The Journal of Geometric Analysis
https://doi.org/10.1007/s12220-019-00351-8

Creating and Flattening Cusp Singularities by Deformations of Bi-conformal Energy

Tadeusz Iwaniec[1] · **Jani Onninen**[1,2] · **Zheng Zhu**[2]

Received: 13 July 2019

Abstract

Mappings of bi-conformal energy form the widest class of homeomorphisms that one can hope to build a viable extension of Geometric Function Theory with connections to mathematical models of Nonlinear Elasticity. Such mappings are exactly the ones with finite conformal energy and integrable inner distortion. It is in this way that our studies extend the applications of quasiconformal homeomorphisms to the degenerate elliptic systems of PDEs. The present paper searches a bi-conformal variant of the Riemann Mapping Theorem, focusing on domains with exemplary singular boundaries that are not quasiballs. We establish the sharp description of boundary singularities that can be created and flattened by mappings of bi-conformal energy.

Keywords Cusp · Bi-conformal energy · Mappings of integrable distortion · quasiball

Mathematics Subject Classification Primary 30C65

T. Iwaniec was supported by the NSF Grant DMS-1802107. J. Onninen was supported by the NSF Grant DMS-1700274. Z. Zhu was supported by the Academy of Finland #307333 and China Scholarship Council fellowship #201506020103. This research was done while Z. Zhu was visiting Mathematics Department at Syracuse University. He wishes to thank SU for the hospitality.

✉ Jani Onninen
jkonnine@syr.edu

Tadeusz Iwaniec
tiwaniec@syr.edu

Zheng Zhu
zheng.z.zhu@jyu.fi

1 Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA

2 Department of Mathematics and Statistics, University of Jyväskylä, P.O.Box 35 (MaD), 40014 Jyväskylä, Finland

Published online: 18 January 2020

1 Introduction

There is a broad literature dealing with a question when a pair of domains $(\mathbb{X}, \mathbb{Y})$ is quasiconformally or even bi-Lipschitz equivalent. Gehring and Väisälä [11] raised the question: *Which domains $D \subset \mathbb{R}^n$ are quasiconformally equivalent with the unit ball $\mathbb{B} \subset \mathbb{R}^n$?* Such domains D are called *quasiballs*. The interested reader is referred to the recent book by Gehring et al. [10]. The Riemann Mapping Theorem gives a complete answer to this question when $n = 2$. If $D \subsetneq \mathbb{C}$ is a simply connected domain, then there exists a conformal mapping $h : \mathbb{B} \xrightarrow{\text{onto}} D$. It is, however, a highly nontrivial question when a domain $D \subset \mathbb{R}^n$ is a quasiball when $n \geqslant 3$. Among geometric obstructions are the inward cusps. Indeed, Gehring and Väisälä [11] proved that a ball with inward cusp is not a quasiball. A ball with outward cusp, however, is always a quasiball. We denote an n-dimensional unit balls with exemplary boundary singularities of the form of cusps by $\mathbb{B}_u$ where $u : [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ is a strictly increasing function and which characterizes the singularity at the origin, see Fig. 1 and Sect. 1.6 for the precise definition.

In this article, we describe boundary singularities that can be created by finite n-*harmonic energy* and return to the original shape by the inverse deformations whose n-harmonic energy is finite as well. This is in accordance with the Hooke's Law, see Sect. 1.4. We remind the reader that there exists a Lipschitz homeomorphism to both directions between $\mathbb{B}$ and $\mathbb{B}_u$. However, it is not always possible to have $\mathscr{W}^{1,n}$-Sobolev bounds to both directions for a single map. We state this as follows.

Theorem 1.1 *Let $n \geqslant 3$ and*

$$u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}} \quad \textit{for } 0 \leqslant t \leqslant 1\,, \quad \textit{where } \alpha > 0. \tag{1.1}$$

Then there exists a homeomorphism $h : \mathbb{B} \xrightarrow{onto} \mathbb{B}_u$ in $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{R}^n)$ whose inverse $f = h^{-1} : \mathbb{B}_u \xrightarrow{onto} \mathbb{B}$ lies in $\mathscr{W}^{1,n}(\mathbb{B}_u, \mathbb{R}^n)$ if and only if $\alpha < n$.

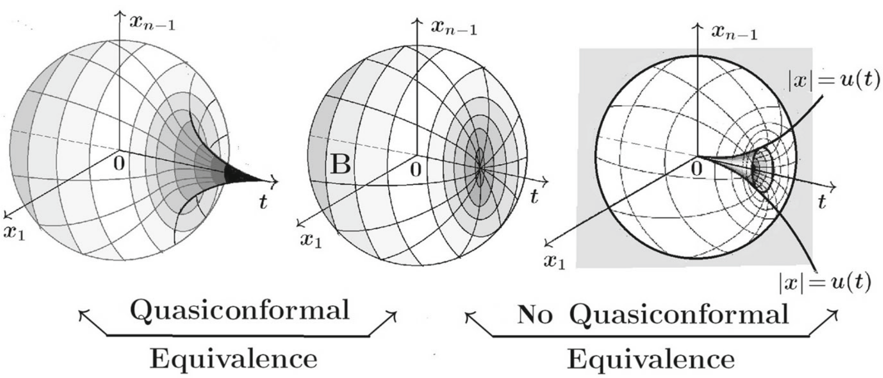


Fig. 1 Quasiconformal mapping can flatten the outward cusp but not the inward cusp

Theorem 1.1 is a special case of our main result (Theorem 1.11). Before formulating Theorem 1.11 we discuss the studied mapping problem in more details.

We are concerned with orientation-preserving homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between bounded domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$, $n \geqslant 2$, of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 \leqslant p \leqslant \infty$.

1.1 Quasiconformal Deformations

Of particular interest are homeomorphisms of finite *n-harmonic energy*; that is, with $p = n$.

$$\mathsf{E}_{\mathbb{X}}[h] \stackrel{\text{def}}{=\!=} \int_{\mathbb{X}} |Dh(x)|^n \,\mathrm{d}x \; < \infty. \tag{1.2}$$

Hereafter the symbol $|Dh(x)|$ stands for the operator norm of the differential matrix $Dh(x) \in \mathbb{R}^{n\times n}$ called the *deformation gradient*. This integral is invariant under the conformal change of variables in the *reference configuration* $\mathbb{X}$ (not in the *deformed configuration* $\mathbb{Y}$). That is, $\mathsf{E}_{\mathbb{X}'}[h'] = \mathsf{E}_{\mathbb{X}}[h]$, where $h' = h \circ \varphi$ for a conformal transformation $\varphi\colon \mathbb{X}' \xrightarrow{\text{onto}} \mathbb{X}$. This motivates us to call $\mathsf{E}_{\mathbb{X}}[h]$ the **conformal energy** of h. Mappings of conformal energy arise naturally in Geometric Function Theory (GFT) for many reasons [2,11,13,16,26].

Definition 1.2 A *Sobolev homeomorphism* $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, that is, of class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$, is said to be *quasiconformal* if there exists a constant $1 \leqslant \mathcal{K} < \infty$ so that for almost every $x \in \mathbb{X}$ it holds:

$$|Dh(x)|^n \leqslant \mathcal{K}\, J_h(x)\ , \qquad \text{where} \quad J_h(x) = \det Dh(x).$$

Every quasiconformal map $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ has finite conformal energy provided $|\mathbb{Y}| < \infty$. Indeed,

$$\mathsf{E}_{\mathbb{X}}[h] = \int_{\mathbb{X}} |Dh(x)|^n \,\mathrm{d}x \;\leqslant \mathcal{K} \int_{\mathbb{X}} J_h(x)\mathrm{d}x \;=\; \mathcal{K}\,|\mathbb{Y}|. \tag{1.3}$$

1.2 Mappings of Bi-conformal Energy

The remarkable feature of a quasiconformal mapping is that its inverse $f \stackrel{\text{def}}{=\!=} h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ is also quasiconformal. In particular, both h and f have finite conformal energy. Their sum

$$\mathsf{E}_{\mathbb{X}\mathbb{Y}}[h] \stackrel{\text{def}}{=\!=} \int_{\mathbb{X}} |Dh(x)|^n \,\mathrm{d}x + \int_{\mathbb{Y}} |Df(y)|^n \,\mathrm{d}y \stackrel{\text{def}}{=\!=} \mathsf{E}_{\mathbb{Y}\mathbb{X}}[f] \tag{1.4}$$

will be called *bi-conformal energy* of h.

