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Smoothing techniques and difference of convex functions algorithms for image reconstructions

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ABSTRACT

In this paper, we study characterizations of differentiability for real-valued functions based on generalized differentiation. These characterizations provide the mathematical foundation for Nesterov's smoothing techniques in infinite dimensions. As an application, we provide a simple approach to image reconstructions based on Nesterov's smoothing and algorithms for minimizing differences of convex (DC) functions that involve the $\ell_1 - \ell_2$ regularization.

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1. Introduction and problem formulation

Gradient-based methods in optimization have been strongly developed over the last decades to solve optimization problems. One of the disadvantages of these methods is the requirement of the differentiability of the objective functions involved, while nondifferentiability appears frequently and naturally in many optimization models. A natural way to cope with the nondifferentiability in optimization is to approximate nonsmooth objective functions by smooth functions that are favourable for applying smooth optimization schemes. In his seminal paper [1], Nesterov proposed a method for approximating a class of nondifferentiable convex functions by smooth convex functions with Lipschitz continuous gradients. It turns out that Nesterov's smoothing is highly important in solving nonsmooth optimization problems in many fields such as facility location, sparse optimization and compressed sensing.

The first goal of this paper is to study Nesterov's smoothing techniques in infinite dimensions. Note that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable at $\bar{x} \in \mathbb{R}^n$ if and only if the subdifferential in the sense of convex analysis $\partial f(\bar{x})$ reduces to a singleton. This is the starting point for further studies of

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Dedicated to Boris S. Mordukhovich on the occasion of his 70th birthday

characterizations of differentiability for both convex and nonconvex functions in infinite dimensions based on generalized differentiation. These characterizations allow us to study Nesterov's smoothing techniques in a more general setting, while having the potential of applying to broader classes of nondifferentiable functions usually considered in optimization.

Along with the difficulty in dealing with nondifferentiability, another challenge in modern optimization is to go from convexity to nonconvexity as nonconvex optimization techniques and algorithms allow us to solve more complex optimization problems arising naturally in many practical applications. This is a motivation of the search for new optimization methods to deal with broader classes of functions and sets where convexity is not assumed. One of the most successful approaches to go beyond convexity is to consider the class of DC (difference of convex) functions. Given a linear space X, a DC program is an optimization problem in which we would like to minimize a function $f : X \to \mathbb{R}$ representable as $f = g - h$, where $g, h : X \to \mathbb{R}$ are convex functions. This extension of convex programming is not too far to take advantage of the available tools from convex analysis and optimization. At the same time, DC programming is sufficiently large to apply to many nonconvex optimization problems faced in recent applications.

Although the role of DC functions had been known earlier in optimization theory, the first algorithmic approach was developed by Pham Dinh Tao in 1985. The algorithm introduced by Pham Dinh Tao for minimizing $f = g - h$, called the DCA, is based on subgradients of the function h and subgradients of the Fenchel conjugate of the function g. This algorithm is summarized as follows: with given $x_1 \in \mathbb{R}^n$, define $y_k \in \partial h(x_k)$ and $x_{k+1} \in \partial g^*(y_k)$. Under suitable conditions on the DC decomposition of the function f, two sequences $\{x_k\}$ and $\{y_k\}$ in the DCA satisfy the monotonicity conditions in the sense that $\{g(x_k) - h(x_k)\}$ and $\{h^*(y_k) - g^*(y_k)\}$ are both decreasing. In addition, the sequences $\{x_k\}$ and $\{y_k\}$ converge to *critical points* of the primal function $g - h$ and the dual function $h^* - g^*$, respectively. In practice, with suitable initialization techniques, the DCA is very effective, producing sequences that converge to global solutions of the problem; see [2–4] and the references therein.

In our recent research, we have been successful in applying Nesterov's smoothing techniques and the DCA to a number of optimization problems in facility location and clustering. This paper continues this effort by providing their applications to image reconstructions. Consider an unknown image M of size $N_1 \times N_2$. After the image is corrupted by a linear operator A and distorted by some noise ξ, we observe only the image $b = A(M) + \xi$, and seek to recover the true image M. The operator may act to simulate blurring, data compression or down-sampling. In the case that the operator is the identity, the problem is called denoising.

We denote the columns of M as $m_1, \ldots, m_{N_2}$, to represent M in *vectorized form* as the $N_1N_2 \times 1$ column vector $M = [m_1^T \; m_2^T \; \cdots \; m_{N_2}^T]^{\mathrm{T}}$. The vectorized form of M can be attained in MATLAB using the `reshape` function and is equivalent to $\sum_{i=1}^{N_2}(e_i \otimes I_{N_1})(M \cdot e_i)$, where e_i is the ith standard basis vector in $\mathbb{R}^{N_2}$ and $\otimes$ is the Kronecker product. Conversely, a vector $M \in \mathbb{R}^{N_1 N_2}$ is reshaped into an $N_1 \times N_2$ matrix.

A vector is referred to as sparse when many of its entries are zeros. An image $x \in \mathbb{R}^n$ (in vectorized form) is said to have a sparse representation y under D if there is some $n \times K$ matrix D, known as a dictionary, and a vector $y \in \mathbb{R}^K$ such that $x = Dy$. In this case, the dictionary D maps a sparse vector to a full image. The columns of D are called *atoms*, and given a suitable dictionary in this model, theoretically any image can be built from a linear combination of the columns (atoms) of the dictionary. Using a clever choice of dictionary allows us to work with sparse vectors, thereby reducing the amount of computer memory needed to store an image. Further, sparse representations tend to capture the true image without extraneous noise.

The method in this paper is based on the following foundational model: given a dictionary D and an observed image b which has been corrupted by a linear operator A, recover a sparse representation of the image by solving the minimization problem

$$\min_y \|y\|_0 \quad \text{s.t.} \quad \|A(Dy) - b\|^2 \le \varepsilon$$

where $\varepsilon > 0$ is some small constraint term. Here $\|y\|_0$ is not a norm, but simply the number of nonzero entries in y. It is common to approximate $\|\cdot\|_0$ with $\|\cdot\|_1$, or with $\|\cdot\|_1 - \|\cdot\|$, known as ℓ_1 and $(\ell_1 - \ell_2)$ regularization, respectively. In this paper, using the DCA and Nesterov's smoothing techniques, we develop a very simple algorithms based on the $(\ell_1 - \ell_2)$ regularization for image reconstructions. The proposed method allows us to avoid solving subproblems in using the DCA for the $(\ell_1 - \ell_2)$ regularization; see [5,6] and the references therein. We also apply this idea to build a simple but effective algorithm for dictionary learning. Our numerical examples show that our algorithms are competitive with state-of-the-art methods for image reconstructions.

Our paper contains two main sections. In Section 2, we study characterizations of different concepts of differentiability and strict differentiability with applications to Nesterov's smoothing techniques in infinite dimensions. In Section 3, we provide a simple method for image reconstructions using Nesterov's smoothing techniques and the DCA. Throughout the paper, we use standard notations of convex and nonsmooth analysis. For a vector $x \in \mathbb{R}^d$, we use $\|x\|_1$ and $\|x\|$ to denote its ℓ^1-norm and Euclidean norm, respectively.

2. Characterizations of differentiability and Nesterov's smoothing techniques

In this section, we study characterizations of strict differentiability and their applications to smoothing techniques. Consider a real normed space X with its topological dual denoted by X^* which consists of all real-valued linear continuous functions defined on X. It is well known that X^* is a normed space with the dual norm given by

$$\|x^*\| = \sup\{\langle x^*, x\rangle \mid \|x\| \leq 1\}, x^* \in X^*,$$

where $\langle x^*, x\rangle = x^*(x)$.

Let $f : X \to \overline{\mathbb{R}} = (-\infty, \infty]$ be an extended real-valued function with the effective domain $\text{dom}\,(f) = \{x \in X \mid f(x) < \infty\}$ and $\bar{x} \in int(dom\,(f))$. We first recall some classical concepts of differentiability. We say that f is *Gâteaux differentiable* at $\bar{x}$ if there exists $x^* \in X^*$ such that

$$\lim_{t\to 0+} \frac{f(\bar{x}+td) - f(\bar{x}) - t\langle x^*, d\rangle}{t} = 0 \text{ for all } d \in X.$$

Such an element x^* is unique if exists and is called the *Gâteaux derivative* of f at $\bar{x}$ denoted by $\nabla_G f(\bar{x})$. It follows directly from the definition that f is Gâteaux differentiable at $\bar{x}$ with $\nabla_G f(\bar{x}) = x^*$ if and only if

$$\lim_{t\to 0} \frac{f(\bar{x}+td) - f(\bar{x}) - t\langle x^*, d\rangle}{t} = 0 \text{ for all } d \in X.$$

We say that f is *Fréchet differentiable* at $\bar{x}$ if there exists $x^* \in X^*$ such that

$$\lim_{h\to 0} \frac{f(\bar{x}+h) - f(\bar{x}) - \langle x^*, h\rangle}{\|h\|} = 0.$$

The element x^* is called the *Fréchet derivative* of f at $\bar{x}$ denoted by $\nabla_F f(\bar{x})$. It follows from the definition that if f is Fréchet differentiable at $\bar{x}$, then it is Gâteaux differentiable at this point.

Let us now discuss three important concepts of strict differentiability. Let $f : X \to \overline{\mathbb{R}}$ be an extended real-valued function with $\bar{x} \in int(dom(f))$.

We say that f is *Fréchet strictly differentiable* at $\bar{x}$ if there exists an element $x^* \in X^*$ such that

$$\lim_{x,y\to\bar{x}, x\neq y} \frac{f(x) - f(y) - \langle x^*, x-y\rangle}{\|x-y\|} = 0.$$

In this case, the element x^* is uniquely defined and is called the *Fréchet strict derivative* of f at $\bar{x}$ denoted by $\nabla_{sF} f(\bar{x})$.

We say that f is *Hadamard strictly differentiable* at $\bar{x}$ if there exists $x^* \in X^*$ such that for any $d \in X$ we have

$$\lim_{t\to 0^+, x\to\bar{x}} \frac{f(x+td) - f(x) - t\langle x^*, d\rangle}{t} = 0,$$

where the convergence is uniform for d in compact subset of X. Similarly, we call x^* the *Hadamard strict derivative* of f at $\bar{x}$ denoted by $\nabla_{sH} f(\bar{x})$.

We say that f is *Gâteaux strictly differentiable* at $\bar{x}$ if there exists an element $x^* \in X^*$ such that for any $d \in X$ we have

$$\lim_{x \to \bar{x}, t \to 0^+} \frac{f(x+td) - f(x) - t\langle x^*, d\rangle}{t} = 0.$$

In this case, the element x^* is uniquely defined and is called the *Gâteaux strict derivative* of f at $\bar{x}$ denoted by $\nabla_{sG} f(\bar{x})$.

Observe from the definition that

$$\boxed{\text{Fréchet strict differentiability}} \Rightarrow \boxed{\text{Hadamard strict differentiability}} \Rightarrow \boxed{\text{Gâteaux strict differentiability}}.$$

The Proposition provides further relation between Gâtaeux differentiability and Hadamard strict differentiability, see [7, Proposition 2.2.1].

Proposition 2.1: *Let X be a normed space and let $f : X \to \overline{\mathbb{R}}$ with $\bar{x} \in int(dom\,(f))$. Then the following properties are equivalent:*

(a) *f is Hadamard strictly differentiable at $\bar{x}$.*
(b) *f is locally Lipschitz continuous around $\bar{x}$ and Gâteaux strictly differentiable at $\bar{x}$.*

Remark 2.2: (a) The Fréchet strict differentiability is equivalent to the Hadamard strict differentiability in finite dimensions.
(b) For locally Lipschitz continuous function, a natural question is whether the Hadamard strict differentiability implies the Fréchet strict differentiability. The answer is no and we can find a counterexample such as $f(x) = \|x\|_1$ for $x \in \ell^1$.

