

Research Article

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The Nitsche phenomenon for weighted Dirichlet energy

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Abstract: The present paper arose from recent studies of energy-minimal deformations of planar domains. We are concerned with the Dirichlet energy. In general the minimal mappings need not be homeomorphisms. In fact, a part of the domain near its boundary may collapse into the boundary of the target domain. In mathematical models of nonlinear elasticity this is interpreted as *interpenetration of matter*. We call such occurrence the *Nitsche phenomenon*, after Nitsche’s remarkable conjecture (now a theorem) about existence of harmonic homeomorphisms between annuli. Indeed the round annuli proved to be perfect choices to grasp the nuances of the problem. Several papers are devoted to a study of deformations of annuli. Because of rotational symmetry it seems likely that the Dirichlet energy-minimal deformations are radial maps. That is why we confine ourselves to radial minimal mappings. The novelty lies in the presence of a weight in the Dirichlet integral. We observe the Nitsche phenomenon in this case as well, see our main results Theorem 1.4 and Theorem 1.7. However, the arguments require further considerations and new ingredients. One must overcome the inherent difficulties arising from discontinuity of the weight. The Lagrange–Euler equation is unavailable, because the outer variation violates the principle of none interpenetration of matter. Inner variation, on the other hand, leads to an equation that involves the derivative of the weight. But our weight function is only measurable which is the main challenge of the present paper.

Keywords: Weighted Dirichlet energy, harmonic mappings, variational integrals

MSC 2010: 31A05, 30C70, 35J50

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1 Introduction

This paper has its origin in the study of energy-minimal homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ between 2-dimensional annuli, where

$$\mathbb{A} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : r < |z| < R\} \quad \text{and} \quad \mathbb{A}^* \stackrel{\text{def}}{=} \{z_* \in \mathbb{C} : r_* < |z_*| < R_*\}.$$

Hereafter,

$$0 < r < R < \infty \quad \text{and} \quad 0 < r_* < R_* < \infty.$$

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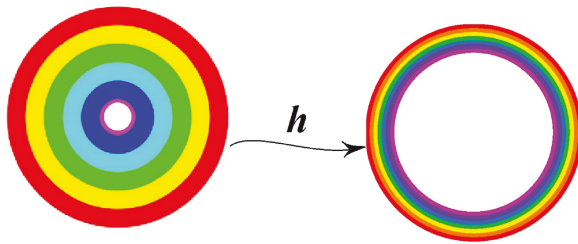


Figure 1: Please provide a caption.

In this paper we are concerned with the so-called λ -Dirichlet energy (or simply, Dirichlet energy if $\lambda \equiv 1$) defined by

$$\mathcal{D}_\lambda[h] \stackrel{\text{def}}{=} \frac{1}{2\pi} \iint_{\mathbb{A}} |Dh(z)|^2 \lambda(z, h) \, dz \quad (\text{denoted by } \mathcal{D}[h] \text{ if } \lambda \equiv 1). \quad (1.1)$$

Throughout this paper our standing assumption on the weight λ is that it is a measurable function of the form $\lambda(z, h) = \lambda(|z|)$, where

$$0 < m \leq \lambda(t) \leq M < \infty \quad \text{for almost every } r < t < R. \quad (1.2)$$

The reader is referred to Section 1.3 for comments concerning more general weights.

We confine ourselves to discussing the radial mappings and their *normal strain functions*

$$H : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$$

given by

$$h(z) = H(|z|) \frac{z}{|z|}.$$

Thus (1.1) reduces to a line integral, called λ -harmonic energy of H ,

$$\mathcal{E}_\lambda[H] \stackrel{\text{def}}{=} \int_r^R \mathbf{E}(t, H) \lambda(t) \, dt, \quad \mathbf{E}(t, H) = t \dot{H}^2(t) + t^{-1} H^2(t). \quad (1.3)$$

The natural domain of definition of $\mathcal{E}_\lambda$ consists of Sobolev functions $H \in \mathcal{W}^{1,2}(r, R) \subset \mathcal{C}[r, R]$. In particular, such an H is absolutely continuous on the closed interval $[r, R]$. Throughout what follows, the values of H at the endpoints will be fixed

$$H(r) = r_* < R_* = H(R). \quad (1.4)$$

We discuss two types of energy-minimization problems. The first one is the classical approach to the Dirichlet boundary value problem. That is, solutions are none other than the critical points of the functional $\mathcal{E}_\lambda$. We include this fact for completeness.

Proposition 1.1 (Unconditional Dirichlet boundary-value problem). *There is unique $\mathcal{H} \in \mathcal{W}^{1,2}(r, R)$ subject to the boundary values (1.4) with smallest λ -energy. Equivalently, $\mathcal{H}$ is the unique solution to the Lagrange–Euler equation for $\mathcal{E}_\lambda$. Such an $\mathcal{H}$ is called λ -harmonic curve.*

1.1 λ -harmonic curves

Definition 1.2. Given an interval $(a, b) \subset [r, R]$, a function $\mathcal{H} \in \mathcal{W}^{1,2}(a, b)$ is called λ -harmonic curve, simply λ -curve, if it satisfies the Lagrange–Euler equation, briefly the λ -harmonic equation

$$\frac{d}{dt} [t \lambda(t) \dot{\mathcal{H}}(t)] = \frac{\lambda(t)}{t} \mathcal{H}(t) \quad \text{almost everywhere in } (a, b). \quad (1.5)$$

Hereafter the differential operator $\frac{d}{dt} : \mathcal{L}^2(a, b) \rightarrow \mathcal{D}'(a, b)$ is understood in the sense of Schwartz distribu-

tions; that is,

$$\int_a^b t\lambda(t)\mathcal{H}(t)\dot{\Theta}(t) dt = - \int_a^b \frac{\lambda(t)\mathcal{H}(t)}{t} \Theta(t) dt \quad (1.6)$$

for every test function $\Theta \in \mathcal{W}_0^{1,2}(a, b) \subset \mathcal{C}_0^\infty(a, b)$.

Throughout this text we reserve the letter $\mathcal{H}$ for λ -harmonic curves. The space of λ -harmonic curves will be designated by the symbol $\mathcal{H}_\lambda(a, b)$.

The existence and uniqueness of a λ -curve $\mathcal{H} \in \mathcal{H}_\lambda(a, b)$ subject to the given boundary values $\mathcal{H}(a) = a_*$ and $\mathcal{H}(b) = b_*$ follows by variational principles, see Section 2.2 for some details. Precisely, such a λ -curve $\mathcal{H} \in \mathcal{H}_\lambda(a, b)$ is uniquely characterized through the minimum-energy property

$$\int_a^b \mathbf{E}(t, \mathcal{H})\lambda(t) dt \leq \int_a^b \mathbf{E}(t, H)\lambda(t) dt \quad \text{for all } H \in \mathcal{W}^{1,2}(a, b) \text{ such that } H(a) = \begin{pmatrix} a_* \\ H(b) \end{pmatrix} = b_*. \quad (1.7)$$

Of particular interest to us will be the nondecreasing curves.

Problem 1.3 (Conditional Dirichlet problem). Among all functions $H \in \mathcal{W}^{1,2}(r, R)$ subject to the boundary values at (1.4) and satisfying

$$\dot{H}(t) \geq 0 \quad \text{almost everywhere in } (r, R)$$

find the one of smallest λ -energy.

Here the additional constraint $\dot{H}(t) \geq 0$, which makes H nondecreasing, is imposed in order to capture the curves whose λ -harmonic energy equals exactly the infimum of energies among all homeomorphisms $H : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$. In general, the infimum energy need not be attained within homeomorphisms, injectivity is lost in passing to the limit of the energy-minimizing sequence. In mathematical models of nonlinear elasticity this is interpreted as *interpenetration of matter* [3, 36]. The infimum, however, is always attained within the $\mathcal{W}^{1,2}$ -closure of homeomorphisms. The latter consists exactly of functions having nonnegative derivative almost everywhere. For more about the limits of Sobolev homeomorphisms we refer to [24, 25].

Our main result reads as follows.

Theorem 1.4. Among all nondecreasing curves $H : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ of Sobolev class $\mathcal{W}^{1,2}(r, R)$ there exists exactly one with smallest λ -harmonic energy. We call it λ -minimal curve, briefly, minimal curve.

The bold-letter font **H** will be used for λ -minimal curves. Figure 2 below illustrates typical noninjective minimal curve.

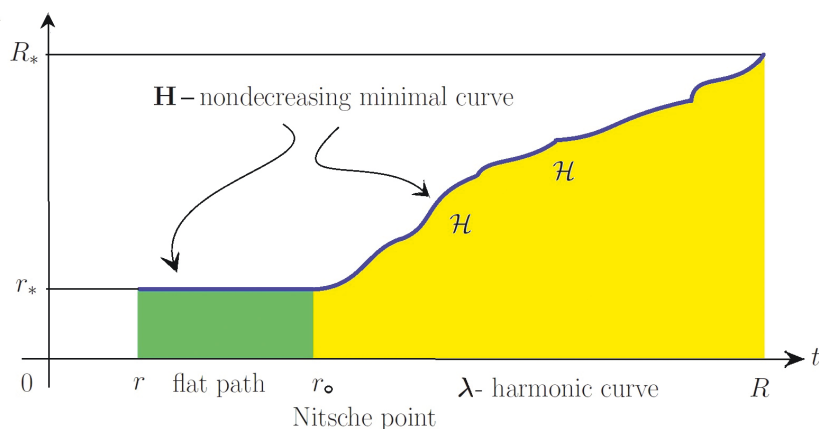


Figure 2: A nondecreasing path of smallest energy. It begins with a flat path and then becomes a strictly increasing λ -harmonic curve.

1.2 Existence

The existence of a λ -minimal curve $\mathbf{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ under the constraint $\dot{\mathbf{H}} \geq 0$ is straightforward by the direct method in the calculus of variations. The key observation is that such a class of functions is weakly closed in $\mathcal{W}^{1,2}(r, R)$. In particular, the weak limit of an energy-minimizing sequence remains nondecreasing and as such becomes a λ -minimal curve. The arguments for uniqueness and the description of the minimal curves are much more involved. There are two inherent difficulties. First, the Lagrange–Euler equation is unavailable, well recognized difficulty in the theory of nonlinear elasticity [4, 5, 35]. Second, the inner variation of the independent variable $t_\varepsilon = t + \varepsilon\tau(t)$ does not lead to a differential equation, unless λ is differentiable. In the literature, inner variation equations are also known as *energy-momentum* or *equilibrium* equations, etc. [7, 31, 37].

1.3 Variational equations

To discuss the weighted Dirichlet integrals in full generality, let us look at the following situation of this kind:

$$\mathcal{E}_\lambda[h] = \int_{\Omega} (|h_z|^2 + |h_{\bar{z}}|^2) \lambda(z, h) \, dz$$

subject to mappings $h : \Omega \xrightarrow{\text{onto}} \Omega^*$ of Sobolev class $\mathcal{W}^{1,2}(\Omega, \mathbb{C})$, where $\lambda : \Omega \times \Omega^* \rightarrow \mathbb{R}_+$ is a given weight function,

$$0 < m \leq \lambda(z, \xi) \leq M < \infty \quad \text{for almost every } (z, \xi) \in \Omega \times \Omega^*.$$

Of particular interest to us are the orientation preserving mappings; that is, whose Jacobian determinant $J_h(z) \geq 0$ a.e. We distinguish two types of variations of a map $h \in \mathcal{W}^{1,2}(\Omega, \mathbb{C})$.