This leads us to a viable extension of GFT with connections to mathematical models of Nonlinear Elasticity (NE) [1,4,6,22].

Definition 1.3 A homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$, whose inverse $f = h^{-1}\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ also belongs to $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{R}^n)$ is called a *mapping of bi-conformal energy*.

It is equivalent to saying that the inner distortion function of h is integrable over $\mathbb{X}$ and the inner distortion function of f is integrable over $\mathbb{Y}$. For a precise statement (Theorem 1.5) we need some definitions.

1.3 Inner Distortion

Consider a Sobolev mapping $h \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$ and its *co-differential* $D^\sharp h(x) \in \mathbb{R}^{n\times n}$ - the matrix determined by Cramer's rule $D^\sharp h \circ Dh = J_h(x)\,\mathbf{I}$.

Definition 1.4 The inner distortion of h is the smallest measurable function $K_I(x) = K_I(x, h) \in [1, \infty]$ such that

$$|D^\sharp h(x)|^n \leqslant K_I(x) J_h(x)^{n-1} \qquad \text{for almost every } x \in \mathbb{X}. \tag{1.5}$$

The question of finite inner distortion merely asks for the co-differential $D^\sharp h(x) = 0$ at the points where the Jacobian $J_h(x) = 0$. However, for $n \geqslant 3$, the differential $Dh(x)$ need not vanish if $D^\sharp h(x) = 0$.

A formal algebraic computation reveals that the pullback of the n-form $K_I(x, h)\, \mathrm{d}x \in \wedge^n \mathbb{X}$ via the inverse mapping $f\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ equals $|Df(y)|^n\, \mathrm{d}y \in \wedge^n \mathbb{Y}$.

This observation is the key to the fundamental equality between the $\mathscr{L}^1$-norm of $K_I(x, h)$ and conformal energy of the inverse map f, which is usually derived under various regularity assumptions [3,7,12,14,24]. We shall state in the following form:

Theorem 1.5 *Let $h\colon \mathbb{X} \to \mathbb{Y}$ be an orientation-preserving homeomorphism in the Sobolev space $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$, $n \geqslant 2$. Then the inner distortion of h is integrable if and only if the inverse mapping $f = h^{-1}\colon \mathbb{Y} \to \mathbb{X}$ has finite conformal energy. Furthermore, we have*

$$\int_{\mathbb{Y}} |Df(y)|^n\, \mathrm{d}y = \int_{\mathbb{X}} K_I(x, h)\, \mathrm{d}x. \tag{1.6}$$

Theorem 1.5 is known among the experts in the field and follows combining a few results in the literature. We will provide a proof for the convenience of the reader in the appendix. The interested reader is referred to [20] for planar mappings with integrable distortion (Stoilow factorization). The following corollary is immediate.

Corollary 1.6 *A homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ is quasiconformal if and only if with $K_I(\cdot, h) \in \mathscr{L}^\infty(\mathbb{X})$.*

1.4 Hooke's Law for Materials of Conformal Stored-Energy

In a different direction, the principle of hyper-elasticity is to minimize the given stored-energy functional subject to deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of domains made of elastic materials, see [1,4,6,22]. Here we take on stage the materials of *conformal stored-energy*. This means that the bodies can endure only deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ whose gradient Dh is integrable with power n (the dimension of the deformed body). A deformation of infinite n-harmonic energy would break the internal structure of the material causing permanent damage. There are examples galore in which one can return the deformed body to its original shape by a deformation of finite conformal energy, but not necessarily via the inverse mapping $f \stackrel{\text{def}}{=} h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$. The inverse map need not even belong to $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{R}^n)$. On the other hand the essence of Hooke's Law is reversibility. Accordingly, we wish that both h and $f = h^{-1}$ have finite conformal energy. We call this model *n-harmonic hyper-elasticity*. It is from this point of view that we arrive at the following n-dimensional variant of the conformal Riemann mapping problem.

1.5 Mapping Problems

Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ be bounded domains of the same topological type. For each of the three problems below find conditions on the pair $(\mathbb{X}, \mathbb{Y})$ to ensure that:

(P1) There exists a bi-Lipschitz deformation $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$.
(P2) There exists a quasiconformal deformation $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$.
(P3) There exists deformation $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of bi-conformal energy.

The implications (P1) $\Longrightarrow$ (P2) $\Longrightarrow$ $(P3)$ are straightforward.

1.6 Ball with Inward Cusp

We shall distinguish a horizontal coordinate axis in $\mathbb{R}^n$,

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t, x) \colon t \in \mathbb{R} \text{ and } x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\}$$

and introduce the notation

$$\rho = |x| \stackrel{\text{def}}{=} \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}.$$

Consider a strictly increasing function $u \colon [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ of class $\mathscr{C}^1(0, \infty) \cap \mathscr{C}[0, \infty)$. We assume that u' is increasing in $(0, \infty)$ and

$$\lim_{\rho \searrow 0} u'(\rho) = 0.$$

To every such function there corresponds an $(n-1)$-dimensional surface of revolution $\mathbf{S}_u \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$

$$\mathbf{S}_u \stackrel{\text{def}}{=} \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{n-1} \colon \ |x| = u(t)\}, \text{ where } \mathbb{R}_+ = [0, \infty).$$

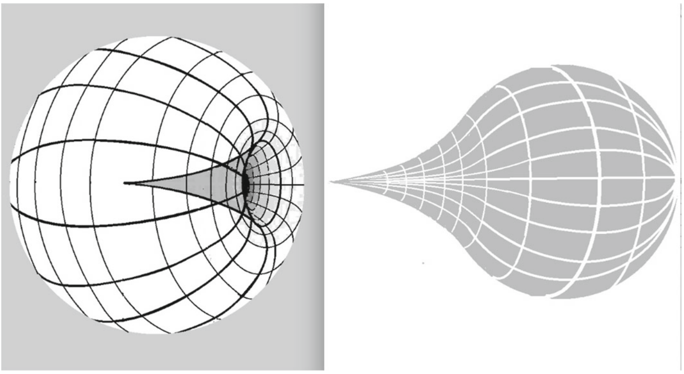


Fig. 2 Inward and outward cusp in a ball

We shall refer to $\mathbf{S}_u$ as *a model cusp* at the origin. Let us emphasize that the case $\limsup_{\rho\searrow 0} u'(\rho) > 0$ is **excluded** from this definition. We may (and do) rescale u so that $u(1) = 1$. The model inward cuspy ball is defined by

$$\mathbb{B}_u \overset{\text{def}}{=\!=} \mathbb{B} \setminus \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{n-1} : \ |x| \leqslant u(t)\},$$

see Fig. 1.

1.7 Bi-Lipschitz Deformations

There is no bi-Lipschitz transformation of a cuspy ball (inward or outward as in Fig. 2) onto a ball without cusp. We say that a cusp cannot be flatten via bi-Lipschitz deformation.

However, there always exists a Lipschitz homeomorphism of a cuspy ball onto a round ball and there is a Lipschitz homeomorphism of the round ball onto the cuspy ball, but these two deformations cannot be inverse to each other. The same pertains to a *degenerate cusp* defined by $u \equiv 0$, as in Fig. 3. In this degenerate case, if there would exist a bi-Lipschitz mapping $h : \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B} \setminus \mathbf{I}$, it would extend as a homeomorphism of $\partial\mathbb{B}$ onto $\partial(\mathbb{B} \setminus \mathbf{I})$, $n \geqslant 3$, see [18] for more details. It is clear that the conflicting topology of the boundaries is an obstruction to the existence of a bi-Lipschitz deformation. This fact is also valid for deformations of bi-conformal energy, but it requires additional arguments.