Given a function $f : X \to \overline{\mathbb{R}}$ that is locally Lipschitz continuous around $\bar{x} \in int(dom\,(f))$, the *Clarke generalized directional derivative* of f at $\bar{x}$ in the direction $d \in X$ is defined by

$$f^\circ(\bar{x}; d) = \limsup_{x \to \bar{x}, t \to 0^+} \frac{f(x+td) - f(x)}{t}.$$

Based on the generalized derivative of f at $\bar{x}$, the *Clarke subdifferential* of f at $\bar{x}$ is defined by

$$\partial_C f(\bar{x}) = \{x^* \in X^* \mid \langle x^*, d\rangle \le f^\circ(\bar{x}; d) \ \text{ for all } d \in X\}.$$

Note that in the case where f is convex,

$$\partial_C f(\bar{x}) = \partial f(\bar{x}) = \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \le f(x) - f(\bar{x}) \ \text{ for all } x \in X\},$$

which is the *subdifferential in the sense of convex analysis* of f at $\bar{x}$. A characterization for Hadamard strict differentiability is given in the proposition below, see [7, Proposition 2.2.4].

Theorem 2.3: *Let X be a normed space and let $f : X \to \overline{\mathbb{R}}$ with $\bar{x} \in int(dom\,(f))$. If f is Hadamard strictly differentiable at $\bar{x}$, then f is locally Lipschitz continuous around $\bar{x}$ and $\partial_C f(\bar{x}) = \{\nabla_{sH} f(\bar{x})\}$. Conversely, if f is locally Lipschitz continuous around $\bar{x}$ and $\partial_C f(\bar{x})$ reduces to a singleton $\{x^*\}$, then f is Hadamard strictly differentiable at $\bar{x}$ and $\nabla_{sH} f(\bar{x}) = x^*$.*

The following is a direct consequence of this result for the convex case.

Corollary 2.4: *Let X be a normed space and let $f : X \to \overline{\mathbb{R}}$ be a convex function with $\bar{x} \in int(dom\,(f))$. Then the following properties are equivalent:*

(a) *f is Hadamard strictly differentiable at $\bar{x}$.*
(b) *f is continuous at $\bar{x}$ and f is Gâteaux differentiable at this point.*
(c) *f is continuous at $\bar{x}$ and $\partial f(\bar{x})$ is a singleton.*

Proof: (a) $\Rightarrow$ (b): This is a direct consequence of Theorem 2.3 taking into account that the strict Gâteaux differentiability implies the Gâteaux differentiability.

(b) $\Rightarrow$ (c): This implication is straight forward.

(c) $\Rightarrow$ (a): By the convexity of f, its continuity at $\bar{x}$ implies the local Lipschitz continuity around $\bar{x}$. Note that in this case the Clarke subdifferential of f at $\bar{x}$ reduces to the subdifferential in the sense of convex analysis at this point, i.e. $\partial_C f(\bar{x}) = \partial f(\bar{x})$. Thus, this implication follows directly from Theorem 2.3. ■

Note that the equivalence of (a) and (b) in Corollary 2.4 does not hold true in the general nonconvex setting as shown in the example below.

Example 2.5: Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then we can show that f is not Hadamard strictly differentiable at 0, but it is Fréchet differentiable and thus continuous and Gâteaux differentiable at this point.

In what follows we study a characterization for Fréchet differentiability based on Clarke subdifferentials. For a nonempty subset Ω and $\bar{x} \in X$, the notation $d(\bar{x}; \Omega)$ is used for the distance from $\bar{x}$ to Ω defined by

$$d(\bar{x}; \Omega) = \inf\{\|\bar{x} - w\| \mid w \in \Omega\}.$$

Given a set-valued mapping $F : X \rightrightarrows X^*$, where both X and X^* are equipped with the strong topology. We say that F is *upper semicontinuous* at $\bar{x} \in \text{dom}\,(F) :=$

$\{x \in X \mid F(x) \neq \emptyset\}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(x) \subset \mathbb{B}(F(\bar{x}); \varepsilon) \quad \text{whenever } x \in \mathbb{B}(\bar{x}; \delta),$$

where $\mathbb{B}(F(\bar{x}); \varepsilon) = \{x^* \in X^* \mid d(x^*; F(\bar{x})) \leq \varepsilon\}$ and $\mathbb{B}(\bar{x}; \delta)$ denotes the closed ball with centre $\bar{x}$ and radius δ in X.

The following version of the mean value theorem (see [7]) is useful in what follows.

Theorem 2.6: *Let X be a normed space and let $f : X \to \mathbb{R}$ be Lipschitz continuous on an open set $G \subset X$. For any $[a, b] \subset G$, there exists $z \in (a, b)$ such that*

$$f(b) - f(a) \in \langle \partial_C f(z), b - a \rangle.$$

Let us now present a characterization of Fréchet strict differentiability based on Clarke subdifferentials, see, e.g. [8]. We provide an alternative detailed proof here for the convenience of the reader.

Theorem 2.7: *Let X be a normed space and let $f : X \to \overline{\mathbb{R}}$ with $\bar{x} \in int(dom\,(f))$. Then the following properties are equivalent:*

(a) *f is Fréchet strictly differentiable at $\bar{x}$.*
(b) *f is locally Lipschitz continuous around $\bar{x}$, $\partial_C f(\bar{x})$ is a singleton, and $\partial_C f(\cdot)$ is upper semicontinuous at $\bar{x}$.*

Proof: (a) $\Rightarrow$ (b): Suppose that f is Fréchet strictly differentiable at $\bar{x}$. Then it is Hadamard strictly differentiable at this point. By Theorem 2.3, the function f is locally Lipschitz continuous around $\bar{x}$ and $\partial_C f(\bar{x})$ is a singleton. It remains to show that $\partial_C f(\cdot)$ is upper semicontinuous at $\bar{x}$. Fix any sequence $\{x_k\}$ in X that converges to $\bar{x}$, and fix any $x_k^* \in \partial_C f(x_k)$. Fix any $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|f(x) - f(u) - \langle \nabla_{sF} f(\bar{x}), x - u \rangle| \leq \varepsilon \|x - u\| \quad \text{whenever } x, u \in \mathbb{B}(\bar{x}; \delta).$$

Since $\{x_k\}$ converges to $\bar{x}$, we can find $k_0 \in \mathbb{N}$ such that $x_k \in \mathbb{B}(\bar{x}; \delta/4)$ for all $k \geq k_0$. Fix any $k \geq k_0$ and any $v \in X$. If $\|x - x_k\| < \delta/4$ and $0 < t < \delta/(2(\|v\| + 1))$, then $\|x - \bar{x}\| \leq \|x - x_k\| + \|x_k - \bar{x}\| < \delta/4 + \delta/4 = \delta/2 < \delta$. It follows that $\|x + tv - \bar{x}\| \leq \|x - \bar{x}\| + t\|v\| < \delta/2 + \delta/2 = \delta$, and so

$$f(x + tv) - f(x) \leq \langle \nabla_s Ff(\bar{x}), tv \rangle + \varepsilon \|tv\|.$$

This implies

$$\frac{f(x + tv) - f(x)}{t} \leq \langle \nabla_{sF} f(\bar{x}), v \rangle + \varepsilon \|v\| \quad \text{for such } x, t.$$

It follows that

$$f^\circ(x_k; v) = \limsup_{t\to 0^+, x\to x_k} \frac{f(x+tv)-f(x)}{t} \leq \langle \nabla_{sF} f(\bar{x}), v\rangle + \varepsilon\|v\|.$$

Now, $\langle x_k^*, v\rangle \leq f^\circ(x_k; v) \leq \langle \nabla_{sF} f(\bar{x}), v\rangle + \varepsilon\|v\|$. Then $\|x_k^* - \nabla_{sF} f(\bar{x})\| \leq \varepsilon$ for all $k \geq k_0$. Therefore, $\{x_k^*\}$ converges strongly to $\nabla_{sF} f(\bar{x})$. The upper semicontinuity of the Clarke subdifferential mapping is now straightforward.

(b) $\Rightarrow$ (a): Let us prove the converse by assuming that (b) is satisfied. Suppose that $\partial_C f(\bar{x}) = \{x^*\}$. Since $\partial_C f(\cdot)$ is upper semicontinuous at $\bar{x}$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\partial_C f(u) \subset \mathbb{B}(x^*, \varepsilon) \text{ whenever } u \in \mathbb{B}(\bar{x}, \delta).$$

We can choose $\delta > 0$ sufficiently small such that f is Lipschitz continuous on the open ball $B(\bar{x}, \delta)$. Fix any $x, y \in B(\bar{x}; \delta)$ with $x \neq y$. By Theorem 2.6, there exist $u \in (x, y)$ and $w^* \in \partial_C f(u)$ such that

$$f(x) - f(y) = \langle w^*, x - y\rangle.$$

Then $\|w^* - x^*\| \leq \varepsilon$, and hence

$$\left|\frac{f(x) - f(y) - \langle x^*, x-y\rangle}{\|x-y\|}\right| = \left|\frac{\langle w^* - x^*, x-y\rangle}{\|x-y\|}\right| \leq \|w^* - x^*\| \leq \varepsilon.$$

Therefore, f is Fréchet strictly differentiable at $\bar{x}$. ■

Let us now derive a corollary for the convex case.

Corollary 2.8: *Let X be a normed space and let $f : X \to \overline{\mathbb{R}}$ be a convex function with $\bar{x} \in int(dom\,(f))$. Then the following properties are equivalent:*

(a) *f is Fréchet strictly differentiable at $\bar{x}$.*
(b) *f is Fréchet differentiable at $\bar{x}$.*
(c) *f is locally Lipschitz continuous around $\bar{x}$, $\partial f(\bar{x})$ is a singleton, and $\partial f(\cdot) : X \rightrightarrows X^*$ is upper semicontinuous at $\bar{x}$.*

Proof: The implication (a) $\Rightarrow$ (b) is obvious. If f is Fréchet differentiable at $\bar{x}$, it is well known that $\partial_C f(\bar{x}) = \partial f(\bar{x}) = \{\nabla_F f(\bar{x})\}$ under the convexity of f. In addition, f is locally bounded around $\bar{x}$, so it is locally Lipschitz continuous around this point. Let us show that the subdifferential mapping is upper semicontinuous at $\bar{x}$. Let $x^* = \nabla_F f(\bar{x})$. We have

$$\lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} = 0.$$

For any $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$\langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \varepsilon \|x - \bar{x}\| \text{ whenever } \|x - \bar{x}\| < \delta.$$

Fix any $x \in \mathbb{B}(\bar{x}; \delta)$ and any $u^* \in \partial f(x)$. Then

$$\begin{aligned}\langle x^* - u^*, x - \bar{x} \rangle &= \langle x^*, x - \bar{x} \rangle + \langle u^*, \bar{x} - x \rangle \\ &\leq f(x) - f(\bar{x}) + \varepsilon \|x - \bar{x}\| + f(\bar{x}) - f(x) = \varepsilon \|x - \bar{x}\|.\end{aligned}$$

This implies $\|x^* - u^*\| \leq \varepsilon$, which justifies the upper semicontinuity of the subdifferential mapping. Thus the implication (b)$\Rightarrow$ (c) holds. Finally, the implication (c)$\Rightarrow$ (a) follows from Theorem 2.7. ∎

