Shuffle-Rational Series: Recognizability and Realizations

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***Abstract*—The notion of a *shuffle-rational* formal power series is introduced. Then two equivalent characterizations are presented, one in terms of an analogue of Schützenberger's recognizability of a series, and the other in the context of state space realizations of nonlinear input-output systems represented by Chen-Fliess series. An underlying computational framework is also described which employs the Hopf algebra associated with the shuffle group. As an application, it is shown how to model a bilinear system with output saturation in this context.**

***Index Terms*—formal power series, Chen-Fliess series, bilinear systems**

I. Introduction

Let $X = \{x_0, x_1, \ldots, x_m\}$ be a fixed set of noncommuting symbols. Let $\mathbb{R}\langle X\rangle$ and $\mathbb{R}\langle\langle X\rangle\rangle$ denote, respectively, the set of polynomials and formal power series over X with real coefficients. Each set forms an $\mathbb{R}$-vector space and an associative $\mathbb{R}$-algebra under the catenation (Cauchy) product. The smallest subset of $\mathbb{R}\langle\langle X\rangle\rangle$ containing $\mathbb{R}\langle X\rangle$ which is closed under addition, scalar multiplication, the Cauchy product, and inversion (in the Cauchy product sense), that is, the *rational closure* of $\mathbb{R}\langle X\rangle$, constitutes the set of *rational series* [2]. Let F_c denote the Chen-Fliess series having $c \in \mathbb{R}\langle\langle X\rangle\rangle$ as its generating series [5], [6]. The following statements are known to be equivalent:

1) Series c is rational.
2) Series c is recognizable (Schützenberger's theorem [20]).
3) Series c has a Hankel matrix with finite rank [4].
4) Operator $y = F_c[u]$ has a bilinear state space realization [5], [6].
5) Operator $y = F_c[u]$ satisfies a linear ordinary differential equation in y of order equal to its Hankel rank and having coefficients which are rational functions of $\{u, \dot{u}, \ddot{u}, \ldots\}$ [7], [8] (see also [24], [25]).

Bilinear systems, of course, play a special role in the theory of nonlinear control systems [15], [17]. The definition of rationality given above, however, is simply one instance of a more general concept. Namely, given any associative algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ and an arbitrary subalgebra $\mathscr{F}$, the corresponding set of rational series is defined as those series in the rational closure of $\mathscr{F}$. Therefore, it is natural to ask whether any other notion of rationality has utility in the context of nonlinear control theory. If so, are there equivalences analogous to those given above in this alterative setting?

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The goal of this paper is to partially answer this question to the affirmative by providing one specific example, namely, by replacing the Cauchy product on $\mathbb{R}\langle\langle X\rangle\rangle$ with the *shuffle product* [18]. The latter forms a commutative algebra and is in some sense the adjoint of the Cauchy product [19]. This product appears naturally in nonlinear control theory when systems are interconnected in parallel (taking the product of the outputs) and in series [5], [10]. This alternative product leads directly to the notion of *shuffle-rationality*. The next objective is to provide two equivalent characterizations of shuffle-rationality, namely, the analogues of statements 2 and 4 above. The concept of shuffle-recognizability will be introduced and then a shuffle version of Schützenberger's theorem is proved. Next, it is shown that there is a correspondence between shuffle-rational series and a class of state space realizations which are bilinear in the state but have rational output functions. A common theme in all of the analysis is the evaluation of rational functions on formal power series. To facilitate these calculations, the underlying Hopf algebra (e.g., see [1], [21]) for the group of non proper formal power series under the shuffle product is introduced. Finally, as an application, it is shown how to model bilinear systems with output saturation in this context. It should be stated that the question of whether there exist analogous versions of statements 3 and 5 in this setting is an open question at present.

The work is presented in six sections. The next section provides a summary of the notation and terminology employed in paper. The subsequent section describes the computational framework used for developing the theory and doing examples. Section IV presents the concept of shuffle-rationality. Section V introduces the notion of shuffle-recognizability and gives the proof of the equivalence of statements analogous to 1 and 2 above. Finally, a state space characterization of shuffle-rationality is presented in Section VI to prove the equivalence of statements analogous to 1 and 4 above. Conclusions and directions for future research are given in the last section.

