

# COMPLEMENTARY INFORMATION AND LEARNING TRAPS\*

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We develop a model of social learning from complementary information: short-lived agents sequentially choose from a large set of flexibly correlated information sources for prediction of an unknown state, and information is passed down across periods. Will the community collectively acquire the best kinds of information? Long-run outcomes fall into one of two cases: (i) efficient information aggregation, where the community eventually learns as fast as possible; (ii) “learning traps,” where the community gets stuck observing suboptimal sources and information aggregation is inefficient. Our main results identify a simple property of the underlying informational complementarities that determines which occurs. In both regimes, we characterize which sources are observed in the long run and how often. *JEL* Codes: D81, D83, D62, O32.

## I. INTRODUCTION

Societies accumulate knowledge over time through the efforts of many individuals. The quality of current knowledge—for example, about science and engineering—translates into the quality of decisions that are made, and so the rates at which societies accumulate knowledge matter for their welfare. These rates vary substantially across contexts. What helps a society efficiently refine its knowledge over time? What, in contrast, can cause the social process of knowledge acquisition to become trapped in slower, less productive paths?

As a leading example, we focus on research as a channel of knowledge accumulation. The process of research is inherently “path-dependent,” because past research affects what current researchers know, and thus which investigations they consider most valuable to undertake. If research were centrally planned,

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then a planner's directives could account for the externalities of researchers' activities. But researchers are often motivated by more immediate and individual goals, such as making the biggest possible discoveries now (rather than enabling others to make better discoveries later). How does the socially optimal information acquisition process compare to a decentralized one that results from the choices of short-lived agents, who do not internalize the externalities of their information acquisitions?

In this article, we consider a sequence of researchers working on a shared scientific problem, which we model as acquiring information over time about the value of an unknown parameter. We study the rate at which the society learns about this parameter. Our main results demonstrate that the structure of the interdependencies across the available signals—more precisely, *informational complementarities*—is crucial in determining whether the decentralized process is essentially efficient, or whether it gets trapped at a learning rate inferior to the optimal one.<sup>1</sup>

To fix ideas, consider as a stylized example a particular research question: whether dopamine levels are predictive of the onset and severity of Parkinson's disease (PD). Each researcher can conduct a study to learn more about the typical level of dopamine in PD patients. A study is modeled as a Gaussian signal, whose realization is informative about the value of the parameter. For example, researchers have various technologies for measuring dopamine levels, such as different imaging methods. We model limits on researchers' resources by restricting each researcher to obtain one signal about the unknown parameter from a finite but potentially large set of signals: in the example, this corresponds to a choice of which measurement to use in the study.<sup>2</sup> The

1. The basic feature that information can be complementary appears in many settings besides research—for example, informational complementarities are relevant for auctions (Milgrom and Weber 1982a, 1982b), for firms (Athey and Schmutzler 1995), for team composition (Chade and Eeckhout 2018), and within markets (Goldstein and Yang 2015; Chen and Waggoner 2016). We build on this literature—which has primarily focused on one-time information acquisitions—by asking how informational complementarities affect learning in a dynamic setting. Prior work studying the rate of learning include Vives (1992), DeMarzo, Vayanos, and Zwiebel (2003), Golub and Jackson (2012), Hann-Caruthers, Martynov, and Tamuz (2017), and Harel et al. (2018) among others.

2. This feature relates to a number of recent papers also studying dynamic information acquisition from different kinds of information—see, for example, Che and Mierendorff (forthcoming) and Mayskaya (2019), who study learning from two Poisson signals, and Sethi and Yildiz (2016) and Fudenberg, Strack, and Strzalecki (2018), who study learning from multiple Gaussian signals. Although

researcher's goal is to maximize immediate reduction in uncertainty about the unknown parameter. This framework is a social learning model, but our article departs from the classic model (Banerjee 1992; Bikhchandani, Hirshleifer, and Welch 1992; Smith and Sørensen 2000) by assuming that all information is *public*, thus turning off the inference problem essential to informational cascades in the previous literature.<sup>3</sup> In addition, we suppose that information is endogenously acquired—as in Burguet and Vives (2000), Mueller-Frank and Pai (2016), and Ali (2018)—with the new feature that agents choose from a finite set of complementary signals.

In our model, the choices of successive researchers are linked because of informational complementarities, which we now describe. Each signal observation is modeled as a linear combination of the parameter of interest, various confounding variables, and idiosyncratic Gaussian noise.<sup>4</sup> How informative the observation is depends on what is known about the confounding variables. For example, sensors used to measure dopamine may also be sensitive to other chemicals in the brain. The informativeness of the sensor's measurement is improved by a better understanding of the extent to which these chemicals bias the reading. In this sense, information about the confounding variables, and observations of measurements confounded by those variables, are complementary.

Formally, we propose a definition for *complementary sets of signals*, which builds on prior work by Börgers, Hernando-Veciana, and Krahmer (2013). A complementary set is a set of signals that (i) if observed infinitely often, reveals the value of the payoff-relevant parameter, and (ii) has no proper subset that also reveals the parameter. Removing any source from such a set therefore makes it impossible to learn the parameter of interest.

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the framework bears some resemblance to classic multi-armed bandit models (Gittins 1979; Easley and Kiefer 1988), the signals in our setting do not directly produce payoffs.

3. A recent paper that also uses a social learning model to study the take-up of new technologies is Wolitzky (2018). This model departs from the classic herding model by assuming that agents observe the outcomes of previous agents and not their actions, which leads to very different learning dynamics relative to our assumption that agents observe the information of past agents.

4. Online Appendix G has a discussion of how our main results might be extended beyond normal signals.

Our main result is that if the smallest complementary set of signals is of size  $K$ —where  $K$  is the number of unknown variables (including both the parameter of interest and all of the confounding variables)—then a decentralized research process achieves efficient learning in the long run. In contrast, if the smallest complementary set contains fewer than  $K$  signals, then early suboptimal information acquisitions can propagate across time, creating persistent inefficiencies in information gathering. We call such outcomes “learning traps.”

The main intuition is as follows. The tension between the interests of short-lived agents and a patient social planner arises because optimal learning may require investments in information about confounding variables that will be useful only later in the learning process. If the smallest complementary set has cardinality equal to the number of unknowns, then the payoff-relevant parameter can only be learned if all confounds are also learned. Thus even short-lived agents will choose to acquire information that reveals the confounding variables, leading them to eventually discover the best set of signals (that leads to efficient learning). By the same logic, if there is a complementary set of size smaller than the number of unknowns, then it is possible to learn the payoff-relevant parameter even if some confounding variables are never learned. This allows the wedge between short- and long-run incentives to persist.

The main technical innovation behind our results is to establish a formal connection between society’s learning dynamics and a limiting dynamical system (see [Section VII](#) for details). We demonstrate that a set of signals is observed in the long run (from some prior beliefs) if and only if it corresponds to a stable point in this system. This connection enables us to completely characterize the possible long-run observation sets.

Next we consider the size of welfare losses associated with learning traps. We show that the rate of information aggregation can be arbitrarily slower than the efficient benchmark. Measured in terms of absolute discounted payoffs, inefficiency can also be arbitrarily large in a learning trap. However, if measured on a per period basis (i.e., considering *average* discounted payoffs), the payoff loss caused by a learning trap vanishes in the patient limit because society eventually learns (albeit slowly) the value of the payoff-relevant parameter. We present a generalization of our model, where the unknown variables are slowly changing over

time.<sup>5</sup> In this model, we show that even the average discounted loss can be large. Thus, except in the special case of perfect persistence, learning traps can lead not only to inefficiencies in the rate of information aggregation but also to inefficiencies in payoff terms.

Having shown that learning traps can exist, and that the welfare loss under these traps can be large, we turn to considering different interventions for avoiding learning traps. We show that policy makers can restore efficient information aggregation by providing information about the relevant confounding variables (for example, a forward-looking funding agency can support research about confounding variables that are not of direct societal interest). Another effective intervention is to reshape the payoff structure so that agents can be rewarded for information acquired over many periods. These observations are consistent with practices that have arisen in academic research, including the establishment of third-party funding agencies (e.g., the National Science Foundation) to support basic science and methodological research, and the evaluation of researchers based on advancements developed across several papers (e.g., tenure and various awards).

The learning traps we demonstrate connect to a body of work regarding dynamic investment in human capital (Jovanovic and Nyarko 1996; Cunha and Heckman 2007; Lizzeri and Siniscalchi 2008) that studies complementarities in production technologies (rather than complementarities in information). There are certain high-level connections. For example, this literature shows that early investment choices affect the incentives to invest in the future and that early misallocations potentially lead to worse long-run outcomes (similar to our learning traps). Related to our Section IX, interventions can push agents onto better skill-acquisition paths. However, the specific form of long-run inefficiencies we obtain follows from the structure of informational complementarities in our setting.

Our framework, and the long-run learning patterns we identify, are particularly related to Sethi and Yildiz (2016, 2019). These papers study a setting where individuals can learn from one

5. This is a technically challenging setting to analyze and correspondingly prior work is limited: Jovanovic and Nyarko (1996), Moscarini, Ottaviani, and Smith (1998), Frongillo, Schoenebeck, and Tamuz (2011), Alatas et al. (2016), and Dasaratha, Golub, and Hak (2018) are the only social learning settings with a dynamic state that we are aware of.

another over time, choosing which of many individuals to listen to in each period. A key force in [Sethi and Yildiz \(2016\)](#) is that repeatedly listening to a given individual is informative about that individual's "perspective," which allows for improved (future) debiasing of that person's information. This pushes individuals to return to those they have listened to before. The same force plays an important role in our article. Repeated acquisition of a given signal helps agents learn not just the payoff-relevant parameter but also all of the variables confounding that signal. Unlike in [Sethi and Yildiz \(2016\)](#), we allow for different signals to share confounding variables. Thus, observations of a given signal not only help agents interpret future observations of signals of the same kind; they also potentially aid in the interpretation of *other* signals. It is exactly the structure of these learning spillovers that determines whether inefficient learning can obtain in the long run. Our finding of learning traps in the present study is related to the observation of long-run homophily in [Sethi and Yildiz \(2019\)](#), where individuals eventually listen only to others whose perspectives are correlated with their own.

Finally, in related work ([Liang, Mu, and Syrgkanis 2017, 2019](#)), we study dynamic learning from correlated Gaussian signals, focusing on informational environments with a single complementary set, where agents must attend to all signals to learn the payoff-relevant parameter. We find there that myopic information acquisition is in some cases efficient from period 1 ([Liang, Mu, and Syrgkanis 2017](#)), and that under conditions on the prior belief, the optimal path of information acquisitions admits a simple and exact characterization ([Liang, Mu, and Syrgkanis 2019](#)). The present article considers a substantially more general class of informational environments—in particular, allowing for society to have multiple ways for eventually learning the state—but focuses on asymptotic efficiency. We find here a necessary and sufficient condition for long-run efficiency, which nests the environment considered in [Liang, Mu, and Syrgkanis \(2017, 2019\)](#). In environments that do not satisfy this condition, we demonstrate that in contrast, learning traps may obtain.

## II. EXAMPLES

Our main results demonstrate that in some informational environments, society maximizes the long-run speed of learning, whereas in others, agents can persistently acquire information

from inefficient sources. The examples below illustrate properties of the informational environment that differentiate these two cases.

### II.A. Existence of Learning Traps

Suppose agents sequentially acquire information to learn about an unknown parameter  $\omega \sim \mathcal{N}(\mu_\omega, \sigma_\omega^2)$  and have access to three kinds of signals. The first is

$$X_1 = 3\omega + b + \varepsilon_1,$$

where  $b \sim \mathcal{N}(\mu_b, \sigma_b^2)$  is a persistent and unknown bias that is independent of  $\omega$ . The noise term  $\varepsilon_1 \sim \mathcal{N}(0, 1)$  is independent of  $\omega$  and  $b$ , and is redrawn each time an agent acquires signal  $X_1$ . The second signal provides information about the bias term  $b$ :

$$X_2 = b + \varepsilon_2.$$

Finally, there is an unbiased signal about the parameter of interest:

$$X_3 = \omega + \varepsilon_3.$$

Like  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_3$  are standard Gaussian noise terms independent of  $\omega$  and  $b$  (and of one another).

Agents are indexed by discrete time and act in order. Each agent chooses to acquire one independent observation of either  $X_1$ ,  $X_2$ , or  $X_3$ , where his choice is determined by which observation would be most informative about  $\omega$  (i.e., maximally reduces the uncertainty about  $\omega$ ). This signal realization is then made public.

Repeated acquisitions of  $X_3$  suffice for agents to eventually learn  $\omega$ . But acquisitions of  $X_1$ , if debiased via acquisitions of  $X_2$ , can lead to much faster learning. This follows from our subsequent Claim 1, but one can also observe that  $X_1 - X_2$  is Blackwell more-informative (in the sense of Blackwell (1951)) than two realizations of  $X_3$ ; thus, agents learn faster by alternating between  $X_1$  and  $X_2$  than by acquiring  $X_3$  exclusively.<sup>6</sup>

Suppose that agents have large initial uncertainty about the bias term  $b$ —specifically, the prior variance  $\sigma_b^2 > 8$ . In this case,

6. Note that  $X_1 - X_2 = 3\omega + \varepsilon$ , where  $\varepsilon \sim \mathcal{N}(0, 2)$ , whereas two independent realizations of  $X_3$  provide the same information as  $2\omega + \varepsilon$ , with  $\varepsilon \sim \mathcal{N}(0, 2)$ . The former is clearly more informative.

the signal  $X_3$  is initially most informative about  $\omega$ , as the biased signal  $X_1$  is noisier due to the uncertainty about  $b$ , and the signal  $X_2$  about  $b$  is completely uninformative (recall that  $b$  is independent of  $\omega$  in the prior). The first agent's acquisition of  $X_3$  does not provide any information about  $b$ , so the foregoing arguments show that  $X_3$  remains most informative for the second agent. Iterating this logic, it follows that every agent chooses to acquire  $X_3$ .

This example demonstrates that although it is socially optimal for agents to invest in understanding the bias  $b$ , short-sighted agents will choose instead to repeatedly acquire  $X_3$ , maximizing immediate gains but leading to inefficiently slow long-run learning about  $\omega$ . We refer to the set  $\{X_3\}$  as a “learning trap.”<sup>7</sup>

## II.B. Efficient Information Aggregation

In contrast, consider the following informational environment with signals

$$\begin{aligned}X_1 &= \omega + b_1 + \varepsilon_1, \\X_2 &= b_1 + b_2 + \varepsilon_2, \\X_3 &= b_2 + \varepsilon_3, \\X_4 &= 10\omega + b_1 + 2b_2 + \varepsilon_4.\end{aligned}$$

The payoff-relevant parameter  $\omega$  and confounding variables  $b_1$ ,  $b_2$  are persistent and jointly normally distributed, while the (independent) noise terms  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\varepsilon_3$ ,  $\varepsilon_4$  are standard Gaussian and i.i.d. across realizations.

As in the previous environment, there are multiple sets of signals that permit long-run learning of  $\omega$ . Specifically, repeated observations of signals in any of the four sets  $\{X_1, X_2, X_3\}$ ,  $\{X_1, X_2, X_4\}$ ,  $\{X_1, X_3, X_4\}$ , and  $\{X_2, X_3, X_4\}$  will lead agents to eventually learn  $\omega$ . In Section IV, we formalize the sense in which these are “complementary” sets of signals. However, the rates of learning permitted by each of these sets are not the same, and society's long-run speed of learning is strictly maximized when agents repeatedly acquire  $X_2$ ,  $X_3$ , and  $X_4$ .<sup>8</sup> It turns out that starting from

7. Online Appendix G contains a similar example of a learning trap with binary states and signals.

8. This follows from the later Proposition 2: it can be checked that  $\text{val}(\{X_2, X_3, X_4\}) = \frac{100}{9}$  is largest across the four sets, so the “best set” is  $S^* = \{X_2, X_3, X_4\}$ .



any prior belief over the unknown variables, agents will eventually discover this best set and exclusively acquire these signals.

The key difference between the two environments is that in this example the inefficient sets  $\{X_1, X_2, X_3\}$ ,  $\{X_1, X_2, X_4\}$ , and  $\{X_1, X_3, X_4\}$  share their confounding variables ( $b_1$  and  $b_2$ ) with the efficient set  $\{X_2, X_3, X_4\}$ . Repeated observation of signals from any inefficient set will thus lead agents to learn  $b_1$  and  $b_2$ , given which they can debias signals in the efficient set. This informational spillover from learning about  $\omega$  to learning about the confounding variables means that inefficient sets fail to be self-reinforcing—the information they generate helps agents realize that there are more informative signals. Put another way, the complementarities between signals in an inefficient set are eventually outweighed by stronger complementarities with signals outside of the set. Only the efficient set  $\{X_2, X_3, X_4\}$  has the property that complementarities within the set remain strongest (see the discussion after Theorem 1 for more detail).

In contrast, in our example in Section II.A, repeated acquisition of the signal  $X_3$  provides no information about the variable  $b_1$  that confounds the signals  $\{X_1, X_2\}$ , and so repeated observation of  $X_3$  is self-reinforcing. Our subsequent main results make this contrast precise, and explain in general how the structure of complementarities across signals determines the efficiency of long-run information aggregation.

### III. SETUP

#### III.A. Informational Environment

There are  $K$  persistent unknowns: a real-valued payoff-relevant state  $\omega$  and  $K-1$  real-valued confounding variables  $b_1, \dots, b_{K-1}$ .<sup>9</sup> We assume that the state vector  $\theta := (\omega, b_1, \dots, b_{K-1})'$  follows a multivariate normal distribution  $\mathcal{N}(\mu^0, \Sigma^0)$ , where  $\mu^0$  is a  $K \times 1$  real-valued vector, and the prior covariance matrix  $\Sigma^0$  has full rank.<sup>10</sup>

There are  $N$  (fixed) kinds or sources of information available at each discrete period  $t \in \mathbb{Z}_+$ . Observation of source  $i$  in period  $t$

9. Online Appendix D.3 discusses the case of multiple payoff-relevant states.

10. The full-rank assumption is without loss of generality. If there is linear dependence across the states, the model can be reduced to a lower-dimensional state space that satisfies full rank.

produces a realization of the random variable

$$X_i^t = \langle c_i, \theta \rangle + \varepsilon_i^t = \omega c_{i1} + b_1 c_{i2} + \cdots + b_{K-1} c_{iK} + \varepsilon_i^t, \quad \varepsilon_i^t \sim \mathcal{N}(0, 1),$$

where  $c_i = (c_{i1}, \dots, c_{iK})'$  is a vector of constants, and the noise terms  $\varepsilon_i^t$  are independent from each other and across periods. Normalizing these noise terms to have unit variance is without loss of generality because the coefficients  $c_i$  are unrestricted. We often drop the time indices on the random variables, associating  $X_i = \langle c_i, \theta \rangle + \varepsilon_i$  with source  $i$  and understanding that the noise term is independently realized with each new observation. We use  $[N]$  to denote the set  $\{1, \dots, N\}$  of all sources.

