On the inverse scattering from anisotropic periodic layers and transmission eigenvalues*

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Abstract

This paper is concerned with the inverse scattering and the transmission eigenvalues for anisotropic periodic layers. For the inverse scattering problem, we study the Factorization method for shape reconstruction of the periodic layers from near field scattering data. This method provides a fast numerical algorithm as well as a unique determination for the shape reconstruction of the scatterer. We present a rigorous justification and numerical examples for the factorization method. The transmission eigenvalue problem in scattering have recently attracted a lot of attentions. Transmission eigenvalues can be determined from scattering data and they can provide information about the material parameters of the scatterers. In this paper we formulate the interior transmission eigenvalue problem and prove the existence of infinitely many transmission eigenvalues for the scattering from anisotropic periodic layers.

Keywords. inverse scattering, Factorization method, transmission eigenvalues, anisotropic periodic structure, TM polarization

AMS subject classification. 35R30, 78A46, 65C20

1 Introduction

We study in this paper the inverse scattering problem and the transmission eigenvalues for anisotropic periodic structures in $\mathbb{R}^2$. The periodic structures of interest

*This work was partially supported by NSF grant DMS-1812693 and the Faculty Enhancement Program Award from Kansas State University

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are supposed to be unboundedly periodic in the horizontal direction and bounded in the vertical direction. This can be considered as the model for one-dimensional photonic crystals. We are mainly concerned with a sampling method for shape reconstruction of the periodic scatterers from near field data, the formulation of the transmission eigenvalue problem and existence of transmission eigenvalues. This study is motivated by applications of nondestructive evaluations for periodic structures in optics. The development of numerical methods for shape reconstruction of periodic structures in inverse scattering has been an active research topic during the past years, see [1–3, 9, 10, 12, 16, 18, 20, 21] for a non exhaustive list of results. However, most of the results focus on the case of isotropic periodic scattering structures. The case of anisotropic periodic structures has not been studied much. The first part of this paper is devoted to a study of the Factorization method for solving the inverse scattering problem in two dimensions. This two-dimensional problem can be considered as the (simplified) TM-polarization case of the full Maxwell problem for anisotropic periodic structures. This is an extension of the results for the half space problem in [18] to the full space one. While we only need to measure scattering data above the periodic layer in the half space problem, the analysis for the Factorization method for the full space case in this paper requires the data measured from both sides of the periodic layer. Therefore, the measurement operator and the analysis of its factorization have to be modified for the theoretical analysis. We want to point out that the inverse scattering problem for anisotropic periodic layers has also been recently studied in the paper [19]. The sampling methods developed in [19] can detect the local perturbation and/or the periodic layer itself. However, it is assumed in the cited paper that the complement of the periodic layer in one period is connected, while our theoretical analysis does not need this assumption.

The interior transmission eigenvalues in scattering theory have recently received a great attention thanks to their mathematical interests and applications. Transmission eigenvalue problems are non self-adjoint as well as non-linear which makes their investigation mathematically interesting. One can determine these transmission eigenvalues from scattering data (see for e.g. [5] and [14]). More importantly, they can provide information about the material parameters of the scattering medium. In general, they are monotone with respect to the material parameters which means they can be used as a target signature to determine changes in the scatterer. The transmission eigenvalue problem for anisotropic medium scattering has been studied in [11, 15]. We also refer to [7] for a study of homogenization of the transmission eigenvalue problem for periodic media. Recent results and developments of the transmission eigenvalues and their applications can be found in [6]. The interior transmission eigenvalue problem is less well understood in the case of periodic media, and has not been studied in the context considered in this paper. We present first in this work a formulation of the transmission eigenvalue problem. Second, we follow the theory in [7] to prove that there exists infinitely many transmission eigenvalues for the periodic layer scattering under certain assumption.

The outline of the paper is as follows. After the introduction, we present in Section 2 a formulation of the direct scattering problem as well as a brief discussion on its variational form. Section 3 is dedicated to the inverse scattering problem and the justification of the Factorization method for solving the inverse problem. In Section 4, we formulate the transmission eigenvalue problem for the scattering from anisotropic periodic structures and prove the existence of infinitely many transmission eigenvalues. We present some numerical examples in Section 5 to demonstrate the performance of the Factorization method for the shape reconstruction, and a short summary of the paper in Section 6.

2 Direct problem formulation

We consider a two-dimensional layer which is 2π-periodic in x_1-direction and bounded in x_2-direction. Let A be a matrix-valued bounded function which is 2π-periodic with respect to x_1. Suppose that this periodic scattering layer is fully characterized by A and that the medium outside of the layer is homogeneous which means $A = I$ in this area. Note that we could assume an arbitrary value for the period of the layer. The period 2π is chosen for the convenience of the presentation.

Suppose that this periodic layer is illuminated by the incident plane wave

$$u_{\rm in}(x) = \mathrm{e}^{\mathrm{i}(d_1 x_1 + d_2 x_2)} \tag{1}$$

where $(d_1, d_2)^\top$ is the wave propagation vector direction satisfying $d_1^2 + d_2^2 = k^2$, $k > 0$ is the wave number and $d_2 \neq 0$. The latter condition means we are only interested in incident plane waves propagating downward or upward toward the layer, see also Figure 1 for a schematic of the periodic scattering. The scattering of this incident plane wave by the anisotropic periodic layer produces the scattered field $u_{\rm sc}$ described by

$$\mathrm{div}(A\nabla u_{\rm sc}) + k^2 u_{\rm sc} = -\mathrm{div}(Q\nabla u_{\rm in}) \quad \text{in } \mathbb{R}^2, \tag{2}$$

where Q is the contrast given by

$$Q = A - I.$$

It is important to note that the incident field $u_{\rm in}$ is α-quasi-periodic in x_1 with period 2π, that means, for $\alpha := d_1$, it satisfies

$$u_{\rm in}(x_1 + 2\pi n, x_2) = \mathrm{e}^{\mathrm{i}2\pi n\alpha} u_{\rm in}(x_1, x_2), \quad n \in \mathbb{Z}, \quad (x_1, x_2)^\top \in \mathbb{R}^2.$$

From now on we call functions with this property quasi-periodic functions for short. It is well known for this scattering problem that the scattered field $u_{\rm sc}$ must also be quasi-periodic (in x_1), and that the direct problem of finding the scattered field can be reduced to one period

$$\Omega := (-\pi, \pi) \times \mathbb{R}.$$

Let $h > 0$ be a positive constant such that

$$h > \sup\left\{|x_2| : (x_1, x_2)^\top \in \operatorname{supp}(Q)\right\}, \tag{3}$$

where $\operatorname{supp}(Q)$ is the support of the contrast Q. The direct scattering problem is completed by the Rayleigh expansion condition for the scattered field

$$u_{\mathrm{sc}}(x) = \begin{cases} \sum_{n\in\mathbb{Z}} \widehat{u}_n^+ \mathrm{e}^{\mathrm{i}\alpha_n x_1 + \mathrm{i}\beta_n(x_2 - h)}, & x_2 > h, \\ \sum_{n\in\mathbb{Z}} \widehat{u}_n^- \mathrm{e}^{\mathrm{i}\alpha_n x_1 - \mathrm{i}\beta_n(x_2 + h)}, & x_2 < -h, \end{cases} \tag{4}$$

where

$$\alpha_n := \alpha + n, \quad \beta_n := \begin{cases} \sqrt{k^2 - \alpha_n^2}, & k^2 \geq \alpha_n^2 \\ \mathrm{i}\sqrt{\alpha_n^2 - k^2}, & k^2 < \alpha_n^2 \end{cases}, \quad n \in \mathbb{Z},$$

and $(\widehat{u}_n^\pm)_{n\in\mathbb{Z}}$ are the Rayleigh sequences given by

$$\widehat{u}_n^\pm := \frac{1}{2\pi} \int_0^{2\pi} u^s(x_1, \pm h) e^{-\mathrm{i}\alpha_n x_1} \, \mathrm{d}x_1 \, .$$

The condition (4) means that the scattered field u_{sc} is an outgoing wave (see e.g. [4]). Note that only a finite number of terms in (4) are propagating plane waves which are called propagating modes, the rest are evanescent modes which correspond to exponentially decaying terms. This also implies the absolute convergence of the series in (4). From now, we call a function satisfying (4) a radiating function. In addition, we also assume that β_n is nonzero for all n which means the Wood anomalies are excluded in our analysis.

