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Flat Outputs in Terms of SISO Operator Compositions

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***Abstract*— The goal of this paper is to use a flat coordinate system to show that a flat output for a SISO flat system can be written in terms of a certain composition of input-output operators. The work is partially motivated by the author's recent work on computing the relative degree of interconnected systems. First the general smooth case is considered, followed by the control affine analytic case. The latter is more amenable to computations in terms of Chen-Fliess series.**

I. Introduction

The concept of differential flatness in control theory was first introduced by Fliess, Lévine, Martin and Rouchon in [4], [5]. It has been used widely in applications involving trajectory tracking and motion planning (see [14] for a survey). In the single-input, single-output (SISO) setting, flatness is exactly equivalent to the solvability of the state space linearizability problem [13]. That is, the existence of an output for which the system has full relative degree.

As motivation for the problem considered here, consider first a SISO linear system $u \mapsto y$ with irreducible transfer function

$$\begin{aligned} H(s) &= K\frac{b(s)}{a(s)} \\ &= K\frac{b_0 + b_1 s + \cdots + b_{n-r-1}s^{n-r-1} + s^{n-r}}{a_0 + a_1 s \cdots + \cdots a_{n-1}s^{n-1} + s^n}, \end{aligned}$$

where $K \neq 0$ and with relative degree $1 \leq r < n$. Define a flat output y_f via the transfer function

$$H_f(s) = \frac{K}{a(s)},$$

so that if $H_i(s) := 1/b(s)$ then

$$H_f(s) = H_i(s)H(s) \tag{1}$$

and

$$\begin{aligned} H(s) &= b_0 H_f(s) + b_1 s H_f(s) + \cdots \\ &\quad + b_{n-r-1}s^{n-r-1}H_f(s) + s^{n-r}H_f(s). \end{aligned} \tag{2}$$

Equation (1) states that the flat output y_f can be written in terms of a composition of two input-output systems, while (2) indicates that the real output y can be written in terms of the flat output and its first $(n-r)$ derivatives.

To view the situation from a state space point of view, first divide $b(s)$ into $a(s)$ so that $a(s) = b(s)p(s) + r(s)$ with $(r(s), b(s))$ being a coprime pair of polynomials

$$\begin{aligned} p(s) &= p_0 + p_1 s + \cdots + p_{r-1}s^{r-1} + s^r \\ r(s) &= r_0 + r_1 s + \cdots + r_{n-r-2}s^{n-r-2} + r_{n-r-1}s^{n-r-1} \end{aligned}$$

and $\deg(r(s)) < \deg(b(s))$. In which case,

$$H(s) = \frac{K}{p(s) + \frac{r(s)}{b(s)}} = \frac{K}{p(s)}\left(1 + \frac{r(s)}{b(s)}\frac{1}{p(s)}\right)^{-1},$$

and thus, $H(s)$ can be viewed as a feedback interconnection with $1/p(s)$ in the forward path, $r(s)/b(s)$ in the feedback path, and K scaling the input. Let (A_1, b_1, c_1) and (A_2, b_2, c_2) be minimal realizations of $1/p(s)$ and $r(s)/b(s)$, respectively. Then a realization of $H(s)$ follows directly from this feedback structure to be

$$\begin{aligned} \dot{z} &= \left[\begin{array}{c|c} A_1 & -b_1 c_2 \\ \hline b_2 c_1 & A_2 \end{array}\right] z + \left[\begin{array}{c} K b_1 \\ \hline 0 \end{array}\right] u, \quad z(0) = z_0 \\ y &= \left[\begin{array}{c|c} c_1 & 0 \end{array}\right] z. \end{aligned}$$

If both realizations are in controller canonical form, then this realization is in the *Byrnes-Isidori normal form*

$$\dot{z}_1 = z_2 \tag{3a}$$
$$\dot{z}_2 = z_3 \tag{3b}$$
$$\vdots$$
$$\dot{z}_{r-1} = z_r \tag{3c}$$
$$\dot{z}_r = P\xi + R\eta + Ku \tag{3d}$$
$$\dot{\eta} = S\xi + Q\eta \tag{3e}$$
$$y = z_1, \tag{3f}$$

where $\xi = [z_1 \cdots z_r]$, $\eta = [z_{r+1} \cdots z_n]$, $P = -[p_0 \cdots p_{r-1}]$, $R = -[r_0 \cdots r_{n-r-1}]$, $S = e_{n-r}(n-r)e_1^T(r)$, and

$$Q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-r-1} \end{bmatrix}.$$

(Here $e_i(j) \in \mathbb{R}^j$ has a one in the i-th position and zero elsewhere. If j is understood then the notation is abbreviated to e_i.) It is immediate that the subsystem $\dot{\eta} = Q\eta + e_{n-r}y$ with input y and output η_1 has transfer function $e_1^T(sI - Q)^{-1}e_n = 1/b(s)$. Thus, it is equivalent to $H_i(s)$ in the cascade structure (1), implying that $y_f = \eta_1$. From (2) it is clear that

$$y = b_0 y_f + b_1 y_f^{(1)} + \cdots + b_{n-r-1}y_f^{(n-r-1)} + y_f^{(n-r)},$$

and thus, in light of the companion form of Q, y can be written in terms of the output function