II. Preliminaries

Let $X = \{x_0, x_1, \ldots, x_m\}$ denote a fixed *alphabet* of noncommuting symbols with m finite. Any finite sequence of *letters* from X, $\eta = x_{i_1} \cdots x_{i_k}$, is a *word* over X having *length* $|\eta| \triangleq k$. The empty word, $\emptyset$, is the word of length zero. The set of all words is written as X^*, and $X^+ \triangleq X^* \backslash \emptyset$. The subset of words having prefix η is denoted by ηX^*. It is easily verified that X^* constitutes a monoid under the catenation product $\mathcal{C} : (\eta, \xi) \mapsto \eta\xi$ with $\emptyset$ acting as the unit. Any mapping $c : X^* \to \mathbb{R}^\ell : \eta \mapsto (c, \eta)$ is called a *formal power series*. It is customary to write c as a formal summation $c = \sum_{\eta \in X^*} (c, \eta)\eta$. If $(c, \emptyset) = 0$, then c is said to be *proper*. Here $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ will represent the set of all such formal power series over X. Extending $\mathcal{C}$ in a natural way to series with coefficients in $\mathbb{R}^\ell$, $(\mathbb{R}^\ell\langle\langle X\rangle\rangle, \mathcal{C})$ is an associative unital $\mathbb{R}$-algebra. $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is also an associative and commutative unital $\mathbb{R}$-algebra under the *shuffle product*. The latter is a product first defined inductively for two words as

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $\eta, \xi \in X^*$, $x_i, x_j \in X$, and $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta \triangleq \eta$. It is then extended bilinearly to series in $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ [5]. The $\mathbb{R}$-algebra of $\mathbb{R}^\ell$-valued polynomials over the set of commutative indeterminates $\{y_1, y_2, \ldots, y_n\}$ is denoted by $\mathbb{R}^\ell[y_1, y_2, \ldots, y_n]$.

Let $L_\mathfrak{p}^m[t_0, t_1]$ be the set of $\mathbb{R}^m$-valued measurable functions on $[t_0, t_1]$ with a finite $\mathfrak{p}$-norm. Every series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ has an associated *Chen-Fliess series*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0), \tag{1}$$

where $E_\emptyset[u] = 1$ and

$$E_{x_i\eta}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_\eta[u](\tau, t_0)\, d\tau$$

with $x_i \in X$, $\eta \in X^*$, and $u_0 = 1$ [5]. Such series are known to converge absolutely and uniformed on a ball of radius R center at the origin of $L_1^m[0, T]$ whenever there exists real numbers $K, M > 0$ such that

$$|(c, \eta)| \leq K M^{|\eta|} |\eta|!, \quad \forall \eta \in X^*,$$

and R and T are sufficiently small [13]. $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ will denote the set of all such *locally convergent* generating series. If instead the coefficients of c satisfy the Gevrey growth condition

$$|(c, \eta)| \leq K M^{|\eta|} (|\eta|!)^s, \quad \forall \eta \in X^*, \quad s \in [0, 1[,$$

then the series is called *globally convergent*, and the corresponding Chen-Fliess series is known to converge for *any* finite R and T [26]. The set of all such series is written as $\mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$. Whenever (1) converges in any sense, F_c is called a *Fliess operator*. Given two Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, their parallel and product connections always produce another Fliess operator, specifically, $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup\!\sqcup d}$ [5]. It is known that addition preserves both local and global convergence, while the shuffle product preserves local convergence and global convergence when $s = 0$ [22], [23].

III. Rational Functions of Formal Series

This section describes the evaluation of commutative polynomial maps and rational functions over noncommutative formal power series. First observe that the set of non proper series in $\mathbb{R}\langle\langle X\rangle\rangle$ constitutes a group under the shuffle product [11]. The shuffle inverse in this case is taken to be

$$c^{\sqcup\!\sqcup -1} = ((c, \emptyset)(1 - c'))^{\sqcup\!\sqcup -1} = (c, \emptyset)^{-1} (c')^{\sqcup\!\sqcup *}, \tag{2}$$

where $c' \triangleq 1 - c/(c, \emptyset)$ is proper, and $(c')^{\sqcup\!\sqcup *} \triangleq \sum_{k \in \mathbb{N}_0} (c')^{\sqcup\!\sqcup k}$. Here $(c')^{\sqcup\!\sqcup k} \triangleq c' \sqcup\!\sqcup (c')^{\sqcup\!\sqcup k-1}$ with $(c')^{\sqcup\!\sqcup 0} = 1$.