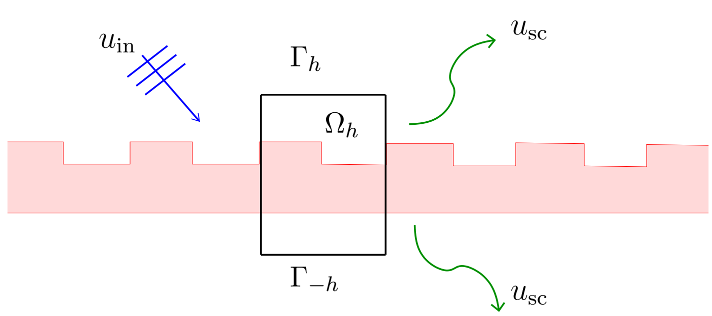


Figure 1: Schematic of the scattering from a penetrable periodic layer

Well-posedness of scattering problem (2)–(4) is well-known, see for instance [4]. For h given in (3), consider a truncation of Ω as

$$\Omega_h := (-\pi, \pi) \times (-h, h), \quad \Gamma_{\pm h} := (-\pi, \pi) \times \{\pm h\}.$$

The variational form of the direct problem is formulated in

$$H^1_\alpha(\Omega_h) := \big\{ u \in H^1(\Omega_h) : \, u = U|_{\Omega_h} \text{ for some } \alpha\text{-quasi-periodic } U \in H^1_{\text{loc}}(\mathbb{R}^2) \big\}.$$

Now the variational problem is to find $u_{\text{sc}} \in H^1_\alpha(\Omega_h)$ such that, for $f = Q\nabla u_{\text{in}}$,

$$\mathcal{B}(u_{\text{sc}}, v; A) = -\int_{\Omega_h} f \cdot \nabla \overline{v} \, \mathrm{d}x \,, \quad \text{for all } v \in H^1_\alpha(\Omega_h), \tag{5}$$

where

$$\mathcal{B}(u_{\text{sc}}, v; A) := \int_{\Omega_h} A \nabla u_{\text{sc}} \cdot \nabla \overline{v} - k^2 u_{\text{sc}} \overline{v} \, \mathrm{d}x \, - \int_{\Gamma_h} T^+(u_{\text{sc}}) \overline{v} \, \mathrm{d}s \, - \int_{\Gamma_{-h}} T^-(u_{\text{sc}}) \overline{v} \, \mathrm{d}s \,.$$

Here the operators $T^\pm : H^{1/2}(\Gamma_{\pm h}) \to H^{-1/2}(\Gamma_{\pm h})$, defined by $T^\pm(\varphi) = \mathrm{i} \sum_{n \in \mathbb{Z}} \beta_n \widehat{\varphi}^\pm_n e^{\mathrm{i}\alpha_n x_1}$, are the exterior Dirichlet-to-Neumann operators. It was proved in [4] that, under uniform ellipticity conditions on A, the direct problem has a unique solution for all but a discrete set of wave number k. In this paper we only consider the wave number $k > 0$ such that the direct problem has a unique solution. We also note that the solution to problem (5) can be extended from Ω_h to $\mathbb{R}^2$ as the solution of the direct scattering problem by using the Rayleigh radiation condition in x_2-direction and the quasi-periodicity in x_1-direction.

3 The inverse problem

In this section we formulate the inverse problem of interest. First we define that

$$\overline{D} \subset \mathbb{R}^2 : \text{ the support of the contrast } Q \text{ in the period } \Omega.$$

The following assumption is important for the analysis of the inverse problem.

Assumption 3.1. *Suppose that D is a Lipschitz domain and that $\Omega \backslash \overline{D}$ has at most two unbounded connected components. The contrast $Q(x)$ is complex-valued and symmetric for almost all $x \in \mathbb{R}^2$. There exist positive constants c_1, c_2 such that $\mathrm{Re}\, Q(x)\xi \cdot \overline{\xi} \geq c_1 |\xi|^2$ and $\mathrm{Im}\, Q(x)\xi \cdot \overline{\xi} \leq -c_2 |\xi|^2$ for all $\xi \in \mathbb{C}^2$ and almost all $x \in D$. Furthermore, the well-defined square root $(\mathrm{Re}\, Q(x))^{1/2}$ is also positive definite with inverse $(\mathrm{Re}\, Q(x))^{-1/2}$, and $(\mathrm{Re}\, Q)^{\pm 1/2}$ belong to $L^\infty(D)^{2\times 2}$.*

Note that by assuming that $\operatorname{Im} Q$ is negative definite we assume absorbing materials for the periodic scatterer. This assumption also excludes the transmission eigenvalues, and is important in our factorization method analysis. However, for the study of transmission eigenvalues in Section 4 we assume that $\operatorname{Im} Q = 0$. Denote by $\ell^2(\mathbb{Z})$ the space of square summable sequences. Thanks to the well-posedness of the direct problem we can define the solution operator $G : [L^2(D)]^2 \to [\ell^2(\mathbb{Z})]^2$ by

$$G(f) = (\widehat{u}_n^+, \widehat{u}_n^-)_{n\in\mathbb{Z}}^\top, \tag{6}$$

where $(\widehat{u}_n^+, \widehat{u}_n^-)_{n\in\mathbb{Z}}^\top$ are the Rayleigh sequences of solution $u \in H_\alpha^1(\Omega_h)$ of

$$\mathcal{B}(u, v; A) = -\int_D Q(\operatorname{Re} Q)^{-1/2} f \cdot \nabla \overline{v} \, \mathrm{d}x \quad \text{for all } v \in H_\alpha^1(\Omega_h). \tag{7}$$

For the inverse problem we consider the quasi-periodic incident plane waves

$$\varphi_n^\pm = \mathrm{e}^{\mathrm{i}(\alpha_n x_1 - \beta_n x_2)} \pm \mathrm{e}^{\mathrm{i}(\alpha_n x_1 + \beta_n x_2)}, \quad n \in \mathbb{Z}. \tag{8}$$

Since problem (7) is linear, a linear combination of several incident fields will lead to a corresponding linear combination of resulting scattered fields. We consider a linear combination using sequences $(a_n)_{n\in\mathbb{Z}} = (a_n^+, a_n^-)_{n\in\mathbb{Z}}^\top \in [\ell^2(\mathbb{Z})]^2$ and define the operator $H : [\ell^2(\mathbb{Z})]^2 \to [L^2(D)]^2$ by

$$H(a_n) = (\operatorname{Re} Q)^{1/2} \sum_{n\in\mathbb{Z}} \left(\frac{a_n^+}{\beta_n w_n^+} \nabla \varphi_n^+ + \frac{a_n^-}{\beta_n w_n^-} \nabla \varphi_n^- \right), \tag{9}$$

where

$$w_n^+ := \begin{cases} \mathrm{i}, & k^2 > \alpha_n^2, \\ \mathrm{e}^{-\mathrm{i}\beta_n h}, & k^2 < \alpha_n^2, \end{cases} \qquad w_n^- := \begin{cases} 1, & k^2 > \alpha_n^2, \\ \mathrm{e}^{-\mathrm{i}\beta_n h}, & k^2 < \alpha_n^2. \end{cases}$$

We use the weights $\beta_n w_n^\pm$ in the linear combination (9) for the convenience of our calculations.

