are ordered by the relation

$$\langle U, V\rangle \le \langle U', V'\rangle \quad \Longleftrightarrow \quad U \subseteq U' \ \wedge \ V \supseteq V' \tag{3}$$

and that they in fact form a lattice, which is a retract of the lattice $\wp A \times \wp^o B$, where $\wp A$ is the set of subsets of A ordered by the inclusion $\subseteq$, while $\wp^o B$ is the set of subsets of B ordered by reverse inclusion $\supseteq$. This is the *concept lattice* $\mathfrak{D}$ induced by the *context matrix* $\widehat{\Phi} \subseteq A \times B$, along the lines of Fig. 3.

In general, the sets A and B may already carry partial orders, e.g. from earlier concept analyses.

The category of context matrices is thus

$$
\begin{aligned}
|\mathsf{Mat}_0| &= \coprod_{A,B\in\mathsf{Pos}} \mathsf{Pos}(A^o \times B, \mathbb{2}) && (4)\\
\mathsf{Mat}_0(\Phi,\Psi) &= \{\langle h,k\rangle \in \mathsf{Pos}(A,C)\times\mathsf{Pos}(B,D) \mid \Phi(a,b) = \Psi(ha,kb)\}
\end{aligned}
$$

where $\Phi \in \mathsf{Pos}(A^o \times B, \mathbb{2})$ and $\Psi \in \mathsf{Pos}(C^o \times D, \mathbb{2})$ are matrices with entries from the poset $\mathbb{2} = \{0 < 1\}$. When working with matrices in general, it is often necessary or convenient to use their *comprehensions*, i.e. to move along the correspondence

$$
\begin{array}{ccc}
\mathsf{Pos}(A^o \times B, \mathbb{2}) & \underset{\chi}{\overset{\widehat{(-)}}{\rightleftarrows}}\ \cong & \mathsf{Sub}\diagup A \times B^o \\
\Phi & \mapsto & \widehat{\Phi} = \{\langle x,y\rangle \in A \times B^o \mid \Phi(x,y) = 1\} \\
\chi_S(x,y) = \begin{cases} 1 & \text{if } \langle x,y\rangle \in S \\ 0 & \text{otherwise} \end{cases} & \mapsfrom & \left(S \subseteq A \times B^o\right)
\end{array}
\tag{5}
$$

A comprehension $\widehat{\Phi}$ of a matrix Φ is thus lower-closed in the first component, and upper-closed in the second:

$$
a \le a' \ \wedge\ a'\widehat{\Phi}b' \ \wedge\ b' \le b \quad\Longrightarrow\quad a\widehat{\Phi}b \tag{6}
$$

To extract the concepts from a context $\widehat{\Phi} \subseteq A \times B$, we thus need to explore the candidate lower-closed subsets of A, and the upper-closed subsets of B, which form complete semilattices $(\Downarrow A, \bigvee)$ and $(\Uparrow B, \bigwedge)$, where

$$
\begin{aligned}
\Downarrow A &= \{L \subseteq A \mid a \le a' \in L \implies a \in L\} && (7)\\
\Uparrow B &= \{U \subseteq B \mid U \ni b' \le b \implies U \ni b\} && (8)
\end{aligned}
$$

so that $\bigvee$ in $\Downarrow A$ and $\bigwedge$ in $\Uparrow B$ are both set union. It is easy to see that the embedding $A \xrightarrow{\blacktriangledown} \Downarrow A$, mapping $a \in A$ into the lower set $\blacktriangledown a = \{x \in A \mid x \le a\}$, is the join completion of the poset A, whereas $B \xrightarrow{\blacktriangle} \Uparrow B$, mapping $b \in B$ into the upper set $\blacktriangle b = \{y \in B \mid b \le y\}$, is the meet completion of the poset B. These semilattice completions support the context matrix extension $\Phi \subseteq \Downarrow A \times \Uparrow B$ defined by

$$
L\Phi U \quad\iff\quad \forall a \in L\ \forall b \in U.\ a\widehat{\Phi}b \tag{9}
$$

As a matrix between complete semilattices, Φ is representable in the form

$$
\Phi^* L \subseteq U \quad\iff\quad L\Phi U \quad\iff\quad L \supseteq \Phi_* U \tag{10}
$$

where the adjoints now capture the *complete-bipartite-subgraph* idea from Fig. 4:

$$\begin{tikzcd} L \arrow[d, mapsto] & \Downarrow A \arrow[d, bend right, "\Phi^*"'] \arrow[d, phantom, "\dashv"] & \bigcap\limits_{y\in U} {}_\bullet\Phi y \\ \bigcap\limits_{x\in L} x\Phi_\bullet & \Uparrow B \arrow[u, bend right, "\Phi_*"'] & U \arrow[u, mapsto] \end{tikzcd} \tag{11}$$

Here ${}_\bullet\Phi y = \{x \in A \mid x\Phi y\}$ and $x\Phi_\bullet = \{y \in B \mid x\widehat{\Phi}y\}$ define the transposes ${}_\bullet\Phi : B \to \Downarrow A$ and $\Phi_\bullet : A \to \Uparrow B$ of $\Phi : A^o \times B \to \mathbb{2}$. Poset adjunctions like (11) are often also called *Galois connections*. They form the category

$$|\mathsf{Adj}_0| \;=\; \coprod_{A,B\in\mathsf{Pos}} \{\langle\Phi^*, \Phi_*\rangle \in \mathsf{Pos}(A,B)\times\mathsf{Pos}(B,A) \mid \Phi^*x \le y \iff x \le \Phi_*y\} \tag{12}$$

$$\mathsf{Adj}_0(\Phi,\Psi) \;=\; \{\langle H,K\rangle \in \mathsf{Pos}(A,C)\times\mathsf{Pos}(B,D) \mid K\Phi^* = \Psi^*H \;\wedge\; H\Phi_* = \Psi_*H\}$$

The first step of concept analysis is thus the matrix extension

$$\begin{aligned} \mathsf{MA}_0 : \mathsf{Mat}_0 &\to \mathsf{Adj}_0 \\ \Phi &\mapsto (\Phi^* \dashv \Phi_* : \Uparrow B \to \Downarrow A) \text{ as in (11)} \end{aligned} \tag{13}$$

To complete the process of concept analysis, we use the full subcategories of Adj_0 spanned by the closure and the interior operators, respectively:

$$\mathsf{Mnd}_0 \;=\; \{(\Phi^* \dashv \Phi_*) \in \mathsf{Adj}_0 \mid \Phi^*\Phi_* = \mathrm{id}\} \tag{14}$$

$$\mathsf{Cmn}_0 \;=\; \{(\Phi^* \dashv \Phi_*) \in \mathsf{Adj}_0 \mid \Phi_*\Phi^* = \mathrm{id}\} \tag{15}$$

It is easy to see that

- Mnd_0 is equivalent with the category of posets A equipped with closure operators, i.e. monotone maps $A \xrightarrow{\overleftarrow{\Phi}} A$ such that $x \le \overleftarrow{\Phi}x = \overleftarrow{\Phi}\overleftarrow{\Phi}x$, for $\overleftarrow{\Phi} = \Phi_*\Phi^*$; while
- Cmn_0 is equivalent with the category of posets B equipped with interior operators, i.e. monotone maps $B \xrightarrow{\overrightarrow{\Phi}} B$ such that $y \ge \overrightarrow{\Phi}y = \overrightarrow{\Phi}\overrightarrow{\Phi}y$, for $\overrightarrow{\Phi} = \Phi^*\Phi_*$.

The functors $\mathsf{AM}_0 : \mathsf{Adj}_0 \twoheadrightarrow \mathsf{Mnd}_0$ and $\mathsf{AC}_0 : \mathsf{Adj}_0 \twoheadrightarrow \mathsf{Cmn}_0$ are thus inclusions, and their resolutions are

$$
\begin{array}{rcl}
\mathsf{EM}_0 : \mathsf{Mnd}_0 & \rightarrowtail & \mathsf{Adj}_0 \\
\left(A \xrightarrow{\overleftarrow{\Phi}} A\right) & \mapsto & \left(\begin{tikzcd} \Downarrow A \arrow[r, bend right, two heads] \arrow[r, phantom, "\top"] & \Downarrow A^{\overleftarrow{\Phi}} \arrow[l, bend right, hook'] \end{tikzcd}\right) \\
& & \text{where} \quad \Downarrow A^{\overleftarrow{\Phi}} = \{U \in \Downarrow A \mid U = \overleftarrow{\Phi} U\}
\end{array}
\tag{16}
$$

$$
\begin{array}{rcl}
\mathsf{KC}_0 : \mathsf{Cmn}_0 & \rightarrowtail & \mathsf{Adj}_0 \\
\left(B \xrightarrow{\overrightarrow{\Phi}} B\right) & \mapsto & \left(\begin{tikzcd} \Uparrow B^{\overrightarrow{\Phi}} \arrow[r, bend right, hook] \arrow[r, phantom, "\top"] & \Uparrow B \arrow[l, bend right, two heads] \end{tikzcd}\right) \\
& & \text{where} \quad \Uparrow B^{\overrightarrow{\Phi}} = \{V \in \Uparrow B \mid \overrightarrow{\Phi} V = V\}
\end{array}
\tag{17}
$$

Mnd_0 thus turns out to be a reflective subcategory of Adj_0, and Cmn_0 coreflective. The category Nuc_0 of concept lattices is their intersection, thus is coreflective in Mnd_0 and reflective in Cmn_0. In fact, these posetal resolutions turn out to be adjoint to the inclusions both on the left and on the right; but that is a peculiarity of the posetal case. Another posetal quirk is that the category Nuc_0 boils down to the category Pos of posets, because an operator that is both a closure and an interior must be an identity. That will not happen in general.

2.3 Summary

Going from left to right through Fig. 3 with the categories defined in (4), (12), (14) and (15), and reflecting everything back into Adj_0, we made the following steps

$$\Phi : A^o \times B \to \mathbb{2}$$

$$\Phi^*_* = \mathsf{MA}_0\Phi = \left(\begin{tikzcd} \Downarrow A \arrow[r, bend right, "\Phi^*"'] \arrow[r, phantom, "\top"] & \Uparrow B \arrow[l, bend right, "\Phi_*"'] \end{tikzcd}\right)$$

$$
\overleftarrow{\mathsf{EM}}_0\Phi^*_* = \left(\begin{tikzcd} \Downarrow A \arrow[r, bend right, two heads] \arrow[r, phantom, "\top"] & \Downarrow A^{\overleftarrow{\Phi}} \arrow[l, bend right, hook'] \end{tikzcd}\right)
\qquad\qquad
\overrightarrow{\mathsf{KC}}_0\Phi^*_* = \left(\begin{tikzcd} \Uparrow B^{\overrightarrow{\Phi}} \arrow[r, bend right, hook] \arrow[r, phantom, "\top"] & \Uparrow B \arrow[l, bend right, two heads] \end{tikzcd}\right)
\tag{18}
$$

$$\overleftarrow{\mathfrak{N}}_0\Phi = \left(\begin{tikzcd} \Uparrow B^{\overrightarrow{\Phi}} \arrow[r, bend right, two heads, "\Phi^\sharp"'] \arrow[r, phantom, "\cong"] & \Downarrow A^{\overleftarrow{\Phi}} \arrow[l, bend right, two heads, "\Phi_\sharp"'] \end{tikzcd}\right)$$

where $\overleftarrow{\mathsf{EM}}_0 = \mathsf{EM}_0 \circ \mathsf{AM}_0$, and $\overrightarrow{\mathsf{KC}}_0 = \mathsf{KC}_0 \circ \mathsf{AC}_0$, and $\overleftarrow{\mathfrak{N}}_0$ defines the poset nucleus (which will be subsumed under the general definition in Sec. 6). For posets, the final step happens to be trivial, because of the order isomorphisms

$$\Downarrow A^{\overleftarrow{\Phi}} \;\cong\; \mathfrak{D} \;\cong\; \Uparrow B^{\overrightarrow{\Phi}} \tag{19}$$

where $\mathfrak{D}$

$$\mathfrak{D} \quad = \quad \{\langle L, U\rangle \in \Downarrow A \times \Uparrow B \mid L = \Phi_* U \;\wedge\; \Phi^* L = U\} \tag{20}$$

is the familiar lattice of *Dedekind cuts*. The images of the context Φ in Mnd_0, Cmn_0 and Nuc_0 thus give three isomorphic views of the concept lattice. But this is a degenerate case.

Comment. The situation when the two resolutions of an adjunction (the one in Mnd and the one in Cmn) are isomorphic is very special. E.g., when $A = B = \mathbb{Q}$ is the field of rational numbers, and $\Phi = (\leq)$ is their partial order, then $\mathsf{MA}_1{}^*\Phi$ is the set of pairs $\langle L, U\rangle$, where L is an open and closed lower interval, U is an open or closed upper interval, and $L \leq U$. The resolutions eliminate the rational points between L and U, by requiring that L contains all lower bounds of U and U all upper bounds of L. The nucleus then comprises the Dedekind cuts. But any Dedekind cut $\langle L, U\rangle$ is also completely determined by L alone, and by U alone. Hence the isomorphisms (19). The same generalizes when $A = B$ is a partial order, and the nucleus yields its Dedekind-MacNeille completion: it adjoins all joins and meets that are missing while preserving those that already exist. When A and B are different posets, and Φ is a nontrivial context between them, we are in the business of concept analysis, and generate the concept lattice — with similar generation and preservation requirements like for the Dedekind-MacNeille completion. In a sense, the posets A and B are "glued together" along the context $\widehat{\Phi} \subseteq A \times B$ into the joint completion $\mathfrak{D}$, where the joins are generated from A, and the meets from B. On the other hand, any meets that may have existed in A are preserved in $\mathfrak{D}$; as are any joins that may have existed in B.

It is a remarkable fact of category theory that no such tight bicompletion exists in general, when the poset P is generalized to a category [55, 40]. It also is well known that this phenomenon is closely related to the idempotent monads induced by adjunctions, and by profunctors in general [1].

The phenomenon is, however, quite general, and in a sense, hides in plain sight.

3 Example 2: Nuclei in linear algebra

3.1 Matrices and linear operators

The nucleus examples in this section take us back to undergraduate linear algebra. The first part is in fact even more basic. To begin, we consider matrices $\dot{A} \times \dot{B} \to R$, where R is an arbitrary ring, and $\dot{A}, \dot{B}$ are *finite* sets. We denote the category of all sets by Set, its full subcategory spanned by finite sets by $\dot{\mathsf{Set}}$, and generally use the dot to mark finiteness, so that $\dot{A}, \dot{B} \in \dot{\mathsf{Set}} \subset \mathsf{Set}$. Viewing both finite sets $\dot{A}, \dot{B}$ and the ring R together in the category of sets, we define

$$\begin{aligned} |\mathsf{Mat}_1| &= \coprod_{\dot{A},\dot{B}\in\dot{\mathsf{Set}}} \mathsf{Set}(\dot{A}\times\dot{B}, R) \\ \mathsf{Mat}_1(\Phi,\Psi) &= \left\{\langle H, K\rangle \in R^{\dot{A}\times\dot{C}} \times R^{\dot{B}\times\dot{D}} \mid K\Phi = \Psi H\right\} \end{aligned} \tag{21}$$

where $R^{\dot{A}\times\dot{C}}$ abbreviates $\mathsf{Set}(\dot{A}\times\dot{C}, R)$, and ditto $R^{\dot{B}\times\dot{D}}$. The matrix composition is written left to right

$$\begin{aligned} R^{\dot{X}\times\dot{Y}} \times R^{\dot{Y}\times\dot{Z}} &\longrightarrow R^{\dot{X}\times\dot{Z}} \\ \langle F, G\rangle &\mapsto (GF)_{ik} = \sum_{j\in B} F_{ij}\cdot G_{jk} \end{aligned}$$

When R is a field, Mat_1 is the arrow category of finite-dimensional R-vector spaces with chosen bases. When R is a general ring, Mat_1 is the arrow category free R-modules with finite generators. When R is not even a ring, but say the *rig* ("a ri*n*g without the *n*egatives") $\mathbb{N}$ of natural numbers, then Mat_1 is the arrow category of free commutative monoids. Sec. 3.2 applies to all these cases, and Sec. 3.3 applies to real closed fields. Since the goal of this part of the paper is to recall familiar examples of the nucleus construction, we can just as well assume that R is the field of real numbers. The full generality of the construction will emerge in the end.

3.2 Nucleus as an automorphism of the rank space of a linear operator

Since finite-dimensional vector spaces always carry a separable inner product, the category Mat_1 over the field of real numbers R is equivalent to the arrow category over finite-dimensional real Hilbert spaces *with chosen bases*. This assumption yields a canonical matrix representation for each linear operator. Starting, on the other hand, from the category $\dot{\mathsf{H}}\mathsf{ilb}$ of finite-dimensional Hilbert spaces *without* chosen bases, we define the category Adj_1 as the arrow category $\dot{\mathsf{H}}\mathsf{ilb}\diagup\dot{\mathsf{H}}\mathsf{ilb}$ of linear operators and their commutative squares, i.e.

$$
\begin{aligned}
|\mathsf{Adj}_1| &= \coprod_{\mathbb{A},\mathbb{B}\in\dot{\mathsf{H}}\mathsf{ilb}} \dot{\mathsf{H}}\mathsf{ilb}(\mathbb{A},\mathbb{B}) \\
\mathsf{Adj}_1(\Phi,\Psi) &= \left\{\langle H,K\rangle \in \dot{\mathsf{H}}\mathsf{ilb}(\mathbb{A},\mathbb{C})\times\dot{\mathsf{H}}\mathsf{ilb}(\mathbb{B},\mathbb{D}) \mid K\Phi = \Psi H\right\}
\end{aligned}
\tag{22}
$$

The finite-dimensional Hilbert spaces $\mathbb{A}$ and $\mathbb{B}$ are still isomorphic to $R^{\dot{A}}$ and $R^{\dot{B}}$ for some finite spaces $\dot{A}$ and $\dot{B}$ of basis vectors; but the particular isomorphisms would choose a standard basis for each of them, so now we are not given such isomorphisms. This means that the linear operators like H and K in (22) do not have standard matrix representations, but are given as linear functions between the entire spaces. The categories Mnd_1 and Cmn_1 will be the full subcategories of Adj_1 spanned by

$$\mathsf{Mnd}_1 = \{\Phi \in \mathsf{Adj}_1 \mid \Phi \text{ is surjective }\} \tag{23}$$

$$\mathsf{Cmn}_1 = \{\Phi \in \mathsf{Adj}_1 \mid \Phi^\ddagger \text{ is surjective }\} \tag{24}$$

where $\Phi^\ddagger$ is the adjoint of $\Phi \in \dot{\mathsf{H}}\mathsf{ilb}(\mathbb{A},\mathbb{B})$, i.e. the operator $\Phi^\ddagger \in \dot{\mathsf{H}}\mathsf{ilb}(\mathbb{B},\mathbb{A})$ satisfying

$$\langle b \mid \Phi a\rangle_{\mathbb{B}} = \langle \Phi^\ddagger b \mid a\rangle_{\mathbb{A}}$$

where $\langle - | - \rangle_{\mathbb{H}}$ denotes the inner product on the space $\mathbb{H}$.

3.2.1 Hilbert space adjoints: Notation and construction

In the presence of inner products[5] $\langle - | - \rangle : \mathbb{A}\times\mathbb{A} \to R$, it is often more convenient to use the bra-ket notation, where a vector $\vec{a} \in \mathbb{A}$ is written as a "bra" $|a\rangle$, and the corresponding linear

[5] If R were not a *real* closed field, the inner product would involve a conjugate in the first argument. Although this is for most people the more familiar situation, the adjunctions here do not depend on conjugations, so we omit them.

functional $\vec{a}^{\ddagger} = \langle\vec{a}|-\rangle \in \mathbb{A}^*$ is written as the "ket" $\langle a|$. If $\mathbb{A}$ is the $\dot{A}$-dimensional space $R^{\dot{A}}$, then the basis vectors $\vec{e}_i$, $i = 1, 2, \ldots, \dot{A}$ are written $|1\rangle, |2\rangle, \ldots, |\dot{A}\rangle$, whereas the basis vectors of $\mathbb{A}^*$ are $\langle 1|, \langle 2|, \ldots, \langle\dot{A}|$, and the base decompositions become

- $|a\rangle = \sum_{i=1}^{\dot{A}} |i\rangle\langle i|a\rangle$ instead of $\vec{a} = \sum_{i=1}^{\dot{A}} a_i\vec{e}_i$, and
- $\langle a| = \sum_{i=1}^{\dot{A}}\langle a|i\rangle\langle i|$ instead of $\vec{a}^{\ddagger} = \sum_{i=1}^{\dot{A}} a_i\vec{e}_i^{\,\ddagger}$.

For convenience, here we assume that the finite sets $\dot{A}, \dot{B}, \ldots \in \dot{\mathsf{Set}}$ are ordered, i.e. reduce $\dot{\mathsf{Set}}$ to $\mathbb{N}$. In practice, the difference between $\mathbb{A}$ and $\mathbb{A}^*$ is often ignored, because any basis induces a linear isomorphism $\mathbb{A}^* \cong \mathbb{A}$, and is uniquely determined by it [19]; but it creeps from under the carpet when vector spaces are combined or aligned with other structures, as we will see further on. Writing $\langle j|\Phi|i\rangle$ for the entries Φ_{ji} of a matrix $\Phi = \left(\Phi_{ji}\right)_{n\times\dot{A}}$ gives

- $\langle j|\Phi|a\rangle = \sum_{i=1}^{\dot{A}}\langle j|\Phi|i\rangle\langle i|a\rangle$ instead of $(\Phi\vec{a})_j = \sum_{i=1}^{\dot{A}} \Phi_{ji}a_i$,
- $\langle b|\Phi|i\rangle = \sum_{j=1}^{\dot{B}}\langle b|j\rangle\langle j|\Phi|i\rangle$ instead of $\left(\vec{b}^{\ddagger}\Phi\right)_i = \sum_{j=1}^{\dot{B}} b_j\Phi_{ji}$, and
- $\langle b|\Phi|a\rangle = \sum_{i=1}^{\dot{A}}\sum_{j=1}^{\dot{B}}\langle b|j\rangle\langle j|\Phi|i\rangle\langle i|a\rangle$ instead of $\vec{b}^{\ddagger}\Phi\vec{a} = \sum_{i=1}^{\dot{A}}\sum_{j=1}^{\dot{B}} b_j\Phi_{ji}a_i$

and hence the inner-product adjunction

$$\langle b|\Phi a\rangle_{\mathbb{B}} \quad = \quad \langle b|\Phi|a\rangle \quad = \quad \langle\Phi^{\ddagger}b|a\rangle_{\mathbb{A}} \tag{25}$$

where we adhere to the usual abuse of notation, and denote both the matrix and the induced linear operator by Φ. The dual matrix and the induced adjoint operator are $\Phi^{\ddagger}$. If (25) is the Hilbert space version of (10), then (11) becomes

$$\begin{tikzcd}
|a\rangle \arrow[d, mapsto] & R^{\dot{A}} \arrow[d, bend right, "\Phi"'] & \sum_{j=1}^{\dot{B}}\langle b|j\rangle\langle j|\Phi_{\bullet} \\
\sum_{i=1}^{\dot{A}} {}_{\bullet}\Phi|i\rangle\langle i|a\rangle & R^{\dot{B}} \arrow[u, bend right, "\Phi^{\ddagger}"'] & \langle b| \arrow[u, mapsto]
\end{tikzcd} \tag{26}$$

Here ${}_{\bullet}\Phi|i\rangle = \sum_{j=1}^{\dot{B}}\langle j|\langle j|\Phi|i\rangle$ is the i-th column of Φ, transposed into a row, whereas $\langle j|\Phi_{\bullet} = \sum_{i=1}^{\dot{A}}\langle j|\Phi|i\rangle|i\rangle$ is its j-th row vector, transposed into a column.

3.2.2 Factorizations

The maps in (26) induce the functor $\mathsf{MA}_1 : \mathsf{Mat}_1 \to \mathsf{Adj}_1$, for $\mathbb{A} = R^{\dot{A}}$ and $\mathbb{B} = R^{\dot{B}}$. This functor is, of course, tacit in the practice of representing linear operators by matrices, and identifying them notationally. The functors $\mathsf{AM}_1 : \mathsf{Adj}_1 \to \mathsf{Mnd}_1$ and $\mathsf{AC}_1 : \mathsf{Adj}_1 \to \mathsf{Cmn}_1$, on the other hand, require factoring linear operators through their rank spaces:

$$
\begin{tikzcd}
 & \mathbb{A} \arrow[dl, two heads, "\mathsf{AM}_1(\Phi)"'] \arrow[d, bend right, "\Phi"'] \arrow[d, phantom, "\dashv\vdash"] & \mathbb{B}^{\overrightarrow{\Phi}} \arrow[l, hook', "U"'] \\
\mathbb{A}^{\overleftarrow{\Phi}} \arrow[r, hook, "V"'] & \mathbb{B} \arrow[u, bend right, "\Phi^\ddagger"'] \arrow[ur, two heads, "\mathsf{AC}_1(\Phi)^\ddagger"'] &
\end{tikzcd}
\tag{27}
$$

where we define

$$
\begin{aligned}
\mathbb{B}^{\overrightarrow{\Phi}} &= \{\Phi^\ddagger|b\rangle \mid |b\rangle \in \mathbb{B}\} \quad \text{with} \quad \langle x|y\rangle_{\mathbb{B}^{\overrightarrow{\Phi}}} = \langle Ux|Uy\rangle_{\mathbb{A}} \\
\mathbb{A}^{\overleftarrow{\Phi}} &= \{\Phi|a\rangle \mid |a\rangle \in \mathbb{A}\} \quad \text{with} \quad \langle x|y\rangle_{\mathbb{A}^{\overleftarrow{\Phi}}} = \langle Vx|Vy\rangle_{\mathbb{B}}
\end{aligned}
$$

It is easy to see that the adjoints $\mathsf{EM}_1 : \mathsf{Mnd}_1 \to \mathsf{Adj}_1$ and $\mathsf{KC}_1 : \mathsf{Cmn}_1 \to \mathsf{Adj}_1$ can be viewed as inclusions. To define $\mathsf{MN}_1 : \mathsf{Mnd}_1 \to \mathsf{Nuc}_1$ and $\mathsf{CN}_1 : \mathsf{Cmn}_1 \to \mathsf{Nuc}_1$, note that

$$
\langle U^\ddagger U x \mid y\rangle_{\mathbb{B}^{\overrightarrow{\Phi}}} \quad = \quad \langle Ux \mid Uy\rangle_{\mathbb{A}} \quad = \quad \langle x \mid y\rangle_{\mathbb{B}^{\overrightarrow{\Phi}}}
$$

Since finite-dimensional Hilbert spaces are separable, this implies that $U^\ddagger U = \mathrm{id}$ and that $U^\ddagger$ is thus a surjection. So we have two factorizations of Φ

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, two heads, "U^\ddagger"] \arrow[d, two heads, "\mathsf{AC}_1(\Phi)"'] & \mathbb{B}^{\overrightarrow{\Phi}} \arrow[d, tail, "\mathsf{AM}_1(\Phi)"] \arrow[dl, dashed, tail, two heads, "\substack{\mathsf{CN}_1\circ\mathsf{AC}_1(\Phi)= \\ \mathsf{MN}_1\circ\mathsf{AM}_1(\Phi)}"'] \\
\mathbb{A}^{\overleftarrow{\Phi}} \arrow[r, tail, "V"'] & \mathbb{B}
\end{tikzcd}
\tag{28}
$$

The definitions of CN_1 and MN_1 for general objects of Cmn_1 and Mnd_1 proceed similarly, by factoring the adjoints.

3.3 Nucleus as matrix diagonalization

When the field R supports spectral decomposition, the above factorizations can be performed directly on matrices. The nucleus of a matrix then arises as its diagonal form. In linear algebra, the process of the nucleus extraction thus boils down to the Singular Value Decomposition (SVD) of a matrix [32, Sec. 2.4], which is yet another tool of concept analysis [5, 20].

To set up this version of the nucleus setting we take $\mathsf{Adj}_2 = \mathsf{Mat}_2 = \mathsf{Mat}_1$ and let $\mathsf{MA}_2 : \mathsf{Mat}_2 \to \mathsf{Adj}_2$ be the identity. The categories Mnd_2 and Cmn_2 will again be full subcategories of Adj_2, this time spanned by

$$\begin{aligned} \mathsf{Mnd}_2 &= \{\Phi \in \mathsf{Set}(\dot{A} \times \dot{B}, R) \mid \langle k|\overrightarrow{\Phi}|\ell\rangle = \lambda_k \langle k|\ell\rangle\} \quad (29)\\ \mathsf{Cmn}_2 &= \{\Phi \in \mathsf{Set}(\dot{A} \times \dot{B}, R) \mid \langle i|\overleftarrow{\Phi}|j\rangle = \lambda_j \langle i|j\rangle\} \quad (30) \end{aligned}$$

where

- $\overrightarrow{\Phi} = \Phi\Phi^{\ddagger}$ and $\overleftarrow{\Phi} = \Phi^{\ddagger}\Phi$, with the entries $\langle k|\overrightarrow{\Phi}|\ell\rangle = \overrightarrow{\Phi}_{k\ell}$ $\langle i|\overleftarrow{\Phi}|j\rangle = \overleftarrow{\Phi}_{ij}$,
- $\langle i|j\rangle = \left\{\begin{matrix} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{matrix}\right\}$, and
- λ_k and λ_j are scalars.

In the theory of Banach spaces, operators that yield to this type of representation have been called nuclear since [34]. Hence our terminology. For finite-dimensional spaces, definitions (29-30) say that for a matrix $\Phi \in \mathsf{Mat}_2$ holds that

$$\begin{aligned} \Phi \in \mathsf{Mnd}_2 &\iff \overrightarrow{\Phi} \text{ is diagonal}\\ \Phi \in \mathsf{Cmn}_2 &\iff \overleftarrow{\Phi} \text{ is diagonal} \end{aligned}$$

Since both $\overleftarrow{\Phi}$ and $\overrightarrow{\Phi}$ are self-adjoint:

$$\begin{aligned} \langle \Phi^{\ddagger}\Phi a \mid a'\rangle &= \langle \Phi a \mid \Phi a'\rangle &&= \langle \Phi^{\ddagger\ddagger} a \mid \Phi a'\rangle &&= \langle a \mid \Phi^{\ddagger}\Phi a'\rangle\\ \langle b \mid \Phi\Phi^{\ddagger} b'\rangle &= \langle \Phi^{\ddagger} b \mid \Phi^{\ddagger} b'\rangle &&= \langle \Phi^{\ddagger} b \mid \Phi^{\ddagger\ddagger\ddagger} b'\rangle &&= \langle \Phi^{\ddagger\ddagger}\Phi^{\ddagger} b \mid b'\rangle = \langle \Phi\Phi^{\ddagger} b \mid b'\rangle \end{aligned}$$

their spectral decompositions yield real eigenvalues λ. Assuming for simplicity that each of their eigenvalues has a one-dimensional eigenspace, we define

$$\begin{aligned} \dot{A}^{\overleftarrow{\Phi}} &= \{|v\rangle \in R^{\dot{B}} \mid \langle v|v\rangle = 1 \wedge \exists \lambda_v.\ \overrightarrow{\Phi}|v\rangle = \lambda_v|v\rangle\} \quad (31)\\ \dot{B}^{\overrightarrow{\Phi}} &= \{|u\rangle \in R^{\dot{B}} \mid \langle u|u\rangle = 1 \wedge \exists \lambda_u.\ \overleftarrow{\Phi}|u\rangle = \lambda_u|u\rangle\} \quad (32) \end{aligned}$$

Hence the matrices

$$\begin{array}{ccccccc} \dot{B}^{\overrightarrow{\Phi}} \times \dot{A} & \xrightarrow{\quad U \quad} & R & & R & \xleftarrow{\quad V \quad} & \dot{A}^{\overleftarrow{\Phi}} \times \dot{B} \\ \\ \big\langle |u\rangle, i \big\rangle & \longmapsto & u_i & & v_\ell & \longleftarrow\!\!\shortmid & \big\langle |v\rangle, \ell \big\rangle \end{array}$$

which isometrically embed $\dot{B}^{\overrightarrow{\Phi}}$ into $\mathbb{A} = R^{\dot{A}}$ and $A^{\overleftarrow{\Phi}}$ into $\mathbb{B} = R^{\dot{B}}$. It is now straightforward to show that $\mathsf{AM}_2 : \mathsf{Adj}_2 \to \mathsf{Mnd}_2$ and $\mathsf{AC}_2 : \mathsf{Adj}_2 \to \mathsf{Cmn}_2$ are still given according to the schema in (27), i.e. by

$$\begin{aligned} \check{\Phi} &= \mathsf{AM}_2(\Phi) &= V^{\ddagger}\Phi \quad (33)\\ \hat{\Phi} &= \mathsf{AC}_2(\Phi) &= \Phi U \quad (34) \end{aligned}$$

They satisfy not only the requirements that $\check\Phi^\dagger\check\Phi$ and $\hat\Phi\hat\Phi^\ddagger$ be diagonal, as required by (29) and (30), but also that

$$\check\Phi\check\Phi^\dagger = \Phi\Phi^\dagger = \overleftarrow{\Phi} \qquad\qquad \hat\Phi^\dagger\hat\Phi = \Phi^\dagger\Phi = \overrightarrow{\Phi}$$

Repeating the diagonalization process on each of them leads to the following refinement of (27):

$$\begin{tikzcd}
\dot A \arrow[d, "\Phi"'] \arrow[dr, two heads, "\check\Phi"'] & \dot B^{\overrightarrow\Phi} \arrow[l, hook', "U"'] \arrow[dl, tail, "\hat\Phi"'] \arrow[d, dashed, two heads, "{\mathsf{MN}_2(\hat\Phi) = \mathsf{CN}_2(\check\Phi)}"] \arrow[dr, tail, two heads, "\hat{\check\Phi}"'] \arrow[r, leftrightarrow, "\sim"] & \left(\dot A^{\overleftarrow\Phi}\right)^{\overrightarrow\Phi} \arrow[dl, tail, two heads, "\check{\hat\Phi}"'] \arrow[d, tail, two heads] \\
\dot B \arrow[r, two heads, "V^\ddagger"'] & \dot A^{\overleftarrow\Phi} \arrow[r, leftrightarrow, "\sim"] & \left(\dot B^{\overrightarrow\Phi}\right)^{\overleftarrow\Phi}
\end{tikzcd} \tag{35}$$

This diagram displays a bijection between the eigenvertors in $\dot B^{\overrightarrow\Phi}$ and $A^{\overleftarrow\Phi}$. The diagonal matrix between them is the nucleus of Φ. The singular values along its diagonal measure, in a certain sense, how much the operators $\overleftarrow\Phi$ and $\overrightarrow\Phi$, induced by composing Φ and $\Phi^\ddagger$, deviate from being projectors onto the respective rank spaces.