Example 3.1: Let $c = 1 - x_1 \in \mathbb{R}\langle\langle X\rangle\rangle$ so that $c' = x_1$. Then $c^{\sqcup\!\sqcup -1} = x_1^{\sqcup\!\sqcup *} = \sum_{k \in \mathbb{N}_0} k!\, x_1^k$. □

Since $\mathbb{R}\langle\langle X\rangle\rangle$ under the shuffle product is a commutative and associative $\mathbb{R}$-algebra, for any $k \in \mathbb{N}$ and $c \in \mathbb{R}\langle\langle X\rangle\rangle$ one can write $c^{\sqcup\!\sqcup k} = y^k(c)$, where $y^k \in \mathbb{R}[y]$. Similarly, if d is a proper series, then

$$\begin{aligned}
(1 - d)^{\sqcup\!\sqcup -1} &= d^{\sqcup\!\sqcup *} \\
&= 1 + d + d^{\sqcup\!\sqcup 2} + d^{\sqcup\!\sqcup 3} + \cdots \\
&= (1 + y + y^2 + y^3 + \cdots)(d) \\
&= \left(\frac{1}{1 - y}\right)(d).
\end{aligned}$$

Therefore, the shuffle inverse of a series can be written as a rational function of the proper part of the series. The notions of a polynomial map and a rational function of a noncommutative formal power series is formalized by the following definition.

Definition 3.1: Let $p, q \in \mathbb{R}[y]$ and $c \in \mathbb{R}\langle\langle X\rangle\rangle$. Assume $p(y) = \sum_{i=0}^{k} a_i y^i$, where $k \in \mathbb{N}_0$, and $q((c, \emptyset)) \neq 0$. The composition of p and c is defined as

$$p(c) = \sum_{i=0}^{k} a_i c^{\sqcup\!\sqcup i}.$$

Extending the definition to rational functions gives

$$\frac{p}{q}(c) = p(c) \sqcup\!\sqcup q(c)^{\sqcup\!\sqcup -1}.$$

These definitions can be generalized to functions on k-tuples of series by first observing that $\mathbb{R}\langle\langle X\rangle\rangle$ is a commutative ring with the shuffle product. Therefore, $\mathbb{R}\langle\langle X\rangle\rangle$ is an $\mathbb{R}\langle\langle X\rangle\rangle$-module, where scalar multiplication of the ring with a module element is also defined by the shuffle product. In other words, $\mathbb{R}\langle\langle X\rangle\rangle \bigotimes_{\mathbb{R}\langle\langle X\rangle\rangle} \mathbb{R}\langle\langle X\rangle\rangle$ is isomorphic to $\mathbb{R}\langle\langle X\rangle\rangle$ in the category of commutative $\mathbb{R}\langle\langle X\rangle\rangle$-modules. Let A, B be $\mathbb{R}$-modules and denote the set of all $\mathbb{R}$-linear morphisms by $\mathrm{Hom}(A, B)$. Recall that $\mathrm{Hom}(A, B)$ forms an $\mathbb{R}$-module by itself. Let $\Gamma \in \mathbb{R}$-module $\mathrm{Hom}\Big(A, \mathrm{Hom}\big(B, C\big)\Big)$, where the modules $A \triangleq \bigoplus_{k \in \mathbb{N}} \bigotimes_{\mathbb{R}\, i=1}^{k} \mathbb{R}[y_i]$, $B \triangleq \bigoplus_{k \in \mathbb{N}} \prod_{i=1}^{k} \mathbb{R}\langle\langle X\rangle\rangle$, and $C \triangleq \bigoplus_{k\in\mathbb{N}} \bigotimes_{\mathbb{R}\langle\langle X\rangle\rangle}{}_{i=1}^{k} \mathbb{R}\langle\langle X\rangle\rangle$. The morphism Γ is defined as

$$\Gamma\left(\bigotimes_{\mathbb{R}}{}_{i=1}^{k} p_i\right)(c_1, c_2, \ldots, c_k) = \bigotimes_{\mathbb{R}\langle\langle X\rangle\rangle}{}_{i=1}^{k} p_i(c_i),$$

where $p_i \in \mathbb{R}[y_i]$ and $c_i \in \mathbb{R}\langle\langle X\rangle\rangle$, $i = 1, 2, \ldots k$. The right-hand side is expanded using the shuffle product as

$$\bigotimes_{\mathbb{R}\langle\langle X\rangle\rangle}{}_{i=1}^{k} p_i(c_i) = p_1(c_1) \sqcup\!\sqcup\, p_2(c_2) \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, p_k(c_k).$$