Next, we will study characterizations of strict differentiability via Mordukhovich/limiting subdifferentials in *Asplund spaces*; see [9]. Consider an extended real-valued function $f : X \to \overline{\mathbb{R}}$. In the sequel, the notation $x \xrightarrow{f} \bar{x}$ means that $x \to \bar{x}$ and $f(x) \to f(\bar{x})$. Given $\varepsilon \geq 0$, the ε-*Fréchetsubdifferential* of f at $\bar{x} \in \operatorname{dom} f$ is the set

$$\widehat{\partial}_\varepsilon f(\bar{x}) = \left\{ x^* \in X^* \;\middle|\; \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\varepsilon \right\}.$$

The set $\widehat{\partial}_0 f(\bar{x})$ ($\varepsilon = 0$) is called the *Fréchet subdifferential* of f at $\bar{x}$ and is denoted simply by $\widehat{\partial} f(\bar{x})$. The *limiting/Mordukhovich subdifferential* of f at $\bar{x}$ is defined by the *Kuratowski upper limit*:

$$\partial_M f(\bar{x}) = \operatorname*{Lim\,sup}_{x \xrightarrow{f} \bar{x}, \varepsilon \to 0^+} \widehat{\partial}_\varepsilon f(x).$$

Proposition 2.9: *Let X be an Asplund space and let $f : X \to \overline{\mathbb{R}}$ with $\bar{x} \in$ int(dom (f)). If f is Hadamard strictly differentiable at $\bar{x}$, then f is locally Lipschitz continuous around $\bar{x}$ and $\partial_M f(\bar{x}) = \{\nabla_{sH} f(\bar{x})\}$. Conversely, if f is locally Lipschitz continuous around $\bar{x}$ and $\partial_M f(\bar{x})$ reduces to a singleton $\{x^*\}$, then f is Hadamard strictly differentiable at $\bar{x}$ and $\nabla_{sH} f(\bar{x}) = x^*$.*

Proof: Suppose that f is Hadamard strictly differentiable at $\bar{x}$. By Theorem 2.3, the function f is locally Lipschitz continuous around $\bar{x}$ and $\partial_C f(\bar{x}) = \{\nabla_{sH} f(\bar{x})\}$. In addition, by [9, Corollary 2.25], $\partial_M f(\bar{x})$ is nonempty and $\partial_M f(\bar{x}) \subset \partial_C f(\bar{x})$. It follows that $\partial_M f(\bar{x})$ is a singleton. Now, suppose that f is locally Lipschitz continuous around $\bar{x}$ and $\partial_M f(\bar{x})$ reduces to a singleton $\{x^*\}$. By [9, Theorem 3.57],

$$\partial_C f(\bar{x}) = \text{cl}^*\text{co}\left(\partial_M f(\bar{x})\right), \tag{1}$$

here cl* stands for the weak* topological closure of a set in X^*. This implies that $\partial_C f(\bar{x})$ is a singleton, and so by Theorem 2.3, f is Hadamard strictly differentiable at $\bar{x}$. ■

Theorem 2.10: *Let X be a normed space and let $f : X \to \overline{\mathbb{R}}$ with $\bar{x} \in int(dom\,(f))$. Consider the following statements:*

(a) *f is Fréchet strictly differentiable at $\bar{x}$.*
(b) *f is locally Lipschitz continuous around $\bar{x}$, $\partial_M f(\bar{x})$ is a singleton, and $\partial_M f(\cdot)$ is upper semicontinuous at $\bar{x}$.*

If X is an Asplund space, then (a) *implies* (b). *The converse also holds true if we assume that X is reflexive.*

Proof: (a) $\Rightarrow$ (b): Suppose that (a) is satisfied. By Theorem 2.7 and the fact that $\emptyset \neq \partial_M f(\bar{x}) \subset \partial_C f(\bar{x})$, the function f is locally Lipschitz continuous around $\bar{x}$ and $\partial_M f(\bar{x})$ is a singleton. Let $x^* = \nabla_{sF} f(\bar{x})$. The upper semicontinuity of $\partial_M f(\cdot)$ follows directly from that of $\partial_C f(\cdot)$, which is also guaranteed by Theorem 2.7.

(b) $\Rightarrow$ (a): Suppose that (b) is satisfied with $\partial_M f(\bar{x}) = \{x^*\}$. By (1), we have $\partial_C f(\bar{x}) = \{x^*\}$. Applying Theorem 2.7, it suffices to show that $\partial_C f(\cdot)$ is upper semicontinuous at $\bar{x}$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|u^* - x^*\| < \varepsilon \quad \text{whenever } \|x - \bar{x}\| < \delta, u^* \in \partial_M f(x).$$

We can choose $\delta > 0$ sufficiently small so that f is locally Lipschitz continuous on $\mathbb{B}(\bar{x}; \delta)$. Fix any $x \in X$ with $\|x - \bar{x}\| < \delta$ and fix any $z^* \in \text{co}(\partial_M f(x))$. Then there exist $\lambda_i \geq 0$ and $u_i^* \in \partial_M f(x)$ for $i = 1, \ldots, m$ with $\sum_{i=1}^m \lambda_i = 1$ such that

$$z^* = \sum_{i=1}^m \lambda_i u_i^*.$$

It follows that

$$\|z^* - x^*\| = \|\sum_{i=1}^m \lambda_i u_i^* - x^*\| = \|\sum_{i=1}^m \lambda_i (u_i^* - x^*)\| \leq \sum_{i=1}^m \lambda_i \|u_i^* - x^*\| \leq \varepsilon.$$

Therefore, $\text{co}(\partial_M f(x)) \subset \mathbb{B}(x^*, \varepsilon)$ whenever $x \in \mathbb{B}(\bar{x}; \delta)$. Under the reflexivity of X, we can apply the celebrated Mazur theorem and get

$$\partial_C f(x) = \text{cl}^*\text{co}\left(\partial_M f(x)\right) = \text{cl}^w\text{co}\left(\partial_M f(x)\right) = \text{clco}\left(\partial_M f(x)\right).$$

Now, for any $u^* \in \partial_C f(x)$, there exists a sequence $\{u_k^*\}$ in $\text{co}(\partial_M f(x))$ that converges to u^* strongly. Thus

$$\|u^* - x^*\| = \lim_{k \to \infty} \|u_k^* - x^*\| \le \varepsilon.$$

This implies the upper semicontinuity of $\partial_C f(\cdot)$ and completes the proof. ■

Given a function $f : X \to \overline{\mathbb{R}}$, recall that the *Fenchel conjugate* of f is given by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) \mid x \in X\}, x^* \in X^*.$$

Proposition 2.11: *Let X be a normed space and let $f : X \to \overline{\mathbb{R}}$. Suppose that f is proper, l.s.c., convex, and coercive in the sense that*

$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = \infty. \tag{2}$$

Then $\text{dom}\,(f^*) = X^*$ *and f^* is continuous on X^*, where X^* is equipped with the strong topology.*

Proof: Since f is proper, l.s.c., and convex, we can find $v^* \in X^*$ and $c \in \mathbb{R}$ such that

$$c + \langle v^*, x\rangle \le f(x) \ \text{ for all } x \in X.$$

Fix any $x^* \in X^*$. Under the coercive property of f, we can find $\delta > 0$ such that

$$\|x\|(\|x^*\| + 1) \le f(x) \ \text{ whenever } \|x\| \ge \delta.$$

It follows that

$$\sup\{\langle x^*, x\rangle - f(x) \mid \|x\| \ge \delta\} \le -\|x\|.$$

We also have

$$\sup\{\langle x^*, x\rangle - f(x) \mid \|x\| \le \delta\} \le \sup\{\langle x^*, x\rangle - \langle v^*, x\rangle - c \mid \|x\| \le \delta\} < \infty.$$

Therefore, $f^*(x^*) < \infty$ and $\text{dom}\,(f^*) = X^*$.

Observe that f^* is convex and lower semicontinuous on X^*, where X^* is equipped with the strong topology generated by the dual norm. Since X^* with the dual norm is a Banach space, f^* is continuous. ■

The following result provides conditions on f ensuring the Gâteaux differentiability of its Fenchel conjugate.

Theorem 2.12: *Let $f : X \to \overline{\mathbb{R}}$ be proper l.s.c. function defined on a reflexive Banach space X. If f is strictly convex and coercive on X, then the conjugate f^* is Gâteaux differentiable on X^*.*

Proof: By Proposition 2.11, the function f^* is convex and continuous on X^*. Fix any $v^* \in X^*$. Let us first prove the Gâteaux differentiability of f^* at v^*. Note that $\bar{x} \in \partial f^*(v^*)$ if and only if $v^* \in \partial f(\bar{x})$, which holds iff

$$f(\bar{x}) - \langle v^*, \bar{x} \rangle \leq f(x) - \langle v^*, x \rangle \quad \text{for all } x \in X.$$

Equivalently, $\bar{x}$ is an absolute minimizer of the function $g(x) := f(x) - \langle v^*, x \rangle$ for $x \in X$. It follows from (2) that $\lim_{\|x\| \to \infty} g(x) = \infty$. Then by the strict convexity of f, the function g has a unique absolute minimizer on X. Thus $\partial f^*(v^*) = \{\bar{x}\}$ is a singleton. Therefore, by Corollary 2.4, the function f^* is Gâteaux differentiable at v^*. ■

We say that a function $f : X \to \overline{\mathbb{R}}$ defined on a normed space X is *strongly convex* with parameter $\sigma > 0$ if

$$f(\lambda x + (1-\lambda)u) + \sigma \lambda (1-\lambda) \frac{\|x-u\|^2}{2} \leq \lambda f(x) + (1-\lambda) f(u) \quad \text{for all } x,\ u \in X,\ \lambda \in (0,1).$$

Given a convex function $f : X \to \overline{\mathbb{R}}$ defined on a normed space X, we say that its subdifferential mapping $\partial f : X \rightrightarrows X^*$ is *strongly monotone* with parameter $\sigma > 0$ (or $\sigma-$strong monotone) if

$$\sigma \|x_1 - x_2\|^2 \leq \langle v_1^* - v_2^*, x_1 - x_2 \rangle \quad \text{whenever} \quad v_i^* \in \partial f(x_i),\ i = 1,2.$$

In particular, it implies that

$$\sigma \|x_1 - x_2\| \leq \|v_1^* - v_2^*\| \quad \text{whenever} \quad v_i^* \in \partial f(x_i),\ i = 1,2.$$

The proposition below shows that the σ-strong convexity of a convex function $f : X \to \overline{\mathbb{R}}$ defined on a Banach space X can be characterized by the σ-strong monotonicity of the subdifferential mapping $\partial f : X \rightrightarrows X^*$, see [10, Corollary 3.5.11].

Proposition 2.13: *Let X be a Banach space and let $f : X \to \overline{\mathbb{R}}$ be proper, l.s.c., and convex. Then the following properties are equivalent:*

(a) *If f is strongly convex with parameter $\sigma > 0$.*
(b) *The subdifferential mapping $\partial f : X \rightrightarrows X^*$ is strongly monotone with parameter $\sigma > 0$.*

Theorem 2.14: *Let X be a reflexive Banach space and let $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. If f is strongly convex with parameter $\sigma > 0$, then f^* is Fréchet differentiable and $\nabla_F f^*$ is Lipschitz continuous with constant $\ell = 1/\sigma$. In addition,*

$$\nabla_F f^*(v^*) = \arg\max\{\langle v^*, x \rangle - f(x) \mid x \in X\}.$$

Proof: It is not hard to prove that the strong convexity of f implies its strict convexity and coercivity. By Proposition 2.11, the function f^* is convex and continuous on X^*. Thus, for any fixed $v^* \in X^*$, the function f^* is Gâteaux differentiable at v^*. Fix any $v_i^* \in X^*$ and $x_i \in X$ with $x_i \in \partial f^*(v_i^*)$ for $i=1,2$. Then $v_i^* \in \partial f(x_i)$ for $i=1,2$ and

$$\|x_1 - x_2\| \leq \frac{1}{\sigma}\|v_1^* - v_2^*\|.$$

Thus, we can easily show that the subdifferential mapping $\partial f^*(\cdot) : X^* \rightrightarrows X$ is upper semicontinuous at v^*. It follows from Corollary 2.8 that f^* is Fréchet differentiable, and in addition,

$$\|\nabla_F f^*(v_1^*) - \nabla_F f^*(v_2^*)\| \leq \frac{1}{\sigma}\|v_1^* - v_2^*\|$$

for all $v_1^*, v_2^* \in X^*$. This completes the proof. ■

For a bounded linear mapping $A : X \to Y$ between normed spaces, we define the *norm* of A as usual:

$$\|A\| = \sup\left\{\|A(x)\| \;\middle|\; \|x\| \leq 1\right\}.$$

It follows from the definition that $\|A(x)\| \leq \|A\|\|x\|$ for all $x \in X$. The *adjoint mapping* of A denoted by $A^* : Y^* \to X^*$ is defined by $A^*(y^*) = y^* \circ A$ for $y^* \in Y^*$. It is well known that if $A : X \to Y$ is a bounded linear mapping, then $\|A\| = \|A^*\|$.