The payoff-irrelevant unknowns  $b_1, \dots, b_{K-1}$  produce correlations across the sources, and can be interpreted for example as confounding variables. The difference between the terms  $b_i$  and the terms  $\varepsilon_i$  is that the former are persistent over time whereas the latter are i.i.d.—so the variances of  $b_i$  can be reduced over time, but the variances of  $\varepsilon_i$  are fixed. Separating these terms allows us to distinguish between reducible and irreducible noise (with respect to learning about  $\omega$ ) in the signals.

### III.B. Decision Environment

Agents are indexed by discrete time  $t$  and move sequentially. Each agent first chooses one of the  $N$  sources, observing an independent realization of the corresponding signal. He then predicts  $\omega$ , selecting an action  $a \in \mathbb{R}$  and receiving the payoff  $-\mathbb{E}[(a - \omega)^2]$ . The agent's optimal prediction is the posterior mean of  $\omega$ , and the payoff is the negative of the posterior variance of  $\omega$ . We note that the action  $a$  is not necessary for specifying the model: the analysis is unchanged if we directly assume that each agent chooses to acquire the signal that maximally reduces posterior variance of  $\omega$ . In some applications, the latter formulation (where the agent acquires information but does not take an action) may be more natural—for example, if the agent's goal is simply to progress scientific understanding—whereas in others, it may be natural to suppose that the agent does take an action on the basis of his information (e.g., recommends a treatment) and receives a higher payoff when his belief is more precise. We also note that the specific payoff function of quadratic loss is not crucial. Our subsequent analysis, with the exception of the result in part (b) of Proposition 2, goes through for arbitrary payoff functions  $u(a, \omega)$ . See [Online Appendix D.1](#) for more detail.

We assume throughout that all signal realizations are public. Thus, each agent  $t$  faces a history  $h^{t-1} \in ([N] \times \mathbb{R})^{t-1} = H^{t-1}$  consisting of all past signal choices and their realizations, and his signal acquisition strategy is a mapping from histories to sources. At every history  $h^{t-1}$ , the agent's payoffs are maximized by choosing the signal that minimizes his posterior variance of  $\omega$ .<sup>11</sup>

### III.C. Society's Information Acquisitions

Because the environment is Gaussian, the posterior variance of  $\omega$  is a deterministic function  $V(q_1, \dots, q_N)$  of the number of times  $q_i$  that each signal  $i$  has been observed so far.<sup>12</sup> Thus, each agent's signal acquisition is a function of the prior and past signal acquisitions only (and does not depend on the signal realizations). This allows us to track society's acquisitions as deterministic count vectors

$$m(t) = (m_1(t), \dots, m_N(t))' \in \mathbb{Z}_+^N,$$

where  $m_i(t)$  is the number of times that signal  $i$  has been observed up to and including period  $t$ . The count vector  $m(t)$  evolves according to the following rule:  $m(0)$  is the zero vector, and for each time  $t \geq 0$  there exists  $i^* \in \operatorname{argmin}_i V(m_i(t) + 1, m_{-i}(t))$  such that

$$m_i(t+1) = \begin{cases} m_i(t) + 1 & \text{if } i = i^* \\ m_i(t) & \text{otherwise} \end{cases}.$$

That is, the count vector increases by 1 in the coordinate corresponding to the signal that yields the greatest immediate reduction in posterior variance. We allow ties to be broken arbitrarily, and there may be multiple possible paths  $\{m(t)\}_{t=0}^\infty$ .

We are interested in the long-run frequencies of observation  $\lim_{t \rightarrow \infty} \frac{m_i(t)}{t}$  for each source  $i$ —that is, the fraction of periods eventually devoted to each source. As we show in [Section VI](#), these limits exist under a weak technical assumption. But note that the limit may depend on the prior belief, as already illustrated by the example in [Section II.A](#). Which long-run outcomes are possible is a central question we seek to understand in this article.

11. [Online Appendix D.2](#) shows that our results generalize to agents who are slightly forward-looking.

12. See [Appendix A.1](#) for the complete closed-form expression for  $V$ , which depends on the prior  $\Sigma^0$  and signal coefficient vectors  $\{c_i\}_{i=1}^N$ .

## IV. COMPLEMENTARY SET OF SOURCES

We first introduce a definition for a complementary set of sources. These sets will play an important role in the subsequent results. As a preliminary step, we assign to each set of sources  $S \subseteq [N] := \{1, \dots, N\}$  an informational value.

## IV.A. Informational Value

Write  $\tau(q_1, \dots, q_N) = \frac{1}{V(q_1, \dots, q_N)}$  for the posterior precision about the payoff-relevant state  $\omega$  given  $q_i$  observations of each source  $i$ , where the prior precision is  $\tau_0 := \tau(0, \dots, 0)$ . The informational value of  $S$ , denoted  $\text{val}(S)$ , is defined to be the largest feasible improvement on precision (averaged across many periods), when signals are acquired from  $S$  alone.<sup>13</sup>

**DEFINITION 1.** The (asymptotic) *informational value* of the set  $S$  is the maximal per period increase in the precision about  $\omega$  over a long horizon:

$$\text{val}(S) = \limsup_{t \rightarrow \infty} \left[ \max_{q^t \in Q_S^t} \left( \frac{\tau(q^t) - \tau_0}{t} \right) \right],$$

where

$$Q_S^t = \left\{ q \in \mathbb{Z}_+^N : \sum_{i=1}^N q_i = t \text{ and } \text{supp}(q) \subset S \right\}$$

is the set of all count vectors that allocate  $t$  observations across (only) the sources in  $S$ .

The informational value is defined with respect to learning about  $\omega$ , but we omit this dependence since the payoff-relevant state is fixed throughout this article. The informational value turns out to be prior-independent, as we show in Proposition 1. Finally, note that  $\text{val}(S)$  exceeds 0 only if it is possible to completely learn  $\omega$  given infinite observations from  $S$ , as otherwise the reduction in variance is finite, and hence the average improvement of

13. This definition of informational value closely resembles the definition of the value of a team in [Chade and Eeckhout \(2018\)](#), although we consider belief precision instead of negative posterior variance. Using posterior variances in Definition 1 would yield a value given by  $\text{val}(S) = \limsup_{t \rightarrow \infty} \left[ \max_{q^t \in Q_S^t} (-tV(q^t)) \right]$ . This together with Definition 2 would return a similar notion of complementarity but would present the technical issue of evaluating  $\infty - \infty$  because value as defined this way (is always negative and) can be  $-\infty$ .

each signal observation (taking the total number of observations to infinity) must be 0.

#### IV.B. Complementary Set

Our definition for a complementary set is based on [Börgers, Hernando-Veciana, and Krahmer \(2013\)](#); see [Online Appendix H](#) for an extended comparison.

**DEFINITION 2.** The set  $S$  is *complementary* if  $\text{val}(S) > 0$  and

$$(1) \quad \text{val}(S) > \text{val}(S') + \text{val}(S \setminus S')$$

for all nonempty proper subsets  $S'$  of  $S$ .

Condition (1) requires that the marginal value of having access to the sources in any  $S' \subset S$  is increased by also having access to sources  $S \setminus S'$ .<sup>14</sup> We note that the first condition that  $\text{val}(S) > 0$  is implied by condition (1) whenever  $S$  is not a singleton.<sup>15</sup>

#### IV.C. Characterization of Complementary Sets

The proposition below shows that a set  $S$  is complementary if and only if its signals uniquely combine to produce an unbiased signal about  $\omega$ .

**PROPOSITION 1.**  $S$  is a complementary set if and only if the first coordinate vector in  $\mathbb{R}^K$  admits a unique decomposition  $(1, 0, \dots, 0)' = \sum_{i \in S} \beta_i^S \cdot c_i$ , where all coefficients  $\beta_i^S$  are nonzero.

This characterization makes clear a second, equivalent way of understanding complementary sets: they are sets with the property that observing all signals within that set infinitely often reveals the value of the payoff relevant state, and moreover, all signals are crucial for this recovery—that is, removing any source(s) from a complementary set makes recovery of the payoff-relevant state impossible.

Proposition 1 allows us to identify complementary sets based on their signal coefficient vectors.

14. Since  $\text{val}(\emptyset) = 0$ , the inequality can be rewritten as  $\text{val}(S) - \text{val}(S \setminus S') > \text{val}(S') - \text{val}(\emptyset)$ .

15. The condition  $\text{val}(S) > 0$  has bite when  $|S| = 1$ . Specifically, it rules out all singleton sets  $S$ , except those consisting of a single unbiased signal  $X = c\omega + \varepsilon$ .

**EXAMPLE 1.** The set  $\{X_1, X_2, X_3\}$  consisting of signals  $X_1 = \omega + b_1 + \varepsilon_1$ ,  $X_2 = b_1 + b_2 + \varepsilon_2$ , and  $X_3 = b_2 + \varepsilon_3$  is complementary. To see this, observe that  $(1, 0, 0)' = c_1 - c_2 + c_3$  (where  $c_1 = (1, 1, 0)'$  is the coefficient vector associated with  $X_1$ ,  $c_2 = (0, 1, 1)'$  is the coefficient vector associated with  $X_2$ , and  $c_3 = (0, 0, 1)'$  is the coefficient vector associated with  $X_3$ ). In contrast, the set of signals  $\{X_4, X_5\}$  with  $X_4 = \omega + \varepsilon_1$  and  $X_5 = 2\omega + \varepsilon_2$  is not complementary, because many different linear combinations of  $X_4$  and  $X_5$  produce an unbiased signal about  $\omega$ . The set  $\{X_1, X_2, X_3, X_4\}$  is also not complementary, although it contains two complementary subsets.

#### IV.D. Best Complementary Set $S^*$

The informational value for any complementary set can be computed using the following claim.

**CLAIM 1.** Let  $S$  be a complementary set. Then,  $\text{val}(S) = \frac{1}{(\sum_{i \in S} |\beta_i^S|)^2}$ , where  $\beta_i^S$  are the ones given in Proposition 1.<sup>16</sup>

More generally,  $\text{val}(S)$  can be determined for an arbitrary set  $S$  as follows: If  $S$  contains at least one complementary subset, then its value is equal to the highest value among its complementary subsets; otherwise the value of  $S$  is zero. This result will follow from Proposition 2 part (a) in the next section.

Throughout the article, we assume that there is at least one complementary set and that complementary sets can be completely ordered based on their informational values.

**ASSUMPTION 1.** There is at least one complementary set  $S \subseteq [N]$ .

**ASSUMPTION 2.** Each complementary set has a distinct informational value; that is,  $\text{val}(S) \neq \text{val}(S')$  for any pair of complementary sets  $S \neq S'$ .

Assumption 1 guarantees that the payoff-relevant parameter  $\omega$  is identifiable given the available signals, and hence it is possible to learn it eventually. Our main results extend even when this assumption fails, and we refer the reader to [Online Appendix C](#)

16. This claim and the definition of informational value together imply that the minimum posterior variance at time  $t$  (when restricting to signals in  $S$ ) vanishes like  $\frac{(\sum_{i \in S} |\beta_i^S|)^2}{t}$  asymptotically.

for details and discussion of some subtleties. Assumption 2 is generically satisfied.

Together, these assumptions imply the existence of a “best” complementary set, whose informational value is largest among complementary sets. This set plays an important role, and we denote it by  $S^*$  in the remainder of this article.

## V. OPTIMAL LONG-RUN OBSERVATIONS

We show next that optimal information acquisitions eventually concentrate on the best complementary set  $S^*$ . Specifically, consider the distribution

$$\lambda_i^* = \begin{cases} \frac{|\beta_i^{S^*}|}{\sum_{j \in S^*} |\beta_j^{S^*}|} & \forall i \in S^* \\ 0 & \text{otherwise,} \end{cases}$$

which assigns zero frequency to sources outside of the best set  $S^*$ , and samples sources within  $S^*$  proportionally to the magnitude of  $\beta_i^{S^*}$ . That is, each signal in  $S^*$  receives frequency proportional to its contribution to an unbiased signal about  $\omega$ , as defined in Proposition 1. The result below shows two senses in which  $\lambda^*$  is the optimal long-run frequency over signals.

PROPOSITION 2.

- (a) Optimal Information Aggregation:  $\text{val}([N]) = \text{val}(S^*)$ . Moreover, for any sequence  $q(t)$  such that  $\lim_{t \rightarrow \infty} \frac{\tau(q^t) - \tau_0}{t} = \text{val}([N])$ , it holds that  $\lim_{t \rightarrow \infty} \frac{q(t)}{t} = \lambda^*$ .
- (b) Social Planner Problem: For any  $\delta$ , let  $d_\delta(t)$  be the vector of signal counts (up to period  $t$ ) associated with any signal path that maximizes the  $\delta$ -discounted average payoff

$$U_\delta := \mathbb{E} \left[ - \sum_{t=1}^{\infty} (1 - \delta) \delta^{t-1} \cdot (a_t - \omega)^2 \right].$$

Then there exists  $\underline{\delta} < 1$  such that  $\lim_{t \rightarrow \infty} \frac{d_\delta(t)}{t} = \lambda^*$  for every  $\delta \geq \underline{\delta}$ .

Part (a) says that the informational value of  $S^*$  is the same as the informational value of the entire set of available sources. In this sense, having access to all available sources does not improve on the speed of learning achievable from the best complementary set  $S^*$  alone. Moreover, this speed of learning is attainable only

if the long-run frequency over sources is the distribution  $\lambda^*$ .<sup>17</sup> Part (b) of Proposition 2 says that a (patient) social planner—who maximizes a discounted average of agent payoffs—will eventually observe sources in the proportions described by  $\lambda^*$ . Based on these results, we subsequently use  $\lambda^*$  as the optimal benchmark.

## VI. MAIN RESULTS

We now ask whether society's acquisitions converge to the optimal long-run frequencies  $\lambda^*$  characterized above. We show that informational environments can be classified into two kinds—those for which efficient information aggregation is guaranteed (long-run frequencies are  $\lambda^*$  from all prior beliefs), and those for which “learning traps” are possible (there are prior beliefs from which agents end up exclusively observing some set of sources different from the efficient set  $S^*$ ). Separation of these two classes depends critically on the size of the smallest complementary set.

### VI.A. *Learning Traps versus Efficiency*

Our first result uses an assumption on the signal structure, which requires that every set of  $k \leq K$  signals are linearly independent:

**ASSUMPTION 3** (Strong linear independence). Every  $k \leq K$  signal coefficient vectors  $c_{i_1}, c_{i_2}, \dots, c_{i_k}$  are linearly independent.

If there are at least  $K$  signals (i.e.,  $N \geq K$ ), then Assumption 3 requires every  $K$  signal coefficient vectors to be linearly independent. If instead  $N < K$ , then all of the signal coefficient vectors should be linearly independent.

Strong linear independence will be assumed in part (a) of the following result, although not in part (b) or in any of our subsequent results.

17. This result builds on Chaloner (1984), who shows that a “ $c$ -optimal simultaneous experiment design” exists on at most  $K$  points. Part (a) also supplies a characterization of the optimal design itself and demonstrates uniqueness, with a minor technical difference that we impose an integer constraint on signal counts. We are not aware of prior work on the discounted payoff criterion studied in part (b).



THEOREM 1.

- (a) Assume strong linear independence. Then for every complementary set  $S$  with  $|S| < K$ , there exists an open set of prior beliefs given which agents exclusively observe signals from  $S$ .
- (b) If there are no complementary sets with fewer than  $K$  sources, then starting from any prior belief,  $\lim_{t \rightarrow \infty} \frac{m_i(t)}{t} = \lambda_i^*$  holds for every signal  $i$ .

Part (a) of the theorem generalizes our example in [Section II.A](#). It says that every small complementary set (fewer than  $K$  signals) is exclusively observed in the long run from some set of priors.<sup>18</sup> When that complementary set is not the best one, it is a “learning trap.”<sup>19</sup>

In contrast, if no complementary sets are smaller than size  $K$ ,<sup>20</sup> then a very different long-run outcome obtains: starting from any prior, society’s information acquisitions eventually approximate the optimal frequency. Thus, even though agents are short-lived (“myopic”), they end up acquiring information efficiently. We mention that the conclusion of part (b) can be strengthened to  $m_i(t) - \lambda_i^* \cdot t$  being bounded as  $t \rightarrow \infty$  (see [Online Appendix B](#)). Thus, in particular, signals outside of the efficient set  $S^*$  are observed only finitely often. This result provides a stronger sense in which society’s information acquisitions will be asymptotically efficient under the stated assumptions in part (b) of Theorem 1.

We now provide a brief intuition for Theorem 1, and in particular for the importance of the number  $K$ . Since each agent chooses the signal whose marginal value (reduction of posterior variance of  $\omega$ ) is highest, any set  $S$  on which signal acquisitions concentrate must satisfy two properties. First, all signals in the set must repeatedly have their turn as “most valuable,” so that agents do not focus on a strict subset of  $S$ . This condition requires that the

18. In the example in [Section II.A](#), the set  $\{X_3\}$  is a complementary set of size 1, while  $K = 2$ .

19. Note that the dynamics here are deterministic. Instead of bad signal realizations causing a failure of learning, efficient learning either fails or succeeds depending on the prior and signal structure. This is a key difference between our result and the more classic learning frictions in [Banerjee \(1992\)](#), [Bikhchandani, Hirshleifer, and Welch \(1992\)](#), and [Smith and Sørensen \(2000\)](#). We thank an anonymous referee for pointing this out.

20. It follows from Proposition 1 that there are no complementary sets with more than  $K$  sources, so this is equivalent to assuming that all complementary sets are of size  $K$ .

long-run outcome is a complementary set, where (by definition) all sources are critical to the value of the set.

Second, the marginal values of signals in that set must be persistently higher than marginal values of other signals. Not all complementary sets satisfy this criterion, because the sources within a given complementary set  $S$  could have even stronger complementarities with sources outside of the set. If that were the case, observation of sources within  $S$  would eventually push agents to acquire information outside of  $S$ .

No complementary set  $S$  consisting of  $K$  sources can satisfy this second property unless it is the best set: as observations accumulate from such a set, agents learn about all of the confounding variables and come to evaluate all sources according to “objective” (i.e., prior-independent) asymptotic values. Repeated acquisitions of signals from  $S$  improve the value of signals in  $S^*$  over the value of signals in  $S$ . Thus, agents eventually turn to the sources in  $S^*$ , achieving efficient information aggregation as predicted by part (b) of Theorem 1.