$$y = \bar{h}(\eta) := [b_0\ b_1\ \cdots\ b_{n-r-1}\ 1]\eta.$$

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In [12] the authors consider a nonlinear generalization of this construction for a smooth flat SISO system

$$\dot{z}(t) = f(z(t), u(t)), \quad z(0) = z_0 \tag{4}$$

defined on a neighborhood $W \times U \subseteq \mathbb{R}^n \times \mathbb{R}$ of (z_0, u_0). In particular, it is shown using a normal form analogous to (3) that for any output $y = h(z)$ with relative degree r there exists an output function $\bar{h} : V \subseteq \mathbb{R}^{(n-r)} \to \mathbb{R}$ and a flat output y_f such that

$$y = \bar{h}\left(y_f, y_f^{(1)}, \ldots, y_f^{(n-r)}\right) = \bar{h}(\eta),$$

where $\eta_i = y_f^{(i-1)}$, $i = 1, 2, \ldots, n-r$. When viewed as a system of $n - r$ differential equations, this can be seen as a diffeomorphic representation of the internal dynamics of (f, h, z_0, u_0). The goal of this paper is to use this setting to more fully develop the nonlinear cascade structure analogous to (1) that renders flat outputs. The work is partially motivated by the recent results in [10], which show how to compute the relative degree of interconnected systems. It also provides some computation tools that will be useful here.

The organization of the paper is as follows. First the general smooth case is considered, followed by the control affine analytic case. The latter is more amenable to computations in terms of Chen-Fliess series [3], [13], [15]. Then a collection of specific examples is considered. The paper's conclusions are summarized in the final section.

II. Smooth Systems

First recall the following definition of relative degree for a smooth system (4) with output $y = h(z)$.

Definition 1: [15, p. 417] The input-output map $y = F[u]$ with smooth realization (f, h, z_0, u_0) has *relative degree* r at (z_0, u_0) if on some neighborhood $W \times U$ of (z_0, u_0):

$$\frac{\partial}{\partial u} L_f^i h(z, u) = 0, \quad i = 0, 1, \ldots, r-1$$
$$\frac{\partial}{\partial u} L_f^r h(z_0, u_0) \neq 0,$$

where $L_f h$ denotes the Lie derivative of h with respect to f.

The following lemma is useful.

Lemma 1: Consider two input-output maps F_1 and F_2 each with a smooth realization $(f_i, h_i, z_{i0}, u_{i0})$ on $W_i \times U_i$ and having relative degree r_i at $(z_{i0}, u_{i0}) \in W_i \times U_i$, where $i = 1, 2$. Then the composed system $F_2 \circ F_1$, provided it is well defined (i.e., $h_1(W_1) \subseteq U_2$ with $u_{20} := h_1(z_{10})$), has relative degree $r_1 + r_2$ at $(\tilde{z} := [z_{10}^T \; z_{20}^T]^T, u_{10})$.

Proof: Apply the definition of relative degree to a realization $(\tilde{f}, \tilde{h}, \tilde{z}_0, u_{10})$ of $F_2 \circ F_1$, namely

$$\tilde{f}(z) = \begin{bmatrix} f_1(z_1, u) \\ f_2(z_2, h_1(z_1)) \end{bmatrix}, \quad z(0) = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}$$
$$\tilde{h}(z) = h_2(z_2),$$

where $z := [z_1^T \; z_2^T]^T$ and $W := W_1 \times W_2$. It follows directly that on $W \times U_1$

$$\frac{\partial}{\partial u} L_{\tilde{f}}^i \tilde{h}(z, u) = 0, \quad i = 0, \ldots, r_2 - 1$$

using the assumption that on $W_2 \times h_1(W_1)$

$$\frac{\partial}{\partial u_2} L_{f_2}^i h_2(z_2, u_2) = 0, \quad i = 0, \ldots, r_2 - 1.$$

Likewise, on $W \times U_1$ it follows that

$$\frac{\partial}{\partial u} L_{\tilde{f}}^{r_2+i} \tilde{h}(z, u) = 0, \quad i = 0, \ldots, r_1 - 1$$

using the assumption that on $W_1 \times U_1$

$$\frac{\partial}{\partial u_1} L_{f_1}^i h_1(z_1, u_1) = 0, \quad i = 0, \ldots, r_2 - 1.$$

Finally, it is clear that

$$\begin{aligned} &\frac{\partial}{\partial u} L_{\tilde{f}}^{r_2+r_1} \tilde{h}(z_0, u_0) \\ &= \frac{\partial}{\partial u_2} L_{f_2}^{r_2} h_2(z_{20}, h_1(z_{10})) \frac{\partial}{\partial u_1} L_{f_1}^{r_1} h_1(z_{10}, u_{10}) \neq 0 \end{aligned}$$

as required. ■

The main result of this section is given next.