3.4 Summary

The path from a matrix to its nucleus can now be summarized by

$$\Phi : \dot A \times \dot B \to R$$

$$\begin{tikzcd}
R^{\dot A} \arrow[r, bend right, "\Phi"'] & R^{\dot B} \arrow[l, bend right, "\Phi^\ddagger"']
\end{tikzcd}$$

$$\begin{tikzcd}
R^{\dot A} \arrow[r, bend right, two heads] & \dot A^{\overleftarrow\Phi} \arrow[l, bend right, tail, "U=\mathfrak{M}_2\Phi"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\dot B^{\overrightarrow\Phi} \arrow[r, bend left, tail, "V=\mathfrak{E}_2\Phi"] & R^{\dot B} \arrow[l, bend left, two heads]
\end{tikzcd}$$

$$\begin{tikzcd}
\dot B^{\overrightarrow\Phi} \arrow[r, tail, two heads, "\overleftarrow{\mathfrak{N}}_2\Phi"] & \dot A^{\overleftarrow\Phi} \arrow[d, tail, "V"] \\
R^{\dot A} \arrow[u, two heads, "U^\ddagger"] \arrow[r, "\Phi"'] & R^{\dot B}
\end{tikzcd}$$

Note that the isomorphisms from (19) are now replaced by the diagonal matrix $\overleftarrow{\mathfrak{N}}_1\Phi : \dot B^{\overrightarrow\Phi} \rightarrowtail\!\!\!\!\twoheadrightarrow \dot A^{\overleftarrow\Phi}$, wich is still invertible as a linear operator, and provides a bijection between the bases $\dot B^{\overrightarrow\Phi}$ and $\dot A^{\overleftarrow\Phi}$ of the rank spaces of Φ and of $\Phi^\ddagger$, respectively. But the singular values along the diagonal of

$\overleftarrow{\mathfrak{N}}_1\Phi$ quantify the relationships between the corresponding elements of $\dot{B}^{\overrightarrow{\Phi}}$ and $\dot{A}^{\overleftarrow{\Phi}}$. This is, on the one hand, the essence of the concept analysis by singular value decomposition [56]. Even richer conceptual correspondences will, on the other hand, emerge in further examples.

4 Example 3: Nuclear Chu spaces

4.1 Abstract matrices

So far we have considered matrices in specific frameworks, first of posets, then of Hilbert spaces. In this section, we broaden the view, and study an abstract framework of matrices. Suppose that $\mathcal{S}$ is a category with finite products, $R \in \mathcal{S}$ is an object, and $\dot{\mathcal{S}} \subseteq \mathcal{S}$ is a full subcategory. The objects of $\dot{\mathcal{S}}$ are also marked by a dot, and are thus written $\dot{A}, \dot{B}, \ldots, \dot{X} \in \dot{\mathcal{S}}$. Now consider the following variation on the theme of (4) and (21):

$$\begin{aligned} |\mathsf{Mat}_3| &= \coprod_{\dot{A},\dot{B}\in\dot{\mathcal{S}}} \mathcal{S}(\dot{A}\times\dot{B}, R) \\ \mathsf{Mat}_3(\Phi,\Psi) &= \left\{\langle f^*, f_*\rangle \in \dot{\mathcal{S}}(\dot{A},\dot{C})\times\dot{\mathcal{S}}(\dot{D},\dot{B}) \mid \Phi(a, f_*d) = \Psi(f^*a, d)\right\} \end{aligned} \tag{36}$$

where $\Psi \in \mathcal{S}(\dot{C}\times\dot{D}, R)$, as illustrated in Fig. 5. We consider a couple of examples.

$$\begin{tikzcd}[column sep=small] & \dot{A}\times\dot{D} \arrow[dl, "\dot{A}\times f_*"'] \arrow[dr, "f^*\times\dot{D}"] & \\ \dot{A}\times\dot{B} \arrow[dr, "\Phi"'] & \xrightarrow{f} & \dot{C}\times\dot{D} \arrow[dl, "\Psi"] \\ & R & \end{tikzcd}$$

Figure 5: A Chu-morphism $f = \langle f^*, f_*\rangle : \Phi \to \Psi$ in Mat_3

4.1.1 Posets

Let the category $\mathcal{S} = \dot{\mathcal{S}}$ be the category Pos of posets, and let R be the poset $\mathbb{2} = \{0 < 1\}$. The poset matrices in $\mathsf{Mat}_3^{\mathsf{Pos}}$ then differ from those in Mat_0 by the fact that they are covariant in both arguments, i.e. they satisfy $a'\widehat{\Phi}b' \;\wedge\; a' \leq a \;\wedge\; b' \leq b \implies a\widehat{\Phi}b$ instead of (6). Any poset A is represented both in Mat_0 and in $\mathsf{Mat}_3^{\mathsf{Pos}}$ by the matrix $(\overset{A}{\leq}) : A^o \times A \to \mathbb{2}$. But they are quite different objects in the different categories. If $(\overset{B}{\leq}) : B^o \times B \to \mathbb{2}$ is another such matrix, then

- in Mat_0, a morphism in the form $\langle h, k\rangle$ is required to satisfy $x \overset{A}{\leq} x' \iff hx \overset{B}{\leq} kx'$ for all $x, x' \in A$, whereas
- in $\mathsf{Mat}_3^{\mathsf{Pos}}$, a morphism in the form $\langle f^*, f_*\rangle$ is required to satisfy $x \leq f_*y \iff f^*x \leq y$ for all $x \in A$ and $y \in B$.

The $\mathsf{Mat}_3^{\mathsf{Pos}}$ isomorphisms are thus the poset adjunctions (a.k.a. Galois connections), whereas the Mat_0-morphisms in the form $\langle h, h\rangle$ are the order isomorphisms.

4.1.2 Linear spaces

Let $\mathcal{S}$ be the category Set of sets, $\dot{\mathcal{S}}$ the category $\dot{\mathsf{Set}}$ of finite sets, and let R be the set of real numbers. Then the objects of $\mathsf{Mat}_3^{\mathsf{Lin}}$ are the real matrices, just like in Mat_1, but the morphisms in $\mathsf{Mat}_3^{\mathsf{Lin}}$ are a very special case of those in Mat_1. A Mat_1-morphism $\langle H, K\rangle$ from (21) boils down to a pair of functions $\langle f^*, f_*\rangle$ from (36) precisely when the matrices H and K comprise of 0s, except that H has precisely one 1 in every row, and K has precisely one 1 in every column. With such constrained morphisms, $\mathsf{Mat}_3^{\mathsf{Lin}}$ does not support the factorizations on which the constructions in Mat_1 were based. The completions will afford it more flexible morphisms. Mat_1's morphisms are already complete matrices, which is why we were able to take $\mathsf{Adj}_2 = \mathsf{Mat}_2 = \mathsf{Mat}_1$.

4.1.3 Categories

Let $\mathcal{S}$ be the category CAT of categories, small or large; let R be the category Set of sets; and let $\dot{\mathcal{S}}$ be the category Cat of small categories. The matrices in $\mathsf{Mat}_3^{\mathsf{CAT}}$ are then distributors [16, Vol. I, Sec. 7.8], also also called profunctors, or bimodules. The $\mathsf{Mat}_3^{\mathsf{CAT}}$-morphisms are generalized adjunctions, as discussed in [47]. Any small category $\dot{\mathbb{A}}$ occurs as the matrix $\hom_{\dot{\mathbb{A}}} \in \mathsf{CAT}(\dot{\mathbb{A}}^o \times \dot{\mathbb{A}}, \mathsf{Set})$ in $\mathsf{Mat}_3^{\mathsf{CAT}}$. The $\mathsf{Mat}_3^{\mathsf{CAT}}$-morphisms between the matrices in the form $\hom_{\dot{\mathbb{A}}}$ and $\hom_{\dot{\mathbb{B}}}$ are precisely the adjunctions between the categories $\dot{\mathbb{A}}$ and $\dot{\mathbb{B}}$.

4.2 Representability and completions

A matrix $\Phi : \dot{A} \times \dot{B} \to R$ is said to be *representable* when there are matrices $\mathbb{A} : \dot{A} \times \dot{A} \to R$ and $\mathbb{B} : \dot{B} \times \dot{B} \to R$ and a morphism $f = \langle f^*, f_*\rangle \in \mathsf{Mat}_3(\mathbb{A}, \mathbb{B})$ such that $\Phi = \mathbb{A} \circ (\dot{A} \times f_*) = \mathbb{B}(f^* \times \dot{B})$. Inside the category Mat_3, this means that the morphism f can be factorized through Φ, as displayed in Fig. 6. Inside $\mathsf{Mat}_3^{\mathsf{CAT}}$, a distributor $\Phi : \dot{\mathbb{A}}^o \times \mathbb{B} \to \mathsf{Set}$ is representable if and only if there is an adjunction $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$ such that $\mathbb{A}(x, F_*y) = \Phi(x, y) = \mathbb{B}(F^*x, y)$.

4.3 Abstract adjunctions

In the category of adjunctions Adj_3, all matrices from Mat_3 become representable. This is achieved by dropping the "finiteness" requirement $\dot{A}, \dot{B}, \dot{C}, \dot{D} \in \dot{\mathcal{S}}$ from Mat_3, and defining

$$\begin{aligned} |\mathsf{Adj}_3| &= \coprod_{A,B\in\mathcal{S}} \mathcal{S}(A \times B, R) \\ \mathsf{Adj}_3(\Phi, \Psi) &= \{\langle f^*, f_*\rangle \in \mathcal{S}(A, C) \times \mathcal{S}(D, B) \mid \Psi(f^*a, d) = \Phi(a, f_*d)\} \end{aligned} \tag{37}$$

$$
\begin{tikzcd}
& \dot{A}\times\dot{B} \arrow[dl, "\mathrm{id}\times f_*"'] \arrow[d, "\mathrm{id}\times\mathrm{id}"] \arrow[dr, "f^*\times\mathrm{id}"] & \\
\dot{A}\times\dot{A} \arrow[r, two heads, "\langle \mathrm{id}, f_*\rangle"] \arrow[dr, "\mathbb{A}"'] & \dot{A}\times\dot{B} \arrow[r, tail, "\langle f^*, \mathrm{id}\rangle"] \arrow[d, "\Phi"] & \dot{B}\times\dot{B} \arrow[dl, "\mathbb{B}"] \\
& R &
\end{tikzcd}
$$

Figure 6: A matrix Φ representable in Mat_3 by factoring $\langle f^*, f_*\rangle = \left(\mathbb{A} \xrightarrow{\langle \mathrm{id}, f_*\rangle} \Phi \xrightarrow{\langle f^*, \mathrm{id}\rangle} \mathbb{B}\right)$

4.3.1 The Chu-construction

The readers familiar with the Chu-construction will recognize Adj_3 as $\mathsf{Chu}(\mathcal{S}, R)$. The Chu-construction is a universal embedding of monoidal categories with a chosen dualizing object into $*$-autonomous categories. It was spelled out by Barr and his student Chu [9], and extensively studied in topological duality theory and in semantics of linear logic [10, 11, 12, 13, 21, 61, 70, 76]. Its conceptual roots go back to the early studies of infinite-dimensional vector spaces [61]. Our category Mat_3 can be viewed as a "finitary" part of a Chu-category, where an abstract notion of "finiteness" is imposed by requiring that the matrices are sized by a "finite" category $\dot{\mathcal{S}} \subset \mathcal{S}$.

4.3.2 Representing matrices as adjunctions

The functor $\mathsf{MA}_3 : \mathsf{Mat}_3 \to \mathsf{Adj}_3$ will be the obvious embedding. When $\dot{\mathcal{S}} = \mathcal{S}$, it boils down to the identity. The difference between (36) and (37) is technically, of course, a minor wrinkle. But when the object R is exponentiable, in the sense that there is a functor $R^{(-)} : \dot{\mathcal{S}}^o \to \mathcal{S}$ such that

$$
\mathcal{S}(\dot{A}\times\dot{B}, R) \;\cong\; \mathcal{S}(\dot{A}, R^{\dot{B}}) \tag{38}
$$

holds naturally in $\dot{A}$ and $\dot{B}$, then the Mat_3-matrices can be represented as Adj_3-morphisms. Each matrix appears in four avatars

$$
\begin{array}{ccccccc}
\mathcal{S}(\dot{A}, R^{\dot{B}}) & \cong & \mathcal{S}(\dot{A}\times\dot{B}, K) & \cong & \mathcal{S}(\dot{B}\times\dot{A}, K) & \cong & \mathcal{S}(\dot{B}, R^{\dot{A}}) \\
\Cup & & \Cup & & \Cup & & \Cup \\
\Phi^* & & \Phi & & \Phi^\ddagger & & \Phi_*
\end{array} \tag{39}
$$

and the leftmost and the rightmost represent it as the abstract adjunction in Fig. 7. The objects $R^{\dot{A}}$ and $R^{\dot{B}}$, that live in $\mathcal{S}$ but not in $\dot{\mathcal{S}}$ will play a similar role to $\Downarrow A$ and $\Uparrow B$ in Sec. 2, and to the eponymous Hilbert spaces Sec. 3. They are the abstract "completions". We come back to this in Sec. 4.5.

$$\begin{tikzcd}
 & \dot{A}\times\dot{B} \arrow[dl, "\dot{A}\times\Phi_*"'] \arrow[dr, "\Phi^*\times\dot{B}"] \arrow[dd, "\Phi"] & \\
\dot{A}\times R^{\dot{A}} \arrow[dr, "\in"'] & & R^{\dot{B}}\times\dot{B} \arrow[dl, "\ni"] \\
 & R &
\end{tikzcd}$$

Figure 7: The adjunction $(\Phi^* \dashv \Phi_*) \in \mathsf{Adj}_3(\in_{\dot{A}}, \ni_{\dot{B}})$ representing the matrix $\Phi : \dot{A} \times \dot{B} \to R$ from Mat_3

4.3.3 Separated and extensional adjunctions

The correspondences in (39) assert that any matrix $\Phi : A \times B \to R$ can be viewed as

- a map $A \xrightarrow{\Phi^*} R^B$, assigning a "matrix row" $\Phi^*(a)$ to each basis element $a \in A$;
- a map $B \xrightarrow{\Phi_*} R^A$, assigning a "matrix column" $\Phi_*(b)$ to each basis element $b \in B$.

The elements a and a' are indistinguishable for Φ if $\Phi^*(a) = \Phi^*(a')$; and the elements b and b' are distinguishable for Φ if $\Phi_*(b) = \Phi_*(b')$. The idea of Barr's *separated-extensional* Chu construction [10, 12] is to quotient out any indistinguishable elements. A Chu space is called

- *separated* if $\Phi^*(a) = \Phi^*(a') \implies a = a'$, and
- *extensional* if $\Phi_*(b) = \Phi_*(b') \implies b = b'$.

To formalize this idea, we assume the category $\mathcal{S}$ is given with a family $\mathcal{M}$ of abstract monics, so that Φ is separated if $\Phi^* \in \mathcal{M}$ and extensional if $\Phi_* \in \mathcal{M}$. To extract such an $\mathcal{M}$-separated-extensional nucleus from any given Φ, the family $\mathcal{M}$ is given as a part of a *factorization system* $\mathcal{E} \wr \mathcal{M}$, such that $R^{\mathcal{E}} \subseteq \mathcal{M}$. For convenience, an overview of factorization systems is given in Appendix A. The construction yields an instance of Fig. 3 for the full subcategories of Adj_3 defined by

$$\begin{aligned}
\mathsf{Mnd}_3 &= \{\Phi \in \mathsf{Adj}_3 \mid \Phi^* \in \mathcal{M}\} &&= \mathsf{Chu}_s(\mathcal{S}, R) && (40)\\
\mathsf{Cmn}_3 &= \{\Phi \in \mathsf{Adj}_3 \mid \Phi_* \in \mathcal{M}\} &&= \mathsf{Chu}_e(\mathcal{S}, R) && (41)\\
\mathsf{Nuc}_3 &= \{\Phi \in \mathsf{Adj}_3 \mid \Phi^*, \Phi_* \in \mathcal{M}\} &&= \mathsf{Chu}_{se}(\mathcal{S}, R) && (42)
\end{aligned}$$

where $\mathsf{Chu}_s(\mathcal{S}, R)$ and $\mathsf{Chu}_e(\mathcal{S}, R)$ are the full subcategories of $\mathsf{Chu}(\mathcal{S}, R)$ spanned, respectively, by the separated and the extensional Chu spaces, as constructed in [10, 12]. The reflections and coreflections, induced by the factorization, have been analyzed in detail there. The separated-extensional nucleus of a matrix is constructed through the factorizations displayed in Fig. 8, where we use Barr's notation. The functor AM_3 corresponds to Barr's Chu_s, the functor AC_3 to Chu_e. Proving that $A' \cong A$" and $B' \cong B$" gives the nucleus $\mathsf{Chu}_{se}(\Phi) = \mathsf{Chu}_{es}(\Phi)$ in Nuc_3.

$$A \times B \xrightarrow{\Phi} R$$

$$A \xrightarrow{\Phi^*} R^B \qquad B \xrightarrow{\Phi_*} R^A$$

$$A \overset{\mathcal{E}(\Phi^*)}{\twoheadrightarrow} A' \overset{\mathsf{Chu}_s(\Phi)}{\rightarrowtail} R^B \qquad B \overset{\mathcal{E}(\Phi_*)}{\twoheadrightarrow} B' \overset{\mathsf{Chu}_e(\Phi)}{\rightarrowtail} R^A$$

$$B \overset{\mathcal{E}(\mathsf{Chu}_s(\Phi))}{\twoheadrightarrow} B'' \overset{\mathsf{Chu}_{se}(\Phi)}{\rightarrowtail} R^{A'} \qquad A \overset{\mathcal{E}(\mathsf{Chu}_e(\Phi))}{\twoheadrightarrow} A'' \overset{\mathsf{Chu}_{es}(\Phi)}{\rightarrowtail} R^{B'}$$

Figure 8: Overview of the separated-extensional Chu construction

4.4 What does the separated-extensional nucleus capture in examples 4.1?

4.4.1 Posets

Restricted to the poset matrices in the form $A^o \times B \xrightarrow{\Phi} \mathbb{2}$, as explained in Sec. 4.1.1, the separated-extensional nucleus construction gives the same output as the concept lattice construction in Sec. 2. The factorizations Chu_s and Chu_e in Fig. 8 correspond to the extensions Φ^* and Φ_* in (11).

4.4.2 Linear spaces

Extended from finite bases to the entire spaces generated by them, the Chu view of the linear algebra example in 4.1.2 captures the rank space factorization and Nuc_1, but the spectral decomposition into Nuc_2 requires a suitable completeness assumption on R.

4.4.3 Categories

The separated-extensional nucleus construction does not seem applicable to the categorical example in 4.1.3 directly, as none of the familiar functor factorization systems satisfy the requirement $R^{\mathcal{E}} \subseteq \mathcal{M}$. This provides an opportunity to explore the role of factorizations in extracting the nuclei. In Sec. 4.5 we explore a variation on the theme of the factorization-based nucleus. In Sec. 4.6 we spell out a modified version of the separated-extensional nucleus construction that does apply to the categorical example in 4.1.3.

4.5 Discussion: Combining factorization-based approaches

Some factorization-based nuclei, in the situations when the requirement $R^{\mathcal{E}} \subseteq \mathcal{M}$ is not satisfied, arise from a combination of the separated-extensional construction from Sec. 4.3.1 and the diagonalization factoring from Sec. 3.

4.5.1 How nuclei depend on factorizations?

As explained in the Appendix, every factorization system $\mathcal{E} \wr \mathcal{M}$ in any category $\mathcal{S}$ can be viewed as an algebra for the Arr-monad, where $\mathsf{Arr}(\mathcal{S}) = \mathcal{S}/\mathcal{S}$ is the category consisting of the $\mathcal{S}$-arrows as objects, and the pairs of arrows forming commutative squares as the morphisms. An arbitrary factorization system $\mathcal{E} \wr \mathcal{M}$ on $\mathcal{S}$ thus corresponds to an algebra $\wr : \mathcal{S}/\mathcal{S} \to \mathcal{S}$; and a factorization system that satisfies the requirements for the separated-extensional Chu construction lifts to an algebra $\wr : \mathsf{Adj}_3/\mathsf{Adj}_3 \to \mathsf{Adj}_3$. To see this, note the natural bijection $\mathcal{S}(A \times B, R) \cong \mathcal{S}(A, R^B)$ induces an isomorphism of $\mathsf{Adj}_3 = \mathsf{Chu}(\mathcal{S}, R)$ with the comma category $\mathcal{S}R = \mathcal{S}/R^{(-)}$, whose arrows are in the form

$$\begin{tikzcd}
A \arrow[r, "f^*"] \arrow[d, "\varphi"'] & C \arrow[d, "\psi"] \\
R^B \arrow[r, "R^{f_*}"] & R^D \\
B & D \arrow[l, "f"]
\end{tikzcd} \tag{43}$$

Such squares permit $\mathcal{E} \wr \mathcal{M}$-factorization whenever $R^{\mathcal{E}} \subseteq \mathcal{M}$. If we now set

$$\mathsf{Mat}_4 = \mathsf{Adj}_3 \tag{44}$$

$$\mathsf{Adj}_4 = \mathsf{Adj}_3/\mathsf{Adj}_3 \tag{45}$$

then the isomorphism $\mathsf{Adj}_3 \cong \mathcal{S}R$ liefts to of $\mathsf{Adj}_4 \cong \mathcal{S}R/\mathcal{S}R$. The objects of Adj_4 can thus be viewed as the squares in the form (43), and the object part of the abstract completion functor $\mathsf{MA}_4 : \mathsf{Mat}_4 \to \mathsf{Adj}_4$ can be defined as in Fig. 9. One immediate consequence is that the two factorization

$$\mathsf{Mat}_4 \xrightarrow{\mathsf{MA}_4} \mathsf{Adj}_4$$

$$\begin{tikzcd}
& A \times D \arrow[dl, "A \times f_*"'] \arrow[dr, "f^* \times D"] & \\
A \times B \arrow[r, phantom, "\xrightarrow{f}"] \arrow[dr, "\Phi"'] & & C \times D \arrow[dl, "\Psi"] \\
& R &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
R^{R^A} \arrow[rrr, "R^{R^{f^*}}"] \arrow[ddd, "R^{\Phi_*}"'] & & & R^{R^C} \arrow[ddd, "R^{\Psi_*}"] \\
& A \arrow[ul, "\eta"'] \arrow[r, "f^*"] \arrow[d, "\Phi^*"] \arrow[l, phantom, "\mathsf{MA}_4(\Phi)"] & C \arrow[ur, "\eta"] \arrow[d, "\Psi^*"'] \arrow[r, phantom, "\mathsf{MA}_4(\Psi)"] & \\
& R^B \arrow[r, "R^{f_*}"'] \arrow[dl, "\mathrm{id}"] & R^D \arrow[dr, "\mathrm{id}"'] & \\
R^B \arrow[rrr, "R^{f_*}"'] & & & R^D
\end{tikzcd}$$

$$\xrightarrow{\mathsf{MA}_4 f}$$

Figure 9: The abstract completion functor $\mathsf{MA}_4 : \mathsf{Mat}_4 \to \mathsf{Adj}_4$

steps of the two-step separated-and-extensional construction $\overleftarrow{\mathfrak{N}}_3 = \mathsf{Chu}_{se}$, summarized in Fig. 8, can now be obtained in a single sweep, by directly composing the completion with the factorization

$$\overleftarrow{\mathfrak{N}}_3 \quad = \quad \left(\mathsf{Adj}_3 \xrightarrow{\mathsf{MA}_4} \mathsf{Adj}_3 \diagup \mathsf{Adj}_3 \xrightarrow{\wr} \mathsf{Adj}_3\right) \tag{46}$$

The fixed points of this functor are just the separated-extensional nuclei. This is, of course, just another presentation of the same thing; and perhaps a wrongheaded one, as it folds the two steps of the nucleus construction into one. These two steps are displayed as the two paths from left to right through Fig. 3, corresponding to the two orders in which the steps can be taken; and of course as the separate part and the extensional part of the separate-extensional Chu-construction. The commutativity of the two steps is, in a sense, the heart of the matter. However, packaging a nucleus construction into one step allows packaging two such constructions into one. What might that be useful for?

When $\mathcal{S}$ is, say, a category of topological spaces, and $\mathcal{E} \wr \mathcal{M}$ the the dense-closed factorization, then it may happen that the separated-extensional nucleus of a space is much bigget than the original space. If the nucleus $\overleftarrow{\mathfrak{N}}_3\Phi : A' \times B' \to R$ of a matrix $\Phi : A \times B \to R$ is constructed by factoring $A \xrightarrow{\Phi^*} R^B$ and $B \xrightarrow{\Phi_*} R^A$ into

$$A \twoheadrightarrow A' \xrightarrow{\overleftarrow{\mathfrak{N}}_3\Phi^*} R^{B'} \rightarrowtail R^B \qquad\qquad B \twoheadrightarrow B' \xrightarrow{\overleftarrow{\mathfrak{N}}_3\Phi_*} R^{A'} \rightarrowtail R^A$$

as in Fig. 8, then A and B can be dense spaces of rational numbers, and A' and B' can be their closures in the space of real numbers, representable within both R^A and R^B for a cogenerator R. The same effect occurs if we take $\mathcal{S}$ to be posets, and in many other situations where the $\mathcal{E}$-maps are not quotients. One way to sidestep the problem might be to strengthen the requirements.

4.5.2 Exercise

Given a matrix $A \times B \xrightarrow{\Phi} R$, find a nucleus $A' \times B' \xrightarrow{L\Phi} R$ such that

(a) $A \twoheadrightarrow A'$ and $B \twoheadrightarrow B'$ are quotients, whereas

(b) $A' \overset{\Phi^*}{\rightarrowtail} R^{B'}$ and $B' \overset{\Phi^*}{\rightarrowtail} R^{A'}$ are closed embeddings.

Requirement (b) is from the separated-extensional construction in Sec. 4.3.1, whereas requirement (a) is from the diagonalization factoring in Sec. 3).

4.5.3 Workout

Suppose that category $\mathcal{S}$ supports two factorization systems:

- $\mathcal{E} \wr \mathcal{M}^\bullet$, where $\mathcal{M}^\bullet \subseteq \mathcal{M}$ are the regular monics (embeddings, equalizers), and
- $\mathcal{E}^\bullet \wr \mathcal{M}$, where $\mathcal{E}^\bullet \subseteq \mathcal{E}$ are the regular epis (quotients, coequalizers).

In balanced categories, these factorizations would coincide, because $\mathcal{M}^\bullet = \mathcal{M}$ and $\mathcal{E}^\bullet = \mathcal{E}$, and we would be back to the situation where the separated-extensional construction applies. In general, the two factorizations can be quite different, like in the category of topological spaces. Nevertheless, since homming into the exponentiable object R is a contravariant right adjoint functor, it maps coequalizers to equalizers. Assuming that R is an injective cogenerator, it also maps general epis to monics, and vice versa. So we have

$$R^{\mathcal{E}^\bullet} \subseteq \mathcal{M}^\bullet \qquad\qquad R^{\mathcal{E}} \subseteq \mathcal{M} \qquad\qquad R^{\mathcal{M}} \subseteq \mathcal{E} \tag{47}$$

However, $\mathcal{E}^\bullet$ and $\mathcal{M}^\bullet$ generally do not form a factorization system, because there are maps that do not have a quotient-embedding decomposition; and $\mathcal{E}$ and $\mathcal{M}$ do not form a factorization system because there are maps whose epi-mono decomposition is not unique. The factorization $\mathcal{E}^\bullet \wr \mathcal{E}$ does satisfy $R^{\mathcal{E}^\bullet} \subseteq \mathcal{M}$, but does not lift from $\mathcal{S}/\mathcal{S} \to \mathcal{S}$ to $\mathsf{Chu}/\mathsf{Chu} \to \mathsf{Chu}$.

Our next nucleus setting will be full subcategories again:

$$\begin{aligned} \mathsf{Mnd}_4 &= \{\langle f^*, f_*\rangle \in \mathsf{Adj}_4 \mid f^* \in \mathcal{M}, f_* \in \mathcal{E}\} && (48)\\ \mathsf{Cmn}_4 &= \{\langle f^*, f_*\rangle \in \mathsf{Adj}_4 \mid f^* \in \mathcal{E}, f_* \in \mathcal{M}\} && (49)\end{aligned}$$

These two categories are dual, just like Mnd_1 and Cmn_1 were dual. In both cases, they are in fact the same category, since switching between Φ and $\Phi^\ddagger$ in (23-24) and between f^* and f_* in (48-49) is a matter of notation. But distinguishing the two copies of the category on the two ends of the duality makes it easier to define one as a reflexive and the other one as a coreflexive subcategory of the category of adjunctions.