Lemma 2.15: *Let $A : X \to Y^*$ be a bounded linear mapping, where Y is a reflexive Banach space, and let $\varphi : Y \to \overline{\mathbb{R}}$ be proper and l.s.c. Consider the function*

$$g(x) = \sup\{\langle Ax, y\rangle - \phi(y) \mid y \in Y\}, x \in X. \tag{3}$$

(a) *If ϕ is strictly convex and coercive, then $g : X \to \mathbb{R}$ is Gâteaux differentiable on X.*
(b) *If ϕ is strongly convex with parameter $\sigma > 0$, then g is Fréchet differentiable and ∇g is Lipschitz continuous on X with constant $\|A\|^2/\sigma$.*

Proof: The function g can be represented as

$$g(x) = \sup\{\langle Ax, y\rangle - \phi(y) \mid y \in Y\} = \phi^*(Ax).$$

If ϕ is strictly convex and coercive, then by Theorem 2.14, the function ϕ^* is Gâteaux differentiable on Y^*. Thus it is straightforward to see that g is Gâteaux differentiable. Now, assume that ϕ is strongly convex with parameter $\sigma > 0$. It follows from Theorem 2.14 that the function ϕ^* is Fréchet differentiable and $\nabla_F \phi^*$ is Lipschitz continuous with constant $1/\sigma$. This implies that g is Fréchet differentiable on X with the derivative representation

$$\nabla g(x) = A^* \nabla \phi^*(Ax) \quad \text{for all} \ \ x \in X,$$

and thus ∇g is Lipschitz continuous on X with constant $\|A\|^2/\sigma$. ■

Using the obtained result together with the conjugate sum rule, we derive now efficient conditions ensuring both Gâteaux and Fréchet differentiability of the constrained version of function (3).

Let X and Y be normed spaces. Given a bounded linear mapping $A : X \to Y^*$ and function $\varphi : Y \to \mathbb{R}$, consider the function

$$f(x) = \sup\{\langle Ax, y\rangle - \varphi(y) \mid y \in Y\}, x \in X. \tag{4}$$

In general, $f : X \to \overline{\mathbb{R}}$ is a nondifferentiable convex function.

Our goal now is to find a differentiable approximation of the function f given by (4) in infinite dimensions. The idea comes from Nesterov in [1] with further studies in [11,12] in finite dimensions. Fix a function $p : Y \to \overline{\mathbb{R}}$. Given $\mu > 0$, define

$$f_\mu(x) = \sup\{\langle Ax, y\rangle - \varphi(y) - \mu p(y) \mid y \in Y\}, x \in X. \tag{5}$$

The theorem below allows us to build a family of differentiable functions based on the structure of the function f.

Theorem 2.16: *Let X be a normed space and let Y be a reflexive Banach space. Consider the function f defined by (4) and the function f_μ defined by (5) in which $A : X \to Y^*$ is a bounded linear mapping and $\varphi : Y \to \mathbb{R}$ is proper, l.s.c., and convex.*

(a) *If p is strictly convex and coercive with* $\text{dom}\,(\varphi) \cap \text{dom}\,(p) \neq \emptyset$, *then f_μ is Gâteaux differentiable.*
(b) *If p is strongly convex with parameter $\sigma > 0$ and* $\text{dom}\,(\varphi) \cap \text{dom}\,(p) \neq \emptyset$, *then f_μ is a $C^{1,1}$ function with the Lipschitz constant for the gradient ∇f_μ calculated by $\|A\|^2/\sigma\mu$.*

Proof: Observe that

$$f_\mu(x) = \sup\{\langle Ax, y\rangle - (\varphi(y) + \mu p(y)) \mid y \in Y\} = (\varphi + \mu p)^*(Ax).$$

The conclusion follows directly from Lemma 2.15. Note that if p is strictly convex and coercive, then the function $h(u) = \varphi(u) + \mu p(u)$ for $u \in Y$ is also strictly convex and coercive. In addition, if p is strongly convex with parameter σ, then h is strongly monotone with parameter $\sigma\mu$. ■

The next step in Nesterov's smoothing techniques involves imposing more properties of the function p to ensure that f_μ provides smooth approximations to the function f.

We say that a function $p : Y \to \overline{\mathbb{R}}$ is a *prox-function* of the function f defined in (4) if the following conditions are satisfied:

(a) p is proper, l.s.c., and σ-strongly convex with some $\sigma > 0$.
(b) $\text{dom}\,(\varphi) \subset \text{dom}\,(p)$.
(c) $p(y) \geq 0$ for all $y \in \text{dom}\,(\varphi)$.
(d) $D = \sup_{y \in \text{dom}(\varphi)} p(y) < \infty$.

Theorem 2.17: *In the setting of Theorem 2.16 suppose that p is a prox-function for f. Then*

$$f_\mu(x) \leq f(x) \leq f_\mu(x) + \mu D \quad \text{for all } x \in X,$$

where $D = \sup_{y \in \text{dom}\,(\varphi)} p(y)$.

Proof: Since $p(y) \geq 0$ for all $y \in \text{dom}\,(\varphi)$ and $\text{dom}\,(\varphi) \subset \text{dom}\,(p)$, we have

$$\begin{aligned} f_\mu(x) &= \sup\{\langle Ax, y\rangle - (\varphi(y) + \mu p(y)) \mid y \in Y\} \\ &= \sup\{\langle Ax, y\rangle - (\varphi(y) + \mu p(y)) \mid y \in \text{dom}\,(\varphi) \cap \text{dom}\,(p)\} \\ &= \sup\{\langle Ax, y\rangle - (\varphi(y) + \mu p(y)) \mid y \in \text{dom}\,(\varphi)\} \\ &\leq \sup\{\langle Ax, y\rangle - \varphi(y) \mid y \in \text{dom}\,(\varphi)\} = f(x). \end{aligned}$$

We also have

$$\begin{aligned} f(x) &= \sup\{\langle Ax, y\rangle - \varphi(y) \mid y \in \text{dom}\,(\varphi)\} \\ &= \sup\{\langle Ax, y\rangle - (\varphi(y) + \mu p(y)) + \mu p(y) \mid y \in \text{dom}\,(\varphi)\} \\ &\leq \sup\{\langle Ax, y\rangle - (\varphi(y) + \mu p(y)) \mid y \in \text{dom}\,(\varphi)\} \\ &\quad + \mu \sup\{p(y) \mid y \in \text{dom}\,(\varphi)\} \\ &= f_\mu(x) + \mu D. \end{aligned}$$

The proof is now complete. ■

Let us continue by providing some examples of the function p that satisfies condition (a) or (b) in Theorem 2.16. Recall that a subset F with nonempty interior of a normed space is called *strictly convex* if for any $x, y \in F$ with $x \neq y$ and for any $t \in (0, 1)$, we have $tx + (1 - t)y \in \text{int}(F)$. The proof of the proposition below is straightforward.

Proposition 2.18: *Let X be a normed space and let F be a nonempty convex set in X that contains the origin in its interior. Suppose that F is strictly convex. Consider* *the Minkowski function associated with F defined by* $\rho_F(x) := \inf\{t > 0 \mid x \in tF\}$ *for* $x \in X$*. Then the function* $p = (\rho_F)^2$ *is continuous, strictly convex and coercive.*

We say that a function $p : X \to \overline{\mathbb{R}}$ is called *F-strongly convex* with parameter $\sigma > 0$ if the function $p - \sigma/2(\rho_F)^2$ is convex. In particular, if p is $\mathbb{B}$-strongly convex, where $\mathbb{B}$ is the closed unit ball of X, then this definition reduces to the well-known definition of strong convexity.

Proposition 2.19: *Let X be a normed space and let* $p : X \to \mathbb{R}$ *be a continuous function that is F-strongly convex, where the set F satisfies the conditions in Proposition 2.18. Then p is also strictly convex and coercive. If we assume in addition that X is a Hilbert space and p is strongly convex with parameter* $\sigma > 0$*, then* $\partial p(\cdot)$ *is strongly monotone with parameter* σ*.*

Proof: Define the function $h = p - \sigma/2(\rho_F)^2$. Then h is a continuous convex function. Thus there exist $w^* \in X^*$ and $b \in \mathbb{R}$ such that

$$\langle w^*, x\rangle + b \le h(x) \text{ for all } x \in X.$$

Since $p = h + \sigma/2(\rho_F)^2$, the conclusions become straightforward. ■

Finally, let us consider a direct corollary of Theorem 2.16 (b).

Corollary 2.20: *Let X and Y be Hilbert spaces. In the setting of Theorem 2.16, let* $p(y) = \frac{1}{2}\|y - y_0\|^2$ *for* $y \in Y$*, where* $y_0 \in Q$ *and Q is bounded. The function* f_μ *given by (5) is Fréchet differentiable and its gradient is Lipschitz continuous on X with Lipschitz constant* $\ell_\mu = \|A\|^2/\mu$*. In addition,*

$$f_\mu(x) \le f(x) \le f_\mu(x) + \frac{\mu}{2}[D(y_0; Q)]^2 \text{ for all } x \in X,$$

where $D(y_0; Q) = \sup\{\|y_0 - y\| \mid y \in Q\} < \infty$*.*

In particular, if $\varphi(y) = \langle b, y\rangle$ *for* $y \in Y$*, where* $b \in Y$*, then the function* f_μ *has the explicit representation*

$$f_\mu(x) = \frac{\|Ax - b\|^2}{2\mu} + \langle Ax - b, y_0\rangle - \frac{\mu}{2}\left[d(y_0 + \frac{Ax - b}{\mu}; Q)\right]^2$$

and is Fréchet differentiable on X with its gradient given by $\nabla_F f_\mu(x) = A^* u_\mu(x)$*, where* u_μ *can be expressed in terms of the Euclidean projection*

$$u_\mu(x) = \Pi\left(y_0 + \frac{Ax - b}{\mu}; Q\right).$$

Proof: The conclusion follows directly from Theorem 2.16 with the observation that p is strongly convex with constant $\sigma = 1$. ■‘

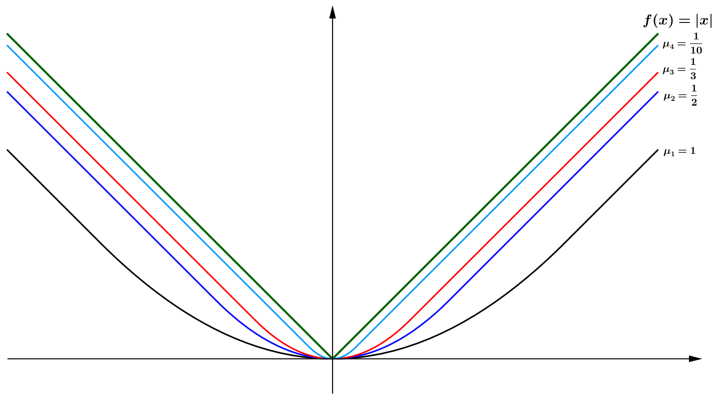


Figure 1. Smooth approximation of the absolute value function

Example 2.21: Let $f(x) = |x|$ for $x \in \mathbb{R}$. Then $f(x) = \sup\{xu \mid |u| \leq 1\}$. Using $p(u) = u^2/2$ for $u \in \mathbb{R}$ gives smooth approximations (Figure 1):

$$f_\mu(x) = \begin{cases} \dfrac{x^2}{2\mu} & \text{if } |x| \leq \mu, \\ |x| - \dfrac{\mu}{2} & \text{if } |x| > \mu. \end{cases}$$

Example 2.22: Let X be a Hilbert space. Given $b \in X$, define $f(x) = \|x - b\|$ for $x \in X$. Then

$$f(x) = \sup\{\langle x - b, y\rangle \mid y \in \mathbb{B}\} = \sup\{\langle x, y\rangle - \langle b, y\rangle \mid y \in \mathbb{B}\}.$$

Using Corollary 2.20 with $p(y) = 1/2\|y\|^2$ gives

$$f_\mu(x) = \frac{\|x - b\|^2}{2\mu} - \frac{\mu}{2}\left[d\left(\frac{x - b}{\mu}; \mathbb{B}\right)\right]^2,$$

where $\mathbb{B}$ is the closed unit ball of X.