In contrast, if agents observe only  $k < K$  sources, then they can have persistent uncertainty about some confounding variables. This may cause society to persistently undervalue those sources confounded by these variables and continually observe signals from a small complementary set. We saw this already in the example in Section II.A, where agents failed to obtain any information about the confounding variable  $b_1$  and thus persistently undervalued the sources  $X_1$  and  $X_2$ . The same intuition applies to part (a) of Theorem 1.

One may argue that the condition that no complementary set has fewer than  $K$  sources is generically satisfied. However, if we expect that sources are endogenous to design or strategic motivations, the relevant informational environments may not fall under this condition. For example, signals that partition into different groups with group-specific confounding variables (as studied in Sethi and Yildiz 2019) are economically interesting but non-generic. Part (a) of Theorem 1 shows that inefficiency is a possible outcome in these cases.

Finally, we use a few examples to illustrate some implications of Theorem 1. First, it is possible that agents end up concentrating on an inefficient set of higher cardinality than the optimal set.

**EXAMPLE 2.** Suppose the available signals are  $X_1 = \omega + b_1 + \varepsilon_1$ ,  $X_2 = b_1 + b_2 + \varepsilon_2$ ,  $X_3 = b_2 + \varepsilon_3$ ,  $X_4 = \omega + b_3 + \varepsilon_4$ , and

$X_5 = b_3 + \varepsilon_5$ ; strong linear independence is satisfied. There are two complementary sets,  $\{X_1, X_2, X_3\}$  and  $\{X_4, X_5\}$ , and the set  $\{X_4, X_5\}$  is efficient. But part (a) of Theorem 1 tells us that both complementary sets are potential long-run outcomes (since  $K = 4$ ). Thus from some set of priors, agents will end up exclusively observing the inefficient set  $\{X_1, X_2, X_3\}$ , which is of larger size than the optimal set.

In Section II.A, we already saw that agents may concentrate on a set of lower cardinality than the efficient complementary set. Thus, we cannot in general compare sizes of learning traps versus efficient sets.

It is also straightforward to see that adding sources can worsen overall learning.

**EXAMPLE 3.** Suppose the available signals are  $X_1 = 3\omega + b_1 + \varepsilon_1$  and  $X_2 = b_1 + \varepsilon_2$ . Then,  $\{X_1, X_2\}$  is the only complementary set. It follows from part (b) of Theorem 1 that agents will achieve the efficient benchmark with a value of  $\text{val}(\{X_1, X_2\}) = \frac{9}{4}$ . Now suppose we add  $X_3 = \omega + \varepsilon_3$ , returning the example in Section II.A. The efficient benchmark does not change, but now there are priors that lead to exclusive observation of  $X_3$ , achieving  $\text{val}(\{X_3\}) = 1$ .

Relatedly, worsening the information content of a signal can improve the speed of long-run learning.<sup>21</sup>

**EXAMPLE 4.** Consider the environment of Section II.A with signals  $X_1 = 3\omega + b_1 + \varepsilon_1$  and  $X_2 = b_1 + \varepsilon_2$ , and  $X_3 = \omega + \varepsilon_3$ . Now suppose we degrade  $X_3$  by replacing it with  $X'_3 = \omega + b_1 + \varepsilon_3$ . In contrast to Section II.A, once signal  $X_3$  has been degraded in this way, all priors lead to long-run information acquisition in the efficient frequency, which concentrates on  $X_1$  and  $X_2$ .

We note that, in general, adjustments to the signal structure can result in changes to the efficient speed of learning, so the welfare comparison is not straightforward.

## VI.B. General Characterization of Long-Run Outcomes

We now generalize Theorem 1, providing a complete characterization of the possible long-run observations (as the prior belief

21. We thank an anonymous referee for this example.

varies) for an arbitrary signal structure. We need a new definition, which strengthens the notion of a complementary set.

**DEFINITION 3.**  $S$  is a *strongly complementary* set if it is complementary, and  $\text{val}(S) > \text{val}(S')$  for all sets  $S'$  such that  $|S - S'| = |S' - S| = 1$ .<sup>22</sup>

This property can be understood as requiring that the set is complementary and also something more: These complementarities are “locally best” in the sense that it is not possible to obtain stronger complementarities by swapping out just one source. We point out that while the definition of complementary sets does not depend on the ambient set (i.e.,  $[N]$ ) of available sources, the notion of strong complementarity does.

**EXAMPLE 5.** Suppose the available signals are  $X_1 = \omega + b_1 + \varepsilon_1$ ,  $X_2 = b_1 + \varepsilon_2$ , and  $X_3 = 2b_1 + \varepsilon_3$ . Then the set  $\{X_1, X_2\}$  is complementary but not strongly complementary, as  $\text{val}(\{X_1, X_3\}) > \text{val}(\{X_1, X_2\})$ .

Theorem 2 says that long-run information acquisitions concentrate on a set  $S$  (starting from some prior belief) if and only if  $S$  is strongly complementary. This generalizes Theorem 1 to signal structures that need not satisfy strong linear independence.

**THEOREM 2.** The set  $S$  is strongly complementary if and only if there exists an open set of prior beliefs given which agents eventually exclusively observe signals from  $S$  (that is, long-run frequencies exist and have support in  $S$ ).<sup>23</sup>

When there is a unique strongly complementary set, then all priors must eventually lead to this set. Part (b) of Theorem 1 provides a sufficient condition that implies uniqueness, and moreover gives that the single strongly complementary set is the best complementary set. When there are multiple strongly complementary sets, then different priors lead to different long-run outcomes,

22. In fact, the requirement that  $S$  is complementary is extraneous. One can show using Proposition 1 that if  $\text{val}(S) > \text{val}(S')$  for all sets  $S'$  with  $|S - S'| = |S' - S| = 1$ , then  $S$  must be complementary.

23. The “if” part of this statement can be strengthened as follows. The set  $S$  is strongly complementary if there exists any prior belief given which agents eventually choose from  $S$  (see Appendix A.6). Thus, the regions of prior beliefs that would lead to different strongly complementary sets cover the whole space.

some of which are inefficient. Part (a) of Theorem 1 describes a sufficient condition for such multiplicity.

Theorem 2 implies that learning must always end in a complementary set, so that (myopic) learners will eventually recover the payoff-relevant state, albeit potentially slowly. This is so even in settings such as the following: Agents have access to  $X_1 = \omega + b_1 + \varepsilon_1$  and  $X_2 = \frac{\omega}{100} + \varepsilon_2$ , where the prior belief over  $\omega$  and  $b_1$  is standard normal.<sup>24</sup> Here, the initial agents will acquire  $X_1$ , viewing  $X_2$  as less informative about  $\omega$ . But what Theorem 2 tells us is that agents must eventually switch over to acquiring (the efficient signal)  $X_2$ —this is because once agents have learned the sum  $\omega + b_1$  very well, the marginal value to learning more about this biased sum will be smaller than the marginal value to learning directly about  $\omega$ . In Section VIII.B, we revisit this observation that agents eventually learn the state starting from all prior beliefs and show that this can fail when we allow for arbitrarily small amounts of evolution in the state.

Obtaining a complete characterization of the sets of prior beliefs associated with different long-run outcomes is challenging, since society’s signal path may in general exhibit complex dynamics—for example, switching multiple times between different complementary sets (including the efficient set). This makes it difficult to relate the initial prior to long-run learning behavior. In the next section we discuss the technical details that go into the proof of Theorem 2. In particular, we explain how we are able to determine the range of possible long-run outcomes despite incomplete knowledge about which occurs under a given prior.<sup>25</sup>

## VII. PROOF OUTLINE FOR THEOREM 2

### VII.A. Limiting Dynamical System

We first introduce the following normalized asymptotic posterior variance function  $V^*$ , which takes frequency vectors  $\lambda \in \Delta^{N-1}$  as input, where the  $i$ th coordinate of  $\lambda$  is the proportion of total

24. We thank an anonymous referee for this example.

25. For some signal structures, such as the example in Section II.A, a partial characterization is feasible. We showed previously that if  $\omega$  and  $b_1$  are independent under the prior, and the prior variance of  $b_1$  exceeds 8, then every agent observes  $X_3$ , leading to a learning trap. If instead the prior variance of  $b_1$  is smaller than 8, then we can show that every agent chooses from the efficient set  $\{X_1, X_2\}$ . See Online Appendix E for the analysis as well as a related example.

acquisitions devoted to source  $i$ .<sup>26</sup>

$$V^*(\lambda) = \lim_{t \rightarrow \infty} t \cdot V(\lambda t).$$

The RHS is well defined because we can extend the domain of the posterior variance function  $V$  so that it takes positive real numbers (and not just natural numbers) as arguments (see [Appendix A.1](#)). The asymptotic variance function  $V^*(\lambda)$  is convex in  $\lambda$  and its unique minimizer is the optimal frequency vector  $\lambda^*$  (see [Lemma 5](#) in [Appendix A.2](#)).

For simplicity of explanation, we assume throughout this section that at large  $t$ , the signal choice that minimizes  $V$  also minimizes  $V^*$ .<sup>27</sup> Then, the frequency vector  $\lambda(t) := \frac{m(t)}{t}$  evolves in the coordinate direction that minimizes  $V^*$ . We refer to this as *coordinate descent*. Unlike the usual gradient descent, coordinate descent is restricted to move in coordinate directions. This restriction reflects our assumption that each agent can only acquire a discrete signal (rather than a mixture of signals).

One case where the rest point of coordinate descent coincides with that of gradient descent is when  $V^*$  is everywhere differentiable, because differentiability ensures that directional derivatives can be written as convex combinations of partial derivatives along coordinate directions. In this case, evolution of  $\lambda(t)$  necessarily ends at the global minimizer  $\lambda^*$ , implying efficient information aggregation.

### VII.B. Differentiability of $V^*$

The function  $V^*$ , however, is not guaranteed to be differentiable everywhere. Consider our example from [Section II.A](#) with signals  $X_1 = 3\omega + b_1 + \varepsilon_1$ ,  $X_2 = b_1 + \varepsilon_2$ ,  $X_3 = \omega + \varepsilon_3$ , and fix the frequency vector to be  $\lambda = (0, 0, 1)$ . It is easy to verify that the asymptotic posterior variance  $V^*(\lambda)$  is increased if we perturb  $\lambda$  by reassigning weight from  $X_3$  to  $X_1$  or from  $X_3$  to  $X_2$ . But  $V^*$  is reduced if we reassign weight from  $X_3$  to both  $X_1$  and  $X_2$  in an even

26. Note that the coordinates of  $\lambda$  must sum to 1.

27. This is not in fact generally correct, and the potential gap is one of the technical challenges in the proof. Nevertheless, we do show that at large  $t$ , the signal choice that minimizes  $V$  approximately minimizes  $V^*$ .

manner.<sup>28</sup> So  $V^*$  is not differentiable at  $\lambda$ . Such points of nondifferentiability are exactly why learning traps are possible: coordinate descent can become stuck at these vectors  $\lambda$ , so agents repeatedly sustain an inefficient frequency of information acquisitions.<sup>29</sup>

A sufficient condition for  $V^*$  to be differentiable at a frequency vector turns out to be that the signals receiving positive frequencies at that vector span all of  $\mathbb{R}^K$ . This explains the result in part (b) of Theorem 1: when each complementary set consists of  $K$  signals, society has to observe  $K$  signals to learn the payoff-relevant state  $\omega$ . Thus, driven by learning about  $\omega$ , agents end up observing signals that span  $\mathbb{R}^K$ , which leads to efficient information aggregation.

### VII.C. Generalization to Arbitrary Subspaces

Now observe that our arguments above were not special to considering the whole space  $\mathbb{R}^K$ . If we restrict the available sources to some subset of  $[N]$ , and look at the subspace of  $\mathbb{R}^K$  spanned by these sources, our previous analysis applies to this restricted space.

Specifically, given any prior belief, define  $\mathcal{S}$  to be the set of sources that agents eventually observe. Let  $\bar{\mathcal{S}}$  be the available signals that can be reproduced as linear combinations of signals from  $\mathcal{S}$ —these sources belong to the “subspace spanned by  $\mathcal{S}$ .” We can consider the restriction of the function  $V^*$  to all frequency vectors with support in  $\bar{\mathcal{S}}$ . Parallel to the foregoing discussion, the restricted version of  $V^*$  is both convex and differentiable in this subspace (at frequency vectors that assign positive weights to

28. This follows from the formula  $V^*(\lambda_1, \lambda_2, \lambda_3) = \lambda_3 + \frac{9}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}}$ . The derivative of  $V^*$  in either direction  $(1, 0, -1)$  or  $(0, 1, -1)$  is positive, while its derivative in the direction  $(\frac{1}{2}, \frac{1}{2}, -1)$  is in fact negative.

29. The above intuition connects to a literature on learning convergence in potential games (Monderer and Shapley 1996; Sandholm 2010). Define an  $N$ -player game where each player  $i$  chooses a number  $\lambda_i \in \mathbb{R}_+$  and receives payoff  $-\left(\sum_{j=1}^N \lambda_j\right) \cdot V^*(\lambda) = -V^*\left(\frac{\lambda}{\sum_{j=1}^N \lambda_j}\right)$ . Then, we have a potential game with (exact) potential function  $-V^*$ , and our long-run observation sets correspond to equilibria of this game. This is an infinite potential game with a nondifferentiable potential function. It is known that Nash equilibria in such games need not occur at extreme points, and this is consistent with our observation of learning traps. Nonetheless, we note that the connection to potential games is not sufficient to derive our main results, because our agents receive payoff  $-V(\lambda t)$  rather than its asymptotic variant  $V^*$ .

signals in  $S$ ). Thus, coordinate descent must lead to the minimizer of  $V^*$  in this subspace.

Just as the overall optimal frequency vector  $\lambda^*$  is supported on the best complementary set  $S^*$ , the frequency vector that minimizes  $V^*$  in the restricted subspace is also supported on the best complementary set within  $\bar{S}$ . So agents can eventually concentrate signal acquisitions on the set  $S$  only if  $S$  is *best in its subspace*; that is,  $\text{val}(S) = \text{val}(\bar{S})$ .

#### VII.D. An Equivalence Result

Next we demonstrate that a set is “best in its subspace” if and only if it is strongly complementary.

LEMMA 1. The following conditions are equivalent for a complementary set  $S$ :

- (a)  $\text{val}(S) = \text{val}(\bar{S})$ .
- (b)  $S$  is strongly complementary.
- (c) For any  $i \in S$  and  $j \notin S$ ,  $\partial_i V^*(\lambda^S) < \partial_j V^*(\lambda^S)$ , where  $\lambda^S$  (proportional to  $|\beta^S|$ ) is the optimal frequency vector supported on  $S$ .

This lemma states that a strongly complementary set  $S$  is “locally best” in three different senses. Condition (a) says such a set has the highest informational value in its subspace. Condition (b) says its informational value is higher than any set obtained by swapping out one source. Condition (c) says that starting from the optimal sampling rule over  $S$ , reallocating frequencies from signals in  $S$  to any other signal increases asymptotic posterior variance and reduces speed of learning. The rest of this subsection is devoted to the proof of Lemma 1.

The implication from (a) to (b) is straightforward. Suppose  $S$  is best in its subspace, and  $S'$  is obtained from  $S$  by removing signal  $i$  and adding signal  $j$ . Then the informational value of  $S'$  is either 0 or equal to the highest value among its complementary subsets. In the latter case, such a complementary subset necessarily includes signal  $j$ , and Proposition 1 implies that  $j$  belongs to the subspace spanned by  $S$ . Thus  $S' \subset \bar{S}$ , and  $\text{val}(S') \leq \text{val}(\bar{S}) = \text{val}(S)$ . The inequality is in fact strict, because complementary sets have different values by Assumption 2.

We show that (b) implies (c). Suppose condition (c) fails, so some perturbation that shifts weight from source  $i \in S$  to source



$j \notin S$  decreases  $V^*$ . Then, by definition of informational value, we would have  $\text{val}(S \cup \{j\}) > \text{val}(S)$ . But as Proposition 2 part (a) suggests, the value of  $S \cup \{j\}$  is equal to the highest value among its complementary subsets. Strong complementarity of  $S$  ensures that  $S$  is the best complementary subset of  $S \cup \{j\}$ . Thus we obtain  $\text{val}(S \cup \{j\}) = \text{val}(S)$ , leading to a contradiction.

Finally, condition (c) implies that  $\lambda^S$  is a local minimizer of  $V^*$  in the subspace spanned by  $S$  (where the restriction of  $V^*$  is differentiable at  $\lambda^S$ ). Because  $V^*$  is convex, the frequency vector  $\lambda^S$  must in fact be a global minimizer of  $V^*$  in this subspace. Hence  $S$  is best in its subspace and (a) holds.

### VII.E. Completing the Argument

These arguments tell us that information acquisitions eventually concentrate on a strongly complementary set, delivering one direction of Theorem 2:  $S$  is a long-run outcome only if  $S$  is strongly complementary.

To prove the “only if” direction, we directly construct priors such that a given strongly complementary set  $S$  is the long-run outcome. The construction generalizes the idea in the example in Section II.A, where we assign high uncertainty to those confounding variables that do not afflict signals in  $S$  and low uncertainty to those that do. This asymmetry guarantees that signals in  $\bar{S}$  have persistently higher marginal values than the remaining signals. Last, we use part (c) of the above Lemma 1 to show that agents focus on observing signals from  $S$ , rather than the potentially larger set  $\bar{S}$ . Indeed, if the historical frequency of acquisitions is close to  $\lambda^S$ , then signals in  $S$  have higher marginal values than the remaining signals in their subspace; and as these signals in  $S$  continue to be chosen, society’s frequency vector remains close to  $\lambda^S$ . This completes the proof of Theorem 2.

## VIII. WELFARE LOSS UNDER LEARNING TRAPS

The previous sections demonstrate that long-run learning is sometimes inefficient; how large can this inefficiency be? In this section, we study the welfare loss under learning traps and, in the process, develop a generalization of our model in which the unknown states evolve over time.

VIII.A. *Welfare Criteria*

Two classic welfare criteria are (1) the speed of information aggregation and (2) the discounted average payoff achieved by agents within the community.

According to the first criterion, the welfare loss under learning traps can be arbitrarily large. Specifically, as the following example shows, the informational value of the best complementary set can be arbitrarily large compared to the set that agents eventually observe (and thus, the achieved speed of learning can be arbitrarily slow compared to what is feasible, see footnote 16).

EXAMPLE 6. There are three available sources:  $X_1 = \omega + b_1 + \varepsilon_1$ ,  $X_2 = b_1 + \varepsilon_2$ , and  $X_3 = \frac{1}{L}\omega + \varepsilon_3$ , where  $L > 0$  is a constant.