The image of Γ, denoted by $\mathrm{Im}(\Gamma)$, is an $\mathbb{R}$-module. In fact, it possesses an $\mathbb{R}$-algebra structure as $\bigotimes_{\mathbb{R}\langle\langle X\rangle\rangle}{}_{i=1}^{k} \mathbb{R}\langle\langle X\rangle\rangle$ is an $\mathbb{R}$-algebra. If $p, p' \in \bigotimes_{\mathbb{R}}{}_{i=1}^{k} \mathbb{R}[y_i]$, then

$$(\Gamma(p)\Gamma(p'))(c_1, \ldots, c_k) = (\Gamma(p))(c_1, \ldots, c_k) \sqcup\!\sqcup\, (\Gamma(p'))(c_1, \ldots, c_k).$$

As the shuffle product has no zero divisors, it is simple to check that the underlying ring structure in $\mathbb{R}$-algebra $\mathrm{Im}(\Gamma)$ is an integral domain. Hence, the quotient field $\mathrm{Im}(\Gamma)$ is the set of rational functions

$$\frac{\Gamma(p)}{\Gamma(p')}(c_1, \ldots, c_k) = \Gamma(p)(c_1, \ldots, c_k) \sqcup\!\sqcup\, (\Gamma(p')(c_1, \ldots, c_k))^{\sqcup\!\sqcup\, -1},$$

where $p, p' \in \bigotimes_{\mathbb{R}}{}_{i=1}^{k} \mathbb{R}[y_i]$. The symbol Γ is suppressed for brevity so that given $p \otimes p' \in \mathbb{R}[y] \otimes \mathbb{R}[y']$,

$$p \otimes p'(c, c') = \big(\Gamma(p \otimes p')\big)(c, c') = p(c) \sqcup\!\sqcup\, p'(c').$$

Likewise, if $p, q \in \bigotimes_{\mathbb{R}}{}_{i=1}^{k} \mathbb{R}[y_i]$, then

$$\frac{p}{q}(c_1, \ldots, c_k) = \frac{\Gamma(p)}{\Gamma(q)}(c_1, \ldots, c_k).$$

The evaluation of rational functions over a noncommutative formal power series requires one to compute shuffle powers and the shuffle inverse of series. Such computations can be implemented algorithmically with the aid of the Hopf algebra corresponding to the shuffle group as described next [9]. Consider the set

$$M \triangleq \{1 + c : c \in \mathbb{R}\langle\langle X\rangle\rangle,\ (c, \emptyset) = 0\} \subset \mathbb{R}\langle\langle X\rangle\rangle.$$

M is an Abelian group under the shuffle product with 1 as the identity element. The shuffle inverse is defined as in (2). The set of coordinate maps on $\mathbb{R}\langle\langle X\rangle\rangle$ is taken to be

$$H = \{a_\eta : M \longrightarrow \mathbb{R},\ \eta \in X^*\},$$

where $a_\eta(c) = (c, \eta)$. H constitutes a commutative $\mathbb{R}$-algebra with addition, scalar multiplication and product defined, respectively, as

$$\begin{aligned}(a_\eta + a_\zeta)(c) &= a_\eta(c) + a_\zeta(c)\\ (ka_\eta)(c) &= k(a_\eta(c))\\ \boldsymbol{m}(a_\eta, a_\zeta)(c) &= a_\eta(c) a_\zeta(c),\end{aligned}$$

where $\eta, \zeta \in X^*, k \in \mathbb{R}$. The unit for the product is $\mathbf{1} \sim a_\emptyset$ so that $\mathbf{1}(c) = 1$, $\forall c \in M$. Define the coproduct $\Delta : H \longrightarrow H \bigotimes H$ as $\Delta a_\eta(c, d) = a_\eta(c \sqcup\!\sqcup\, d)$, where $c, d \in M$ and $\eta \in X^*$. It can be computed inductively as

$$\begin{aligned}\Delta\mathbf{1} &= \mathbf{1} \otimes \mathbf{1}\\ \Delta \circ \theta_i &= (\theta_i \otimes \mathbf{1} + \mathbf{1} \otimes \theta_i) \circ \Delta,\end{aligned}$$

where θ_i denotes the vector space endomorphism on H specified by $\theta_i a_\eta = a_{x_i\eta}$, $i = 0, 1, \ldots, m$. The counit map ϵ is defined as

$$\epsilon(a_\eta) = \begin{cases} k : & a_\eta = k a_\emptyset \\ 0 : & \text{otherwise.}\end{cases}$$