(a) Outer variation:

$$h_\varepsilon(z) = h(z) + \varepsilon\eta(z) \quad \text{for } \eta \in \mathcal{C}_0^\infty(\Omega, \mathbb{C}), \quad \varepsilon \in \mathbb{C}, \quad \varepsilon \approx 0.$$

(b) Inner variation:

$$h^\varepsilon(z) = h(z + \varepsilon\eta(z)) \quad \text{for } \eta \in \mathcal{C}_0^\infty(\Omega, \mathbb{C}), \quad \varepsilon \in \mathbb{C}, \quad \varepsilon \approx 0.$$

Thus we look at the equalities

$$\frac{\partial}{\partial \varepsilon} \mathcal{E}_\lambda[h_\varepsilon] = 0 = \frac{\partial}{\partial \varepsilon} \mathcal{E}_\lambda[h^\varepsilon]$$

Here $\frac{\partial}{\partial \varepsilon}$ stands for the Cauchy–Riemann complex derivative. Under reasonable differentiability assumptions on the weight we obtain:

- The outer variational equation reads as

$$\frac{\partial}{\partial \bar{z}} (\lambda h_z) + \frac{\partial}{\partial z} (\lambda h_{\bar{z}}) = (|h_z|^2 + |h_{\bar{z}}|^2) \frac{\partial \lambda}{\partial \bar{\xi}}, \quad (1.8)$$

where the complex derivatives on the left hand side are understood in the sense of distributions.

- The inner variation equation reads as

$$2 \frac{\partial}{\partial \bar{z}} (\lambda h_z \overline{h_{\bar{z}}}) = -(|h_z|^2 + |h_{\bar{z}}|^2) \frac{\partial \lambda}{\partial z}. \quad (1.9)$$

The outer variational equation (1.8) makes sense if the weight $\lambda = \lambda(z, \xi)$ depends smoothly on ξ . Unfortunately, even if it does depend smoothly, the outer variation at (a) cannot be performed under the requirement that $J_h(z) \geq 0$, so is still unavailable. Thus we are left with the inner variational equation (1.9). But it requires λ to be $\mathcal{C}^1$ -smooth with respect to z , so does not apply because λ is only measurable. The case $\lambda = \lambda(|h|)$ is treated by the methods of free Lagrangians [2, 19–21], see [26]. In summary, none of the variational equations is available in this paper. The challenge is when λ in (1.2) is only a measurable function.

The case $\lambda \equiv 1$ reduces to the familiar Hopf–Laplace equation [8, 12–15, 18, 22, 34]

$$\frac{\partial}{\partial \bar{z}} (h_z \overline{h_{\bar{z}}}) = 0 \quad \text{for } h \in \mathcal{W}^{1,2}(\mathbb{A}).$$

For the radial minimal mappings $h(z) = H(|z|) \frac{z}{|z|}$ equation (1.9) becomes a second-order equation for the map $H = H(t)$, $r < t < R$,

$$\frac{d}{dt}(H^2 - t^2 \dot{H}^2) = 2 \frac{\lambda'(t)}{\lambda(t)} (t \dot{H})^2, \quad (1.10)$$

which in case $\lambda \equiv 1$ reduces to

$$H^2 - t^2 \dot{H}^2 \equiv C \quad (\text{a constant}) \quad (1.11)$$

All solutions of (1.11) take the form

$$H(t) = \alpha t + \frac{\beta}{t}.$$

1.4 Flattened λ -curves

Figure 2 illustrates typical λ -minimal curve. In order to state our results succinctly, we introduce the following concept.

Definition 1.5. The term *flattened λ -curve* pertains to any continuous function $\mathcal{F} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ defined by the following rule: There is $r_0 \in (r, R)$ such that

- $\mathcal{F}$ is equal to r_* everywhere in $[r, r_0]$,
- $\mathcal{F}$ is differentiable at r_0 and $\dot{\mathcal{F}}(r_0) = 0$,
- $\mathcal{F} : [r_0, R] \xrightarrow{\text{onto}} [r_*, R_*]$ is strictly increasing and λ -harmonic.

Hereafter the constant function $\mathcal{F} : [r, r_0] \rightarrow \{r_*\}$ is referred to as the initial *flat segment* and its right endpoint $r_0 \in (r, R)$ as the *Nitsche point* of $\mathcal{F}$. To this class of curves we shall also include λ -harmonic homeomorphisms $\mathcal{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$; that is, increasing λ -curves. Their flat segment degenerates to the endpoint $\{r\}$. In this case, however, the second condition in the above definition will not be required, though we continue to call $r_0 = r$ the Nitsche point. In either case r_0 is a point where $\mathcal{F}$ assumes its minimum value, equal to r_* .

Remark 1.6. Ultimately, all flattened λ -curves will turn out to be λ -minimal, but it takes some efforts to prove this fact. Precisely, we aim to prove the following theorem and its corollary.

Theorem 1.7. *The existence of a minimal curve follows by variational principles for convex energy-functionals, so we have the following:*

- (α) *There exists a λ -minimal curve $\mathbf{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$.*
- (β) *Every λ -minimal curve $\mathbf{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ arises as a flattened λ -curve.*
- (γ) *There is at most one flattened λ -curve $\mathcal{F} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$.*
- (δ) *Thus both $\mathbf{H}$ and $\mathcal{F}$ do exist, are unique, and $\mathbf{H} = \mathcal{F}$.*

This theorem just amounts to saying that the intervals $[r, R]$ and $[r_*, R_*]$, together with the weight function λ , settle uniquely the Nitsche critical point r_0 (possibly $r_0 = r$) and the λ -harmonic homeomorphism $\mathcal{F} : [r_0, R] \xrightarrow{\text{onto}} [r_*, R_*]$.

Corollary 1.8. *The ascent function $A_{\mathbf{H}}(t) \stackrel{\text{def}}{=} t\lambda(t)\dot{\mathbf{H}}(t)$ is Lipschitz continuous on $[r, R]$. It is strictly positive and increasing after crossing the Nitsche point.*

Examples abound in which the weight λ is discontinuous. The simplest example of this is furnished by a single jump in which λ is constant in a subinterval $[r, \rho]$ and a different constant in $(\rho, R]$, where $r < \rho < R$. The Nitsche point r_0 is unrelated to the jump point at $t = \rho$, see Section 3.2.1. There are also interesting unsolved questions, see Section 4.

2 Existence and properties of minimal curves

Let us begin with the classical case of the Dirichlet integral without weight; that is, when $\lambda \equiv 1$.

2.1 Harmonic curves

The normal strain function $H = H(t)$ of a complex harmonic radial map $h(z) = H(|z|)\frac{z}{|z|}$ is called *harmonic curve*. It is not difficult to see that every such harmonic function h admits a unique harmonic extension into the punctured plane $\mathbb{C} \setminus \{0\}$. More precisely, we have:

Proposition 2.1 (Harmonics). *Harmonic curves extend to the entire positive line $(0, \infty)$ where they take the form*

$$H(t) = \alpha t + \frac{\beta}{t} \quad \text{for some coefficients } \alpha, \beta \in \mathbb{R}. \quad (2.1)$$

Let $\mathcal{H} = \mathcal{H}(0, \infty)$ denote the linear space of all harmonic curves. We shall use the notation $\mathcal{H}[a, b]$, $0 < a < b < \infty$, to indicate that the harmonic curves are considered only in a closed interval $[a, b] \Subset (0, \infty)$. Since $\ddot{H}(t) = 2\beta t^{-3}$, we see that H is strictly convex if $\beta > 0$, linear if $\beta = 0$ and strictly concave if $\beta < 0$. In particular, H has no inflection points.

2.1.1 Variational equation

Harmonic curves (simply harmonics) are just solutions to the Lagrange–Euler equation of the energy functional (1.3) with $\lambda \equiv 1$. Namely,

$$[t\dot{\mathcal{H}}(t)]' = t^{-1}\mathcal{H}(t). \quad (2.2)$$

Clearly, all solutions $\mathcal{H} \in \mathcal{W}^{1,2}(a, b)$ are $\mathcal{C}^\infty$ -smooth and take the form $\mathcal{H}(t) = \alpha t + \beta/t$. The Lagrange–Euler equation implies the inner variational equation (1.11); namely,

$$t^2\dot{\mathcal{H}}^2(t) - \mathcal{H}^2(t) \equiv \gamma, \quad \text{where } \gamma = -4\alpha\beta. \quad (2.3)$$

However, in order to deduce (2.2) from (2.3), we restrict ourselves to solutions which are at least $\mathcal{C}^2$ -smooth. For example, the flattened harmonic curve

$$\mathbf{H} = \frac{1}{2} \begin{cases} 1 & \text{if } r \leq t \leq 1, \\ t + \frac{1}{t} & \text{if } 1 \leq t \leq R, \end{cases}$$

which satisfies (2.3), are only $\mathcal{C}^{1,1}$ -smooth. For a more general discussion about when the inner-variational equation implies the corresponding Lagrange–Euler equation we refer to [23].

2.1.2 Graphs

Consider harmonic curves in (2.1) with $\alpha + \beta \neq 0$. It simplifies the list of such harmonics, and causes no loss of generality, to assume that $H(1) = 1$. With this normalization we have a family $\{H_k\}_{k \in \mathbb{R}}$ which is parameterized by the slope at $t = 1$; that is, by $k = \dot{H}(1) \in \mathbb{R}$,

$$H_k(t) \stackrel{\text{def}}{=} \frac{1}{2}[(1+k)t + (1-k)t^{-1}].$$

2.1.3 Minimal curves for $\lambda \equiv 1$

Theorems 1.4 and 1.7 were already known in case of Dirichlet energy, see [1, 2, 21]. Accordingly, there are three types of minimal curves $\mathbf{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$.

(I) $\mathbf{H}$ is a harmonic curve, $\mathbf{H}(t) = \alpha t + \beta t^{-1}$, of positive derivative. This, together with the condition that $\mathbf{H}(r) > 0$, yields that $|\beta| < \alpha r^2$. Hence, such harmonic curves occur if and only if

$$\frac{R_*}{r_*} > \frac{1}{2} \left(\frac{R}{r} + \frac{r}{R} \right) \quad (\text{call it Nitsche lower bound}).$$

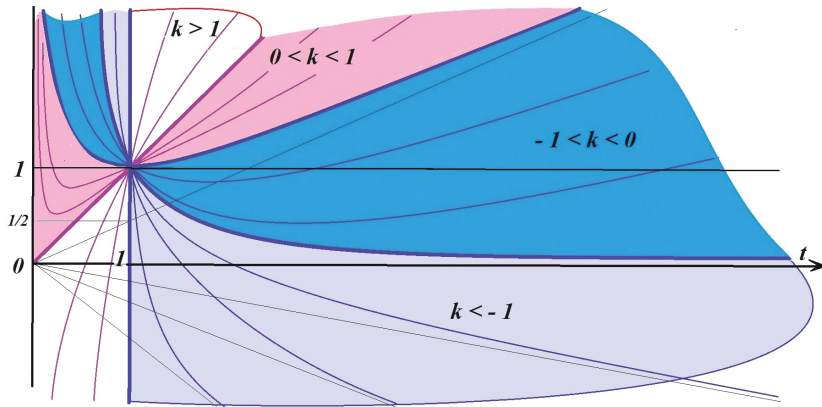


Figure 3: One-parameter family of harmonic curves.

This name praises J. C. C. Nitsche, because of his conjecture [28]. See also [29, § 878], [11, p. 138], [1, Conjecture 21.3.2] and the surveys [6, 27, 32].