The functors $\mathsf{EM}_4 : \mathsf{Mnd}_4 \hookrightarrow \mathsf{Adj}_4$ and $\mathsf{KC}_4 : \mathsf{Cmn}_4 \hookrightarrow \mathsf{Adj}_4$ are again the obvious inclusions. The reflection $\mathsf{AM}_4 : \mathsf{Adj}_4 \twoheadrightarrow \mathsf{Mnd}_4$ and the coreflection $\mathsf{AC}_4 : \mathsf{Adj}_4 \twoheadrightarrow \mathsf{Cmn}_4$ are constructed in Fig. 10. The factoring triangles on are related in a similar way to the two factoring triangles in (27). The nucleus is obtained by composing them, in either order. More precisely, the coreflection $\mathsf{NM}_4 : \mathsf{Mnd}_4 \twoheadrightarrow \mathsf{Nuc}_4$ is obtained by restricting the coreflection $\mathsf{AC}_4 : \mathsf{Adj}_4 \twoheadrightarrow \mathsf{Cmn}_4$ along the inclusion $\mathsf{EM}_4 : \mathsf{Mnd}_4 \hookrightarrow \mathsf{Adj}_4$; the reflection $\mathsf{NC}_4 : \mathsf{Cmn}_4 \twoheadrightarrow \mathsf{Nuc}_4$ is obtained by restricting $\mathsf{AM}_4 : \mathsf{Adj}_4 \twoheadrightarrow \mathsf{Mnd}_4$ along the inclusion $\mathsf{KC}_4 : \mathsf{Mnd}_4 \hookrightarrow \mathsf{Adj}_4$. The outcome is in Fig. 11. The category of nuclear Chu spaces is thus the full subcategory spanned by

$$\mathsf{Nuc}_4 = \{\langle f^*, f_*\rangle \in \mathsf{Adj}_4 \mid f^*, f_* \in \mathcal{E} \cap \mathcal{M}\} \tag{50}$$

If a factorization does not support the separated-extensional Chu-construction because it is not stable under dualizing, but if it is dual with another factorization, like e.g. the isometric-diagonal factorization in the category if finite-dimensional Hilbert spaces in Sec. 3, then the nucleus can still be constructed, albeit not as a subcategory of the original category, but of its arrow category. While the original separated-extensional Chu-construction yields a full subcategory $\mathsf{Chu}_{se} \subseteq \mathsf{Chu}$, here we get the Chu-nucleus as a full subcategory $\overleftarrow{\mathfrak{N}}_4 \subseteq \mathsf{Chu}/\mathsf{Chu}$. A Chu-nucleus is thus an arrow $\langle \mathcal{EM}(\Phi^*), \mathcal{EM}(\Phi_*)\rangle \in \mathsf{Chu}(\Phi', \Phi'')$, as seen in Fig. 11, such that

(a) $A \twoheadrightarrow A'$ and $B \twoheadrightarrow B''$ are in $\mathcal{E}^\bullet$,

(b) $B' \overset{\widetilde{\Phi}'}{\hookrightarrow} R^{A'}$ and $A'' \overset{\Phi''}{\hookrightarrow} R^{B''}$ are in $\mathcal{M}^\bullet$,

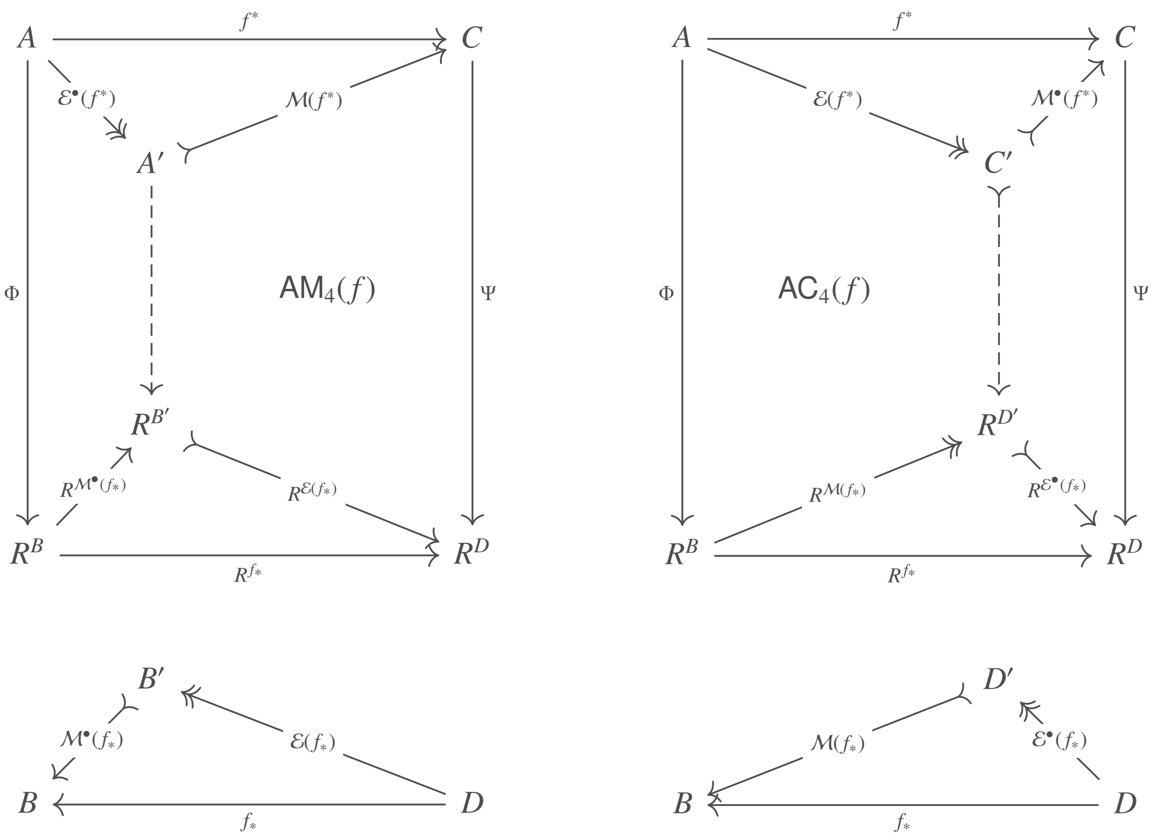

Figure 10: The object parts of the functors $\mathsf{AM}_4 : \mathsf{Adj}_4 \twoheadrightarrow \mathsf{Mnd}_4$ and $\mathsf{AC}_4 : \mathsf{Adj}_4 \twoheadrightarrow \mathsf{Cmn}_4$

(c) $A' \overset{\Phi'}{\rightarrowtail} R^{B'}$ and $B'' \overset{\widetilde{\Phi}''}{\rightarrowtail} R^{A''}$ are in $\mathcal{M}$,

(d) $\mathcal{EM}(\Phi^*)$ and $\mathcal{EM}(\Phi_*)$ are in $\mathcal{E} \cap \mathcal{M}$.

where $B' \xrightarrow{\widetilde{\Phi}'} R^{A'}$ is the transpose of $A' \xrightarrow{\Phi'} R^{B'}$, and $B'' \xrightarrow{\widetilde{\Phi}''} R^{A''}$ is the transpose of $A'' \xrightarrow{\Phi''} R^{B''}$. According to (d), Chu spaces $\mathcal{EM}(\Phi^*)$ and $\mathcal{EM}(\Phi_*)$ are thus monics in one factorization system and epis in another one, like the diagonalizations were in diagram (28) in Sec. 3. According to (a) and (b), $\mathcal{EM}(\Phi^*)$ and $\mathcal{EM}(\Phi_*)$ are moreover the best such approximations of Φ^* and Φ_*, as their largest quotients and embeddings, like the diagonalizations were, according to (27) and (35). The difference between the current situation and the one in one in Sec. 3, is that the diagonal nucleus there was self-dual, whereas $\mathcal{EM}(\Phi^*)$ and $\mathcal{EM}(\Phi_*)$ are not, but they are rather dual to one another. It also transposes Φ' and Φ'', and the transposition does not preserve regularity, but in this case it switches the $\mathcal{M}^\bullet$-map with the $\mathcal{M}$-map. Intuitively, the nucleus $\overleftarrow{\mathfrak{N}}_4\Phi$ can thus be thought as the best approximation of a diagonalization, in situations when the spectra of the two self-adjoints induced by a matrix are not the same; or the best approximation of a separated-extensional core when Chu_{se} and Chu_{es} do not coincide.

$$
\begin{tikzcd}
A \arrow[rrr, "\Phi^*"] \arrow[dr, two heads, "\mathcal{E}^\bullet(\Phi^*)"'] \arrow[ddd, "\eta"'] & & & R^B \arrow[ddd, "\mathrm{id}"] \\
& A' \arrow[r, tail, two heads, "\mathcal{E}\mathcal{M}(\Phi^*)"] \arrow[d, tail, dashed, "\Phi'"'] \arrow[dr, phantom, "\overleftarrow{\mathfrak{N}}_4\Phi"] & A'' \arrow[ur, tail, "\mathcal{M}^\bullet(\Phi^*)"'] \arrow[d, tail, dashed, "\Phi''"] & \\
& R^{B'} \arrow[r, tail, two heads, "R^{\mathcal{E}\mathcal{M}(\Phi_*)}"] & R^{B''} \arrow[dr, tail, two heads, "R^{\mathcal{E}(\Phi_*)}"] & \\
R^{R^A} \arrow[ur, "R^{\mathcal{M}(\Phi_*)}"] \arrow[rrr, "R^{\Phi_*}"'] & & & R^B
\end{tikzcd}
$$

$$
\begin{tikzcd}
& B' \arrow[dl, tail, "\mathcal{M}^\bullet(\Phi_*)"'] & B'' \arrow[l, tail, two heads, "\mathcal{E}\mathcal{M}(\Phi_*)"'] & \\
R^A & & & B \arrow[ul, two heads, "\mathcal{E}^\bullet(\Phi_*)"'] \arrow[lll, "\Phi_*"]
\end{tikzcd}
$$

Figure 11: The Chu-nucleus of the matrix $\Phi : A \times B \to R$

4.6 Towards the categorical nucleus

Although the categorical example 4.1.3 does not yield to the separated-extensional nucleus construction, a suitable modification of the example suggests the suitable modification of the construction.

Consider a distributor $\Phi : \mathbb{A}^o \times \mathbb{B} \to \mathsf{Set}$, representable in the form $\mathbb{A}(x, F_* y) = \Phi(x, y) = \mathbb{B}(F^* x, y)$ for some adjunction $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$. The factorization of representable matrices displayed in Fig. 6 induces in Adj_3 the diagrams in Fig. 12. Here the representation $\mathbb{A}(x, F_* y) = \Phi(x, y) = \mathbb{B}(F^* x, y)$ induces

$$
\begin{aligned}
\Phi_* : \mathbb{B} &\to \mathsf{Set}^{\mathbb{A}^o} \\
b &\mapsto \lambda x.\, \mathbb{A}(x, F_* b)
\end{aligned}
\qquad\qquad
\begin{aligned}
\Phi^* : \mathbb{A}^o &\to \mathsf{Set}^{\mathbb{B}} \\
a &\mapsto \lambda y.\, \mathbb{B}(F^* a, y)
\end{aligned}
$$

i.e. $\Phi_* = (\mathbb{B} \xrightarrow{F_*} \mathbb{A} \xrightarrow{\blacktriangledown} \mathsf{Set}^{\mathbb{A}^o})$ and $\Phi^* = (\mathbb{A} \xrightarrow{F^*} \mathbb{B} \xrightarrow{\blacktriangle} \left(\mathsf{Set}^{\mathbb{B}}\right)^o)$. So the Chu view of a distributor Φ representable by an adjunction $F^* \dashv F_*$ is based on the Kan extensions of the adjunction. The point of this packaging is that the separated-extensional nucleus of the distributor Φ for the factorization system $(\mathsf{Ess} \wr \mathsf{Ffa})$ in CAT where[6]

[6]This basic factorization takes scene in the final moments of the paper, in Sec. 11.4.

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, equal] \arrow[dd, "\blacktriangledown"] & \mathbb{A} \arrow[rr, "F^*"] \arrow[dd, "(\Phi^*)^o"] \arrow[dr, two heads] & & \mathbb{B} \arrow[rr, equal] \arrow[dd, "\blacktriangledown"] & & \mathbb{B} \arrow[r, "F_*"] \arrow[dd, "\Phi_*"] \arrow[dl, two heads] & \mathbb{A} \arrow[dd, "\blacktriangle"] \\
& & \mathbb{A}_{\overleftarrow{F}} \arrow[dl, tail] \arrow[rr, dashed, bend left, "F_\flat"] & & \mathbb{B}_{\overrightarrow{F}} \arrow[dr, tail] & & \\
\mathsf{Set}^{\mathbb{A}^o} \arrow[r, "\mathrm{Lan}_{\blacktriangledown}(\Phi^*)^o"'] & \left(\mathsf{Set}^{\mathbb{B}}\right)^o \arrow[rr, equal] & & \left(\mathsf{Set}^{\mathbb{B}}\right)^o \arrow[rr, "\mathrm{Ran}_{\blacktriangledown}\Phi_*"'] & & \mathsf{Set}^{\mathbb{A}^o} \arrow[r, equal] & \mathsf{Set}^{\mathbb{A}^o}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{B} \arrow[dd, "\blacktriangle"] & \mathbb{A} \arrow[l, "F^*"'] \arrow[rr, equal] \arrow[dd, "(\Phi^*)^o"] \arrow[dr, two heads] & & \mathbb{A} \arrow[dd, "\blacktriangledown"] & & \mathbb{B} \arrow[ll, "F_*"'] \arrow[r, equal] \arrow[dd, "\Phi_*"] \arrow[dl, two heads] & \mathbb{B} \arrow[dd, "\blacktriangle"] \\
& & \mathbb{A}_{\overleftarrow{F}} \arrow[dl, tail] & & \mathbb{B}_{\overrightarrow{F}} \arrow[dr, tail] \arrow[ll, dashed, bend left, "F^\flat"] & & \\
\left(\mathsf{Set}^{\mathbb{B}}\right)^o \arrow[r, equal] & \left(\mathsf{Set}^{\mathbb{B}}\right)^o & & \mathsf{Set}^{\mathbb{A}^o} \arrow[ll, "\mathrm{Lan}_{\blacktriangledown}(\Phi^*)^o"] \arrow[rr, equal] & & \mathsf{Set}^{\mathbb{A}^o} & \left(\mathsf{Set}^{\mathbb{B}}\right)^o \arrow[l, "\mathrm{Ran}_{\blacktriangledown}\Phi_*"]
\end{tikzcd}
$$

Figure 12: Separated-extensional nucleus + Kan extensions = Kleisli resolutions

- $\mathcal{E} = \mathsf{Ess}$ = essentially surjective functors,
- $\mathcal{M} = \mathsf{Ffa}$ = full-and-faithful functors

gives rise to the Kleisli categories $\mathbb{A}_{\overleftarrow{F}}$ and $\mathbb{B}_{\overrightarrow{F}}$ for the monad $\overleftarrow{F} = F_*F^*$ and the comonad $\overrightarrow{F} = F^*F_*$, since

$$
\begin{aligned}
\left|\mathbb{A}_{\overleftarrow{F}}\right| &= |\mathbb{A}| & \qquad \left|\mathbb{A}_{\overleftarrow{F}}\right| &= |\mathbb{B}| \\
\mathbb{A}_{\overleftarrow{F}}(x, x') &= \mathbb{B}(F^*x, F^*x') & \qquad \mathbb{B}_{\overrightarrow{F}}(y, y') &= \mathbb{A}(F_*y, F_*y')
\end{aligned}
\tag{51}
$$

It is easy to see that this is equivalent to the usual Kleisli definitions, since $\mathbb{B}(F^*x, F^*x') \cong \mathbb{A}(x, F_*F^*x')$ and $\mathbb{A}(F_*y, F_*y') \cong \mathbb{B}(F^*F_*y, y')$. The functors $F_\flat$ and $F^\flat$ induced in Fig. 12 by the factorization form the adjunction displayed in Fig. 13, because

$$
\mathbb{A}_{\overleftarrow{F}}(F_*y, x) \;=\; \mathbb{B}(F^*F_*y, F^*x) \;\cong\; \mathbb{A}(F_*y, F_*F^*x) \;=\; \mathbb{B}_{\overrightarrow{F}}(y, F^*x)
$$

While this construction is universal, it is not idempotent, as the adjunctions between the categories of free algebras over cofree coalgebras and of cofree coalgebras over free algebras often form transfinite embedding chains. The idempotent nucleus construction is just a step further. Remarkably, categorical localizations turn out to arise beyond factorizations.

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathbb{A} \arrow[loop above, "\overleftarrow{F}"] \arrow[d, bend right, "F^*"'] \arrow[d, phantom, "\dashv"] \arrow[dr, two heads, "\mathcal{E}(F^*)" near end] & \mathbb{B}_{\overrightarrow{F}} \arrow[l, tail, "\mathcal{M}(F_*)"'] \arrow[d, bend right, "F^\flat"'] \arrow[d, phantom, "\dashv"] \\
\mathbb{B} \arrow[loop below, "\overrightarrow{F}"] \arrow[u, bend right, "F_*"'] \arrow[ur, "\mathcal{E}(F_*)" near end] & \mathbb{A}_{\overleftarrow{F}} \arrow[l, tail, "\mathcal{M}(F^*)"] \arrow[u, bend right, "F_\flat"']
\end{tikzcd}
$$

Figure 13: A nucleus $F^\flat \dashv F_\flat$ spanned by the initial resolutions of the adjunction $F^* \dashv F_*$

5 Example ∞: Nuclear adjunctions, monads, comonads

5.1 The categories

The general case of Fig. 3 involves the following categories:

- matrices between categories, or distributors (also called profunctors, or bimodules):

$$
\begin{aligned}
|\mathsf{Mat}| &= \coprod_{\mathbb{A},\mathbb{B}\in\mathsf{CAT}} \mathsf{CAT}(\mathbb{A}^o \times \mathbb{B}, \mathsf{Set}) \qquad (52)\\
\mathsf{Mat}(\Phi, \Psi) &= \{\langle H, K\rangle \in \mathsf{CAT}(\mathbb{A}, \mathbb{C}) \times \mathsf{CAT}(\mathbb{B}, \mathbb{D}) \mid \Phi(a, b) \cong \Psi(Ha, Kb)\}
\end{aligned}
$$

- adjoint functors:

$$
\begin{aligned}
|\mathsf{Adj}| &= \coprod_{\mathbb{A},\mathbb{B}\in\mathsf{CAT}} \coprod_{\substack{F^*\in\mathsf{CAT}(\mathbb{A},\mathbb{B})\\ F_*\in\mathsf{CAT}(\mathbb{B},\mathbb{A})}} \{\langle \eta, \varepsilon\rangle \in \mathrm{Nat}(\mathrm{id}, F_*F^*) \times \mathrm{Nat}(F^*F_*, \mathrm{id}) \mid \qquad (53)\\
&\qquad\qquad \varepsilon F^* \circ F^*\eta = F^* \;\wedge\; F_*\varepsilon \circ \eta F_* = F_*\}\\
\mathsf{Adj}(F, G) &= \{\langle H, K\rangle \in \mathsf{CAT}(\mathbb{A}, \mathbb{C}) \times \mathsf{CAT}(\mathbb{B}, \mathbb{D}) \mid KF^* \overset{\upsilon^*}{\cong} G^*H \;\wedge\; HF_* \overset{\upsilon_*}{\cong} G_*K \;\wedge\\
&\qquad\qquad H\eta^F \overset{\upsilon_*\upsilon^*}{\cong} \eta^G H \;\wedge\; K\varepsilon^F \overset{\upsilon^*\upsilon_*}{\cong} \varepsilon^G K\}
\end{aligned}
$$

- monads (also called triples):

$$|\mathsf{Mnd}| = \coprod_{\mathbb{A}\in\mathsf{CAT}} \coprod_{\overleftarrow{T}\in\mathsf{CAT}(\mathbb{A},\mathbb{A})} \{\langle\eta,\mu\rangle \in \mathrm{Nat}(\mathrm{id},\overleftarrow{T}) \times \mathrm{Nat}(\overleftarrow{T}\overleftarrow{T},\overleftarrow{T}) \mid \mu\circ\overleftarrow{T}\mu = \mu\circ\mu\overleftarrow{T} \wedge \mu\circ\overleftarrow{T}\eta = \overleftarrow{T} = \mu\circ\eta\overleftarrow{T}\} \tag{54}$$

$$\mathsf{Mnd}\left(\overleftarrow{T},\overleftarrow{S}\right) = \{H\in\mathsf{CAT}(\mathbb{A},\mathbb{C}) \mid H\overleftarrow{T} \overset{\chi}{\cong} \overleftarrow{S}H \ \wedge \ H\eta^{\overleftarrow{T}} \overset{\chi}{\cong} \eta^{\overleftarrow{S}}H \ \wedge \ H\mu^{\overleftarrow{T}} \overset{\chi}{\cong} \mu^{\overleftarrow{S}}H\}$$

- comonads (or cotriples):

$$|\mathsf{Cmn}| = \coprod_{\mathbb{B}\in\mathsf{CAT}} \coprod_{\overrightarrow{T}\in\mathsf{CAT}(\mathbb{B},\mathbb{B})} \{\langle\varepsilon,\nu\rangle \in \mathrm{Nat}(\overrightarrow{T},\mathrm{id}) \times \mathrm{Nat}(\overrightarrow{T},\overrightarrow{T}\overrightarrow{T}) \mid \overrightarrow{T}\nu\circ\nu = \nu\overrightarrow{T}\circ\nu \wedge \overrightarrow{T}\varepsilon\circ\nu = \overrightarrow{T} = \varepsilon\overrightarrow{T}\circ\nu\} \tag{55}$$

$$\mathsf{Cmn}\left(\overrightarrow{S},\overrightarrow{T}\right) = \{K\in\mathsf{CAT}(\mathbb{B},\mathbb{D}) \mid K\overrightarrow{S} \overset{\kappa}{\cong} \overrightarrow{T}K \ \wedge \ K\varepsilon^{\overrightarrow{S}} \overset{\kappa}{\cong} \varepsilon^{\overrightarrow{T}}K \ \wedge \ K\nu^{\overrightarrow{S}} \overset{\kappa}{\cong} \nu^{\overrightarrow{T}}K\}$$

- The category Nuc can be equivalently viewed as a full subcategory of Adj, Mnd or Cmn, and the three versions will be discussed later.

Remark. The above definitions follow the pattern from the preceding sections. The difference is that the morphisms, which are still structure-preserving pairs, this time of functors, now satisfy the preservation requirements up to isomorphism. In each case, there may be many different isomorphisms witnessing the structure preservation. We leave them out of picture, under the pretext that they are preserved under the compositions. This simplification does not change the nucleus construction itself, but it does project away information about the morphisms. Moreover, the construction also applies to a richer family of morphisms, with non-trivial 2-cells. The chosen presentation framework thus incurs a loss of information and generality. We believe that this is the unavoidable price of not losing the sight of the forest for the trees, at least in this presentation. Some aspects of the more general framework of the results are sketched in Appendix B. We leave further explanations for the final section of the paper.

5.2 Assumption: Idempotents can be split.

An endomorphism $\varphi : X \to X$ is *idempotent* if it satisfies $\varphi\circ\varphi = \varphi$. A *retraction* is a pair of morphisms $e : X \to R$ and $m : R \to X$ such that $e\circ m = \mathrm{id}_R$. We often write retractions in the form $R \underset{e}{\overset{m}{\rightleftarrows}} X$, or $m : R \leftrightarrows X : e$. Note that $\varphi = m\circ e$ is an idempotent. Given an idempotent φ, any retraction with $\varphi = m\circ e$ is called the *splitting* of φ. It is easy to see that the component $m : R \rightarrowtail X$ of a retraction is an equalizer of φ and the identity on X; and that $e : X \twoheadrightarrow R$ is a coequalizer of φ

and the identity. It follows that all splittings of an idempotent are isomorphic. An idempotent on X is thus resolved by a splitting into a projection and an injection of an object R, which is called its *retract*. When φ is a function on sets, then its idempotency means that φ picks in X a representative of each equivalence class modulo the equivalence relation $(x \sim y) \iff (\varphi(x) = \varphi(y))$, and thus represents the quotient $X/\sim$ as a subset $R \subseteq X$.

The assumption that all idempotents split is the weakest categorical *completeness* requirement. A categorical limit or colimit is said to be *absolute* if it is preserved by all functors. Since all functors preserve equations, they map idempotents to idempotents, and preserve their splittings. Since a splitting of an idempotent consist of its equalizer and a coequalizer with the identity, the idempotent splittings are absolute limits and colimits. It was proved in [66] that all absolute limits and colimits must be in this form. *The concepts of absolute limit, absolute colimit, and retraction coincide.*

The absolute completion $\mathbb{A}$ of a given category $\mathbb{A}$ consists of the idempotents in $\mathbb{A}$ as the objects. A morphism $f \in \mathbb{A}(\varphi, \psi)$ between the idempotents $\varphi : X \to X$ and $\psi : Y \to Y$ in $\mathbb{A}$ is an arrow $f \in \mathbb{A}(X, Y)$ such that $\psi \circ f \circ \varphi = f$, or equivalently $\psi \circ f = f = f \circ \varphi$. A morphism from φ to ψ thus coequalizes φ with the identity, and equalizes ψ with the identity. If φ and ψ split in $\mathbb{A}$ into retracts R and S, then the set $\mathbb{A}(\varphi, \psi)$ is in a bijective correspondence with $\mathbb{A}(R, S)$. It follows that $\mathbb{A}$ embeds into $\mathbb{A}$ fully and faithfully, and that they are equivalent if and only if $\mathbb{A}$ is absolutely complete. While the assumption that the idempotents split can presently be taken as a matter of convenience, we argue in Sec. 11.4, at the very end of the paper, that the absolute completeness is not a side condition, but a central feature of categories observed through the lense of adjunctions.

The assumption that the idempotents *can* split does not mean that they *must* split. Like any assumption, the above assumption should not be taken as a constraint. Applying it blindly would eliminate, e.g., the categories of free algebras and coalgebras from consideration, since they are not absolutely complete. This could be repaired by completing them, which would leave us with the category of projective algebras on one hand, and the category of injective coalgebras on the other hand[7]. This is, however, not only unnecessary, but also undesirable. Assuming that an equation has a solution does not mean that it can only be viewed in the solved form. Assuming that the idempotents split makes their retractions available, not mandatory. This expands our toolkit, but it should not be misunderstood to narrow our perspective by banishing any subjects of interest.

5.3 Tools

5.3.1 Extending matrices to adjunctions

Any matrix $\Phi\colon \mathbb{A}^o \times \mathbb{B} \to \mathsf{Set}$ from small categories $\mathbb{A}$ and $\mathbb{B}$ can be extended along the Yonda embeddings $\mathbb{A} \overset{\blacktriangledown}{\to} \mathsf{Set}^{\mathbb{A}^o}$ and $\mathbb{B} \overset{\blacktriangle}{\to} \left(\mathsf{Set}^{\mathbb{B}}\right)^o$ into an adjunction $\Phi^* \dashv \Phi_* : \left(\mathsf{Set}^{\mathbb{B}}\right)^o \to \mathsf{Set}^{\mathbb{A}^o}$ as

[7] An algebra is projective if it is a retract of a free algebra. Dually, a coalgebra is injective if it is a retract of a cofree coalgebra [75, Sec. II].

follows:

$$
\begin{array}{cc}
\multicolumn{2}{c}{\Phi\colon \mathbb{A}^o \times \mathbb{B} \to \mathsf{Set}} \\
\Phi_\bullet\colon \mathbb{A}^o \to \mathsf{Set}^{\mathbb{B}} & {}_\bullet\Phi\colon \mathbb{B} \to \mathsf{Set}^{\mathbb{A}^o} \\
\Phi^*\colon \mathsf{Set}^{\mathbb{A}^o} \to \left(\mathsf{Set}^{\mathbb{B}}\right)^o & \Phi_*\colon \left(\mathsf{Set}^{\mathbb{B}}\right)^o \to \mathsf{Set}^{\mathbb{A}^o}
\end{array}
\tag{56}
$$

The second step brings us to Kan extensions. In the current context, the path to extensions leads through comprehensions.

5.3.2 Comprehending presheaves as discrete fibrations

Following the step from (4) to (52), the comprehension correspondence (5) now lifts to

$$
\begin{array}{ccc}
\mathsf{Cat}(\mathbb{A}^o \times \mathbb{B}, \mathsf{Set}) & \underset{\Xi}{\overset{\widehat{(-)}}{\cong}} & \mathsf{Dfib} \diagup A \times B^o \\
\left(\mathbb{A}^o \times \mathbb{B} \xrightarrow{\Phi} \mathsf{Set}\right) & \mapsto & \left(\int \Phi \xrightarrow{\widehat{\Phi}} \mathbb{A} \times \mathbb{B}^o\right) \\
\left(\mathbb{A}^o \times \mathbb{B} \xrightarrow{\Xi_E} \mathsf{Set}\right) & \leftarrow\!\shortmid & \left(\mathbb{E} \xrightarrow{E} \mathbb{A} \times \mathbb{B}^o\right)
\end{array}
\tag{57}
$$

Transposing the arrow part of Φ, which maps every pair $f \in \mathbb{A}(a, a')$ and $g \in \mathbb{B}(b', b)$ into $\Phi(a', b') \xrightarrow{\Phi_{fg}} \Phi(a, b)$, the closure property expressed by the implication in (6) becomes the mapping

$$
\mathbb{A}(a, a') \times \Phi(a', b') \times \mathbb{B}(b', b) \quad \to \quad \Phi(a, b) \tag{58}
$$

The *lower-upper* closure property expressed by (6) is now captured as the structure of the total category $\int \Phi$, defined as follows:

$$
\begin{aligned}
\left|\int \Phi\right| &= \coprod_{\substack{a \in \mathbb{A} \\ b \in \mathbb{B}}} \Phi(a, b) \\
\int \Phi\left(x_{ab}, x'_{a'b'}\right) &= \left\{\langle f, g\rangle \in \mathbb{A}(a, a') \times \mathbb{B}(b', b) \mid x = \Phi_{fg}(x')\right\}
\end{aligned}
\tag{59}
$$

It is easy to see that the obvious projection

$$
\begin{array}{ccc}
\int \Phi & \xrightarrow{\widehat{\Phi}} & \mathbb{A} \times \mathbb{B}^o \\
x_{ab} & \mapsto & \langle a, b\rangle
\end{array}
\tag{60}
$$

is a discrete fibration, i.e., an object of $\mathsf{Dfib} \diagup \mathbb{A} \times \mathbb{B}^o$. In general, a functor $\mathbb{F} \xrightarrow{F} \mathbb{C}$ is a discrete fibration over $\mathbb{C}$ when for all $x \in \mathbb{F}$ the obvious induced functors $\mathbb{F}/x \xrightarrow{F_x} \mathbb{C}/Fx$ are isomorphisms. In other words, for every $x \in \mathbb{F}$ and every morphism $c \xrightarrow{t} Fx$ in $\mathbb{C}$, there is a unique lifting $t^!x \xrightarrow{\vartheta^t} x$

of t to $\mathbb{F}$, i.e., a unique $\mathbb{F}$-morphism into x such that $F(\theta^t) = t$. For a discrete fibration $\mathbb{E} \xrightarrow{E} \mathbb{A} \times \mathbb{B}^o$, such liftings induce the arrow part of the corresponding presheaf

$$\begin{array}{rcl} \Xi_E : \mathbb{A}^o \times B & \to & \mathsf{Set} \\ \langle a, b\rangle & \mapsto & \{x \in \mathbb{E} \mid Ex = \langle a, b\rangle\} \end{array}$$

because any pair of morphisms $\langle f, g\rangle \in \mathbb{A}(a, a') \times \mathbb{B}^o(b, b')$ lifts to a function $\Xi_E(f, g) = \langle f, g\rangle^! : \Xi_E(a', b') \to \Xi_E(a, b)$. Fibrations go back to Grothendieck [35, 36]. Overviews can be found in [42, 67]. With (4) generalized to (52), and (5) to (57), (7–8) become

$$\Downarrow\mathbb{A} \;=\; \mathsf{Dfib} \diagup \mathbb{A} \;\simeq\; \mathsf{Set}^{\mathbb{A}^o} \tag{61}$$

$$\Uparrow B \;=\; (\mathsf{Dfib} \diagup \mathbb{B}^o)^o \;\simeq\; \left(\mathsf{Set}^{\mathbb{B}}\right)^o \tag{62}$$

Just like the poset embeddings $A \xrightarrow{\blacktriangledown} \Downarrow A$ and $B \xrightarrow{\blacktriangle} \Uparrow B$ were the join and the meet completions, the Yoneda embeddings $\mathbb{A} \xrightarrow{\blacktriangledown} \Downarrow\mathbb{A}$ and $\mathbb{B} \xrightarrow{\blacktriangle} \Uparrow\mathbb{B}$, where $\blacktriangledown a = \left(\mathbb{A}/a \xrightarrow{\text{Dom}} \mathbb{A}\right)$ amd $\blacktriangle b = \left(b/\mathbb{B} \xrightarrow{\text{Cod}} \mathbb{B}\right)$ are the colimit and the limit completions, respectively.

5.4 The functors

5.4.1 The functor $\mathsf{MA} : \mathsf{Mat} \to \mathsf{Adj}$

The adjunction $\mathsf{MA}(\Phi) = (\Phi^* \dashv \Phi_*)$ induced by a matrix $\Phi : \mathbb{A}^o \times \mathbb{B} \to \mathsf{Set}$ is defined by lifting (11) from posets to categories:

$$\begin{array}{ccc} \mathbb{L} \xrightarrow{L} \mathbb{A} & \Downarrow\mathbb{A} & \varprojlim\left(\mathbb{U} \xrightarrow{U} \mathbb{B} \xrightarrow{\bullet\Phi} \Downarrow\mathbb{A}\right) \\ \big\downarrow & \Phi^* \big\downarrow \dashv \big\uparrow \Phi_* & \big\uparrow \\ \varprojlim\left(\mathbb{L}^o \xrightarrow{L^o} \mathbb{A}^o \xrightarrow{\Phi\bullet} (\Uparrow\mathbb{B})^o\right) & \Uparrow\mathbb{B} & \mathbb{U} \xrightarrow{U} \mathbb{B} \end{array} \tag{63}$$

The fact that $\mathbb{A} \xrightarrow{\blacktriangledown} \Downarrow\mathbb{A}$ is a colimit completion means that every $L \in \Downarrow\mathbb{A}$ is generated by the representables, i.e. $L = \varinjlim\left(\mathbb{L} \xrightarrow{L} \mathbb{A} \xrightarrow{\blacktriangledown} \Downarrow\mathbb{A}\right)$. Any $\varinjlim$-preserving functor $\Phi^* : \Downarrow\mathbb{A} \to \Uparrow\mathbb{B}$ thus satisfies

$$\Phi^*(L) \;=\; \Phi^*\left(\varinjlim\left(\mathbb{L} \xrightarrow{L} \mathbb{A} \xrightarrow{\blacktriangledown} \Downarrow\mathbb{A}\right)\right) \;=\; \varinjlim\left(\mathbb{L} \xrightarrow{L} \mathbb{A} \xrightarrow{\Phi^o_\bullet} \Uparrow\mathbb{B}\right) \;=\; \varprojlim\left(\mathbb{L}^o \xrightarrow{L^o} \mathbb{A}^o \xrightarrow{\Phi\bullet} (\Uparrow\mathbb{B})^o\right)$$

Analogous reasoning goes through for Φ_*. This completes the definition of the object part of $\mathsf{MA} : \mathsf{Mat} \to \mathsf{Adj}$. The arrow part is completely determined by the object part.