Motivated by applications in near field optics we consider near field measurements in our inverse problem. More precisely, we define the *near field* operator $N : [\ell^2(\mathbb{Z})]^2 \to [\ell^2(\mathbb{Z})]^2$ mapping sequence $(a_n)_{n\in\mathbb{Z}}$ to the Rayleigh sequences of the scattered field generated by the linear combinations of the incident plane waves in (8), i.e.

$$N(a_n) := (\widehat{u}_j^+, \widehat{u}_j^-)_{j\in\mathbb{Z}}^\top, \tag{10}$$

where $u \in H_\alpha^1(\Omega_h)$ is the radiating solution to (7) for $f = H(a_n)$. Here we note that the Rayleigh sequences in (10) are given by the solution operator G acting on the function $H(a_n)$, which also means that the near field operator can be factorized as

$$N = GH.$$

Now the inverse scattering problem can be stated as follows.
Inverse problem: find the support $\overline{D}$ of the periodic contrast Q given near field operator N.

3.1 The adjoint operator H^*

We solve the inverse problem using the factorization method. Factorizing the near field operator is one of the important steps of this method. Before doing that, in the next lemma, we find the adjoint H^* of operator H in (9).

Lemma 3.2. *For $f \in [L^2(D)]^2$, the adjoint $H^* : [L^2(D)]^2 \to [\ell^2(\mathbb{Z})]^2$ of operator H in* (9) *satisfies*

$$(H^* f)_n = 4\pi \begin{pmatrix} \widetilde{w}_n^+ & \widetilde{w}_n^+ \\ \widetilde{w}_n^- & -\widetilde{w}_n^- \end{pmatrix} \begin{pmatrix} \widehat{u}_n^+ \\ \widehat{u}_n^- \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{11}$$

where

$$\widetilde{w}_n^+ = \begin{cases} e^{-\mathrm{i}\beta_n h}, & k^2 > \alpha_n^2, \\ \mathrm{i}, & k^2 < \alpha_n^2, \end{cases} \qquad \widetilde{w}_n^- = \begin{cases} e^{-\mathrm{i}\beta_n h}, & k^2 > \alpha_n^2, \\ 1, & k^2 < \alpha_n^2, \end{cases}$$

and $\widehat{u}_n^\pm$ are the Rayleigh sequences of radiating solution $u \in H^1_\alpha(\Omega_h)$ to

$$\mathcal{B}(u, v; I) = \int_D (\mathrm{Re}\, Q)^{1/2} f \cdot \nabla \overline{v} \,\mathrm{d}x\,, \quad \textit{for all } v \in H^1_\alpha(\Omega_h). \tag{12}$$

Proof. First, the problem (12) is uniquely solvable for all wave numbers $k > 0$ (see [4]). Now we compute

$$\int_D H(a_n) \cdot \overline{f} \,\mathrm{d}x = \sum_{n \in \mathbb{Z}} \left[\frac{a_n^+}{\beta_n w_n^+} \int_D (\mathrm{Re}\, Q)^{1/2} \nabla \varphi_n^+ \cdot \overline{f} \,\mathrm{d}x + \frac{a_n^-}{\beta_n w_n^-} \int_D (\mathrm{Re}\, Q)^{1/2} \nabla \varphi_n^- \cdot \overline{f} \,\mathrm{d}x \right]$$

$$= \left\langle (a_n), \left(\int_D (\mathrm{Re}\, Q)^{1/2} \nabla \overline{\left(\frac{\varphi_n^+}{\beta_n w_n^+} \right)} \cdot f \,\mathrm{d}x\,, \int_D (\mathrm{Re}\, Q)^{1/2} \nabla \overline{\left(\frac{\varphi_n^-}{\beta_n w_n^-} \right)} \cdot f \,\mathrm{d}x \right)^\top \right\rangle_{[\ell^2(\mathbb{Z})]^2} .$$

Setting $g_n = \varphi_n^+/(\beta_n w_n^+)$ we now compute $\int_D (\mathrm{Re}\, Q)^{1/2} \nabla \overline{g_n} \cdot f \,\mathrm{d}x$ in the inner product above.

Let u be the radiating solution to (12). Since $(\mathrm{Re}\, Q)^{1/2}$ is symmetric,

$$\int_D (\mathrm{Re}\, Q)^{1/2} \nabla \overline{g_n} \cdot f \,\mathrm{d}x = \int_D (\mathrm{Re}\, Q)^{1/2} f \cdot \nabla \overline{g_n} \,\mathrm{d}x\,.$$

Letting $v = g_n$ in (12), using Green's theorems and the fact that $\Delta g_n + k^2 g_n = 0$ we obtain

$$\int_D (\mathrm{Re}\, Q)^{1/2} f \cdot \nabla \overline{g_n} \,\mathrm{d}x = \int_{\Gamma_h} u \partial_{x_2} \overline{g_n} - T^+(u) \overline{g_n} \,\mathrm{d}s - \int_{\Gamma_{-h}} u \partial_{x_2} \overline{g_n} + T^-(u) \overline{g_n} \,\mathrm{d}s\,. \tag{13}$$

From a straightforward calculation we further have

$$\overline{g_n}|_{\Gamma_h} = \overline{g_n}|_{\Gamma_{-h}} = -\frac{1}{\beta_n w_n^+}(\mathrm{e}^{\mathrm{i}\beta_n h} + \mathrm{e}^{-\mathrm{i}\beta_n h})e^{-\mathrm{i}\alpha_n x_1},$$
$$\partial_{x_2}\overline{g_n}|_{\Gamma_h} = -\partial_{x_2}\overline{g_n}|_{\Gamma_{-h}} = -\frac{\mathrm{i}}{w_n^+}(\mathrm{e}^{\mathrm{i}\beta_n h} - \mathrm{e}^{-\mathrm{i}\beta_n h})e^{-\mathrm{i}\alpha_n x_1}.$$

Substituting these equations and the radiation condition $u|_{\Gamma_{\pm h}} = \sum_{j\in\mathbb{Z}} \widehat{u}_j^{\pm}\mathrm{e}^{\mathrm{i}\alpha_j x_1}$ in (13), and doing some calculations we obtain

$$\begin{aligned}\int_D (\operatorname{Re} Q)^{1/2} f\cdot\nabla\overline{g_n}\,\mathrm{d}x &= \frac{2\mathrm{i}\widehat{u}_n^+}{w_n^+}\int_{\Gamma_h}\mathrm{e}^{-\mathrm{i}\beta_n h}\,\mathrm{d}s + \frac{2\mathrm{i}\widehat{u}_n^-}{w_n^+}\int_{\Gamma_{-h}}\mathrm{e}^{-\mathrm{i}\beta_n h}\,\mathrm{d}s\\ &= \begin{cases} 4\pi\mathrm{e}^{-\mathrm{i}\beta_n h}(\widehat{u}_n^+ + \widehat{u}_n^-), & k^2>\alpha_n^2\\ 4\pi\mathrm{i}(\widehat{u}_n^+ + \widehat{u}_n^-), & k^2<\alpha_n^2\end{cases} = 4\pi\widetilde{w}_n^+(\widehat{u}_n^+ + \widehat{u}_n^-).\end{aligned}$$

Similarly we obtain

$$\int_D (\operatorname{Re} Q)^{1/2}\nabla\left(\overline{\frac{\varphi_n^-}{\beta_n w_n^-}}\right)\cdot f\,\mathrm{d}x = 4\pi\widetilde{w}_n^-(\widehat{u}_n^+ - \widehat{u}_n^-).$$