Theorem 1: Consider an input-output map $F : u \mapsto y$ with smooth realization (f, h, z_0, u_0) on $W \times U$ and relative degree r at $(z_0, u_0) \in W \times U$. If (f, z_0, u_0) is flat then any flat output $y_f = h_f(z)$ can be written in terms of a composition $y_f = F_i \circ F[u]$, where F_i corresponds to the input-output map for the internal dynamics of F with well defined relative degree $n - r$ at some point $(\eta_0, y(0))$ in the flat normal form coordinates (defined below). Furthermore, the realization $(\tilde{f}, \tilde{h}_f, \tilde{z}_0, u_0)$ of $F_i \circ F[u]$ has well defined relative degree n at $(\tilde{z}_0, u_0)$.

Proof: It was shown in [12, Theorem 7] by passing through the flat coordinates $\bar{z}_i = y_f^{(i-1)}$, $i = 1, 2, \ldots, n$ that there exists a smooth diffeomorphism $[\xi^T \; \eta^T]^T := \phi(z)$ which transforms (f, h, z_0, u_0) into the *flat* normal form

$$\dot{\xi}_1 = \xi_2 \tag{5a}$$
$$\dot{\xi}_2 = \xi_3 \tag{5b}$$
$$\vdots$$
$$\dot{\xi}_{r-1} = \xi_r \tag{5c}$$
$$\dot{\xi}_r = p(\xi, \eta, u) \tag{5d}$$
$$\dot{\eta}_1 = \eta_2 \tag{5e}$$
$$\dot{\eta}_2 = \eta_3 \tag{5f}$$
$$\vdots$$
$$\dot{\eta}_{n-r-1} = \eta_{n-r} \tag{5g}$$
$$\dot{\eta}_{n-r} = q(\eta, \xi_1), \tag{5h}$$
$$y = \xi_1, \tag{5i}$$

where $\partial q / \partial \xi_1(\eta_0, y(0)) \neq 0$, and $y_f = \eta_1$. Therefore, the realization $(q, h_q(\eta) := \eta_1, \eta_0, y_0)$ of $F_i : y \mapsto \eta_1$ has relative degree $n - r$ at $(\eta_0, y(0))$. By assumption (5) as a realization of $F : u \mapsto y$ has relative degree r at $([\xi_0^T \; \eta_0^T]^T, u_0)$. Hence, from the structure of (5) and Lemma 1, it is immediate that the realization $(\tilde{f}, \tilde{h}_f, \tilde{z}_0, u_0)$ of $F_i \circ F : u \mapsto y_f$ has relative degree n at $(\tilde{z}_0, u_0)$. ■

III. Control Affine Analytic Systems

In the case where $f(z,u)$ in (4) is control affine, it is not true in general that F_i in Theorem 1 has a control affine state space realization. So in this section, conditions are given under which a flat output of a control affine analytic system

$$\dot{z} = g_0(z) + g_1(z)u, \;\; z(0) = z_0,$$

can be written in terms of the composition of two operators, where each operator has a control affine analytic realization. The analyticity ensures that for any analytic output $y = h(z)$, the corresponding input-output map $u \mapsto y$ can be written in terms of a convergent Chen-Fliess series or Fliess operator [3], [13], [15]. In this setting, one can do explicit calculations purely in an input-output setting.

Let $X = \{x_0, x_1, \ldots, x_m\}$ be a set of noncommuting letters with $m \in \mathbb{N}$. The set of all words having finite length, X^*, forms a monoid under catenation, where the identity element is the empty word, $\emptyset$. The set of all words with prefix $\eta \in X^*$ is written as ηX^*. A formal power series over X is any mapping $c : X^* \to \mathbb{R}^\ell$, where c evaluated at $\eta \in X^*$ is written as (c, η). It is customary to represent c as the formal sum $c = \sum_{\eta \in X^*}(c,\eta)\eta$. A series c is said to be *proper* when $(c, \emptyset) \neq 0$. The *support* of c is defined as $\mathrm{supp}(c) = \{\eta \in X^* : (c,\eta) \neq 0\}$.

Any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ can be used as a generating series to define a causal m-input, ℓ-output operator, F_c. First fix $\mathfrak{p} \geq 1$ and $t_0 < t_1$. For a Lebesgue measurable function $u_i : [t_0, t_1] \to \mathbb{R}$, let $\|u_i\|_\mathfrak{p}$ denote the usual $L_\mathfrak{p}$ function norm. For any measurable vector-valued function $u : [t_0, t_1] \to \mathbb{R}^m$ define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : \; 1 \leq i \leq m\}$. Let $L_\mathfrak{p}^m[t_0, t_1]$ denote the set of all such functions having finite $\|\cdot\|_\mathfrak{p}$ norm. A closed ball of radius R at the origin of $L_\mathfrak{p}^m[t_0, t_1]$ is written as $B_\mathfrak{p}^m(R)[t_0, t_1]$. Let $C[t_0, t_1] \subset L_1^m[t_0, t_1]$ be the set of continuous functions. For any $\eta \in X^*$ define the iterated integral $E_\eta : L_1^m[t_0, t_1] \to C[t_0, t_1]$ inductively by first setting $E_\emptyset[u] = 1$ and then letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output map associated with c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0).$$

If $y = F_c[u]$ has a control affine analytic realization (g, h, z_0) in local coordinates, then

$$(c, \eta) = L_{g_{i_1}} \cdots L_{g_{i_k}} h(z_0) \tag{6}$$

for any word $\eta = x_{i_k} \cdots x_{i_1} \in X^*$. Furthermore, it is shown in [11] that F_c converges in a local sense and constitutes a well defined mapping from $B_\mathfrak{p}^m(R)[t_0,\, t_0 + T]$ into $B_\mathfrak{q}^\ell(S)[t_0,\, t_0 + T]$ for sufficiently small $R, T > 0$ and $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ satisfying $1/\mathfrak{p} + 1/\mathfrak{q} = 1$.