It is simple to check that $(H, \boldsymbol{m}, \mathbf{1}, \Delta, \epsilon)$ forms a commutative and cocommutative bialgebra structure. The bialgebra is graded based on word length. Hence, $H = \bigoplus_{k\in\mathbb{N}_0} H_k$ with $a_\eta \in H_k$ if and only if $|\eta| = k$. Since $\mathbb{R} \cong H_0$ in the category of algebras with ϵ acting as the isomorphism, H is a connected and graded bialgebra, and thus a Hopf algebra [3]. The reduced coproduct Δ' is defined as

$$\Delta'(a_\eta) = \begin{cases} \Delta(a_\eta) - a_\eta \otimes \mathbf{1} - \mathbf{1} \otimes a_\eta & : a_\eta \neq a_\emptyset \\ 0 & : a_\eta = a_\emptyset.\end{cases}$$

Using Sweedler's notation, the coproduct can be written as

$$\Delta(a_\eta) = \sum a_{(\eta_1)} \otimes a_{(\eta_2)},$$

where the sum is over all words η_1, η_2 such that $\eta_1 \sqcup\!\sqcup\, \eta_2 = \eta$ [21]. The antipode map $S : H \longrightarrow H$ is given by $S(a_\eta)(c) = a_\eta(c^{\sqcup\!\sqcup\, -1})$. It can be computed inductively for any $a \in H^+$ (where $H^+ \triangleq \bigoplus_{k\geq 1} H_k$) by

$$S(a_\eta) = -a_\eta - \sum a'_{(\eta_1)} S(a'_{(\eta_2)}),$$

where the summation is taken over all the components of the reduced coproduct $\Delta'(a_\eta)$.

The coproduct Δ is useful for computing shuffle powers of formal power series. For example, if $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is non proper, then $c = (c, \emptyset)c'$, where $c' \in M$. For any $\eta \in X^*$, it follows that

$$\begin{aligned}(c^{\sqcup\!\sqcup\, 2}, \eta) &= (c, \emptyset)^2 (c'^{\sqcup\!\sqcup\, 2}, \eta)\\ &= (c, \emptyset)^2 a_\eta(c'^{\sqcup\!\sqcup\, 2})\\ &= (c, \emptyset)^2 \Delta a_\eta(c', c').\end{aligned}$$

In the case where c is proper, one can use the corresponding group element $(1 + c)$ and compute the reduced coproduct since $(c^{\sqcup\!\sqcup\, 2}, \eta) = \Delta' a_\eta((1 + c), (1 + c))$. The shuffle inverse of a non proper series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ can be computed directly using the antipode S as

$$\begin{aligned}(c^{\sqcup\!\sqcup\, -1}, \eta) &= (c, \emptyset)^{-1} (c'^{\sqcup\!\sqcup\, -1}, \eta)\\ &= (c, \emptyset)^{-1} S(a_\eta)(c').\end{aligned}$$

The coproduct can be linearly extended to computing the polynomial map of arbitrary formal power series. Let $\Delta^{\circ k}$ denote the composition of the coproduct Δ with itself k times where $k \geq 1$. If $c \in M$ and $\eta \in X^*$, for brevity $\Delta^{\circ k} a_\eta(c, c, \ldots, c)$ with the argument c repeated $(k+1)$ times is written as $\Delta^{\circ k} a_\eta(c)$. Suppose $p \in \mathbb{R}[x]$ is written as $p(x) = \sum_{i=0}^{m} a_i x^i$. Then for $c \in \mathbb{R}\langle\langle X \rangle\rangle$ observe

$$\begin{aligned}(p(c), \eta) &= a_\eta(p(c)) \\ &= \{a_0\epsilon + a_1 + a_2\Delta + \cdots \\ &\quad + a_{m-1}\Delta^{\circ(m-2)} + a_m\Delta^{\circ(m-1)}\}(a_\eta)(c).\end{aligned}$$

Now assume $p(x), q(x) \in \mathbb{R}[x]$ such that

$$\frac{p}{q}(x) = \frac{\sum_{i=0}^{m} a_i x^i}{\sum_{j=0}^{n} b_j x^j},$$

and $q(1) = 1$ without loss of generality. The computation of $(p/q)(c)$ is done as follows

$$\left(\frac{p}{q}(c), \eta\right) = \big(P(\epsilon, \Delta) \otimes Q(\epsilon, \Delta, S)\big) \circ \Delta a_\eta(c),$$

where

$$\begin{aligned}P(\epsilon, \Delta) &= a_0\epsilon + a_1 + a_2\Delta + \cdots \\ &\quad + a_{m-1}\Delta^{\circ(m-2)} + a_m\Delta^{\circ(m-1)}. \qquad \text{(3a)}\\ Q(\epsilon, \Delta, S) &= b_0(\epsilon \circ S) + b_1 S + b_2(\Delta \circ S) + \cdots \\ &\quad + b_{n-1}(\Delta^{\circ(n-2)} \circ S) + b_n(\Delta^{\circ(n-1)} \circ S). \qquad \text{(3b)}\end{aligned}$$