This shows that H^* is given by (11). □

We need the following operators in our analysis. Let $W : [\ell^2(\mathbb{Z})]^2 \to [\ell^2(\mathbb{Z})]^2$ defined by

$$\left(W\begin{pmatrix} a_n\\ b_n\end{pmatrix}\right)_n = 4\pi\begin{pmatrix}\widetilde{w}_n^+ & \widetilde{w}_n^-\\ \widetilde{w}_n^+ & -\widetilde{w}_n^-\end{pmatrix}\begin{pmatrix} a_n\\ b_n\end{pmatrix},\quad n\in\mathbb{Z}, \tag{14}$$

and $E : [L^2(D)]^2 \to [\ell^2(\mathbb{Z})]^2$ defined by

$$(Ef)_n = \begin{pmatrix}\widehat{u}_n^+\\ \widehat{u}_n^-\end{pmatrix},\quad n\in\mathbb{Z}, \tag{15}$$

where $\widehat{u}_n^{\pm}$ are the Rayleigh sequences of the radiating solution $u\in H^1_\alpha(\Omega_h)$ to

$$\mathcal{B}(u,v;I) = \int_D (\operatorname{Re} Q)^{1/2} f\cdot\nabla\overline{v}\,\mathrm{d}x \quad\text{ for all }\quad v\in H^1_\alpha(\Omega_h).$$

It is easy to see that these are linear bounded operators and that $H^* = WE$. Moreover, W has a bounded inverse because

$$\det\begin{pmatrix}\widetilde{w}_n^+ & \widetilde{w}_n^-\\ \widetilde{w}_n^+ & -\widetilde{w}_n^-\end{pmatrix} = -2\widetilde{w}_n^+\widetilde{w}_n^- \neq 0 \quad\text{ for all } n\in\mathbb{Z}.$$

Next, we show that the range of the adjoint operator H^*, denoted by Range(H^*), can characterize D. To this end, we first need the quasi-periodic Green function of the direct problem (2)–(4) (see e.g. [17])

$$\mathcal{G}(x,z) = \frac{\mathrm{i}}{4\pi} \sum_{n\in\mathbb{Z}} \frac{1}{\beta_n} \mathrm{e}^{\mathrm{i}\alpha_n(x_1-z_1)+\mathrm{i}\beta_n|x_2-z_2|}, \quad x,z\in\Omega,\ x_2\neq z_2. \tag{16}$$

It is easy to check that, for a fixed z, the Rayleigh sequences of $\mathcal{G}(x,z)$ are given by

$$r_n^{\pm}(z) = \frac{\mathrm{i}}{4\pi\beta_n} \mathrm{e}^{-\mathrm{i}\alpha_n z_1 \pm \mathrm{i}\beta_n(z_2\mp h)}. \tag{17}$$

We have the following characterization of D.

Lemma 3.3. *A point z in Ω belongs to D if and only if*

$$W \begin{pmatrix} r_n^+(z) \\ r_n^-(z) \end{pmatrix} \in \mathrm{Range}(H^*).$$

Proof. For $z\in D$, let $\rho>0$ such that the ball $B(z,\rho)$ belongs to D. Consider a smooth function ξ which is 2π-periodic in x_1 and satisfies $\xi(x)=0$ in $B(z,\rho/2)$ and $\xi(x)=1$ for $|x-z|\geq\rho$. Then the function

$$\Phi(x) := \frac{1}{k^2}\Delta(\xi(x)\mathcal{G}(x,z))$$

is a quasi-periodic smooth function and $\Phi(x) = -\mathcal{G}(x,z)$ for $|x-z|\geq\rho$. For $v\in H^1_\alpha(\Omega_h)$, substituting $\Phi=\frac{1}{k^2}\Delta(\xi\mathcal{G}(\cdot,z))$ in the zero-order term of $\mathcal{B}(\Phi,v;I)$ and using Green's identities we obtain

$$\begin{aligned}
\mathcal{B}(\Phi,v;I) &= \int_{\Omega_h} \nabla\Phi\cdot\nabla\overline{v} - \Delta(\xi\mathcal{G}(\cdot,z))\overline{v}\,\mathrm{d}x - \int_{\Gamma_h} T^+(\Phi)\overline{v}\,\mathrm{d}s - \int_{\Gamma_{-h}} T^-(\Phi)\overline{v}\,\mathrm{d}s \\
&= \int_{\Omega_h} [\nabla\Phi + \nabla(\xi\mathcal{G}(\cdot,z))]\cdot\nabla\overline{v}\,\mathrm{d}x + \int_{\Gamma_h} [-\partial_{x_2}\mathcal{G}(\cdot,z) + T^+(\mathcal{G}(\cdot,z))]\overline{v}\,\mathrm{d}s \\
&\quad + \int_{\Gamma_{-h}} [\partial_{x_2}\mathcal{G}(\cdot,z) + T^-(\mathcal{G}(\cdot,z))]\overline{v}\,\mathrm{d}s
\end{aligned}$$

The boundary terms are zero because of the definition of $T^{\pm}$. Let

$$f = (\mathrm{Re}\,Q)^{-1/2}[\nabla\Phi + \nabla(\xi\mathcal{G}(\cdot,z))] \in [L^2(D)]^2.$$

Then f is supported in D since $\Phi = -\mathcal{G}(\cdot,z)$ in $\Omega\setminus\overline{D}$. Therefore, we have proven that

$$\mathcal{B}(\Phi,v;I) = \int_D (\mathrm{Re}\,Q)^{1/2} f\cdot\nabla\overline{v}\,\mathrm{d}x\,, \quad \text{for all } v\in H^1_\alpha(\Omega_h).$$

This means that there exists $f \in [L^2(D)]^2$ such that

$$(Ef)_n = (\widehat{\Phi}_n^+(\cdot, z), \widehat{\Phi}_n^+(\cdot, z))^\top = (r_n^+(z), r_n^-(z))^\top.$$

Together with $H^* = WE$, this implies that

$$W(r_n^+(z), r_n^-(z))^\top \in \mathrm{Range}(H^*).$$

Now suppose that $z \notin D$ and $(r_n^+(z), r_n^-(z))^\top \in \mathrm{Range}(E)$. Then there exists $u \in H^1_\alpha(\Omega_h)$ solving problem (12) for some $f \in [L^2(D)]^2$ on the right-hand side, and $\widehat{u}_n^\pm = r_n^\pm(z)$ for all $n \in \mathbb{Z}$. This implies that $u = \mathcal{G}(\cdot, z)$ in $\Omega \setminus \overline{(-h, h)}$. Since u and $\mathcal{G}(\cdot, z)$ are respectively analytic functions in $\Omega \setminus D$ and $\Omega \setminus \{z\}$, the analytic continuation implies that $u = \mathcal{G}(\cdot, z)$ in $\Omega \setminus (D \cup \{z\})$. However, it is well known that $\mathcal{G}(\cdot, z)$ is singular at z which leads to a contradiction since $u \in H^1(O)$ for some neighborhood O of z but $\mathcal{G}(\cdot, z) \notin H^1(O)$ due to the singularity at z. □

We can't find D yet since H^* is defined on $[L^2(D)]^2$. One of the most important steps of the factorization method is to connect $\mathrm{Range}(H^*)$ to something related to the near field operator N that is given. This is the content of the next section.

3.2 Shape reconstruction by the factorization method

The following operator is crucial in the factorization method. Let $T : [L^2(D)]^2 \to [L^2(D)]^2$ defined by

$$Tf = (\mathrm{Re}\, Q)^{-1/2} Q((\mathrm{Re}\, Q)^{-1/2} f + \nabla u), \tag{18}$$

where $u \in H^1_\alpha(\Omega_h)$ is the solution to (7). It is not difficult to see that this is a linear bounded operator. Now we factorize the operator N in the following theorem.