Remark. The limits in $\Downarrow\mathbb{A} \simeq \mathsf{Set}^{\mathbb{A}^o}$ and in $(\Uparrow\mathbb{B})^o \simeq \mathsf{Set}^{\mathbb{B}}$ are pointwise, which means that for any $b \in \mathbb{B}$ and diagram $\mathbb{D} \xrightarrow{D} \mathsf{Set}^{\mathbb{B}}$, the Yoneda lemma implies

$$\left(\varprojlim D\right) b \quad = \quad \mathsf{Set}^{\mathbb{B}}\left(\blacktriangle b, \varprojlim D\right) \quad = \quad \mathrm{Cones}(b, \widehat{D})$$

In words, the limit of D at a point b is the set of commutative cones in $\mathbb{B}$ from b to a diagram $\widehat{D} : \int D \to \mathbb{B}$ constructed by a lifting like (59).

5.4.2 From adjunctions to monads and comonads, and back

The projections of adjunctions onto monads and comonads, and the embeddings that arise as their left and right adjoints, all displayed in Fig. 14, are one of the centerpieces of the categorical toolkit. The displayed functors are well known, but we list them for naming purposes:

$$\begin{tikzcd}
\mathsf{Cmn} \arrow[rr, bend left, "\mathsf{EC}"] \arrow[rr, bend right, "\mathsf{KC}"'] & & \mathsf{Adj} \arrow[ll, two heads, "\mathsf{AC}"'] \arrow[rr, two heads, "\mathsf{AM}"] & & \mathsf{Mnd} \arrow[ll, bend right, "\mathsf{EM}"'] \arrow[ll, bend left, "\mathsf{KM}"]
\end{tikzcd}$$

Figure 14: Relating adjunctions, monads and comonads

- $\mathsf{EC}\left(\overrightarrow{F} : \mathbb{B} \to \mathbb{B}\right) = \left(V^* \dashv V_* : \mathbb{B} \to \mathbb{B}^{\overrightarrow{F}}\right)$ $\leftsquigarrow$ all coalgebras (Eilenberg-Moore)
- $\mathsf{AC}\left(F^* \dashv F_* : \mathbb{B} \to \mathbb{A}\right) = \left(\overrightarrow{F} = F^*F_* : \mathbb{B} \to \mathbb{B}\right)$ $\leftsquigarrow$ adjunction-induced comonad
- $\mathsf{KC}\left(\overrightarrow{F} : \mathbb{B} \to \mathbb{B}\right) = \left(U^* \dashv U_* : \mathbb{B} \to \mathbb{B}_{\overrightarrow{F}}\right)$ $\leftsquigarrow$ cofree coalgebras (Kleisli)
- $\mathsf{EM}\left(\overleftarrow{F} : \mathbb{A} \to \mathbb{A}\right) = \left(V^* \dashv V_* : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{A}\right)$ $\leftsquigarrow$ all algebras (Eilenberg-Moore)
- $\mathsf{AM}\left(F^* \dashv F_* : \mathbb{B} \to \mathbb{A}\right) = \left(\overleftarrow{F} = F_*F^* : \mathbb{A} \to \mathbb{A}\right)$ $\leftsquigarrow$ adjunction-induced monad
- $\mathsf{KM}\left(\overleftarrow{F} : \mathbb{A} \to \mathbb{A}\right) = \left(U^* \dashv U_* : \mathbb{A}_{\overrightarrow{F}} \to \mathbb{A}\right)$ $\leftsquigarrow$ free algebras (Kleisli)

Here $\mathbb{A}^{\overleftarrow{F}}$ is the category of all algebras and $\mathbb{A}_{\overleftarrow{F}}$ is the category of free algebras for the monad $\overleftarrow{F}$ on $\mathbb{A}$; and dually $\mathbb{B}^{\overrightarrow{F}}$ is the category of all coalgebras for the comonad $\overrightarrow{F}$ on $\mathbb{B}$, whereas $\mathbb{B}_{\overrightarrow{F}}$ is the category of cofree coalgebras. As the right adjoints, the Eilenberg-Moore constructions of all algebras and all coalgebras thus provide the final resolutions for their respective monad and comonad, whereas the Kleisli constructions of free algebras and cofree coalgebras as the left adjoints provide the initial resolutions.

Note that the nucleus setting in Fig. 3 only uses parts of the above reflections: the final resolution $\mathsf{AM} \dashv \mathsf{EM}$ of monads, and the initial resolution $\mathsf{KC} \dashv \mathsf{AC}$ of comonads. Dually, we could use $\mathsf{KM} \dashv \mathsf{AM}$ and $\mathsf{AC} \dashv \mathsf{EC}$. Either choice induces a composite adjunction, with an induced monad on one side, and a comonad on the other side, as displayed in Fig. 15.

6 Theorem

The Street *monads* $\overleftarrow{\mathfrak{M}} : \mathsf{Mnd} \to \mathsf{Mnd}$ *and* $\overleftarrow{\mathfrak{E}} : \mathsf{Cmn} \to \mathsf{Cmn}$*, defined by*

$$\overleftarrow{\mathfrak{M}} \;=\; \mathsf{AM} \circ \mathsf{EC} \circ \mathsf{AC} \circ \mathsf{KM} \tag{64}$$

$$\overleftarrow{\mathfrak{E}} \;=\; \mathsf{AC} \circ \mathsf{EM} \circ \mathsf{AM} \circ \mathsf{KC} \tag{65}$$

as illustrated in Fig. 15, are idempotent in the strong sense: iterating them leads to natural equiv-

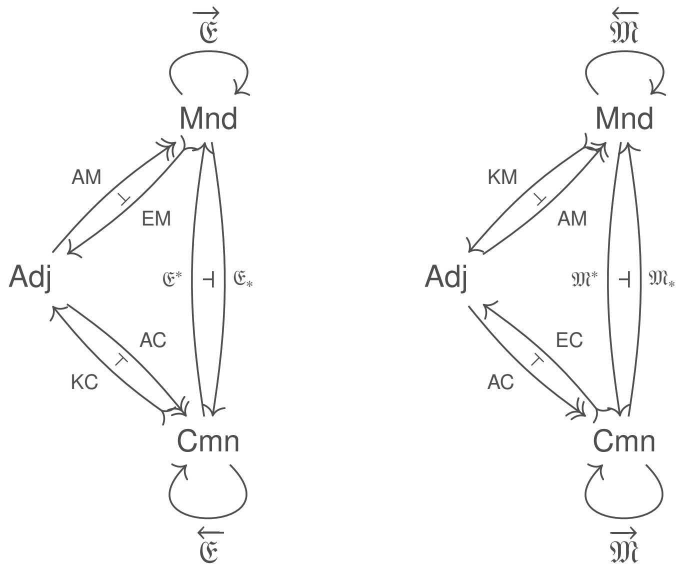

Figure 15: Monads and comonads on Cmn and Mnd induced by the localizations in Fig. 14

alences

$$\overleftarrow{\mathfrak{M}} \overset{\eta}{\simeq} \overleftarrow{\mathfrak{M}} \circ \overleftarrow{\mathfrak{M}} \qquad\qquad \overleftarrow{\mathfrak{E}} \overset{\eta}{\simeq} \overleftarrow{\mathfrak{E}} \circ \overleftarrow{\mathfrak{E}}$$

Moreover, the induced categories of algebras coincide. More precisely, there are equivalences

$$\mathsf{Cmn}^{\overleftarrow{\mathfrak{E}}} \;\simeq\; \mathsf{Nuc} \;\simeq\; \mathsf{Mnd}^{\overleftarrow{\mathfrak{M}}} \tag{66}$$

where

$$\mathsf{Cmn}^{\overleftarrow{\mathfrak{E}}} \;=\; \left\{ \overrightarrow{F} \in \mathsf{Cmn} \mid \overrightarrow{F} \overset{\eta}{\cong} \overleftarrow{\mathfrak{E}}\overrightarrow{F} \right\} \tag{67}$$

$$\mathsf{Nuc} \;=\; \left\{ F \in \mathsf{Adj} \mid \; F \overset{\eta}{\cong} \overleftarrow{\mathsf{EM}}(F) \;\wedge\; F \overset{\eta}{\cong} \overleftarrow{\mathsf{EC}}(F) \right\} \tag{68}$$

$$\mathsf{Mnd}^{\overleftarrow{\mathfrak{M}}} \;=\; \left\{ \overleftarrow{F} \in \mathsf{Mnd} \mid \overleftarrow{F} \overset{\eta}{\cong} \overleftarrow{\mathfrak{M}}\overleftarrow{F} \right\} \tag{69}$$

for $\overleftarrow{\mathsf{EM}} = \mathsf{EM} \circ \mathsf{AM}$ *and* $\overleftarrow{\mathsf{EC}} = \mathsf{EC} \circ \mathsf{AC}$.

Terminology. The objects of the equivalent categories $\mathsf{Nuc} \subset \mathsf{Adj}$, $\mathsf{Mnd}^{\overleftarrow{\mathfrak{M}}} \subset \mathsf{Mnd}$, and $\mathsf{Cmn}^{\overleftarrow{\mathfrak{C}}} \subset \mathsf{Cmn}$ are *nuclear* adjunctions, monads, or comonads, respectively. They are the *nuclei* of the corresponding adjunctions, monads, comonads.

Remark. For an adjunction $F = (F^* \dashv F_*)$, the condition $F \overset{\eta}{\cong} \overleftarrow{\mathsf{EM}}(F)$ implies that F_* is monadic, and $F \overset{\eta}{\cong} \overleftarrow{\mathsf{EC}}(F)$ implies that F^* is comonadic. Equation (68) thus provides a more formal view of nuclear adjunctions, where the right adjoint is monadic and the left adjoint is comonadic, as discussed the Introduction. Although defined slightly more formally than in the Introduction, the category Nuc is still specified as an intersection of two reflective subcategories. To ensure the soundness of such a definition, one should prove that the two reflections commute, i.e., that the two monads distribute over one another. Otherwise, the two reflections could alternate mapping an object outside each other's range, and generate chains. In the case at hand, this does not happen: the distributive law $\overleftarrow{\mathsf{EM}} \circ \overleftarrow{\mathsf{EC}} \cong \overleftarrow{\mathsf{EC}} \circ \overleftarrow{\mathsf{EM}}$ is spelled out in Corollary 7.8. It arises from the nucleus monad $\overleftarrow{\mathfrak{N}} : \mathsf{Adj} \to \mathsf{Adj}$, which we will work on in the next section. We swept it under the carpet just for a moment, to keep the theorem shorter.

7 Propositions

Proposition 7.1 *Let $F = (F^* \dashv F_* : \mathbb{B} \to \mathbb{A})$ be an arbitrary adjunction, which induces*

- *the monad $\overleftarrow{F} = F_*F^*$ with the (Eilenberg-Moore) category of algebras $\mathbb{A}^{\overleftarrow{F}}$ and the final adjunction resolution $U = \left(U^* \dashv U_* : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{A}\right)$, and*
- *the comonad $\overrightarrow{F} = F^*F_*$ with the (Eilenberg-Moore) category of coalgebras $\mathbb{B}^{\overrightarrow{F}}$ and the final resolution $V = \left(V^* \dashv V_* : \mathbb{B} \to \mathbb{B}^{\overrightarrow{F}}\right)$.*

The fact that U and V are final resolutions of the monad $\overleftarrow{F}$ and the comonad $\overrightarrow{F}$, respectively, means that there are unique comparison functors from the adjunction F to each of them, and these functors are:

- *$H^0 : \mathbb{A} \to \mathbb{B}^{\overrightarrow{F}}$, such that $F^* = V^* \circ H^0$ and $F_* \circ H^0 = V_*$,*
- *$H_1 : \mathbb{B} \to \mathbb{A}^{\overleftarrow{F}}$, such that $F_* = U_* \circ H_1$ and $H_1 \circ F^* = U^*$.*

Then the functors $F^\sharp = H_1 \circ V^*$ and $F_\sharp = H_0 \circ U_*$ defined in Fig. 16 form the adjunction $F^\sharp \dashv F_\sharp : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{B}^{\overrightarrow{F}}$.

Proof. The object parts of the definitions of the functors $F_\sharp$ and $F^\sharp$ are unfolded in Fig. 17. The arrow part of $F_\sharp$ is F^* and the arrow part of $F^\sharp$ is F_*. For these $F^\sharp$ and $F_\sharp$, we shall prove that the correspondence

$$\begin{aligned} \mathbb{A}^{\overleftarrow{F}}(F^\sharp\beta, \alpha) \quad &\cong \quad \mathbb{B}^{\overrightarrow{F}}(\beta, F_\sharp\alpha) \\ f &\mapsto f = F^* f \circ \beta \end{aligned} \tag{70}$$

$$\begin{tikzcd}
\mathbb{A} \arrow[loop above, "\overleftarrow{F}"] \arrow[rr, dashed, "H^0"] \arrow[d, bend right, "F^*"'] \arrow[d, phantom, "\dashv"] & & \mathbb{B}^{\overrightarrow{F}} \arrow[lld, "V^*"] \arrow[d, bend right, "F^\sharp"'] \arrow[d, phantom, "\dashv"] \\
\mathbb{B} \arrow[loop below, "\overrightarrow{F}"] \arrow[u, bend right, "F_*"'] \arrow[rr, dashed, "H_1"'] & & \mathbb{A}^{\overleftarrow{F}} \arrow[llu, "U_*"] \arrow[u, bend right, "F_\sharp"']
\end{tikzcd}$$

Figure 16: The nucleus $F^\sharp \dashv F_\sharp$ of $F^* \dashv F_*$ consists of $F^\sharp = H_1 \circ V^*$ and $F_\sharp = H^0 \circ U_*$

$$\begin{tikzcd}
x \arrow[r, mapsto, dashed] & \left(\begin{array}{c} F^*x \\ \downarrow F^*\eta \\ F^*F_*F^*x \end{array}\right) & \left(\begin{array}{c} F_*F^*x \\ \downarrow \alpha \\ x \end{array}\right) \arrow[l, mapsto] \\
\mathbb{A} \arrow[r, dashed, "H^0"] & \mathbb{B}^{\overrightarrow{F}} & \mathbb{A}^{\overleftarrow{F}} \arrow[l, "F_\sharp"'] \arrow[ll, bend left, "U_*"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{B} \arrow[r, dashed, "H_1"] & \mathbb{A}^{\overleftarrow{F}} & \mathbb{B}^{\overrightarrow{F}} \arrow[l, "F^\sharp"'] \arrow[ll, bend right, "V^*"'] \\
y \arrow[r, mapsto, dashed] & \left(\begin{array}{c} F_*F^*F_*y \\ \downarrow F_*\varepsilon \\ F_*y \end{array}\right) & \left(\begin{array}{c} y \\ \downarrow \beta \\ F^*F_*y \end{array}\right) \arrow[l, mapsto]
\end{tikzcd}$$

Figure 17: The definitions of $F_\sharp$ and $F^\sharp$

is a natural bijection. More precisely, the claim is that

a) f is an algebra homomorphism if and only if f is a coalgebra homomorphism: each of the following squares commutes if and only if the other one commutes

$$\begin{tikzcd}
F_*F^*F_*y \arrow[r, "F_*F^*f"] \arrow[d, "F_*\varepsilon \,=F^\sharp\beta"'] & F_*F^*x \arrow[d, "\alpha"] \\
F_*y \arrow[r, "f"'] & x
\end{tikzcd}
\iff
\begin{tikzcd}
F^*F_*y \arrow[r, "F^*F_*f"] & F^*F_*F^*x \\
y \arrow[u, "\beta"] \arrow[r, "f"'] & F^*x \arrow[u, "F_\sharp\alpha= \,F^*\eta"']
\end{tikzcd}
\tag{71}$$

b) the map $f \mapsto f$ is a bijection, natural along the coalgebra homomorphisms on the left and along the algebra homomorphisms on the right.

Claim (a) is proved as Lemma 7.3. The bijection part of claim (b) is proved as Lemma 7.2. The naturality part is straightforward. □

Lemma 7.2 *For an arbitrary adjunction $F = F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$, any algebra $F_*F^*x \xrightarrow{\alpha} x$, and any coalgebra $y \xrightarrow{\beta} F^*F_*y$ in $\mathbb{B}$, the mappings*

$$\begin{tikzcd} \mathbb{A}(F_*y, x) \arrow[r, bend left, "(-)"] & \mathbb{B}(y, F^*x) \arrow[l, bend left, "(-)"] \end{tikzcd}$$

defined by

$$f = F^*f \circ \beta \qquad\qquad g = \alpha \circ F_*g$$

induce a bijection between the subsets

$$\{f \in \mathbb{A}(F_*y, x) \mid f = \alpha \circ F_*F^*f \circ F_*\beta\} \quad \cong \quad \{g \in \mathbb{B}(y, F^*x) \mid g = F^*\alpha \circ F^*F_*g \circ \beta\}$$

illustrated in the following diagram.

$$\begin{tikzcd} F_*F^*F_*y \arrow[r, "F_*F^*f"] & F_*F^*x \arrow[d, "\alpha"] \\ F_*y \arrow[u, "F_*\beta"] \arrow[ur, "F_*g"] \arrow[r, "f"'] & x \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} F^*F_*y \arrow[r, "F^*F_*g"] \arrow[dr, "F^*f"] & F^*F_*F^*x \arrow[d, "F_*\alpha"] \\ y \arrow[u, "\beta"] \arrow[r, "g"'] & F^*x \end{tikzcd}$$

Proof. Following each of the mappings "there and back" gives

$$\begin{array}{ccccc} f & \longmapsto & f = F^*f \circ \beta & \longmapsto & f = \alpha \circ F_*F^*f \circ F_*\beta = f \\ g & \longmapsto & g = \alpha \circ F_*g & \longmapsto & g = F^*\alpha \circ F^*F_*g \circ \beta = g \end{array}$$

□

Lemma 7.3 *For any adjunction $F = F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$, algebra $F_*F^*x \xrightarrow{\alpha} x$ in $\mathbb{A}$, coalgebra $y \xrightarrow{\beta} F^*F_*y$ in $\mathbb{B}$, arrow $f \in \mathbb{A}(F_*y, x)$ and $f = F^*f \circ \beta \in \mathbb{B}(x, F^*y)$, if any of the squares (1-4) in Fig. 18 commutes, then they all commute. In particular, a square on one side of any of the equivalences (a–c) commutes if and only if the square on the other side of the equivalence commutes.*

Proof. The claims are established as follows.

$$
\begin{tikzcd}
F_*F^*F_*y \arrow[r, "F_*F^*f"] \arrow[d, "F_*\varepsilon"'] \arrow[dr, phantom, "(1)"] & F_*F^*x \arrow[d, "\alpha"] \\
F_*y \arrow[r, "f"'] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
F^*F_*y \arrow[r, "F^*F_*f"] \arrow[dr, phantom, "(4)"] & F^*F_*F^*x \\
y \arrow[u, "\beta"] \arrow[r, "f"'] & F^*x \arrow[u, "F^*\eta"']
\end{tikzcd}
$$

$(a) \Updownarrow \qquad\qquad \Updownarrow (c)$

$$
\begin{tikzcd}
F_*F^*F_*y \arrow[r, "F_*F^*f"] \arrow[dr, phantom, "(2)"] & F_*F^*x \arrow[d, "\alpha"] \\
F_*y \arrow[u, "F_*\beta"] \arrow[r, "f"'] & x
\end{tikzcd}
\quad \overset{(b)}{\Leftrightarrow} \quad
\begin{tikzcd}
F^*F_*y \arrow[r, "F^*F_*f"] \arrow[dr, phantom, "(3)"] & F^*F_*F^*x \arrow[d, "F_*\alpha"] \\
y \arrow[u, "\beta"] \arrow[r, "f"'] & F^*x
\end{tikzcd}
$$

Figure 18: Proof schema for (71)

(1) $\overset{(a)}{\Rightarrow}$ (2): Using the commutativity of (1) and $(*)$ the counit equation $\varepsilon \circ \beta = \mathrm{id}$ for the coalgebra β, we derive (2) as

$$\alpha \circ F_*F^*f \circ F_*\beta \overset{(1)}{=} f \circ F_*\varepsilon \circ F_*\beta \overset{(*)}{=} f$$

$(2) \overset{(a)}{\Rightarrow} (1)$ is proved by chasing the following diagram:

$$
\begin{tikzcd}
F_*F^*F_*F^*F_*y \arrow[rrr, "F_*F^*F_*F^*f"] \arrow[dddd, "F_*\varepsilon"'] & & & F_*F^*F_*F^*x \arrow[dl, "F_*F^*\alpha"] \arrow[dddd, "F_*\varepsilon"] \\
& F_*F^*F_*y \arrow[ul, "F_*F^*F_*\beta"'] \arrow[r, "F_*F^*f"] \arrow[dd, "F_*\varepsilon"] \arrow[ur, phantom, "(2)"] \arrow[ddr, phantom, "(1)"] \arrow[ddl, phantom, "(\dagger)"] & F_*F^*x \arrow[dd, "\alpha"'] \arrow[ddr, phantom, "(\ddagger)"] & \\
& & & \\
& F_*y \arrow[r, "f"'] \arrow[ddl, "F_*\beta"'] \arrow[ddr, phantom, "(2)"] & x & \\
& & & \\
F_*F^*F_*y \arrow[rrr, "F_*F^*f"'] & & & F_*F^*x \arrow[uul, "\alpha"']
\end{tikzcd}
$$

The top and the bottom trapezoids commute by assumption (2), whereas the left hand trapezoid (denoted (†)) and the outer square (denoted (□)) commute by the naturality of ε. The right hand trapezoid (denoted (‡)) commutes by the cochain condition for the algebra α. It follows that the inner square (denoted (1)) must also commute:

$$
\begin{aligned}
f \circ F_*\varepsilon &\overset{(2)}{=} \alpha \circ F_*F^*f \circ F_*\beta \circ F_*\varepsilon \\
&\overset{(\dagger)}{=} \alpha \circ F_*F^*f \circ F_*\varepsilon \circ F_*F^*F_*\beta \\
&\overset{(\square)}{=} \alpha \circ F_*\varepsilon \circ F_*F^*F_*F^*f \circ F_*F^*F_*\beta \\
&\overset{(\ddagger)}{=} \alpha \circ F_*\alpha^* \circ F_*F^*F_*F^*f \circ F_*F^*F_*\beta \\
&\overset{(2)}{=} \alpha \circ F_*F^*f
\end{aligned}
$$

$(4) \overset{(c)}{\Leftrightarrow} (3)$ is proven dually to $(1) \overset{(a)}{\Leftrightarrow} (2)$ above. The duality consists of reversing the arrows, switching F_* and F^*, and also α and β, and replacing ε with η.

$(2) \overset{(b)}{\Leftrightarrow} (3)$ follows from Lemma 7.2. □

Proposition 7.4 *The adjunction $F^\sharp \dashv F_\sharp : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{B}^{\overrightarrow{F}}$ constructed in Prop. 7.1 is nuclear:*

- *$F^\sharp : \mathbb{B}^{\overrightarrow{F}} \to \mathbb{A}^{\overleftarrow{F}}$ is comonadic*
- *$F_\sharp : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{B}^{\overrightarrow{F}}$ is monadic*

This construction induces the idempotent monad

$$\begin{array}{rcl} \overleftarrow{\mathfrak{N}} : \mathsf{Adj} & \to & \mathsf{Adj} \\ (F^* \dashv F_*) & \mapsto & (F^\sharp \dashv F_\sharp) \end{array}$$

Proof. It is easy to see that the construction of $\overleftarrow{\mathfrak{N}}F = (F^\sharp \dashv F_\sharp)$ in Prop. 7.1 is functorial, and that the comparison functors as used in Fig. 16 provide the monad unit $F \xrightarrow{\eta} \overleftarrow{\mathfrak{N}}F$. We show that $\overleftarrow{\mathfrak{N}}F \xrightarrow{\overleftarrow{\mathfrak{N}}\eta} \overleftarrow{\mathfrak{N}}\overleftarrow{\mathfrak{N}}F$ is always an equivalence. This means that the comparison functors from $\overleftarrow{\mathfrak{N}}F$ to $\overleftarrow{\mathfrak{N}}\overleftarrow{\mathfrak{N}}F$ are equivalences. These comparison functors are constructed in Fig. 19, still under the names H^0 and H_1, lifting the construction from Fig. 16. is an equivalence of categories. We prove

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{B}^{\overrightarrow{F}} \arrow[loop above, "\overset{\Leftarrow}{F}"] \arrow[r, dashed, "H^0"] \arrow[d, bend right, "F^\sharp"'] \arrow[d, phantom, "\dashv"] & \left(\mathbb{A}^{\overrightarrow{F}}\right)^{\overset{\Leftarrow}{F}} \arrow[d, bend right, "F^{\sharp\sharp}"'] \arrow[d, phantom, "\dashv"] \arrow[dl, "V^*"' near end] \\
\mathbb{A}^{\overleftarrow{F}} \arrow[loop below, "\overset{\Rightarrow}{F}"] \arrow[u, bend right, "F_{\sharp\sharp}"'] \arrow[r, dashed, "H_1"'] & \mathbb{A}^{\overleftarrow{F}} \arrow[u, bend right, "F_\sharp"'] \arrow[ul, "U_*"' near end]
\end{tikzcd}$$

Figure 19: The construction of the nucleus $\overleftarrow{\mathfrak{N}}\overleftarrow{\mathfrak{N}}F = \left(F^{\sharp\sharp} \dashv F_{\sharp\sharp}\right)$ of nucleus $\overleftarrow{\mathfrak{N}}F = \left(F^\sharp \dashv F_\sharp\right)$

this only for H^0. The argument for H_1 is dual.

Instantiating the usual definition of the comparison functor for the comonad $\overset{\Rightarrow}{F} : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{A}^{\overleftarrow{F}}$ to the resolution $F^\sharp \dashv F_\sharp$, we get

$$
\begin{tikzcd}
\mathbb{B}^{\overrightarrow{F}} \arrow[rr, "H^0"] & & \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}}
\end{tikzcd}
\tag{72}
$$

$$
\begin{tikzcd}
y \arrow[d, "\beta"] \\
F^*F_*y
\end{tikzcd}
\qquad \longmapsto \qquad
\begin{tikzcd}[column sep=huge, row sep=huge]
F_*F^*F_*y \arrow[r, "F_*\varepsilon_y \,=\, F^\sharp\beta"] \arrow[d, "F_*F^*F_*\beta"'] \arrow[dr, phantom, "H^0\beta \,=\, F^\sharp\eta_\beta"] & F_*y \arrow[d, "F_*\beta"] \\
F_*F^*F_*F^*F_*y \arrow[r, "F^\sharp F_\sharp F^\sharp\beta", "= F_*\varepsilon_{\overrightarrow{F}y}"'] & F_*F^*F_*y
\end{tikzcd}
$$

Since by assumption the idempotents split in $\mathbb{B}$, the comparison functor H^0 also has a right adjoint H_0, which must be in the form

$$
\begin{tikzcd}
\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \arrow[rr, "H_0"] & & \mathbb{B}^{\overrightarrow{F}}
\end{tikzcd}
\tag{73}
$$

$$
\begin{tikzcd}[column sep=large, row sep=huge]
F_*F^*x \arrow[r, "\alpha"'] \arrow[d, "F_*F^*d"'] \arrow[dr, phantom, "\delta"] & x \arrow[d, "d"] \\
F_*F^*F_*F^*x \arrow[r, "F^\sharp F_\sharp\alpha", "= F_*\varepsilon_{F^*x}"'] & F_*F^*x
\end{tikzcd}
\qquad \longmapsto \qquad
\begin{tikzcd}[column sep=large, row sep=huge]
y \arrow[r, tail, "e"] \arrow[d, dashed, "H_0\delta"'] & F^*x \arrow[l, bend left, "r"] \arrow[r, shift left, tail, "F^*d"] \arrow[r, shift right, tail, "F^*\eta"'] \arrow[d, "F_\sharp\alpha"', "=F^*\eta"] & F^*F_*F^*x \arrow[l, bend right, "\varepsilon"'] \arrow[d, "F_\sharp F^\sharp F_\sharp\alpha \,="', "F^*\eta"] \\
F^*F_*y \arrow[r, tail, "F^*F_*e"] & F^*F_*F^*x \arrow[l, bend left, "F^*F_*r"] \arrow[r, shift left, tail, "F^*F_*F^*d"] \arrow[r, shift right, tail, "F^*F_*F^*\eta"'] & F^*F_*F^*F_*F^*x \arrow[l, bend right, "F^*F_*\varepsilon"']
\end{tikzcd}
$$

where y is defined by splitting the idempotent $\varepsilon \circ F^*d$, and d is the structure map of the coalgebra $\alpha \xrightarrow{d} F^\sharp F_\sharp\alpha$ in $\mathbb{A}^{\overleftarrow{F}}$.

To show that the adjunction $H^0 \dashv H_0 : \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \to \mathbb{B}^{\overrightarrow{F}}$ is an equivalence, we construct natural isomorphisms $H_0H^0 \cong \mathrm{id}$ and $H^0H_0 \cong \mathrm{id}$.

Towards the isomorphism $H_0H^0 \cong \mathrm{id}$, note that instantiating $H^0\beta : F^\sharp\beta \to F^\sharp F_\sharp F^\sharp\beta$ (the right-hand square in (72)) as $\delta : \alpha \to F^\sharp F_\sharp\alpha$ (the left-hand square in (73)) reduces the right-hand equalizer of (73) to the following form:

$$
\begin{tikzcd}
y \arrow[r, tail, "\beta"'] \arrow[d, dashed, "H_0H^0\beta = \beta"'] & F^*F_*y \arrow[l, bend left, "\varepsilon"] \arrow[r, tail, shift left, "F^*F_*\beta"] \arrow[r, tail, shift right, "F^*\eta"'] \arrow[d, "F^*\eta"] & F^*F_*F^*F_*y \arrow[l, bend right, "\varepsilon"'] \arrow[d, "F^*\eta"] \\
F^*F_*y \arrow[r, tail, "F^*F_*\beta"'] & F^*F_*F^*F_*y \arrow[l, bend left, "F^*F_*\varepsilon"] \arrow[r, tail, shift left, "F^*F_*F^*F_*\beta"] \arrow[r, tail, shift right, "F^*F_*F^*\eta"'] & F^*F_*F^*F_*y \arrow[l, bend right, "F^*F_*\varepsilon"']
\end{tikzcd}
\tag{74}
$$

It is a basic fact of (co)monad theory that every coalgebra β in $\mathbb{B}^{\overrightarrow{F}}$ makes diagram (74) commute [14, Sec. 3.6].

Towards the isomorphism $H^0H_0 \cong \mathrm{id}$, take an arbitrary coalgebra $\alpha \xrightarrow{\delta} F^\sharp F_\sharp\alpha$ from $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overrightarrow{\overrightarrow{F}}}$ and consider (72) instantiated to $\beta = H_0\delta$. By extending the right-hand side of this instance of (72) by the F_*-image of the right-hand side of (73), we get the following diagram

$$
\begin{tikzcd}
F_*F^*F_*y \arrow[r, "F_*\varepsilon"] \arrow[d, "F_*F^*F_*H_0\delta"'] \arrow[dr, phantom, "H^0H_0\delta"] & F_*y \arrow[r, tail, "F_*e"] \arrow[d, dashed, "F_*H_0\delta"] & F_*F^*x \arrow[l, bend left, "F_*r"] \arrow[r, tail, shift left, "F_*F^*d"] \arrow[r, tail, shift right, "F_*F^*\eta"'] \arrow[d, "F_*F^*\eta"] & F_*F^*F_*F^*x \arrow[l, bend right, "F_*\varepsilon"'] \arrow[d, "F_*F^*\eta"] \\
F_*F^*F_*F^*F_*y \arrow[r, "F_*\varepsilon"'] & F_*F^*F_*y \arrow[r, tail, "F_*F^*F_*e"] & F_*F^*F_*F^*x \arrow[l, bend left, "F_*F^*F_*r"] \arrow[r, tail, shift left, "F_*F^*F_*F^*d"] \arrow[r, tail, shift right, "F_*F^*F_*F^*\eta"'] & F_*F^*F_*F^*x \arrow[l, bend right, "F_*F^*F_*\varepsilon"']
\end{tikzcd}
\tag{75}
$$

The claim is now that $x \overset{d}{\rightarrowtail} F_*F^*x$ equalizes the parallel pair $\langle F_*F^*\eta, F_*F^*d\rangle$ in the first row. Since $y \overset{e}{\rightarrowtail} F^*x$ was defined in (73) as a split equalizer of the pair $\langle F^*\eta, F^*d\rangle$, and all functors preserve split equalizers, it follows that $F_*y \overset{F_*e}{\rightarrowtail} F_*F^*x$ is also an equalizer of the same pair $\langle F_*F^*\eta, F_*F^*d\rangle$. Hence the isomorphism $x \cong F_*y$, which gives $H^0H_0\delta \cong \delta$.