1.8 Inward Slit in a Ball (the case $u \equiv 0$)

We will discuss the degenerate cups separately ($u \equiv 0$). Let us take a look at the pair $(\mathbb{B}, \mathbb{B} \setminus \mathbf{I})$ of a unit ball and the ball with a slit along the line segment

$$\mathbf{I} \overset{\text{def}}{=\!=} \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 \leqslant t < 1 \text{ and } |x| = 0\},$$

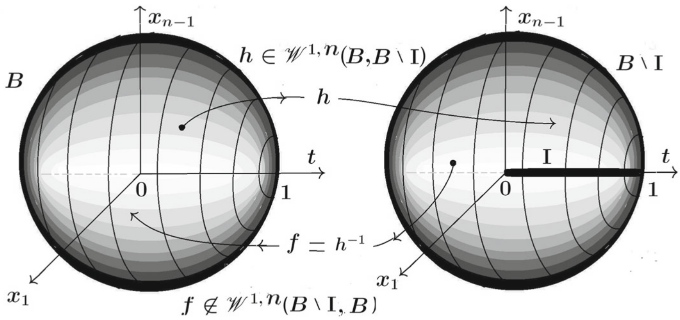


Fig. 3 Two domains which are not of the same bi-conformal energy type

see Fig. 3.

We have already mentioned that there exists a Lipschitz homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B} \setminus \mathbf{I}$; in particular, $h \in \mathscr{W}^{1,n}(\mathbb{B}, \mathbb{B} \setminus \mathbf{I})$. The question arises whether there exists a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B} \setminus \mathbf{I}$ of finite conformal energy whose inverse $f = h^{-1} : \mathbb{B} \setminus \mathbf{I} \xrightarrow{\text{onto}} \mathbb{B}$ also has finite conformal energy. Theorem 1.1 answer to this question is in the negative.

Theorem 1.7 *In dimension $n \geqslant 3$ the domains $\mathbb{B}$ and $\mathbb{B} \setminus \mathbf{I}$ are not of the same bi-conformal energy type; that is, there is no homeomorphism $h : \mathbb{B} \xrightarrow{onto} \mathbb{B} \setminus \mathbf{I}$ of finite bi-conformal energy.*

On one hand we have:

Example 1.8 There is a homeomorphism $f : \mathbb{B} \setminus \mathbf{I} \xrightarrow{\text{onto}} \mathbb{B}$ of finite conformal energy such that $h = f^{-1} \in \mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$ for all exponents $p < n$.

On the other hand, Theorem 1.7 is a special case of the following.

Theorem 1.9 *For $p > n - 1 \geqslant 2$ there is no homeomorphism $h : \mathbb{B} \xrightarrow{onto} \mathbb{B} \setminus \mathbf{I}$ of finite conformal energy with inverse $h^{-1} = f \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{R}^n)$.*

The lower bound for the Sobolev exponent in this theorem is essentially sharp. More precisely, we have

Theorem 1.10 *For every $p < n - 1$ there is a homeomorphism $h : \mathbb{B} \xrightarrow{onto} \mathbb{B} \setminus \mathbf{I}$ of finite conformal energy with inverse $f = h^{-1} \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{R}^n)$.*

The borderline case $p = n - 1$ remains open.

1.9 Main Result

Our central question is when the unit ball and the ball with a model inward cusp $\mathbf{S}_u$ are of the same bi-conformal energy type. Let $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ be a deformation of bi-conformal energy. To predict what cusps $\mathbf{S}_u$ can be created, i.e., to predict that u is given by (1.1) it is natural to combine the estimates of the modulus of continuity of h near 0 with those for the inverse deformation $f = h^{-1} : \mathbb{B} \setminus \mathbf{I} \xrightarrow{\text{onto}} \mathbb{B}$. From this point of view, deformations of bi-conformal energy are very different from quasiconformal mappings. The latter behave singular-like radial stretchings/squeezing; a poor modulus of continuity is always balanced by a better modulus of continuity of its inverse. Surprisingly, a deformation of bi-conformal energy and its inverse may exhibit the same optimal modulus of continuity [19], locally at a given point. Recall that a homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ satisfies the following estimate of the modulus of continuity:

$$|h(x_1) - h(x_2)|^n \leqslant C_n \left(\int_{2\mathbf{B}} |Dh|^n \right) \log^{-1} \left(e + \frac{\operatorname{diam} \mathbf{B}}{|x_1 - x_2|} \right) , \tag{1.7}$$

where $x_1, x_2 \in \mathbf{B} \overset{\text{def}}{=\!=} B(x_\circ, R) \subset B(x_\circ, 2R) \overset{\text{def}}{=\!=} 2\mathbf{B} \Subset \mathbb{X}$.

Applying the estimates in (1.7) would give us a nonexistence of a deformation of bi-conformal energy from $\mathbb{B}$ onto $\mathbb{B}_u$ with $u(t) = \exp^{-1}(\exp^\alpha(1/t))$, where $\alpha > n$ (applied to both h and f on the boundaries, see Theorem 3.1). This seemingly natural approach does not lead to a sharp result. Creating and flatting cusp singularities through mappings of bi-conformal energy is in a whole different scale, as stated in (1.1). Even more, Theorem 1.1 is a corollary of the following result.

Theorem 1.11 *(Main Theorem) Let* $n \geqslant 3$ *and*

$$u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^\alpha} \quad \text{for } 0 \leqslant t \leqslant 1\,, \ \text{where } \alpha > 0.$$

For $\alpha \geqslant n$ *there is no homeomorphism* $h\colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ *with finite conformal energy whose inverse* $h^{-1} = f \in \mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$*,* $p > n-1$*. If* $\alpha < n$*, then there exists a homeomorphism* $h\colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ *with finite conformal energy such that* f *is Lipschitz.*

2 Prerequisites

Our notation is fairly standard. Throughout the paper $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$. We write $C, C_1, C_2, \ldots$ as generic positive constants. These constants may change even in a single string of estimates. The dependence of constant on a parameter p is expressed by the notation $C = C(p) = C_p$ if needed.

We will appeal to the Sobolev embedding on spheres, see [13, Lemma 2.19].

Lemma 2.1 *Let* $h\colon \mathbb{B} \to \mathbb{R}^n$ *be a continuous mapping in the Sobolev class* $\mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$*, for some* $p > n-1$*. Then for almost every* $0 < t < 1$ *and every* $x, y \in \partial\mathbb{B}(0,t) = \mathbb{S}_t$*, we have*

$$|h(x) - h(y)| \leqslant C\, t^{1-\frac{n-1}{p}} \left(\int_{\mathbb{S}_t} |Dh(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} .$$

Here the constant C *depends only on* n *and* p*.*

It is relatively easy to conclude from this estimate that a $\mathscr{W}^{1,p}$-homeomorphism when $p > n-1$ is differentiable almost everywhere. It also follows that a homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ satisfies Lusin's condition (N). This simply means, by definition, that $|h(E)| = 0$ whenever $|E| = 0$.

Lemma 2.2 *Let $\mathbb{X}, \mathbb{Y}$ be domains in $\mathbb{R}^n$ and $h\colon \mathbb{X} \xrightarrow{onto} \mathbb{Y}$ be a homeomorphism in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. Then h is differentiable almost everywhere and satisfies Lusin's condition (N).*

Due to Lusin's condition (N) we have the following version of change of variables formula, see, e.g., [16, Theorem 6.3.2] or [13, Corollary A.36].

Lemma 2.3 *Let $h\colon \mathbb{X} \xrightarrow{onto} \mathbb{Y}$ be a homeomorphism in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$. If η is a nonnegative Borel measurable function on $\mathbb{R}^n$ and A is a Borel measurable set in $\mathbb{X}$, then we have*

$$\int_A \eta(h(x))|J_h(x)|\,\mathrm{d}x = \int_{h(A)} \eta(y)\,\mathrm{d}y. \tag{2.1}$$

Next, we recall a well-known fact that a function in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R})$, $\mathbb{X} \subset \mathbb{R}^n$, has a representative which is locally Hölder continuous with exponent $1 - p/n$, provided $p > n$. More precisely, we have the following oscillation lemma.

Lemma 2.4 *Let $u \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R})$ where $\mathbb{X} \subset \mathbb{R}^n$ and $p > n$. Then*

$$|u(x) - u(y)| \leqslant C\, r^{1-\frac{n}{p}} \left(\int_{\mathbb{B}_r} |\nabla u|^p \right)^{\frac{1}{p}}$$

for every $x, y \in \mathbb{B}_r = \mathbb{B}(z, r) \subset \mathbb{X}$.

We will employ a higher dimension version of the classical Jordan curve theorem due to Brouwer [5], see also [25, Theorem 6.35].