- (II) When equality occurs in the above lower bound, $\mathbf{H}$ is still harmonic and strictly increasing, but $\dot{\mathbf{H}}(r) = 0$. This yields $0 < \beta = ar^2$ and reads as

$$\frac{R_*}{r_*} = \frac{1}{2} \left(\frac{R}{r} + \frac{r}{R} \right) \quad (\text{critical Nitsche condition}).$$

Explicitly,

$$\mathbf{H}(t) = \frac{r^*}{2} \left(\frac{t}{r} + \frac{r}{t} \right).$$

- (III) However, squeezing phenomenon occurs if

$$\frac{R_*}{r_*} < \frac{1}{2} \left(\frac{R}{r} + \frac{r}{R} \right) \quad (\text{below the Nitsche bound}). \quad (2.4)$$

In this case $\mathbf{H}$ is a flattened harmonic curve. Its Nitsche point r_0 is determined through the critical Nitsche condition

$$\frac{R_*}{r_*} = \frac{1}{2} \left(\frac{R}{r_0} + \frac{r_0}{R} \right),$$

so $\mathbf{H}$ takes the form,

$$\mathbf{H}(t) = \begin{cases} r_* & \text{for } r \leq t \leq r_0, \\ \frac{r^*}{2} \left(\frac{t}{r_0} + \frac{r_0}{t} \right) & \text{for } r_0 \leq t \leq R, \end{cases} \quad \dot{\mathbf{H}}(r_0) = 0. \quad (2.5)$$

Remark 2.2. Although it is not obvious, it is true that the Dirichlet energy (defined by (1.1) with $\lambda \equiv 1$) of the radial map $\mathbf{h}(z) \stackrel{\text{def}}{=} \mathbf{H}(|z|) \frac{z}{|z|}$ equals exactly the infimum of Dirichlet energies among all $\mathcal{W}^{1,2}$ -homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$, not necessarily rotationally invariant [1, 2, 21]. Another related result merits mentioning here as well. Namely, in case below the Nitsche bound at (2.4), there is actually no harmonic homeomorphism of $\mathbb{A}$ onto $\mathbb{A}^*$ (energy-minimal or not). This fact, stated in the theory of doubly connected minimal surfaces [9, 10, 30] as *Nitsche Conjecture*, has been solved in [16, 17]

2.2 λ -harmonics

The ideas and our proofs follow very closely the classical variational approach to PDEs.

2.2.1 Uniqueness

For completeness, we include the standard proof of uniqueness of λ -harmonics subject to given boundary values. Suppose two λ -harmonics H_1 and H_2 share the same values at the endpoints; that is, $H_1 - H_2 \in \mathcal{W}_0^{1,2}(r, R)$.

We subtract the Lagrange equations at (1.6) for H_1 and H_2 to obtain

$$\int_r^R t\lambda(t)[\dot{H}_1(t) - \dot{H}_2(t)]\dot{\Theta}(t) dt = - \int_r^R \frac{\lambda(t)[H_1(t) - H_2(t)]}{t} \Theta(t) dt.$$

Then we apply the test function $\Theta = H_1 - H_2 \in \mathcal{W}_\circ^{1,2}(r, R)$, which results in the equality

$$\int_r^R t\lambda(t)[\dot{H}_1(t) - \dot{H}_2(t)]^2 dt = - \int_r^R \frac{\lambda(t)[H_1(t) - H_2(t)]^2}{t} dt.$$

This makes it obvious that $H_1 = H_2$.

2.2.2 Minimum-energy property

For the characterization of λ -harmonics through minimum-energy property at (1.7), we argue as follows:

$$\begin{aligned} \mathcal{E}_\lambda[H] - \mathcal{E}_\lambda[\mathcal{H}] &= \int_a^b \left\{ t[\dot{H}^2(t) - \dot{\mathcal{H}}^2(t)] + \frac{[H^2(t) - \mathcal{H}^2(t)]}{t} \right\} \lambda(t) dt \\ &\geq 2 \int_a^b \left\{ t[\dot{H}(t) - \dot{\mathcal{H}}(t)]\dot{\mathcal{H}}(t) + \frac{[H(t) - \mathcal{H}(t)]\mathcal{H}(t)}{t} \right\} \lambda(t) dt \\ &= 2 \int_a^b \left\{ t\dot{\Theta}(t)\dot{\mathcal{H}}(t) + \frac{\Theta(t)\mathcal{H}(t)}{t} \right\} \lambda(t) dt = 0, \end{aligned}$$

as claimed. Here, we applied the integral form of the Lagrange equation in (1.6), where the test function $\Theta = H - \mathcal{H} \in \mathcal{W}_\circ^{1,2}(a, b)$.

2.2.3 Regularity

Consider a λ -harmonic curve $h \in \mathcal{W}^{1,2}(a, b)$ in an interval $(a, b) \subset (r, R)$. The imbedding $\mathcal{W}^{1,2}(a, b) \subset \mathcal{C}[a, b]$ shows that $\mathcal{H}$ is Hölder continuous (of Hölder exponent $\frac{1}{2}$) on the closed interval $[a, b]$. Then the Lagrange equation

$$\frac{d}{dt}[t\lambda(t)\dot{\mathcal{H}}(t)] = \frac{\lambda(t)}{t}\mathcal{H}(t) \in \mathcal{L}^\infty(a, b)$$

tells us that the ascent function

$$\mathcal{A}_{\mathcal{H}}(t) \stackrel{\text{def}}{=} t\lambda(t)\dot{\mathcal{H}}(t)$$

is Lipschitz continuous. This in turn shows that $\dot{\mathcal{H}} \in \mathcal{L}^\infty[a, b]$.

Corollary 2.3. *Every λ -harmonic curve is Lipschitz continuous, so is the ascent function $\mathcal{A}_{\mathcal{H}}$. In particular, as a consequence of Definition 1.5, all flattened λ -harmonics are also Lipschitz continuous.*

Remark 2.4. One should be aware that this corollary does not imply (yet) Corollary 1.8 because the minimal curve $\mathbf{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ need not be λ -harmonic on the entire interval $[r, R]$. Nevertheless, this will indeed imply Corollary 1.8 once we prove that λ -minimal curves are the same as flattened λ -harmonics, see Section 2.5.

Remark 2.5. The interested reader may observe that if $\lambda \in \mathcal{C}^1(a, b)$, then both $\mathcal{H}$ and $\mathcal{A}_{\mathcal{H}}$ are $\mathcal{C}^2$ -smooth in $[a, b]$. However, the minimal curves need not be so smooth. For example, the flattened harmonic curve in (2.5), where $\lambda \equiv 1$, is only $\mathcal{C}^{1,1}$ -smooth.

2.2.4 Stored energy function and its antiderivative

There is another interesting differential expression to consider, namely,

$$\mathbf{E}_{\mathcal{H}}(t) \stackrel{\text{def}}{=} \mathcal{A}(t)\mathcal{H}(t) \in \text{Lip}[a, b], \quad \text{where } \mathcal{A}(t) = t\lambda(t)\dot{\mathcal{H}}(t). \quad (2.6)$$

We see that for almost every $t \in (a, b)$ it holds that

$$\frac{d}{dt} \mathbf{E}_{\mathcal{H}}(t) = \mathcal{A}(t)\dot{\mathcal{H}}(t) + \dot{\mathcal{A}}(t)\mathcal{H}(t) = [t\dot{\mathcal{H}}^2(t) + t^{-1}\mathcal{H}^2(t)]\lambda(t) \geq 0.$$

The latter term is none other than the integrand of the λ -harmonic energy functional for $\mathcal{H}$, called *stored energy function*, see (1.3). In particular, we have the following formula for the energy of $\mathcal{H}$ over the interval $[a, b]$:

$$\mathcal{E}_a^b[\mathcal{H}] \stackrel{\text{def}}{=} \int_a^b [t\dot{\mathcal{H}}^2(t) + t^{-1}\mathcal{H}^2(t)]\lambda(t) dt = \mathbf{E}_{\mathcal{H}}(b) - \mathbf{E}_{\mathcal{H}}(a). \quad (2.7)$$

The following corollary will prove extremely useful.

Corollary 2.6. *For every λ -harmonic curve $\mathcal{H}$ the differential expression $\mathbf{E}_{\mathcal{H}}(t)$ is nondecreasing.*

2.2.5 Unique continuation property

Lemma 2.7. *If two λ -harmonics, defined in an interval $[a, b] \subset [r, R]$, coincide at two different points in $[a, b]$, they coincide everywhere in $[a, b]$.*

Proof. The assertion reduces to showing that if a λ -harmonic curve $\mathcal{H} \in \mathcal{H}_\lambda[a, b]$ vanishes at two points, say $\mathcal{H}(t_1) = \mathcal{H}(t_2) = 0$, where $a \leq t_1 < t_2 \leq b$, then $\mathcal{H}$ vanishes everywhere in $[a, b]$. For this, we examine the ascent function $\mathcal{A}_{\mathcal{H}}(t) = t\lambda(t)\dot{\mathcal{H}}(t)$ (which is Lipschitz continuous) and the product $\mathbf{E}_{\mathcal{H}}(t) = \mathcal{A}_{\mathcal{H}}(t) \cdot \mathcal{H}(t)$ (which is nondecreasing in $[a, b]$ by Corollary 2.6). Accordingly,

$$0 = \mathbf{E}_{\mathcal{H}}(t_1) \leq \mathbf{E}_{\mathcal{H}}(t) \leq \mathbf{E}_{\mathcal{H}}(t_2) = 0 \quad \text{for every } t \in [t_1, t_2].$$

Hence $t\lambda(t)\dot{\mathcal{H}}(t) \cdot \mathcal{H}(t) = \mathbf{E}_{\mathcal{H}}(t) \equiv 0$ in $[t_1, t_2]$. This yields $\frac{d}{dt}(\mathcal{H}^2) = 2\dot{\mathcal{H}}\mathcal{H} \equiv 0$ almost everywhere in $[t_1, t_2]$. Hence, $\mathcal{H} \equiv 0$ everywhere in $[t_1, t_2]$. (Actually, the above arguments also give a proof of the uniqueness statement in Proposition 1.1.) Now consider the largest closed interval $[\alpha, \beta]$ on which $\mathcal{H}$ vanishes. Thus $[t_1, t_2] \subset [\alpha, \beta] \subset [a, b]$. We aim to show that $\alpha = a$ and $\beta = b$. Suppose that, on the contrary, $\beta < b$. The case $a < \alpha$ is analogous. Choose and fix $\beta' > \beta$ close enough to β to satisfy

$$\log \frac{\beta'}{\beta} < \sqrt{\frac{2m}{M}}, \quad \beta' \in (\beta, b],$$

where we recall the lower and upper bounds of the weight function, namely $m \leq \lambda(t) \leq M$ for all $t \in [r, R]$. We define $\Lambda = \max_{\beta \leq \tau \leq \beta'} |\mathcal{H}(\tau)| > 0$. Since $\mathcal{H}(\beta) = 0$ and $\mathcal{A}_{\mathcal{H}}(\tau)$ (being a Lipschitz function on $[a, b]$) also vanishes at $\tau = \beta$, we can write

$$\mathcal{H}(t) = \int_{\beta}^t \dot{\mathcal{H}}(s) ds = \int_{\beta}^t \frac{1}{s\lambda(s)} \mathcal{A}_{\mathcal{H}}(s) ds = \int_{\beta}^t \frac{1}{s\lambda(s)} \left(\int_{\beta}^s \dot{\mathcal{A}}_{\mathcal{H}}(\tau) d\tau \right) ds = \int_{\beta}^t \frac{1}{s\lambda(s)} \left(\int_{\beta}^s \frac{\lambda(\tau)\mathcal{H}(\tau)}{\tau} d\tau \right) ds.$$

Therefore,

$$|\mathcal{H}(t)| \leq \frac{M}{m} \int_{\beta}^t \frac{1}{s} \left(\int_{\beta}^s \frac{d\tau}{\tau} \right) ds \cdot \Lambda = \frac{M}{2m} \log^2 \frac{t}{\beta} \cdot \Lambda < \Lambda.$$

This holds for all $t \in [\beta, \beta']$, which contradicts the definition of Λ . □

The unique continuation property strengthens Corollary 2.6.