The following definition describes the relative degree of a generating series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ when $X = \{x_0, x_1\}$. It is equivalent to the usual definition whenever F_c has an analytic realization (g, h, z_0) [6], [7]. It employs the language of *linear words*

$$L = \{\eta \in X^* : \eta = x_0^{n_1} x_1 x_0^{n_0}, \; n_1, n_0 \geq 0\}.$$

In addition, the decomposition of $c \in \mathbb{R}\langle\langle X \rangle\rangle$ into its natural and forced components is useful, i.e., $c = c_N + c_F$, where $c_N := \sum_{k \geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$.

Definition 2: [6] Given $c \in \mathbb{R}\langle\langle X \rangle\rangle$, let $r \geq 1$ be the largest integer such that $\mathrm{supp}(c_F) \subseteq x_0^{r-1}X^*$. Then c has *relative degree* r if the linear word $x_0^{r-1}x_1 \in \mathrm{supp}(c)$, otherwise it is not well defined.

It can be verified that c has relative degree r if and only if there exists some proper $e \in \mathbb{R}\langle\langle X \rangle\rangle$ with $x_1 \notin \mathrm{supp}(e)$ such that

$$c = c_N + c_F = c_N + Kx_0^{r-1}x_1 + x_0^{r-1}e \tag{7}$$

with $K \neq 0$.

The cascade connection of two convergent Fliess operators is known to always yield another operator in this class, independent of whether any of these operators are realizable [2], [8]. To compute the corresponding generating series, first observe that under the catenation product $\mathbb{R}^\ell\langle\langle X \rangle\rangle$ is an associative $\mathbb{R}$-algebra. It also forms an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, which is defined inductively on words by

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$. The product is then extended linearly to formal power series. If F_c and F_d are two Fliess operators with $c, d \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, then the input-output maps for the parallel sum and parallel product connections are given by $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup\!\sqcup d}$, respectively [3]. The cascade connection $F_c \circ F_d$, where $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$, yields the Fliess operator $F_{c \circ d}$. Here the *composition product*

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \psi_d(\eta)(1) \tag{8}$$

is defined in terms of the continuous (in the ultrametric topology) algebra homomorphism ψ mapping $\mathbb{R}\langle\langle X \rangle\rangle$ to the set of vector space endomorphisms on $\mathbb{R}\langle\langle X \rangle\rangle$, $\mathrm{End}(\mathbb{R}\langle\langle X \rangle\rangle)$. It is uniquely determined by $\psi_d(x_i\eta) = \psi_d(x_i) \circ \psi_d(\eta)$, where

$$\psi_d(x_i)(e) = x_0(d_i \sqcup\!\sqcup e), \;\; e \in \mathbb{R}\langle\langle X \rangle\rangle$$

for $i = 0, 1, \ldots, m$ with d_i being the i-th component series of d ($d_0 := 1$). For the empty word, $\psi_d(\emptyset)$ is taken to be the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$. It is known that the composition product is associative and $\mathbb{R}$-linear in its left argument. In [10] it is shown that if c and d have relative degree r_c and r_d, respectively, then $c \circ d$. has relative degree $r_c + r_d$. Of course, this fact is expected from Lemma 1 in the case where both operators are realizable.

Now the main results of this section are developed. Henceforth, it is assumed that $X = \{x_0, x_1\}$. The following lemmas will be needed.

Lemma 2: Let $c \in \mathbb{R}\langle\langle X \rangle\rangle$ be proper with relative degree r. If F_c is convergent, and $\gamma : \mathbb{R} \to \mathbb{R}$ is real analytic on a neighborhood of the origin, then there exists a $c_\gamma \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $F_{c_\gamma} = \gamma(F_c)$, and F_{c_γ} is convergent. In addition, c_γ has relative degree r.

Proof: Letting $\gamma(z) = \sum_{k\geq 0} \gamma_k z^k/k!$, it follows directly by substitution that

$$c_\gamma = \gamma \circ c := \sum_{k=0}^{\infty} \gamma_k \frac{c^{\sqcup\!\sqcup\, k}}{k!}, \tag{9}$$

where $c^{\sqcup\!\sqcup\, k}$ denotes the shuffle power of c. It was shown in [9] using the properness of c that c_γ is always well defined (locally finite). In this same work it is also shown that the convergence property of F_c is preserved by (9). To see that c_γ has relative degree r, it can be verified for $k > 1$ that $\mathrm{supp}(c^{\sqcup\!\sqcup\, k}) \subseteq x_0^{r-1}X^*$ (see [10, Lemma 4]) and that $x_0^{r-1}x_1 \notin \mathrm{supp}(c^{\sqcup\!\sqcup\, k})$. In which case, the claim follows directly from the assumption that c has relative degree r. ∎

Lemma 3: Let $c \in \mathbb{R}\langle\langle X\rangle\rangle$ be nonproper with relative degree r. If F_c is convergent, and $\gamma : \mathbb{R} \to \mathbb{R}$ is a polynomial, then there exists a $c_\gamma \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $F_{c_\gamma} = \gamma(F_c)$, and F_{c_γ} is convergent. If $\gamma'((c, \emptyset)) \neq 0$, then c_γ has relative degree r.