Lemma 2.5 (Jordan–Brouwer separation theorem) *A topological $(n-1)$-sphere $\mathcal{S}$ disconnects $\mathbb{R}^n$ into a bounded component $\mathcal{S}_\circ$ and an unbounded component $\mathcal{S}_\infty$. Their common boundary is $\overline{\mathcal{S}_\circ} \cap \overline{\mathcal{S}_\infty} = \mathcal{S}$.*

A homeomorphism $h\colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ of finite conformal energy extends as a continuous map $h\colon \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}}_u$. This follows from the following result, see [17, Theorem 1.3].

Lemma 2.6 *Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded domains of finite connectivity. Suppose $\partial\mathbb{X}$ is locally quasiconformally flat and $\partial\mathbb{Y}$ is a neighborhood retract. Then every homeomorphism $h\colon \mathbb{X} \xrightarrow{onto} \mathbb{Y}$ in the class $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ extends to a continuous map $h\colon \overline{\mathbb{X}} \xrightarrow{onto} \overline{\mathbb{Y}}$.*

The assumed boundary regularities are defined as follows.

Definition 2.7 The boundary $\partial\mathbb{Y}$ is a *neighborhood retract*, if there is a neighborhood $\mathbb{U} \subset \mathbb{R}^n$ of $\partial\mathbb{Y}$ and a continuous map $\chi : \mathbb{U} \to \partial\mathbb{Y}$ which is an identity on $\partial\mathbb{Y}$.

Definition 2.8 The boundary $\partial\mathbb{X}$ is said to be *locally quasiconformally flat* if every point in $\partial\mathbb{X}$ has a neighborhood $\mathbb{U} \subset \mathbb{R}^n$ and a homeomorphism $g : \mathbb{U} \cap \overline{\mathbb{X}} \xrightarrow{\text{onto}} \mathbb{B} \cap (\mathbb{R}^{n-1} \times \mathbb{R}^+)$ which is quasiconformal on $\mathbb{U} \cap \mathbb{X}$; see [27].

Recall that $\mathbb{R}^+ = [0, \infty)$. It is also known that a mapping of bi-conformal energy between domains with locally quasiconformally flat boundaries has a homeomorphic extension up to the boundary, see [17, Corollary 1.1]. Note that $\partial\mathbb{B}_u$ is not locally quasiconformally flat and this result does not apply in our case.

Nevertheless, Lemma 2.6 tells us that h extends as a continuous mapping $h : \overline{\mathbb{B}} \to \overline{\mathbb{B}_u}$. Since $h(\overline{\mathbb{B}})$ is a compact subset of $\overline{\mathbb{B}_u}$, it follows that h takes $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}_u}$. Second, it is a topological fact that such a continuous extension is a monotone mapping $h : \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$:

Proposition 2.9 *[8] Suppose that there is a continuous extension $G : \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}}$ of a homeomorphism $g : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}$. Then $G : \partial\mathbb{B} \xrightarrow{\text{onto}} \partial\mathbb{B}$ is monotone.*

By the definition, monotonicity, the concept of Morrey [23], simply means that for a continuous $h : \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ the preimage $h^{-1}(y_\circ)$ of a point $y_\circ \in \overline{\mathbb{Y}}$ is a connected set in $\overline{\mathbb{X}}$. It is worth noting that the converse statement of Proposition 2.9 is also valid when $n = 2, 3$. Such an elegant characterization of monotone mappings of a 2-sphere onto itself was obtained by Floyd and Fort [9].

In the next lemmas we will analyze the boundary behavior of continuous extension of homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ with finite conformal energy which we will still denote by $h : \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$. The following claim follows from Lemma 2.6 and Proposition 2.9.

Lemma 2.10 *Suppose a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ lies in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{R}^n)$. Then for every $x \in \partial\mathbb{B}_u$ the preimage $h^{-1}(x)$ is a nonempty continuum in $\partial\mathbb{B}$.*

Simplifying writing we set $o' \stackrel{\text{def}}{=} (1, 0, ..., 0) \in \partial\mathbb{B}$ and $o \stackrel{\text{def}}{=} (0, 0, ..., 0) \in \partial\mathbb{B}_u$. Without loss of generality, we may and will assume that $h(o') = o$. For every $0 < t < 1$, we define

$$S_t \stackrel{\text{def}}{=} \{x \in \mathbb{B}_u : |x| = t\} \quad \text{and} \quad C_t \stackrel{\text{def}}{=} \{x \in \partial\mathbb{B}_u : |x| = t\} ,$$

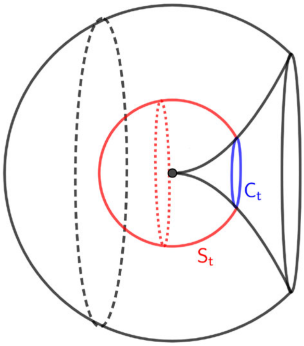
see Fig. 4. Note that here $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^n$.

Furthermore, let $S_t' \stackrel{\text{def}}{=} h^{-1}(S_t)$ and $C_t' \stackrel{\text{def}}{=} \overline{S_t'} \cap \partial\mathbb{B}$. Since $h : S_t' \xrightarrow{\text{onto}} S_t$ and $\overline{S_t'}$ is compact, the extension of h is also surjective and we have $h : \overline{S_t'} \xrightarrow{\text{onto}} \overline{S_t}$. We state this fact as a lemma.

Lemma 2.11 *Suppose a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ lies in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{R}^n)$. Then we have $h(C_t') = C_t$.*

The next lemma shows that the Sobolev embedding on spheres, Lemma 2.1, also holds on S_t. In particular, we will need its variant on C_t, see Fig. 4.

Fig. 4 S_t and C_t



Lemma 2.12 *Suppose that a homeomorphism* $h\colon \mathbb{B} \xrightarrow{onto} \mathbb{B}_u$ *has finite conformal energy. If the inverse mapping* $f = h^{-1}\colon \mathbb{B}_u \to \mathbb{B}$ *belongs to the Sobolev class* $\mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$ *for some* $p > n-1$*, then for almost every* $0 < t < 1$ *and every* $x'_t, y'_t \in C'_t$ *we have*

$$|x'_t - y'_t| \leqslant C|x_t - y_t|^{1-\frac{n-1}{p}} \left(\int_{S_t} |Df|^p \mathrm{d}x\right)^{\frac{1}{p}}. \tag{2.2}$$

Here $x_t = h(x'_t)$ *and* $y_t = h(y'_t)$ *and* C *is a positive constant independent of* t, x_t *and* y_t.

Proof Let $x'_t, y'_t \in C'_t$. By Lemma 2.11 there are two sequences $\{x'_{t,i}\}_{i=1}^{\infty}$ and $\{y'_{t,i}\}_{i=1}^{\infty}$ in S'_t such that

$$\lim_{i\to\infty} x'_{t,i} = x'_t, \quad \lim_{i\to\infty} y'_{t,i} = y'_t$$

and

$$\lim_{i\to\infty} x_{t,i} = x_t \in C_t, \quad \lim_{i\to\infty} y_{t,i} = y_t \in C_t.$$

Here,

$$x_{t,i} = h(x'_{t,i}), \quad y_{t,i} = h(y'_{t,i}), \quad x_t = h(x'_t) \text{ and } \quad y_t = h(y'_t).$$

By the classical Sobolev embedding on sphere, Lemma 2.1, we have

$$|x'_{t,i} - y'_{t,i}| \leqslant C|x_{t,i} - y_{t,i}|^{1-\frac{n-1}{p}} \left(\int_{S_t} |Df|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Passing to the limit, we obtain

$$|x'_t - y'_t| \leqslant C|x_t - y_t|^{1-\frac{n-1}{p}} \left(\int_{S_t} |Df|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

If $f \in \mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$, $p > n-1$, then there is a decreasing sequence $\{t_i\}_{i=1}^{\infty}$ with $0 < t_1 < 1$, which converges to 0, and satisfies (2.2) and □

$$\int_{S_{t_i}} |Df|^p \,\mathrm{d}x < \frac{1}{t_i}.$$

Indeed, if not, then by Fubini's theorem for some $T \in (0,1)$ we have

$$\int_{\mathbb{B}_u} |Df(x)|^p \,\mathrm{d}x \geqslant \int_0^T \int_{S_t} |Df(x)|^p \,\mathrm{d}x\mathrm{d}t \geqslant \int_0^T \frac{1}{t}\,\mathrm{d}t = \infty.$$

Without loss of generality, we may also assume that $\operatorname{diam} C'_{t_i}$ is decreasing with respect to t_i and $\operatorname{diam} C'_{t_1} < \frac{1}{4}$.