Corollary 2.8. *The differential expression $\mathbf{E}_{\mathcal{H}}(t)$ is actually strictly increasing if $\mathcal{H} \neq 0$.*

2.2.6 Maximum principle

Lemma 2.9. *If a nonnegative λ -harmonic curve $\mathcal{H} \in \mathcal{H}_\lambda[a, b]$ assumes its maximum value in the open interval (a, b) , then $\mathcal{H} \equiv 0$ in $[a, b]$.*

Proof. Under the above assumption we can find two points $\alpha < \beta$ in $[a, b]$ such that $\mathcal{H}(t) \geq \mathcal{H}(\alpha) = \mathcal{H}(\beta) \geq 0$ for all $t \in [\alpha, \beta]$. Truncate $\mathcal{H}$ as follows:

$$\widetilde{\mathcal{H}}(t) \stackrel{\text{def}}{=} \begin{cases} \mathcal{H}(\alpha) = \mathcal{H}(\beta) & \text{whenever } \alpha \leq t \leq \beta, \\ \mathcal{H}(t) & \text{otherwise.} \end{cases}$$

Clearly $\widetilde{\mathcal{H}}$ has smaller energy than $\mathcal{H}$, unless $\widetilde{\mathcal{H}} \equiv \mathcal{H}$. Thus $\mathcal{H} \equiv 0$ in the interval (α, β) . The Lagrange equation (1.5) implies that $\mathcal{H} \equiv 0$ on $[\alpha, \beta]$. Finally, by the unique continuation property in Lemma 2.7, we conclude that $\mathcal{H} \equiv 0$ on $[a, b]$, as claimed. $\square$

2.2.7 Minimum principle

We consider the unique $\mathcal{W}^{1,2}(a, b)$ -solution of the boundary value problem

$$\begin{cases} \frac{d}{dt}[t\lambda(t)\dot{\mathcal{H}}(t)] = \frac{\lambda(t)}{t}\mathcal{H}(t) & \text{almost everywhere in } (a, b), \\ \mathcal{H}(a) \stackrel{\text{def}}{=} a_* \quad \text{and} \quad \mathcal{H}(b) \stackrel{\text{def}}{=} b_*, & \text{where } [a, b] \subset [r, R], \quad 0 < a_* \leq b_*. \end{cases} \quad (2.8)$$

Proposition 2.10 (Minimum point). *Let $\mathcal{H}$ satisfy (2.8). Then the following statements hold true:*

- (i) *The solution is nonnegative in $[a, b]$. See (vi) below for a slightly stronger statement.*
- (ii) *$\mathcal{H}$ assumes its minimum value at exactly one point, denoted by $t_{\min} \in [a, b]$ and called the Nitsche point of $\mathcal{H}$ (by analogy with the case of harmonic curves).*
- (iii) *If $t_{\min} = a$, then $\mathcal{H}$ is strictly increasing from a_* to b_* . Therefore, $\mathcal{H}$ is the unique λ -minimal curve.*
- (iv) *If $a < t_{\min} < b$, then $\mathcal{H}$ is strictly decreasing in the interval $[a, t_{\min}]$ and strictly increasing in the interval $[t_{\min}, b]$.*
- (v) *In case $a < t_{\min} < b$ the ascent function $\mathcal{A}_{\mathcal{H}}$ vanishes at $t_{\min}$.*
- (vi) *In case $a < t_{\min} < b$ the solution $\mathcal{H}$ is differentiable at $t_{\min}$ and $\dot{\mathcal{H}}(t_{\min}) = 0$.*
- (vii) *The solution $\mathcal{H}$ is positive in $[a, b]$.*

Proof. (i) The possibility $\min_{[a,b]} \mathcal{H} < 0$ is ruled out by the maximum principle applied to $-\mathcal{H}$ in a small neighborhood of $t_{\min}$ in which $-\mathcal{H}$ is nonnegative. Thus $\mathcal{H}(t) \geq 0$ for all $t \in [a, b]$.

(ii) Next, $\mathcal{H}$ cannot assume its smallest value at two distinct points, say $a \leq t_{\min} < t'_{\min} \leq b$. For otherwise $\mathcal{H}$ restricted to $[t_{\min}, t'_{\min}]$ would assume its maximum value inside this interval. Again by the maximum principle $\mathcal{H}$ would vanish in this interval and, by the unique continuation property, $\mathcal{H}$ would vanish in the entire interval $[a, b]$. This contradicts the fact that $\mathcal{H}(a) < \mathcal{H}(b)$.

(iii) This is again a consequence of the maximum principle applied to subintervals $[a', b'] \subset [a, b]$. In fact, any continuous function $\mathcal{H} : [a, b] \rightarrow \mathbb{R}$ which assumes its smallest value at the left endpoint $t = a$ and satisfies the maximum principle on every subinterval is strictly increasing.

(iv) The proof is in all respects similar to that of (iii). We leave the details to the reader.

(v) By (iv), for almost every $t \in [a, t_{\min})$ we have $\dot{\mathcal{H}}(t) \leq 0$, so $\mathcal{A}_{\mathcal{H}}(t) \leq 0$. Similarly $\mathcal{A}_{\mathcal{H}}(t) \geq 0$ almost everywhere in $(t_{\min}, b]$. Since $\mathcal{A}_{\mathcal{H}}$ is continuous, we conclude that $\mathcal{A}_{\mathcal{H}}(t_{\min}) = 0$.

(vi) For δ sufficiently close to 0 (positive or negative), we can write

$$\mathcal{H}(t_{\min} + \delta) - \mathcal{H}(t_{\min}) = \int_{t_{\min}}^{t_{\min} + \delta} \dot{\mathcal{H}}(s) ds = \int_{t_{\min}}^{t_{\min} + \delta} \frac{\mathcal{A}_{\mathcal{H}}(s)}{s\lambda(s)} ds = \int_{t_{\min}}^{t_{\min} + \delta} \frac{\mathcal{A}_{\mathcal{H}}(s) - \mathcal{A}_{\mathcal{H}}(t_{\min})}{s\lambda(s)} ds = o(\delta),$$

because $\mathcal{A}_{\mathcal{H}}$ is continuous, as desired.

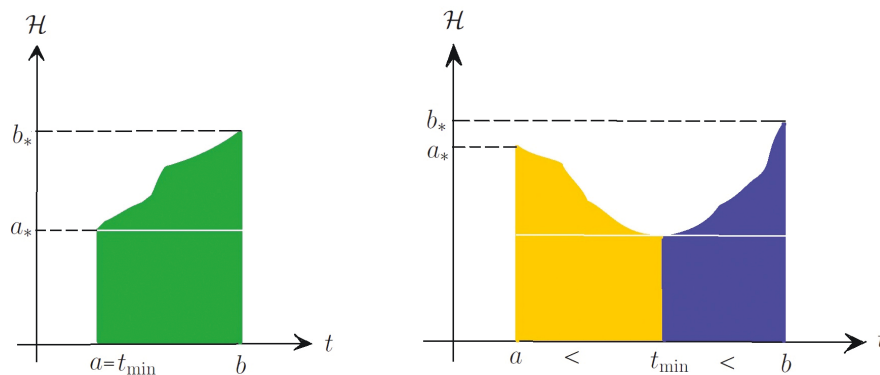


Figure 4: The point of minimum value of a λ -harmonic curve is the critical Nitsche point.

(vii) Suppose that $\mathcal{H}(t_{\min}) = 0$. We recall the lower and upper bounds $0 < m \leq \lambda(t) \leq M$ for $0 < r \leq t \leq R$. Choose and fix an ε such that

$$0 < \varepsilon < r \sqrt{\frac{m}{M}}, \quad 0 < \varepsilon < t_{\min} - a,$$

and define

$$\Lambda \stackrel{\text{def}}{=} \max\{\mathcal{H}(t) : t_{\min} - \varepsilon \leq t \leq t_{\min}\}.$$

We integrate the Lagrange equation $\mathcal{A}'_{\mathcal{H}}(\tau) = \frac{\lambda(\tau)}{\tau} \mathcal{H}(\tau)$ to compute the ascent function

$$\mathcal{A}_{\mathcal{H}}(t) = - \int_t^{t_{\min}} \frac{\lambda(\tau)}{\tau} \mathcal{H}(\tau) d\tau.$$

Hence $|\mathcal{A}_{\mathcal{H}}(t)| \leq \varepsilon \frac{M}{r} \Lambda$ for all $t_{\min} - \varepsilon \leq t \leq t_{\min}$. On the other hand $\dot{\mathcal{H}}(t) = \mathcal{A}_{\mathcal{H}}(t)/(t\lambda(t))$, thus $|\dot{\mathcal{H}}(t)| \leq \frac{\varepsilon M}{mr^2} \Lambda$. Finally, it follows from $\mathcal{H}(t) = - \int_t^{t_{\min}} \dot{\mathcal{H}}(\tau) d\tau$ that $|\mathcal{H}(t)| \leq \frac{\varepsilon^2 M}{mr^2} \Lambda$ for all $t_{\min} - \varepsilon \leq t \leq t_{\min}$. Taking maximum yields $\Lambda \leq \frac{\varepsilon^2 M}{mr^2} \Lambda$, where the factor in front of Λ is smaller than 1. Thus $\Lambda = 0$. This means that $\mathcal{H}(t) = 0$ for all $t_{\min} - \varepsilon \leq t \leq t_{\min}$, contrary to the unique continuation property. $\square$

2.3 Integration of the Lagrange equation

This term pertains to a reduction of the second-order equation to a family of the first-order ODEs parameterized by constants. There are many ways to achieve this goal. The keen reader may see a resemblance with the idea in PDEs when we multiply a general differential form (in 2D) by a suitable integrating factor to make the form exact. Let us first illustrate the case $\lambda \equiv 1$. Thus we begin with the second-order Lagrange equation

$$[t\dot{\mathcal{H}}(t)]' = t^{-1}\mathcal{H}(t). \quad (2.9)$$

Backward algebraic/differential operations lead us, equivalently, to a one-parameter family of the first-order nonlinear equations

$$t^2\dot{\mathcal{H}}^2(t) - \mathcal{H}^2(t) \equiv c, \quad \text{where } c \text{ can be any constant.} \quad (2.10)$$

The general $\mathcal{C}^2$ -smooth solutions take the form $\mathcal{H}(t) = at + \frac{b}{t}$, where $c = -4ab$. Interesting enough, equation (2.10) also admits solutions which are only $\mathcal{C}^{1,1}$ -smooth. This happens if $c < 0$ and $r < t_* \stackrel{\text{def}}{=} \sqrt{\frac{b}{a}} < R$. Indeed, we have a solution defined by the rule

$$\mathbf{H}(t) = \begin{cases} at_* + \frac{b}{t_*} & \text{for } r \leq t \leq t_* \text{ (flat segment),} \\ at + \frac{b}{t} & \text{for } t_* \leq t \leq R \text{ (harmonic curve).} \end{cases}$$

Note that t_* is the point at which $\dot{\mathcal{H}}(t_*) = 0$; that is, the Nitsche point.

Another way to reduce the Lagrange equation (2.9) to a one-parameter family of first-order equations is as follows:

$$t\dot{\mathcal{H}}(t) = \mathcal{H}(t) \frac{ct^2 - 1}{ct^2 + 1}, \quad \text{where } c \text{ is a constant.} \quad (2.11)$$

Here, we capture the harmonic curve $\mathcal{H}(t) = at + \frac{b}{t}$ by taking $c = \frac{a}{b}$.

One major advantage of equation (2.11) over (2.10) is its linearity. But there is a price to pay for this. Namely, as opposed to (2.10), the $\mathcal{C}^{1,1}$ -solutions with a flat segment no longer satisfy equation (2.11). Consequently, the minimal curves are lost. The same holds for λ -harmonic equations. Nevertheless, integration of $\frac{d}{dt}[t\lambda(t)\dot{\mathcal{H}}(t)] = \frac{\lambda(t)}{t}\mathcal{H}(t)$ proves useful in understanding the λ -harmonics and the λ -minimal curves, as well.