Lemma 3.4. *Suppose that the Assumption 3.1 holds true. Then near field operator N can be factorized as*

$$WN = H^* T H.$$

Proof. In the definition of the operator G in (6) we observe that the variational problem (7) can be written as

$$\mathcal{B}(u, v; I) = \int_D (\mathrm{Re}\, Q)^{1/2} (\mathrm{Re}\, Q)^{-1/2} Q((\mathrm{Re}\, Q)^{-1/2} f + \nabla u) \cdot \nabla \overline{v} \, \mathrm{d}x\,.$$

This means that for all $f \in [L^2(D)]^2$

$$Gf = ETf.$$

Thanks to the facts that $N = GH$ and $H^* = WE$ we have

$$WN = WGH = WETH = H^* TH,$$

which completes the proof. □

Let $T_0 : [L^2(D)]^2 \to [L^2(D)]^2$ be defined by

$$T_0 f = (\mathrm{Re}\, Q)^{-1/2} Q((\mathrm{Re}\, Q)^{-1/2} f + \nabla \widetilde{u})$$

where $\widetilde{u} \in H^1_\alpha(\Omega_h)$ solves (7) for $k = \mathrm{i}$. We have the following analytical properties of the operators in the factorization obtained above.

Lemma 3.5. *Suppose that Assumption 3.1 holds true. Then operators H and T satisfy the following:*
(a) H is compact and injective.
(b) T is injective, $\langle \mathrm{Im}\, Tf, f\rangle \leq 0$ for all $f \in [L^2(D)]^2$, and $\langle \mathrm{Im}\, Tf, f\rangle < 0$ for all $f \neq 0$ in $[L^2(D)]^2$.
(c) $T - T_0$ is compact and $\mathrm{Re}\,(T_0)$ is coercive in $[L^2(D)]^2$.

Proof. The proofs for these properties can be done following their analogues of the half-space case [18] and therefore are omitted here. □

From the range identity theorem [13], these analytical properties and the factorization in Lemma 3.4 allow us to obtain that $\mathrm{Range}(H^*) = \mathrm{Range}(WN)^{1/2}_\sharp$ where

$$(WN)_\sharp = |\mathrm{Re}\, WN| - \mathrm{Im}\, WN$$

is a positive definite operator. Therefore, from Lemma 3.3 we now have a necessary and sufficient characterization of D in terms of $\mathrm{Range}(WN)^{1/2}_\sharp$. Since $(WN)_\sharp$ is a compact and self-adjoint operator, we can exploit its eigensystem for imaging of D from the near field data. This is the content of the following theorem.

Theorem 3.6. *Suppose that Assumption 3.1 holds true. For $j \in \mathbb{Z}$, denote by $(\lambda_j, \psi_{n,j})_{j\in\mathbb{N}}$ an orthonormal eigensystem of $(WN)_\sharp$. Then a point $z \in \Omega$ belongs to $\overline{D}$ if and only if*

$$\sum_{j=1}^{\infty} \frac{|\langle r_n(z), \psi_{n,j}\rangle_{[\ell^2(\mathbb{Z})]^2}|^2}{\lambda_j} < \infty. \tag{19}$$

Proof. The proof is similar to that of the half space case [18]. □

4 The transmission eigenvalue problem

In this section, we derive and study the corresponding transmission eigenvalue problem for the scattering by an anisotropic periodic layer. In general, the real eigenvalues can be recovered from the scattering data and can be used to determine the material properties of the anisotropic periodic layer. We wish to prove the existence of infinity many real transmission eigenvalues. See for e.g. [7] for the estimation of the effective material properties for a highly oscillatory media and [11] for the recovery of the transmission eigenvalues for an anisotropic media from the scattering data. Since, in general it is known that absorbing materials do not have real transmission eigenvalues we will assume in this section that the scatterer is non-absorbing (i.e. $\text{Im}\, Q = 0$) for the study of transmission eigenvalues.

We now derive our transmission eigenvalue problem which corresponds to the wave numbers k for which the scattered field vanishes away from the object for some non-trivial quasi-periodic incident field. This means that there is a quasi-periodic incident field $u_{\text{in}} \neq 0$ that is a solution to the Helmholtz equation such that the scattered field $u_{\text{sc}} = 0$ for all $|x_2| > h$ by Holmgren's theorem and the Rayleigh expansion condition (4). By appealing to Holmgren's theorem again we obtain that $u_{\text{sc}} = 0$ for all $x \in \Omega_h \setminus \overline{D}$. Now assuming that

$$\sup\big\{|x_1| \, : \, (x_1, x_2)^\top \in \text{supp}(Q) \cap \Omega_h\big\} < \pi$$

then we have that $\partial\Omega_h \cap \partial D$ is empty. Therefore, we have that $w = u_{\text{in}} + u_{\text{sc}}$ and $v = u_{\text{in}}$ are in $H^1(D)$ satisfying

$$\text{div}(A\nabla w) + k^2 w = 0 \qquad \text{and} \qquad \Delta v + k^2 v = 0 \quad \text{in } \; D \tag{20}$$

$$w = v \qquad \text{and} \qquad \frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad \text{on } \; \partial D \tag{21}$$

where for a generic function $\partial\varphi/\partial\nu_A = \nu \cdot A\nabla\varphi$. Notice, that the transmission eigenvalue problem (20)–(21) has already been studied (see for e.g. [6]). Therefore, we now assume that

$$\sup\big\{|x_1| \, : \, (x_1, x_2)^\top \in \text{supp}(Q) \cap \Omega_h\big\} = \pi$$

which implies that

$$D = \big\{(x_1, x_2)^\top \in \mathbb{R}^2 \, : \, -\pi < x_1 < \pi \;\; \text{and} \;\; f_-(x_1) < x_2 < f_+(x_1)\big\} \tag{22}$$

where $f_\pm \in C^{0,1}[-\pi, \pi]$. We further denote

$$\Gamma_\pm = \{(x_1, x_2)^\top \in \overline{D} : x_2 = f_\pm(x_1)\}.$$

Now if $u_{\text{sc}} = 0$ for any $x_2 > f_+(x_1)$ and $x_2 < f_-(x_1)$ for all $x_1 \in (-\pi, \pi)$, then we have that $w = u_{\text{in}} + u_{\text{sc}}$ and $v = u_{\text{in}}$ are in $H^1_\alpha(D)$ satisfying

$$\text{div}(A\nabla w) + k^2 w = 0 \qquad \text{and} \qquad \Delta v + k^2 v = 0 \quad \text{in } \; D \tag{23}$$

$$w = v \qquad \text{and} \qquad \frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad \text{on } \; \Gamma_\pm. \tag{24}$$

Notice, that here to completely formulate the eigenvalue problem we must enforce the quasi-periodic boundary condition which is not needed for the problem in the previous case.

We say that k is a *transmission eigenvalue* provided that there is a non-trivial solution $(w, v) \in H^1_\alpha(D)^2$ satisfying (23)–(24). The transmission eigenvalue problem (23)–(24) is new since the case where the eigenfunctions are quasi-periodic has not been studied. Now following the analysis in [8, 11] we will prove the existence of transmission eigenvalues. Notice that a 4-th order formulation is not used as is done for the standard transmission eigenvalue problem (see for e.g. [6]). To this end, we will need the following Poincaré inequality result for $H^1_\alpha(D)$.