According to Lemmas 2.11 and 2.12 we have that $h \colon C'_t \xrightarrow{\text{onto}} C_t$ is a homeomorphism. Now, Jordan–Brouwer Separation Theorem, Lemma 2.5, yields the following result.

Lemma 2.13 *Suppose that a homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ has finite conformal energy and the inverse mapping $f = h^{-1} \colon \mathbb{B}_u \to \mathbb{B}$ belongs to the Sobolev class $\mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$ for some $p > n-1$. Then $\partial\mathbb{B} \setminus C'_t$ consists of two disjoint connected open sets whose common boundary is C'_t.*

The boundary mapping $h \colon \partial\mathbb{B} \xrightarrow{\text{onto}} \partial\mathbb{B}_u$ is monotone. We can say more about the preimage of the singular point o.

Lemma 2.14 *Suppose that a homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ has finite conformal energy and the inverse mapping $f = h^{-1} \colon \mathbb{B}_u \to \mathbb{B}$ belongs to the Sobolev class $\mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$ for some $p > n-1$. Then we have $h^{-1}(o) = o'$.*

Proof According to Lemma 2.13, $\partial\mathbb{B} \setminus C'_t$ consists of two disjoint connected open sets in $\partial\mathbb{B}$ whose common boundary is C'_t. We denote the one with smaller diameter by $\mathbb{U}_t$. Now, for $0 < t < \tau < t_1$, we have $U_t \subset U_\tau$ and we denote $U_\circ \overset{\text{def}}{=\!=} \lim_{t\to 0} \overline{U_t}$. Combining this with continuity of $h \colon \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$, we obtain

$$h(U_\circ) = \lim_{t\to 0} h(\overline{U_t}). \tag{2.3}$$

By Lemma 2.11 $h(C'_t) = C_t$. Since further $C'_t \subset \overline{U_t}$ and $\lim_{t\to 0} C_t = o$ we have $o \in h(U_\circ) \subset h(\overline{U_t})$ for every $0 < t < t_1$. By Lemma 2.10 $h^{-1}(o)$ is connected. Thus we obtain that $h^{-1}(o) \subset \overline{U_t}$ for every $0 < t < t_1$. By Lemma 2.12, $\operatorname{diam} C'_t$ will converge to 0 as t goes to 0. Therefore, also the diameter of $\overline{U_t}$ approaches 0. Hence $h^{-1}(o) = o'$. □

Our last lemma in this section gives a precise modulus of continuity estimate for a homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ with finite conformal energy. Recall that such a homeomorphism has a continuous extension up to the boundary. Furthermore, the boundary mapping $h \colon \partial\mathbb{B} \xrightarrow{\text{onto}} \partial\mathbb{B}_u$ is monotone in the sense of Morrey, see Lemma 2.10.

Monotone mappings enjoy a property which is commonly known in literature also as monotonicity. This notion goes back to H. Lebesgue [21] in 1907. To avoid confusion, in the following definition we use the term monotone in the sense of Lebesgue.

Definition 2.15 Let $\mathbb{X}$ be an open subset of $\mathbb{R}^n$. A continuous mapping $h\colon \overline{\mathbb{X}} \to \mathbb{R}^n$ is *monotone in the sense of Lebesgue* if for every compact set $K \subset \overline{\mathbb{X}}$ we have

$$\operatorname{diam} h(K) = \operatorname{diam} h(\partial K). \tag{2.4}$$

Note that for real-valued functions (2.4) can be stated as

$$\min_K h = \min_{\partial K} h \leqslant \max_{\partial K} h = \max_K h.$$

Remark 2.16 A folding map is a characteristic example of continuous nonmonotone mapping which is monotone in the sense of Lebesgue.

Lemma 2.17 *Let $h\colon \mathbb{B} \to \mathbb{B}_u$ be a homeomorphism with finite conformal energy. If $h(o') = o$, then there exists an increasing function $\varepsilon\colon [0,1) \to [0,\infty)$ with $\lim\limits_{t\to 0+} \varepsilon(t) = 0$ such that for $x' \in \overline{\mathbb{B}}$ with $0 < |x'-o'| < 1$ we have*

$$|h(x') - h(o')| \leqslant \frac{\varepsilon(|x'-o'|)}{\log^{\frac{1}{n}}\left(\frac{1}{|x'-o'|}\right)}. \tag{2.5}$$

Proof Set

$$\mathcal{S}_t \stackrel{\text{def}}{=\!=} \partial\mathbb{B}(o',t) \cap \overline{\mathbb{B}},$$

and

$$\operatorname{osc}(h,\mathcal{S}_t) \stackrel{\text{def}}{=\!=} \max_{x'_t, y'_t \in \mathcal{S}_t} |h(x'_t) - h(y'_t)|.$$

Since $h\colon \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$ is continuous and belongs to the Sobolev class $\mathscr{W}^{1,n}(\mathbb{B},\mathbb{R}^n)$, applying a slightly modified version of the Sobolev embedding on sphere, Lemma 2.1 for almost every $0 < t < 1$ we have

$$(\operatorname{osc}(h,\mathcal{S}_t))^n \leqslant C t \int_{\mathcal{S}_t} |Dh(x)|^n \, \mathrm{d}x. \tag{2.6}$$

Here C is a positive constant, independent of t. Fix $x' \in \mathbb{B}$ such that $\tau \stackrel{\text{def}}{=\!=} |x'-o'| < 1$. We write

$$\mathcal{B}(o',t) \stackrel{\text{def}}{=\!=} \mathbb{B} \cap \mathbb{B}(o',t) \quad \text{for } 0 < t < 1.$$

Choose $t \in [\tau, \sqrt{\tau}]$. Then

$$\operatorname{osc}(h, \overline{\mathcal{B}(o', \tau)}) \leqslant \operatorname{osc}(h, \overline{\mathcal{B}(o', t)}) \leqslant \operatorname{osc}(h, \partial\overline{\mathcal{B}(o', t)}) \,,$$

where the latter inequality follows from the fact that h is monotone in the sense of Lebesgue. Since $\mathcal{S}_t = \partial\overline{\mathcal{B}(o', t)} \cap \mathbb{B}$ and h is monotone in the sense of Lebesgue, we have

$$\operatorname{osc}(h, \partial\overline{\mathcal{B}(o', t)}) = \operatorname{osc}(h, \mathcal{S}_t).$$

Combining this with (2.6) for almost every $t \in [\tau, \sqrt{\tau}]$ we have

$$\frac{\left(\operatorname{osc}(h, \overline{\mathcal{B}(o', \tau)})\right)^n}{t} = C \int_{\mathcal{S}_t} |Dh(x)|^n \, \mathrm{d}x.$$

Integrating this from τ to $\sqrt{\tau}$ with respect to the variable t, the claimed inequality (2.5) follows with

$$\varepsilon(\tau) = C \cdot \left(\int_{\mathcal{B}(o', \sqrt{\tau})} |Dh(x)|^n \mathrm{d}x \right)^{\frac{1}{n}}, \qquad \tau = |x' - o'|. \tag{2.7}$$

□

3 Homeomorphic Boundary Extension

Lemma 2.6 shows that a homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ of finite conformal energy can be extended as a continuous mapping from $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}_u}$. In this section we will prove that a homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ of bi-conformal energy extends as a homeomorphism up to the boundary.

Theorem 3.1 *Let $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ be a homeomorphism of finite bi-conformal energy. Then h admits a homeomorphic extension to the boundary, again denoted by $h \colon \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$.*

The existence of such an extension is known [17, Corollary 1.1] if the reference and deformed configurations have locally quasiconformally flat boundaries, see Definition 2.8. Obviously, $\partial\mathbb{B}_u$ is not locally quasiconformally flat.

Proof of Theorem 3.1 By Lemma 2.6 a homeomorphism $h \colon \overline{\mathbb{B}} \to \overline{\mathbb{B}_u}$ with finite conformal energy extends as a continuous mapping $h \colon \overline{\mathbb{B}} \to \overline{\mathbb{B}_u}$. Since $h(\overline{\mathbb{B}})$ is a compact subset of $\overline{\mathbb{B}_u}$, it follows that $h \colon \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$. Furthermore, by Lemma 2.10 the boundary map $h \colon \partial\mathbb{B} \xrightarrow{\text{onto}} \partial\mathbb{B}_u$ is monotone.