To prove the claim that $x \overset{d}{\rightarrowtail} F_*F^*x$ equalizes the first row, note that, just like the coalgebra $y \xrightarrow{\beta} F^*F_*y$ in $\mathbb{B}^{\overrightarrow{F}}$ was determined up to isomorphism by the split equalizer in $\mathbb{B}$, shown in (74), the coalgebra $\alpha \xrightarrow{\delta} F^\sharp F_\sharp\alpha$ in $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overrightarrow{\overrightarrow{F}}}$ is determined up to isomorphism by the following split equalizer in $\mathbb{A}^{\overleftarrow{F}}$

$$\begin{tikzcd}
\alpha \arrow[r, tail, "\delta"] & F^\sharp F_\sharp \alpha \arrow[l, two heads, bend left, "\varepsilon"] \arrow[r, shift left, "F^\sharp F_\sharp \delta"] \arrow[r, tail, shift right, "F^\sharp \eta"'] & F^\sharp F_\sharp F^\sharp F_\sharp \alpha \arrow[l, two heads, bend right, "\varepsilon"']
\end{tikzcd} \tag{76}$$

In $\mathbb{A}$, the split equalizer (76) unfolds to the lower squares of the following diagram

$$\begin{tikzcd}
x \arrow[r, tail, "d"] \arrow[d, tail, dashed, "d"] & F_*F^*x \arrow[l, two heads, bend left, "\alpha"] \arrow[r, shift left, "F_*F^*d"] \arrow[r, tail, shift right, "F_*F^*\eta"'] \arrow[d, tail, "F_*F^*\eta"] & F_*F^*F_*F^*x \arrow[l, two heads, bend right, "F_*\varepsilon"'] \arrow[d, tail, "F_*F^*\eta"] \\
F_*F^*x \arrow[r, tail, "F_*F^*d"] \arrow[d, two heads, dashed, "\alpha"] & F_*F^*F_*x \arrow[l, two heads, bend left, "F_*F^*\alpha"] \arrow[r, shift left, "F_*F^*F_*F^*d"] \arrow[r, tail, shift right, "F_*F^*F_*F^*\eta"'] \arrow[d, two heads, "F_*\varepsilon"] & F_*F^*F_*F^*F_*x \arrow[l, two heads, bend right, "F^*F_*\varepsilon"'] \arrow[d, two heads, "F_*\varepsilon"] \\
x \arrow[r, tail, "d"] & F_*F^*x \arrow[l, two heads, bend left, "\alpha"] \arrow[r, shift left, "F_*F^*d"] \arrow[r, tail, shift right, "F_*F^*\eta"'] & F_*F^*F_*F^*x \arrow[l, two heads, bend right, "F_*\varepsilon"']
\end{tikzcd} \tag{77}$$

Since the upper right-hand squares also commute (by the naturality of η), they also induce the factoring of the split equalizers in the upper left-hand square. But the upper right-hand squares in (77) are identical to the right-hand squares in (75). The fact that both $F_*y \overset{F_*e}{\rightarrowtail} F_*F^*x$ and $x \overset{d_*}{\rightarrowtail} F_*F^*x$ are split equalizers of the same pair yields the isomorphism $F_*y \overset{\iota}{\to} x$ in $\mathbb{A}$, which turns out to be a coalgebra isomorphism $H^0H_0\delta \underset{\sim}{\xrightarrow{\iota}} \delta$ in $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overrightarrow{\overrightarrow{F}}}$, as shown in (78).

$$\begin{tikzcd}
F_*F^*x \arrow[rrr, "\alpha"] \arrow[ddd, "F_*F^*d"'] & & & x \arrow[ddd, "d"] \\
& F_*F^*F_*y \arrow[ul, "F_*F^*\iota"'] \arrow[r, "F_*\varepsilon"] \arrow[d, "F_*F^*F_*H_0\delta"] & F_*y \arrow[ur, "\iota"'] \arrow[d, "F_*H_0\delta"'] & \\
& F_*F^*F_*F^*F_*y \arrow[dl, "F_*F^*F_*F^*\iota"'] \arrow[r, "F_*\varepsilon"] & F_*F^*F_*y \arrow[dr, "F_*F^*\iota"] & \\
F_*F^*F_*F^*x \arrow[rrr, "F_*\varepsilon"'] & & & F_*F^*x
\end{tikzcd} \tag{78}$$

Here the outer square is δ, as in (73) on the left, whereas the inner square is $H^0H_0\delta$, as in (75) on the left. The right-hand trapezoid commutes because the middle square in (75) commutes, and can be chased down to (79) using the fact that ι is defined by $F_*e = d \circ \iota$.

$$\begin{tikzcd}
F_*y \arrow[r, "\iota"'] \arrow[rr, bend left, "F_*e"] \arrow[d, "F_*H_0\delta"'] & x \arrow[r, tail, "\delta"'] & F_*F^*x \arrow[d, tail, "F_*F^*\eta"] \arrow[dl, equal, bend right] \\
F_*F^*F_*F^*y \arrow[r, "F_*F^*\iota"] \arrow[rr, bend right, "F_*F^*F_*e"'] & F_*F^*x \arrow[r, tail, "F_*F^*\delta"] & F_*F^*F_*F^*x \arrow[l, two heads, bend right, "F_*\varepsilon"']
\end{tikzcd} \tag{79}$$

The commutativity of the left-hand trapezoid in (78) follows, because it is an F_*F^*-image of the right-hand trapezoid. The bottom trapezoid commutes by the naturality of ε. The top trapezoid commutes because everything else commutes, and d is a monic. The commutative diagram in (78) thus displays the claimed isomorphism $H^0H_0\delta \xrightarrow{\iota} \delta$.

This completes the proof that $H^0H_0 \cong \mathrm{id}$. Together with the proof that $H_0H^0 \cong \mathrm{id}$, as seen in (74), this also completes the proof that $H = \left(H^0 \dashv H_0 : \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overrightarrow{\overrightarrow{F}}} \to \mathbb{B}^{\overrightarrow{F}}\right)$ is an equivalence. We have thus shown that $F^\sharp : \mathbb{B}^{\overrightarrow{F}} \to \mathbb{A}^{\overleftarrow{F}}$ is comonadic. The proof that $F_\sharp : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{B}^{\overrightarrow{F}}$ can be constructed as a mirror image. $\square$

Corollary 7.5 *For any adjunction $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$ with the nucleus $F^\sharp \dashv F_\sharp : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{B}^{\overrightarrow{F}}$ it holds that the induced monad $\overset{\Longleftarrow}{F} = F_\sharp F^\sharp$ on $\mathbb{B}^{\overrightarrow{F}}$ and comonad $\overset{\Longrightarrow}{F} = F^\sharp F_\sharp$ on $\mathbb{A}^{\overleftarrow{F}}$ are isomorphic with those induced by the final resolutions:*

$$\begin{aligned} \overset{\Longleftarrow}{F} &\cong \left(\mathbb{B}^{\overrightarrow{F}} \xrightarrow{V^*} \mathbb{B} \xrightarrow{V_*} \mathbb{B}^{\overrightarrow{F}}\right) \\ \overset{\Longrightarrow}{F} &= \left(\mathbb{A}^{\overleftarrow{F}} \xrightarrow{U_*} \mathbb{B} \xrightarrow{U^*} \mathbb{A}^{\overleftarrow{F}}\right) \end{aligned}$$

The monad $\overset{\Longleftarrow}{F}$ on $\mathbb{B}^{\overrightarrow{F}}$ thus only depends on the comonad $\overrightarrow{F}$ on $\mathbb{B}$, whereas the cononad $\overset{\Longrightarrow}{F}$ on $\mathbb{A}^{\overleftarrow{F}}$ only depends on the monad $\overleftarrow{F}$ on $\mathbb{A}$. Neither depends on the particular adjunction from which the nucleus originates.

Proof. Using the definitions $F_\sharp = H^0 U_*$ and $F^\sharp = H_1 F^*$, and chasing Fig. 16 gives

$$\begin{aligned} \overset{\Longleftarrow}{F} &= F_\sharp F^\sharp = H^0 U_* H_1 V^* = H^0 F_* V^* = V_* V^* \\ \overset{\Longrightarrow}{F} &= F^\sharp F_\sharp = H_1 V^* H^0 U_* = H_1 F^* V_* = U^* U_* \end{aligned}$$

□

Corollary 7.6 *All resolutions of a monad induce equivalent categories of coalgebras. More precisely, for any given monad $\overleftarrow{T} : \mathbb{A} \to \mathbb{A}$ any pair of adjunctions $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$ and $G^* \dashv G_* : \mathbb{D} \to \mathbb{A}$ holds*

$$\overleftarrow{F} \cong \overleftarrow{T} \cong \overleftarrow{G} \quad \Longrightarrow \quad \mathbb{B}^{\overrightarrow{F}} \simeq \mathbb{D}^{\overrightarrow{G}} \tag{80}$$

where $\overleftarrow{F} = F^* F_*$, $\overrightarrow{F} = F^* F_*$, $\overleftarrow{G} = G_* G^*$ and $\overrightarrow{G} = G^* G_*$. The equivalences are natural with respect to the monad morphisms. Comonads satisfy the dual claim.

Proof. By Corollary 7.5, the comonads $\overset{\Longrightarrow}{F}$ and $\overset{\Longrightarrow}{G}$ on the category $\mathbb{A}^{\overleftarrow{F}} \simeq \mathbb{A}^{\overleftarrow{T}} \simeq \mathbb{A}^{\overleftarrow{G}}$ do not depend on the particular resolutions $F^* \dashv F_*$ and $G^* \dashv G_*$, but depend only on the monad $\overleftarrow{F} \cong \overleftarrow{T} \cong \overleftarrow{G}$, and must be in the form $\overset{\Longrightarrow}{F} \cong \overset{\Longrightarrow}{G} \cong \overset{\Longrightarrow}{T} = \left(\mathbb{A}^{\overleftarrow{T}} \xrightarrow{U_*} \mathbb{A} \xrightarrow{U^*} \mathbb{A}^{\overleftarrow{T}}\right)$. Hence

$$\mathbb{B}^{\overrightarrow{F}} \simeq \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \simeq \left(\mathbb{A}^{\overleftarrow{T}}\right)^{\overset{\Longrightarrow}{T}} \simeq \left(\mathbb{A}^{\overleftarrow{G}}\right)^{\overset{\Longrightarrow}{G}} \simeq \mathbb{C}^{\overrightarrow{G}}$$

where Prop. 7.4 is used at the first and at the last step, and Corollary 7.5 in the middle. □

Corollary 7.7 *For any adjunction $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$, monad $\overleftarrow{F}$, and comonad $\overrightarrow{F}$ holds*

$$\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \simeq \left(\mathbb{A}_{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \qquad\qquad \left(\mathbb{B}^{\overrightarrow{F}}\right)^{\overset{\Longleftarrow}{F}} \simeq \left(\mathbb{B}_{\overrightarrow{F}}\right)^{\overset{\Longleftarrow}{F}}$$

where $\mathbb{A}_{\overleftarrow{F}}$ is the (Kleisli-)category of free $\overleftarrow{F}$-algebras, $\mathbb{A}^{\overleftarrow{F}}$ is the (Eilenberg-Moore-)category of all $\overleftarrow{F}$-algebras, and similarly $\mathbb{B}_{\overrightarrow{F}}$ and $\mathbb{B}^{\overrightarrow{F}}$. These equivalences are natural, and thus induce

$$\mathsf{EC} \circ \mathsf{AC} \circ \mathsf{KM} \cong \mathsf{EC} \circ \mathsf{AC} \circ \mathsf{EM} \tag{81}$$

$$\mathsf{EM} \circ \mathsf{AM} \circ \mathsf{KC} \cong \mathsf{EM} \circ \mathsf{AM} \circ \mathsf{EC} \tag{82}$$

Proof. The claims are special cases of Corollary 7.6, obtained by taking pairs of resolutions considered there to be the initial resolution, into free algebras (or cofree coalgebras), and the final resolution, into all algebras (resp. coalgebras). □

Corollary 7.8 *The idempotent monads $\overleftarrow{\mathsf{EM}} = \mathsf{EM} \circ \mathsf{AM}$ and $\overleftarrow{\mathsf{EC}} = \mathsf{EC} \circ \mathsf{AC}$ on Adj distribute over one another, and*

$$\overleftarrow{\mathsf{EM}} \circ \overleftarrow{\mathsf{EC}} \cong \overleftarrow{\mathfrak{N}} \cong \overleftarrow{\mathsf{EC}} \circ \overleftarrow{\mathsf{EM}} \tag{83}$$

Proof. The distributivity law is displayed in Fig. 20. The comonad on $\mathbb{A}^{\overleftarrow{F}}$ and the monad on $\mathbb{B}^{\overrightarrow{F}}$ are not displayed, since they have just been spelled out in Corollary 7.5. The isomorphisms claimed in (83) follow from the fact that they coincide. □

Remark. Fig. 20 internalizes in Adj the commutative square of the nucleus schema in Fig. 3.

Proof of Thm. 6. The monads $\overleftarrow{\mathfrak{M}}$ and $\overleftarrow{\mathfrak{E}}$ are in fact retracts of the monad $\overleftarrow{\mathfrak{N}}$ from Adj to Mnd and to Cmn, respectively:

$$\begin{aligned}
\overleftarrow{\mathfrak{M}} &= \mathsf{AM} \circ \mathsf{EC} \circ \mathsf{AC} \circ \mathsf{KM} \\
&\overset{(81)}{\cong} \mathsf{AM} \circ \mathsf{EC} \circ \mathsf{AC} \circ \mathsf{EM} \\
&\overset{(\dagger)}{\cong} \mathsf{AM} \circ \mathsf{EC} \circ \mathsf{AC} \circ \mathsf{EM} \circ \mathsf{AM} \circ \mathsf{EM} \\
&= \mathsf{AM} \circ \overleftarrow{\mathsf{EC}} \circ \overleftarrow{\mathsf{EM}} \circ \mathsf{EM} \\
&\overset{(83)}{\cong} \mathsf{AM} \circ \overleftarrow{\mathfrak{N}} \circ \mathsf{EM}
\end{aligned}
\qquad\qquad
\begin{aligned}
\overleftarrow{\mathfrak{E}} &= \mathsf{AC} \circ \mathsf{EM} \circ \mathsf{AM} \circ \mathsf{KC} \\
&\overset{(82)}{\cong} \mathsf{AC} \circ \mathsf{EM} \circ \mathsf{AM} \circ \mathsf{EC} \\
&\overset{(\dagger)}{\cong} \mathsf{AC} \circ \mathsf{EM} \circ \mathsf{AM} \circ \mathsf{EC} \circ \mathsf{AC} \circ \mathsf{EC} \\
&= \mathsf{AC} \circ \overleftarrow{\mathsf{EM}} \circ \overleftarrow{\mathsf{EC}} \circ \mathsf{EC} \\
&\overset{(83)}{\cong} \mathsf{AC} \circ \overleftarrow{\mathfrak{N}} \circ \mathsf{EC}
\end{aligned}$$

At step (†), we use the fact that the monads $\overleftarrow{\mathsf{EM}} = \mathsf{EM} \circ \mathsf{AM}$ and $\overleftarrow{\mathsf{EC}} = \mathsf{EC} \circ \mathsf{AC}$ are idempotent. The natural isomorphisms $\overleftarrow{\mathfrak{M}} \overset{\eta}{\cong} \overleftarrow{\mathfrak{M}} \circ \overleftarrow{\mathfrak{M}}$ and $\overleftarrow{\mathfrak{E}} \overset{\eta}{\cong} \overleftarrow{\mathfrak{E}} \circ \overleftarrow{\mathfrak{E}}$ are derived from $\overleftarrow{\mathfrak{N}} \overset{\eta}{\cong} \overleftarrow{\mathfrak{N}} \circ \overleftarrow{\mathfrak{N}}$, by $\overleftarrow{\mathsf{EM}} \overset{\eta}{\cong} \overleftarrow{\mathsf{EM}} \circ \overleftarrow{\mathsf{EM}}$ or $\overleftarrow{\mathsf{EC}} \overset{\eta}{\cong} \overleftarrow{\mathsf{EC}} \circ \overleftarrow{\mathsf{EC}}$, and retracting into Mnd or Cmn, respectively. The equivalences $\mathsf{Mnd}^{\overleftarrow{\mathfrak{M}}} \simeq \mathsf{Adj}^{\overleftarrow{\mathfrak{N}}} \simeq \mathsf{Cmn}^{\overleftarrow{\mathfrak{E}}}$ arise from these derivations. The fact that $\mathsf{Adj}^{\overleftarrow{\mathfrak{N}}}$ is equivalent with the category Nuc, defined in (68), and used in (66), follows from Corollary 7.8. □

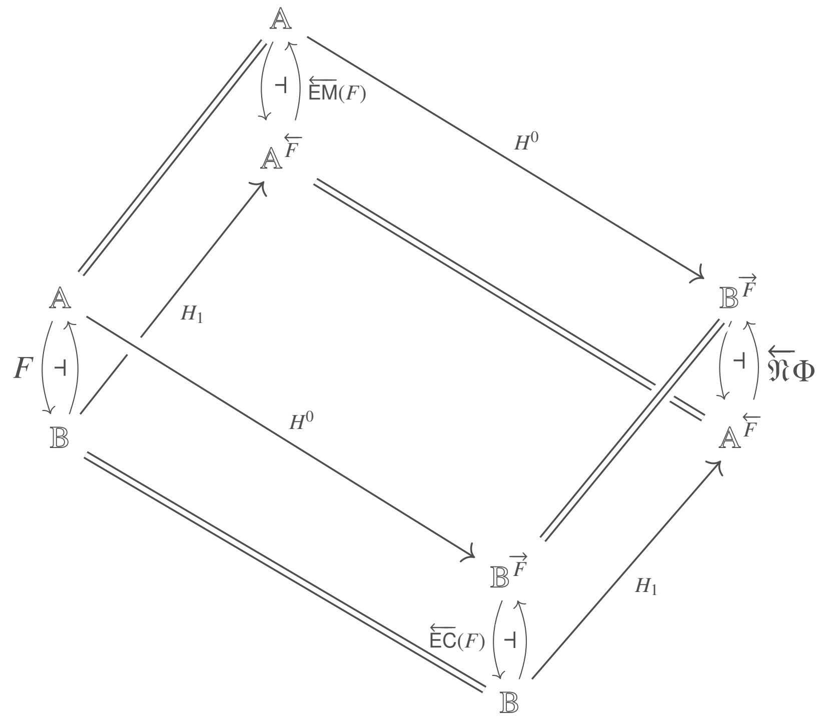

Figure 20: The nucleus construction $\overleftarrow{\mathfrak{N}}$ factorized into $\overleftarrow{\mathsf{EM}} \circ \overleftarrow{\mathsf{EC}} \cong \overleftarrow{\mathsf{EC}} \circ \overleftarrow{\mathsf{EM}}$

8 Simple nucleus

The main idea of monads and comonads is that they capture algebra and coalgebra. For any monad $\overleftarrow{F} : \mathbb{A} \to \mathbb{A}$, the categories $\mathbb{A}^{\overleftarrow{F}}$ of all $\overleftarrow{F}$-algebras and $\mathbb{A}_{\overleftarrow{F}}$ of free $\overleftarrow{F}$-algebras frame all analyses, since all resolutions lie in-between them [14, 27, 49]. Corollary 7.6 says that all these resolutions induce equivalent categories of coalgebras, which lie in-between the categories $\left(\mathbb{A}^{\overleftarrow{F}}\right)_{\overrightarrow{\overrightarrow{F}}}$ and $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overrightarrow{\overrightarrow{F}}}$. So it also makes sense to talk about the coalgebras for a monad, and analogously about the algebras for a comonad. But the categories $\left(\mathbb{A}^{\overleftarrow{F}}\right)_{\overrightarrow{\overrightarrow{F}}}$ and $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overrightarrow{\overrightarrow{F}}}$ of coalgebras over algebras are built in two layers of structure, one on top of the other. When their interactions are discharged and the reduncancies removed, the composite structure turns out to be simpler than either of the components. This was spelled out in [75], with an eye on applications. The nucleus construction explains the simplicity of this structure, and uses it as a pivot point for further developments.

Proposition 8.1 *Given an adjunction $F = (F^* \dashv F_* : \mathbb{B} \to \mathbb{A})$, consider the categories*

$$
|\mathbb{A}^{\overrightarrow{F}}| \;=\; \coprod_{x\in|\mathbb{A}|} \left\{ \alpha_x \in \mathbb{B}(F^*x, F^*x) \;\middle|\; \begin{tikzcd} F^*x \arrow[d, "\alpha_x"'] \arrow[dr, "\alpha_x"] & F_*F^*x \arrow[r, two heads] \arrow[dr, "F_*\alpha_x"] & x \arrow[d, tail, "\tilde{\alpha}_x"] \\ F^*x \arrow[r, "\alpha_x"'] & F^*x & F_*F^*x \end{tikzcd} \right\} \tag{84}
$$

$$
\mathbb{A}^{\overrightarrow{F}}(\alpha_x, \gamma_z) \;=\; \left\{ f \in \mathbb{A}(x,z) \;\middle|\; \begin{tikzcd} F^*x \arrow[r, "F^*f"] \arrow[d, "\alpha_x"'] & F^*z \arrow[d, "\gamma_z"] \\ F^*x \arrow[r, "F^*f"'] & F^*z \end{tikzcd} \right\}
$$

$$
|\mathbb{B}^{\overleftarrow{F}}| \;=\; \coprod_{u\in|\mathbb{B}|} \left\{ \beta^u \in \mathbb{A}(F_*u, F_*u) \;\middle|\; \begin{tikzcd} F_*x \arrow[d, "\beta^u"'] \arrow[dr, "\beta^u"] & F^*F_*u \arrow[r, two heads, "\tilde{\beta}^u"] \arrow[dr, "F^*\beta^u"] & u \arrow[d] \\ F_*u \arrow[r, "\beta^u"'] & F_*u & F^*F_*u \end{tikzcd} \right\} \tag{85}
$$

$$
\mathbb{B}^{\overleftarrow{F}}(\beta^u, \delta^w) \;=\; \left\{ g \in \mathbb{B}(u,w) \;\middle|\; \begin{tikzcd} F_*u \arrow[r, "F_*g"] \arrow[d, "\beta^u"'] & F_*w \arrow[d, "\delta^w"] \\ F_*u \arrow[r, "F_*g"'] & F_*w \end{tikzcd} \right\}
$$

where $x \xrightarrow{\tilde{\alpha}_x} F_*F^*x$ is the transpose of $F^*x \xrightarrow{\alpha_x} F^*x$, and $F^*F_*u \xrightarrow{\tilde{\beta}^u} u$ is the transpose of $F_*u \xrightarrow{\beta} F_*u$. The adjunction $F^\natural \dashv F_\natural : \mathbb{B}^{\overleftarrow{F}} \to \mathbb{A}^{\overrightarrow{F}}$ defined in Fig. 21 with the comparison functors

$$
\begin{array}{rcl}
K^0 \;:\; \mathbb{A} & \longrightarrow & \mathbb{A}^{\overrightarrow{F}} \\[1em]
x & \longmapsto & \left\langle F_*F^*x, \; \begin{array}{c} F^*F_*F^*x \\ \varepsilon F^* \big\downarrow \\ F^*x \\ F^*\eta \big\downarrow \\ F^*F_*F^*x \end{array} \right\rangle
\end{array}
\qquad\qquad
\begin{array}{rcl}
K_1 \;:\; \mathbb{B} & \longrightarrow & \mathbb{B}^{\overleftarrow{F}} \\[1em]
u & \longmapsto & \left\langle F^*F_*u, \; \begin{array}{c} F_*F^*F_*u \\ F_*\varepsilon \big\downarrow \\ F_*u \\ \eta F_* \big\downarrow \\ F_*F^*F_*u \end{array} \right\rangle
\end{array}
$$

is equivalent to the nucleus, i.e.

$$
\overleftarrow{\mathfrak{N}}\,(F^* \dashv F_*) \;\cong\; \left(F^\natural \dashv F_\natural\right)
$$

Lemma 8.2 *If $\varphi \circ \varphi = \varphi = m \circ e$, where m is a monic and e is an epi, then $e \circ m = \mathrm{id}$.*

Proof. $m \circ e \circ m \circ e = \varphi \circ \varphi = \varphi = m \circ e$ implies $e \circ m \circ e = e$ because m is a monic, and $e \circ m = \mathrm{id}$ because e is epi. □

$$
\begin{tikzcd}[row sep=huge]
\mathbb{A} \arrow[loop above, "\overleftarrow{F}"] \arrow[rrr, dashed, "K^0"] \arrow[d, bend right, "F^*"'] \arrow[d, phantom, "\dashv"] & & \langle x, \alpha_x \rangle \arrow[d, mapsto] & \mathbb{A}^{\overset{\Longrightarrow}{F}} \arrow[d, bend right, "F^\natural"'] \arrow[d, phantom, "\dashv"] & \langle F_*u, F^*\beta^u \rangle \\
\mathbb{B} \arrow[loop below, "\overrightarrow{F}"] \arrow[u, bend right, "F_*"'] \arrow[rrr, dashed, "K_1"'] & & \langle F^*x, F_*\alpha_x \rangle & \mathbb{B}^{\overset{\Longleftarrow}{F}} \arrow[u, bend right, "F_\natural"'] & \langle u, \beta^u \rangle \arrow[u, mapsto]
\end{tikzcd}
$$

Figure 21: The simple nucleus $F^\natural \dashv F_\natural$ of $F^* \dashv F_*$

Lemma 8.3 *If* $\left(F_*F^*x \xrightarrow{F_*\alpha_x} F_*F^*x\right) = \left(F_*F^*x \overset{\alpha^x}{\twoheadrightarrow} x \overset{\tilde{\alpha}_x}{\rightarrowtail} F_*F^*x\right)$*, where* $\tilde{\alpha}_x = F_*\alpha_x \circ \eta_x$ *is a monic and* α^x *is an epi, then* $\alpha^x \circ \eta_x = \mathrm{id}$*.*

Proof. $\tilde{\alpha}_x \circ \alpha^x = F_*\alpha_x \circ \eta_x \circ \alpha^x = \tilde{\alpha}_x \circ \alpha^x \circ \eta_x \circ \alpha^x$ implies $\alpha^x = \alpha^x \circ \eta_x \circ \alpha^x$ because $\tilde{\alpha}_x$ is a monic, and $\mathrm{id} = \alpha^x \circ \eta_x$ because α^x is epi. □

Proof of Prop. 8.1. Still writing $F_*F^*x \overset{\alpha^x}{\twoheadrightarrow} x \overset{\tilde{\alpha}_x}{\rightarrowtail} F_*F^*x$ for the decomposition of $F_*F^*x \xrightarrow{F_*\alpha_x} F_*F^*x$, we have

$$
\alpha^x \circ \tilde{\alpha}_x = \mathrm{id}_x \qquad \text{and} \qquad \alpha^x \circ \eta_x = \mathrm{id}_x \tag{86}
$$

from Lemma 8.2 and Lemma 8.3, respectively. Similar lemmata lead to the equations

$$
\begin{aligned}
\alpha^x \circ F_*F^*\alpha^x &= \alpha^x \circ F_*\varepsilon_{F_*x} \\
\tilde{\alpha}_x \circ \alpha^x &= F_*\varepsilon_{F_*x} \circ F_*F^*\tilde{\alpha}_x
\end{aligned}
$$

which, together with (86), say that $F_*F^*x \xrightarrow{\alpha^x} x$ is an algebra in $\mathbb{A}^{\overleftarrow{F}}$ and that $\tilde{\alpha}_x \in \mathbb{A}^{\overleftarrow{F}}(\alpha^x, \mu_x)$ is an algebra homomorphism, and in fact a coalgebra over α^x in $\left(\mathbb{A}^{\overrightarrow{F}}\right)^{\overset{\Longleftarrow}{F}}$. Hence the functor from $\mathbb{A}^{\overset{\Longrightarrow}{F}}$ to $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}}$, which turns out to be an equivalence upon straightforward checks. A similar argument leads to a similar functor from $\mathbb{B}^{\overset{\Longleftarrow}{F}}$ to $\left(\mathbb{B}^{\overrightarrow{F}}\right)^{\overset{\Longleftarrow}{F}}$. Hence the equivalences

$$
\mathbb{A}^{\overset{\Longrightarrow}{F}} \simeq \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \qquad\qquad \mathbb{B}^{\overset{\Longleftarrow}{F}} \simeq \left(\mathbb{B}^{\overrightarrow{F}}\right)^{\overset{\Longleftarrow}{F}}
$$

On the other hand, the equivalences

$$\mathbb{A}^{\overset{\rightrightarrows}{F}} \simeq \mathbb{B}^{\overrightarrow{F}} \qquad\qquad\qquad \mathbb{B}^{\overset{\leftleftarrows}{F}} \simeq \mathbb{A}^{\overleftarrow{F}}$$

are spelled out and verified in [75]. Every object $\langle x, F^*x \xrightarrow{\alpha_x} F^*x\rangle \in \mathbb{A}^{\overset{\rightrightarrows}{F}}$ is shown to be isomorphic to one in the form $\langle F_*y, F^*F_* \xrightarrow{\varepsilon} y \xrightarrow{\beta} F^*F_*y\rangle$ for some $y \in \mathbb{B}$ and a coalgebra $\beta \in \mathbb{B}^{\overrightarrow{F}}$. It follows that both squares in the following diagram commute

$$\begin{tikzcd}
F^*x \arrow[rrr, "F^*\eta"] \arrow[dddd, "\alpha_x"'] \arrow[dr, leftrightarrow, "F^*\iota"] & & & F^*F_*F^*x \arrow[dddd, "\alpha_x"] \arrow[dl, leftrightarrow, "F^*F_*F^*\iota"'] \\
& F^*F_*y \arrow[r, "F^*\eta F_*"] \arrow[d, two heads, "\varepsilon"] & F^*F_*F^*F_*y \arrow[d, two heads, "F^*F_*\varepsilon"] & \\
& y \arrow[r, tail, "\beta"] \arrow[d, tail, "\beta"] & F^*F_*y \arrow[d, "F^*F_*\beta"] & \\
& F^*F_*y \arrow[r, "F^*\eta F_*"] \arrow[dl, leftrightarrow, "F^*\iota"'] & F^*F_*F^*F_*y \arrow[dr, leftrightarrow, "F^*F_*F^*\iota"] & \\
F^*x \arrow[rrr, "F^*\eta"] & & & F^*F_*F^*x
\end{tikzcd} \tag{87}$$

for $x \xleftrightarrow{\iota} F_*y$ an isomorphism in $\mathbb{A}$. Transferring the nuclear adjunction $F^\sharp \dashv F_\sharp : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{B}^{\overrightarrow{F}}$ along the equivalences yields the nuclear adjunction $F^\natural \dashv F_\natural : \mathbb{B}^{\overset{\leftleftarrows}{F}} \to \mathbb{A}^{\overset{\rightrightarrows}{F}}$, with the natural correspondence

$$\begin{array}{rcl}
\mathbb{B}^{\overset{\leftleftarrows}{F}}(F^\natural\alpha_x, \beta^u) & \cong & \mathbb{A}^{\overset{\rightrightarrows}{F}}(\alpha_x, F_\natural\beta^u) \\
\left(F^*x \xrightarrow{f} u\right) & \mapsto & \tilde{f} = \left(x \xrightarrow{\eta} F_*F^*x \xrightarrow{F_*f} F_*u\right)
\end{array}$$

The adjunction correspondence $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$ lifts to $F^\natural \dashv F_\natural : \mathbb{B}^{\overset{\leftleftarrows}{F}} \to \mathbb{A}^{\overset{\rightrightarrows}{F}}$ because each of the following squares commutes if and only if the other one does:

$$\begin{tikzcd}
F_*F^*x \arrow[r, "F_*f"] \arrow[d, "F_*\alpha_x"'] & F_*u \arrow[d, "\beta^u"] \\
F_*F^*x \arrow[r, "F_*f"'] & F_*u
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
F^*x \arrow[r, "F^*(\tilde{f})"] \arrow[d, "\alpha_x"'] & F^*F_*u \arrow[d, "F^*\beta^u"] \\
F^*x \arrow[r, "F^*(\tilde{f})"'] & F^*F_*u
\end{tikzcd} \tag{88}$$