Lemma 2.23: *For $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, define $p(y) = \sum_{i=1}^m y_i \ln(y_i) + \ln(m)$ if $y_i > 0$ for all $i = 1, \ldots, m$ and $\sum_{i=1}^m y_i = 1$, and $p(y) = \infty$ otherwise. Then p is strongly convex and $p(y) \geq 0$ for all $y \in \mathbb{R}^m$.*

Proof: Define the function $p^+(y) = \sum_{i=1}^m y_i \ln(y_i) + \ln(m)$ if $y_i > 0$ for $i = 1, \ldots, m$, where $y := (y_1, \ldots, y_m)$. We can show that $\nabla^2 p^+(y) \succeq 0$ for all $y \in \mathbb{R}^m$ with $y_i > 0$ for all $y = 1, \ldots, m$. Thus p is strongly convex on $G = \{(x_1, \ldots, x_m) \mid x_i > 0 \text{ for } i = 1, \ldots, m\}$. It follows that p is strongly convex. An elementary method of Lagrange multiplier shows that $p(y) \geq 0$ for all $y \in \mathbb{R}^m$. ■

Example 2.24: Consider the function

$$f(x) = \max\{x_1, \ldots, x_n\} \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

The function f can be represented by

$$\begin{aligned} f(x) &= \sup\left\{x_1y_1 + x_2y_2 + \cdots + x_ny_n \mid y_1, y_2, \ldots, y_n \geq 0, \sum_{i=1}^{n} y_i = 1\right\} \\ &= \sup\{\langle x, y\rangle - \delta_\Delta(y) \mid y \in Y\}, \end{aligned}$$

where $\Delta = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1, y_2, \ldots, y_n \geq 0, \sum_{i=1}^n y_i = 1\}$.

Define the function

$$p(y) = \begin{cases} \sum_{i=1}^n y_i \ln(y_i) + \ln(n) & \text{if } y_1, \ldots, y_n \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

In this definition we use the convention that $y_i \ln(y_i) = 0$ if $y_i = 0$ for $i = 1, \ldots, n$. With the method of Lagrange multipliers, we can easily show that p is a prox-function for f and

$$\begin{aligned} f_\mu(x) &= \sup\left\{\langle x, y\rangle - \delta_\Delta(y) - \mu p(y) \mid y \in Y\right\} \\ &= \sup\{\langle x, y\rangle - \delta_\Delta(y) - \mu p(y) \mid y \in \Delta\} \\ &= \sup\left\{\sum_{i=1}^n x_iy_i - \mu\sum_{i=1}^n y_i \ln(y_i) - \mu\ln(n) \mid y \in \Delta\right\} \\ &= \mu\ln\left(\sum_{i=1}^n \mathrm{e}^{x_i/\mu}\right) - \mu\ln(n) = \mu\ln\left(\frac{\sum_{i=1}^n \mathrm{e}^{x_i/\mu}}{n}\right). \end{aligned}$$

The smoothing technique obtained can be used to solve the smallest enclosing ball problem:

$$\text{minimize } \phi(x) = \max\{[d(x; \Omega_i)]^2 \mid i = 1, \ldots, m\}, x \in \mathbb{R}^n,$$

where Ω_i for $i = 1, \ldots, m$ are nonempty closed convex sets in $\mathbb{R}^n$. Indeed, a smoothing approximation of ϕ is given by

$$\phi_\mu(x) = \mu\ln\left(\frac{\sum_{i=1}^m \mathrm{e}^{[d(x;\Omega_i)]^2/\mu}}{m}\right).$$

This function can be minimized using accelerated first-order optimization methods.

3. Applications to image reconstructions

In this section, we consider an unknown image M of size $N_1 \times N_2$. After the image is corrupted by a linear operator A and distorted by some noise ε, we observe only the image $b = A(M) + \xi$, and seek to recover the true image M.

3.1. Patching an image

In this section, we describe an optimization problem which models the image reconstruction problem by expressing an image as the sum of sparse representations of distinct 'patches' of the image, see, e.g. [13]. Given an $N_1 \times N_2$ image matrix M, let $\mathcal{P}$ be a collection of submatrices $P_{i,j}$ of M with size $n_1 \times n_2$, which cover M. We henceforth refer to these submatrices as patches, and sometimes identify $P_{i,j}$ with its index (i,j). The covering condition ensures that every pixel of M appears in some patch, and we will use collections only of non-overlapping patches, so that $\mathcal{P}$ partitions M.

Define $R_{i,j}$ as the function that maps the image M to patch $P_{i,j}$, that is, $R_{i,j}(M) = P_{i,j}$ (Figure 2). If M is in vectorized form, $R_{i,j}$ can be expressed as an $n_1 n_2 \times N_1 N_2$ matrix with exactly one 1 in each row and zeros elsewhere. In particular, $[R_{i,j}]_{ab}$ is 1 if the bth entry of the image M appears in the ath entry of the (vectorized) patch $P_{i,j}$, and 0 otherwise. For example, if $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$ is a 3×3 image and patch $P_{2,2}$ is the 2×2 bottom right corner, then

$$R_{2,2}(M) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{12} \\ m_{22} \\ m_{32} \\ m_{13} \\ m_{23} \\ m_{33} \end{bmatrix}$$

$$= \begin{bmatrix} m_{22} \\ m_{32} \\ m_{23} \\ m_{33} \end{bmatrix} = P_{2,2} \sim \begin{bmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{bmatrix},$$

where $\sim$ represents reshaping a vectorized form to a matrix. It is now straightforward to write a MATLAB code `buildRij([N1 N2],[n1 n2],[s t])` to construct a matrix $R_{i,j}$ which operates on a vectorized $N_1 \times N_2$ image M to produce the vectorized $n_1 \times n_2$ patch whose upper-left index in M is (s,t).

For a collection $\mathcal{P}$ of patches of an image M, define $T_{\mathcal{P}} = \sum_{(i,j)\in\mathcal{P}} R_{i,j}^T R_{i,j}$.

$$R_{1,1} \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & 18 & 19 & 20 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 6 & 7 \\ \hline \end{array} \qquad R_{1,1}^T \begin{array}{|c|c|} \hline w & x \\ \hline y & z \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline w & x & 0 & 0 & 0 \\ \hline y & z & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

Figure 2. *Left*: $R_{1,1}$ acts on a 4×5 matrix to extract the 2×2 patch $(1,1)$. *Right*: $R_{1,1}^T$ embeds a 2×2 patch into patch $(1,1)$ of a zero matrix.

Lemma 3.1: *For all $P_{i,j} \in \mathcal{P}$, $R_{i,j}^T R_{i,j}$ is diagonal and $T_\mathcal{P}$ is invertible. If the patches are non-overlapping, then $T_\mathcal{P}$ is the identity matrix.*

Proof: Let $P_{i,j}$ be some patch of M, both in vector form. Then $[R_{i,j}]_{rc} = 1$ iff $M_c = [P_{i,j}]_r$. Because $[P_{i,j}]_r$ must contain exactly one element of M, each row of $R_{i,j}$ contains exactly 1 nonzero entry. As each element of M appears at most once in $P_{i,j}$, each column of $R_{i,j}$ contains at most one nonzero element. This means that if $a \neq b$ then columns a and b of $R_{i,j}$ cannot have a nonzero element in the same row, so $[R_{i,j}^\mathrm{T} R_{i,j}]_{a,b}$ (being the dot product of columns a and b of $R_{i,j}$) must be zero. So $R_{i,j}^\mathrm{T} R_{i,j}$ is diagonal. Along the diagonal, $[R_{i,j}^T R_{i,j}]_{a,a}$ is the dot product of the ath column of $R_{i,j}$ with itself, and is thus 1 iff the ath entry of M appears in patch $P_{i,j}$. Since each entry of M appears in at least one patch, it follows that summing $R_{i,j}^\mathrm{T} R_{i,j}$ over all patch indices (i,j) ensures $T_\mathcal{P}$ has nonzero diagonals and is therefore invertible. When the patches are nonoverlapping, each entry of M appears in only one patch, so $[R_{i,j} R_{i,j}^\mathrm{T}]_{a,a} = 1$ for all $a = 1, \ldots, N_1 N_2$, by which we find $T_\mathcal{P}$ to be the identity matrix. ∎

We then express the image M as

$$M = (T_\mathcal{P})^{-1} \left(\sum_{(i,j)\in\mathcal{P}} R_{i,j}^T R_{i,j}(M) \right).$$

Because in this paper we use only nonoverlapping patches, $T_\mathcal{P}$ is the identity, thus

$$M = \sum_{(i,j)\in\mathcal{P}} {R_{i,j}}^\mathrm{T} R_{i,j}(M).$$

For any $(i,j) \in \mathcal{P}$, let $y_{i,j} \in \mathbb{R}^K$ be a sparse representation of the patch $R_{i,j}(M) = P_{i,j}$ under an $n_1 n_2 \times K$ dictionary D, so that $P_{i,j} = D y_{i,j}$. We thus say

$$M = \sum_{(i,j)\in\mathcal{P}} {R_{i,j}}^\mathrm{T} D y_{i,j}.$$

To reconstruct the image with sparsely represented patches in a way which fits the observed data, we solve the following:

$$\min_{\substack{y_{i,j} \\ (i,j)\in\mathcal{P}}} \sum_{(i,j)\in\mathcal{P}} \|y_{i,j}\|_1 - \sum_{(i,j)\in\mathcal{P}} \|y_{i,j}\| + \frac{\nu}{2} \left\| A \left(\sum_{(i,j)\in\mathcal{P}} {R_{i,j}}^\mathrm{T} D y_{i,j} \right) - b \right\|^2, \quad (6)$$

and immediately note that (6) can be expressed as

$$\min_y \quad \frac{\nu}{2} \left\| \mathcal{A} y - b \right\|^2 + \|y\|_1 - \|y\|, \quad (7)$$

where $y = \left[y_{11}^\mathrm{T} \; y_{12}^\mathrm{T} \cdots y_{\bar{i}\bar{j}}^\mathrm{T} \right]^\mathrm{T}$ is the concatenation of all the sparse representations of patches under D, $\bar{i}$ and $\bar{j}$ the final row and column, respectively, of the patch partition, and finally

$$\mathcal{A} = A\left[R_{11}{}^{\mathrm{T}}D \quad R_{12}{}^{\mathrm{T}}D \quad \dots \quad R_{\bar{i}\bar{j}}{}^{\mathrm{T}}D\right].$$