Now, we need to show that the boundary mapping is injective. We again use the notation $o = (0, 0, ..., 0)$ and $o' = (1, 0, ..., 0)$ and assume, without loss of generality, that $h(o') = o$. First, $h^{-1}(o) = o'$ by Lemma 2.14. Second let $y \in \partial\mathbb{B}_u \setminus \{o\}$. Choosing

$0 < r_y < |y-o|$, then $\mathbb{B}(y, r_y) \cap \mathbb{B}_u$ is locally quasiconformally flat. By Lemma 2.6, the homeomorphism $f : \mathbb{B}(y, r_y) \cap \mathbb{B}_u \xrightarrow{\text{onto}} f(\mathbb{B}(y, r_y) \cap \mathbb{B}_u)$ has a continuous extension $f : \overline{\mathbb{B}(y, r_y) \cap \mathbb{B}_u} \xrightarrow{\text{onto}} \overline{f(\mathbb{B}(y, r_y) \cap \mathbb{B}_u)}$. The extension of f is still an inverse of h in the quasiconformally flat part of the boundary; that is, $h^{-1}(y) = f(y)$ is a single point. Now we know that $h : \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$ is a continuous bijection, and therefore it is a homeomorphism. $\square$

4 Construction of Example 1.8

Here we show that there exists a homeomorphism from $\mathbb{B} \setminus \mathbf{I}$ onto $\mathbb{B}$ with finite conformal energy, actually Lipschitz continuous, whose inverse lies in $\mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$ for every $p < n$. To simplify our construction, we may and do replace $\mathbb{B}$ by a bi-Lipschitz equivalent domain; namely,

$$\mathbb{Y} = \{(s, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |y| < 1 \text{ and } -1 < s < |y|\}.$$

As for the reference configuration we replace $\mathbb{B} \setminus \mathbf{I}$ by a cylinder $\mathbf{C} = (-1, 1) \times \mathbb{B}^{n-1}$ with the line segment $\mathbf{I}$ removed from it. Consider the Lipschitz homeomorphism $h : \mathbf{C} \setminus \mathbf{I} \xrightarrow{\text{onto}} \mathbb{Y}$ defined by the rule

$$h(t, x) = \begin{cases} (t|x|,\ x) & \text{for } t > 0, \\ (t, x) & \text{for } t < 0. \end{cases} \tag{4.1}$$

Its inverse mapping $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbf{C} \setminus \mathbf{I}$ takes the form

$$f(s, y) = \begin{cases} \left(\frac{s}{|y|},\ y\right) & \text{for } s \geqslant 0, \\ (s, y) & \text{for } s < 0. \end{cases} \tag{4.2}$$

It is easy to see that

$$|Df(s, y)| \leqslant \frac{C_n}{|y|}.$$

Therefore,

$$\int_{\mathbb{Y}} |Df|^p < \infty \qquad \text{for every } 1 \leqslant p < n$$

as desired.

5 Proof of Theorem 1.9

5.1 The Nonexistence Part of Theorem 1.9

First, we will prove the nonexistence part of Theorem 1.9.

Theorem 5.1 *If $p > n-1$, then there is no homeomorphism $h\colon \mathbb{B} \overset{onto}{\longrightarrow} \mathbb{B} \setminus \mathbf{I}$ with $h \in \mathscr{W}^{1,n}(\mathbb{B}, \mathbb{B} \setminus \mathbf{I})$ whose inverse $f = h^{-1} \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{B})$.*

Proof Suppose to the contrary that there is a homeomorphism $h\colon \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B} \setminus \mathbf{I}$ in the Sobolev class $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{B} \setminus \mathbf{I})$ such that $f \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{B})$. Since $\partial(\mathbb{B} \setminus \mathbf{I})$ is a neighborhood retract, Lemma 2.6 tells us that the homeomorphism $h\colon \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B} \setminus \mathbf{I}$ extends as a continuous mapping $h\colon \overline{\mathbb{B}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{B}}$. We denote

$$S_t = \partial B_t \setminus \{x_t\}\,, \qquad \text{where } x_t \overset{\text{def}}{=\!=} (t, 0, \dots, 0) \quad 0 < t < s < 1.$$

Here $B_t = B(0,t)$. Fubini's theorem implies that for almost every $t \in (0,1)$, $f\big|_{S_t} \in \mathscr{W}^{1,p}(S_t, \mathbb{R}^n)$. Since $p > n-1$ and $n \geqslant 3$, the possible singularity of f at x_t is removable. For such t, applying Lemma 2.4, $f\big|_{S_t}$ extends as a homeomorphism $f\colon \overline{S_t} \overset{\text{onto}}{\longrightarrow} f(\overline{S_t})$. Write $x_t' = f(x_t) \in \partial\mathbb{B}$. Now, Jordan–Brouwer Separation Theorem (Lemma 2.5) tells us that $\mathbb{R}^n \setminus f(\overline{S_t})$ consists of two disjoint connected open sets whose common boundary is $f(\overline{S_t})$. Let us denote the bounded one by U_t. Note that $U_t \subset \mathbb{B}$ and $\overline{U_t} \cap \partial\mathbb{B} = \{x_t'\}$. Since for almost every $t < s \in (0,1)$ we have $B_t \setminus \mathbf{I} \subset B_s \setminus \mathbf{I}$ then $U_t = h^{-1}(B_t \setminus \mathbf{I}) \subset h^{-1}(B_s \setminus \mathbf{I}) = U_s$.

Now comes an elementary topological fact: *given two domains $U \subset V \subset \mathbb{B}$ such that $\overline{U} \cap \partial\mathbb{B} = \{x_\nu\}$ and $\overline{V} \cap \partial\mathbb{B} = \{x_\mu\}$, then $x_\nu = x_\mu$* (Fig. 5).

Now, we have $x_s' = x_t'$. This, however, is impossible since $h(x_s') = (s, 0, \dots, 0)$ and $h(x_t') = (t, 0, \dots, 0)$.

□

5.2 The Existence Part of Theorem 1.9

Here we verify the existence part of Theorem 1.9. Namely,

Theorem 5.2 *There exists a Lipschitz homeomorphism $h\colon \mathbb{B} \to \mathbb{B} \setminus \mathbf{I}$ whose inverse $f \in \mathscr{W}^{1,p}(\mathbb{B} \setminus \mathbf{I}, \mathbb{B})$ for every $1 \leqslant p < n-1$.*

Proof We shall view $\mathbb{R}^n$ as

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t,x)\colon t \in \mathbb{R}\,,\ x \in \mathbb{R}^{n-1}\}.$$

To simplify our construction, we may and do replace $\mathbb{B}$ by a bi-Lipschitz equivalent domain; namely $\mathbb{X} = \mathbb{X}_- \cup \mathbb{X}_+$, where

$$\mathbb{X}_- = \{(t,x)\colon \ -1 < t < 0 \text{ and } |x| < 1\}$$

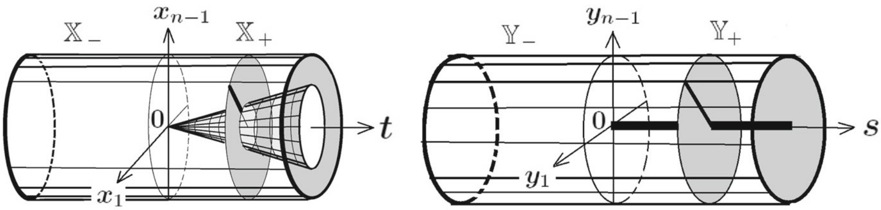


Fig. 5 The domains $\mathbb{X}$ and $\mathbb{Y}$

and

$$\mathbb{X}_+ = \{(t,x)\colon 0 \leqslant t < 1 \text{ and } \frac{t}{2} < |x| < 1\}.$$

As for the reference configuration we consider $\mathbb{Y} = \mathbb{Y}_+ \cup \mathbb{Y}_-$ where $\mathbb{Y}_-$ is the open unit cylinder

$$\mathbb{Y}_- = \{(s,y)\colon \; -1 < s < 0 \text{ and } |y| < 1\}$$

and

$$\mathbb{Y}_+ = \{(s,y)\colon 0 \leqslant s < 1 \text{ and } 0 < |y| < 1\}.$$

We define a Lipschitz map $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by the rule

$$h(t,x) = \begin{cases} (t,x) & \text{in } \mathbb{X}_-, \\ \left(t, \left[\frac{2|x|}{2-t} - \frac{t}{2-t}\right] \frac{x}{|x|}\right) & \text{in } \mathbb{X}_+. \end{cases}$$