In this example, the ratio  $\frac{\text{val}(\{X_1, X_2\})}{\text{val}(\{X_3\})} = \frac{L^2}{4}$  increases without bound as  $L \rightarrow \infty$ . But for every choice of  $L$ , there is a set of priors given which  $X_3$  is exclusively observed.<sup>30</sup>

For the second criterion, define

$$U_\delta^M = \mathbb{E}_M \left[ - \sum_{t=1}^{\infty} (1-\delta)\delta^{t-1} \cdot (a_t - \omega)^2 \right]$$

to be the  $\delta$ -discounted average payoff across agents who follow a “myopic” signal acquisition strategy with optimal predictions  $a_t$ . Also define  $U_\delta^{SP}$  to be the maximum  $\delta$ -discounted average payoff, where the social planner can use any signal acquisition strategy. Note that both payoff sums are negative, since flow payoffs are quadratic loss at every period.

Again from Example 6, we see that for every constant  $c > 0$ , there is a signal structure and prior belief such that the limiting payoff ratio satisfies<sup>31</sup>

$$\liminf_{\delta \rightarrow 1} \frac{U_\delta^M}{U_\delta^{SP}} > c.$$

30. The region of inefficient priors (that result in suboptimal learning) does decrease in size as the level of inefficiency increases. Specifically, as  $L$  increases, the prior variance of  $b_1$  has to increase correspondingly for the first agent to choose  $X_3$ .

31. Example 6 implies the ratio of flow payoffs at late periods can be arbitrarily large. As  $\delta \rightarrow 1$ , these later payoffs (see footnote 16) dominate the total payoffs from the initial periods (since the harmonic series diverges). So the ratio of aggregate discounted payoffs is also large.

Thus, the payoff ratio can be arbitrarily large. Note that because payoffs are negative, larger values of the ratio  $\frac{U_\delta^M}{U_\delta^{SP}}$  correspond to greater payoff inefficiencies.

On the other hand, the payoff difference vanishes in the patient limit, that is,

$$\lim_{\delta \rightarrow 1} (U_\delta^{SP} - U_\delta^M) = 0$$

in all environments. To see this, note that agents eventually learn  $\omega$  even while in a learning trap, albeit slowly. Thus flow payoffs converge to 0 at large periods, implying  $\lim_{\delta \rightarrow 1} U_\delta^{SP} = \lim_{\delta \rightarrow 1} U_\delta^M = 0$ .

In what follows, we show that this conclusion critically depends on the assumption that unknown states are perfectly persistent. We outline a sequence of autocorrelated models that converge to our main model (with perfect state persistence). At near perfect persistence, welfare losses under learning traps can be large according to all of the above measures.

### VIII.B. Extension: Autocorrelated Model

In our main model, the state vector  $\theta = (\omega, b_1, \dots, b_{K-1})'$  is persistent across time. Consider now a state vector  $\theta^t$  that evolves according to the following law:

$$\begin{aligned} \theta^1 &\sim \mathcal{N}(0, \Sigma^0); \quad \theta^{t+1} = \sqrt{\alpha} \cdot \theta^t + \sqrt{1-\alpha} \cdot \eta^t, \\ &\text{where } \eta^t \sim \mathcal{N}(0, M). \end{aligned}$$

Above, means are normalized to 0, and the prior covariance matrix of the state vector at time  $t = 1$  is  $\Sigma^0$ . We restrict the autocorrelation coefficient  $\sqrt{\alpha}$  to belong to  $(0,1)$ . Choice of  $\alpha = 1$  returns our main model, and we will be interested in approximations where  $\alpha$  is close to but strictly less than 1. The innovation  $\eta^t \sim \mathcal{N}(0, M)$  captures the additional noise terms that emerge under state evolution, which we assume to be i.i.d. across time.<sup>32</sup> Fixing signal

32. The coefficient  $\sqrt{1-\alpha}$  in front of  $\eta^t$  is chosen so that when no signals are observed, society's posterior covariance matrix about  $\theta^t$  will converge to  $M$ . This allows us to meaningfully consider the limit as  $\alpha \rightarrow 1$  while keeping  $M$  fixed.

coefficients  $\{c_i\}$ , every autocorrelated model is indexed by the triple  $(M, \Sigma^0, \alpha)$ .

In each period, the available signals are

$$X_i^t = \langle c_i, \theta^t \rangle + \varepsilon_i^t, \quad \varepsilon_i^t \sim \mathcal{N}(0, 1).$$

The signal noises  $\varepsilon_i^t$  are i.i.d. and further independent from the innovations in state evolution. The agent in period  $t$  chooses the signal that minimizes the posterior variance of  $\omega^t$ , and the social planner seeks to minimize a discounted sum of such posterior variances.

We have the following result.

**THEOREM 3.** Suppose  $\mathcal{S}$  is strongly complementary. Then there exist  $M$  and  $\Sigma^0$  such that for every  $\varepsilon > 0$ , there is an  $\underline{\alpha}(\varepsilon) < 1$  with the following property: in each autocorrelated model  $(M, \Sigma^0, \alpha)$  with  $\alpha > \underline{\alpha}(\varepsilon)$ ,

- (a) all agents only observe signals in  $\mathcal{S}$ ;
- (b) the resulting discounted average payoff satisfies

$$\limsup_{\delta \rightarrow 1} U_\delta^M \leq -(1 - \varepsilon) \cdot \sqrt{(1 - \alpha) \left( \frac{M_{11}}{\text{val}(\mathcal{S})} \right)},$$

while it is feasible to achieve a patient payoff of

$$\liminf_{\delta \rightarrow 1} U_\delta^{SP} \geq -(1 + \varepsilon) \cdot \sqrt{(1 - \alpha) \left( \frac{M_{11}}{\text{val}(\mathcal{S}^*)} \right)}$$

by sampling from  $\mathcal{S}^*$ .

Part (a) generalizes Theorem 2, showing that every strongly complementary set is a potential long-run observation set given imperfect persistence. This suggests that the notion of strong complementarity and its importance extend beyond our main model with unchanging states.

Part (b) shows that whenever  $\mathcal{S}$  is different from the best complementary set  $\mathcal{S}^*$ , then social acquisitions result in significant payoff inefficiency as measured by the payoff ratio. Indeed, for  $\alpha$  close to 1 the ratio  $\lim_{\delta \rightarrow 1} \frac{U_\delta^M}{U_\delta^{SP}}$  is at least  $\sqrt{\frac{\text{val}(\mathcal{S}^*)}{\text{val}(\mathcal{S})}}$ , which can be arbitrarily large depending on the signal structure.

The following proposition strengthens this statement, using Example 6 to show that the payoff difference between optimal and social acquisitions can also be arbitrarily large.

PROPOSITION 3. For every  $\varepsilon > 0$ , there exists a signal structure as in Example 6 and a corresponding autocorrelated model  $(M, \Sigma^0, \alpha)$  such that

$$\liminf_{\delta \rightarrow 1} U_{\delta}^{SP} \geq -\varepsilon;$$

$$\limsup_{\delta \rightarrow 1} U_{\delta}^M \leq -\frac{1}{\varepsilon}.$$

From this analysis, we take away that learning traps can result in (potentially large) average payoff losses as long as unknown states are not perfectly persistent over time.

## IX. INTERVENTIONS

We have shown that learning traps are possible and can lead to large welfare loss. This naturally suggests a question of what kinds of policies could preclude learning traps, or break agents free from an inefficient path of learning. We compare several possible policy interventions in this section: increasing the quality of information acquisition (so that each signal realization is more informative); restructuring incentives so that agents' payoffs are based on information obtained over several periods (equivalent to acquisition of multiple signals each period); and providing a one-shot release of free information, which can guide subsequent acquisitions.

### IX.A. More Precise Information

Consider first an intervention in which the precision of each signal draw is uniformly increased. We model this intervention by supposing that each signal acquisition now produces  $B$  independent observations from that source (where the main model is nested as  $B = 1$ ). The result below shows that providing more informative signals is of limited effectiveness: all potential learning traps for  $B = 1$  remain potential learning traps under arbitrary improvements to signal precision.

COROLLARY 1. Suppose that for  $B = 1$ , there is a set of priors given which signals in  $S$  are exclusively viewed in the long run.

Then, for every  $B \in \mathbb{Z}_+$ , there is a set of priors given which these signals are exclusively viewed in the long run.<sup>33</sup>

This corollary follows directly from Theorem 2.<sup>34</sup>

### IX.B. Batches of Signals

Another possibility is to restructure the incentive scheme so that agents' payoffs are based on information acquired from multiple signals. In practice, this might mean that payoffs are determined after a given time interval: For example, researchers may be evaluated based on a set of papers, so that they maximize the impact of the entire set. Alternatively, agents might be given the means to acquire multiple signals each period. For example, researchers may be arranged in labs, with a principal investigator directing the work of multiple people at once.

Formally, suppose that each agent can allocate  $B$  observations across the sources (where  $B = 1$  returns the main model). Note the key difference from the previous intervention: It is now possible for the  $B$  observations to be allocated across different signals. This distinction enables agents to take advantage of the presence of complementarities, and we show that efficient information aggregation can be guaranteed in this case:

**PROPOSITION 4.** For sufficiently large  $B$ , if each agent acquires  $B$  signals every period, then society's long-run frequency vector is  $\lambda^*$  starting from every prior belief.

33. However, the set of prior beliefs that yield  $S$  as a long-run outcome need not be the same as  $B$  varies. For a fixed prior belief, subsidizing higher quality acquisitions may or may not move society out of a learning trap. To see this, consider the signal structure and prior belief from the example in [Section II.A](#). Increasing the precision of signals is ineffective there: as long as the prior variance on  $b$  is larger than 8, each agent still chooses signal  $X_3$  regardless of signal precision. In [Online Appendix F](#), we provide a contrasting example in which increasing the precision of signals indeed breaks agents out of a learning trap from a specified prior.

34. To see this, observe that  $B$  independent observations reduce the noise variance of each signal to  $\frac{1}{B}$ . Thus the model with signal coefficient vectors  $\{c_i\}$  and  $B > 1$  observations is equivalent to our main model ( $B = 1$ ) with scaled coefficient vectors  $\{\sqrt{B} \cdot c_i\}$ . Because scaling does not change the family of strongly complementary sets, this model produces the same set of learning traps as in our main model.

Thus, given sufficiently many observations each period, agents will allocate observations in a way that approximates the optimal frequency.

The number of observations needed for long-run efficiency, however, depends on details of the informational environment. In particular, the required  $B$  cannot be bounded as a function of the number of states  $K$  and number of signals  $N$ .<sup>35</sup> See [Appendix A.7.1](#) for further details.

### IX.C. Free Information

Finally, we consider provision of free information to the agents. We can interpret this either as release of information that a policy maker knows, or as a reduced form for funding specific kinds of research, the results of which are then made public.

Formally, the policy maker chooses several signals  $Y_j = \langle p_j, \theta \rangle + \mathcal{N}(0, 1)$ , where each  $\|p_j\|_2 \leq \gamma$  so that signal precisions are bounded by  $\gamma^2$ . At time  $t = 0$ , independent realizations of these signals are made public. All subsequent agents update their prior beliefs based on this free information in addition to the history of signal acquisitions thus far.

We show that given a sufficient number of (different kinds of) signals, efficient learning can be guaranteed. Specifically, if  $k \leq K$  is the size of the best set  $S^*$ , then  $k-1$  precise signals are sufficient to guarantee efficient learning.

**PROPOSITION 5.** Let  $k := |S^*|$ . There exists a  $\gamma < \infty$ , and  $k-1$  signals  $Y_j = \langle p_j, \theta \rangle + \mathcal{N}(0, 1)$  with  $\|p_j\|_2 \leq \gamma$ , such that with these free signals provided at  $t = 0$ , society's long-run frequency vector is  $\lambda^*$  starting from every prior belief.

The proof is by construction. We show that as long as agents understand those confounding variables that appear in the best set of signals (these variables have dimension  $k-1$ ), they will come to acquire information from this set.<sup>36</sup>

35. The required  $B$  depends on two properties. First, it depends on how well the optimal frequency  $\lambda^*$  can be approximated by  $B$  (discrete) observations. Second, it depends on the difference in learning speed between the best set and the next best complementary set, which determines the slack that is permitted in the approximation of  $\lambda^*$ .

36. This intervention requires knowledge of the full correlation structure as well as which set  $S^*$  is best. An alternative intervention, with higher demands on

We point out the following converse to Proposition 5: whenever agents begin with sufficiently low prior uncertainty about the confounding variables that afflict the signals in  $S^*$ , it is impossible for a malevolent third party to provide free information and induce a learning trap as the long-run outcome.

## X. CONCLUSION

We conclude with brief mention of additional directions and interpretations of the model. First, although we have focused on a sequence of decision-makers with a common prior, we might alternatively consider multiple communities of decision makers, each seeded with a different prior belief. For example, in the absence of a global research community, researchers in different countries may share different prior beliefs and pass down their information within their country. Under this setup, our results can be interpreted as answering the question: will individuals from different communities end up observing the same (best) set of sources, or will they persistently acquire information from different sources? Our main results show that when there is a unique strongly complementary set of sources, then different priors wash out. Otherwise, different priors can result in persistent differences across communities in what sources are listened to and consequently differences in beliefs.<sup>37</sup>

Second, although we have focused on research as the leading interpretation of the framework, the model is relevant to other settings of knowledge acquisition where (i) information is passed down across time/generations, and (ii) information acquisition is myopic at each period. For example, we may consider a sequence of managers within a company (or politicians within a state) who seek only to maximize profits during their tenure, but acquire information that has externalities for future managers. Alternatively, we may consider knowledge acquisition by a single decision maker over time, for example, an aspiring computer programmer's choices of what classes to take or blogs to read.

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information provision but lower demands on knowledge of the environment, is to provide  $K-1$  (sufficiently precise) signals about all of the confounding variables.

37. In our main model with persistent states, beliefs about  $\omega$  end up converging across the population to the truth. However, beliefs about other confounding variables need not converge. And when states are not fully persistent (as in Section VIII.B), even beliefs about  $\omega$  can diverge across communities.



Here, too, investment in certain skills (e.g., abstract math classes) may not be immediately useful but may allow the individual to learn faster in future courses (e.g., an algorithms course). Our article characterizes the cases in which a student who “only learns things that are useful right now” will nevertheless end up developing his abilities as fast as a student who recognized in advance the complementarities across courses.

Finally, although we consider choice between information sources, a more general model may consider choice between complementary actions. The concepts of efficient information aggregation and learning traps have natural generalizations (i.e., actions that maximize society’s long-term welfare, versus those that do not). Relative to the general setting, we study a class of complementarities that are micro-founded in correlated signals. It is an interesting question of whether and how the forces we find here generalize to other kinds of complementarities.

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#### SUPPLEMENTARY MATERIAL

An [Online Appendix](#) for this article can be found at *The Quarterly Journal of Economics* online.

#### APPENDIX A: PROOFS FOR THE MAIN MODEL

The structure of the appendix follows that of the article. We provide proofs for the results in our main model, where states are perfectly persistent. [Appendix B](#) provides proofs for the autocorrelated model as discussed in [Section VIII.B](#). The only exception is that the proof of part (b) of [Proposition 2](#) is more technical, so it is given in a separate [Online Appendix](#), which also contains additional results and examples.

##### A.1. Preliminaries

*1. Posterior Variance Function.* Throughout, let  $C$  denote the  $N \times K$  matrix of signal coefficients, whose  $i$ th row is the vector  $c_i'$  associated with signal  $i$ . Here we review and extend a basic result from [Liang, Mu, and Syrgkanis \(2017\)](#). Specifically, we show that the posterior variance of  $\omega$  weakly decreases over time and the marginal value of any signal decreases in its signal count.

LEMMA 2. Given prior covariance matrix  $\Sigma^0$  and  $q_i \in \mathbb{Z}_+$  observations of each signal  $i$ , society's posterior variance of  $\omega$  is

$$(2) \quad V(q_1, \dots, q_N) = [((\Sigma^0)^{-1} + C'QC)^{-1}]_{11},$$

where  $Q = \text{diag}(q_1, \dots, q_N)$ . This function  $V$  admits an extension to the larger domain of nonnegative real numbers  $q_i$  (beyond integers), and the extended function is decreasing and convex in each  $q_i$ .

*Proof.* Note that  $(\Sigma^0)^{-1}$  is the prior precision matrix and  $C'QC = \sum_{i=1}^N q_i \cdot [c_i c_i']$  is the total precision from the observed signals. Thus equation (2) simply represents the fact that for Gaussian prior and signals, the posterior precision matrix is the sum of the prior and signal precision matrices. The RHS of equation (2) can be evaluated for any  $q_i \in \mathbb{R}_+$ , providing an extension of the function  $V$  to noninteger arguments.

To prove the monotonicity of  $V$ , consider the partial order  $\succeq$  on positive semidefinite matrices where  $A \succeq B$  if and only if  $A - B$  is positive semidefinite. As  $q_i$  increases, the matrix  $Q$  and  $C'QC$  increase in this order. Thus the posterior covariance matrix  $((\Sigma^0)^{-1} + C'QC)^{-1}$  decreases in this order, which implies that the posterior variance of  $\omega$  decreases.

To prove that  $V$  is convex, it suffices to prove that  $V$  is midpoint-convex because the function is clearly continuous.<sup>38</sup> Take  $q_1, \dots, q_N, r_1, \dots, r_N \in \mathbb{R}_+$  and let  $s_i = \frac{q_i + r_i}{2}$ . Define the corresponding diagonal matrices to be  $Q, R, S$ . Note that  $Q + R = 2S$ . Thus by the AM-HM inequality for positive-definite matrices, we have

$$((\Sigma^0)^{-1} + C'QC)^{-1} + ((\Sigma^0)^{-1} + C'RC)^{-1} \succeq 2((\Sigma^0)^{-1} + C'SC)^{-1}.$$

Using equation (2), we conclude that  $V(q_1, \dots, q_N) + V(r_1, \dots, r_N) \geq 2V(s_1, \dots, s_N)$ . This proves the (midpoint) convexity of  $V$ .  $\square$

*2. Inverse of Positive Semidefinite Matrices.* For future use, we provide a definition of  $[X^{-1}]_{11}$  for positive semidefinite matrices  $X$ . When  $X$  is positive definite, its eigenvalues are strictly positive, and its inverse matrix is defined as usual. In general, we can apply the spectral theorem to write  $X = UDU'$ , where  $U$  is a

38. A function  $V$  is midpoint-convex if the inequality  $V(a) + V(b) \geq 2V(\frac{a+b}{2})$  always holds. Every continuous function that is midpoint-convex is also convex.