Lemma 4.1. *For all $u \in H^1_\alpha(D)$ we have that $\|u\|^2_{L^2(D)} \leq C_\alpha \|\nabla u\|^2_{L^2(D)}$ provided $\alpha \notin \mathbb{Z}$ where C_α is a positive constant that is independent of k.*

Proof. Assume on the contrary that $H^1_\alpha(D)$ does not satisfy a Poincaré inequality. This implies that we can find a sequence u_n such that $\|u_n\|^2_{L^2(D)} = 1$ for all $n \in \mathbb{N}$ and $\|\nabla u_n\|^2_{L^2(D)} \to 0$ as $n \to \infty$. By Rellichs compact embedding we can conclude that u_n (up to a subsequence) converges weakly in $H^1(D)$ to u such that the weak limit satisfies $\|u\|^2_{L^2(D)} = 1$ and $\|\nabla u\|^2_{L^2(D)} = 0$. We have that u is a non-zero constant and quasi-periodic which implies that $1 = \mathrm{e}^{2\pi \mathrm{i}\alpha}$ which can not be since $\alpha \notin \mathbb{Z}$. Therefore, we have proven that there is a constant $C_\alpha > 0$ such that $\|u\|^2_{L^2(D)} \leq C_\alpha \|\nabla u\|^2_{L^2(D)}$ for all $u \in H^1_\alpha(D)$. □

In order to insure that the space $H^1_\alpha(D)$ satisfies the above Poincaré inequality we will now assume that $\alpha \notin \mathbb{Z}$ for the rest of the section. We now reformulate the quasi-periodic transmission eigenvalue problem (23)–(24) as a problem for $u = v - w \in H^1_{0,\alpha}(D)$ where the Hilbert space

$$H^1_{0,\alpha}(D) = \big\{ u \in H^1_\alpha(D) \, : \, u = 0 \quad \text{on } \Gamma_\pm \big\}$$

equipped with the $H^1(D)$ norm. By subtracting the equations and boundary conditions in (23)–(24) for the eigenfunctions w and v we obtain

$$\operatorname{div}(A\nabla u) + k^2 u = \operatorname{div}(Q\nabla v) \qquad \text{in} \quad D \tag{25}$$

$$\frac{\partial u}{\partial \nu_A} = \nu \cdot Q\nabla v \qquad \text{on} \quad \Gamma_\pm. \tag{26}$$

In order to completely reformulate the problem for the difference u we must show that (25)–(26) defines a bounded linear mapping $u \mapsto v$ from $H^1_{0,\alpha}(D) \longmapsto H^1_\alpha(D)$. Notice that this implies that (23)–(24) and (25)–(26) are equivalent provided that v satisfies (23) by taking $w = v - u$ since $H^1_{0,\alpha}(D) \subset H^1_\alpha(D)$. The variational formulation of (25)–(26) is given by

$$\int_D Q\nabla v \cdot \nabla \overline{\varphi} \, \mathrm{d}x = \int_D A\nabla u \cdot \nabla \overline{\varphi} - k^2 u \overline{\varphi} \, \mathrm{d}x \qquad \text{for all} \;\; \varphi \in H^1_\alpha(D). \tag{27}$$

Due to the Poincaré inequality and the fact that Q is a uniformly positive definite matrix with bounded entries give that (27) is well-posed by the Lax-Milgram theorem. Next, we define the operator $\mathbb{L}_k$ that maps $H^1_{0,\alpha}(D)$ into itself via the Riesz representation theorem such that

$$\big(\mathbb{L}_k u, \varphi\big)_{H^1(D)} = \int_D \nabla v_u \cdot \nabla \overline{\varphi} - k^2 v_u \overline{\varphi} \, \mathrm{d}x \quad \text{ for all } \; \varphi \in H^1_{0,\alpha}(D). \tag{28}$$

where v_u is the unique solution to (27). This operator $\mathbb{L}_k$ is a key ingredient to the study of the transmission eigenvalue problem. Note that the right hand side of (28) is designed to take into account the fact that v_u satisfies (23). It is obvious that $\mathbb{L}_k$ depends continuously on k. More importantly, we have that if u is in the kernel of $\mathbb{L}_k$ for some $k > 0$, then $w = v_u - u$ and v_u are quasi-periodic transmission eigenfunctions with transmission eigenvalue k. Vice versa, if w and v are quasi-periodic transmission eigenfunctions with eigenvalue k, then $u = v - w$ belongs to the kernel of $\mathbb{L}_k$.

In order to prove the existence of the transmission eigenvalues k we need to determine some properties of the operator $\mathbb{L}_k$. To this end, we will denote v_j to be the unique solution to (27) for a given u_j and $w_j = u_j - v_j$ for $j = 1, 2$. Then similar calculations as in [8] gives that

$$\big(\mathbb{L}_k u_1, u_2\big)_{H^1(D)} = \int_D \nabla u_1 \cdot \nabla \overline{u}_2 - k^2 u_1 \overline{u}_2 \, \mathrm{d}x + \int_D Q \nabla w_1 \cdot \nabla \overline{w}_2 \, \mathrm{d}x\,. \tag{29}$$

Notice, that since Q is a real symmetric matrix we have that the sesquilinear form in (29) is Hermitian giving that $\mathbb{L}_k$ is a selfadjoint operator. Now, taking $k = 0$ we obtain that

$$\big(\mathbb{L}_0 u, u\big)_{H^1(D)} = \int_D |\nabla u|^2 \, \mathrm{d}x + \int_D Q \nabla w \cdot \nabla \overline{w} \, \mathrm{d}x$$

which gives that $\mathbb{L}_0$ is a coercive operator due to the Poincaré inequality and the fact that Q is a positive definite matrix. We now show that the operator $\mathbb{L}_k - \mathbb{L}_0$ is compact. Indeed, let the sequence u_n in $H^1_{0,\alpha}(D)$ weakly converge to zero as $n \to \infty$. Therefore, we have that the sequence of solutions to (27) denoted $v_{n,k}$ in $H^1_{0,\alpha}(D)$ (where we explicitly denote the dependance on k) weakly converges to zero as $n \to \infty$ for all $k \in \mathbb{R}$ by the well-posedness of equation (27). Rellich's compact embedding implies that u_n and $v_{n,k}$ converges to zero in the $L^2(D)$ norm. By subtracting equation (27) for $k \neq 0$ and $k = 0$ gives that

$$\int_D Q \nabla (v_{n,k} - v_{n,0}) \cdot \nabla \overline{\varphi} \, \mathrm{d}x = -k^2 \int_D u_n \overline{\varphi} \, \mathrm{d}x \quad \text{ for any } \; \varphi \in H^1_\alpha(D)$$

which implies that $v_{n,k} - v_{n,0}$ converges to zero in the $H^1(D)$ norm by letting $\varphi = v_{n,k} - v_{n,0}$ and appealing to fact that Q is uniformly positive definite. We now have that

$$\Big((\mathbb{L}_k - \mathbb{L}_0)u_n, \varphi \Big)_{H^1(D)} = \int_D \nabla (v_{n,k} - v_{n,0}) \cdot \nabla \overline{\varphi} - k^2 v_{n,k} \overline{\varphi} \, \mathrm{d}x$$

and by the Cauchy-Schwartz inequality

$$\Big\| (\mathbb{L}_k - \mathbb{L}_0)u_n \Big\|_{H^1(D)} \leq \Big(||v_{n,k} - v_{n,0}||_{H^1(D)} + k^2 ||v_{n,k}||_{L^2(D)} \Big) \to 0 \quad \text{as} \quad n \to \infty.$$

This gives that $\mathbb{L}_k - \mathbb{L}_0$ is compact. From the above analysis we have the following result.

Lemma 4.2. *For any $k \in \mathbb{R}$ the operator $\mathbb{L}_k : H^1_{0,\alpha}(D) \longmapsto H^1_{0,\alpha}(D)$ satisfies:*

1. *$\mathbb{L}_k$ is self-adjoint*
2. *$\mathbb{L}_0$ is coercive*
3. *$\mathbb{L}_k - \mathbb{L}_0$ is a compact.*

By appealing to the theory developed in [8] in order to prove the existence of transmission eigenvalues we now need to show that $\mathbb{L}_k$ is positive on $H^1_{0,\alpha}(D)$ for some k_m and is non-positive on a M–dimensional subspace of $H^1_{0,\alpha}(D)$ for some k_M. This would imply that there are M transmission eigenvalues by the arguments in Section 2.1 of [8].