Suppose that the left-hand side square commutes. To see that the right-hand side square commutes as well, take its F^*-image and precompose it with the outer square from (87), as in the following diagram.

$$\begin{tikzcd}
F^*x \arrow[r, tail, "F^*\eta"] \arrow[rr, bend left, "F^*(\tilde{f})"] \arrow[d, "\alpha_x"] & F^*F_*F^*x \arrow[r, "F^*F_*f"'] \arrow[d, "F^*F_*\alpha_x"] & F^*F_*u \arrow[d, "F^*\beta^u"] \\
F^*x \arrow[r, tail, "F^*\eta"'] \arrow[rr, bend right, "F^*(\tilde{f})"'] & F^*F_*F^*x \arrow[r, "F^*F_*f"] & F^*F_*u
\end{tikzcd} \tag{89}$$

The two outer paths around this diagram are the paths around right-hand square in (88). The implication is analogous. $\square$

Remarks. The constructions $\mathbb{A}^{\overset{\rightrightarrows}{F}}$ and $\mathbb{B}^{\overset{\leftleftarrows}{F}}$ are given with respect to an adjunction $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$, rather than just a monad $\overleftarrow{F}$ or just a comonad $\overrightarrow{F}$. The constructions for a monad or a comonad alone can be extrapolated by applying the above constructions to their Kleisli or Eilenberg-Moore resolutions. Corollary 7.6 says that all resolutions lead to equivalent categories. The Kleisli resolution gives a smaller object class, but that is not always an advantage. Some adjunctions give simpler simple nuclei than other adjunctions. The objects of the category $\mathbb{A}^{\overset{\rightrightarrows}{F}}$ built over the Eilenberg-Moore resolution of a monad $\overleftarrow{F}$ turn out to be the projective $\overleftarrow{F}$-algebras, but the morphisms are not just the $\overleftarrow{F}$-algebra homomorphisms, but also the homomorphisms of $\overset{\rightrightarrows}{F}$-coalgebras. The objects can be viewed as triples in the form $\langle x, \alpha^x, \tilde{\alpha}_x \rangle$ which make the following diagrams commute.

$$\begin{tikzcd}
x \arrow[r, shift left, "\tilde{\alpha}_x"] \arrow[r, shift right, "\eta"'] \arrow[dr, "\mathrm{id}"'] & \overleftarrow{F}x \arrow[d, "\alpha^x"] \\
& x
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\overleftarrow{F}\overleftarrow{F}x \arrow[r, "\overleftarrow{F}\alpha^x"] \arrow[d, "\mu"'] & \overleftarrow{F}x \arrow[r, "\overleftarrow{F}\tilde{\alpha}_x"] \arrow[d, "\alpha^x"] & \overleftarrow{F}\overleftarrow{F}x \arrow[d, "\mu"] \\
\overleftarrow{F}x \arrow[r, "\alpha^x"] & x \arrow[r, "\tilde{\alpha}_x"] & \overleftarrow{F}x
\end{tikzcd} \tag{90}$$

Here we do not display just (84) instantiated to $U^* \dashv U_* : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{A}$, but also the data that are implied: the middle filling in the rectangle on the right must be α^x because $\overleftarrow{F}\eta$ is the splitting of both $\overleftarrow{F}\alpha^x$ and μ. This makes it clear that α^x is an $\overleftarrow{F}$-algebra, whereas $\tilde{\alpha}_x$ is an algebra homomorphism that embeds it as a subalgebra of the free $\overleftarrow{F}$-algebra μ. So α^x is a projective algebra. On the other hand, $\tilde{\alpha}_x$ is also an $\overset{\rightrightarrows}{F}$-coalgebra structure over the $\overleftarrow{F}$-algebra α^x. An $\mathbb{A}^{\overset{\rightrightarrows}{F}}$-morphism from

$\langle x, \alpha^x, \tilde{\alpha}_x\rangle$ to $\langle z, \gamma^x, \tilde{\gamma}_x\rangle$ is an arrow $f \in \mathbb{A}(x,z)$ that makes the following diagram commute.

$$
\begin{tikzcd}
\overleftarrow{F}x \arrow[r, "\alpha^x"] \arrow[d, "\overleftarrow{F}f"'] & x \arrow[r, "\tilde{\alpha}_x"] \arrow[d, "f"] & \overleftarrow{F}x \arrow[d, "\overleftarrow{F}f"] \\
\overleftarrow{F}z \arrow[r, "\gamma^z"] & z \arrow[r, "\tilde{\gamma}_z"] & \overleftarrow{F}z
\end{tikzcd}
\tag{91}
$$

The left-hand square says that f is an $\overleftarrow{F}$-algebra homomorphism. The right-hand square says that it is also an $\overrightarrow{\overrightarrow{F}}$-coalgebra homomorphism. So we are not looking at a category of projective algebras in $\mathbb{A}^{\overleftarrow{F}}$, but at a category of $\overrightarrow{\overrightarrow{F}}$-coalgebras over it, which turns out to be equivalent to $\mathbb{B}^{\overrightarrow{F}}$, as Prop. 7.4 established. The conundrum that $\overrightarrow{\overrightarrow{F}}$-coalgebras boil down to projective $\overleftarrow{F}$-algebras, but that the $\overrightarrow{\overrightarrow{F}}$-coalgebra homomorphisms satisfy just two out of three conditions required from the $\overleftarrow{F}$-algebra homomorphisms was discussed and used in [75].

Corollary 8.4 *If a given adjunction $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$ is nuclear, then*

a) every object $x \in \mathbb{A}$ is a retract of F_*F^*x, and thus of an image along F_*;

b) every object $u \in \mathbb{B}$ is a retract of F^*F_*x, and thus of an image along F^*.

Proof. By Thm. 6, $F = (F^* \dashv F_*)$ is nuclear if and only if $\overleftarrow{\mathfrak{N}}(F) \cong F$. By Prop. 8.1, $\mathsf{Nuc}(F) \cong \left(F^\natural \dashv F_\natural\right)$. The claim thus boils down to proving that *(a)* every $\alpha_x \in \mathbb{A}^{\overrightarrow{\overrightarrow{F}}}$ is a retract of an $F_\natural F^\natural \alpha_x$, and *(b)* every $\beta_u \in \mathbb{B}^{\overleftarrow{\overleftarrow{F}}}$ is a retract of an $F^\natural F_\natural \beta_u$. The following derivation establishes *(a)*. Case *(b)* is analogous.

$$
\begin{tikzcd}
\Big\langle\, x\,, & F^*x \arrow[r, "\alpha_x"] & F^*x \,\Big\rangle
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Big\langle F^*x\,, \arrow[d, "\alpha_x"'] & F_*F^*x \arrow[r, "F_*\alpha_x"] \arrow[d, "F_*\alpha_x"'] & F_*F^*x\Big\rangle \arrow[d, "F_*\alpha_x"] \\
\Big\langle F^*x\,, & F_*F^*x \arrow[r, "F_*\alpha_x"] & F_*F^*x\Big\rangle
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Big\langle F_*F^*x\,, \arrow[d, two heads, "\alpha^x"'] & F^*F_*F^*x \arrow[r, "F^*F_*\alpha_x"] \arrow[d, two heads, "F^*\alpha^x"'] & F^*F_*F^*x\Big\rangle \arrow[d, "F^*\alpha^x"] \\
\Big\langle\, x\,, \arrow[d, tail, "\tilde{\alpha}_x"'] & F^*x \arrow[r, "\alpha_x"] \arrow[d, tail, "F^*\tilde{\alpha}_x"'] & F^*x\,\Big\rangle \arrow[d, tail, "F^*\tilde{\alpha}_x"] \\
\Big\langle F_*F^*x\,, & F^*F_*F^*x \arrow[r, "F^*F_*\alpha_x"] & F^*F_*F^*x\Big\rangle
\end{tikzcd}
$$

$\square$

Discussion. We know that an $\overleftarrow{F}$-algebra structure $\overleftarrow{F}x \overset{\alpha}{\twoheadrightarrow} X$ makes the $\overleftarrow{F}$-algebra α into a quotient of the free algebra $\overleftarrow{F}\overleftarrow{F}x \overset{\mu}{\twoheadrightarrow} \overleftarrow{F}x$, but that the epimorphism $\overleftarrow{F}x \overset{\alpha}{\twoheadrightarrow} X$ only splits by the unit $x \overset{\eta}{\rightarrowtail} \overleftarrow{F}x$ when projected down into $\mathbb{A}$ by the forgetful functor $U_* : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{A}$, and generally not in the category of algebras $\mathbb{A}^{\overleftarrow{F}}$ itself. In other words, the $\overleftarrow{F}$-algebra homomorphism $\alpha \in \mathbb{A}^{\overleftarrow{F}}(\mu, \alpha)$ induces a retraction $U_*\alpha \in \mathbb{A}(\overleftarrow{F}x, x)$, with $\eta \in \mathbb{A}(\overleftarrow{F}x, x)$ as its inverse *only within* $\mathbb{A}$, but η is not an $\overleftarrow{F}$-algebra homomorphism, and the splitting does not live in $\mathbb{A}^{\overleftarrow{F}}$. This is where concept of reflecting the U_*-split coequalizers in Beck's characterization of monadicity of U_* comes from. It is not hard to show that η is an $\overleftarrow{F}$-algebra homomorphism only when it is an isomorphism, which makes α into a free $\overleftarrow{F}$-algebra. More generally, when the algebra homomorphism $\alpha \in \mathbb{A}^{\overleftarrow{F}}(\mu, \alpha)$ has a splitting $\tilde{\alpha} \in \mathbb{A}^{\overleftarrow{F}}(\alpha, \mu)$, with the underlying map that may be different from $\eta \in \mathbb{A}(x, \overleftarrow{F}x)$, then the algebra α is projective. This thread was pursued in [75].

It may seem curious that Corollary 8.4 now says that x is *always* a retract of $\overleftarrow{F}x$ in $\mathbb{A}^{\overset{\rightrightarrows}{F}}$. More precisely, any $\overleftarrow{F}$-algebra $\overleftarrow{F}x \xrightarrow{\alpha^x} x$ with which x may appear in $\mathbb{A}^{\overset{\rightrightarrows}{F}}$, is a retraction, and has a splitting $x \xrightarrow{\tilde{\alpha}_x} \overleftarrow{F}x$. Does this not say that all $\overleftarrow{F}$-algebras are projective? The following explains that *it does not*.

Remember that the category $\mathbb{A}^{\overset{\rightrightarrows}{F}}$ is the simple form of the category $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\rightrightarrows}{F}}$, and that it is equivalent with the category of $\overrightarrow{F}$-coalgebras $\mathbb{B}^{\overrightarrow{F}}$, and certainly *not* with the category of $\overleftarrow{F}$-algebras $\mathbb{A}^{\overleftarrow{F}}$. In the category of $\overset{\rightrightarrows}{F}$-coalgebras over $\overleftarrow{F}$-algebras, an object $\tilde{\alpha}_x \in \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\rightrightarrows}{F}}$, like any coalgebra, comes with the coalgebra monic $\tilde{\alpha}_x \in \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\rightrightarrows}{F}}(\tilde{\alpha}_x, \nu)$ into the cofree coalgebra ν. This monic generally does not split in $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\rightrightarrows}{F}}$, but the forgetful functor $V^* : \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\rightrightarrows}{F}} \to \mathbb{A}^{\overleftarrow{F}}$ maps it into a split monic, and its splitting in $\mathbb{A}^{\overleftarrow{F}}$ is the comonad counit $\overset{\rightrightarrows}{F}\alpha^x \to \alpha^x$. The underlying map of this counit is the structure map $\overleftarrow{F}x \overset{\alpha^x}{\twoheadrightarrow} x$ in $\mathbb{A}$. The underlying map of the $\overset{\rightrightarrows}{F}$-coalgebra $\tilde{\alpha}_x$ has the form $x \overset{\tilde{\alpha}_x}{\rightarrowtail} \overleftarrow{F}x \overset{\alpha^x}{\twoheadrightarrow} x$, and the fact that it is a V^*-split equalizer means that $\alpha^x \circ \tilde{\alpha}_x$ holds in $\mathbb{A}$. Just as the forgetful functor $\mathbb{B}^{\overrightarrow{F}} \to B$ makes the $\overrightarrow{F}$-coalgebra embeddings into split equalizers in $\mathbb{B}$, the forgetful functor $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\rightrightarrows}{F}} \to \mathbb{A}^{\overleftarrow{F}}$ makes the $\overset{\rightrightarrows}{F}$-coalgebra embeddings into split equalizers in $\mathbb{A}^{\overleftarrow{F}}$. But there, the split equalizers display some $\overleftarrow{F}$-algebras as retracts of free $\overleftarrow{F}$-algebras. The equivalence $\left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\rightrightarrows}{F}} \simeq \mathbb{B}^{\overrightarrow{F}}$ thus presents $\overrightarrow{F}$-coalgebras as projective $\overleftarrow{F}$-algebras. Corollary 8.4 therefore does not say that all $\overleftarrow{F}$-algebras are projective, but that all $\overrightarrow{F}$-coalgebras can be presented by some projective $\overleftarrow{F}$-algebras. This was the pivot point of [75].

Note, however, that this representation does not imply that the $\overrightarrow{F}$-coalgebra category $\mathbb{B}^{\overrightarrow{F}}$ is equivalent with the category of projective $\overleftarrow{F}$-algebras, viewed, e.g., as a subcategory of $\mathbb{A}^{\overleftarrow{F}}$. It is not, because the $\overleftarrow{F}$-algebra morphisms between projective algebras are strictly more constrained than the $\overset{\Longrightarrow}{F}$-coalgebra homomorphisms between the same projective algebras. This was explained in [75].

The coequalizers that become split when projected along the forgetful functor from algebras are crux of Beck's Monadicity Theorem [15, 14, Sec. 3.3]. The equalizers that split along the forgetful functor from coalgebras play the analogous role in the dual theorem, characterizing comonadicity. The fact that such a peculiar structure is so prominent in such fundamental theorems has been a source of wonder and mystery. In his seminal early work [65, 66], Paré explained it as an avatar of a fundamental phenomenon: of reflecting absolute colimits into coeqalizers (in the case of monadic functors), or of absolute limits into equalizers (in the case of comonadic functors). In the framework of simple nuclei, such reflections are finally assigned the role of first-class citizens that they deserve, and made available for categorical concept analysis.

9 Little nucleus

We define the *little nucleus* to be the initial (Kleisli) resolution $\overrightarrow{\mathfrak{N}}F$ of the (big) nucleus $\overleftarrow{\mathfrak{N}}F$ of an adjunction $F = (F^* \dashv F_*)$. The little nucleus of a monad $\overleftarrow{F}$ (and of a comonad $\overrightarrow{F}$) will be the monad $\overrightarrow{\mathfrak{C}}\overleftarrow{F}$ (resp. the comonad $\overrightarrow{\mathfrak{M}}\overrightarrow{F}$) induced by the little nucleus of any of the resolutions of $\overleftarrow{F}$ (resp. of $\overrightarrow{F}$). The constructions $\overrightarrow{\mathfrak{C}}$ and $\overrightarrow{\mathfrak{M}}$ are the comonads on Mnd and Cmn, respectively, constructed in Fig. 14, displayed in the statement of Thm. 6.

We say that an adjunction $F^* \dashv F_* : \mathbb{B} \to \mathbb{A}$ is *subnuclear* if the categories can be reconstructed from each other as initial resolutions of the induced monad and comonad: $\mathbb{A}$ is equivalent to the Kleisli category $\mathbb{B}_{\overrightarrow{F}}$ for the comonad $\overrightarrow{F} = F^*F_* : \mathbb{B} \to \mathbb{B}$, and $\mathbb{B}$ is equivalent to the Kleisli category $\mathbb{A}_{\overleftarrow{F}}$ for the monad $\overleftarrow{F} = F_*F^* : \mathbb{A} \to \mathbb{A}$. More precisely, the comparison functors $\mathbb{B}_{\overrightarrow{F}} \xrightarrow{E_0} \mathbb{A}$ and $\mathbb{A}_{\overleftarrow{F}} \xrightarrow{E^1} \mathbb{B}$ are required to be equivalences. If the two Kleisli constructions are construed as essentially surjective / fully faithful factorizations

$$
\begin{aligned}
F^* &= \left(\mathbb{A} \overset{U^\flat}{\twoheadrightarrow} \mathbb{A}_{\overleftarrow{F}} \overset{E^1}{\rightarrowtail} \mathbb{B}\right)\\
F_* &= \left(\mathbb{B} \overset{V_\flat}{\twoheadrightarrow} \mathbb{B}_{\overrightarrow{F}} \overset{E_0}{\rightarrowtail} \mathbb{A}\right)
\end{aligned}
$$

(see Fig. 13), then the requirement that E^1 and E_0 are equivalences means that F^* and F_* in a subnuclear adjunction must be essentially surjective. But, as mentioned at the end of Sec. 4, while the adjunction between the Kleisli categories is subnuclear itself, its resolutions may not be. The upshot is that the little nucleus must be extracted from the big nucleus. The situation is summarized in Fig. 22. The little nucleus arises as the initial resolution

$$
\overrightarrow{\mathfrak{N}}\left(F^* \dashv F_* : \mathbb{B} \to \mathbb{A}\right) \;=\; \left(F^{\sharp\flat} \dashv F_{\sharp\flat} \;:\; \left(\mathbb{A}^{\overleftarrow{F}}\right)_{\overset{\Longrightarrow}{F}} \to \left(\mathbb{B}^{\overrightarrow{F}}\right)_{\overset{\Longleftarrow}{F}}\right)
$$

$$
\begin{tikzcd}
\mathbb{B}_{\overrightarrow{F}} \arrow[r, tail, dashed, "E_0"] \arrow[d, bend right, "F^\flat"'] \arrow[d, phantom, "\dashv"] \arrow[rr, tail, dashed, bend left] & \mathbb{A} \arrow[r, dashed, "H^0"] \arrow[d, bend right, "F^*"'] \arrow[d, phantom, "\dashv"] \arrow[dl, "U^\flat"] & \mathbb{B}^{\overrightarrow{F}} \arrow[d, bend right, "F^\sharp"'] \arrow[d, phantom, "\dashv"] \arrow[dl, "V^*"] \\
\mathbb{A}_{\overleftarrow{F}} \arrow[r, tail, dashed, "E^1"] \arrow[u, bend right, "F_\flat"'] \arrow[rr, tail, dashed, bend right] & \mathbb{B} \arrow[r, dashed, "H_1"] \arrow[u, bend right, "F_*"'] \arrow[ul, "V_\flat"] & \mathbb{A}^{\overleftarrow{F}} \arrow[u, bend right, "F_\sharp"'] \arrow[ul, "U_\sharp"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(\mathbb{A}^{\overleftarrow{F}}\right)_{\overset{\Longrightarrow}{F}} \arrow[r, tail, dashed, "E_0"] \arrow[d, bend right, "F^{\sharp\flat}"'] \arrow[d, phantom, "\dashv"] \arrow[rr, tail, dashed, bend left] & \mathbb{B}^{\overrightarrow{F}} \arrow[r, dashed, "\sim"] \arrow[d, bend right, "F^\sharp"'] \arrow[d, phantom, "\dashv"] \arrow[dl, "U^\flat"] & \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \arrow[d, bend right, "F^{\sharp\sharp}"'] \arrow[d, phantom, "\dashv"] \arrow[dl, "V^*"] \\
\left(\mathbb{B}^{\overrightarrow{F}}\right)_{\overset{\Longleftarrow}{F}} \arrow[r, tail, dashed, "E^1"] \arrow[u, bend right, "F_{\sharp\flat}"'] \arrow[rr, tail, dashed, bend right] & \mathbb{A}^{\overleftarrow{F}} \arrow[r, dashed, "\sim"] \arrow[u, bend right, "F_\sharp"'] \arrow[ul, "V_\flat"] & \left(\mathbb{B}^{\overrightarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \arrow[u, bend right, "F_{\sharp\sharp}"'] \arrow[ul, "U_\sharp"]
\end{tikzcd}
$$

Figure 22: The resolutions of an adjunction $F = (F^* \dashv F_*)$ and of its nucleus $\overleftarrow{\mathfrak{N}}F = (F^\sharp \dashv F_\sharp)$

of the (big) nucleus, which is the final resolution

$$
\overleftarrow{\mathfrak{N}}\left(F^* \dashv F_* : \mathbb{B} \to \mathbb{A}\right) \quad = \quad \left(F^\sharp \dashv F_\sharp : \mathbb{A}^{\overleftarrow{F}} \to \mathbb{B}^{\overrightarrow{F}}\right)
$$

Since Corollary 7.7 implies $\overleftrightarrow{\mathfrak{N}\mathfrak{N}}(F) \simeq \overleftarrow{\mathfrak{N}}\overleftarrow{\mathfrak{N}}(F)$, and Prop. 7.4 says that $\overleftarrow{\mathfrak{N}}$ is idempotent, tracking the equivalences through

$$\begin{tikzcd}
\overrightarrow{\mathfrak{N}}\overrightarrow{\mathfrak{N}}(F) \arrow[r, tail] \arrow[dd, "\simeq"] & \overleftarrow{\mathfrak{N}}\overrightarrow{\mathfrak{N}}(F) \arrow[d, dashed, "\simeq"] \\
& \overleftarrow{\mathfrak{N}}\overleftarrow{\mathfrak{N}}(F) \arrow[d, dashed, "\simeq"] \\
\overrightarrow{\mathfrak{N}}(F)b \arrow[r, tail] & \overleftarrow{\mathfrak{N}}(F)
\end{tikzcd} \tag{92}$$

yields a natural family of equivalences $\overrightarrow{\mathfrak{N}}\overrightarrow{\mathfrak{N}}(F) \simeq \overrightarrow{\mathfrak{N}}(F)$. But spelling out these equivalences, of categories of coalgebras over algebras and algebras over coalgebras, is an unwieldy task. The flood of structure can be dammed by reducing the (big) nucleus to the simple form from Sec. 8

$$\overleftarrow{\mathfrak{N}}\left(F^* \dashv F_* : \mathbb{B} \to \mathbb{A}\right) \;=\; \left(F^\sharp \dashv F_\sharp : \mathbb{B}^{\overset{\Leftarrow}{F}} \to \mathbb{A}^{\overset{\Rightarrow}{F}}\right)$$

and defining the little nucleus in the form

$$\overrightarrow{\mathfrak{N}}\left(F^* \dashv F_* : \mathbb{B} \to \mathbb{A}\right) \;=\; \left(F^\flat \dashv F_\flat : \mathbb{A}_{\overset{\Rightarrow}{F}} \to \mathbb{B}_{\overset{\Rightarrow}{F}}\right)$$

where the categories $\mathbb{A}_{\overset{\Rightarrow}{F}}$ and $\mathbb{B}_{\overset{\Leftarrow}{F}}$ are defined by the factorizations in Fig. 23. The category $\mathbb{B}_{\overset{\Leftarrow}{F}}$

$$\begin{tikzcd}
\mathbb{A}_{\overset{\Rightarrow}{F}} \arrow[rr, tail] \arrow[dd, bend right, "\overrightarrow{\mathfrak{N}}F"'] & & \mathbb{A}^{\overset{\Rightarrow}{F}} \arrow[ddll] \arrow[dd, bend left, "\overleftarrow{\mathfrak{N}}F"] \\
& & \\
\mathbb{B}_{\overset{\Leftarrow}{F}} \arrow[rr, tail] \arrow[uu, bend right] & & \mathbb{B}^{\overset{\Leftarrow}{F}} \arrow[uull, two heads] \arrow[uu, bend left]
\end{tikzcd}$$

Figure 23: Little nucleus $\overrightarrow{\mathfrak{N}}F$ defined by factoring simple nucleus $\overleftarrow{\mathfrak{N}}F$

thus consists of $\mathbb{A}^{\overset{\Rightarrow}{F}}$-objects and $\mathbb{B}^{\overset{\Leftarrow}{F}}$-morphisms, whereas $\mathbb{A}_{\overset{\Rightarrow}{F}}$ is the other way around[8]. Unpacking

[8] A very careful reader may at this point think that we got the notation wrong way around, because $\mathbb{B}_{\vec{F}}$ consists of $\mathbb{B}$-objects and $\mathbb{A}$-morphisms, whereas $\mathbb{A}_{\overset{\leftarrow}{F}}$ consists of $\mathbb{A}$-objects and $\mathbb{B}$-morphisms. Fig. 24 explains this choice of notation.

the definitions gives:

$$|\mathbb{B}_{\overset{\Leftarrow}{F}}| \;=\; \coprod_{x\in|\mathbb{A}|} \left\{ \alpha_x \in \mathbb{B}(F^*x, F^*x) \;\middle|\; \begin{tikzcd} F^*x \arrow[d, "\alpha_x"'] \arrow[dr, "\alpha_x"] & F_*F^*x \arrow[r, two heads] \arrow[dr, "F_*\alpha_x"] & x \arrow[d, tail, "\tilde{\alpha}_x"] \\ F^*x \arrow[r, "\alpha_x"'] & F^*x & F_*F^*x \end{tikzcd} \right\} \tag{93}$$

$$\mathbb{B}_{\overset{\Leftarrow}{F}}(\alpha_x, \gamma_z) \;=\; \left\{ g \in \mathbb{B}(F^*x, F^*z) \;\middle|\; \begin{tikzcd} F_*F^*x \arrow[r, "F_*g"] \arrow[d, "F_*\alpha_x"'] & F_*F^*z \arrow[d, "F_*\gamma_z"] \\ F_*F^*x \arrow[r, "F_*g"] & F_*F^*z \end{tikzcd} \right\}$$

$$|\mathbb{A}_{\overset{\Rightarrow}{F}}| \;=\; \coprod_{u\in|\mathbb{B}|} \left\{ \beta^u \in \mathbb{A}(F_*u, F_*u) \;\middle|\; \begin{tikzcd} F_*x \arrow[d, "\beta^u"'] \arrow[dr, "\beta^u"] & F^*F_*u \arrow[r, two heads, "\tilde{\beta}^u"] \arrow[dr, "F^*\beta^u"] & u \arrow[d, tail] \\ F_*u \arrow[r, "\beta^u"'] & F_*u & F^*F_*u \end{tikzcd} \right\} \tag{94}$$

$$\mathbb{A}_{\overset{\Rightarrow}{F}}(\beta^u, \delta^w) \;=\; \left\{ f \in \mathbb{A}(F_*u, F_*w) \;\middle|\; \begin{tikzcd} F^*F_*u \arrow[r, "F^*f"] \arrow[d, "F^*\beta^u"'] & F^*F_*w \arrow[d, "F^*\delta^w"] \\ F_*u \arrow[r, "F^*f"] & F^*F_*w \end{tikzcd} \right\}$$

The adjunction $F^\flat \dashv F_\flat : \mathbb{B}_{\overset{\Leftarrow}{F}} \to \mathbb{A}_{\overset{\Rightarrow}{F}}$ is obtained by restricting $F^\sharp \dashv F_\sharp : \mathbb{B}^{\overset{\Leftarrow}{F}} \to \mathbb{A}^{\overset{\Rightarrow}{F}}$ along the embeddings $\mathbb{B}_{\overset{\Leftarrow}{F}} \rightarrowtail \mathbb{B}^{\overset{\Leftarrow}{F}}$ and $\mathbb{A}_{\overset{\Rightarrow}{F}} \rightarrowtail \mathbb{A}^{\overset{\Rightarrow}{F}}$. Hence the functor

$$\overrightarrow{\mathfrak{N}}\left(F^* \dashv F_* : \mathbb{B} \to \mathbb{A}\right) \;=\; \left(F^\flat \dashv F_\flat : \mathbb{B}_{\overset{\Leftarrow}{F}} \to \mathbb{A}_{\overset{\Rightarrow}{F}}\right) \tag{95}$$

To see that it is an idempotent comonad, in addition to the natural equivalences $\overrightarrow{\mathfrak{N}}\overrightarrow{\mathfrak{N}}(F) \simeq \overrightarrow{\mathfrak{N}}(F)$ from (92), we need a counit $\overrightarrow{\mathfrak{N}}(F) \xrightarrow{\varepsilon} F$. The salient feature of the presentation in (94–93) is that it shows the forgetful functors $\mathbb{B}_{\overset{\Leftarrow}{F}} \to \mathbb{A}_{\overleftarrow{F}}$ and $\mathbb{A}_{\overset{\Rightarrow}{F}} \to \mathbb{B}_{\overrightarrow{F}}$, which complement the equivalences $\mathbb{B}^{\overset{\Leftarrow}{F}} \simeq \mathbb{A}^{\overleftarrow{F}}$ and $\mathbb{A}^{\overset{\Rightarrow}{F}} \simeq \mathbb{B}^{\overrightarrow{F}}$ in Fig. 24.

Proposition 9.1 *The little nucleus construction*

$$\begin{aligned} \overrightarrow{\mathfrak{N}} \;:\; \mathsf{Adj} \;&\longrightarrow\; \mathsf{Adj} & (96)\\ \left(F^* \dashv F_* : \mathbb{B} \to \mathbb{A}\right) \;&\longmapsto\; \left(F^\flat \dashv F_\flat : \mathbb{B}_{\overset{\Leftarrow}{F}} \to \mathbb{A}_{\overset{\Rightarrow}{F}}\right) & (97) \end{aligned}$$

$$
\begin{tikzcd}
\mathbb{A}_{\overset{\Rightarrow}{F}} \arrow[r] \arrow[rrrr, tail, bend left] & \mathbb{B}_{\vec{F}} \arrow[r, tail] \arrow[rr, tail, bend left] & \mathbb{A} \arrow[r] & \mathbb{B}^{\vec{F}} \arrow[r, "\sim"] & \mathbb{A}^{\overset{\Rightarrow}{F}} \\
\overrightarrow{\mathfrak{N}}F \arrow[u, shift right] \arrow[d, shift left] \arrow[rr, Rightarrow, "\varepsilon"] & & F \arrow[u, shift right] \arrow[d, shift left] \arrow[rr, Rightarrow, "\eta"] & & \overleftarrow{\mathfrak{N}}F \arrow[u, shift right] \arrow[d, shift left] \\
\mathbb{B}_{\overset{\Leftarrow}{F}} \arrow[r] \arrow[rrrr, tail, bend right] & \mathbb{A}_{\overset{\leftarrow}{F}} \arrow[r, tail] \arrow[rr, tail, bend right] & \mathbb{B} \arrow[r] & \mathbb{A}^{\overset{\leftarrow}{F}} \arrow[r, "\sim"] & \mathbb{B}^{\overset{\Leftarrow}{F}}
\end{tikzcd}
$$

Figure 24: The counit $\overrightarrow{\mathfrak{N}}F \xrightarrow{\varepsilon} F$ and the unit of $F \xrightarrow{\eta} \overleftarrow{\mathfrak{N}}F$ in Adj

is an idempotent comonad. An adjunction is subnuclear if and only if it is fixed by this comonad. The category of subnuclear adjunctions

$$\mathsf{Luc} \quad = \quad \left\{F \in \mathsf{Adj} \mid \overrightarrow{\mathfrak{N}}(F) \overset{\varepsilon}{\cong} F\right\} \tag{98}$$

is equivalent to the category of nuclear adjunctions:

$$\mathsf{Luc} \quad \simeq \quad \mathsf{Nuc}$$

Proof. The only claim not proved before the statement is the equivalence $\mathsf{Luc} \simeq \mathsf{Nuc}$. The functor $\mathsf{Luc} \to \mathsf{Nuc}$ can be realized by restricting $\overleftarrow{\mathfrak{N}}$ from Adj to $\mathsf{Luc} \subset \mathsf{Adj}$. The functor $\mathsf{Nuc} \to \mathsf{Luc}$ can be realized by restricting $\overrightarrow{\mathfrak{N}}$ from Adj to $\mathsf{Nuc} \subset \mathsf{Adj}$. The idempotency of both restricted functors implies that they form an equivalence. □

Theorem 9.2 *The comonads* $\overrightarrow{\mathfrak{M}} : \mathsf{Cmn} \to \mathsf{Cmn}$ *and* $\overrightarrow{\mathfrak{E}} : \mathsf{Mnd} \to \mathsf{Mnd}$, *defined*

$$\overrightarrow{\mathfrak{E}} \quad = \quad \mathsf{AM} \circ \mathsf{KC} \circ \mathsf{AC} \circ \mathsf{EM} \tag{99}$$

$$\overrightarrow{\mathfrak{M}} \quad = \quad \mathsf{AC} \circ \mathsf{EM} \circ \mathsf{AM} \circ \mathsf{EC} \tag{100}$$

are idempotent. Iterating them leads to the natural equivalences

$$\overrightarrow{\mathfrak{M}} \circ \overrightarrow{\mathfrak{M}} \overset{\varepsilon}{\simeq} \overrightarrow{\mathfrak{M}} \qquad\qquad \overrightarrow{\mathfrak{E}} \circ \overrightarrow{\mathfrak{E}} \overset{\varepsilon}{\simeq} \overrightarrow{\mathfrak{E}}$$

Moreover, their categories of coalgebras are equivalent:

$$\mathsf{Cmn}^{\overrightarrow{\mathfrak{M}}} \;\simeq\; \mathsf{Luc} \;\simeq\; \mathsf{Mnd}^{\overrightarrow{\mathfrak{E}}} \tag{101}$$

with Luc *as defined in* (98)*, and*

$$\mathsf{Cmn}^{\overrightarrow{\mathfrak{M}}} \;=\; \left\{\overrightarrow{F} \in \mathsf{Cmn} \mid \overrightarrow{\mathfrak{E}}\left(\overrightarrow{F}\right) \stackrel{\varepsilon}{\cong} \overrightarrow{F}\right\} \tag{102}$$

$$\mathsf{Mnd}^{\overrightarrow{\mathfrak{E}}} \;=\; \left\{\overleftarrow{F} \in \mathsf{Mnd} \mid \overrightarrow{\mathfrak{M}}\left(\overleftarrow{F}\right) \stackrel{\varepsilon}{\cong} \overleftarrow{F}\right\} \tag{103}$$

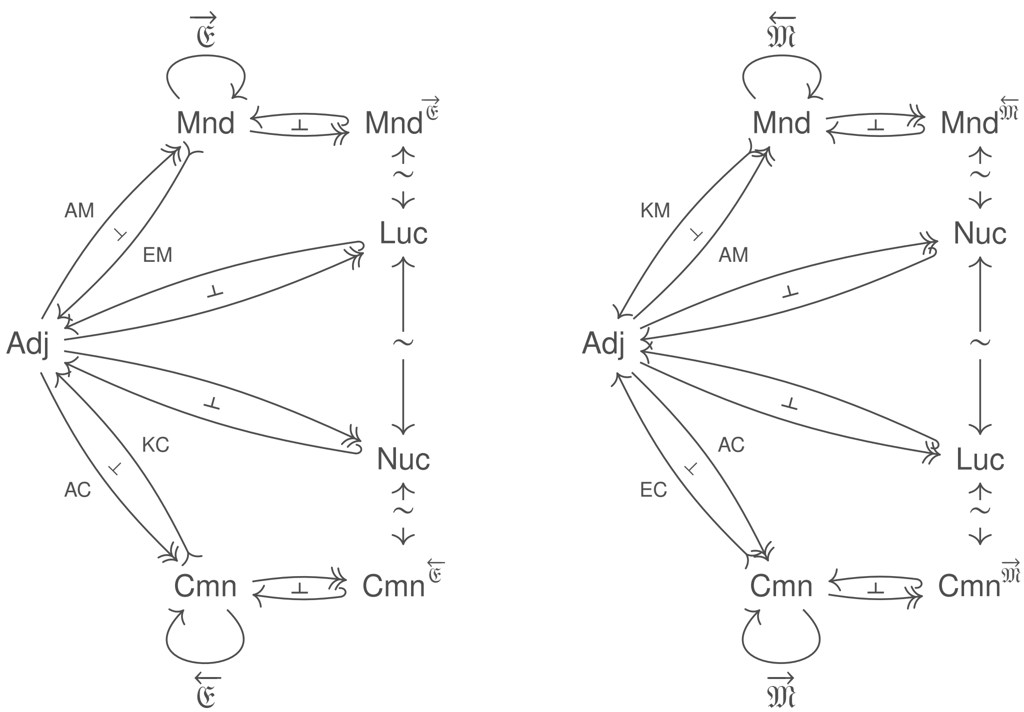

Figure 25: Relating little and big nuclei

The **proof** boils down to straightforward verifications with the simple nucleus formats. Fig. 25 summarizes and aligns the claims of Theorems 6 and 9.2.