Proposition 2.11. *The Lagrange equation (2.9) reduces to*

$$t\lambda(t)\dot{\mathcal{H}}(t) = \mathcal{H}(t) \cdot \Phi(t), \quad (2.12)$$

where $\Phi \in \text{Lip}[r, R]$ is found by solving a nonlinear first-order ODE,

$$\lambda^2(t) - \Phi^2(t) = t\lambda(t)\dot{\Phi}(t). \quad (2.13)$$

Note that solutions to this latter equation form a one-parameter family, each of which is determined by a certain constant, say c . The natural constant, which has a geometric meaning, is the conformal modulus of the target annulus $\mathbb{A}^*$:

$$c \stackrel{\text{def}}{=} \int_r^R \frac{\Phi(t)}{t\lambda(t)} dt = \int_r^R \frac{\dot{\mathcal{H}}(t)}{\mathcal{H}(t)} dt = \log \mathcal{H}(t) \Big|_r^R = \log \frac{R_*}{r_*} \stackrel{\text{def}}{=} \text{Mod } \mathbb{A}^*.$$

Proof. Let Φ be defined by equation (2.12), so the Lagrange equation (2.9) reads as $(\mathcal{H}\Phi)' = \frac{\lambda\dot{\mathcal{H}}}{t}$. Apply the product rule $\dot{\mathcal{H}}\Phi + \mathcal{H}\dot{\Phi} = \frac{\lambda\dot{\mathcal{H}}}{t}$ and multiply this equation by $t\lambda$ and use the fact that $t\lambda \cdot \dot{\mathcal{H}} = \mathcal{H}\Phi$, to conclude that $\mathcal{H}\Phi^2 + t\lambda\dot{\Phi} = \lambda^2\mathcal{H}$. This is the desired equation (2.13). Conversely, equation (2.13) implies (2.12), which is just as easy to see. $\square$

2.4 Uniqueness of flattened λ -curves

Here we proof statement (y) in Theorem 1.7. The statement asserts that there can be at most one flattened λ -curve $\mathcal{F} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$. Suppose to the contrary that there are two such curves

$$\mathcal{F}_a(t) \stackrel{\text{def}}{=} \begin{cases} r_* & \text{for } r \leq t \leq a < R \text{ where } a \in [r, R], \\ \text{increasing } \lambda\text{-curve} & \text{for } a \leq t \leq R, \end{cases} \quad \dot{\mathcal{F}}_a(a) = 0,$$

and

$$\mathcal{F}_b(t) \stackrel{\text{def}}{=} \begin{cases} r_* & \text{for } r \leq t \leq b < R \text{ where } b \in [r, R], \\ \text{increasing } \lambda\text{-curve} & \text{for } b \leq t \leq R, \end{cases} \quad \dot{\mathcal{F}}_b(b) = 0.$$

We may assume that $a \leq b$. The case $a = b$ is immediate. To see this, just look at the λ -harmonic function $\mathcal{H} = \mathcal{F}_a(t) - \mathcal{F}_b(t)$ in the interval $[a, R]$, which vanishes at the endpoints. By the unique continuation property $\mathcal{H} \equiv 0$, so $\mathcal{F}_a \equiv \mathcal{F}_b$ in $[r, R]$.

Now let $a < b$. The function $\mathcal{H} = \mathcal{F}_a(t) - \mathcal{F}_b(t)$ equals zero in the interval $[r, a]$. It increases in the interval $[a, b]$. This is because $\mathcal{F}_a : [a, R] \xrightarrow{\text{onto}} [r_*, R_*]$, being a λ -harmonic homeomorphism, is increasing whereas $\mathcal{F}_b$ remains constant. Now the key observation is that $\mathcal{H}$ is still increasing in the interval $[b, R]$. For this, we invoke the function $\mathbf{E}_{\mathcal{H}}(t)$ defined for $b \leq t \leq R$ in (2.6).

$$\mathbf{E}_{\mathcal{H}}(t) \stackrel{\text{def}}{=} \mathcal{A}_{\mathcal{H}}(t)\mathcal{H}(t) \in \text{Lip}[b, R], \quad \mathbf{E}_{\mathcal{H}}(t) = t\lambda(t)\dot{\mathcal{H}}(t)\mathcal{H}(t), \text{ a.e.}$$

Since $\mathcal{H} \neq 0$, by Corollary 2.8, this function is strictly increasing. Both $\mathcal{A}_{\mathcal{H}}(b)$ and $\mathcal{H}(b)$ are nonnegative, so $\mathbf{E}_{\mathcal{H}}(t) > 0$, for $b < t \leq R$. Thus $\mathcal{H}$, being positive at $t = b$, remains positive in $[b, R]$ (it cannot vanish). This implies that the ascent function $\mathcal{A}_{\mathcal{H}}(t) = t\lambda(t)\dot{\mathcal{H}}(t)$ is also positive in $(b, R]$, so $\dot{\mathcal{H}}(t) > 0$ almost everywhere in $[b, R]$.

In conclusion: $\mathcal{H}$ is strictly increasing in the interval $[a, R]$. In particular, $\mathcal{H}$ cannot vanish at the endpoint $t = R$, which contradicts $\mathcal{F}_a(R) = \mathcal{F}_b(R) = R_*$.

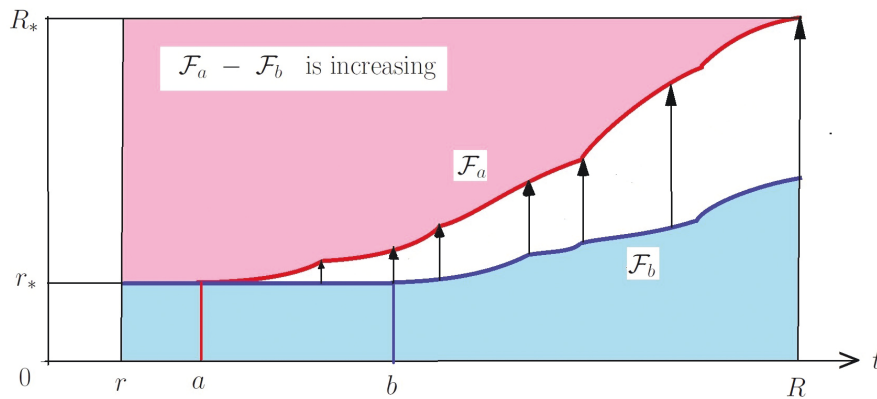


Figure 5: Once two flattened λ -curves split (at $t = a$) the distance between them increases. They cannot meet again, contradicting the equality $\mathcal{F}_a(R) = \mathcal{F}_b(R) = R_*$.

2.5 All minimal curves are flattened λ -harmonics

Here we proof statement (β) in Theorem 1.7. The idea is to perturb the λ -minimal curve $\mathbf{H}$ by suitable local λ -harmonics. Since $\mathbf{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ is nondecreasing we let $[r, r_0] \subset [r, R]$ denote the largest interval in which $\mathbf{H} \equiv r_*$. We do not rule out the possibility that $r_0 = r$. Thus $\mathbf{H}(t) > r_*$ whenever $r_0 < t \leq R$. Our first objective is to show the following.

2.5.1 $\mathbf{H}$ is λ -harmonic in $[r_0, R]$

It suffices to prove that $\mathbf{H}$ satisfies the λ -harmonic equation (1.5) in a neighborhood of any point $t' \in (r_0, R)$. Since $\mathbf{H}(t') > r_*$, and $\mathbf{H}$ is continuous, there is an open interval $(r', R') \subset (r_0, R)$ containing t' whose length $|R' - r'|$ is small enough to satisfy the inequality

$$(R' - r') \cdot \mathcal{E}_\lambda[\mathbf{H}] < (\mathbf{H}(t') - r_*)^2 m r. \quad (2.14)$$

Here we recall that m stands for a lower bound of λ . We solve (uniquely) the following Dirichlet boundary value problem for $\mathcal{G} \in \mathcal{W}^{1,2}(r', R')$:

$$\begin{cases} \frac{d}{dt}[t\lambda(t)\dot{\mathcal{G}}(t)] = \frac{\lambda(t)}{t}\mathcal{G}(t) & \text{in } (r', R'), \\ \mathcal{G}(r') = \mathbf{H}(r') & \text{and } \mathcal{G}(R') = \mathbf{H}(R'). \end{cases}$$

Thus $\mathcal{G}$ has smallest λ -energy among all functions in $\mathcal{W}^{1,2}(r', R')$ (with the above prescribed values at the endpoints). Accordingly, we have the inequality

$$\int_{r'}^{R'} \mathbf{E}(t, \mathcal{G})\lambda(t) dt \leq \int_{r'}^{R'} \mathbf{E}(t, \mathbf{H})\lambda(t) dt.$$

Equality occurs if and only if $\mathcal{G} \equiv \mathbf{H}$ in $[r', R']$.

We begin with the case when $\mathcal{G}$ is nondecreasing in $[r', R']$. In this case the opposite inequality also holds. Otherwise, replacing $\mathbf{H}$ by $\mathcal{G}$ over the interval $[r', R']$, we would obtain a nondecreasing curve over $[r, R]$ with energy lower than that of $\mathbf{H}$. Therefore $\mathbf{H} = \mathcal{G}$, which means that $\mathbf{H}$ is λ -harmonic in $[r', R']$, as desired.

We are left with the task of showing that $\mathcal{G}$ is indeed nondecreasing in (r', R') . Suppose otherwise. Striving for a contradiction, we appeal to Proposition 2.10 for $\mathcal{G}$ in place of $\mathcal{H}$ and $[r', R']$ in place of $[a, b]$, where $0 < \mathcal{G}(r') = \mathbf{H}(r') \leq \mathbf{H}(R') = \mathcal{G}(R')$. Accordingly, there is unique point $t_{\min} \in (r', R')$ at which $\mathcal{G}$ assumes its minimum value. Moreover, $\mathcal{G}$ is strictly decreasing in the interval $[r', t_{\min}]$ and strictly increasing in the interval $[t_{\min}, R']$. Moreover, $\mathcal{G}(t_{\min}) > 0$. In particular, $\dot{\mathcal{G}}(t) \geq 0$ almost everywhere in $[t_{\min}, R']$.

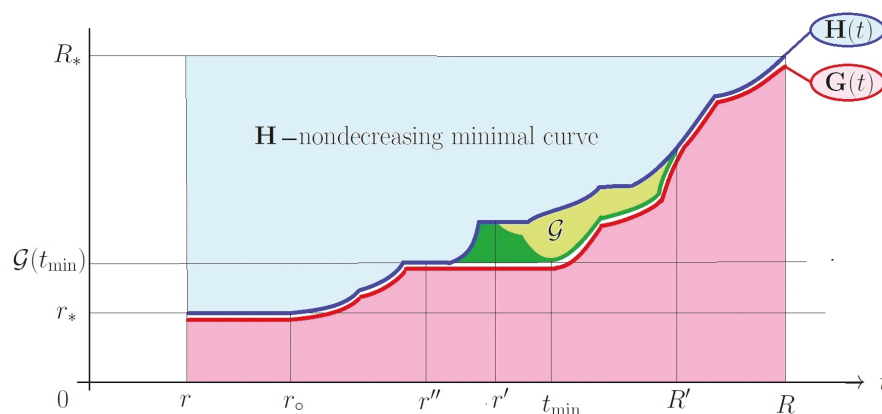


Figure 6: Perturbation of a minimal curve beyond its initial flat segment (possibly $r_0 = r$).