Then the inverse map $f = h^{-1}\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes the form

$$f(s,y) = \begin{cases} (s,y) & \text{in } \mathbb{Y}_-, \\ \left(s, \left[\frac{2-s}{2}|y| + \frac{s}{2}\right] \frac{y}{|y|}\right) & \text{in } \mathbb{Y}_+ \end{cases}.$$

It is the identity map on $\mathbb{Y}_-$ while on $\mathbb{Y}_+$ we write it as

$$f(s,y) = \left(s, \frac{2-s}{2} y\right) + \left(0, \frac{sy}{2|y|}\right),$$

where the first term is $\mathscr{C}^\infty$-smooth. It is now easy to verify the estimate

$$|Df(s,y)| \leqslant C \cdot \left(1 + \frac{s}{|y|}\right),$$

where $|s| < 1$ and $y \in \mathbb{R}^{n-1}$, $0 < |y| < 1$. Hence

$$\int_{\mathbb{Y}_+} |Df|^p < \infty \quad \text{for every } 1 \leqslant p < n-1$$

as desired. □

6 Proof of Theorem 1.11

6.1 The Nonexistence Part of Theorem 1.11

Here we give a proof of the nonexistence part of Theorem 1.11. We recall the statement for the convenience of the reader.

Theorem 6.1 *Let $\alpha \geqslant n$ and $p > n-1$ be fixed and $u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}}$. Then there does not exist a homeomorphism $h : \mathbb{B} \to \mathbb{B}_u$ with $h \in \mathscr{W}^{1,n}(\mathbb{B}, \mathbb{B}_u)$ and $h^{-1} \in \mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{B})$.*

Proof Fix $\alpha \geqslant n$ and $p > n-1$. Suppose to the contrary that there exists a homeomorphism $h \colon \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_u$ with finite conformal energy such that its inverse f is in $\mathscr{W}^{1,p}(\mathbb{B}_u, \mathbb{R}^n)$. According to Lemma 2.6, h extends as a continuous mapping $h \colon \overline{\mathbb{B}} \xrightarrow{\text{onto}} \overline{\mathbb{B}_u}$. Furthermore, by Lemma 2.10 the boundary mapping $h \colon \partial\mathbb{B} \xrightarrow{\text{onto}} \partial\mathbb{B}_u$ is monotone.

We follow the notation introduced in Sect. 2 and set $o = (0, 0, \dots, 0)$ and $o' = (1, 0, \dots, 0)$. We may and do assume that $h(o') = o$. Moreover, for every $0 < t < 1$,

$$S_t = \{x \in \mathbb{B}_u \colon |x| = t\} \ \text{ and } \ C_t = \{x \in \partial\mathbb{B}_u \colon |x| = t\}$$

and

$$S'_t = h^{-1}(S_t) \quad \text{and} \quad C'_t = \overline{S'_t} \cap \partial\mathbb{B}.$$

Lemma 2.13 tells us that C'_t divides $\partial\mathbb{B}$ into two disjoint components. We denote the component which contains o' by U'_t. Accordingly, we also have

$$\partial U'_t = C'_t. \tag{6.1}$$

Since

$$\int_{\mathbb{B}_u} |Df(x)|^p \mathrm{d}x < \infty,$$

there exists a decreasing sequence $\{t_i\}$, which converges to 0 and satisfies

$$\int_{S_{t_i}} |Df(x)|^p \mathrm{d}x < \frac{1}{t_i}. \tag{6.2}$$

Indeed, by Fubini's theorem we have

$$\int_0^1 \int_{S_t} |Df(x)|^p \, \mathrm{d}x < \infty.$$

Hence,

$$\liminf_{t \to 0} t \int_{S_t} |Df(x)|^p = 0.$$

Now, by Lemma 2.11, we have $h(C_t') = C_t$. Combining this with Lemma 2.12 we obtain

$$\begin{aligned} \operatorname{diam} C_{t_i}' &\leqslant C \cdot \left(2\, u(t_i)\right)^{1-\frac{n-1}{p}} \left(\int_{S_{t_i}} |Df(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leqslant C \cdot \left(u(t_i)\right)^{1-\frac{n-1}{p}} \left(\frac{1}{t_i}\right)^{\frac{1}{p}} . \end{aligned} \tag{6.3}$$

Here $u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}}$. Especially, this shows that $\operatorname{diam}(C_{t_i}') \to 0$ as $i \to \infty$ and, therefore, U_{t_i}' lies on the half sphere $\partial\mathbb{B}_+$. We now appeal to the geometric fact if $x, a \in U_{t_i}'$, then $|x - a| \leqslant \operatorname{diam} \partial U_{t_i}'$. Now, for large enough i, by (6.1) we fix $x_{t_i}' \in C_{t_i}'$ and then

$$|x_{t_i}' - o'| \leqslant \operatorname{diam} C_{t_i}'. \tag{6.4}$$

According to Lemma 2.17 and (6.4) we obtain

$$t_i \leqslant |h(x_{t_i}') - o| \leqslant \varepsilon(t_i) \log^{-\frac{1}{n}} \frac{1}{|x_{t_i}' - o'|} \leqslant \varepsilon(t_i) \log^{-\frac{1}{n}} \frac{1}{\operatorname{diam} C_{t_i}'} , \tag{6.5}$$

where $\varepsilon(t)$ is a positive function defined in (2.7) which converges to 0 as t goes to 0. The estimates (6.3) and (6.5) imply

$$C \cdot u(t_i) \geqslant \left(\frac{t_i^{\frac{1}{p}}}{\exp\left(\frac{\varepsilon(t_i)}{t_i}\right)^n} \right)^{\frac{p}{p+1-n}} . \tag{6.6}$$

Since $\alpha \geqslant n$ we have $\exp(1/t^n) \leqslant \exp(1/t^\alpha)$ for $0 < t \leqslant 1$ and therefore

$$\frac{C \cdot e}{\exp\left(t^{-n}\right)} \geqslant \left(\frac{t_i^{\frac{1}{p}}}{\exp\left(\frac{\varepsilon(t_i)}{t_i}\right)^n} \right)^{\frac{p}{p+1-n}} .$$

This means that there are constants $C_1, C_2 > 0$ satisfying

$$\varepsilon(t_i) \geqslant C_1 \cdot t_i^n \log\left(C_2\, t_i^{\beta} \exp(t_i^{-n})\right), \quad \beta = \frac{1}{p-n+1}.$$

Letting $i \to \infty$, the right-hand side converses to C_1 and $\varepsilon(t_i) \to 0$. This contradiction completes the proof.

□

6.2 The Existence Part of Theorem 1.11

Theorem 6.2 *Let $u(t) = \frac{e}{\exp(1/t)^{\alpha}}$ for some $0 < \alpha < n$. Then there exists a homeomorphism $h\colon \mathbb{B} \to \mathbb{B}_u$ with finite conformal energy whose inverse $f = h^{-1}\colon \mathbb{B}_u \to \mathbb{B}$ is Lipschitz regular.*

Proof Fix $0 < \alpha < n$ and the corresponding cusp domain $\mathbb{B}_u$ with $u(t) = \frac{e}{\exp(t^{-1})^{\alpha}}$. As in the proof of Theorem 5.2 we write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t, x)\colon t \in \mathbb{R},\ x \in \mathbb{R}^{n-1}\}$$

and replace $\mathbb{B}$ by a bi-Lipschitz equivalent domain, $\mathbb{X} = \mathbb{X}_- \cup \mathbb{X}_+$, where

$$\mathbb{X}_- = \{(t, x)\colon\ -1 < t \leqslant 0 \text{ and } |x| < 1\}$$

and

$$\mathbb{X}_+ = \{(t, x)\colon 0 < t < 1 \text{ and } t < |x| < 1\}.$$

We replace the cusp domain $\mathbb{B}_u$ by the following bi-Lipschitz equivalent domain $\mathbb{Y} = \mathbb{Y}_- \cup \mathbb{Y}_+$, where

$$\mathbb{Y}_- = \{(s, y)\colon\ -1 < s \leqslant 0 \text{ and } |y| < 1\}$$

and

$$\mathbb{Y}_+ = \{(s, y)\colon 0 < s < 1 \text{ and } u(s) < |y| < 1\}.$$

We define $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by

$$h(t, x) = \begin{cases} (t, x) & \text{in } \mathbb{X}_-, \\ \left(\frac{u^{-1}(|x|)}{|x|} t,\ x\right) & \text{in } \mathbb{X}_+. \end{cases}$$