Here $P(\epsilon, \Delta)$ and $Q(\epsilon, \Delta, S)$ are the operator polynomials corresponding to the rational function p/q. This computation is abbreviated as

$$\left(\frac{p}{q}(c), \eta\right) = \Upsilon(\epsilon, \Delta, S)(a_\eta)(c),$$

where $\Upsilon(\epsilon, \Delta, S) \triangleq \big(P(\epsilon, \Delta) \bigotimes Q(\epsilon, \Delta, S)\big) \circ \Delta$. The operator Υ is viewed as the computational block for the rational function p/q. The computation of a rational function of a series in M is naturally extended when $p, q \in \bigotimes_{\mathbb{R}\, i=1}^{k} \mathbb{R}[y_i]$. Let $p = p_1 \otimes p_2 \otimes \cdots \otimes p_k$ and $q = q_1 \otimes q_2 \otimes \cdots \otimes q_k$, where $p_i, q_i \in \mathbb{R}[y_i]$. Let $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_k$ be the corresponding computational blocks. Therefore,

$$\left(\frac{p}{q}(c_1, \ldots, c_k), \eta\right) = \left(\bigotimes_{i=1}^{k} \Upsilon_i\right) \circ \Delta^{\circ(k-1)} a_\eta(c_1, \ldots, c_k), \qquad (4)$$

where $c_1, c_2, \ldots, c_k \in M$.

The computational framework above can be further extended to rational functions of arbitrary non proper formal power series. Let $c \in \mathbb{R}\langle\langle X \rangle\rangle$ be non proper with $(c, \emptyset) = \alpha \neq 0$ and $c = \alpha c'$. Fix $p(x), q(x) \in \mathbb{R}[x]$ such that

$$\frac{p}{q}(x) = \frac{\sum_{i=0}^{m} a_i x^i}{\sum_{j=0}^{n} b_j x^j}$$

and $q(\alpha) = 1$ without loss of generality. The computation of $(p/q)(c)$ is done as

$$\begin{aligned}\left(\frac{p}{q}(c), \eta\right) = \Big(&\big(\alpha P(\alpha^{-1}\epsilon, (\alpha\Delta))\big) \otimes \\ &\big(\alpha Q(\alpha^{-1}\epsilon, (\alpha\Delta), S)\big) \circ \Delta\Big)(a_\eta)(c'),\end{aligned}$$

where the operator polynomials P and Q are defined as in (3). The computation is abbreviated as the computational block $\overline{\Upsilon}$ so that

$$\left(\frac{p}{q}(c), \eta\right) = \overline{\Upsilon}(\epsilon, \Delta, S)(a_\eta)(c')$$

with $\overline{\Upsilon}(\epsilon, \Delta, S) \triangleq \big(\alpha P(\alpha^{-1}\epsilon, (\alpha\Delta)) \bigotimes \alpha Q(\alpha^{-1}\epsilon, (\alpha\Delta), S)\big) \circ \Delta$. The computation of a rational function of a non proper series is naturally extended when $p, q \in \bigotimes_{\mathbb{R}\, i=1}^{k} \mathbb{R}[y_i]$. Let $p = p_1 \otimes p_2 \otimes \cdots \otimes p_k$ and $q = q_1 \otimes q_2 \otimes \cdots \otimes q_k$, where $p_i, q_i \in \mathbb{R}[y_i]$. Let $\overline{\Upsilon}_1, \overline{\Upsilon}_2, \ldots, \overline{\Upsilon}_k$ be the corresponding computational blocks and $c_1, c_2, \ldots, c_k$ be all non proper series such that $(c_i, \emptyset) = \alpha_i \neq 0$, $i = 1, 2, \ldots, k$. Let $c'_i \in M$ be the corresponding group element of c_i defined as $c_i = \alpha_i c'_i$. In which case,

$$\left(\frac{p}{q}(c_1, \ldots, c_k), \eta\right) = \left(\bigotimes_{i=1}^{k} \overline{\Upsilon}_i\right) \circ \Delta^{\circ(k-1)} a_\eta(c'_1, \ldots, c'_k).$$

The framework for the computation of a rational function of a proper series c with $q(0) = 1$ (without loss of generality) is extended from (3) similarly except that the coproduct Δ in the operator polynomials is replaced with the reduced coproduct Δ'. The extension to the computation of rational functions of ordered collections of proper series $c_1, c_2, \ldots, c_k$ is immediate with respect to (4) except that again the coproduct is replaced with the reduced coproduct. The case where $c_1, c_2, \ldots, c_k$ is a *mixture* of proper and non proper series is difficult and does not fit well in the current scheme.