3.2. DCA for $\ell_1 - \ell_2$ regularization

The use of $(\ell_1 - \ell_2)$ regularization causes the objective function to no longer convex (Figure 3), and so we adopt the Difference of Convex Functions Algorithm (DCA), used to minimize $g-h$ where $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are both convex. The algorithm, developed by Pham Dinh and Le Thi in [3,4] is as follows:

DC Algorithm

INPUT: $x_1, N \in \mathbb{N}$
for $k = 1, \dots, N$ **do**
  Find $y_k \in \partial h(x_k)$
  Find $x_{k+1} \in \partial g^*(y_k)$ (equivalently, $x_{k+1} \in \operatorname{argmin}\{g(x) - \langle x, y_k\rangle : x \in \mathbb{R}^n\}$)
end for
OUTPUT: x_{N+1}

Before using the DCA, we first apply Nesterov's smoothing from Corollary 2.20. Given a function of the form

$$f_0(x) = \max_{u \in Q}\{\langle Ax, u\rangle - \psi(u)\}$$

where $Q \subset \mathbb{R}^m$ is a convex, closed, and bounded set, ψ is a convex map from $\mathbb{R}^m$ to $\mathbb{R}$, and A is an $m \times n$ matrix, for any $\mu > 0$ we may obtain a smooth approximation f_μ using

$$f_\mu(x) = \max_{u \in Q}\{\langle Ax, u\rangle - \psi(u) - \frac{\mu}{2}\|u\|^2\}.$$

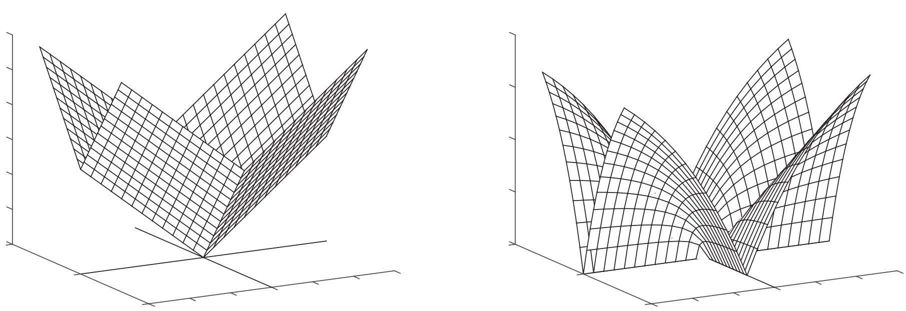


Figure 3. Surface plots of $\|x\|_1$ *(left)* and $\|x\|_1 - \|x\|$ *(right)* on $\mathbb{R}^2$. Minimizing an objective function containing these terms drives solutions towards the axes, simulating sparsity. Note that $\ell_1 - \ell_2$ regularization is not convex.

We now use this to smooth the ℓ_1 norm. Let $p(x) = \|x\|_1$, and note that this is equivalent to $p(x) = \max_{u \in Q}\{\langle x, u\rangle\}$ when Q is the unit box, $Q = \{x \in \mathbb{R}^n \mid |x_i| \le 1\,,\, i = 1, \ldots, n\}$. In the above general setting, this corresponds to A being the identity matrix and ψ the zero map. Then $p_\mu(x)$ is a smooth approximation to $p(x) = \|x\|_1 = \max_{u \in Q}\{\langle x, u\rangle\}$ and can be expressed as

$$p_\mu(x) = \frac{1}{2\mu}\|x\|^2 - \frac{\mu}{2}\left(d\left(\frac{x}{\mu}, Q\right)\right)^2,$$

where $d(x; Q)$ is the Euclidean distance from x to Q.

Now let A be a real $m \times n$ matrix and $b \in \mathbb{R}^m$. Using the above smooth approximation for $\|x\|_1$ we approximate $f(x) = \frac{\nu}{2}\|Ax - b\|^2 + \|x\|_1 - \|x\|$ with

$$\begin{aligned}
f_\mu(x) &= \frac{\nu}{2}\|Ax - b\|^2 + \frac{1}{2\mu}\|x\|^2 - \frac{\mu}{2}\left(d\left(\mu^{-1}x, Q\right)\right)^2 - \|x\| \\
&= \frac{1}{2\mu}\|x\|^2 - \left(\frac{\mu}{2}\left(d\left(\mu^{-1}x, Q\right)\right)^2 - \frac{\nu}{2}\|Ax - b\|^2 + \|x\|\right) \\
&= \frac{1}{2\mu}\|x\|^2 + \frac{\gamma}{2}\|x\|^2 - \left(\frac{\mu}{2}\left(d\left(\mu^{-1}x, Q\right)\right)^2 - \frac{\nu}{2}\|Ax - b\|^2 + \frac{\gamma}{2}\|x\|^2 + \|x\|\right).
\end{aligned}$$

Note this is the difference of convex functions $g - h$ for

$$\begin{aligned}
g(x) &= \left(\frac{1 + \mu\gamma}{2\mu}\right)\|x\|^2 \quad \text{and} \\
h(x) &= \frac{\mu}{2}\left(d\left(\mu^{-1}x, Q\right)\right)^2 - \frac{\nu}{2}\|Ax - b\|^2 + \frac{\gamma}{2}\|x\|^2 + \|x\|,
\end{aligned}$$

assuming that $\gamma > 0$ is sufficiently large so that $\frac{\gamma}{2}\|x\|^2 - \frac{\nu}{2}\|Ax - b\|^2$ is convex. Note this is satisfied when γ is greater than ν times the largest eigenvalue of $A^{\mathrm{T}}A$.

To use the DCA algorithm, we will need y_k in the subdifferential of h at x_k. Using

$$\nabla\|Ax - b\|^2 = 2A^{\mathrm{T}}(Ax - b)$$

and

$$\nabla\left(d(x, Q)\right)^2 = 2(x - \Pi_Q(x)),$$

where $\Pi_Q(x)$ is the projection of x onto Q, along with the chain rule, we have a subgradient of h at x given by

$$\partial_w h(x) = \mu^{-1}x - \Pi_Q\left(\mu^{-1}x\right) - \nu A^{\mathrm{T}}(Ax - b) + \gamma x + \omega(x),$$

where $\omega(x)$ is a subgradient of $\|\cdot\|$ at x. We point out that the projection onto the unit box can be defined componentwise as $[\Pi_Q(x)]_i = \max(-1, \min(x_i, 1))$.

To find $x_{k+1} \in \partial^* g(y_k)$, we use the fact that $u \in \partial^* g(v)$ iff $v \in \partial g(u)$. The subdifferential of g is simply the singleton set containing its gradient (see [14]), so $v \in \partial g(u)$ iff $v = \frac{1+\mu\gamma}{\mu} u$ iff $u = \frac{\mu}{1+\mu\gamma} v$.

We combine these results to implement the DCA algorithm in order to minimize a μ-smoothing approximation to $f(x) = \frac{\nu}{2}\|Ax - b\|^2 + \|x\|_1 - \|x\|$, as outlined below.

Algorithm 1 DCA for smoothed $\ell_1 - \ell_2$ regularization.

INPUT: $\mu > 0$, sufficiently large γ, starting point x

repeat

  Find $\omega = \frac{x}{\|x\|}$ if $x \neq 0$, $\omega = 0$ otherwise

  $y \longleftarrow \mu^{-1}x - \Pi_Q\left(\mu^{-1}x\right) - \nu A^\top(Ax - b) + \gamma x + \omega$

  $x \longleftarrow \frac{\mu}{1+\mu\gamma} y$

until convergence

OUTPUT: x

Experiments suggest that incrementally decreasing μ over the course of the algorithm induces better performance.

3.3. Choosing partitions

This section describes how to obtain t different partitions of the image, following the approach described in [13]. Given an $N_1 \times N_2$ image matrix, choose a general patch size $n_1 \times n_2$. We then choose a size $c_1 \times c_2$ of the upper left-most patch, P_{11}, where $c_i \leq n_i$ for $i = 1,2$. All of the patches not on the boundary of the image will have size $n_1 \times n_2$. The left boundary noncorner patches of M are size $n_1 \times c_2$, the upper boundary noncorner patches have size $c_1 \times n_2$, and the remaining patch sizes are chosen to ensure their borders align with those patches already defined (Figure 4).

If patch $P_{i,j}$ has size less than $n_1 \times n_2$, the patch extraction operator $R_{i,j}$ still creates a patch of size $n_1 \times n_2$, in which $P_{i,j}$ sits in the proper orientation, and the remaining entries are zeros. Similarly, $R_{i,j}^{\mathrm{T}}$ will embed an $n_1 \times n_2$ patch into the corresponding patch in the image, but zero out all entries which do not lie in the

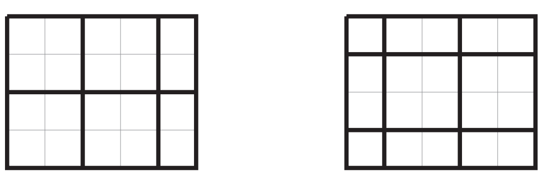


Figure 4. Two partitions of an $(N_1 \times N_2) = (4 \times 5)$ image with patch size $(n_1, n_2) = (2, 2)$. Left: The top left corner has size $(c_1, c_2) = (2, 2)$. Right: The top left patch has size $(c_1, c_2) = (1, 1)$.

smaller patch. For example, if the general patch size is 8×8 but the corner patch P_{11} is 5×5, then R_{11} embeds the top left 5×5 patch into an 8×8 patch of zeros. We say P_{11} has 'virtual size' 5×5. Note that the cell array of patch extraction matrices does not need to be constructed every time a problem is solved. Once it has been constructed for some partition of a given size image, it can be saved and reused. The general algorithm given in [13] is as follows: Given a dictionary D, choose some t different patch partitions of the image, $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t$. For each $k = 1, \ldots, t$, find the solution M_k to the unconstrained problem (7) using partition $\mathcal{P}_k$. Then use the average of those solutions, $\overline{M} = \frac{1}{t}\sum M_k$, as the final reconstruction.

3.4. Dictionaries

In this summary, we use two types of dictionary. One is constructed from the discrete cosine transform (DCT). The other is a 'learned dictionary', constructed using a collection of images as training data, and for which the learned dictionary allows sparse representations. The i,j entry of an $M \times N$ discrete cosine transform (DCT-II) matrix D is given by

$$D_{i,j} = \begin{cases} \sqrt{\frac{1}{N}} & j = 1, \\ \sqrt{\frac{2}{N}} \cos\left(\frac{\pi}{N}(j-1)(i+\frac{1}{2})\right) & j = 2, \ldots, N. \end{cases}.$$

Alternatively, a 'wavelet' dictionary can be called using MATLAB's `wmpdictionary( )` function, with argument equal to the number of atoms.

We find better results when we 'learn' a dictionary from a training data. Consider a training matrix $X = [x_1, \ldots, x_L] \in \mathbb{R}^{n\times L}$ of L images of size n in vectorized form. We seek a dictionary $D = [d_1, \ldots, d_K] \in \mathbb{R}^{n\times K}$ of K atoms of size n and a corresponding coefficient matrix $W = [w_1, \ldots, w_L] \in \mathbb{R}^{K\times L}$ so that $x_i \approx Dw_i$ and w_i is as sparse as possible, for all $i = 1, \ldots, L$.