$K \times K$  orthogonal matrix whose columns are eigenvectors of  $X$ , and  $D = \text{diag}(d_1, \dots, d_K)$  is a diagonal matrix consisting of nonnegative eigenvalues. When these eigenvalues are strictly positive, we have

$$X^{-1} = (UDU')^{-1} = UD^{-1}U' = \sum_{j=1}^K \frac{1}{d_j} \cdot [u_j u_j'],$$

where  $u_j$  is the  $j$ th column vector of  $U$ . In this case

$$(3) \quad [X^{-1}]_{11} = e_1' \cdot X^{-1} \cdot e_1 = \sum_{j=1}^K \frac{(\langle u_j, e_1 \rangle)^2}{d_j}$$

is well defined. Even if some  $d_j$  are 0, we can still use the RHS to define  $[X^{-1}]_{11}$ , applying the convention that  $\frac{0}{0} = 0$  and  $\frac{z}{0} = \infty$  for any  $z > 0$ . Note that by this definition,

$$[X^{-1}]_{11} = \lim_{\varepsilon \rightarrow 0_+} \left( \sum_{j=1}^K \frac{(\langle u_j, e_1 \rangle)^2}{d_j + \varepsilon} \right) = \lim_{\varepsilon \rightarrow 0_+} [(X + \varepsilon I_K)^{-1}]_{11},$$

since the matrix  $X + \varepsilon I_K$  has the same set of eigenvectors as  $X$  (with eigenvalues increased by  $\varepsilon$ ). Hence our definition of  $[X^{-1}]_{11}$  is a continuous extension of the usual definition to positive semidefinite matrices.

**3. Asymptotic Posterior Variance.** We can approximate the posterior variance as a function of the frequencies with which each signal is observed. Specifically, as mentioned in [Section VII](#), we can define

$$V^*(\lambda) := \lim_{t \rightarrow \infty} t \cdot V(\lambda t)$$

for any  $\lambda \in \mathbb{R}_+^N$ . The following result shows  $V^*$  to be well defined and computes its value.

LEMMA 3. Let  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ . Then<sup>39</sup>

$$(4) \quad V^*(\lambda) = [(C' \Lambda C)^{-1}]_{11}.$$

The value of  $[(C' \Lambda C)^{-1}]_{11}$  is well defined, see [equation \(3\)](#).

*Proof.* Recall that  $V(q_1, \dots, q_N) = [((\Sigma^0)^{-1} + C'QC)^{-1}]_{11}$  with  $Q = \text{diag}(q_1, \dots, q_N)$ . Thus

$$t \cdot V(\lambda_1 t, \dots, \lambda_N t) = \left[ \left( \frac{1}{t} (\Sigma^0)^{-1} + C' \Lambda C \right)^{-1} \right]_{11}.$$

Hence the lemma follows from the continuity of  $[X^{-1}]_{11}$  in the matrix  $X$ .  $\square$

## A.2. Key Object $\phi$

We now define an object that will play a central role in the proofs. For each set of signals  $S$ , consider writing the first coordinate vector  $e_1 \in \mathbb{R}^K$  (corresponding to the payoff-relevant state  $\omega$ ) as a linear combination of signals in  $S$ :

$$e_1 = \sum_{i \in S} \beta_i^S \cdot c_i.$$

DEFINITION 4.  $\phi(S) := \min_{\beta} \sum_{i \in S} |\beta_i^S|$ .

That is,  $\phi(S)$  measures the size of the “smallest” (in the  $l_1$  norm) linear combination of the signals in  $S$  to produce an unbiased estimate of the payoff-relevant state. In case  $\omega$  is not spanned by  $S$ , this definition sets  $\phi(S) = \infty$ .

When  $S$  *minimally spans*  $\omega$  (so that no subset spans), the coefficients  $\beta_i^S$  are unique and nonzero. In this case  $\phi(S)$  is easy to compute. In general, we have the following characterization:

LEMMA 4. For any set  $S$  that spans  $\omega$ ,  $\phi(S) = \min_{T \subset S} \phi(T)$  where the minimum is over subsets  $T$  that “minimally span”  $\omega$ .

This lemma is a standard result in linear programming, so we omit the proof.

39. Note that  $C' \Lambda C$  is the Fisher information matrix when signals are observed according to frequencies  $\lambda$ . So this lemma can also be seen as an application of the Bayesian central limit theorem.

As a corollary,  $\phi([N]) = \phi(S^*)$  where  $S^*$  is the set of signals that minimize  $\phi$  among all sets that minimally span  $\omega$ . The following proposition, which generalizes Claim 1, makes clear that this set  $S^*$  also has the greatest informational value.

PROPOSITION 6. For any set of signals  $\mathcal{S}$ ,  $\text{val}(\mathcal{S}) = \frac{1}{\phi(\mathcal{S})^2}$ .

In what follows (before Proposition 6 is proved), we abuse definition and let  $S^*$  denote the minimal spanning set of signals that minimizes  $\phi$ . Accordingly,  $\lambda^*$  denotes the frequency vector supported on  $S^*$  that is proportional to  $|\beta^{S^*}|$ . Once we prove Proposition 6 and Proposition 1, it will follow that  $S^*$  is exactly the best complementary set defined in the main text, and there will be no confusion.

The proof of Proposition 6 uses the following three lemmata:

LEMMA 5.  $\lambda^*$  is the unique minimizer of  $V^*(\lambda)$  as  $\lambda$  varies in  $\Delta^{N-1}$ .

LEMMA 6.  $V^*(\lambda^*) = \phi(S^*)^2$ .

LEMMA 7. Suppose Lemma 5 and Lemma 6 hold. Then  $\text{val}([N]) = \frac{1}{\phi(S^*)^2}$ .

To see why these lemmata imply Proposition 6, recall that Lemma 4 gives  $\phi(S^*) = \phi([N])$ . So Lemma 7 implies

$$\text{val}([N]) = \frac{1}{\phi([N])^2}.$$

More generally, if we take any set of signals  $\mathcal{S}$  that span  $\omega$  as the set of all available signals “[ $N$ ],” then the same analysis yields

$$\text{val}(\mathcal{S}) = \frac{1}{\phi(\mathcal{S})^2}.$$

This proves Proposition 6 whenever  $\mathcal{S}$  spans  $\omega$ . But in case  $\mathcal{S}$  does not span  $\omega$ , the posterior variance of  $\omega$  is bounded away from 0 when agents are constrained to observe from  $\mathcal{S}$ . Thus the posterior precision  $\tau(q^t)$  is bounded above and  $\text{val}(\mathcal{S}) = \limsup_{t \rightarrow \infty} \frac{\tau(q^t) - \tau_0}{t} = 0$ , which is also equal to  $\frac{1}{\phi(\mathcal{S})^2}$  because in this case  $\phi(\mathcal{S}) = \infty$  by definition.

*1. Proof of Proposition 1.* We can use Proposition 6 to show Proposition 1. Take any set of signals  $\mathcal{S}$ . If these signals do not

span  $\omega$ , then Proposition 6 implies  $\text{val}(S) = 0$  and  $S$  is not complementary by Definition 2. If a proper subset of  $S$  spans  $\omega$ , then Proposition 6 together with Lemma 4 implies that the informational value of  $S$  is equal to the highest value among its subsets that minimally span  $\omega$ . Let  $S'$  denote this subset that achieves this highest value. For this  $S'$  the inequality in Definition 2 is violated, and  $S$  is again not complementary.

Finally, suppose  $S$  itself minimally spans  $\omega$ . In this case any nonempty proper subset of  $S$  does not span  $\omega$  and has zero informational value, whereas  $S$  has positive value. So Definition 2 is satisfied and such sets  $S$  are complementary, as described in Proposition 1.

2. *Proof of Lemma 6.* It remains to establish Lemmata 5–7. Here we prove Lemma 6; that is,  $V^*(\lambda^*) = \phi(S^*)^2$ . This proof will illustrate why the  $\phi$  function (i.e., the  $\ell_1$  norm of  $\beta$ ) plays an important role.

Without loss of generality we assume  $S^* = \{1, \dots, k\}$ , which minimally span  $\omega$  (so  $k \leq K$ ). For  $1 \leq i \leq k$ , define a “transformed state”  $\tilde{\theta}_i = \langle c_i, \theta \rangle$ . Then the signals in  $S^*$  are individual  $\tilde{\theta}_i$  plus standard normal noise. The payoff-relevant state  $\omega$  can be written as

$$\omega = \sum_{1 \leq i \leq k} \beta_i \cdot \tilde{\theta}_i.$$

Because we are currently interested in the value of  $V^*(\lambda^*)$  and  $\lambda^*$  is supported on  $S^*$ , only the  $k$  signals in  $S^*$  matter. Thus we can work with this transformed model and focus on the beliefs about the transformed states. Note that the prior covariance matrix  $\Sigma^0$  of the original state vector  $\theta \in \mathbb{R}^K$  induces a prior covariance matrix  $\tilde{\Sigma}^0$  of the transformed state vector  $\tilde{\theta} \in \mathbb{R}^k$ .  $\tilde{\Sigma}^0$  has full rank because  $\Sigma^0$  does and the signal coefficient vectors in  $S^*$  are linearly independent.

Working in this transformed model, we have that the posterior covariance matrix of  $\tilde{\theta}$  is given by

$$\left[ (\tilde{\Sigma}^0)^{-1} + \text{diag}(q_1, \dots, q_k) \right]^{-1},$$

where  $q_i$  is the number of observations of signal  $i$ . Thus the posterior variance of  $\omega$  is

$$V(q_1, \dots, q_k, 0, \dots, 0) = \beta' \cdot \left[ (\tilde{\Sigma}^0)^{-1} + \text{diag}(q_1, \dots, q_k) \right]^{-1} \cdot \beta.$$

It follows that for any frequency vector  $\lambda$  supported on  $\mathcal{S}^*$ ,

$$\begin{aligned} V^*(\lambda) &= \lim_{t \rightarrow \infty} t \cdot V(\lambda t) = \lim_{t \rightarrow \infty} t \cdot \beta' \cdot \left[ (\tilde{\Sigma}^0)^{-1} + \text{diag}(\lambda_1 t, \dots, \lambda_k t) \right]^{-1} \cdot \beta \\ &= \lim_{t \rightarrow \infty} \beta' \cdot \left[ \frac{(\tilde{\Sigma}^0)^{-1}}{t} + \text{diag}(\lambda_1, \dots, \lambda_k) \right]^{-1} \cdot \beta \\ &= \beta' \cdot \left[ \text{diag}(\lambda_1, \dots, \lambda_k) \right]^{-1} \cdot \beta \\ &= \sum_{i=1}^k \frac{\beta_i^2}{\lambda_i}. \end{aligned}$$

By the Cauchy-Schwartz inequality, whenever  $\lambda_1, \dots, \lambda_k$  are nonnegative and sum to 1, it holds that

$$\sum_{i=1}^k \frac{\beta_i^2}{\lambda_i} \geq \left( \sum_{i=1}^k |\beta_i| \right)^2 = \phi(\mathcal{S}^*)^2.$$

Moreover, equality holds if and only if each  $\lambda_i$  is proportional to  $|\beta_i|$ ; that is, when  $\lambda = \lambda^*$ . We thus deduce that

$$V^*(\lambda^*) = \phi(\mathcal{S}^*)^2,$$

and that  $\lambda^*$  uniquely minimizes the value of  $V^*$  when the frequency vector is restricted to be supported on  $\mathcal{S}^*$ . Later we prove Lemma 5, which shows that  $\lambda^*$  remains the unique minimizer without the restriction.

*3. Proof of Lemma 7 and Proposition 2 Part (a).* We now prove Lemma 7. Intuitively, maximizing average increase in precision is equivalent to minimizing asymptotic posterior variance, leading to the relation  $\text{val}[N] = \frac{1}{\min_{\lambda} V^*(\lambda)}$ , which is in turn equal to  $\frac{1}{\phi(\mathcal{S}^*)^2}$  by Lemma 5 and 6.

Toward Lemma 7, we first show  $\text{val}([N]) \geq \frac{1}{\phi(\mathcal{S}^*)^2}$ . By Definition 1,  $\text{val}([N])$  is the maximal average increase in the precision about  $\omega$  given all of the available signals. Choose a sequence of

count vectors  $q^t$  such that  $\lim_{t \rightarrow \infty} \frac{q^t}{t} = \lambda^*$ , then by definition of the function  $V^*$  and by Lemma 6,

$$\lim_{t \rightarrow \infty} t \cdot V(q^t) = V^*(\lambda^*) = \phi(\mathcal{S}^*)^2.$$

Thus  $\tau(q^t) = \frac{1}{V(q^t)} = \frac{(1+o(1))t}{\phi(\mathcal{S}^*)^2}$ . It follows that

$$\text{val}([N]) \geq \limsup_{t \rightarrow \infty} \frac{\tau(q^t) - \tau_0}{t} = \frac{1}{\phi(\mathcal{S}^*)^2}.$$

In the opposite direction, take any sequence  $q^t$  with  $\limsup_t \frac{\tau(q^t) - \tau_0}{t} = \text{val}([N])$ . Because  $\tau_0$  is a constant, we equivalently have  $\limsup_t \frac{\tau(q^t)}{t} = \text{val}([N])$ , which gives

$$\liminf_{t \rightarrow \infty} t \cdot V(q^t) = \frac{1}{\text{val}([N])}.$$

Passing to a subsequence if necessary, we may assume the frequency vector  $\lambda := \lim_{t \rightarrow \infty} \frac{q^t}{t}$  exists. Then by definition of  $V^*$ , the LHS of the above display is simply  $V^*(\lambda)$ . We therefore deduce  $\text{val}([N]) = \frac{1}{V^*(\lambda)}$  for some  $\lambda \in \Delta^{N-1}$ . Since  $\lambda^*$  minimizes  $V^*$ , we conclude that

$$\text{val}([N]) = \frac{1}{V^*(\lambda)} \leq \frac{1}{V^*(\lambda^*)} = \frac{1}{\phi(\mathcal{S}^*)^2}.$$

This proves Lemma 7.

The second half of the analysis proves part (a) of Proposition 2. Indeed, the inequality in  $\text{val}([N]) = \frac{1}{V^*(\lambda)} \leq \frac{1}{V^*(\lambda^*)}$  holds equal only if  $\lambda = \lambda^*$ , since  $\lambda^*$  is the unique minimizer of  $V^*$  by Lemma 5.

### A.3. Proof of Lemma 5

*1. Case 1:  $|\mathcal{S}^*| = K$ .* In this subsection, we prove that  $\lambda^*$  is the unique minimizer of  $V^*$  whenever the set  $\mathcal{S}^*$  contains exactly  $K$  signals. Later we prove the same result even when  $|\mathcal{S}^*| < K$ , but that proof requires additional techniques.

First, we assume  $\mathcal{S}^* = \{1, \dots, K\}$  and let  $C^*$  be the  $K \times K$  submatrix of  $C$  corresponding to the first  $K$  signals. Replacing  $c_i$  with  $-c_i$  if necessary, we can assume  $[(C^*)^{-1}]_{ii}$  is positive for  $1 \leq i \leq K$ . The following technical lemma is key to the argument.



LEMMA 8. Suppose  $S^* = \{1, \dots, K\}$  uniquely minimizes  $\phi$ . Define  $C^*$  as above and further suppose  $[(C^*)^{-1}]_{1i}$  is positive for  $1 \leq i \leq K$ . Then for any signal  $j > K$ , if we write  $c_j = \sum_{i=1}^K \alpha_i \cdot c_i$  (which is a unique representation), then  $|\sum_{i=1}^K \alpha_i| < 1$ .

*Proof of Lemma 8.* By assumption, we have the vector identity

$$e_1 = \sum_{i=1}^K \beta_i \cdot c_i \quad \text{with} \quad \beta_i = [(C^*)^{-1}]_{1i} > 0.$$

Suppose for contradiction that  $\sum_{i=1}^K \alpha_i \geq 1$  (the opposite case where the sum is  $\leq -1$  can be similarly treated). Then some  $\alpha_i$  must be positive. Without loss of generality, we assume  $\frac{\alpha_1}{\beta_1}$  is the largest among such ratios. Then  $\alpha_1 > 0$  and

$$e_1 = \sum_{i=1}^K \beta_i \cdot c_i = \left( \sum_{i=2}^K \left( \beta_i - \frac{\beta_1}{\alpha_1} \cdot \alpha_i \right) \cdot c_i \right) + \frac{\beta_1}{\alpha_1} \cdot \left( \sum_{i=1}^K \alpha_i \cdot c_i \right).$$

This represents  $e_1$  as a linear combination of the vectors  $c_2, \dots, c_K$  and  $c_j$ , with coefficients  $\beta_2 - \frac{\beta_1}{\alpha_1} \cdot \alpha_2, \dots, \beta_K - \frac{\beta_1}{\alpha_1} \cdot \alpha_K$  and  $\frac{\beta_1}{\alpha_1}$ . Note that these coefficients are nonnegative: for each  $2 \leq i \leq K$ ,  $\beta_i - \frac{\beta_1}{\alpha_1} \cdot \alpha_i$  is clearly positive if  $\alpha_i \leq 0$  (since  $\beta_i > 0$ ). If  $\alpha_i > 0$ ,  $\beta_i - \frac{\beta_1}{\alpha_1} \cdot \alpha_i$  is again nonnegative by the assumption that  $\frac{\alpha_i}{\beta_i} \leq \frac{\alpha_1}{\beta_1}$ .

By definition,  $\phi(\{2, \dots, K, j\})$  is the sum of the absolute value of these coefficients. This sum is

$$\sum_{i=2}^K \left( \beta_i - \frac{\beta_1}{\alpha_1} \cdot \alpha_i \right) + \frac{\beta_1}{\alpha_1} = \sum_{i=1}^K \beta_i + \frac{\beta_1}{\alpha_1} \cdot \left( 1 - \sum_{i=1}^K \alpha_i \right) \leq \sum_{i=1}^K \beta_i.$$

But then  $\phi(\{2, \dots, K, j\}) \leq \phi(\{1, 2, \dots, K\})$ , contradicting the unique minimality of  $\phi(S^*)$ . Hence the lemma must be true.  $\square$

*Proof of Lemma 5 using Lemma 8.* Because  $V(q_1, \dots, q_N)$  is convex in its arguments,  $V^*(\lambda) = \lim_{t \rightarrow \infty} t \cdot V(\lambda_1 t, \dots, \lambda_N t)$  is also convex in  $\lambda$ . To show  $\lambda^*$  uniquely minimizes  $V^*$ , we only need to show  $\lambda^*$  is a local minimum. In other words, it suffices to show  $V^*(\lambda^*) < V^*(\lambda)$  for any  $\lambda$  that belongs to an  $\varepsilon$ -neighborhood of  $\lambda^*$ . By definition,  $S^*$  minimally spans  $\omega$ , and so its signals are linearly independent. Under the additional assumption that  $S^*$  has size

$K$ , we deduce that its signals span the entire space  $\mathbb{R}^K$ . From this it follows that the  $K \times K$  matrix  $C' \Lambda^* C$  is positive definite, and by equation (4) the function  $V^*$  is differentiable near  $\lambda^*$ .