Lemma 2.12. *The point to make here is that*

$$\min_{r' \leq t \leq R'} \mathcal{G}(t) = \mathcal{G}(t_{\min}) \stackrel{\text{def}}{=} t_{\min}^* \in (r_*, R_*) \quad (2.15)$$

Proof. Clearly we have $t_{\min}^* = \mathcal{G}(t_{\min}) < \mathcal{G}(R') = \mathbf{H}(R') \leq R_*$. On the other hand, we have the following chain of inequalities:

$$\begin{aligned} [\mathcal{G}(R') - \mathcal{G}(t_{\min})]^2 &= \left[\int_{t_{\min}}^{R'} \dot{\mathcal{G}}(s) \, ds \right]^2 \leq (R' - t_{\min}) \int_{t_{\min}}^{R'} |\dot{\mathcal{G}}(s)|^2 \, ds \\ &\leq \frac{R' - t_{\min}}{mr} \int_{t_{\min}}^{R'} s \lambda(s) |\dot{\mathcal{G}}(s)|^2 \, ds \leq \frac{R' - t_{\min}}{mr} \int_{r'}^{R'} \mathbf{E}(t, \mathcal{G}) \lambda(t) \, dt \\ &\leq \frac{R' - t_{\min}}{mr} \int_{r'}^{R'} \mathbf{E}(t, \mathbf{H}) \lambda(t) \, dt < \frac{R' - r'}{mr} \mathcal{E}_{\lambda}[\mathbf{H}] \\ &\leq [\mathbf{H}(t') - r_*]^2, \end{aligned}$$

by the assumption at (2.14). Hence we conclude that $\mathcal{G}(R') - \mathcal{G}(t_{\min}) < \mathbf{H}(t') - r_* \leq \mathbf{H}(R') - r_* = \mathcal{G}(R') - r_*$, and so $\mathcal{G}(t_{\min}) > r_*$, as claimed. $\square$

Having (2.15), we now proceed to a construction of a small perturbation of $\mathbf{H}$, which leads to a nondecreasing curve of smaller energy. To this effect, we choose and fix a point $r'' \in (r, t_{\min})$ such that $\mathbf{H}(r'') = \mathcal{G}(t_{\min})$ (at this point we cannot claim that r'' is unique; though this will become obvious later on). Then we set

$$\mathbf{G}(t) = \begin{cases} \mathbf{H}(t) & \text{in } [r, r''], \\ \mathcal{G}(t_{\min}) & \text{in } [r'', t_{\min}], \\ \mathcal{G}(t) & \text{in } [t_{\min}, R'], \\ \mathbf{H}(t) & \text{in } [R', R]. \end{cases}$$

Certainly $\dot{\mathbf{G}}(t) \geq 0$ almost everywhere in (r, R) . Also $\mathbf{G}(r) = r_*$ and $\mathbf{G}(R) = R_*$. To reach the desired contradiction, it remains to show that the λ -energy of $\mathbf{G}$ is smaller than that of $\mathbf{H}$. Recall the notation in (2.7):

$$\mathcal{E}_a^b[H] = \int_a^b \mathbf{E}(t, H) \lambda(t) \, dt, \quad \mathbf{E}(t, H) = t \dot{H}^2(t) + t^{-1} H^2(t).$$

We have

- $\mathcal{E}_{R'}^R[\mathbf{G}] = \mathcal{E}_{R'}^R[\mathbf{H}]$ and $\mathcal{E}_{r''}^{r''}[\mathbf{G}] = \mathcal{E}_{r''}^{r''}[\mathbf{H}]$, because $\mathbf{G} \equiv \mathbf{H}$.
- $\mathcal{E}_{r''}^{t_{\min}}[\mathbf{G}] < \mathcal{E}_{r''}^{t_{\min}}[\mathbf{H}]$, because of the pointwise inequality $\mathbf{E}(t, \mathbf{G}) \leq \mathbf{E}(t, \mathbf{H})$.

There remains the interval $[r', R'] = [r', t_{\min}] \cup [t_{\min}, R']$. We argue as follows:

- $\mathcal{E}_{r'}^{t_{\min}}[\mathbf{G}] < \mathcal{E}_{r'}^{t_{\min}}[\mathcal{G}]$, because $\mathbf{E}(t, \mathbf{G}) \leq \mathbf{E}(t, \mathcal{G})$.
- $\mathcal{E}_{t_{\min}}^{R'}[\mathbf{G}] < \mathcal{E}_{t_{\min}}^{R'}[\mathcal{G}]$, because $\mathbf{G} = \mathcal{G}$.

Summing up, we see that

$$\mathcal{E}_{r'}^{R'}[\mathbf{G}] \leq \mathcal{E}_{r'}^{R'}[\mathcal{G}] \leq \mathcal{E}_{r'}^{R'}[\mathbf{H}].$$

The latter holds because $\mathcal{G}$ is λ -harmonic in $[r', R']$ and agrees with $\mathbf{H}$ at the endpoints r' and R' .

In conclusion, we constructed a nondecreasing curve $\mathbf{G}$ satisfying the boundary conditions $\mathbf{G}(r) = \mathbf{H}(r)$ and $\mathbf{G}(R) = \mathbf{H}(R)$, which has smaller energy than $\mathbf{H}$. This contradiction proves statement (β) in Theorem 1.7.

2.5.2 Harmonic perturbations near r_* reveal that $\dot{\mathbf{H}}(r_*) = 0$

Actually, our arguments give even more precise conclusion:

Lemma 2.13. *Suppose that $r < r_* < R$. Then there is a subset $E \subset [r, R]$ of full measure such that*

$$\lim_{\substack{t \rightarrow r_* \\ t \in E}} \dot{\mathbf{H}}(t) = 0. \quad (2.16)$$

Moreover, $\mathbf{H}$ is differentiable at $t = r_*$ and $\dot{\mathbf{H}}(r_*) = 0$.

We emphasize that the statement $\dot{\mathbf{H}}(r_*) = 0$, for a general Lipschitz function, does not guarantee (2.16).

Proof. This is certainly true if $r \leq t \leq r_* < R$, because $\mathbf{H}$ is constant. Thus we will concern ourselves only with $r < r_* \leq t < R$. We have already proved that in this interval $\mathbf{H}(t)$ is a λ -harmonic curve, so we set $\mathcal{H}(t) \stackrel{\text{def}}{=} \mathbf{H}(t)$ for $r_* \leq t < R$. Invoke a Lipschitz continuous representative of the ascending function $\mathcal{A}_{\mathcal{H}}(t) = t\lambda(t)\dot{\mathcal{H}}(t)$ in $[r_*, R]$. We are going to prove that

$$0 \leq \mathcal{A}_{\mathcal{H}}(t) \leq M\mathcal{H}(t) \log \frac{t}{r_*} \quad \text{whenever } r_* \leq t \leq R,$$

where M stands for the upper bound of the weight function λ in (1.2). Obviously, such an inequality will remain valid for all $r \leq t \leq R$. That is,

$$0 \leq t\lambda(t)\dot{\mathbf{H}}(t) \leq M\mathbf{H}(t) \left| \log \frac{t}{r_*} \right| \quad \text{whenever } r \leq t \leq R; \quad (2.17)$$

this will give us Corollary 1.8.

For the proof of (2.17), we first show that $\mathcal{A}_{\mathcal{H}}(r_*) = 0$. To this effect, we test $\mathbf{H}$ with a one-parameter family of harmonics

$$H_\varepsilon(t) = \alpha t + \frac{\beta}{t}, \quad 0 < \varepsilon < \frac{r_*}{r}(r_* - r),$$

where

$$\alpha = \frac{(r_* + \varepsilon)r_*}{2r_*^2} \quad \text{and} \quad \beta = \frac{r_* r_*^2}{2(r_* + \varepsilon)}.$$

Thus H_ε is convex whose minimum values r_* is attained at the point $r_\varepsilon \stackrel{\text{def}}{=} \frac{r_*^2}{r_* + \varepsilon} \in (r, r_*)$:

$$H_\varepsilon(r_\varepsilon) = r_* \quad \text{and} \quad \dot{H}_\varepsilon(r_\varepsilon) = 0.$$

Thus $\dot{H}_\varepsilon(t) > 0$ for $t > r_\varepsilon$. Moreover,

$$H_0(t) \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0} H_\varepsilon(t) = \frac{r_*}{2} \left[\frac{t}{r_*} + \frac{r_*}{t} \right].$$

Clearly, $\dot{H}_\varepsilon(t) > 0$ for all $0 < t < \infty$. The points r_ε converge to r_* as $\varepsilon \rightarrow 0$. Actually, for $t \in (r_*, R]$, the function $\varepsilon \mapsto H_\varepsilon(t)$ increases, because

$$\frac{d}{d\varepsilon} H_\varepsilon(t) = \frac{r_*}{2} \left(\frac{t}{r_*^2} - \frac{r_*^2}{(r_* + \varepsilon)^2 t} \right) > 0.$$

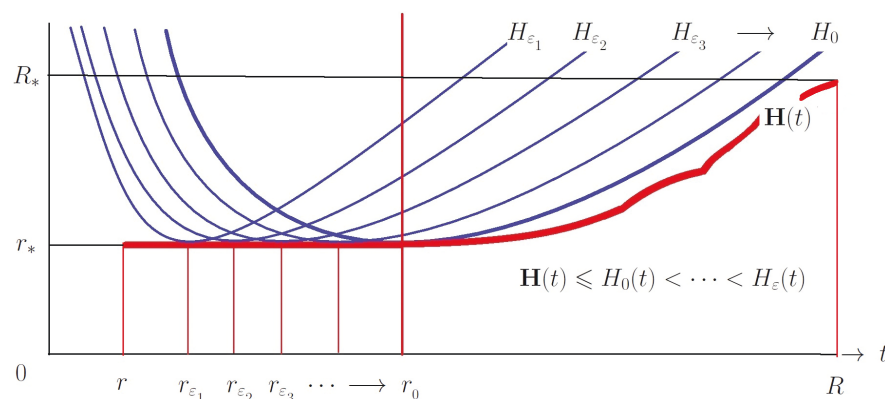


Figure 7: Testing near the Nitsche point to conclude that $\dot{\mathbf{H}}(t_0) = 0$.

The key observation is that

$$r_* \leq \mathbf{H}(t) \leq H_\varepsilon(t) \quad \text{for all } r \leq t \leq R \text{ and } 0 < \varepsilon < \frac{r_0}{r}(r_0 - r). \quad (2.18)$$

This is certainly true for $r \leq t \leq r_0$. Suppose that, contrary to (2.18), for some $\varepsilon > 0$ and $t' \in (r_0, R)$, we have $H_\varepsilon(t') < \mathbf{H}(t')$. Since $H_\varepsilon(r_0) > r_* = \mathbf{H}(r_*)$, by the intermediate value theorem, there is $R' \in (r_0, R)$ at which $H_\varepsilon(R') = \mathbf{H}(R')$. Replacing a segment of $\mathbf{H}$ that lies over the interval $[r_\varepsilon, R']$ by that of H_ε , we lower the energy, contrary to $\mathbf{H}$ being a λ -minimal map.

With inequality (2.18) in hands, we let ε go to zero to obtain

$$r_* \leq \mathbf{H}(t) \leq H_0(t) = \frac{r_*}{2} \left[\frac{t}{r_0} + \frac{r_0}{t} \right] \quad \text{for all } r \leq t \leq R.$$

Hence

$$0 \leq \mathbf{H}(t) - \mathbf{H}(r_0) \leq \frac{r_*}{2} \left[\frac{t}{r_0} - 2 + \frac{r_0}{t} \right] = \frac{r_*(t - r_0)^2}{2r_0 t}, \quad r \leq t \leq R. \quad (2.19)$$

This implies that $\mathbf{H}$ is differentiable at $t = r_0$ and $\dot{\mathbf{H}}(r_0) = 0$.

Now go back to the continuous function $\mathcal{A}_{\mathcal{H}} \in \mathcal{C}[r_0, R]$ which, for almost every $t \in [r_0, R]$, is defined by the rule $\mathcal{A}_{\mathcal{H}}(t) = t\lambda(t)\mathcal{J}\mathcal{H}(t) \leq RM\dot{\mathbf{H}}(t)$. We find that

$$0 \leq \mathcal{A}_{\mathcal{H}}(r_0) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{r_0}^{r_0+\varepsilon} \mathcal{A}_{\mathcal{H}}(t) dt \leq RM \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{r_0}^{r_0+\varepsilon} \mathcal{J}\mathcal{H}(t) dt \leq RM \lim_{\varepsilon \searrow 0} \frac{\mathcal{H}(r_0 + \varepsilon) - \mathcal{H}(r_0)}{\varepsilon} = 0,$$

by (2.19). Finally, estimate (2.17) is immediate from the Lagrange equation as follows:

$$\mathcal{A}_{\mathcal{H}}(t) = \int_{r_0}^t [s\lambda(s)\mathcal{J}\mathcal{H}(s)]' ds = \int_{r_0}^t \lambda(s)\mathcal{H}(s) \frac{ds}{s} \leq M\mathcal{H}(t) \log \frac{t}{r_0},$$

as desired. $\square$

2.5.3 Uniqueness of the minimal curves revisited

Since the minimal curve is a flattened λ -curve and the latter is unique, we conclude that there is exactly one minimal curve, as asserted in Theorem 1.7. Nevertheless, knowing that minimal curves are flattened λ -curves, there is a slightly simpler direct proof of the uniqueness through an energy argument. Suppose, for a contradiction, that there are two λ -energy minimal curves $\mathbf{H}_\pm : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$. Both are flattened λ -harmonics. Let $[r, r_+] \subset [r, R]$ and $[r, r_-] \subset [r, R]$ be their flat segments, respectively. We may (and do) assume that $r \leq r_- \leq r_+ < R$.