There exist several methods for learning a dictionary. One of the most popular algorithms is the K-SVD proposed in [15] which can be modelled as

$$\min_{D,W} \quad \|DW - X\|_F^2$$
$$\text{subject to} \quad \|d_i\| = 1 \ \text{ for all } i = 1, \ldots, K \quad \text{and} \quad \|w_j\|_0 \le s \ \text{ for all } j = 1, \ldots, L,$$

where s is a parameter to control the sparsity. Another popular method is the Online Dictionary Learning (OLM) proposed in [16] which solves the following problem:

$$\begin{aligned} \min_{D,W} \quad & \frac{\lambda}{2}\|DW - X\|_F^2 + \|W\|_1 \\ \text{subject to} \quad & \|d_i\| = 1 \ \text{ for all } i = 1, \ldots, K, \end{aligned} \tag{8}$$

where $\|W\|_1 = \sum_{i=1}^{L} \|w_i\|_1 = \sum_{i=1}^{L} \sum_{j=1}^{K} |w_{ij}|$ and λ is a trade-off parameter to balance data fitting and sparsity level.

To promote the sparsity, our approach is to use the $\ell_1 - \ell_2$ regularization by solving the following problem:

$$\begin{aligned} \min_{D,W} \quad & \frac{\lambda}{2}\|DW - X\|_F^2 + \|W\|_1 - \|W\|_{2,1} \\ \text{subject to} \quad & \|d_i\| \le 1 \;\; \text{for all } i = 1, \ldots, K, \end{aligned} \tag{9}$$

where $\|W\|_{2,1} = \sum_{i=1}^{L} \|w_i\| = \sum_{i=1}^{L} \sqrt{\sum_{j=1}^{K} w_{ji}^2}$. This is a nonconvex problem whose nonconvexity comes from two sources: the sparsity promotion $\ell_1 - \ell_2$ and the bilinearity between the dictionary D and the code W in the fitting term.

For solving this problem, we alternatively update W and D by using the DCA and Nesterov's smoothing.

1. **Sparse coding phase**: In this phase, we fix a dictionary D and try to update the code W by solving (9). The objective function is now a DC function with respect to W:

$$f(W) = \frac{\lambda}{2}\|DW - X\|_F^2 + \|W\|_1 - \|W\|_{2,1}.$$

Let $P(W) = \|W\|_1$. Using the smoothing technique as before, we can approximate the function $P(W)$ by

$$\begin{aligned} P_\mu(W) &= \sum_{i=1}^{L} \left[\frac{1}{2\mu}\|w_i\|^2 - \frac{\mu}{2}\left[d\left(\frac{w_i}{\mu}; Q\right)\right]^2 \right] \\ &= \frac{1}{2\mu}\|W\|_F^2 - \frac{\mu}{2}\sum_{i=1}^{L}\left[d\left(\frac{w_i}{\mu}; Q\right)\right]^2, \end{aligned}$$

where $Q = \{w \in \mathbb{R}^K \mid \|w\|_\infty \le 1\}$. Recall that the ith component of the Euclidean projection from $w \in \mathbb{R}^K$ onto the box Q can be computed as

$$[\Pi_Q(w)]_i = \max(-1, \min(1, w_i)). \tag{10}$$

To process further, we denote $\mathcal{Q} = Q \times Q \times \ldots \times Q \subset \mathbb{R}^{K \times L}$. For an $K \times L$ matrix W, the projection from $W = [w_1, \ldots, w_L]$ onto $\mathcal{Q}$ is defined by

$$\Pi\,(W, \mathcal{Q}) = [\Pi(w_1; Q), \ldots, \Pi(w_L; Q)] \in \mathbb{R}^{K \times L}.$$

We thus have

$$[d(W; \mathcal{Q})]^2 = \|W - \Pi(W, \mathcal{Q})\|_F^2 = \sum_{i=1}^{L}[d(w_i; Q)]^2.$$

The function $f(W)$ can be approximated by the DC function $f_\mu(W) = g_\mu(W) - h_\mu(W)$, where

$$g_\mu(W) = \left(\frac{1}{2\mu} + \frac{\gamma_1}{2}\right) \|W\|_F^2,$$
$$h_\mu(W) = \frac{\mu}{2}\left[d\left(\frac{W}{\mu}; \mathcal{Q}\right)\right]^2 - \frac{\lambda}{2}\|DW - X\|_F^2 + \frac{\gamma_1}{2}\|W\|_F^2 + \|W\|_{2,1},$$

and γ_1 is chosen such that $\frac{\gamma_1}{\lambda}$ is greater than the spectral radius of the symmetric matrix $D^\top D$ in order to guarantee the convexity of the function $h_\mu(W)$.

A subgradient Y of h_μ at W is given by

$$Y = \frac{W}{\mu} - \Pi\left(\frac{W}{\mu}; \mathcal{Q}\right) - \lambda D^\top (DW - X) + \gamma_1 W + \eta(W),$$

where $\eta(W)$ is an $K \times L$ matrix whose i^{th} column is defined via the i^{th} column of W by

$$[\eta(W)]_i = \begin{cases} \dfrac{w_i}{\|w_i\|} & \text{if } w_i \neq 0, \\ 0_{\mathbb{R}^K} & \text{if } w_i = 0. \end{cases} \tag{11}$$

The DCA for solving the sparse coding phase can be outlined as follows.

Algorithm 2 DCA for sparse coding phase.

INPUT: $X \in \mathbb{R}^{n\times L}$, $D \in \mathbb{R}^{n\times K}$, $\mu > 0$, $\lambda > 0$ sufficiently small,
$\gamma_1 > 0$ sufficiently large and starting code $W \in \mathbb{R}^{K\times L}$.
repeat
Find $\Pi(W, \mathcal{Q}) = [\Pi(w_1; Q), \ldots, \Pi(w_L; Q)]$ according to (10)
Find $\eta(W)$ according to (11)
$Y \longleftarrow \frac{W}{\mu} - \Pi\left(\frac{W}{\mu}; \mathcal{Q}\right) - \lambda D^\top (DW - X) + \gamma_1 W + \eta(W)$
$W \longleftarrow \dfrac{\mu}{1 + \mu\gamma_1} Y$
until convergence
OUTPUT: W

2. **Dictionary updating phase.** Now we fix the sparse code W that has been found from the previous phase and update the dictionary D by solving

$$\min_D \quad \|DW - X\|_F^2 \quad \text{subject to} \quad \|d_i\| \le 1 \text{ for all } i = 1, \ldots, K.$$

For solving this nonconvex problem, we use the DCA by reformulating it as a DC programming problem as follows:

$$\min_D \qquad \widetilde{f}(D) = \left[\frac{\gamma_2}{2}\|D\|_F^2 + I_C(D)\right] - \left[\frac{\gamma_2}{2}\|D\|_F^2 - \|DW - X\|_F^2\right],$$

where $\mathcal{C} = \{D = [d_1, \ldots, d_K] \in \mathbb{R}^{n \times K} \mid \|d_i\| \leq 1 \text{ for all } i = 1, \ldots, K\}$ is the constraint. Here γ_2 is chosen greater than the spectral radius of the matrix $WW^\top$ to ensure the convexity of the function $\widetilde{h}(D) = \frac{\gamma_2}{2}\|D\|_F^2 - \|DW - X\|_F^2$.

This function $\widetilde{h}$ is differentiable and its gradient given by

$$\nabla\widetilde{h}(D) = \gamma_2 D - [DW - X]W^\top.$$

Note that the ith component of the Euclidean projection from D onto the constraint $\mathcal{C}$ can be computed by

$$[\Pi_{\mathcal{C}}(D)]_i = \frac{d_i}{\max\{1, \|d_i\|\}}, \text{ for } i = 1, \ldots, K.$$

Thus, the DCA iterative sequence in this phase is simply defined by $D_{k+1} = \Pi_{\mathcal{C}}\left(\frac{\nabla h(D_k)}{\gamma_2}\right)$.

Algorithm 3 DCA for dictionary updating phase.

INPUT: $X \in \mathbb{R}^{n \times L}$, $W \in \mathbb{R}^{K \times L}$, $\gamma_2 > 0$ sufficiently large,
starting dictionary $D \in \mathbb{R}^{n \times K}$.
repeat
$Y \longleftarrow \gamma_2 D - [DW - X]W^\top$
$D \longleftarrow \Pi_{\mathcal{C}}\left(\frac{Y}{\gamma_2}\right)$
until convergence
OUTPUT: D

In practice, when alternatively perform Algorithm 2 and Algorithm 3 to solve (9), we can use a value $\gamma > 0$ sufficiently large to play the role of both γ_1 and γ_2. In addition, we also gradually decrease the value of smoothing parameter μ until a preferred μ_∞ is attained. The final scheme for $\ell_1 - \ell_2$ dictionary learning can be outlined as follows.

3.5. Implementation

Our goal is to restore an unknown image M of size $N_1 \times N_2$ from its corrupted linear measurements of the form $b = A(M) + \xi$. We first choose a general patch size $n_1 \times n_2$ with $n_i \ll N_i$ for $i = 1,2$. Then we generate a dictionary of size $n_1 n_2 \times K$ by using DCT or learning from a training data set X of size $n_1 n_2 \times L$ with $n_1 n_2 \leq K \ll L$. Let $\mathcal{P}$ be a patch partition associated with some choice of upper left-most patch and let S be the number of patches in $\mathcal{P}$. For any $(i, j) \in$

Algorithm 4 DCA for $\ell_1 - \ell_2$ dictionary learning.

INPUT: training set $X \in \mathbb{R}^{n \times L}$, $\lambda > 0$ sufficiently small, $\gamma > 0$ sufficiently large,
starting dictionary $D^0 \in \mathbb{R}^{n\times K}$, starting code $W^0 \in \mathbb{R}^{K\times L}$
$\mu_0 > 0, \sigma \in (0,1)$ and μ_∞ sufficiently small.
$k \longleftarrow 0$
repeat
Compute $W^{k+1} \longleftarrow$ **Algorithm 2**$(X, D^k, W^k, \lambda, \gamma, \mu_k)$
Compute $D^{k+1} \longleftarrow$ **Algorithm 3**$(X, D^k, W^{k+1}, \lambda, \gamma, \mu_k)$
Update $\mu_{k+1} \longleftarrow \sigma\mu_k$
Set $k \longleftarrow k+1$
until $\mu < \mu_\infty$.
OUTPUT: D

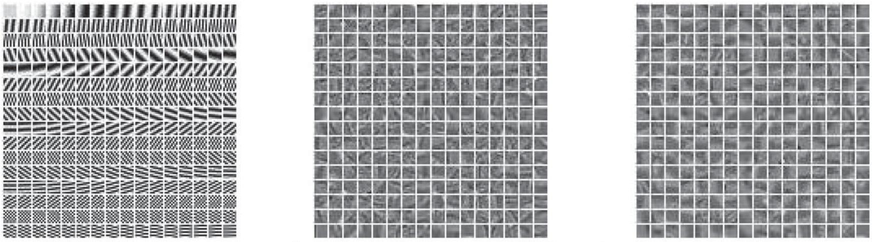
$\{(1,1),\dots,(\bar{i},\bar{j})\}$, we find the extraction operator R_{ij} and form the matrices

$$R = \begin{bmatrix} R_{11} \\ R_{12} \\ \vdots \\ R_{\bar{i}\bar{j}} \end{bmatrix} \quad \text{and} \quad R^{\mathrm{T}} = \begin{bmatrix} R^{\mathrm{T}}{}_{11} & R^{\mathrm{T}}{}_{12} & \dots & R_{\bar{i}\bar{j}}{}^{\mathrm{T}} \end{bmatrix}.$$

We continue by solving (7) to find $y \in \mathbb{R}^{KS}$. Then express $y \in \mathbb{R}^{KS}$ as an $K \times S$ matrix Y of patch representations under D, so $Y = [y_{11}\ y_{12}\ \dots\ y_{\bar{i}\bar{j}}]$ and DY is an $n_1n_2 \times S$ matrix whose columns are vectorized patches. Finally, reshaping DY into $n_1n_2S \times 1$ vectorized form $\overline{Dy}$, we have $R^{\mathrm{T}}\overline{Dy} = \sum_{i,j} R^{\mathrm{T}}_{i,j}Dy_{i,j}$ is an $N_1 \times N_2$ image in vectorized form.