Theorem 4.3. *There exists infinitely many transmission eigenvalues.*

Proof. We begin by showing that for all k sufficiently small the operator $\mathbb{L}_k$ is positive. To this end, notice that by (29) and the fact that Q is a positive definite matrix we have that

$$\big(\mathbb{L}_k u, u\big)_{H^1(D)} \geq \int_D |\nabla u|^2 - k^2 |u|^2 \, \mathrm{d}x \, .$$

By appealing to the Poincaré inequality we obtain the estimate

$$\big(\mathbb{L}_k u, u\big)_{H^1(D)} \geq \big(1 - k^2 C_\alpha\big) \int_D |\nabla u|^2 \, \mathrm{d}x \, .$$

We can then conclude that $\mathbb{L}_k$ is positive on $H^1_{0,\alpha}(D)$ for all $k^2 < 1/C_\alpha$.

Now the goal is to prove that for some subspace of $H^1_{0,\alpha}(D)$ and value k that $\mathbb{L}_k$ is non-positive. Therefore, we let $\overline{B} \subset D$ be a ball of radius ε centering at some point $x \in D$. Define $k_\varepsilon > 0$ to be the smallest transmission eigenvalue of

$$A_{\min} \Delta w_\varepsilon + k_\varepsilon^2 w_\varepsilon = 0 \qquad \text{and} \qquad \Delta v_\varepsilon + k_\varepsilon^2 v_\varepsilon = 0 \quad \text{in} \;\; B \tag{30}$$

$$w_\varepsilon = v_\varepsilon \qquad \text{and} \qquad \frac{\partial w_\varepsilon}{\partial \nu_A} = \frac{\partial v_\varepsilon}{\partial \nu} \quad \text{on} \;\; \partial B \tag{31}$$

where

$$\inf_{x\in D}\inf_{|\xi|=1}\overline{\xi}\cdot A(x)\xi = A_{\min} \quad \text{which gives that} \quad \inf_{x\in D}\inf_{|\xi|=1}\overline{\xi}\cdot Q(x)\xi = A_{\min}-1.$$

We can define $u_\varepsilon = v_\varepsilon - w_\varepsilon$ in $H_0^1(B)$ and its extension by zero to all of D by u in $H_0^1(D)$. Now let v in $H^1_\alpha(D)$ be the solution to (27) with k_ε and u and $w = v - u$. Using Green's theorem and some simple calculations give that (see for e.g. [8])

$$\int_B \big(A_{\min}-1\big)\nabla w_\varepsilon \cdot \nabla \overline{\varphi}\,\mathrm{d}x = \int_B \nabla u_\varepsilon \cdot \nabla\overline{\varphi} - k_\varepsilon^2 u_\varepsilon \overline{\varphi}\,\mathrm{d}x$$

and

$$\int_D Q\nabla w\cdot\nabla\overline{\varphi}\,\mathrm{d}x = \int_D \nabla u\cdot\nabla\overline{\varphi} - k_\varepsilon^2 u\overline{\varphi}\,\mathrm{d}x \quad \text{for all} \quad \varphi\in H^1_\alpha(D).$$

Now, notice that by the definition of u

$$\begin{aligned}\int_D Q\nabla w\cdot\nabla\overline{\varphi}\,\mathrm{d}x = \int_D \nabla u\cdot\nabla\overline{\varphi} - k_\varepsilon^2 u\overline{\varphi}\,\mathrm{d}x &= \int_B \nabla u_\varepsilon\cdot\nabla\overline{\varphi} - k_\varepsilon^2 u_\varepsilon\overline{\varphi}\,\mathrm{d}x \\ &= \int_B \big(A_{\min}-1\big)\nabla w_\varepsilon\cdot\nabla\overline{\varphi}\,\mathrm{d}x\end{aligned}$$

Letting $\varphi = w$ and estimating gives

$$\begin{aligned}\int_D Q\nabla w\cdot\nabla\overline{w}\,\mathrm{d}x &= \int_B \big(A_{\min}-1\big)\nabla w_\varepsilon\cdot\nabla\overline{w}\,\mathrm{d}x \\ &\leq \left(\int_B \big(A_{\min}-1\big)|\nabla w_\varepsilon|^2\,\mathrm{d}x\right)^{1/2}\left(\int_B \big(A_{\min}-1\big)|\nabla w|^2\,\mathrm{d}x\right)^{1/2} \\ &\leq \left(\int_B \big(A_{\min}-1\big)|\nabla w_\varepsilon|^2\,\mathrm{d}x\right)^{1/2}\left(\int_D Q\nabla w\cdot\nabla\overline{w}\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\big(\mathbb{L}_{k_\varepsilon}u,u\big)_{H^1(D)} &= \int_D |\nabla u|^2 - k_\varepsilon^2|u|^2\,\mathrm{d}x + \int_D Q\nabla w\cdot\nabla\overline{w}\,\mathrm{d}x \\ &\leq \int_B |\nabla u_\varepsilon|^2 - k_\varepsilon^2|u_\varepsilon|^2\,\mathrm{d}x + \int_B \big(A_{\min}-1\big)|\nabla w_\varepsilon|^2\,\mathrm{d}x = 0.\end{aligned}$$

This implies that $\mathbb{L}_{k_\varepsilon}$ is non-positive on the subspace of $H^1_{0,\alpha}(D)$ which is the span of u proving the existence of a transmission eigenvalue.

We now wish to construct an infinite dimensional subspace of $H^1_{0,\alpha}(D)$ for which $\mathbb{L}_{k_\varepsilon}$ is non-positive. To this end, we let B_j be the ball centered at $x_j\in D$ with radius $\varepsilon>0$. Here we define M_ε to be the supremum of the number of disjoint balls B_j such that

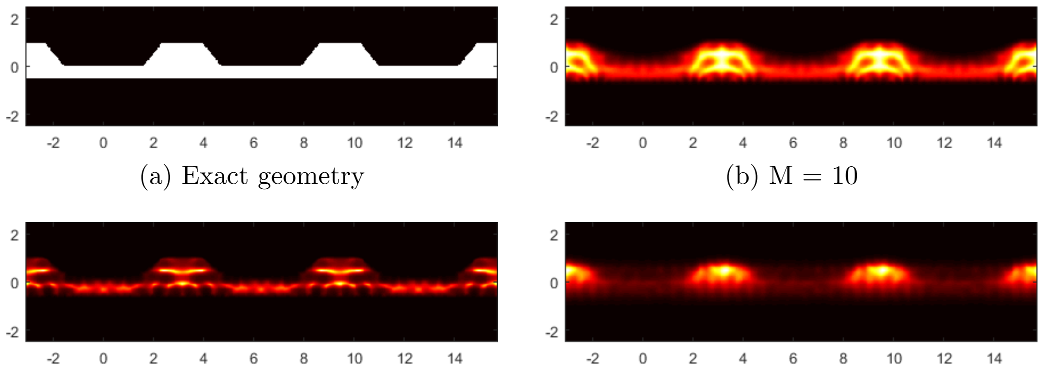
$\overline{B_j} \subset D$. Notice that since the coefficient $A_{\min}$ is constant we have k_ε being the smallest transmission eigenvalue of (30)–(31) is the same for each B_j. Defining u_j in $H^1_{0,\alpha}(D)$ for $j = 1, \ldots, M_\varepsilon$ just as above and we have that since the supports are disjoint u_j is orthogonal to u_i for all $i \neq j$. Therefore, we can conclude that $\text{span}\{u_1, u_2, \ldots, u_{M_\varepsilon}\}$ is a M_ε–dimensional subspace of $H^1_{0,\alpha}(D)$. Due to the disjoint support of the basis functions we can follow the analysis above to show that $\mathbb{L}_{k_\varepsilon}$ is non-positive for any u in this M_ε–dimensional subspace of $H^1_{0,\alpha}(D)$. Since $M_\varepsilon \to \infty$ as $\varepsilon \to 0$ we can conclude that there are infinitely many transmission eigenvalues. $\square$

5 Numerical examples for the shape reconstruction

In this section, we present some numerical examples examining the performance of the factorization method for different types of periodic structures and data perturbed by artificial noise. We also show the dependence of the reconstructions on the number of the incident fields used.