We claim that the partial derivatives of  $V^*$  satisfy the following inequality:

$$(5) \quad \partial_K V^*(\lambda^*) < \partial_j V^*(\lambda^*) \leq 0, \forall j > K.$$

Once this is proved, we will have, for  $\lambda$  close to  $\lambda^*$ ,

$$(6) \quad \begin{aligned} V^*(\lambda_1, \dots, \lambda_K, \lambda_{K+1}, \dots, \lambda_N) &\geq V^* \left( \lambda_1, \dots, \lambda_{K-1}, \sum_{k=K}^N \lambda_k, 0, \dots, 0 \right) \\ &\geq V^*(\lambda^*). \end{aligned}$$

The first inequality is based on inequality (5) and differentiability of  $V^*$ , whereas the second inequality is because  $\lambda^*$  uniquely minimizes  $V^*$  when restricting to the first  $K$  signals.<sup>40</sup> Moreover, when  $\lambda \neq \lambda^*$ , one of these inequalities is strict so that  $V^*(\lambda) > V^*(\lambda^*)$  holds strictly.

To prove inequality (5), we recall that

$$V^*(\lambda) = e'_1 (C' \Lambda C)^{-1} e_1.$$

Because  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ , its derivative is  $\partial_i \Lambda = \Delta_{ii}$ , which is an  $N \times N$  matrix whose  $(i, i)$ th entry is 1 with all other entries equal to 0. Using properties of matrix derivatives, we obtain

$$\partial_i V^*(\lambda) = -e'_1 (C' \Lambda C)^{-1} C' \Delta_{ii} C (C' \Lambda C)^{-1} e_1.$$

As the  $i$ th row vector of  $C$  is  $c'_i$ ,  $C' \Delta_{ii} C$  is the  $K \times K$  matrix  $c_i c'_i$ . This simplifies to

$$\partial_i V^*(\lambda) = -[e'_1 (C' \Lambda C)^{-1} c_i]^2.$$

At  $\lambda = \lambda^*$ , the matrix  $C' \Lambda C$  further simplifies to  $(C^*)' \cdot \text{diag}(\lambda_1^*, \dots, \lambda_K^*) \cdot (C^*)$ , which is a product of  $K \times K$  invertible matrices. We thus deduce that

$$\partial_i V^*(\lambda^*) = - \left[ e'_1 \cdot (C^*)^{-1} \cdot \text{diag} \left( \frac{1}{\lambda_1^*}, \dots, \frac{1}{\lambda_K^*} \right) \cdot ((C^*)')^{-1} \cdot c_i \right]^2.$$

40. See the proof of Lemma 6 before.

Crucially, note that the term in the brackets is a linear function of  $c_i$ . To ease notation, we write  $v' = e'_1 \cdot (C^*)^{-1} \cdot \text{diag}\left(\frac{1}{\lambda_1^*}, \dots, \frac{1}{\lambda_K^*}\right) \cdot ((C^*)')^{-1}$  and  $\gamma_i = \langle v, c_i \rangle$ . Then

$$(7) \quad \partial_i V^*(\lambda^*) = -\gamma_i^2, \quad 1 \leq i \leq N.$$

For  $1 \leq i \leq K$ ,  $((C^*)')^{-1} \cdot c_i$  is just  $e_i$ . Thus, using the assumption  $[(C^*)^{-1}]_{1i} > 0, \forall i$ , we have

$$(8) \quad \gamma_i = e'_1 \cdot (C^*)^{-1} \cdot \text{diag}\left(\frac{1}{\lambda_1^*}, \dots, \frac{1}{\lambda_K^*}\right) \cdot e_i = \frac{[(C^*)^{-1}]_{1i}}{\lambda_i^*} = \frac{\beta_i}{\lambda_i^*} = \beta_1 + \dots + \beta_K = \phi(S^*).$$

On the other hand, choosing any signal  $j > K$ , we can uniquely write the vector  $c_j$  as a linear combination of  $c_1, \dots, c_K$ . By Lemma 8,

$$(9) \quad \gamma_j = \langle v, c_j \rangle = \sum_{i=1}^K \alpha_i \cdot \langle v, c_i \rangle = \sum_{i=1}^K \alpha_i \cdot \gamma_i = \phi(S^*) \cdot \sum_{i=1}^K \alpha_i,$$

where the last equality uses equation (8). Since  $|\sum_{i=1}^K \alpha_i| < 1$ , the absolute value of  $\gamma_j$  is strictly smaller than the absolute value of  $\gamma_K$  for any  $j > K$ . This together with equation (7) proves the desired inequality (5), and Lemma 5 follows.  $\square$

**2. A Perturbation Argument.** We have shown that when  $\phi$  is uniquely minimized by a set  $S^*$  containing exactly  $K$  signals,

$$\min_{\lambda \in \Delta^{N-1}} V^*(\lambda) = V^*(\lambda^*) = \phi(S^*)^2 = \phi([N])^2.$$

We now use a perturbation argument to show this equality holds more generally.

**LEMMA 9.** For any coefficient matrix  $C$ ,

$$(10) \quad \min_{\lambda \in \Delta^{N-1}} V^*(\lambda) = \phi([N])^2.$$

*Proof.* In general, the set  $S^*$  that minimizes  $\phi$  may not be unique or involve exactly  $K$  signals. However, we always have

(by Lemma 4 and Lemma 6)

$$\min_{\lambda \in \Delta^{N-1}} V^*(\lambda) \leq V^*(\lambda^*) = \phi(S^*)^2 = \phi([N])^2.$$

It remains to prove  $V^*(\lambda) \geq \phi([N])^2$  for every  $\lambda \in \Delta^{N-1}$ . Below we fix  $\lambda$ . By Lemma 3, we need to show  $[(C' \wedge C)^{-1}]_{11} \geq \phi([N])^2$ .

Note that we already proved this inequality for generic coefficient matrices  $C$ : specifically, when  $\phi$  is uniquely minimized by a set of  $K$  signals, Lemma 5 holds and we have  $V^*(\lambda) \geq V^*(\lambda^*) = \phi(S^*)^2 = \phi([N])^2$ . But even if  $C$  is “nongeneric,” we can approximate it by a sequence of generic matrices  $C_m$ .<sup>41</sup> Along this sequence, we have

$$[(C'_m \wedge C_m)^{-1}]_{11} \geq \phi_m([N])^2,$$

where  $\phi_m$  is the analogue of  $\phi$  for the coefficient matrix  $C_m$ .

As  $m \rightarrow \infty$ , the LHS approaches  $[(C' \wedge C)^{-1}]_{11}$ . We show that on the RHS

$$\limsup_{m \rightarrow \infty} \phi_m([N]) \geq \phi([N]),$$

which then implies  $[(C' \wedge C)^{-1}]_{11} \geq \phi([N])^2$  and the lemma. Indeed, suppose  $e_1 = \sum_i \beta_i^{(m)} \cdot c_i^{(m)}$  along the convergent sequence, then  $e_1 = \sum_i \beta_i \cdot c_i$  for any limit point  $\beta$  of  $\beta^{(m)}$ . Using the definition of  $\phi$ , this enables us to conclude  $\liminf_{m \rightarrow \infty} \phi_m([N]) \geq \phi([N])$ , which is more than sufficient.  $\square$

3. *Case 2:  $|S^*| < K$ .* We now consider the case where  $S^* = \{1, \dots, k\}$  with  $k < K$ . We will show that  $\lambda^*$  is still the unique minimizer of  $V^*(\cdot)$ . Because  $V^*(\lambda^*) = \phi(S^*)^2 = \phi([N])^2$ , we know from Lemma 9 that  $\lambda^*$  does minimize  $V^*$ . It remains to show  $\lambda^*$  is the unique minimizer.

To do this, we consider a perturbed informational environment in which signals  $k + 1, \dots, N$  are made slightly more

41. First, we may add repetitive signals to ensure  $N \geq K$ . This does not affect the value of  $\min_{\lambda} V^*(\lambda)$  or  $\phi([N])$ . Whenever  $N \geq K$ , it is generically true that every set that minimally spans  $\omega$  contains exactly  $K$  signals. Moreover, the equality  $\phi(S) = \phi(\tilde{S})$  for  $S \neq \tilde{S}$  induces a nontrivial polynomial equation over the entries in  $C$ . This means we can always find  $C_m$  close to  $C$  such that for each coefficient matrix  $C_m$ , different subsets  $S$  of size  $K$  attain different values of  $\phi$ , so that  $\phi$  is uniquely minimized.

precise. Specifically, let  $\eta > 0$  be a small positive number. Consider an alternative signal coefficient matrix  $\tilde{C}$  with  $\tilde{c}_i = c_i$  for  $i \leq k$  and  $\tilde{c}_i = (1 + \eta)c_i$  for  $i > k$ . Let  $\tilde{\phi}(S)$  be the analogue of  $\phi$  for this alternative environment. It is clear that  $\tilde{\phi}(S^*) = \phi(S^*)$ , while  $\tilde{\phi}(S)$  is slightly smaller than  $\phi(S)$  for  $S \neq S^*$ . Thus with sufficiently small  $\eta$ , the set  $S^*$  remains the unique minimizer of  $\phi$  (among sets that minimally span  $\omega$ ) in this perturbed environment, and the definition of  $\lambda^*$  is also maintained.

Let  $\tilde{V}^*$  be the perturbed asymptotic posterior variance function; our previous analysis shows that  $\tilde{V}^*$  has minimum value  $\phi(S^*)^2$  on the simplex. Taking advantage of the connection between  $V^*$  and  $\tilde{V}^*$ , we thus have

$$\begin{aligned} V^*(\lambda_1, \dots, \lambda_N) &= \tilde{V}^*\left(\lambda_1, \dots, \lambda_k, \frac{\lambda_{k+1}}{(1+\eta)^2}, \dots, \frac{\lambda_N}{(1+\eta)^2}\right) \\ &\geq \frac{\phi(S^*)^2}{\sum_{i \leq k} \lambda_i + \frac{1}{(1+\eta)^2} \sum_{i > k} \lambda_i}. \end{aligned}$$

The equality uses [equation \(4\)](#) and  $C' \wedge C = \sum_i \lambda_i c_i c_i' = \sum_{i \leq k} \lambda_i c_i c_i' + \sum_{i > k} \frac{\lambda_i}{(1+\eta)^2} \tilde{c}_i \tilde{c}_i'$ . The inequality follows from the homogeneity of  $\tilde{V}^*$ .

The display implies that at any frequency vector  $\lambda$ ,

$$(11) \quad V^*(\lambda) \geq \frac{\phi(S^*)^2}{1 - \frac{2\eta + \eta^2}{(1+\eta)^2} \sum_{i > k} \lambda_i} \geq \frac{\phi(S^*)^2}{1 - \eta \sum_{i > k} \lambda_i} \quad \text{for some } \eta > 0.$$

Hence  $V^*(\lambda) > \phi(S^*)^2 = V^*(\lambda^*)$  whenever  $\lambda$  puts positive weight outside of  $S^*$ . But as shown already,  $V^*(\lambda) > V^*(\lambda^*)$  also holds when  $\lambda$  is supported on  $S^*$  and different from  $\lambda^*$ .

We conclude that  $\lambda^*$  is the unique minimizer of  $V^*$  over the whole simplex. This proves [Lemma 5](#), which also completes the proof of [Proposition 6](#) and [Proposition 1](#) as we showed before.

#### A.4. Proof of Theorem 1

We show here that [Theorem 1](#) follows from [Theorem 2](#), which we prove in the next appendix. Indeed, as we explained in [Lemma 1](#), the set  $S$  is strongly complementary if and only if it has the greatest informational value among signals in its subspace. Under the assumption of [Theorem 1](#) part (a), every

complementary set  $S$  of size less than  $K$  is strongly complementary because no other signal is in that subspace (otherwise there would be linear dependence). Thus by the “only if” part of Theorem 2, there exist prior beliefs that lead to exclusive observation of signals from  $S$ . As for Theorem 1 part (b), the assumption implies that the subspace spanned by every complementary set is the whole space, and thus the only strongly complementary set is  $S^*$ . Hence the “if” part of Theorem 2 implies that long-run efficiency is guaranteed.

#### A.5. Proof of Theorem 2: “Only If” Part

Let signals  $1, \dots, k$  (with  $k \leq K$ ) be a strongly complementary set; by Lemma 1 in the main text, these signals are best in their subspace. We demonstrate an open set of prior beliefs with the property that starting from any prior in this set, all agents only observe these  $k$  signals. Because these signals are complementary, Proposition 1 implies they must be linearly independent. Thus we can consider linearly transformed states  $\tilde{\theta}_1, \dots, \tilde{\theta}_K$  such that these  $k$  signals are simply  $\tilde{\theta}_1, \dots, \tilde{\theta}_k$  plus standard Gaussian noise. This linear transformation is invertible, so any prior over the original states is bijectively mapped to a prior over the transformed states. Thus it is without loss to work with the transformed model and look for prior beliefs over the transformed states.

The payoff-relevant state  $\omega$  becomes a linear combination  $\lambda_1^* \tilde{\theta}_1 + \dots + \lambda_k^* \tilde{\theta}_k$  (up to a scalar multiple). Since the first  $k$  signals are best in their subspace, Lemma 8 implies that any other signal belonging to this subspace can be written as  $\sum_{i=1}^k \alpha_i \tilde{\theta}_i + \mathcal{N}(0, 1)$  with  $|\sum_{i=1}^k \alpha_i| < 1$ . On the other hand, if a signal does not belong to this subspace, it must take the form of  $\sum_{i=1}^K \beta_i \tilde{\theta}_i + \mathcal{N}(0, 1)$  with  $\beta_{k+1}, \dots, \beta_K$  not all equal to 0.

Now consider any prior belief with precision matrix  $P$ ; the inverse of  $P$  is the prior covariance matrix (in terms of the transformed states). Suppose  $\varepsilon$  is a very small positive number, and  $P$  satisfies the following conditions:

- (i) for  $1 \leq i \leq k$ ,  $P_{ii} \geq \frac{1}{\varepsilon^2}$ ;
- (ii) for  $1 \leq i \neq j \leq k$ ,  $\frac{P_{ij}}{\lambda_i^*} \leq (1 + \varepsilon) \cdot \frac{P_{ji}}{\lambda_j^*}$ ;
- (iii) for  $k + 1 \leq i \leq K$ ,  $P_{ii} \in [\varepsilon, 2\varepsilon]$ ;
- (iv) for  $1 \leq i \neq j \leq K$ ,  $|P_{ij}| \leq \varepsilon^2$ .

Clearly any such  $P$  is positive definite, since on each row the diagonal entry has dominant size.<sup>42</sup> Moreover,  $P$  contains an open subset. Below we show that given any such prior, the myopic signal choice is among the first  $k$  signals, and that the posterior precision matrix also satisfies the same four conditions. As such, all agents will choose from the first  $k$  signals.

Let  $V = P^{-1}$  be the prior covariance matrix. Applying Cramer's rule for the matrix inverse, the above conditions on  $P$  imply the following conditions on  $V$ :

- (i) for  $1 \leq i \leq k$ ,  $V_{ii} \leq 2\varepsilon^2$ ;
- (ii) for  $1 \leq i \neq j \leq k$ ,  $V_{ii}\lambda_i^* \leq (1 + L\varepsilon) \cdot V_{jj}\lambda_j^*$ ;
- (iii) for  $k + 1 \leq i \leq K$ ,  $V_{ii} \in [\frac{1}{4\varepsilon}, \frac{2}{\varepsilon}]$ ;
- (iv) for  $1 \leq i \neq j \leq K$ ,  $|V_{ij}| \leq L\varepsilon \cdot V_{ii}$ .

Here  $L$  is a constant depending only on  $K$  (but not on  $\varepsilon$ ). For example, the last condition is equivalent to  $\det(P_{-ij}) \leq L\varepsilon \cdot \det(P_{-ii})$ . This is proved by expanding both determinants into multilinear sums and using the fact that on each row of  $P$  the off-diagonal entries are at most  $\varepsilon$ -fraction of the diagonal entry.

Given this matrix  $V$ , the variance reduction of  $\omega = \sum_{i=1}^k \lambda_i^* \tilde{\theta}_i$  by any signal  $\sum_{i=1}^k \alpha_i \tilde{\theta}_i + \mathcal{N}(0, 1)$  can be computed as

$$\frac{\left(\sum_{i,j=1}^k \alpha_i \lambda_j^* V_{ij}\right)^2}{1 + \sum_{i,j=1}^k \alpha_i \alpha_j V_{ij}},$$

where the denominator is the variance of the signal and the numerator is the covariance between the signal and  $\omega$ . By the first and last conditions on  $V$ , the denominator here is  $1 + O(\varepsilon^2)$ . By

42. Suppose  $P$  is a symmetric matrix s.t.  $P_{ii} > \sum_{j \neq i} |P_{ij}|$ , then for any vector  $x \in \mathbb{R}^K$ , it holds that

$$\begin{aligned} x'Px &= \sum_{i=1}^K P_{ii}x_i^2 + \sum_{1 \leq i < j \leq K} 2P_{ij}x_ix_j \geq \sum_{i=1}^K P_{ii}x_i^2 - \sum_{1 \leq i < j \leq K} |P_{ij}|(x_i^2 + x_j^2) \\ &= \sum_{i=1}^K \left(P_{ii} - \sum_{j \neq i} |P_{ij}|\right)x_i^2 \geq 0, \end{aligned}$$

with equality only if  $x$  is the zero vector. This shows  $P$  is positive definite.

the second and last condition, the numerator is

$$\left( \left( \sum_{i=1}^k \alpha_i + O(\varepsilon) \right) \cdot \lambda_1^* V_{11} \right)^2.$$

Because  $|\sum_{i=1}^k \alpha_i| < 1$ , we deduce that any other signal belonging to the subspace of the first  $k$  signals is myopically worse than signal 1, whose variance reduction is  $\frac{(\lambda_1^* V_{11})^2}{1 + V_{11}}$ .