We now use the above scheme to solve in-painting problems, where A is the sampling operator. In-painting is a process wherein missing information in an image is recovered, namely when some known set of pixels of an image have been lost. Let $M \in \mathbb{R}^{N_1N_2}$ be a vectorization of an $N_1 \times N_2$ image, Ω a subset of $\{1,\dots,N_1N_2\}$ and A be the $|\Omega| \times N_1N_2$ matrix formed by removing all row i from the identity matrix $I_{N_1N_2}$ for all $i \notin \Omega$. Then we call A a sampling operator with sampling rate $\mathsf{SR} = \frac{|\Omega|}{N_1N_2}$, and $A(M)$ is a vectorization of the original image, containing only those pixels indexed by Ω.

The patching approach developed by Xu and Yin [13] is implemented to minimize (7) using the DCA with Nesterov's smoothing. In all settings, we compare the discrete cosine transform (DCT) dictionary with two different learned dictionaries: ℓ_1 regularization by solving (8) with block proximal gradient (BPG) proposed in [13,17] and $\ell_1 - \ell_2$ regularization by minimizing (9) with Algorithm 4. For all learned dictionaries, we use a training set of 10,000 greyscale patches of size 8×8, chosen randomly from 100 images taken from the Berkeley Segmentation Dataset[1]; see [18]. The training matrix X is of size $64 \times 10,000$. The



DCT dictionary ℓ_1 learned dictionary by BPG $\ell_1 - \ell_2$ learned dictionary by Alg 4

Figure 5. Three different types of dictionaries: (a) DCT dictionary, (b) ℓ_1 learned dictionary by BPG, (c) $\ell_1 - \ell_2$ learned dictionary by Alg 4.

number of atoms for learned dictionaries is set to be $K = 256$ and thus all learned dictionaries are of size 64×256.

A technical step before performing the DCA-based learning algorithm is to set each column of the training matrix X to zero mean. For the ℓ_1 regularization, we solve (8) with $\lambda = 0.1$ by the BPG method using the same parameters as in [13, Algorithm 3]. For $\ell_1 - \ell_2$ regularization, we randomly choose K columns from the training matrix X and normalize them to form a starting dictionary D^0 when solving (9) by Algorithm 4 with $W^0 = \texttt{pinv}(D)X$, $\lambda = 1$, $\gamma = 2000$, $\sigma = 0.8$, $\mu_\infty = 10^{-5}$. The obtained dictionaries are shown in Figure 5.

For all tests, we use the 512×512 standard reference image *Lena*, and choose $n_1 \times n_2 = 8 \times 8$ patches. Before running the test, a column of all ones is added to the DCT and learned dictionaries. As discussed in [13], patching artefacts which appear in the solution are mitigated by processing the image three times, each with a different partition. The solution is then taken to be the average of the three trials. Our partitions were determined by choosing upper left corner patches of

Table 1. Results for in-painting with three different dictionaries. FISTA and DCA are ℓ_1 and $(\ell_1 - \ell_2)$ regularization, respectively. Best results are in bold.

SR			Rel. error (%)	PSNR
		Corrupted image	70.72	8.458
50%	*DCT*	FISTA	**4.81**	**31.83**
		DCA	6.06	29.81
	ℓ_1 *learned*	FISTA	**3.45**	**34.699**
		DCA	4.11	33.168
	$\ell_1 - \ell_2$ *learned*	FISTA	**3.48**	**34.613**
		DCA	4.22	33.937
		Corrupted image	83.77	6.987
30%	*DCT*	FISTA	**7.01**	**28.501**
		DCA	8.27	27.101
	ℓ_1 *learned*	FISTA	**5.21**	**31.113**
		DCA	5.89	30.048
	$\ell_1 - \ell_2$ *learned*	FISTA	**5.02**	**31.438**
		DCA	5.72	30.303

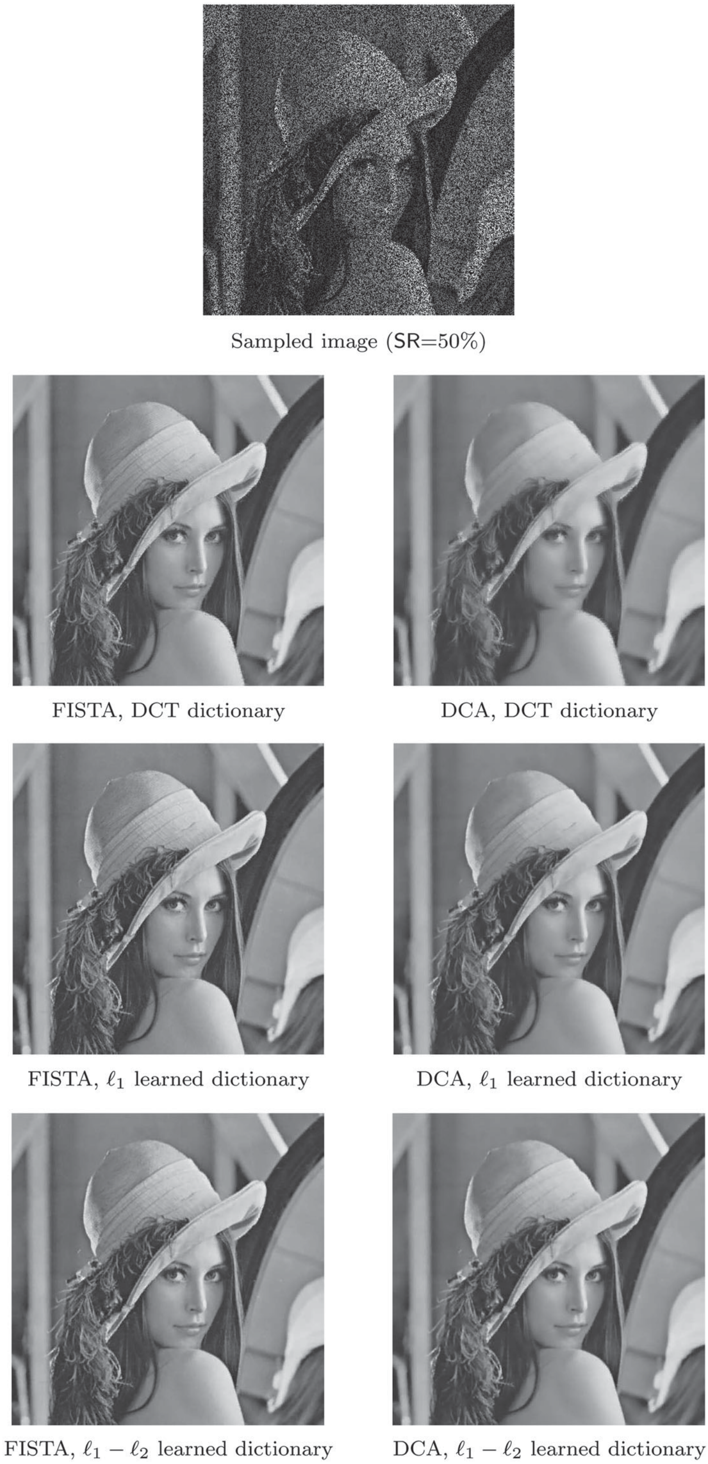


Figure 6. In-painting result on *Lena512* with DCT and learned dictionaries. FISTA and DCA are ℓ_1 and $(\ell_1 - \ell_2)$ regularization, respectively. (a) Sampled image (SR = 50%), (b) FISTA, DCT dictionary, (c) DCA, DCT dictionary, (d) FISTA, ℓ_1 learned dictionary, (e) DCA, ℓ_1 learned dictionary, (f) FISTA, $\ell_1 - \ell_2$ learned dictionary, (g) DCA, $\ell_1 - \ell_2$ learned dictionary.

size 8×8, 5×5, and 2×2. Corrupted images were defined as $b = A(M) + \sigma\xi$, where ξ is a matrix of noise with standard normal distribution scaled by $\sigma = c\frac{\|A(M)\|_2}{\|\xi\|_2}$.

In our experiments, we fix the noise level $c = 1\%$ and use $\nu = \frac{1}{2\sigma}$ for ℓ_1 regularization with FISTA [19] and $\nu = \frac{3}{20\sigma}$ for $\ell_1 - \ell_2$ regularization with Algorithm 1.

We measure error of the solution $\widetilde{M}$ relative to the true image M by relative error, $\mathsf{RE} = \frac{\|\mathsf{M}-\widetilde{\mathsf{M}}\|_F}{\|\mathsf{M}\|_F}$, and peak signal to noise ratio as $\mathsf{PSNR} = 20 \cdot \log_{10}\left(\frac{\sqrt{N_1N_2}}{\|\mathsf{M}-\widetilde{\mathsf{M}}\|_F}\right)$. See Table 1 for a comparison of the PSNR values and relative errors of the in-painting result with different sampling rates and different dictionaries. Figure 6 gives a visual illustration for the case SR = 50%. Given these results, it is evident that $\ell_1 - \ell_2$ learned dictionary obtained from Algorithm 4 yields results very close to the one constructed by the BPG method. Moreover, it can be seen that the performance of DCA with smoothing technique is nearly comparable to that of the FISTA on learned dictionaries.

3.6. Discussion

The fast patch dictionary method given by Xu and Yin [13] was qualitatively successful in reconstructing corrupted images, using both ℓ_1 regularization with FISTA, and $(\ell_1 - \ell_2)$ regularization with DCA in combination with Nesterov's smoothing. In every case, learned dictionaries improve results compared to a DCT dictionary.

The FISTA approach converges after fewer iterations (Figure 7), but DCA required less time per iteration. The optimal choice of μ and γ parameters in the DCA method is unknown, and allows for the possibility of future improvement. Similarly, implementing FISTA without a backtracking line search is likely to induce better results, in cases where the Lipschitz constant of the gradient can

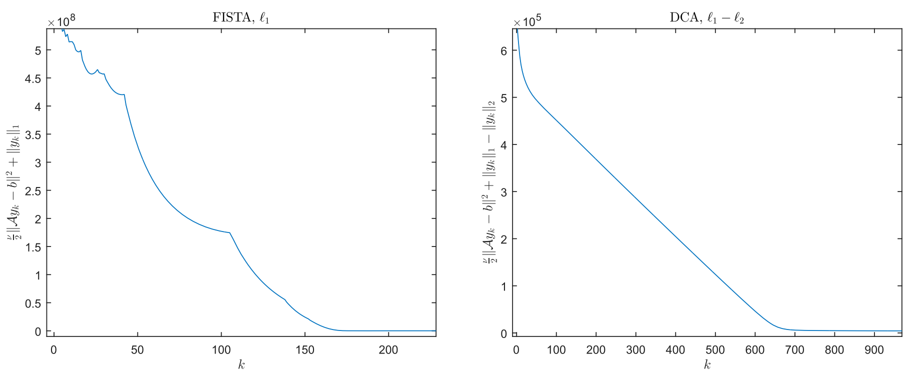


Figure 7. Objective function value versus iteration for FISTA and DCA with learned dictionary.

be determined. Also, it is not known which choice of ν (used to weight data-fitting versus sparsity) leads to the best solution. Future work may explore optimal parameter choice as well as characterize which problems benefit from ℓ_1 versus $(\ell_1 - \ell_2)$ regularization.

Note

1. Available at https://www2.eecs.berkeley.edu/Research/Projects/CS/vision/bsds/

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