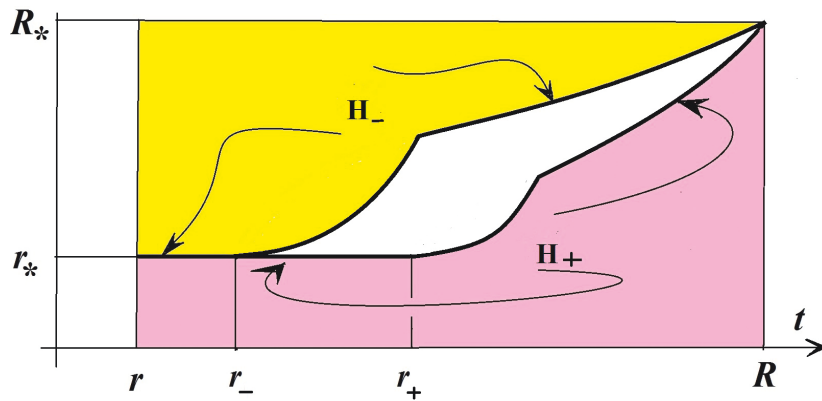


Figure 8: The shorter flat segment, the smaller energy.

Now $\mathbf{H}_-$ is λ -harmonic in the interval $[r_-, R]$, whereas $\mathbf{H}_+$ is a curve that coincides with $\mathbf{H}_-$ at the endpoint of this interval. Therefore,

$$\mathcal{E}_\lambda[\mathbf{H}_-] \leq \mathcal{E}_\lambda[\mathbf{H}_+]$$

and equality occurs if and only if $\mathbf{H}_- \equiv \mathbf{H}_+$, as desired.

3 Samples of the minimal curves

3.1 The power weight $\lambda = t^p$

As an illustration of our results, we consider two examples. The first concerns the power function

$$\lambda = \lambda(t) = t^p, \quad -\infty < p < \infty \text{ and } 0 < t < \infty,$$

which is a natural extension of the harmonic curves ($p = 0$). The λ -harmonic curves $\mathcal{H} \in \mathcal{W}^{1,2}(a, b)$ are solutions to the Lagrange–Euler equation

$$\frac{d}{dt}[t^{p+1}\dot{\mathcal{H}}(t)] = t^{p-1}\mathcal{H}(t).$$

This second-order equation has two fundamental solutions

$$\mathcal{H}_+ = t^A, A = \frac{-p + \sqrt{p^2 + 4}}{2} \quad \text{and} \quad \mathcal{H}_- = t^B, B = \frac{-p - \sqrt{p^2 + 4}}{2}.$$

Therefore, the general λ -harmonic curve takes the form

$$\mathcal{H}(t) = \alpha t^A + \beta t^B \quad \text{for some coefficients } \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions $\mathcal{H}(r) = r_*$ and $\mathcal{H}(R) = R_*$, one finds α and β . Explicitly, we have the formula

$$\mathcal{H}(t) = \frac{r^B R_* - r_* R^B}{r^B R^A - r^A R^B} t^A + \frac{r_* R^A - r^A R_*}{r^B R^A - r^A R^B} t^B,$$

where we note that

$$r^B R^A - r^A R^B = r^A R^B \left[\left(\frac{R}{r} \right)^{A-B} - 1 \right] > 0.$$

Now we invoke Proposition 2.10 which tells us, among other things, that $\mathcal{H}$ is λ -minimal (meaning that $\dot{\mathcal{H}}(t) \geq 0$ in $[r, R]$) if and only if $\dot{\mathcal{H}}(r) \geq 0$. Thus $\mathcal{H}$ is λ -minimal if and only if

$$A(r^B R_* - r_* R^B)r^{A-1} + B(r_* R^A - r^A R_*)r^{B-1} \geq 0.$$

This can be expressed equivalently as

$$\frac{R_*}{r_*} \geq \frac{-B}{A-B} \left(\frac{R}{r} \right)^A + \frac{A}{A-B} \left(\frac{r}{R} \right)^{-B}. \quad (3.1)$$

It is tempting to express p in trigonometric terms:

$$p = 2 \cot 2\phi, \quad \text{where } 0 < \phi < \frac{\pi}{2}.$$

By means of the angle ϕ , our formula (3.1) reads as

$$\frac{R_*}{r_*} \geq \sin^2 \phi \left(\frac{R}{r} \right)^{\tan \phi} + \cos^2 \phi \left(\frac{r}{R} \right)^{\cot \phi}. \quad (3.2)$$

We just proved the following theorem.

Theorem 3.1 (Nitsche criterion for $\lambda(t) = t^p$). *A λ -harmonic curve $\mathcal{H} \in \mathcal{W}^{1,2}(r, R)$, subject to the boundary constraints $\mathcal{H}(r) = r_*$ and $\mathcal{H}(R) = R_*$, is λ -minimal if and only if the target interval $[r_*, R_*]$ is conformally wide enough to satisfy inequality (3.2).*

Remark 3.2. This inequality, in case $\phi = \frac{\pi}{4}$, agrees with the familiar Nitsche criterion for harmonic curves, see Section 2.1.3. Surprisingly, such a more general Nitsche criterion (3.2) is expressed in terms of the conformal modulus of the target interval (that is, in terms of the ratio $\frac{R}{r}$). This phenomenon, however, does not hold in general, as illustrated by our next example in which the weight λ is piecewise constant and exhibits a jump.

3.2 Discontinuous weight

The next example of the Nitsche phenomenon for discontinuous weights is furnished by a step function. In this case the energy-minimal solutions are obtained by gluing two harmonic mappings. One may use this example to model other discontinuities of λ . Indeed, most of the minimal curves can be described using step functions as weights and the corresponding *piecewise harmonics* as the energy-minimal solutions.

3.2.1 Gluing two harmonic curves

We invoke *harmonic curves* that are normalized by the condition $H(1) = 1$:

$$H(t) = \frac{1}{2}[(1+k)t + (1-k)t^{-1}], \quad \text{where } k = \dot{H}(1) \in \mathbb{R}.$$

Figure 9 assembles such harmonics into four classes:

$$(-\infty < k < -1), \quad (-1 < k < 0), \quad (0 < k < 1), \quad (1 < k < \infty).$$

Consider a weight $\lambda = \lambda(t)$ defined by the rule

$$\lambda(t) = \begin{cases} \text{a positive constant} \stackrel{\text{def}}{=} \lambda_- & \text{if } 0 < t < 1, \\ \text{a positive constant} \stackrel{\text{def}}{=} \lambda_+ & \text{if } 1 < t < \infty. \end{cases}$$

Thus the λ -harmonics $H = H(t)$, $0 < t < \infty$, which are normalized at $t = 1$ by the condition $H(1) = 1$, take the form

$$H(t) = \begin{cases} H_+(t) = \frac{1}{2}[(1+k_-)t + (1-k_-)t^{-1}] & \text{if } 0 < t < 1, \\ H_-(t) = \frac{1}{2}[(1+k_+)t + (1-k_+)t^{-1}] & \text{if } 1 < t < \infty. \end{cases}$$

Here the parameters k_- and k_+ must be synchronized to ensure that the ascent function $\mathcal{A}_H(t) = t \cdot \lambda(t) \cdot \dot{H}(t)$ be continuous; that is,

$$\lambda_- \dot{H}_-(1) = \lambda_+ \dot{H}_+(1); \quad \text{equivalently, } \lambda_- k_- = \lambda_+ k_+.$$

Therefore, except for the case when $k_- = k_+ = 0$, we have

$$\lambda_- = ck_+ \quad \text{and} \quad \lambda_+ = ck_- \quad \text{for some constant } c \in \mathbb{R}.$$

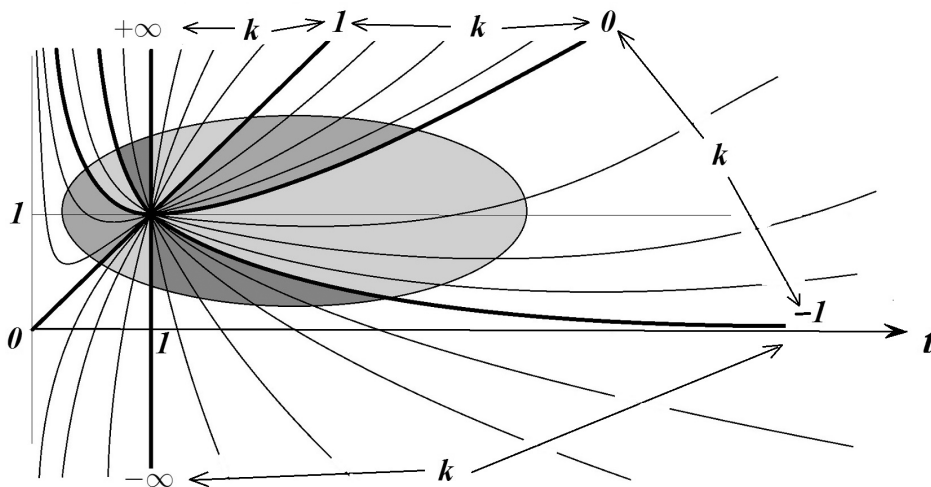


Figure 9: Four classes of normalized harmonic curves.

The case $k_- = k_+ = 0$ is especially interesting. In this case we have $H_-(t) = \frac{1}{2}[t + t^{-1}]$ for $0 < t \leq 1$ and $H_+(t) = \frac{1}{2}[t + t^{-1}]$ for $1 \leq t < \infty$. Therefore, the so-called Nitsche function

$$H(t) \stackrel{\text{def}}{=} \frac{1}{2}[t + t^{-1}]$$

solves the λ -harmonic equation (1.5) in $\mathbb{R}_+$ for all weights $\lambda_- > 0$ and $\lambda_+ > 0$. In all other cases $\dot{H}_-(1)$ and $\dot{H}_+(1)$ must have the same sign. In particular, if we consider only nondecreasing λ -curves then both slopes $\dot{H}_-(1)$ and $\dot{H}_+(1)$ must be positive.

3.2.2 The λ -Nitsche condition

We address the following question:

Question. When is a λ -minimal curve $\mathcal{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ injective?

It follows from what has been established before that $\mathcal{H}$ must be a nondecreasing λ -curve, so that $\dot{\mathcal{H}} \geq 0$ almost everywhere. We point out that the stronger inequality $\dot{\mathcal{H}} > 0$, almost everywhere, is automatic.

Let us discuss in detail a weight λ which assumes exactly two values, namely

$$\lambda(t) \equiv \begin{cases} m & \text{if } r \leq t < \rho, \\ M & \text{if } \rho < t \leq R. \end{cases}$$

Here m and M are positive numbers (not necessary $m < M$). Thus the ratio $\nu \stackrel{\text{def}}{=} \frac{m}{M}$ can be any positive number.

Theorem 3.3 (λ -Nitsche condition). *A λ -minimal curve $\mathcal{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ is injective (homeomorphism) if and only if*

$$\begin{aligned} 4 \frac{R_*}{r_*} &\geq (1 + \nu) \left(\frac{R}{r} + \frac{r}{R} \right) + (1 - \nu) \left(\frac{rR}{\rho^2} + \frac{\rho^2}{rR} \right) \left(\nu = \frac{m}{M} \right) \\ &= \left(\frac{R}{\rho} + \frac{\rho}{R} \right) \left(\frac{r}{\rho} + \frac{\rho}{r} \right) + \nu \left(\frac{R}{\rho} - \frac{\rho}{R} \right) \left(\frac{\rho}{r} - \frac{r}{\rho} \right). \end{aligned} \quad (3.3)$$

The case $\nu = 1$ reduces to the familiar Nitsche condition $\frac{R_*}{r_*} \geq \frac{1}{2} \left(\frac{R}{r} + \frac{r}{R} \right)$.