Meanwhile, take any signal outside of the subspace. The variance reduction by such a signal  $\sum_{i=1}^K \beta_i \tilde{\theta}_i + \mathcal{N}(0, 1)$  is

$$\frac{\left( \sum_{i=1}^K \sum_{j=1}^k \beta_i \lambda_j^* V_{ij} \right)^2}{1 + \sum_{i,j=1}^K \beta_i \beta_j V_{ij}}.$$

By the second and last condition on  $V$ , the numerator here is  $O((\lambda_1^* V_{11})^2)$ . If we can show that the denominator is very large, then such a signal would also be myopically worse than signal 1. Indeed, since  $V_{ij} = O(\varepsilon^2)$  whenever  $i \leq k$  or  $j \leq k$ , it is sufficient to show  $\sum_{i,j > k} \beta_i \beta_j V_{ij}$  is large. This holds by the last two conditions on  $V$  and the assumption that  $\beta_{k+1}, \dots, \beta_K$  are not all 0.<sup>43</sup>

Hence, we have shown that given any prior precision matrix  $P$  satisfying the foregoing conditions, the myopic signal choice is among the first  $k$  signals. It remains to check whether the resulting posterior precision matrix  $\hat{P}$  also satisfies those four conditions. If the signal acquired is signal  $i$  ( $1 \leq i \leq k$ ), then  $\hat{P} = P + \Delta_{ii}$ . Therefore we only need to show the second condition holds for  $\hat{P}$ ; that is,  $\frac{P_{ii} + 1}{\lambda_i^*} \leq (1 + \varepsilon) \cdot \frac{P_{jj}}{\lambda_j^*}$  for each  $1 \leq j \leq k$ . To this end, we note that since signal  $i$  is myopically best given  $V$ , the following must hold:

$$\frac{(\lambda_i^* V_{ii})^2}{1 + V_{ii}} \geq \frac{(\lambda_j^* V_{jj})^2}{1 + V_{jj}}.$$

43. Formally, we can without loss assume  $\beta_K^2 V_{KK}$  is largest among  $\beta_i^2 V_{ii}$  for  $i > k$ . Then for any  $i \neq j$ , the last condition implies

$$\beta_i \beta_j V_{ij} \geq -L\varepsilon \cdot \beta_i \beta_j \sqrt{V_{ii} V_{jj}} \geq -L\varepsilon \cdot \beta_K^2 V_{KK}.$$

This trivially also holds for  $i = j = K$ . Summing across all pairs  $(i, j)$  yields  $\sum_{i,j > k} \beta_i \beta_j V_{ij} > (1 - K^2 L\varepsilon) \beta_K^2 V_{KK}$ , which must be large by the third condition on  $V$ .



As  $0 \leq V_{ii}, V_{jj} \leq 2\varepsilon^2$ , this implies  $\lambda_i^* V_{ii} \geq (1 - \varepsilon^2) \lambda_j^* V_{jj}$ . Now applying Cramer's rule to  $V = P^{-1}$  again, we can deduce  $V_{ii} = \frac{1+O(\varepsilon^2)}{P_{ii}}$ . So for  $\varepsilon$  small it holds that  $\frac{P_{ii}}{\lambda_i^*} \leq (1 + \frac{\varepsilon}{2}) \cdot \frac{P_{jj}}{\lambda_j^*}$ . As  $P_{jj} \geq \frac{1}{\varepsilon^2}$ , we also have  $\frac{1}{\lambda_i^*} \leq \frac{\varepsilon}{2} \cdot \frac{P_{jj}}{\lambda_j^*}$ . Adding up these two inequalities yields the second condition for  $\hat{P}$  and completes the proof.

#### A.6. Proof of Theorem 2: "If" Part

*1. Restated Version.* Given any prior belief, let  $\mathcal{A} \subset [N]$  be the set of signals that are observed by infinitely many agents. Our goal is to show that  $\mathcal{A}$  is strongly complementary. Toward that goal, we first show  $\mathcal{A}$  spans  $\omega$ .

Indeed, by definition we can find some period  $t$  after which agents exclusively observe signals from  $\mathcal{A}$ . Note that the variance reduction of any signal approaches 0 as its signal count gets large. Thus, along society's signal path, the variance reduction is close to 0 at sufficiently late periods. If  $\mathcal{A}$  does not span  $\omega$ , society's posterior variance remains bounded away from 0. Thus in the limit where each signal in  $\mathcal{A}$  has infinite signal counts, there still exists some signal  $j$  outside of  $\mathcal{A}$  whose variance reduction is strictly positive.<sup>44</sup> By continuity, we deduce that at any sufficiently late period, observing signal  $j$  is better than observing any signal in  $\mathcal{A}$ . This contradicts our assumption that later agents only observe signals in  $\mathcal{A}$ .

Now that  $\mathcal{A}$  spans  $\omega$ , we can take  $\mathcal{S}$  to be the best complementary set in  $\bar{\mathcal{A}}$ , which is the subspace spanned by  $\mathcal{A}$ . By Lemma 1,  $\mathcal{S}$  is strongly complementary. To prove Theorem 2 "if" part, we show that long-run frequencies are positive precisely for the signals in  $\mathcal{S}$ . By ignoring the initial periods, we can assume without loss that only signals in  $\bar{\mathcal{A}}$  are available. It thus suffices to show that whenever the signals observed infinitely often span a subspace, agents eventually focus on the best complementary set  $\mathcal{S}$  in that subspace. To ease notation, we assume this subspace is the entire  $\mathbb{R}^K$  and prove the following result.

44. To see this, let  $s_1, \dots, s_N$  denote the limit signal counts, where  $s_i = \infty$  if and only if  $i \in \mathcal{A}$ . We need to find some signal  $j$  such that  $V(s_j + 1, s_{-j}) < V(s_j, s_{-j})$ . If such a signal does not exist, then all partial derivatives of  $V$  at  $s$  are 0. Since  $V$  is always differentiable (unlike  $V^*$ ), this would imply that all directional derivatives of  $V$  are also 0. By the convexity of  $V$ ,  $V$  must be minimized at  $s$ . However, the minimum value of  $V$  is 0 because there exists a complementary set. This contradicts  $V(s) > 0$ .

**Theorem 2 “If” Part Restated.** Suppose that the signals observed infinitely often span  $\mathbb{R}^K$ . Then society’s long-run frequency vector is  $\lambda^*$ .

The next sections are devoted to the proof of this restatement.

2. *Estimates of Derivatives.* We introduce a few technical lemmata:

LEMMA 10. For any  $q_1, \dots, q_N$  and any  $1 \leq j \leq N$ , we have

$$\left| \frac{\partial_{jj} V(q_1, \dots, q_N)}{\partial_j V(q_1, \dots, q_N)} \right| \leq \frac{2}{q_j}.$$

*Proof.* Recall that  $V(q_1, \dots, q_N) = e'_1 \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot e_1$ . Thus

$$\partial_j V = -e'_1 \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j \cdot c'_j \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot e_1,$$

and

$$\begin{aligned} \partial_{jj} V &= 2e'_1 \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j \cdot c'_j \cdot [(\Sigma^0)^{-1} \\ &\quad + C'QC]^{-1} \cdot c_j \cdot c'_j \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot e_1. \end{aligned}$$

Let  $\gamma_j = e'_1 \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j$ , which is a number. Then the above becomes

$$\partial_j V = -\gamma_j^2; \quad \partial_{jj} V = 2\gamma_j^2 \cdot c'_j \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j.$$

Note that  $(\Sigma^0)^{-1} + C'QC \succeq q_j \cdot c_j c'_j$  in the matrix norm. Thus the number  $c'_j \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j$  is bounded above by  $\frac{1}{q_j}$ .<sup>45</sup> This proves the lemma.  $\square$

Since the second derivative is small compared to the first derivative, we deduce that the variance reduction of any discrete signal can be approximated by the partial derivative of  $V$ . This property is summarized in the following lemma.

45. Formally, we need to show that for any  $\varepsilon > 0$ , the number  $c'_j [c_j c'_j + \varepsilon I_K]^{-1} c_j$  is at most 1. Using the trace identity  $\text{tr}(AB) = \text{tr}(BA)$ , we can rewrite this number as

$$\text{tr}([c_j c'_j + \varepsilon I_K]^{-1} c_j c'_j) = \text{tr}(I_K - [c_j c'_j + \varepsilon I_K]^{-1} \varepsilon I_K) = K - \varepsilon \cdot \text{tr}([c_j c'_j + \varepsilon I_K]^{-1}).$$

The matrix  $c_j c'_j$  has rank 1, so  $K-1$  of its eigenvalues are 0. Thus the matrix  $[c_j c'_j + \varepsilon I_K]^{-1}$  has eigenvalue  $\frac{1}{\varepsilon}$  with multiplicity  $K-1$ , and the remaining eigenvalue is positive. This implies  $\varepsilon \cdot \text{tr}([c_j c'_j + \varepsilon I_K]^{-1}) > K-1$ , and then the above display yields  $c'_j \cdot [(\Sigma^0)^{-1} + C'QC]^{-1} \cdot c_j < 1$  as desired.

LEMMA 11. For any  $q_1, \dots, q_N$  and any  $1 \leq j \leq N$ , we have<sup>46</sup>

$$V(q) - V(q_j + 1, q_{-j}) \geq \frac{q_j}{q_j + 1} |\partial_j V(q)|.$$

*Proof.* We show the more general result:

$$V(q) - V(q_j + x, q_{-j}) \geq \frac{q_j x}{q_j + x} \cdot |\partial_j V(q)|, \forall x \geq 0.$$

This clearly holds at  $x = 0$ . Differentiating with respect to  $x$ , we only need to show

$$-\partial_j V(q_j + x, q_{-j}) \geq \frac{q_j^2}{(q_j + x)^2} |\partial_j V(q)|, \forall x \geq 0.$$

Equivalently, we need to show

$$-(q_j + x)^2 \cdot \partial_j V(q_j + x, q_{-j}) \geq -q_j^2 \cdot \partial_j V(q), \forall x \geq 0.$$

Again, this inequality holds at  $x = 0$ . Differentiating with respect to  $x$ , it becomes

$$-2(q_j + x) \cdot \partial_j V(q_j + x, q_{-j}) - (q_j + x)^2 \cdot \partial_{jj} V(q_j + x, q_{-j}) \geq 0.$$

This is exactly the result of Lemma 10.  $\square$

**3. Lower Bound on Variance Reduction.** Our next result lower bounds the directional derivative of  $V$  along the “optimal” direction  $\lambda^*$ .

LEMMA 12. For any  $q_1, \dots, q_N$ , we have  $|\partial_{\lambda^*} V(q)| \geq \frac{V(q)^2}{\phi(S^*)^2}$ .

*Proof.* To compute this directional derivative, we think of agents acquiring signals in fractional amounts, where a fraction of a signal is just the same signal with precision multiplied by that fraction. Consider an agent who draws  $\lambda_i^*$  realizations of each signal  $i$ . Then he essentially obtains the following signals:

$$Y_i = \langle c_i, \theta \rangle + \mathcal{N}\left(0, \frac{1}{\lambda_i^*}\right), \forall i.$$

46. Note that the convexity of  $V$  gives  $V(q) - V(q_j + 1, q_{-j}) \leq |\partial_j V(q)|$ . This lemma provides a converse that we need for the subsequent analysis.

This is equivalent to

$$\lambda_i^* Y_i = \langle \lambda_i^* c_i, \theta \rangle + \mathcal{N}(0, \lambda_i^*), \forall i.$$

Such an agent receives at least as much information as the sum of these signals:

$$\sum_i \lambda_i^* Y_i = \sum_i \langle \lambda_i^* c_i, \theta \rangle + \sum_i \mathcal{N}(0, \lambda_i^*) = \frac{\omega}{\phi(\mathcal{S}^*)} + \mathcal{N}(0, 1).$$

Hence the agent's posterior precision about  $\omega$  (which is the inverse of his posterior variance  $V$ ) must increase by at least  $\frac{1}{\phi(\mathcal{S}^*)^2}$  along the direction  $\lambda^*$ . The chain rule of differentiation yields the lemma.  $\square$

We can now bound the variance reduction at late periods.

**LEMMA 13.** Fix any  $q_1, \dots, q_N$ . Suppose  $L$  is a positive number such that  $(\Sigma^0)^{-1} + C'QC \succeq Lc_jc_j'$  holds in the matrix norm for each signal  $j \in \mathcal{S}^*$ . Then we have

$$\min_{j \in \mathcal{S}^*} V(q_j + 1, q_{-j}) \leq V(q) - \frac{L}{L+1} \cdot \frac{V(q)^2}{\phi(\mathcal{S}^*)^2}.$$

*Proof.* Fix any signal  $j \in \mathcal{S}^*$ . Using the condition  $(\Sigma^0)^{-1} + C'QC \succeq Lc_jc_j'$ , we can deduce the following variant of Lemma 11.<sup>47</sup>

$$V(q) - V(q_j + 1, q_{-j}) \geq \frac{L}{L+1} |\partial_j V(q)|.$$

Since  $V$  is always differentiable,  $\partial_{\lambda^*} V(q)$  is a convex combination of the partial derivatives of  $V$ .<sup>48</sup> Thus  $\max_{j \in \mathcal{S}^*} |\partial_j V(q)| \geq |\partial_{\lambda^*} V(q)|$ . These inequalities, and Lemma 12, complete the proof.  $\square$

47. Even though we are not guaranteed  $q_j \geq L$ , we can modify the prior and signal counts such that the precision matrix  $(\Sigma^0)^{-1} + C'QC$  is unchanged, and signal  $j$  has been observed at least  $L$  times. This is possible thanks to the condition  $(\Sigma^0)^{-1} + C'QC \succeq Lc_jc_j'$ . Then, applying Lemma 11 to this modified problem yields the result here.

48. Although this may be a surprising contrast with  $V^*$ , the difference arises because the formula for  $V$  always involves a full-rank prior covariance matrix, whereas its asymptotic variant  $V^*$  corresponds to a flat prior.

4. *Proof of the Restated Theorem 2 “If” Part.* We show  $t \cdot V(m(t)) \rightarrow \phi(S^*)^2$ , so that society eventually approximates the optimal speed of learning. Because  $\lambda^*$  is the unique minimizer of  $V^*$ , this will imply the desired conclusion  $\frac{m(t)}{t} \rightarrow \lambda^*$  via the second half of Proposition 2 part (a).

To estimate  $V(m(t))$ , we note that for any fixed  $L$ , society’s acquisitions  $m(t)$  eventually satisfy the condition  $(\Sigma^0)^{-1} + C'QC \geq Lc_jc'_j$ . This is due to our assumption that the signals observed infinitely often span  $\mathbb{R}^K$ , which implies that  $C'QC$  becomes arbitrarily large in the matrix norm. Hence, we can apply Lemma 13 to find that

$$V(m(t+1)) \leq V(m(t)) - \frac{L}{L+1} \cdot \frac{V(m(t))^2}{\phi(S^*)^2}$$

for all  $t \geq t_0$ , where  $t_0$  depends only on  $L$ .

We introduce the auxiliary function  $g(t) = \frac{V(m(t))}{\phi(S^*)^2}$ . Then the above simplifies to

$$g(t+1) \leq g(t) - \frac{L}{L+1}g(t)^2.$$

Inverting both sides, we have

$$\frac{1}{g(t+1)} \geq \frac{1}{g(t)(1 - \frac{L}{L+1}g(t))} = \frac{1}{g(t)} + \frac{\frac{L}{L+1}}{1 - \frac{L}{L+1}g(t)} \geq \frac{1}{g(t)} + \frac{L}{L+1}. \quad (12)$$

This holds for all  $t \geq t_0$ . Thus by induction,  $\frac{1}{g(t)} \geq \frac{L}{L+1}(t - t_0)$  and so  $g(t) \leq \frac{L+1}{L(t-t_0)}$ . Going back to the posterior variance function  $V$ , this implies

$$V(m(t)) \leq \frac{L+1}{L} \cdot \frac{\phi(S^*)^2}{t - t_0}. \quad (13)$$

Hence, by choosing  $L$  sufficiently large and then considering large  $t$ , we find that society’s speed of learning is arbitrarily close to the optimal speed  $\phi(S^*)^2$ . This completes the proof.

We comment that the above argument leaves open the possibility that some signals outside of  $S^*$  are observed infinitely often, yet with zero long-run frequency. In [Online Appendix B](#), we show this does not happen.

### A.7. Proofs for Interventions (Section IX)

1. *Proof of Proposition 4.* Given any history of observations, an agent can always allocate his  $B$  observations as follows: he draws  $\lfloor B \cdot \lambda_i^* \rfloor$  realizations of each signal  $i$ , and samples arbitrarily if there is any capacity remaining. Here  $\lfloor \cdot \rfloor$  denotes the floor function.

Fix any  $\varepsilon > 0$ . If  $B$  is sufficiently large, then the strategy acquires at least  $(1 - \varepsilon) \cdot B \cdot \lambda_i^*$  observations of each signal  $i$ . Adapting the proof of Lemma 12, we see that the agent's posterior precision about  $\omega$  must increase by  $\frac{(1-\varepsilon)B}{\phi(S^*)^2}$  under this strategy. Thus the same must hold for his optimal strategy, so that society's posterior precision at time  $t$  is at least  $\frac{(1-\varepsilon)Bt}{\phi(S^*)^2}$ . This implies that average precision per signal is at least  $\frac{1-\varepsilon}{\phi(S^*)^2}$ , which can be arbitrarily close to the optimal precision  $\text{val}([N]) = \frac{1}{\phi(S^*)^2}$  with appropriate choice of  $\varepsilon$ .

Because  $\lambda^*$  is the unique minimizer of  $V^*$ , society's long-run frequencies must be close to  $\lambda^*$ . In particular, with  $\varepsilon$  sufficiently small, we can ensure that each signal in  $S^*$  is observed with positive frequencies. The restated Theorem 2 "if" part extends to the current setting and implies that society's long-run frequency vector must be  $\lambda^*$ . This yields the proposition.<sup>49</sup>

2. *Proof of Proposition 5.* Suppose without loss that the best complementary set  $S^*$  is  $\{1, \dots, k\}$ . By taking a linear transformation, we further assume each of the first  $k$  signals only involves  $\omega$  and the first  $k-1$  confounding terms  $b_1, \dots, b_{k-1}$ . We show that whenever  $k-1$  sufficiently precise signals are provided about each of these confounding terms, the long-run frequency vector converges to  $\lambda^*$  regardless of the prior.