Note that the inverse function $u^{-1}(\eta) = \log^{-\frac{1}{\alpha}}\left(\frac{e}{\eta}\right)$. Then the inverse mapping $f = h^{-1} \colon \mathbb{Y} \overset{\text{onto}}{\longrightarrow} \mathbb{X}$ takes the form

$$f(s, y) = \begin{cases} (s, y) & \text{in } \mathbb{Y}_-, \\ \left(\frac{|y|}{u^{-1}(|y|)} s,\ y\right) & \text{in } \mathbb{Y}_+. \end{cases}$$

Now, f is a Lipschitz regular mapping. Furthermore, we have

$$|Dh(t, x)| \leqslant \frac{C}{|x| \log^{\frac{1}{\alpha}}\left(\frac{e}{|x|}\right)}.$$

Therefore,

$$\int_{\mathbb{X}} |Dh|^n < \infty$$

as claimed. $\square$

Acknowledgements We thank the referee for the valuable comments which were a great help in improving the manuscript.

7 Appendix: Proof of Theorem 1.5

Proof First, we assume that $K_I(\cdot, h) \in \mathscr{L}^1(\mathbb{X})$. Then, Theorem 9.1 in [3] states that a homeomorphism $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ satisfies the claimed identity (1.6) if h has a finite (outer) distortion; that is, there is a function $1 \leqslant K_O(x) < \infty$ such that

$$|Dh(x)|^n \leqslant K_O(x)\, J_h(x) \qquad \text{for almost every } x \in \mathbb{X}. \tag{7.1}$$

The proof, however, only uses a consequence of (7.1) the finite inner inequality (1.5) which is stated in [3, (9.10)].

Second, we assume that $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ and $f = h^{-1} \in \mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{R}^n)$. Then

$$K_I(x, h) = |Df\big(h(x)\big)|^n J_h(x) \quad \text{a.e. } x \in \mathbb{X}. \tag{7.2}$$

Indeed, by Lemma 2.2 both h and f are differentiable almost everywhere. Now, the identity $(f \circ h)(x) = x$, after differentiation, implies that

$$Df(h(x))Dh(x) = \mathbf{I} \quad \text{a.e. in } \mathbb{X}. \tag{7.3}$$

Since both h and f satisfy Lusin's condition (N); that is, preserve sets of zero measure, see Lemma 2.2. This shows that $J_h(x) > 0$ and $J_f(y) > 0$ almost everywhere again we used the fact that h satisfies Lusin's condition (N). Now, the formula (7.2) is a direct consequence of the definition of the inner distortion, Cramer's rule $Dh(x)D^\sharp h(x) = J_h(x)\mathbf{I}$ and (7.3). Indeed,

$$K_I(x,h) = \frac{|D^\sharp h(x)|^n}{|J_h(x)|^{n-1}} = |(Dh(x))^{-1}|^n J_h(x) = |Df\big(h(x)\big)|^n J_h(x).$$

Now the change of variables formula (2.1) gives

$$\int_{\mathbb{X}} K_I(x,h)\,\mathrm{d}x = \int_{\mathbb{Y}} |Df(y)|^n\,\mathrm{d}y.$$

□

Proof of Corollary 1.6 By [16, §6.4] for every $x \in \mathbb{X}$ with $J_h(x) > 0$, we have

$$K_I^{\frac{1}{n-1}}(x,h) \leqslant K_O(x,h) \leqslant K_I^{n-1}(x,h)\,. \tag{7.4}$$

Here $K_O(x,h)$ stands for the smallest function satisfying (7.1). Now, Corollary 1.6 follows immediately from (7.4). □

References

1. Antman, S.S.: Nonlinear Problems of Elasticity. Applied Mathematical Sciences, vol. 107. Springer-Verlag, New York (1995)
2. Astala, K., Iwaniec, T., Martin, G.: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton University Press, Princeton (2009)
3. Astala, K., Iwaniec, T., Martin, G.J., Onninen, J.: Extremal mappings of finite distortion. Proc. Lond. Math. Soc. **91**(3), 655–702 (2005)
4. Ball, J. M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. **63**(4), 337–403 (1976/1977)
5. Brown, M.: A proof of the generalized Schoenflies theorem. Bull. Am. Math. Soc. (N.S.) **66**, 74–76 (1960)
6. Ciarlet, P.G.: Mathematical Elasticity, Vol. I. Three-Dimensional Elasticity. Studies in Mathematics and its Applications. North-Holland Publishing Co, Amsterdam (1988)
7. Csörnyei, M., Hencl, S., Malý, J.: Homeomorphisms in the Sobolev space $W^{1,n-1}$. J. Reine Angew. Math. **644**, 221–235 (2010)
8. Floyd, E.E.: The extension of homeomorphisms. Duke Math. J. **16**, 225–235 (1949)
9. Floyd, E.E., Fort, M.K.: A characterization theorem for monotone mappings. Proc. Am. Math. Soc. **4**, 828–830 (1953)
10. Gehring, F.W., Martin, G.J., Palka, B.P.: An Introduction to the Theory of Higher-dimensional Quasiconformal Mappings, Mathematical Surveys and Monographs, vol. 216. American Mathematical Society, Providence, RI (2017)
11. Gehring, F.W., Väisälä, J.: The coefficients of quasiconformality of domains in space. Acta Math. **114**, 1–70 (1965)
12. Hencl, S., Koskela, P.: Regularity of the inverse of a planar Sobolev homeomorphism. Arch. Ration. Mech. Anal. **80**(1), 75–95 (2006)
13. Hencl, S., Koskela, P.: Lectures on Mappings of Finite Distortion. Lecture Notes in Mathematics, vol. 2096. Springer, Cham (2014)
14. Hencl, S., Koskela, P., Onninen, J.: A note on extremal mappings of finite distortion. Math. Res. Lett. **12**(2–3), 231–237 (2005)
15. Hencl, S., Rajala, K.: Optimal assumptions for discreteness. Arch. Ration. Mech. Anal. **207**(3), 775–783 (2013)

16. Iwaniec, T., Martin, G.: Geometric Function Theory and Non-linear Analysis. Oxford Mathematical Monographs. Oxford University Press, Oxford (2001)
17. Iwaniec, T., Onninen, J.: Deformations of finite conformal energy: boundary behavior and limit theorems. Trans. Am. Math. Soc. **363**(11), 5605–5648 (2011)
18. Iwaniec, T., Onninen, J.: Variational Integrals in Geometric Function Theory. Book in progress
19. Iwaniec, T., Onninen, J., Zhu, Z.: Deformations of bi-conformal energy and a new characterization of quasiconformality. arXiv:1904.03793
20. Iwaniec, T., Šverák, V.: On mappings with integrable dilatation. Proc. Am. Math. Soc. **118**(1), 181–188 (1993)
21. Lebesgue, H.: Sur le probléme de Dirichlet. Rend. Circ. Palermo **27**, 371–402 (1907)
22. Marsden, J.E., Hughes, T.J.R.: Mathematical Foundations of Elasticity. Dover Publications Inc, New York (1994)
23. Morrey, C.B.: The topology of (path) surfaces. Am. J. Math. **57**(1), 17–50 (1935)
24. Onninen, J.: Regularity of the inverse of spatial mappings with finite distortion. Calc. Var. Partial Differ. Equ. **26**(3), 331–341 (2006)
25. Rotman, J.J.: An Introduction to Algebraic Topology. Graduate Texts in Mathematics, vol. 119. Springer-Verlag, New York (1988)
26. Reshetnyak, YuG: Space Mappings with Bounded Distortion. American Mathematical Society, Providence, RI (1989)
27. Väisälä, J.: Lectures on n-Dimensional Quasiconformal Mappings. Lecture Notes in Mathematics, vol. 229. Springer-Verlag, Berlin, New York (1971)