Proof. Every λ -harmonic curve takes the form

$$\mathcal{H}(t) = \begin{cases} \mathcal{H}_m(t) \stackrel{\text{def}}{=} at + bt^{-1} & \text{if } r \leq t \leq \rho, \\ \mathcal{H}_M(t) \stackrel{\text{def}}{=} At + Bt^{-1} & \text{if } \rho \leq t \leq R. \end{cases} \quad (3.4)$$

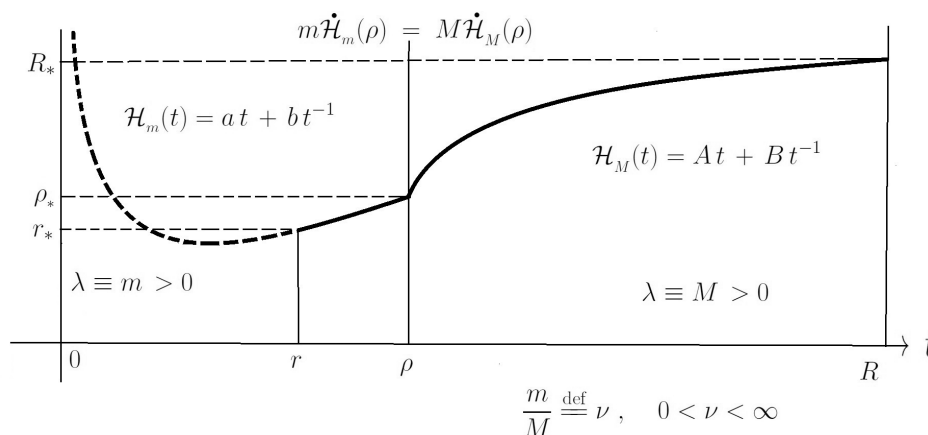


Figure 10: Increasing minimal curve composed of two harmonics.

We have the following linear system of equations for the coefficients a, b, A, B :

$$\begin{cases} ar + br^{-1} = r_*, & \text{where } r_* \stackrel{\text{def}}{=} \mathcal{H}(r), \\ a\rho + b\rho^{-1} = \rho_*, & \text{where } \rho_* \stackrel{\text{def}}{=} \mathcal{H}(\rho), \\ A\rho + B\rho^{-1} = \rho_*, & \text{where } \rho_* \stackrel{\text{def}}{=} \mathcal{H}(\rho), \\ AR + BR^{-1} = R_*, & \text{where } R_* \stackrel{\text{def}}{=} \mathcal{H}(R). \end{cases}$$

This system of equations yields the following formulas for the coefficients a, b, A, B , by means of ρ_* (which is still unknown):

$$\begin{cases} a = \frac{\rho\rho_* - rr_*}{\rho^2 - r^2}, & \text{where } \rho > r, \\ b = \frac{\rho r_* - r\rho_*}{\rho^2 - r^2}r\rho, & \text{where } \rho > r, \\ A = \frac{RR_* - \rho\rho_*}{R^2 - \rho^2}, & \text{where } R > \rho, \\ B = \frac{R\rho_* - \rho R_*}{R^2 - \rho^2}R, & \text{where } R > \rho. \end{cases} \quad (3.5)$$

We have one more linear equation. Namely, the λ -harmonic equation yields $m\dot{\mathcal{H}}_m(\rho) = M\dot{\mathcal{H}}_M(\rho)$. Equivalently,

$$m(a - b\rho^{-2}) = M(A - B\rho^{-2}).$$

Substituting a, b, A, B into this equation yields a linear equation for ρ_* . We express the solution as

$$\rho_* = \frac{2v\alpha(\beta^2 - 1)r_* + 2\beta(1 - \alpha^2)R_*}{v(\beta^2 - 1)(1 + \alpha^2) + (1 - \alpha^2)(1 + \beta^2)}, \quad (3.6)$$

where

$$\alpha \stackrel{\text{def}}{=} \frac{r}{\rho} < 1 \quad \text{and} \quad \beta \stackrel{\text{def}}{=} \frac{R}{\rho} > 1, \quad \text{thus} \quad \frac{R}{r} = \frac{\beta}{\alpha}.$$

Inserting these expressions into (3.4), we arrive at the explicit formula for the desired λ -harmonic function (piecewise harmonic). There is no need to write this formula down here. The necessary and sufficient condition for $\mathcal{H}$ to be strictly increasing is that $\dot{\mathcal{H}}(t) \geq 0$. Indeed, since $\mathcal{L}(t) = t\lambda(t)\dot{\mathcal{H}}(t)$ has positive derivative, it follows that $\dot{\mathcal{H}}(t) > 0$ almost everywhere. It is therefore necessary and sufficient to impose the following *starting-condition*:

$$\dot{\mathcal{H}}(r) \geq 0.$$

This, in view of (3.4), reads as $ar^2 \geq b$, which in turn is equivalent (due to the first two equations in (3.5)) to the so-called *opening-Nitsche condition*

$$\frac{\rho_*}{r_*} \geq \frac{1}{2} \left(\frac{r}{\rho} + \frac{\rho}{r} \right) = \frac{1}{2} \left(\alpha + \frac{1}{\alpha} \right).$$

At this stage we appeal to formula (3.6) to deduce that

$$\frac{\rho_*}{r_*} = \frac{2\nu\alpha(\beta^2 - 1) + 2\beta(1 - \alpha^2)\frac{R_*}{r_*}}{\nu(\beta^2 - 1)(1 + \alpha^2) + (1 - \alpha^2)(1 + \beta^2)} \geq \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right).$$

This inequality gives the following desired lower bound (necessary and sufficient) of the ratio $\frac{R_*}{r_*}$:

$$\begin{aligned} 4\beta(1 - \alpha^2)\frac{R_*}{r_*} &\geq (\alpha + \alpha^{-1})[\nu(\beta^2 - 1)(1 + \alpha^2) + (1 - \alpha^2)(1 + \beta^2)] - 4\nu\alpha(\beta^2 - 1) \\ &= (1 - \alpha^2)(\beta^2 - 1)(\alpha^{-1} - \alpha)\nu + (1 - \alpha^2)(1 + \beta^2)(\alpha + \alpha^{-1}). \end{aligned}$$

Hence

$$4\frac{R_*}{r_*} \geq \left(\beta - \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \alpha\right)\nu + \left(\beta + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} + \alpha\right) = (1 + \nu)\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) + (1 - \nu)\left(\alpha\beta + \frac{1}{\alpha\beta}\right).$$

This is the same as (3.3). $\square$

3.2.3 Critical λ -Harmonic Homeomorphisms

A λ -harmonic homeomorphism $\mathcal{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ is said to be *critical* if (3.3) holds as equality. We denote it by $\mathcal{H}_{\text{cr}}$. The critical curve is composed of one or two harmonics in which the left one, defined on the subinterval $[r, \rho]$, has vanishing derivative at $t = r$. In other words, its ascending (Lipschitz continuous) function $\mathcal{A}_{\text{cr}}(t) \stackrel{\text{def}}{=} t\lambda(t)\dot{\mathcal{H}}_{\text{cr}}(t)$ vanishes at $t = r$, and is continuously increasing.

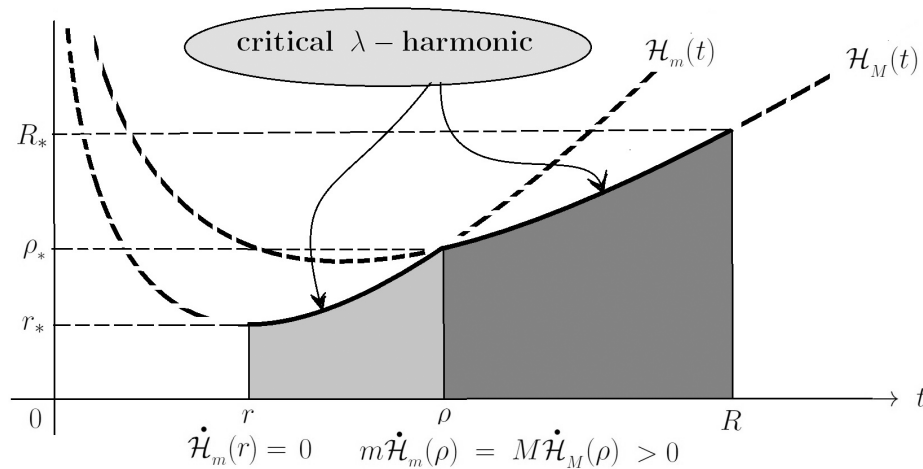


Figure 11: A critical energy-minimal homeomorphism.

3.2.4 Below the λ -Nitsche condition

If the ratio $\frac{R_*}{r_*} > 1$ is too close to 1, then the energy-minimal curve begins with a flat segment. To determine the Nitsche point (where the curve begins to increase), consider a function given by the right-hand side of the λ -Nitsche condition in (3.3); namely,

$$\phi(r) \stackrel{\text{def}}{=} \left(\frac{R}{\rho} + \frac{\rho}{R}\right)\left(\frac{r}{\rho} + \frac{\rho}{r}\right) + \nu\left(\frac{R}{\rho} - \frac{\rho}{R}\right)\left(\frac{\rho}{r} - \frac{r}{\rho}\right) \quad 0 < r < \rho$$

We see that its derivative is negative

$$\phi'(r) = \left(\frac{R}{\rho} + \frac{\rho}{R}\right)\left(\frac{1}{\rho} - \frac{\rho}{r^2}\right) + \nu\left(\frac{R}{\rho} - \frac{\rho}{R}\right)\left(-\frac{\rho}{r^2} - \frac{1}{\rho}\right) < 0.$$

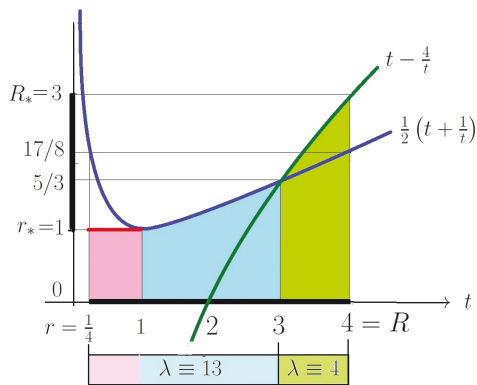


Figure 12: Shaded areas under the λ -minimal curve.

Now, given a critical homeomorphism $\mathcal{H}_{\text{cr}} : [r_0, R] \xrightarrow{\text{onto}} [r_*, R_*]$ so that its left endpoint $0 < r_0 < \rho$ satisfies the equation

$$4 \frac{R_*}{r_*} = \left(\frac{R}{\rho} + \frac{\rho}{R} \right) \left(\frac{r_0}{\rho} + \frac{\rho}{r_0} \right) + v \left(\frac{R}{\rho} - \frac{\rho}{R} \right) \left(\frac{\rho}{r_0} - \frac{r_0}{\rho} \right). \quad (3.7)$$

For every λ -minimal curve $\mathbf{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$, with $0 < r < r_0$, we have an inequality in opposite direction than that in Theorem 3.3. The conclusion is immediate.

Corollary 3.4. *Below the Nitsche bound in Theorem 3.3 the critical Nitsche point of a λ -minimal curve*

$$\mathbf{H} : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$$

is determined, uniquely, by equation (3.7).

See an example in Figure 12.

4 A question

It has been recognized that within the $\mathcal{W}^{1,2}$ -closure of diffeomorphisms $h : \mathbb{A}(r, R) \xrightarrow{\text{onto}} \mathbb{A}(r_*, R_*)$ there is a map of smallest λ -Dirichlet energy (1.1). Call it an *absolute minimizer*.

Question 4.1. Does an absolute minimizer coincide, up to rotation, with the radial map $h(z) = \mathbf{H}(|z|) \frac{z}{|z|}$, where $\mathbf{H}$ is the λ -minimal curve established in this paper?

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