Proof: If $\deg(\gamma) = N$ then clearly

$$c_\gamma = \sum_{k=0}^{N} \gamma_k \frac{c^{\sqcup\!\sqcup\, k}}{k!}.$$

Again, the convergence property of F_{c_γ} was proved in [9]. The relative degree claim is more subtle than in the previous lemma. It was shown in [10, Theorem 4.2] that if c and d have relative degree r then $c \sqcup\!\sqcup\, d$ has relative degree r if and only if $(c \sqcup\!\sqcup\, d, x_0^{r-1}x_1) = (c, \emptyset)(d, x_0^{r-1}x_1) + (c, x_0^{r-1}x_1)(d, \emptyset) \neq 0$. Therefore, since c is nonproper and has relative degree r, $c^{\sqcup\!\sqcup\, k}$ has relative degree r for all $k \geq 1$. (Note this claim was not made in the proof of the previous lemma.) In particular,

$$(c^{\sqcup\!\sqcup\, k}, x_0^{r-1}x_1) = k(c, \emptyset)^{k-1}(c, x_0^{r-1}x_1) \neq 0.$$

In addition, it was shown in [10, Theorem 4.1] that if c and d have relative degree r then $c+d$ has relative degree r if and only if $(c+d, x_0^{r-1}x_1) = (c, x_0^{r-1}x_1) + (d, x_0^{r-1}x_1) \neq 0$. Therefore, assuming $N > 0$ (otherwise $\gamma' = 0$), c_γ has relative degree r if and only if

$$\begin{aligned}(c_\gamma, x_0^{r-1}x_1) &= (c, x_0^{r-1}x_1) \sum_{k=1}^{N} (c, \emptyset)^{k-1} \frac{\gamma_k}{(k-1)!} \\ &= (c, x_0^{r-1}x_1)\gamma'((c, \emptyset)) \neq 0,\end{aligned}$$

which proves the final assertion. ∎

Lemma 4: If $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree $r > 1$, and $y = F_c[u]$ is convergent, then there exists $c' \in \mathbb{R}\langle\langle X\rangle\rangle$ with relative degree $r - 1$ such that $dy/dt = F_{c'}[u]$ is also convergent. If $r = 1$ then dy/dt corresponds to the relative degree zero case.

Proof: It is immediate that

$$\frac{dy}{dt} = F_{x_0^{-1}(c)} + uF_{x_1^{-1}(c)},$$

where $x_i^{-1}(\cdot)$ denotes the left-shift operator. If $r > 1$ then directly $x_1^{-1}(c) = 0$, and $c' = x_0^{-1}(c)$ must have relative degree $r - 1$. The convergence of $F_{c'}$ follows directly from that of F_c [11]. If $r = 1$ then clearly dy/dt depends explicitly on u, so the relative degree is defined to be zero. ∎

Lemma 5: If $y = F[u]$ has an analytic realization (f, h, z_0) with relative degree r at (z_0, u_0) then the augmented realization

$$\begin{aligned}\dot{z} &= f(z, w), \;\; z(0) = z_0 \\ \dot{w} &= v, \;\; w(0) = u_0 \\ y &= h(z)\end{aligned}$$

is control affine and has relative degree $r + 1$ at $[z_0^T, u_0]^T$.

Proof: It is clear that the input-output map $v \mapsto y$ is control affine. (See [17] for further applications of this type of dynamic extension.) The relative degree claim follows by a direct calculation showing that on a neighborhood of $[z_0^T \; u_0]^T$:

$$\begin{aligned}L_{g_1}L_{g_0}^i h(z, w) &= \frac{\partial}{\partial w} L_f^i h(z, w) = 0, \;\; i = 0, 1, \ldots, r-1 \\ L_{g_1}L_{g_0}^r h(z, w) &= \frac{\partial}{\partial w} L_f^r h(z_0, u_0) \neq 0,\end{aligned}$$

where

$$g_0(z, w) = \begin{bmatrix} f(z, w) \\ 0 \end{bmatrix} \;\; g_1(z, w) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{10}$$

∎

The main result of this section is given next.