IV. Rational Series

This section describes the concept of rationally closed subalgebras, and hence, the notion of rational series in the broadest sense. Next, the classical example is briefly reviewed followed by the notion of a shuffle-rational series.

Definition 4.1: [4] An $\mathbb{R}$-subalgebra $\mathscr{F}$ of an $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X \rangle\rangle$ is said to be *rationally closed* if and only if the inverse of all invertible elements of $\mathscr{F}$ belongs to $\mathscr{F}$. The *rational closure* of an $\mathbb{R}$-subalgebra $\mathscr{F}'$ of an $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X \rangle\rangle$ is the smallest rationally closed subalgebra $\mathscr{F}$ containing $\mathscr{F}'$.

Classically, *rational series* are defined to be those in the rational closure of the $\mathbb{R}$-subalgebra of polynomials $\mathbb{R}\langle X \rangle$, where the $\mathbb{R}$-algebra structure on $\mathbb{R}\langle\langle X \rangle\rangle$ is under Cauchy product [2]. This noncommutative algebra of rational series is denoted by $\mathbb{R}\langle(X)\rangle$. Since $\mathbb{R}\langle\langle X \rangle\rangle$ also forms a commutative $\mathbb{R}$-algebra under the shuffle product, a corresponding notion of rationality is possible as described next.

Definition 4.2: The rational closure of the $\mathbb{R}$-subalgebra $\mathbb{R}\langle X \rangle$ of the shuffle algebra on $\mathbb{R}\langle\langle X \rangle\rangle$ is called the algebra of *shuffle-rational series* and denoted by $\mathbb{R}_{\sqcup\!\sqcup}\langle(X)\rangle$.

In other words, $\mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$ under the shuffle product that contains $\mathbb{R}\langle X\rangle$. The next example establishes that $\mathbb{R}\langle(X)\rangle \not\subset \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$, while the subsequent example shows that $\mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle \not\subset \mathbb{R}\langle(X)\rangle$.

Example 4.1: Let c be the rational series

$$\begin{aligned} c &= (1-x_1)^{-1} \\ &= 1 + x_1 + x_1^2 + x_1^3 + \cdots \\ &= 1 + \frac{x_1}{1!} + \frac{x_1^{\sqcup\!\sqcup 2}}{2!} + \frac{x_1^{\sqcup\!\sqcup 3}}{3!} + \cdots \\ &=: \exp(x_1^{\sqcup\!\sqcup}). \end{aligned}$$

Observe that c cannot be represented by a finite number of shuffle products as the exponential map is an entire function and cannot be represented by a finite number of terms or as a rational function. Hence, it is not shuffle-rational. □

Example 4.2: Let c be the shuffle-rational series

$$\begin{aligned} c &= (1-x_1)^{\sqcup\!\sqcup -1} \\ &= 1 + x_1 + x_1^{\sqcup\!\sqcup 2} + x_1^{\sqcup\!\sqcup 3} + \cdots \\ &= 1 + x_1 + 2!x_1^2 + 3!x_1^3 + \cdots \end{aligned}$$

Clearly, $c \notin \mathbb{R}\langle(X)\rangle$ since all rational series have Gevrey order $s = 0$ [2]. □

In the case of a single indeterminate $X = \{x\}$, it is simple verify that

$$\mathbb{R}\langle(X)\rangle = \left\{\frac{p(x)}{q(x)} : p(x), q(x) \in \mathbb{R}[X]\right\}$$
$$\mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle = \left\{p(x) \sqcup\!\sqcup\, q(x)^{\sqcup\!\sqcup -1} : p(x), q(x) \in \mathbb{R}[X]\right\},$$

where $q(x) \neq 0$ in both the cases.

V. Recognizable Series

In this section, the classical definition of a recognizable series is first introduced. Schützenberger showed in [20] that a series is rational under the Cauchy product if and only if it is recognizable. Next, the shuffle analogue of Schützenberger's theorem is stated and proved.

Definition 5.1: [2] A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is said to be *recognizable* if $\exists N \in \mathbb{N}$, a monoid morphism $\mu : X^* \longrightarrow \mathbb{R}^{N\times N}$, and vectors $\lambda, \gamma \in \mathbb{R}^N$ such that $(c, w) = \lambda^T \mu(w)\gamma$, $\forall w \in X^*$. Note that $\mathbb{R}^{N\times N}$ is considered to be a multiplicative monoid. The tuple (λ, μ, γ) is called a *representation* of c with dimension N. The set of all recognizable series is denoted by $\mathbb{R}^{rec}\langle\langle X\rangle\rangle$.