The synthetic scattering data are generated by solving the direct problem with the spectral Galerkin method studied in [17]. We solve the direct problem for incident fields $\varphi_n^\pm$ in (8) where $n = -M, \ldots, M$ ($M \in \mathbb{N}$). For each incident field we collect Rayleigh coefficients $\widehat{u}_j^\pm$ of the corresponding scattered field for $j = -M, \ldots, M$. The near field operator N is then a 2×2 block matrix. Each block is an $(2M+1) \times (2M+1)$ matrix whose (n, j)-entry is the jth Rayleigh coefficient of the scattered field generated by the nth incident field. Two blocks of the block matrix correspond to φ_n^+ and φ_n^- and the other two are for $\widehat{u}^+$ and $\widehat{u}^-$. As in the case of half space [18] using the standard tools of linear algebra we can easily construct the matrix $(WN)_\sharp^{1/2}$ and its eigensystem. To simulate the case of noisy data we add a noise matrix to data matrix N. This noise matrix contains complex random numbers that are uniformly distributed in (0,1). To regularize the imaging functional for noisy data we truncate the singular values of $(WN)_\sharp^{1/2}$. More precisely, we drop the singular values that are less that 5×10^{-4}. We note that Tikhonov regularization can also be applied (as in the half space case [18]) and would give similar results.

As described above, $M = 10$ in the pictures means that we use 21 $(2M+1)$ incident plane waves and 21 Rayleigh coefficients of the corresponding scattered fields. It also means that the series in (19) is truncated with $2(2M+1)$ terms. We use the wave number $k = 5.85$ for all the examples. This means that we have 11 propagating modes for the examples and 10 evanescent modes for $M = 10$ and 30 evanescent modes for $M = 20$. The pictures show that the imaging functional based on the factorization method is able to provide reasonable reconstructions for the shape of several types of periodic layers. As in the previous results for the factorization method for the periodic inverse scattering, the evanescent modes are quite important to have



(a) Exact geometry (b) M = 10

(c) M = 20 (d) M = 20,5% noise

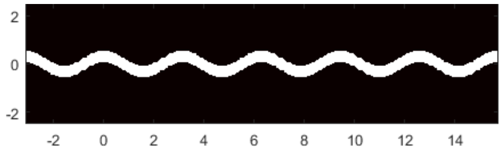
Figure 2: Shape reconstruction for the periodic layer of piecewise linear type.

reasonable reconstruction results. Here, for all four examples, we have respectively 10 and 30 evanescent modes in the scattering data when $M = 10$ and $M = 30$.

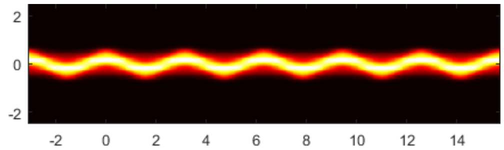
We also observe that the reconstruction results are quite stable with respect to noise in the data for the last two examples (Figures 4 and 5). However, the reconstructions for the first two examples are pretty sensitive to noise, see [2] for a similar situation. The results in Figures 2(d) and 3(d) are chosen as the best results out of 10 numerical experiments. We can't see anything reasonable in the worst cases of these numerical reconstructions. The imaging functional seems to have more stability in the numerical reconstructions when the complement of periodic layer in one period is connected but we have no justification for this behavior.

6 Summary

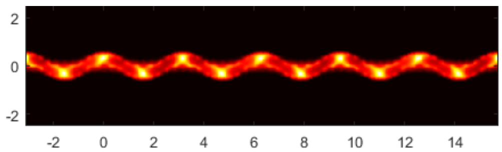
We study the inverse scattering problem for anisotropic periodic layers and the transmission eigenvalues associated to the problem. The Factorization method is investigated as a tool to solve the inverse scattering problem with near field scattering data. This method provides both the unique determination and a fast imaging algorithm for the shape of the periodic scatterer. We present a justification of the Factorization method for the case of absorbing materials and some numerical examples to verify its performance. The interior transmission eigenvalue problem for the scattering from anisotropic periodic layers is formulated. We prove the existence of infinitely many transmission eigenvalues in the case of non-absorbing materials. The periodic scatterer in this paper is assumed to be given by Lipchitz domains and does not have holes. An extension of the results to the case in which the scatterer has holes or is given by non-Lipchitz domains is still an open problem.



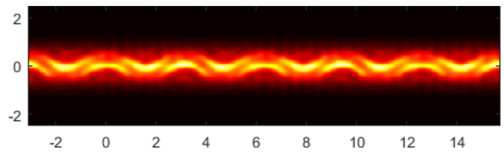
(a) Exact geometry



(b) M = 10

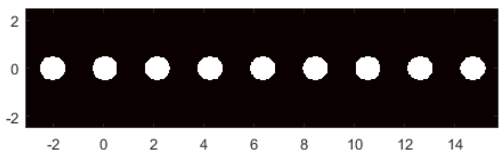


(c) M = 20

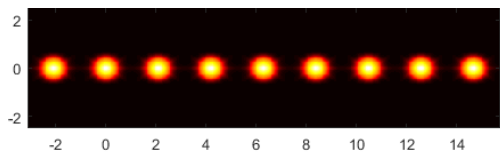


(d) M = 20, 5% noise

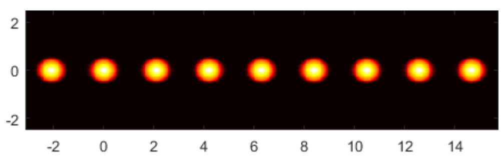
Figure 3: Shape reconstruction for the periodic layer of sinusoidal type.



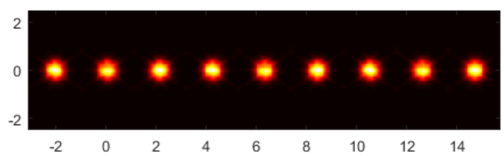
(a) Exact geometry



(b) M = 10

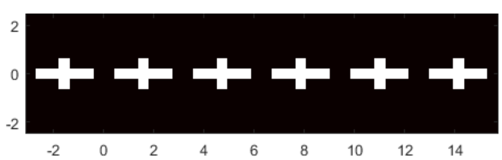


(c) M = 20

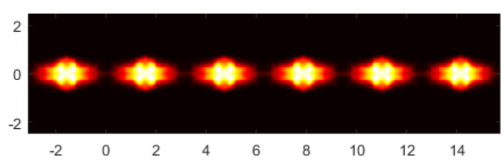


(d) M = 20, 5% noise

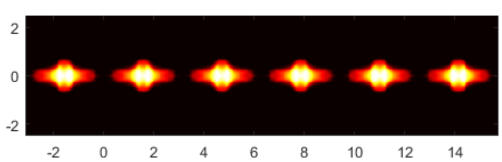
Figure 4: Shape reconstruction for the periodic layer of ball type.



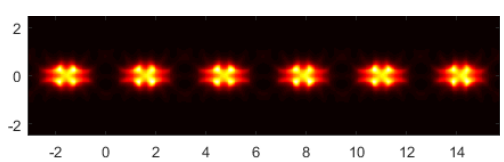
(a) Exact geometry



(b) M = 10



(c) M = 20



(d) M = 20, 5% noise

Figure 5: Shape reconstruction for the periodic layer of cross type.

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