Theorem 2: Consider an input-output map $F_c : u \mapsto y$ with $c \in \mathbb{R}\langle\langle X\rangle\rangle$ and an analytic realization (g, h, z_0) on W with relative degree r at $z_0 \in W$. Assume (g, z_0) is flat, and let y_f denote any flat output. Then the following hold:

1) If c is nonproper, q in (5h) is *separable* in y, i.e., $q(\eta, y) = q_0(\eta) + q_1(\eta)\gamma(y)$, and γ is a polynomial such that $\gamma'((c, \emptyset)) \neq 0$, then $y_f = F_{c_i} \circ F_{c_\gamma}[u]$, where F_{c_i} corresponds to the input-output map of the internal dynamics $(q_0, q_1, \eta_0, h_q(\eta) := \eta_1)$ and has relative degree $n - r$ at η_0. Furthermore, $c_f := c_i \circ c_\gamma$ has relative degree n.
2) If c is proper, and q in (5h) is separable in y, then $y_f = F_{c_i} \circ F_{c_\gamma}[u]$, where F_{c_i} corresponds to the input-output map of the internal dynamics (q_0, q_1, η_0, h_q) and has relative degree $n-r$ at η_0. Furthermore, $c_f := c_i \circ c_\gamma$ has relative degree n.
3) If c has relative degree $r > 1$, then $y_f = F_{c_e} \circ F_{c'}[u]$, where F_{c_e} corresponds to the input-output map of the extended internal dynamics $(q_{e,0}, q_{e,1}, [\eta_0^T, y(0)]^T, h_q)$ and has relative degree $n - r + 1$ at $[\eta_0^T, y(0)]^T$, while $F_{c'} : u \mapsto y'$ and $c' = x_0^{-1}(c)$ has relative degree $r - 1$. Furthermore, $c_f := c_e \circ c'$ has relative degree n.

Proof:

1) In the normal form coordinates (5), the input to the internal dynamics enters linearly as $\gamma(y)$, while $y_f = \eta_1$. Therefore, $y_f = F_{c_i}[\gamma(y)] = F_{c_i}[\gamma(F_c[u])] = F_{c_i} \circ F_{c_\gamma}[u]$. Under the stated conditions, Lemma 3 applies,

so the relative degree claims follows from this result and Theorem 1.

2) The argument here is similar to that of the previous item except Lemma 2 is applied.
3) This claim follows directly from Lemmas 4-5.

■

IV. Examples

The main results of the previous sections are illustrated in this section by examples. The first example applies Theorem 1. The remaining examples exercise all the cases in Theorem 2.

Example 1: The normalized dynamics of a SISO isothermal continuous stirred tank reactor (CSTR) for the Van de Vusse example are given by

$$\dot{z}_1 = -z_1 - z_1^2 + (1 - z_1)u \quad \text{(11a)}$$

$$\dot{z}_2 = z_1 - z_2 - z_2 u, \quad \text{(11b)}$$

where z_i is the concentration of the i-th reactant [16]. The output $y = z_i$ has relative degree 1 at $z_0 = [2\,2]^T$ for $i = 1, 2$. The system is flat since $y_f = z_2/(1 - z_1)$ has relative degree 2 at z_0. Taking, for example, $y = z_1$, the change of coordinates

$$\bar{z}_1 = z_1$$
$$\bar{z}_2 = \frac{z_2}{1 - z_1}$$

gives normal form (5) with

$$q(\bar{z}_2, y) = \frac{y^2 \bar{z}_2 - y + \bar{z}_2}{y - 1}.$$

Even though (11) is control affine, q does not fit any of the cases in Theorem 2. However, the nonaffine factorization in Theorem 1 is still available. It is easily verified that $y_f = \bar{z}_2 = F_i[y]$ with realization $(q, -2, h_q)$ has relative degree 1 at $(\bar{z}_{20}, y_0) = (-2, 2)$ so that the realization of $F_i \circ F_c$ has relative degree 2 at $([z_0^T\,\bar{z}_2]^T, u_0)$ for any u_0. □

Example 2: A normalized three species Lotka-Volterra system with an exogenous input u is

$$\dot{z}_1 = z_1 - z_1 z_2$$
$$\dot{z}_2 = -z_2 + z_1 z_2 - z_2 z_3$$
$$\dot{z}_3 = -z_3 + z_2 z_3 + u,$$

where z_i denotes the biomass of the i-th species [1]. The relative degree for output $y = z_i$ at $z_0 = [1\,1\,1]^T$ is $r_i = 4 - i$, $i = 1, 2, 3$. So, in particular, the output $y = z_1$ is flat. For the nonflat output $y = z_3$ the coordinate transformation

$$\bar{z}_1 = z_3$$
$$\bar{z}_2 = z_1$$
$$\bar{z}_3 = z_1 - z_1 z_2,$$

puts the system into normal form (5) with $q(\bar{z}_2, \bar{z}_3, y)$ separable, specifically,

$$q_0(\bar{z}_2, \bar{z}_3) = \begin{bmatrix} \bar{z}_3 \\ -\bar{z}_2^2 + \bar{z}_2\bar{z}_3 + \frac{\bar{z}_3^2}{\bar{z}_2} \end{bmatrix},$$

$$q_1(\bar{z}_2, \bar{z}_3) = \begin{bmatrix} 0 \\ \bar{z}_2 - \bar{z}_3 \end{bmatrix},$$

and $\gamma(y) = 1 + y$. Furthermore, since $(c, \emptyset) = h(z_0) = 1 \neq 0$ and $\gamma'(y) = 1 \neq 0$, case 1 of Theorem 2 applies. The generating series for the given output and the flat output are, respectively,