10 Example 0: The Kan adjunction

Our final example of a nucleus construction arises from the first example of an adjoint pair of functors. The concept of adjunctions goes back, of course, at least to Évariste Galois, or, depending on how you conceptualize it, as far back as to Heraclitus [54], and into the roots of logics [57];

yet the definition of an adjoint pair of functors between genuine categories goes back to the late 1950s, to Daniel Kan's work in homotopy theory [44]. Kan defined the Kan extensions to capture a particular adjunction, perhaps like Eilenberg and MacLane defined categories and functors to define certain natural transformations.

10.1 Simplices and the simplex category

One of the seminal ideas of algebraic topology arose from Eilenberg's computations of homology groups of topological spaces by decomposing them into simplices [24]. An m-simplex is the set

$$\Delta_{[m]} \quad = \quad \left\{ \vec{x} \in [0,1]^{m+1} \mid \sum_{i=0}^{m} x_i = 1 \right\} \tag{104}$$

with the product topology induced by the open intervals on $[0,1]$. The relevant structure of a topological space X is captured by families of continuous maps $\Delta_m \to X$, for all $m \in \mathbb{N}$. Some such maps do not *embed* simplices into a space, like triangulations do, but contain degeneracies, or singularities. Nevertheless, considering the entire family of such maps to X makes sure that any simplices that can be embedded into X will be embedded by some of them. Since the simplicial structure is captured by each $\Delta_{[m]}$'s projections onto all $\Delta_{[\ell]}$s for $\ell < m$, and by $\Delta_{[m]}$'s embeddings into all $\Delta_{[n]}$s for $n > m$, a coherent simplicial structure corresponds to a functor of the form $\Delta_{[-]} : \Delta \to \mathsf{Esp}$, where Esp is the category of topological spaces and continuous maps[9], and Δ is the simplex category. Its objects are finite ordinals

$$[m] \quad = \quad \{0 < 1 < 2 < \cdots < m\}$$

while its morphisms are the order-preserving functions [28]. All information about the simplicial structure of topological spaces is thus captured in the matrix

$$\begin{aligned} \Upsilon : \Delta^o \times \mathsf{Esp} \quad &\to \quad \mathsf{Set} \\ [m] \times X \quad &\mapsto \quad \mathsf{Esp}\left(\Delta_{[m]}, X\right) \end{aligned} \tag{105}$$

This is, in a sense, the *"context matrix"* of homotopy theory, if it were to be translated to the language of Sec. 2, and construed as a geometric *"concept analysis"*.

10.2 Kan adjunctions and extensions

Daniel Kan's work was mainly concerned with computing homotopy groups in combinatorial terms [45]. That led to the discovery of categorical adjunctions as a tool for Kan's extensions of the

[9]We denote the category of topological spaces by the abbreviation Esp of the French word *espace*, not just because there are other things called Top in the same contexts, but also as authors' reminder-to-self of the tacit sources of the approach [36, 3].

simplicial approach [44]. Applying the toolkit from Sec. 5.3, the matrix Υ from (105) gives rise to the following functors

$$\begin{gathered}\Upsilon\colon \Delta^o \times \mathsf{Esp} \to \mathsf{Set}\\ \Upsilon_\bullet\colon \Delta \to \Uparrow\!\mathsf{Esp} \qquad\qquad {}_\bullet\Upsilon\colon \mathsf{Esp} \to \Downarrow\!\Delta\\ \Upsilon^*\colon \Downarrow\!\Delta \to \Uparrow\!\mathsf{Esp} \qquad\qquad \Upsilon_*\colon \Uparrow\!\mathsf{Esp} \to \Downarrow\!\Delta\end{gathered} \tag{106}$$

where

- $\Downarrow\!\Delta = \mathsf{Dfib}/\Delta \simeq \mathsf{Set}^{\Delta^o}$ is the category of simplicial sets $K : \Delta^o \to \mathsf{Set}$, or equivalently of complexes $\int K : \widehat{K} \to \Delta$, comprehended along the lines of Sec. 5.3.2;
- $\Uparrow\!\mathsf{Esp} = (\mathsf{Ofib}/\mathsf{Esp})^o$ is the opposite category of discrete opfibrations over Esp, i.e. of functors $\mathcal{D} \xrightarrow{D} \mathsf{Esp}$ which establish isomorphisms between the coslices $x/\mathcal{D} \overset{D_x}{\cong} Dx/\mathsf{Esp}$.

The Yoneda embedding $\Delta \xrightarrow{\blacktriangledown} \Downarrow\!\Delta$ makes $\Downarrow\!\Delta$ into a colimit-completion of Δ, and induces the extension $\Upsilon^*: \Downarrow\!\Delta \to \Uparrow\!\mathsf{Esp}$ of $\Upsilon_\bullet : \Delta \to \Uparrow\!\mathsf{Esp}$. The Yoneda embedding $\mathsf{Esp} \xrightarrow{\blacktriangle} \Uparrow\!\mathsf{Esp}$ makes $\Uparrow\!\mathsf{Esp}$ into a limit-completion of Esp, and induces the extension $\Upsilon_* : \Uparrow\!\mathsf{Esp} \to \Downarrow\!\Delta$ of ${}_\bullet\Upsilon\colon \mathsf{Esp} \to \Downarrow\!\Delta$.

However, Esp is a large category, and the category $\Uparrow\!\mathsf{Esp}$ lives in another universe. Moreover, Esp already has limits, and completing it to $\Uparrow\!\mathsf{Esp}$ obliterates them, and adjoins the formal ones. Kan's original extension was defined using the original limits in Esp, and there was no need to form $\Uparrow\!\mathsf{Esp}$. Using the standard notation sSet for simplicial sets Set^{Δ^o}, or equivalently for complexes $\Downarrow\!\Delta$, Kan's original adjunction boils down to

$$\begin{tikzcd} \mathbb{K} \xrightarrow{K} \Delta \arrow[d, mapsto] & \mathsf{sSet} \arrow[d, bend right, "\Upsilon^*"'] \arrow[d, phantom, "\dashv"] & \left(\Delta_{[-]}/X \xrightarrow{\mathrm{Dom}} \Delta\right) \\ \varinjlim\left(\mathbb{K} \xrightarrow{K} \Delta \xrightarrow{\Delta_{[-]}} \mathsf{Esp}\right) & \mathsf{Esp} \arrow[u, bend right, "\Upsilon_*"'] & X \arrow[u, mapsto] \end{tikzcd} \tag{107}$$

where

- $\Upsilon_\bullet = \left(\Delta \xrightarrow{\Delta_{[-]}} \mathsf{Esp} \xrightarrow{\blacktriangle} \Uparrow\!\mathsf{Esp}\right)$, is truncated to $\Delta \xrightarrow{\Delta_{[-]}} \mathsf{Esp}$;
- ${}_\bullet\Upsilon\colon \Uparrow\!\mathsf{Esp} \to \Downarrow\!\Delta$ from (63), restricted to Esp leads to

$$\varprojlim\left(1 \xrightarrow{X} \mathsf{Esp} \xrightarrow{{}_\bullet\Upsilon} \mathsf{Dfib}\,/\Delta\right) \quad = \quad \left(\Delta_{[-]}/X \xrightarrow{\mathrm{Dom}} \Delta\right)$$

The adjunction $\mathsf{MA}(\Upsilon) = (\Upsilon^* \dashv \Upsilon_* : \mathsf{Esp} \to \mathsf{sSet})$, displayed in (107), has been studied for many years. The functor $\Upsilon^* : \mathsf{sSet} \to \mathsf{Esp}$ is usually called the geometric realization [64], whereas $\Upsilon_* : \mathsf{Esp} \to \mathsf{sSet}$ is the singular decomposition on which Eilenberg's singular homology was based [24]. Kan spelled out the concept of adjunction from the relationship between these two functors [44, 46].

The overall idea of the approach to homotopies through adjunctions was that recognizing this abstract relationship between Υ^* and Υ_* should provide a general method for transferring the invariants of interest between a geometric and an algebraic or combinatorial category. For a geometric realization $\Upsilon^* K \in \mathsf{Esp}$ of a complex $K \in \mathsf{sSet}$, the homotopy groups can be computed in purely combinatorial terms, from the structure of K alone [45]. Indeed, the spaces in the form $\Upsilon^* K$ boil down to Whitehead's CW-complexes [64, 83]. What about the spaces that do not happen to be in this form?

10.3 Troubles with localizations

The upshot of Kan's adjunction $\Upsilon^* \dashv \Upsilon_* \colon \mathsf{Esp} \to \mathsf{sSet}$ is that for any space X, we can construct a CW-complex $\overrightarrow{\Upsilon} X = \Upsilon^* \Upsilon_* X$, with a continuous map $\overrightarrow{\Upsilon} X \xrightarrow{\varepsilon} X$, that arises as the counit of Kan's adjunction. In a formal sense, this counit is the best approximation of X by a CW-complex. When do such approximations preserve the geometric invariants of interest? By the late 1950s, it was already known that such combinatorial approximations work in many special cases, certainly whenever ε is invertible. But in general, even $\overrightarrow{\Upsilon}\overrightarrow{\Upsilon} X \xrightarrow{\varepsilon} \overrightarrow{\Upsilon} X$ is not always invertible.

The idea of approximating topological spaces by combinatorial complexes thus grew into a quest for making the units or the counits of adjunctions invertible. Which spaces have the same invariants as the geometric realizations of their singular[10] decompositions? For particular invariants, there are direct answers [25, 26]. In general, though, localizing at suitable spaces along suitable reflections or coreflections aligns (106) with (18) and algebraic topology can be construed as a geometric extension of concept analysis from Sec. 2, extracting concept nuclei from context matrices as the invariants of adjunctions that they induce. Some of the most influential methods of algebraic topology can be interpreted in this way. Grossly oversimplifying, we mention three approaches.

The direct approach [29, 16, Vol. I, Ch. 5] was to enlarge the given category by formal inverses of a family of arrows, usually called weak equivalences, and denoted by Σ. They are thus made invertible in a calculus of fractions, generalizing the one for making the integers, or the elements of an integral domain, invertible in a ring. When applied to a large category, like Esp, this calculus of fractions generally involves manipulating proper classes of arrows, and the resulting category may even have large hom-sets.

Another approach [22, 78] is to factor out the Σ-arrows using two factorization systems. This approach is similar to the constructions outlined in Sections 3 and 4.5.3, but the factorizations of continuous maps that arise in this framework are not unique: they comprise families of fibrations and cofibrations, which are orthogonal by lifting and descent, thus only weakly. Abstract homotopy models in categories thus lead to pairs of *weak* factorization systems. Sticking with the notation $\mathcal{E}^\bullet \wr \mathcal{M}$ and $\mathcal{E} \wr \mathcal{M}^\bullet$ for such weak factorization systems, the idea is thus that the family Σ is now generated by composing the elements of $\mathcal{E}^\bullet$ and $\mathcal{M}^\bullet$. Localizing at the arrows from $\mathcal{E} \cap \mathcal{M}$, that are orthogonal to both $\mathcal{M}^\bullet$ and $\mathcal{E}^\bullet$, makes Σ invertible. It turns out that suitable factorizations can be found both in Esp and in sSet, to make the adjunction between spaces and complexes into an

[10]The word "singular" here means that the simplices, into which space may be decomposed, do not have to be embedded into it, which would make the decomposition *regular*, but that the continuous maps from their geometric realizations may have *singularities*.

equivalence. This was Dan Quillen's approach [77, 78].

The third approach [1, 2] tackles the task of making the arrows $\overrightarrow{\Upsilon}X \xrightarrow{\varepsilon} X$ invertible by modifying the comonad $\overrightarrow{\Upsilon}$ until it becomes idempotent, and then localizing at the coalgebras of this idempotent comonad. Note that this approach does not tamper with the continuous maps in Esp, be it to make some of them formally invertible, or to factor them out. The idea is that an idempotent comonad, call it $\overrightarrow{\Upsilon}_\infty : \mathsf{Esp} \to \mathsf{Esp}$, should localize any space X at a space $\overrightarrow{\Upsilon}_\infty X$ such that $\overrightarrow{\Upsilon}_\infty\overrightarrow{\Upsilon}_\infty X \overset{\varepsilon}{\cong} \overrightarrow{\Upsilon}_\infty X$. That means that Υ_∞ is an idempotent monad. The quest for such a monad is illustrated in Fig. 26. $\mathsf{Esp}^{\overrightarrow{\Upsilon}}$ denotes the category of coalgebras for the comonad $\overrightarrow{\Upsilon} = \Upsilon^*\Upsilon_*$, the

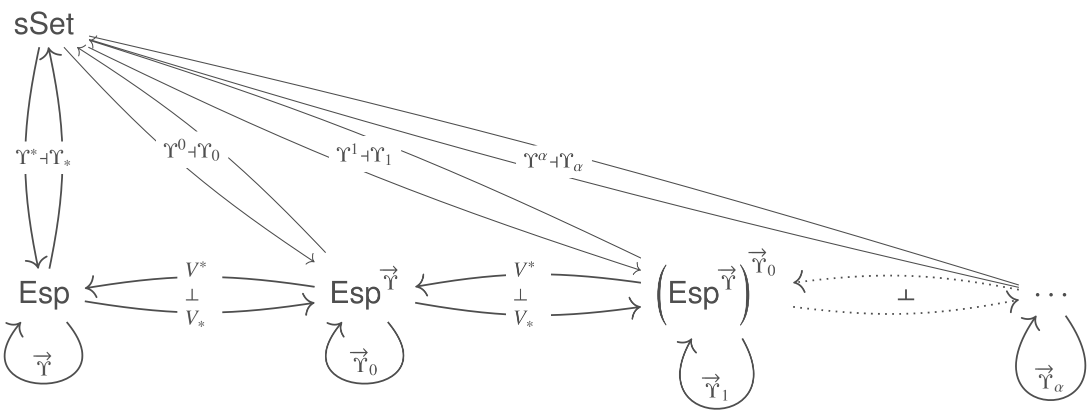

Figure 26: Iterating the comonad resolutions for $\overrightarrow{\Upsilon}$

adjunction $V^* \dashv V_* : \mathsf{Esp} \to \mathsf{Esp}^{\overrightarrow{\Upsilon}}$ is the final resolution of this comonad, and Υ^0 is the couniversal comparison functor into this resolution, mapping a complex K to the coalgebra $\Upsilon^*K \xrightarrow{\eta^*} \Upsilon^*\Upsilon_*\Upsilon^*K$. Since sSet is a complete category, Υ^0 has a right adjoint Υ_0, and they induce the comonad $\overrightarrow{\Upsilon}_0$ on $\mathsf{Esp}^{\overrightarrow{\Upsilon}}$. If $\overrightarrow{\Upsilon}$ was idempotent, then the final resolution $V^* \dashv V_*$ would be a coreflection, and the comonad $\overrightarrow{\Upsilon}_0$ would be (isomorphic to) the identity. But $\overrightarrow{\Upsilon}$ is not idempotent, and the construction can be applied to $\overrightarrow{\Upsilon}_0$ again, leading to $\left(\mathsf{Esp}^{\overrightarrow{\Upsilon}}\right)^{\overrightarrow{\Upsilon}_0}$, with the final resolution generically denoted $V^* \dashv V_* : \mathsf{Esp}^{\overrightarrow{\Upsilon}} \to \left(\mathsf{Esp}^{\overrightarrow{\Upsilon}}\right)^{\overrightarrow{\Upsilon}_0}$, and the comonad $\overrightarrow{\Upsilon}_1$ on $\left(\mathsf{Esp}^{\overrightarrow{\Upsilon}}\right)^{\overrightarrow{\Upsilon}_0}$. Remarkably, Applegate and Tierney [1] found that the process needs to be repeated *transfinitely* before the idempotent monad $\overrightarrow{\Upsilon}_\infty$ is reached. At each step, some parts of a space that are not combinatorially approximable are eliminated, but that causes some other parts, that were previously approximable, to cease being so. And this may still be the case after infinitely many steps. A transfinite induction becomes necessary. The situation is similar to Cantor's quest for accumulation points of the convergence domains of Fourier series, which led him to discover transfinite induction in the first place.

The nucleus of the same adjunction is displayed in Fig. 27. The category $\mathsf{Esp}^{\overrightarrow{\Upsilon}}$ comprises

$$
\begin{tikzcd}
\mathsf{sSet} \arrow[loop above, "\overleftarrow{\Upsilon}"] \arrow[r, shift left] \arrow[d, shift left] & \arrow[l, shift left, "\Upsilon^0 \dashv \Upsilon_0"'] \mathsf{Esp}^{\overrightarrow{\Upsilon}} \arrow[loop above, "\overset{\Leftarrow}{\Upsilon}"] \arrow[dl, shift left] \arrow[d, shift left] \arrow[r, leftrightarrow, "\simeq"] & \mathsf{sSet}^{\overset{\Rightarrow}{\Upsilon}} \arrow[d, shift left] \\
\mathsf{Esp} \arrow[loop below, "\overrightarrow{\Upsilon}"] \arrow[u, shift left, "\Upsilon^* \dashv \Upsilon_*"] \arrow[ur, shift left, "V^* \dashv V_*"] \arrow[r, shift left, "H^1 \dashv H_1"] & \mathsf{sSet}^{\overleftarrow{\Upsilon}} \arrow[loop below, "\overset{\Rightarrow}{\Upsilon}"] \arrow[l, shift left] \arrow[u, shift left, "\Upsilon^\sharp \dashv \Upsilon_\sharp"] \arrow[r, leftrightarrow, "\simeq"] & \mathsf{Esp}^{\overset{\Leftarrow}{\Upsilon}} \arrow[u, shift left, "\Upsilon^\sharp \dashv \Upsilon_\sharp"]
\end{tikzcd}
$$

Figure 27: The nucleus of the Kan adjunction

spaces that may not be homeomorphic with a geometric realization of a complex, but are their retracts, projected along the counit $\overrightarrow{\Upsilon}X \overset{\varepsilon}{\twoheadrightarrow} X$, and included along the structure coalgebra $X \rightarrowtail \overrightarrow{\Upsilon}X$. But the projection does not preserve simplicial decompositions; i.e., it is not an $\overrightarrow{\Upsilon}$-coalgebra homomorphism. The transfinite construction of the idempotent monad $\overrightarrow{\Upsilon}_\infty$ was thus needed to extract just those spaces where the projection boils down to a homeomorphism. But Prop. 8.1 implies that simplicial decompositions of spaces in $\mathsf{Esp}^{\overrightarrow{\Upsilon}}$ can be equivalently viewed as objects of the simple nucleus category $\mathsf{sSet}^{\overset{\Rightarrow}{\Upsilon}}$. Any space X decomposed along a coalgebra $X \rightarrowtail \overrightarrow{\Upsilon}X$ in $\mathsf{Esp}^{\overrightarrow{\Upsilon}}$ can be equivalently viewed in $\mathsf{sSet}^{\overset{\Rightarrow}{\Upsilon}}$ as a complex K with an idempotent $\Upsilon^*K \xrightarrow{\varphi} \Upsilon^*K$. This idempotent secretly splits on X, but the category $\mathsf{sSet}^{\overset{\Rightarrow}{\Upsilon}}$ does not know that. It does know Corollary 8.4, though, which says that the object $\varphi_K = \left\langle K, \Upsilon^*K \xrightarrow{\varphi} \Upsilon^*K \right\rangle$ is a retract of $\overset{\Rightarrow}{\Upsilon}\varphi_K$; and $\overset{\Rightarrow}{\Upsilon}\varphi_K$ secretly splits on $\overrightarrow{\Upsilon}X$. The space X is thus represented in the category $\mathsf{sSet}^{\overset{\Rightarrow}{\Upsilon}}$ by the idempotent φ_K, which is a retract of $\overset{\Rightarrow}{\Upsilon}\varphi_K$, representing $\overrightarrow{\Upsilon}X$. Simplicial decompositions of spaces along coalgebras in $\mathsf{Esp}^{\overrightarrow{\Upsilon}}$ can thus be equivalently captured as idempotents over simplicial sets within the simple nucleus category $\mathsf{sSet}^{\overset{\Rightarrow}{\Upsilon}}$. The idempotency of the nucleus construction can be interpreted as a suitable completeness claim for such representations.

To be continued. How is it possible that X is not a retract of $\overrightarrow{\Upsilon}X$ in $\mathsf{Esp}^{\overrightarrow{\Upsilon}}$, but the object φ_K, representing X in the equivalent category $\mathsf{sSet}^{\overset{\Rightarrow}{\Upsilon}}$, is recognized as a retract of the object $\overset{\Rightarrow}{\Upsilon}\varphi_K$, representing $\overrightarrow{\Upsilon}X$? The answer is that the retractions occur at different levels of the representation. Recall, first of all, that $\mathsf{sSet}^{\overset{\Rightarrow}{\Upsilon}}$ is a simplified form of $\left(\mathsf{sSet}^{\overleftarrow{\Upsilon}}\right)^{\overset{\Rightarrow}{\Upsilon}}$. The reader familiar with Beck's Theorem, this time applied to comonadicity, will remember that X can be extracted from $\overrightarrow{\Upsilon}X$ using an equalizer that splits in Esp, when projected along a forgetful functor $V^* : \mathsf{Esp}^{\overrightarrow{\Upsilon}} \to \mathsf{Esp}$. This split equalizer in Esp lifts back along the comonadic V^* to an equalizer in $\mathsf{Esp}^{\overrightarrow{\Upsilon}}$, which is generally not split. On the other hand, the splitting of this equalizer occurs in $\left(\mathsf{sSet}^{\overleftarrow{\Upsilon}}\right)^{\overrightarrow{\Upsilon}}$ as the algebra carrying the corresponding coalgebra. In $\mathsf{sSet}^{\overset{\Longrightarrow}{\Upsilon}}$, this splitting is captured as the idempotent that it induces. We have shown, of course, that all three categories are equivalent. But $\mathsf{sSet}^{\overset{\Longrightarrow}{\Upsilon}}$ internalizes the absolute limits that get reflected along the forgetful functor V^*. It makes them explicit, and available for computations. But they have to be left for after the break.

11 What?

11.1 What we did

We studied nuclear adjunctions. To garner intuition, we considered some examples. Since every adjunction has a nucleus, the reader's favorite adjunctions provide additional examples and applications. Our favorite example is in [74]. In any case, the abstract concept arose from concrete applications, so there are many [47, 71, 72, 73, 75, 82, 84]. Last but not least, the nucleus construction itself is an example of itself, as it provides the nuclei of the adjunctions between monads and comonads.

11.2 What we did not do

We studied adjunctions, monads, and comonads in terms of adjunctions, monads, and comonads. We took category theory as a language and analyzed it in that same language. We preached what we practice. There is, of course, nothing unusual about that. There are many papers about the English language that are written in English.

However, self-applications of category theory get complicated. They sometimes cause chain reactions. Categories and functors form a category, but natural transformations make them into a 2-category. 2-categories form a 3-category, 3-categories a 4-category, and so on. Unexpected things already happen at level 3 [33, 38]. Strictly speaking, the theory of categories is not a part of category theory, but of *higher* category theory [6, 59, 60, 79]. Grothendieck's *homotopy hypothesis* [37, 62] made higher category theory into an expansive geometric pursuit, subsuming homotopy theory. While most theories grow to be simpler as they solve their problems, and dimensionality reduction is, in fact, the main tenet of statistics, machine learning, and concept analysis, higher category theory makes the dimensionality increase into a principle of the method. This opens up the realm of applications in modern physics but also presents a significant new challenge for the language of mathematical constructions.

Category theory reintroduced diagrams and geometric interactions as first-class citizens of the mathematical discourse, after several centuries of the prevalence of algebraic prose, driven by the facility of printing. Categories were invented to dam the flood of structure in algebraic topology, but they also geometrized algebra. In some areas, though, they produced their own flood of structure. Since the diagrams in higher categories are of higher dimensions, and the compositions are not mere sequences of arrows, diagram chasing became a problem. While it is naturally extended into cell pasting by filling 2-cells into commutative polygons, diagram pasting does not boil down to a directed form of diagram chasing, as one would hope. The reason is that 1-cell composition does not extend into 2-cell composition freely, but modulo the *middle-two interchange* law (a.k.a. *Godement's naturality* law). A 2-cell can thus have many geometrically different representatives. This factoring is easier to visualize using string diagrams, which are the Poincaré duals of the pasting diagrams. The duality maps 2-cells into vertices, and 0-cells into faces of string diagrams. Chasing 2-categorical string diagrams is thus a map-coloring activity.

In the earlier versions of this paper, the nucleus was presented as a 2-categorical construction. We spent several years validating some of the results at that level of generality, and drawing colored maps to make them communicable. Introducing a new idea in a new language is a bootstrapping endeavor. It may be possible when the boots are built and strapped, but not before that. At least in our early presentations, the concept of nucleus and the diagrams of its 2-categorical context evolved two narratives. This paper became possible when we gave up on one of the narratives, and factored out the 2-categorical aspects.

11.3 What needs to be done

In view of Sec. 10, a higher categorical analysis of the nucleus construction seems to be of interest. The standard reference for the 2-categories of monads and comonads is [80], extended in [52]. The adjunction morphisms were introduced in [4]. Their 1-cells, which we sketch in the Appendix, are the lax versions of the morphisms we use in Sec. 5. The 2-cells are easy to derive from the structure preservation requirement, though less easy to draw, and often even more laborious to read. Understanding is a process that unfolds at many levels. The language of categories facilitates understanding by its flexibility, but it is can also obscure its subject when imposed rigidly. The quest for categorical methods of geometry has grown into a quest for geometric methods of category theory. There is a burgeoning new scene of diagrammatic tools [18, 39]. If pictures help us understand categories, then categories will help us to speak in pictures, and the nuclear methods will help us mine concepts as invariants.

11.4 What are categories and what are their model structures?

The spirit of category theory is that the objects should be studied as black boxes, in terms of the morphisms coming in and out of them. If categories themselves are studied in the spirit of category theory, then they should be studied in terms of the functors coming in and out. A functor is defined by specifying an object part and an arrow part, and confirms that a category consists of objects and arrows. Any functor $G : \mathbb{A} \to \mathbb{B}$ can be decomposed[11], as displayed in Fig. 28, into a surjection on the objects, and an injection on the arrows, through the category $\mathbb{A}_G$, with the objects of $\mathbb{A}$ and the arrows of $\mathbb{B}$. The orthogonality of the essentially surjective functors $E \in \mathsf{Ess}$ and full-and-faithful

[11] An overview of the basic structure of factorization systems is in Appendix A.

$$
\begin{array}{rcl}
|\mathbb{A}_G| & = & |\mathbb{A}| \\
\mathbb{A}_G(u, v) & = & \mathbb{B}(Gu, Gv)
\end{array}
\qquad
\begin{tikzcd}
\mathbb{A}_G \arrow[d, tail, "\mathsf{Ffa}(G)"'] & & \mathbb{A} \arrow[ll, two heads, "\mathsf{Ess}(G)"'] \arrow[dll, "G"] \\
\mathbb{B}
\end{tikzcd}
$$

Figure 28: Factoring of an arbitrary functor G through $(\mathsf{Ess} \wr \mathsf{Ffa})$

functors $M \in \mathsf{Ffa}$, is displayed in Fig. 29. Since E is essentially surjective, for any object y in $\mathbb{B}$

$$
\begin{array}{rcl}
HE \cong U & \rightsquigarrow & Hy = UE^{-1}(y) \\
MH \cong V & \rightsquigarrow & Hf = M^{-1}V(f)
\end{array}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[r, "U"] \arrow[d, two heads, "E"'] & \mathbb{C} \arrow[d, tail, "M"] \\
\mathbb{B} \arrow[r, "V"'] \arrow[ur, dotted, "H"] & \mathbb{D}
\end{tikzcd}
$$

Figure 29: The orthogonality of an essential surjection $E \in \mathsf{Ess}$ and a full-and-faithful $M \in \mathsf{Ffa}$

there is some x in $\mathbb{A}$ such that $Ex \cong y$, so we take $Hy = Ux$. If $Ex' \cong y$ also holds for some other x' in $\mathbb{A}$ then $MUx \cong VEx \cong Vy \cong VEx' \cong MUx'$ implies $Ux \cong Ux'$, because M is full-and-faithful. The arrow part is defined using the bijections between the hom-sets provided by M. The factorization system $(\mathsf{Ess} \wr \mathsf{Ffa})$ can be used as a stepping stone into category theory. It confirms that functors see categories as comprised of objects and arrows.

Functors are not the only available morphisms between categories. Many mathematical theories study objects that are instances of categories, but require morphisms for which the functoriality is not enough. E.g., a topology is a lattice of open sets, and a lattice is, of course, a special kind of category. A continuous map between two topological spaces is an adjunction between the lattices of opens: the requirement that the inverse image of a continous map preserves the unions of the opens means that it has a right adjoint. The general functors between topologies, i.e. merely monotone maps between the lattices of opens, are seldom studied because they do not capture continuity, which is the subject of topology. For an even more general example, consider basic set theory. Functions are defined as total and single-valued relations. A total and single-valued relation between two sets is an adjunction between the two lattices of subsets: the totality is the unit of the adjunction, and the single-valuedness is the counit [69]. A general relation induces a monotone map, i.e. a functor between the lattices of subsets. But studying functions means studying adjunctions. There are many mathematical theories where the objects of study are categories of some sort, and the morphisms between them are adjunctions.