Fix any positive real number  $L$ . Since the  $k-1$  free signals are very precise, it is as if the prior precision matrix (after taking

49. This proof also suggests that how small  $\varepsilon$  (and how large  $B$ ) need to be depends on the distance between the optimal speed of learning and the "second-best" speed of learning from any other complementary set. Intuitively, to achieve long-run efficient learning, agents need to allocate  $B$  observations in the best set to approximate the optimal frequencies. If another set of signals offers a speed of learning that is only slightly worse, we need  $B$  sufficiently large for the approximately optimal frequencies in the best set to beat this other set.

into account these free signals) satisfies

$$(\Sigma^0)^{-1} \succeq L^2 \sum_{i=2}^k \Delta_{ii},$$

where  $\Delta_{ii}$  is the  $K \times K$  matrix that has 1 at the  $(i, i)$  entry and 0 otherwise. Recall also that society eventually learns  $\omega$ . Thus at some late period  $t_0$ , society's acquisitions must satisfy

$$C'QC \succeq L^2 \Delta_{11}.$$

Adding up the displays, we have

$$(\Sigma^0)^{-1} + C'QC \succeq L^2 \sum_{i=1}^k \Delta_{ii} \geq Lc_jc'_j, \forall 1 \leq j \leq k.$$

The last inequality uses the fact that each  $c_j$  only involves the first  $k$  coordinates.

This is exactly the condition we need to apply Lemma 13: crucially, whether the condition is met for signals  $j$  outside of  $S^*$  does not affect the argument there. Thus we can follow the proof of the restated Theorem 2 “if” part to deduce inequality (13). That is, for fixed  $L$  and corresponding free information, society's long-run precision per signal is at least  $\frac{L}{(L+1)\phi(S^*)^2}$ . This can be made arbitrarily close to the optimal average precision. Identically to the previous proof, we deduce that for large  $L$ , society's long-run frequency vector must be close to  $\lambda^*$ . The restated Theorem 2 “if” part allows us to conclude that the frequency is exactly  $\lambda^*$ .

## APPENDIX B: PROOFS FOR THE AUTOCORRELATED MODEL (SECTION VIII.B)

### B.1. Proof of Theorem 3

We work with the transformed model such that the signals in  $S$  become the first  $k$  transformed states  $\tilde{\theta}_1, \dots, \tilde{\theta}_k$ . The payoff-relevant state becomes a certain linear combination  $\beta_1\tilde{\theta}_1 + \dots + \beta_k\tilde{\theta}_k$  with positive weights  $\beta_1, \dots, \beta_k$ . Choose  $M$  so that the innovations corresponding to the transformed states are independent from each other. Specifically,  $\tilde{M}$  (the transformed version of  $M$ ) is given by  $\text{diag}(\frac{x}{\beta_1}, \dots, \frac{x}{\beta_k}, y_{k+1}, \dots, y_K)$ . Here  $x$  is a small positive number, whereas  $y_{k+1}, \dots, y_K$  are large positive

numbers. We further choose  $\Sigma^0 = M$ , which is the stable belief without learning.

With these choices, it is clear that if all agents only sample from  $\mathcal{S}$ , society's beliefs about the transformed states remain independent at every period. Let  $v_i^{t-1}$  denote the prior variance of  $\tilde{\theta}_i^t$  at the beginning of period  $t$  (before the signal acquisition in that period). Then as long as agent  $t$  continues to sample a signal  $\tilde{\theta}_j + \mathcal{N}(0, 1)$  in  $\mathcal{S}$ , these prior variances evolve as follows:  $v_i^0 = \frac{x}{\beta_i}$  for  $1 \leq i \leq k$  and  $v_i^0 = y_i$  for  $i > k$ . For  $t \geq 1$ ,

$$v_i^t = \begin{cases} \alpha \cdot v_i^{t-1} + (1 - \alpha)\tilde{M}_{ii}, & \text{if } i \neq j; \\ \alpha \cdot \frac{v_i^{t-1}}{1 + v_i^{t-1}} + (1 - \alpha)\tilde{M}_{ii} & \text{if } i = j. \end{cases}$$

The particular signal  $j$  maximizes the reduction in the posterior variance of  $\omega^t = \sum_{i=1}^k \beta_i \tilde{\theta}_i^t$ . That is,  $j \in \operatorname{argmax}_{1 \leq i \leq k} \frac{(\beta_i \cdot v_i^{t-1})^2}{1 + v_i^{t-1}}$ .

By induction, it is clear that  $v_i^t \leq \tilde{M}_{ii}$  holds for all pairs  $i, t$ , with equality for  $i > k$ . Thus at the beginning of each period  $t$ , assuming that all previous agents have sampled from  $\mathcal{S}$ , agent  $t$ 's prior uncertainties about  $\tilde{\theta}_1, \dots, \tilde{\theta}_k$  are small while his uncertainties about  $\tilde{\theta}_{k+1}, \dots, \tilde{\theta}_K$  are large. As such, our previous proof for the existence of learning traps with persistent states carries over, and we deduce that agent  $t$  continues to observe from  $\mathcal{S}$ .

Note that for  $\alpha$  close to 1, agents sample each signal  $i$  with frequency close to  $\frac{\beta_i}{\beta_1 + \dots + \beta_k} = \lambda_i$ . It follows that the prior variances  $v_i^t$  approximately satisfy the following fixed-point equation (see the above display):

$$v_i = \alpha \cdot \left( v_i - \frac{\beta_i}{\beta_1 + \dots + \beta_k} \cdot \frac{v_i^2}{1 + v_i} \right) + (1 - \alpha) \cdot \frac{x}{\beta_i}.$$

This yields the first-order approximation  $v_i^t \sim \frac{\sqrt{(1-\alpha)x \cdot (\beta_1 + \dots + \beta_k)}}{\beta_i}$  as  $\alpha \rightarrow 1$  and  $t \rightarrow \infty$ . The posterior variance of  $\omega^t$  is therefore approximated as

$$\sum_{i=1}^k \beta_i^2 \cdot v_i^t \sim \sqrt{(1 - \alpha)x \cdot \left( \sum_{i=1}^k \beta_i \right)^3}.$$



This is exactly  $\sqrt{(1-\alpha)(\frac{M_{11}}{\text{val}(\mathcal{S})})}$  because  $M_{11} = \sum_{i=1}^k \beta_i^2 \cdot \tilde{M}_{ii} = x \cdot \sum_{i=1}^k \beta_i$  and  $\text{val}(\mathcal{S}) = \frac{1}{\phi(\mathcal{S})^2} = \frac{1}{(\sum_{i=1}^k \beta_i)^2}$ . We thus deduce the first payoff estimate in the theorem.

It remains to prove the second payoff estimate. For that we just need to show society can achieve posterior variances smaller than  $(1+\varepsilon) \cdot \sqrt{(1-\alpha)(\frac{M_{11}}{\text{val}(\mathcal{S}^*)})}$  at every late period. In fact, we show that myopically choosing from the best set  $\mathcal{S}^*$  achieves this.

Let  $V^t$  denote society's prior covariance matrix at the beginning of period  $t+1$ , under this alternative sampling strategy. Note that for  $\alpha$  close to 1, each signal in  $\mathcal{S}^*$  is observed with positive frequency. Thus, for any  $L > 0$  we have

$$[V^t]^{-1} \geq L \cdot c_j c'_j, \quad \forall j \in \mathcal{S}^*$$

for  $\alpha$  close to 1 and  $t$  large.

Now take  $v(t) := [V^t]_{11}$  to be the prior variance of  $\omega^{t+1}$ . Then myopic sampling together with Lemma 13 implies the posterior variance of  $\omega^{t+1}$  is bounded above by

$$v(t) - \frac{L}{L+1} \cdot \frac{v(t)^2}{\phi(\mathcal{S}^*)^2}.$$

Together with the innovation terms given by  $M$ , the next prior variance  $v(t+1)$  admits the following upper bound:

$$v(t+1) \leq \alpha \cdot \left( v(t) - \frac{L}{L+1} \cdot \frac{v(t)^2}{\phi(\mathcal{S}^*)^2} \right) + (1-\alpha) \cdot M_{11}.$$

Clearly, this implies

$$v(t+1) < v(t) + (1-\alpha) \cdot M_{11} - \frac{L}{L+1} \cdot \frac{v(t)^2}{\phi(\mathcal{S}^*)^2}.$$

As a consequence,  $\limsup_{t \rightarrow \infty} v(t) \leq \sqrt{1 + \frac{1}{L}} \cdot \sqrt{(1-\alpha)M_{11} \cdot \phi(\mathcal{S}^*)^2}$ . Because  $\text{val}(\mathcal{S}^*) = \frac{1}{\phi(\mathcal{S}^*)^2}$ , this yields the desired estimate if we choose  $L > \frac{1}{\varepsilon}$  in the first place. We have thus completed the proof of Theorem 3.

## B.2. Proof of Proposition 3

The environment in Example 6 is equivalent to one with three signals  $\frac{\omega+b}{2}$ ,  $\frac{\omega-b}{2}$  and  $\frac{1}{L}\omega$ , each with standard Gaussian noise

(just let  $b = \omega + 2b_1$ ). We assume  $L$  is large, so that the best complementary set consists of the former two signals.

For the autocorrelated model, we choose  $M = \Sigma^0 = \text{diag}(x, x)$  with  $x \geq L^2$  (this is the covariance matrix for the innovations associated with  $\omega$  and  $b$ ). Then assuming that all previous agents have chosen the third (unbiased) signal, agent  $t$ 's prior variance of  $b^t$  remains  $x \geq L^2$ . As such, he (and in fact each agent) continues to observe the third signal. In this case the prior variance  $v^t$  about  $\omega^{t+1}$  evolves according to

$$v^t = \alpha \cdot \frac{L^2 \cdot v^{t-1}}{L^2 + v^{t-1}} + (1 - \alpha)x.$$

It is not difficult to show that  $v^t$  must converge to the (positive) fixed point of the above equation. Let us in particular take  $\alpha = 1 - \frac{1}{L^3}$  and  $x = L^2$ , then the long-run prior variance  $v$  solves  $v = \frac{(L^2 - \frac{1}{L})v}{L^2 + v} + \frac{1}{L}$ . This yields exactly that  $v = \sqrt{L}$ . Hence long-run posterior variance is  $\frac{L^2 \cdot v}{L^2 + v} > \frac{\sqrt{L}}{2}$ , which implies  $\limsup_{\delta \rightarrow 1} U_\delta^M \leq -\frac{\sqrt{L}}{2}$ .

Let us turn to the optimal sampling strategy. Write  $\tilde{\theta}_1 = \frac{\omega+b}{2}$  and  $\tilde{\theta}_2 = \frac{\omega-b}{2}$ . In this transformed model,  $\tilde{M} = \tilde{\Sigma}^0 = \text{diag}(\frac{x}{2}, \frac{x}{2})$ , and the payoff-relevant state is the sum of  $\tilde{\theta}_1$  and  $\tilde{\theta}_2$ . Consider now a strategy that observes the first two signals alternatively. Then the beliefs about  $\tilde{\theta}_1$  and  $\tilde{\theta}_2$  remain independent (as in  $\tilde{M}$  and  $\tilde{\Sigma}^0$ ), and their variances evolve as follows:  $v_1^0 = v_2^0 = \frac{x}{2}$ ; in odd periods  $t$

$$v_1^t = \alpha \cdot \frac{v_1^{t-1}}{1 + v_1^{t-1}} + (1 - \alpha)\frac{x}{2} \text{ and } v_2^t = \alpha \cdot v_2^{t-1} + (1 - \alpha)\frac{x}{2},$$

and symmetrically for even  $t$ .

These imply that for odd  $t$ ,  $v_1^t$  converges to  $v_1$  and  $v_2^t$  converges to  $v_2$  below (while for even  $t$   $v_1^t \rightarrow v_2$  and  $v_2^t \rightarrow v_1$ ):

$$v_1 = \alpha \cdot \frac{\alpha v_1 + (1 - \alpha)\frac{x}{2}}{1 + \alpha v_2 + (1 - \alpha)\frac{x}{2}} + (1 - \alpha)\frac{x}{2};$$

$$v_2 = \alpha^2 \cdot \frac{v_2}{1 + v_2} + (1 - \alpha^2) \cdot \frac{x}{2}.$$

From the second equation, we obtain  $(1 - \alpha^2)(\frac{x}{2} - v_2) = \alpha^2 \cdot \frac{(v_2)^2}{1+v_2}$ . With  $\alpha = 1 - \frac{1}{L^3}$  and  $x = L^2$ , it follows that

$$v_2 = (1 + o(1)) \frac{1}{\sqrt{L}}.$$

where  $o(1)$  is a term that vanishes as  $L \rightarrow \infty$ . Thus we also have

$$v_1 = \alpha \frac{v_2}{1 + v_2} + (1 - \alpha) \frac{x}{2} = (1 + o(1)) \frac{1}{\sqrt{L}}.$$

Hence under this alternating sampling strategy, long-run posterior variances about  $\tilde{\theta}_1$  and  $\tilde{\theta}_2$  are both bounded above by  $\frac{2}{\sqrt{L}}$ . Since  $\omega = \tilde{\theta}_1 + \tilde{\theta}_2$ , we conclude that  $\liminf_{\delta \rightarrow 1} U_\delta^{SP} \geq -\frac{4}{\sqrt{L}}$ . Choosing  $L$  large proves the proposition.

## REFERENCES

- Alatas, Vivi, Abhijit Banerjee, Arun Chandrasekhar, Rema Hanna, and Ben Olken, "Network Structure and the Aggregation of Information: Theory and Evidence from Indonesia," *American Economic Review*, 106 (2016), 1663–1704.
- Ali, Nageeb, "Herding with Costly Information," *Journal of Economic Theory*, 175 (2018), 713–720.
- Athey, Susan, and Armin Schmutzler, "Product and Process Flexibility in an Innovative Environment," *RAND Journal of Economics*, 26 (1995), 557–574.
- Banerjee, Abhijit, "A Simple Model of Herd Behavior," *Quarterly Journal of Economics*, 107 (1992), 797–817.
- Bikhchandani, Sushil, David Hirshleifer, and Ivo Welch, "A Theory of Fads, Fashion, Custom, and Cultural Change as Information Cascades," *Journal of Political Economy*, 100 (1992), 992–1026.
- Blackwell, David, "Comparison of Experiments," in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley: University of California Press, 1951), 93–102.
- Börgers, Tilman, Angel Hernando-Veciana, and Daniel Krahmer, "When Are Signals Complements or Substitutes?," *Journal of Economic Theory*, 148 (2013), 165–195.
- Burguet, Roberto, and Xavier Vives, "Social Learning and Costly Information," *Economic Theory*, 15 (2000), 185–205.
- Chade, Hector, and Jan Eeckhout, "Matching Information," *Theoretical Economics*, 13 (2018), 377–414.
- Chaloner, Kathryn, "Optimal Bayesian Experimental Design for Linear Models," *Annals of Statistics*, 12 (1984), 283–300.
- Che, Yeon-Koo, and Konrad Mierendorff, "Optimal Dynamic Allocation of Attention," *American Economic Review* (forthcoming).
- Chen, Yiling, and Bo Waggoner, "Informational Substitutes," in proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science (FOCS), New Brunswick, NJ, 2016.
- Cunha, Flavio, and James Heckman, "The Technology of Skill Formation," *American Economic Review*, 97 (2007), 31–47.
- Dasaratha, Krishna, Ben Golub, and Nir Hak, "Social Learning in a Dynamic Environment," Harvard University Working Paper, 2018.

- DeMarzo, Peter, Dimitri Vayanos, and Jeffrey Zwiebel, "Persuasion Bias, Social Influence, and Unidimensional Opinions," *The Quarterly Journal of Economics*, 118 (2003), 909–968.
- Easley, David, and Nicolas M. Kiefer, "Controlling a Stochastic Process with Unknown Parameters," *Econometrica*, 56 (1988), 1045–1064.
- Frongillo, Rafael, Grant Schoenebeck, and Omer Tamuz, "Social Learning in a Changing World," *WINE'11 Proceedings of the 7th International Conference on Internet and Network Economics*, (2011).
- Fudenberg, Drew, Philip Strack, and Tomasz Strzalecki, "Speed, Accuracy, and the Optimal Timing of Choices," *American Economic Review*, 108 (2018), 3651–3684.
- Gittins, J. C., "Bandit Processes and Dynamic Allocation Indices," *Journal of the Royal Statistical Society, Series B* (1979), 148–177.
- Goldstein, Itay, and Liyan Yang, "Information Diversity and Complementarities in Trading and Information Acquisition," *Journal of Finance*, 70 (2015), 1723–1765.
- Golub, Benjamin, and Matthew Jackson, "How Homophily Affects the Speed of Learning and Best-Response Dynamics," *Quarterly Journal of Economics*, 127 (2012), 1287–1338.
- Hann-Caruthers, Wade, Vadim Martynov, and Omer Tamuz, "The Speed of Sequential Asymptotic Learning," *Journal of Economic Theory*, 173 (2018), 383–409.
- Harel, Matan, Elchanan Mossel, Philipp Strack, and Omer Tamuz, "Groupthink and the Failure of Information Aggregation in Large Groups," Caltech Working Paper, 2018.
- Jovanovic, Boyan, and Yaw Nyarko, "Learning by Doing and the Choice of Technology," *Econometrica*, 64 (1996), 1299–1310.
- Liang, Annie, Xiaosheng Mu, and Vasilis Syrgkanis, "Optimal and Myopic Information Acquisition," University of Pennsylvania Working Paper, 2017.
- , "Dynamically Aggregating Diverse Information," University of Pennsylvania Working Paper, 2019.
- Lizzeri, Alessandro, and Marciano Siniscalchi, "Parental Guidance and Supervised Learning," *Quarterly Journal of Economics*, 123 (2008), 1161–1195.
- Mayskaya, Tatiana, "Dynamic Choice of Information Sources," Higher School of Economics Working Paper, 2019.
- Milgrom, Paul, and Rober J. Weber, "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50 (1982a), 1089–1122.
- , "The Value of Information in a Sealed-Bid Auction," *Journal of Mathematical Economics*, 10 (1982b), 105–114.
- Monderer, Dov, and Lloyd Shapley, "Potential Games," *Games and Economic Behavior*, 14 (1996), 124–143.
- Moscarini, Giuseppe, Marco Ottaviani, and Lones Smith, "Social Learning in a Changing World," *Economic Theory*, 11 (1998), 657–665.
- Mueller-Frank, Manuel, and Mallesh Pai, "Social Learning with Costly Search," *American Economic Journal: Microeconomics*, 8 (2016), 83–109.
- Sandholm, William, *Population Games and Evolutionary Dynamics* (Cambridge, MA: MIT Press, 2010).
- Sethi, Rajiv, and Muhamet Yildiz, "Communication with Unknown Perspectives," *Econometrica*, 84 (2016), 2029–2069.
- , "Culture and Communication," MIT Working Paper, 2019.
- Smith, Lones, and Peter Sørensen, "Pathological Outcomes of Observational Learning," *Econometrica*, 68 (2000), 371–398.
- Vives, Xavier, "How Fast Do Rational Agents Learn?," *Review of Economic Studies*, 60 (1992), 329–347.
- Wolitzky, Alex, "Learning from Others' Outcomes," *American Economic Review*, 108 (2018), 2763–2801.