$$\begin{aligned} c = {} & 1 + x_1 - x_0^2 + x_0^3 - 2x_0^2x_1 - x_0x_1x_0 + 4x_0^4 + \\ & 3x_0^3x_1 + 2x_0^2x_1x_0 - 2x_0^2x_1^2 + x_0x_1x_0^2 - x_0x_1x_0x_1 - \\ & 19x_0^5 + 10x_0^4x_1 + 6x_0^3x_1x_0 + 6x_0^3x_1^2 + 3x_0^2x_1x_0^2 + \cdots \\ c_f = {} & 1 + x_0^2 - x_0^3 + x_0^2x_1 + 2x_0^4 - 2x_0^3x_1 - x_0^2x_1x_0 - x_0^5 + \\ & 6x_0^4x_1 + 4x_0^3x_1x_0 - 2x_0^3x_1^2 + 2x_0^2x_1x_0^2 - x_0^2x_1x_0x_1 - \\ & 17x_0^6 - 9x_0^5x_1 - 8x_0^4x_1x_0 + 10x_0^4x_1^2 - 6x_0^3x_1x_0^2 + \cdots \end{aligned}$$

The relative degrees $r_c = 1$ and $r_{c_f} = 3$ are readily apparent in light of (7). The generating series $c_\gamma = 1 + c$, and from the internal dynamics $(q_0, q_1, [1\,0]^T, h_q)$

$$\begin{aligned} c_i = {} & 1 - x_0^2 + x_0x_1 - x_0^3 + x_0^2x_1 + x_0x_1x_0 - x_0x_1^2 + 3x_0^4 - \\ & 3x_0^3x_1 - x_0^2x_1x_0 + x_0^2x_1^2 - x_0x_1^2x_0 + x_0x_1^3 + 14x_0^5 - \\ & 14x_0^4x_1 - 11x_0^3x_1x_0 + 11x_0^3x_1^2 - 7x_0^2x_1x_0^2 + \cdots \end{aligned}$$

As expected, $r_{c_\gamma} = 1$ and $r_{c_i} = 2$, and a direct application of (8) gives $c_f = c_i \circ c_\gamma$. □

Example 3: Consider system

$$\dot{z}_1 = z_1 z_3 + z_1 u$$
$$\dot{z}_2 = -z_1^2 + z_2 z_3 + z_2 u$$
$$\dot{z}_3 = (z_1 - z_2)\frac{z_3}{z_2}.$$

The output $y = 1 - (z_2/z_1)$ has relative degree 2 at $z_0 = [1\,1\,1]^T$, while the output $y = z_3$ has relative degree 3 at z_0. The change of coordinates for the nonflat output

$$\bar{z}_1 = 1 - \frac{z_2}{z_1}$$
$$\bar{z}_2 = z_1$$
$$\bar{z}_3 = z_3$$

puts the system into normal form (5) with $q(\bar{z}_3, y)$ separable, where $q_0(\bar{z}_3) = -\bar{z}_3$, $q_1(\bar{z}_3) = \bar{z}_3$, and $\gamma(y) = 1/(1 - y) = \sum_{k\geq 0} y^k$. Since $(c, \emptyset) = h(z_0) = 0$, the generating series c is proper and case 2 of Theorem 2 applies. The generating series for the given output and the flat output are, respectively,

$$\begin{aligned} c = {} & x_0 + x_0^2 + x_0x_1 + x_0^3 + x_0^2x_1 + x_0x_1x_0 + x_0x_1^2 + \\ & 2x_0^4 + x_0^3x_1 + x_0^2x_1x_0 + x_0^2x_1^2 + x_0x_1x_0^2 + \\ & x_0x_1x_0x_1 + x_0x_1^2x_0 + x_0x_1^3 + 8x_0^5 + 3x_0^4x_1 + \cdots \\ c_f = {} & 1 + x_0^2 + 3x_0^3 + x_0^2x_1 + 16x_0^4 + 5x_0^3x_1 + 3x_0^2x_1x_0 + \\ & x_0^2x_1^2 + 106x_0^5 + 37x_0^4x_1 + 26x_0^3x_1x_0 + 9x_0^3x_1^2 + \\ & 14x_0^2x_1x_0^2 + 5x_0^2x_1x_0x_1 + 3x_0^2x_1^2x_0 + x_0^2x_1^3 + \cdots \end{aligned}$$

Observe that $r_c = 2$ and $r_{c_f} = 3$. The generating series $c_\gamma = \sum_{k\geq 0} c^{\sqcup\!\sqcup\, k}$ and c_i are found to be

$$c_\gamma = 1 + x_0 + 3x_0^2 + x_0x_1 + 13x_0^3 + 5x_0^2x_1 + 3x_0x_1x_0 +$$

$$x_0x_1^2 + 52x_0^4 + 31x_0^3x_1 + 23x_0^2x_1x_0 + 9x_0^2x_1^2 + 13x_0x_1x_0^2 + 5x_0x_1x_0x_1 + 3x_0x_1^2x_0 + x_0x_1^3 + \cdots$$

$$c_i = 1 - x_0 + x_1 + x_0^2 - x_0x_1 - x_1x_0 + x_1^2 - x_0^3 + x_0^2x_1 + x_0x_1x_0 - x_0x_1^2 + x_1x_0^2 - x_1x_0x_1 - x_1^2x_0 + x_1^3 + x_0^4 - x_0^3x_1 - x_0^2x_1x_0 + x_0^2x_1^2 - x_0x_1x_0^2 + \cdots$$