What are categories in terms of adjunctions? We saw in Sec. 4.6 that applying the factorization system $(\mathsf{Ess} \wr \mathsf{Ffa})$ to a pair of adjoint functors gives rise to the two initial resolutions of the adjunction: the (Kleisli) categories of free algebras and coalgebras. Completing them to the final

$$\begin{aligned} \left|\mathbb{A}^{\overset{\Longrightarrow}{F}}\right| &= \coprod_{x\in|\mathbb{A}|}\left\{F^*x \xrightarrow{\alpha_x} F^*x \mid (84)\right\} \\ \mathbb{A}^{\overset{\Longrightarrow}{F}}(\alpha_x,\gamma_z) &= \mathbb{B}^{\overrightarrow{F}}(R^*\alpha_x, R^*\gamma_z) \end{aligned}$$

$$\begin{tikzcd}
\mathbb{A}^{\overset{\Longrightarrow}{F}} & \mathbb{A} \arrow[l, two heads, "C^\bullet(F)"'] \arrow[dl, bend right, "F^*"'] \arrow[dl, phantom, "\dashv"] \\
\mathbb{B} \arrow[from=1-1, tail, "\mathcal{F}(F)"'] \arrow[r, two heads, "\mathcal{F}^\bullet(F)"'] \arrow[ur, bend right, "F_*"'] & \mathbb{B}^{\overset{\Longleftarrow}{F}} \arrow[u, tail, "C(F)"']
\end{tikzcd}$$

$$\begin{aligned} \left|\mathbb{B}^{\overset{\Longleftarrow}{F}}\right| &= \coprod_{u\in|\mathbb{B}|}\left\{F_*u \xrightarrow{\beta^u} F_*u \mid (85)\right\} \\ \mathbb{B}^{\overset{\Longleftarrow}{F}}(\beta^u,\delta^w) &= \mathbb{A}^{\overleftarrow{F}}(L_*\beta^y, L_*\delta^w) \end{aligned}$$

Figure 30: Factoring the adjunction $F = (F^* \dashv F_*)$ through $(C^\bullet \wr \mathcal{F})$ and $(C \wr \mathcal{F}^\bullet)$

resolutions lifts Fig. 28 to Fig. 30. This lifting is yet another perspective on the equivalences $R^* : \mathbb{A}^{\overset{\Longrightarrow}{F}} \to \mathbb{B}^{\overrightarrow{F}}$ and $L_* : \mathbb{B}^{\overset{\Longleftarrow}{F}} \to \mathbb{A}^{\overleftarrow{F}}$ from Sec. 8 and [75, Theorems III.2 and III.3]. Note that the adjunctions are taken here as morphisms in the direction of their lefth-hand component (like functions, and unlike the continuous maps), so that the functors $C(F)$ and $\mathcal{F}^\bullet(F)$ in Fig. 30, as components of a right adjoint, are displayed in the opposite direction. That is why the $C^\bullet$-component is drawn with a tail, although in the context of left-handed adjunctions it plays the role of an abstract epi. The weak factorization systems $(C^\bullet \wr \mathcal{F})$ and $(C \wr \mathcal{F}^\bullet)$ are comprised of the families

- ∼ $\mathcal{F} = \{(F^* \dashv F_*) \mid F^* \text{ is comonadic}\}$,
- ∼ $C^\bullet = \{(F^* \dashv F_*) \mid F^* \text{ is a comparison functor for a comonad}\}$,
- ∼ $C = \{(F^* \dashv F_*) \mid F_* \text{ is monadic}\}$,
- ∼ $\mathcal{F}^\bullet = \{(F^* \dashv F_*) \mid F_* \text{ is a comparison functor for a monad}\}$.

To see how these factorizations are related with $(\mathsf{Ess} \wr \mathsf{Ffa})$, and how Fig. 30 arises from Fig. 28, recall from Sec. 4.6 that the $(\mathsf{Ess}\wr\mathsf{Ffa})$-decomposition of F^* gives the initial resolution $\mathbb{A}_{\overleftarrow{F}}$, whereas the $(\mathsf{Ess} \wr \mathsf{Ffa})$-decomposition of F_* gives the initial resolution $\mathbb{B}_{\overrightarrow{F}}$. However, $\mathbb{A}_{\overleftarrow{F}} \hookrightarrow \mathbb{A}^{\overleftarrow{F}} \simeq \mathbb{B}^{\overset{\Longleftarrow}{F}}$ factors through the $(C \wr \mathcal{F}^\bullet)$-decomposition of F_*, whereas $\mathbb{B}_{\overrightarrow{F}} \hookrightarrow \mathbb{B}^{\overrightarrow{F}} \simeq \mathbb{A}^{\overset{\Longrightarrow}{F}}$ factors through the $(C^\bullet \wr \mathcal{F})$-decomposition of F^*. In particular, while

a) the Ess-image $\mathbb{A}_{\overleftarrow{F}}$ of $\mathbb{A}$ in $\mathbb{B}$ along F^* is spanned by the isomorphisms $y \cong F^*x$,

b) the $C^\bullet$-image $\mathbb{A}^{\overset{\Longrightarrow}{F}}$ of $\mathbb{A}$ in $\mathbb{B}$ along F^* is spanned by the retractions $y \leftrightarrows F^*x$.

It is easy to check that such retractions in $\mathbb{B}$ correspond to $\overrightarrow{F}$-coalgebras. Worked out in full detail, this correspondence is the equivalence $R^* \colon \mathbb{A}^{\overset{\Longrightarrow}{F}} \simeq \mathbb{B}^{\overrightarrow{F}}$. Looking at the $(C^\bullet\wr\mathcal{F})$-decompositions from the two sides of this equivalence aligns the orthogonality of $C^\bullet$ and $\mathcal{F}$ with the orthogonality of Ess and Ffa, as indicated in Fig. 31. Since any object α_x of $\mathbb{A}^{\overset{\Longrightarrow}{F}}$ induces a retraction $\overleftarrow{F}x \overset{\tilde{\alpha}_x}{\twoheadrightarrow} x \overset{\alpha^x}{\rightarrowtail} \overleftarrow{F}x$, and the comparison functor E maps x to $Ex = \left\langle \overleftarrow{F}x, F^*\overleftarrow{F}x \overset{\varepsilon F_*}{\twoheadrightarrow} F^*x \overset{F^*\eta}{\rightarrowtail} F^*\overleftarrow{F}x \right\rangle$, the image $V\alpha_x$ splits

$$\begin{array}{lll} HE \cong U & \rightsquigarrow & H\alpha_x = \left(V\alpha_x \underset{V\alpha^x}{\overset{V\tilde{\alpha}_x}{\leftrightarrows}} VEx \cong MUx \right) \\ MH \cong V & \rightsquigarrow & Hf = V(f) \end{array}
\qquad
\begin{tikzcd} \mathbb{A} \arrow[r, "U"] \arrow[d, two heads, "E"'] & \mathbb{D}^{\overrightarrow{G}} \arrow[d, tail, "M"] \\ \mathbb{A}^{\overrightarrow{F}} \arrow[r, "V"'] \arrow[ur, dotted, "H"] & \mathbb{D} \end{tikzcd}$$

Figure 31: The orthogonality of a comparison functor $E \in \mathbb{C}^\bullet$ and a comonadic $M \in \mathcal{F}$

into $VEx \overset{V\tilde{\alpha}_x}{\twoheadrightarrow} V\alpha_x \overset{V\alpha^x}{\rightarrowtail} VEx$. But the isomorphism $VEx \cong MUx$ and the comonadicity of M imply that the M-split equalizer $V\alpha_x \overset{V\alpha^x}{\rightarrowtail} VEx \cong MUx$ lifts to $\mathbb{D}^{\overrightarrow{G}}$. This lifting determines $H\alpha_x$. The conservativity of M assures that H is well-defined, and that the V-images of the $\mathbb{A}^{\overrightarrow{F}}$-morphisms in $\mathbb{D}$ lift to coalgebra homomorphisms in $\mathbb{D}^{\overrightarrow{G}}$.

Moral. Lifting the canonical factorization $(\mathsf{Ess} \wr \mathsf{Ffa})$ of functors to the canonical factorizations $(C^\bullet \wr \mathcal{F})$ and $(C \wr \mathcal{F}^\bullet)$ of adjunctions thus boils down to *generalizing from isomorphisms to retractions*. If the $(\mathsf{Ess} \wr \mathsf{Ffa})$-factorization confirmed that a category, from the standpoint of functors, consists of objects and arrows, then the factorizations $(C^\bullet \wr \mathcal{F})$ and $(C \wr \mathcal{F}^\bullet)$ suggest that from the standpoint of adjunctions, a category also comprises the absolute limits and colimits, a.k.a. retractions. In summary,

$$\frac{\text{functors}}{\text{category = objects + arrows}} = \frac{\text{adjunctions}}{\text{category = objects + arrows + retractions}}$$

This justifies the assumption that all idempotents can be split, announced and explained in Sec. 5.2. The readers familiar with Quillen's homotopy theory [78] may notice a homotopy model structure lurking behind the weak factorizations $(C^\bullet \wr \mathcal{F})$ and $(C \wr \mathcal{F}^\bullet)$. Corollary 8.4 suggests that the family of weak equivalences, split by the nucleus construction, consists of the functors which do not only preserve, but also reflect the absolute limits and colimits.

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A Appendix: Factorizations

Definition A.1 *A factorization system $(\mathcal{E} \wr \mathcal{M})$ in a category C a pair of subcategories $\mathcal{E}, \mathcal{M} \subseteq C$, which contain all isomorphisms, and satisfy the following requirements:*

- $C = \mathcal{M} \circ \mathcal{E}$*: for every $f \in C$ there are $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $f = m \circ e$, and*
- $\mathcal{E} \perp \mathcal{M}$*: for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and for any $f, g \in C$ such that $mu = ve$ there is a unique $h \in C$ such that $u = he$ and $v = mh$, as displayed in* (108).

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, two heads, "e"'] & C \arrow[d, tail, "m"] \\ B \arrow[r, "v"'] \arrow[ur, dotted, "h"] & D \end{tikzcd} \tag{108}$$

If h is not uniquely determined by this property, then the factorization system is weak. *The elements of $\mathcal{E}$ and of $\mathcal{M}$ are respectively called (abstract) epis amd monics.*

Proposition A.2 *In every factorization system $\mathcal{E} \wr \mathcal{M}$, the families of abstract epis and monics determine each other by*

$$\mathcal{E} \;=\; {}^{\perp}\mathcal{M} \;=\; \{e \in C \mid e \perp \mathcal{M}\} \qquad\qquad \textit{and} \qquad\qquad \mathcal{M} \;=\; \mathcal{E}^{\perp} \;=\; \{m \in C \mid \mathcal{E} \perp m\}$$

where $e \perp m$ means that e and m satisfy (108) *for all u, v, and $e \perp \mathcal{X}$ and $\mathcal{X} \perp m$ mean that $e \perp x$ and $x \perp m$ hold for all $x \in \mathcal{X}$.*

Proposition A.3 *Factorization systems in any category form a complete lattice with respect to the ordering*

$$(\mathcal{E} \wr \mathcal{M}) \leq (\mathcal{E}' \wr \mathcal{M}') \quad \Longleftrightarrow \quad \mathcal{E} \subseteq \mathcal{E}' \ \wedge \ \mathcal{M} \supseteq \mathcal{M}' \tag{109}$$

The suprema and the infima in this lattice are respectively in the forms

- $\bigwedge_{j\in J} \left(\mathcal{E}_j \wr \mathcal{M}_j\right) = \left(\hat{\mathcal{E}} \wr \hat{\mathcal{M}}\right)$ *where* $\hat{\mathcal{E}} = \bigcap_{j\in J} \mathcal{E}_j$, *and* $\hat{\mathcal{M}} = \hat{\mathcal{E}}^{\perp}$,
- $\bigvee_{j\in J} \left(\mathcal{E}_j \wr \mathcal{M}_j\right) = \left(\check{\mathcal{E}} \wr \check{\mathcal{M}}\right)$ *is determined by* $\check{\mathcal{M}} = \bigcap_{j\in J} \mathcal{M}_j$ *and* $\check{\mathcal{E}} = {}^{\perp}\check{\mathcal{M}}$.

Remark. If the category C is large, the lattice of its factorization systems is also large.

Definition A.4 *The* arrow *monad* $\mathsf{Arr} : \mathsf{CAT} \to \mathsf{CAT}$ *maps every category* C *to the induced arrow category* $\mathsf{Arr}(C) = C/C$*, supported by the monad structure*

$$\begin{tikzcd}
C \arrow[r, "\eta"] & \mathsf{Arr}(C) & & \mathsf{Arr}\left(\mathsf{Arr}(C)\right) \arrow[ll, "\mu"'] \\
A \arrow[r, mapsto] & {\begin{matrix} A \\ \downarrow \mathrm{id} \\ A \end{matrix}} & {\begin{matrix} A \\ \downarrow g\varphi = \psi f \\ D \end{matrix}} & {\begin{matrix} A & \xrightarrow{f} & C \\ \varphi\downarrow & & \downarrow\psi \\ B & \xrightarrow[g]{} & D \end{matrix}} \arrow[l, mapsto]
\end{tikzcd}$$

Proposition A.5 *Algebras for the arrow monad* $\mathsf{Arr}(C) = C \diagup C$ *[51, 68] monad* $\mathsf{Arr} : \mathsf{CAT} \to \mathsf{CAT}$ *correspond to factorization systems.*

Proof. The free Arr-algebra $C \diagup C$ comes with the canonical factorization system $\vartriangle \wr \triangledown$, where

$$\vartriangle = \{\langle \iota, f\rangle \in C^2 \mid \iota \in \mathsf{Iso}\} \qquad\qquad \triangledown = \{\langle f, \iota\rangle \in C^2 \mid \iota \in \mathsf{Iso}\}$$

where Iso is the family of all isomorphisms in C. The canonical factorization of a morphism $\langle f, g\rangle \in \mathsf{Arr}(C)(\varphi, \psi)$ thus splits its commutative square into two triangles, along the main diagonal $g \circ \varphi = \psi \circ f$, which is the canonical $(\vartriangle, \triangledown)$-image of the factored morphism:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\varphi"'] & A \arrow[r, "f^*"] \arrow[d, "R^{f_*}\circ\Phi = " left, "\Psi\circ f^*"] & C \arrow[d, "\Psi"] \\
R^B \arrow[r, "R^{f_*}"'] & R^D \arrow[r, equal] & R^D \\
B & D \arrow[l, "f_*"'] \arrow[r, equal] & D
\end{tikzcd} \tag{110}$$

A Chu-algebra $\mathsf{Chu}(\mathbb{C}) \xrightarrow{\alpha} \mathbb{C}$ determines a matrix factorization in $\mathbb{C}$ by

$$\mathcal{E} = \{\alpha(e) \mid e \in \vartriangle\} \qquad\qquad \mathcal{M} = \{\alpha(m) \mid m \in \triangledown\}$$

The other way around, any matrix $\Phi \in \mathbb{C}(A, R^B)$ lifts to $\mathsf{Chu}(\mathbb{C})$ as the morphism $\langle \Phi, \Phi^o \rangle \in \mathsf{Chu}\mathbb{C}(\eta_A, \mathrm{id}_{R^A}$, which is factorized in the form

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\eta"'] & A \arrow[r, "\Phi"] \arrow[d, "\Phi"] & R^B \arrow[d, "\mathrm{id}"] \\
R^{R^A} \arrow[r, "R^{\Phi^o}"'] & R^B \arrow[r, equal] & R^B \\
R^A & B \arrow[l, "\Phi^o"] \arrow[r, equal] & B
\end{tikzcd} \tag{111}$$

The factorization of Φ in $\mathbb{C}$ is now induced by the algebra $\mathsf{Chu}(\mathbb{C}) \xrightarrow{\alpha} \mathbb{C}$. The cochain condition for this algebra gives

$$\alpha(A, A \xrightarrow{\eta} R^{R^A}, R^A) = A \qquad \text{and} \qquad \alpha(R^B, R^B \xrightarrow{\mathrm{id}} R^B, B) = B$$

The factorization $\eta_A \xrightarrow{\langle \mathrm{id}, \Phi^o \rangle} \!\!\!\!\twoheadrightarrow \Phi \rightarrowtail^{\langle \Phi, \mathrm{id} \rangle} \mathrm{id}_{R^B}$ is then projected by α from $\mathsf{Chu}(\mathbb{C})$ to $\mathbb{C}$, and the induced factorization is thus

$$\begin{tikzcd}
A \arrow[rr, "\Phi"] \arrow[dr, two heads, "\alpha(\mathrm{id}, \Phi^o)"'] & & R^B \\
& \alpha(\Phi) \arrow[ur, tail, "\alpha(\Phi, \mathrm{id})"'] &
\end{tikzcd} \tag{112}$$

□

For a more detailed overview of abstract factorization systems, see [16, Vol. I, Sec. 5.5].

B Appendix: Adjunctions, monads, comonads

B.1 Matrices (a.k.a. distributors, profunctors, bimodules)

$$|\mathsf{Mat}| = \coprod_{\mathbb{A}, \mathbb{B} \in \mathsf{CAT}} \mathsf{Dfib} \diagup \mathbb{A} \times \mathbb{B}^o \tag{113}$$

$$\mathsf{Mat}(\Phi, \Psi) = \coprod_{\substack{H \in \mathsf{CAT}(\mathbb{A}, \mathbb{C}) \\ K \in \mathsf{CAT}(\mathbb{B}, \mathbb{D})}} \left(\mathsf{Dfib} \diagup \mathbb{A} \times \mathbb{B}^o \right) \left(\Phi, (H \times K^o)^* \Psi \right)$$

where $\Psi \in \mathsf{Dfib} \diagup \mathbb{C} \times \mathbb{D}$, and $(H \times K^o)^* \Psi$ is its pullback along $(H \times K^o) : \mathbb{A} \times \mathbb{B}^o \longrightarrow \mathbb{C} \times \mathbb{D}^o$. Obviously, $\Phi \in \mathsf{Dfib} \diagup \mathbb{A} \times \mathbb{B}^o$.

B.2 Adjunctions

$$
\begin{aligned}
|\mathsf{Adj}| &= \coprod_{\mathbb{A},\mathbb{B}\in\mathsf{CAT}} \coprod_{\substack{F^*\in\mathsf{CAT}(\mathbb{A},\mathbb{B})\\ F_*\in\mathsf{CAT}(\mathbb{B},\mathbb{A})}} \Big\{\langle\eta,\varepsilon\rangle \in \mathrm{Nat}(\mathrm{id}, F_*F^*)\times\mathrm{Nat}(F^*F_*,\mathrm{id}) \;\Big| \\
&\qquad\qquad \varepsilon F^*\circ F^*\eta = F^* \;\wedge\; F_*\varepsilon\circ\eta F_* = F_*\Big\}
\end{aligned}
\tag{114}
$$

$$
\begin{aligned}
\mathsf{Adj}(F,G) &= \coprod_{\substack{H\in\mathsf{CAT}(\mathbb{A},\mathbb{C})\\ K\in\mathsf{CAT}(\mathbb{B},\mathbb{D})}} \Big\{\langle\upsilon^*,\upsilon_*\rangle \in \mathrm{Nat}(KF^*, G^*H)\times\mathrm{Nat}(HF_*, G_*K) \;\Big| \\
&\qquad \varepsilon^G K\circ G^*\upsilon_*\circ\upsilon^*F_* = K\varepsilon^F \;\wedge\; \eta^G H = G_*\upsilon^*\circ\upsilon_*F^*\circ H\eta^F\Big\}
\end{aligned}
$$

B.3 Monads

$$
\begin{aligned}
|\mathsf{Mnd}| &= \coprod_{\mathbb{A}\in\mathsf{CAT}} \coprod_{\overleftarrow{T}\in\mathsf{CAT}(\mathbb{A},\mathbb{A})} \{\langle\eta,\mu\rangle \in \mathrm{Nat}(\mathrm{id},\overleftarrow{T})\times\mathrm{Nat}(\overleftarrow{T}\overleftarrow{T},\overleftarrow{T}) \mid \\
&\qquad \mu\circ\overleftarrow{T}\mu = \mu\circ\mu\overleftarrow{T} \wedge \mu\circ\overleftarrow{T}\eta = \overleftarrow{T} = \mu\circ\eta\overleftarrow{T}\}
\end{aligned}
\tag{115}
$$

$$
\begin{aligned}
\mathsf{Mnd}\left(\overleftarrow{T},\overleftarrow{S}\right) &= \coprod_{H\in\mathsf{CAT}(\mathbb{A},\mathbb{C})} \Big\{\chi \in \mathrm{Nat}(\overleftarrow{T}H, H\overleftarrow{S}) \;\Big| \\
&\qquad \chi\circ\eta^T H = H\eta^S \;\wedge\; H\mu^S\circ\chi S\circ T\chi = \chi\circ\mu^T H\}
\end{aligned}
$$

B.4 Comonads

$$
\begin{aligned}
|\mathsf{Cmn}| &= \coprod_{\mathbb{B}\in\mathsf{CAT}} \coprod_{\overrightarrow{T}\in\mathsf{CAT}(\mathbb{B},\mathbb{B})} \{\langle\varepsilon,\nu\rangle \in \mathrm{Nat}(\overrightarrow{T},\mathrm{id})\times\mathrm{Nat}(\overrightarrow{T},\overrightarrow{T}\overrightarrow{T}) \mid \\
&\qquad \overrightarrow{T}\nu\circ\nu = \nu\overrightarrow{T}\circ\nu \wedge \overrightarrow{T}\varepsilon\circ\nu = \overrightarrow{T} = \varepsilon\overrightarrow{T}\circ\nu\}
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
\mathsf{Cmn}\left(\overrightarrow{S},\overrightarrow{T}\right) &= \coprod_{K\in\mathsf{CAT}(\mathbb{B},\mathbb{D})} \Big\{\kappa \in \mathrm{Nat}(K\overrightarrow{S},\overrightarrow{T}K) \;\Big| \\
&\qquad \varepsilon^T K\circ\kappa = K\varepsilon^S \;\wedge\; \overrightarrow{T}\kappa\circ\kappa\overrightarrow{S}\circ K\nu^S = \nu^S K\circ\kappa\Big\}
\end{aligned}
$$

B.5 The initial (Kleisli) resolutions $\mathsf{KM} : \mathsf{Mnd} \to \mathsf{Adj}$ and $\mathsf{KC} : \mathsf{Cmn} \to \mathsf{Adj}$

The Kleisli construction assigns to the monad $T : \mathbb{A} \to \mathbb{A}$ the resolution $\overleftarrow{\mathcal{K}}T = \left(T^\flat \dashv T_\flat : \mathbb{A}_{\overleftarrow{T}} \to \mathbb{A}\right)$ where the category $\mathbb{A}_{\overleftarrow{T}}$ consists of

- *free algebras* as objects, which boil down to $|\mathbb{A}_{\overleftarrow{T}}| = |\mathbb{A}|$;

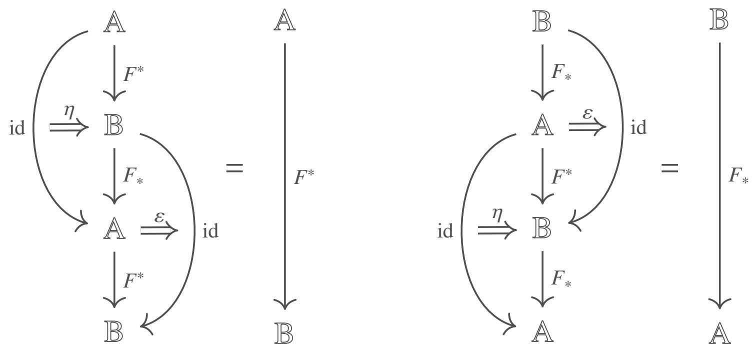

Figure 32: Pasting equations for adjunction $F^* \dashv F_*$.

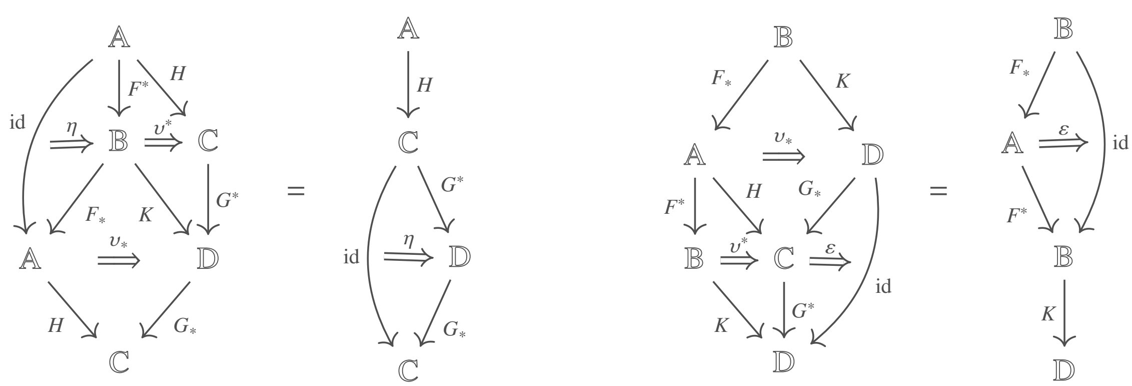

Figure 33: Pasting equations for adjunction 1-cell $\langle H, K, \upsilon^*, \upsilon_* \rangle : F \to G$.

$$
\begin{tikzcd}
\mathbb{A} \arrow[d, "T"'] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}" near start] & \mathbb{A} \arrow[l, "T"'] \arrow[dl, "T"] \arrow[dd, "T"] & & & \mathbb{A} \arrow[dl, "T"'] \arrow[dd, "T"] \\
\mathbb{A} \arrow[dr, "T"'] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & & = & \mathbb{A} \arrow[d, "T"'] \arrow[dr, "T"] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & \\
& \mathbb{A} & & \mathbb{A} \arrow[r, "T"'] \arrow[ur, phantom, "\overset{\mu}{\Rightarrow}" near start] & \mathbb{A}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{A} \arrow[dl, "T"] \arrow[dl, bend right=60, "\mathrm{id}"'] \arrow[dl, phantom, bend right=30, "\overset{\eta}{\Rightarrow}"] \arrow[dd, "T"] & & \mathbb{A} \arrow[dd, "T"] & & & \mathbb{A} \arrow[dl, "T"'] \arrow[dd, "T"] \\
\mathbb{A} \arrow[dr, "T"'] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & & = & & = & \mathbb{A} \arrow[dr, "T"] \arrow[dr, bend right=60, "\mathrm{id}"'] \arrow[dr, phantom, bend right=30, "\overset{\eta}{\Rightarrow}"] \arrow[r, phantom, "\overset{\mu}{\Rightarrow}"] & \\
& \mathbb{A} & & \mathbb{A} & & & \mathbb{A}
\end{tikzcd}
$$

Figure 34: Pasting equations for monad $\overleftarrow{T}$ on $\mathbb{A}$.

$$
\begin{tikzcd}
& \overleftarrow{T}H \arrow[dd, "\chi"] & \overleftarrow{T}\overleftarrow{T}H \arrow[l, "\mu^T H"'] \arrow[d, "\overleftarrow{T}\chi"] \\
H \arrow[ur, "\eta^T H"] \arrow[dr, "H\eta^S"'] & & \overleftarrow{T}H\overleftarrow{S} \arrow[d, "\chi\overleftarrow{S}"] \\
& H\overleftarrow{S} & H\overleftarrow{S}\overleftarrow{S} \arrow[l, "H\mu^S"]
\end{tikzcd}
$$

Figure 35: Commutative diagrams for monad 1-cell $\langle H, \chi \rangle : \overleftarrow{T} \to \overleftarrow{S}$.

$$
\begin{tikzcd}
\mathbb{A} \arrow[d, "S"'] \arrow[dr, phantom, "\Uparrow\nu" near start] & \mathbb{A} \arrow[l, "S"'] \arrow[dl, "S"] \arrow[dd, "S"] & & & \mathbb{A} \arrow[dl, "S"'] \arrow[dd, "S"] \\
\mathbb{A} \arrow[dr, "S"'] \arrow[r, phantom, "\Leftarrow\nu"] & & = & \mathbb{A} \arrow[r, phantom, "\Leftarrow\nu"] \arrow[d, "S"'] \arrow[dr, "S"] & \\
& \mathbb{A} & & \mathbb{A} \arrow[r, "S"'] \arrow[ur, phantom, "\Downarrow\nu" near start] & \mathbb{A}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{A} \arrow[dl, "S"] \arrow[dd, "S"] \arrow[dl, bend right=60, "\mathrm{id}"'] \arrow[dl, phantom, "\Uparrow\varepsilon", shift right=3] & & \mathbb{A} \arrow[dd, "S"] & & & \mathbb{A} \arrow[dl, "S"'] \arrow[dd, "S"] \\
\mathbb{A} \arrow[dr, "S"'] \arrow[r, phantom, "\Leftarrow\nu"] & & = & & = & \mathbb{A} \arrow[r, phantom, "\Leftarrow\nu"] \arrow[dr, "S"] \arrow[dr, bend right=60, "\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\varepsilon", shift right=3] & \\
& \mathbb{A} & & \mathbb{A} & & & \mathbb{A}
\end{tikzcd}
$$

Figure 36: Pasting equations for comonad $\overrightarrow{S}$ on $\mathbb{B}$

$$
\begin{tikzcd}
& \overrightarrow{T}K \arrow[dl, "\varepsilon^T K"'] \arrow[r, "\nu^T H"] & \overrightarrow{T}\,\overrightarrow{T}K \\
K & & \overrightarrow{T}H\overrightarrow{S} \arrow[u, "\overrightarrow{T}\kappa"] \\
& K\overrightarrow{S} \arrow[ul, "K\varepsilon^S"] \arrow[uu, "\kappa"] \arrow[r, "K\nu^S"'] & K\overrightarrow{S}\,\overrightarrow{S} \arrow[u, "\kappa\overrightarrow{S}"]
\end{tikzcd}
$$

Figure 37: Commutative diagrams for comonad 1-cell $\langle K, \kappa \rangle : \overrightarrow{S} \to \overrightarrow{T}$.

- *algebra homomorphisms* as arrows, which boil down to $\mathbb{A}_{\overleftarrow{T}}(x,x') = \mathbb{A}(x,Tx')$;

with the composition

$$\begin{array}{ccc} \mathbb{A}_{\overleftarrow{T}}(x,x') \times \mathbb{A}_{\overleftarrow{T}}(x',x'') & \xrightarrow{\circ} & \mathbb{A}_{\overleftarrow{T}}(x,x'') \\ \langle x \xrightarrow{f} Tx' \,,\; x' \xrightarrow{g} Tx'' \rangle & \longmapsto & (x \xrightarrow{f} Tx' \xrightarrow{Tg} TTx'' \xrightarrow{\mu} Tx'') \end{array}$$

and with the identity on x induced by the monad unit $\eta : x \to Tx$

B.6 The final (Eilenberg-Moore) resolutions $\mathsf{EM} : \mathsf{Mnd} \to \mathsf{Adj}$ and $\mathsf{EC} : \mathsf{Cmn} \to \mathsf{Adj}$

The Eilenberg-Moore construction assigns to the monad $T : \mathbb{A} \to \mathbb{A}$ the resolution $\overleftarrow{\mathcal{E}}T = \left(T^{\sharp} \dashv T_{\sharp} : \mathbb{A}^{\overleftarrow{T}} \to \mathbb{A}\right)$ where the category $\mathbb{A}^{\overleftarrow{T}}$ consists of

- *all algebras* as objects:

$$|\mathbb{A}^{\overleftarrow{T}}| \;=\; \sum_{x \in |\mathbb{A}|} \{\alpha \in \mathbb{A}(Tx,x) \mid \alpha \circ \eta = \mathrm{id} \;\wedge\; \alpha \circ T\alpha = \alpha \circ \mu\}$$

- *algebra homomorphisms* as arrows:

$$\mathbb{A}^{\overleftarrow{T}}(Tx \xrightarrow{\alpha} x, Tx' \xrightarrow{\gamma} x') \;=\; \{f \in \mathbb{A}(x,x') \mid f \circ \alpha = \gamma \circ Tf\}$$

C Appendix: Split equalizers

Split equalizers and coequalizers[14, 15] are conventionally written as *partially* commutative diagrams: the straight arrows commute, the epi-mono splittings compose to identities on the quotient side, and to equal idempotents on the other side.

Proposition C.1 *Consider the split equalizer diagram*

$$\begin{tikzcd} A \arrow[r, tail, "i"] & B \arrow[l, two heads, bend left, "q"] \arrow[r, tail, shift left, "f"] \arrow[r, tail, shift right, "j"'] & C \arrow[l, two heads, bend left, "r"] \end{tikzcd} \tag{117}$$

where

$$q \circ i = \mathrm{id}_A \qquad r \circ j = \mathrm{id}_B \qquad f \circ r \circ f = j \circ r \circ f$$

Then

- *$r \circ f$ is idempotent and*
- *i is the equalizer of f and j if and only if $i \circ q = r \circ f$.*