Here c_γ has relative degree 2, and c_i has relative degree 1. By direct computation it is found that $c_f = c_i \circ c_\gamma$. □

Example 4: Consider Example 4.3.4 in [13] where

$$\dot{z}_1 = z_3 - z_2^3$$
$$\dot{z}_2 = -z_2 - u$$
$$\dot{z}_3 = z_1^2 - z_3 + u.$$

The output $y = z_1$ has relative degree 2 at $z_0 = [1\,1\,1]^T$, while the outputs $y = z_2$ and $y = z_3$ both have relative degree 1 at z_0. The system is flat since the output $y = z_2 + z_3$ has relative degree 3 at z_0. For the $y = z_1$ case, the change of coordinates

$$\bar{z}_1 = z_1$$
$$\bar{z}_2 = z_3 - z_2^3$$
$$\bar{z}_3 = z_2 + z_3$$

puts the system into normal form (5) with $q(\bar{z}_3, y)$ separable, namely, $q_0(\bar{z}_3) = -\bar{z}_3$, $q_1(\bar{z}_3) = 1$, and $\gamma(y) = y^2$. Since $(c, \emptyset) = h(z_0) = 1 \neq 0$ and $\gamma'(1) = 2 \neq 0$, the factorization in case 1 of Theorem 2 is available. The generating series for the given output and the flat output are, respectively,

$$c = 1 + 3x_0^2 + 4x_0x_1 - 9x_0^3 - 10x_0^2x_1 - 6x_0x_1x_0 - 6x_0x_1^2 + 33x_0^4 + 36x_0^3x_1 + 18x_0^2x_1x_0 + 18x_0^2x_1^2 + 12x_0x_1x_0^2 + 12x_0x_1x_0x_1 + 6x_0x_1^2x_0 + 6x_0x_1^3 + \cdots$$

$$c_f = 2 - x_0 + x_0^2 + 5x_0^3 + 8x_0^2x_1 - 23x_0^4 - 28x_0^3x_1 - 12x_0^2x_1x_0 - 12x_0^2x_1^2 + 143x_0^5 + 172x_0^4x_1 + 96x_0^3x_1x_0 + 112x_0^3x_1^2 + 48x_0^2x_1x_0^2 + \cdots$$

Observe that $r_c = 2$ and $r_{c_f} = 3$. The generating series $c_\gamma = c \sqcup\!\sqcup\, c$ and c_i are found to be

$$c_\gamma = 1 + 6x_0^2 + 8x_0x_1 - 18x_0^3 - 20x_0^2x_1 - 12x_0x_1x_0 - 12x_0x_1^2 + 120x_0^4 + 144x_0^3x_1 + 84x_0^2x_1x_0 + 100x_0^2x_1^2 + 48x_0x_1x_0^2 + 56x_0x_1x_0x_1 + 12x_0x_1^2x_0 + 12x_0x_1^3 - \cdots$$

$$c_i = 2 - 2x_0 + x_1 + 2x_0^2 - x_0x_1 - 2x_0^3 + x_0^2x_1 + 2x_0^4 - x_0^3x_1 - 2x_0^5 + x_0^4x_1 + 2x_0^6 - x_0^5x_1 - 2x_0^7 + x_0^6x_1 + \cdots$$

Again, as expected, c_γ has relative degree 2, and c_i has relative degree 1, By direct computation it is verified that $c_f = c_i \circ c_\gamma$.

Alternatively, since $r > 1$, the factorization in case 3 of Theorem 2 is also available. Observe that from (6) and (10)

$$c_e = 2 - x_0 + x_0^2 + 2x_0x_1 - x_0^3 - 2x_0^2x_1 + 2x_0x_1^2 + x_0^4 + 2x_0^3x_1 - 2x_0^2x_1^2 - x_0^5 - 2x_0^4x_1 + 2x_0^3x_1^2 + x_0^6 + 2x_0^5x_1 - 2x_0^4x_1^2 - x_0^7 - 2x_0^6x_1 + 2x_0^5x_1^2 + x_0^8 + \cdots$$

$$c' = x_0^{-1}(c) = 3x_0 + 4x_1 - 9x_0^2 - 10x_0x_1 - 6x_1x_0 - 6x_1^2 + 33x_0^3 + 36x_0^2x_1 + 18x_0x_1x_0 + 18x_0x_1^2 + 12x_1x_0^2 + 12x_1x_0x_1 + 6x_1^2x_0 + 6x_1^3 - 105x_0^4 - \cdots$$

Therefore, $c_f = c_e \circ c'$, where c_e has relative degree 2, and c' has relative degree 1. □

V. Conclusions

Using a flat coordinate system, it was shown that a flat output for a SISO flat system can be written in terms of a composition of two input-output operators having realizations with relative degrees summing to the relative degree of the flat output. First the general smooth case was considered, followed by the control affine analytic case. The latter was subdivided into three commonly encountered cases, but this list is likely not exhaustive. These factorizations could be written in terms of Fliess operators. Each case presented was illustrated by at least one example.

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