The following lemma will be useful in the work that follows.

Lemma 5.1: A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is a polynomial if and only if it has a representation (λ, μ, γ) with $\mu(x_i)$ being a strictly upper triangular matrix $\forall x_i \in X$.

Proof: If c has a representation with $\mu(x_i)$ strictly upper triangular $\forall x_i \in X$, then $\exists k \in \mathbb{N}$ such that $\mu(w) = 0$ when $|w| \geq k$, as strictly upper triangular matrices are always nilpotent. Hence, c is a polynomial. Conversely, if c is a polynomial, then the underlying vector fields of any realization of F_c form a nilpotent distribution [16]. Since $\mathbb{R}\langle X\rangle \subset \mathbb{R}\langle(X)\rangle$, and the underlying vector fields associated with any generating series in $\mathbb{R}\langle(X)\rangle$ comes from the Lie algebra $\mathfrak{gl}(\mathbb{R}^N)$, the fact that the subalgebra of strictly upper triangular matrices is a nilpotent Lie subalgebra of the Lie algebra $\mathfrak{gl}(\mathbb{R}^N)$ completes the proof [14]. ■

Example 5.1: It is easily checked that $c = x_0x_1$ has the representation

$$\mu(x_0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu(x_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$\lambda = e_1 = [1\ 0\ 0]^T$, and $\gamma = e_3 = [0\ 0\ 1]^T$. Note that $\mu(x_0)$ and $\mu(x_1)$ are strictly upper triangular matrices and nilpotent of index 2. □

Next the definition of shuffle-recognizability is given.

Definition 5.2: Let $k \in \mathbb{N}$ such that $\{N_1, N_2, \ldots N_k\}$ is a multiset of k positive integers. Let $\{\lambda_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k$ be ordered collections of k vectors such that $\lambda_i, \gamma_i \in \mathbb{R}^{N_i}$. Assume $\{\mu_i\}_{i=1}^k$ is an ordered collection of k monoid morphisms $\mu_i : X^* \longrightarrow \mathbb{R}^{N_i\times N_i}$ such that $\mu_i(x_j)$ is a strictly upper triangular matrix $\forall x_j \in X$, $i = 1, \ldots, k$. Define two polynomials $p, q \in \bigotimes_{\mathbb{R}\,i=1}^{\ k}\mathbb{R}[y_i]$ such that $q(\lambda_i^T\gamma_i) \neq 0$, $i = 1, \ldots, k$. A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is said to be *shuffle-recognizable* if $c = p/q\left(\sum_{w\in X^*} \lambda^T\mu(w)\gamma\, w\right)$, where $\lambda^T = (\lambda_1^T \times \lambda_2^T \times \cdots \times \lambda_k^T)$, $\mu = (\mu_1 \times \mu_2 \times \cdots \times \mu_k)$, and $\gamma = (\gamma_1 \times \gamma_2 \times \cdots \times \gamma_k)$. The tuple $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$ is called a k order *shuffle-representation* of c. The set of all such shuffle-recognizable series is denoted by $\mathbb{R}^{\sqcup\!\sqcup\, rec}\langle\langle X\rangle\rangle$.

Given a shuffle-recognizable series c with shuffle-representation $(p, q, \{\lambda\}_{i=1}^k, \{\mu\}_{i=1}^k, \{\gamma\}_{i=1}^k)$, the computation of (c, η), $\eta \in X^*$ can be made algorithmic using the Hopf algebra corresponding to the shuffle group. By Lemma 5.1, observe that the expression $\sum_{w\in X^*} \lambda^T\mu(w)\gamma$ is a Cartesian product of k polynomials, say $d_1, d_2, \cdots d_k$. Hence, for all $\eta \in X^*$

$$(c, \eta) = \left(\frac{p}{q}(d_1, d_2, \ldots, d_k), \eta\right),$$

which can be computed directly using (4). In addition,

$$\mathbb{R}^{\sqcup\!\sqcup\, rec}\langle\langle X\rangle\rangle = \{p(c_1, c_2, \ldots, c_k) \sqcup\!\sqcup\, q(c_1, c_2, \ldots, c_k)^{\sqcup\!\sqcup -1} : p, q \in \mathscr{G}_k\}, \tag{5}$$

where $c_1, \ldots, c_k \in \mathbb{R}\langle X\rangle$, $(q(c_1, \ldots, c_k), \emptyset) \neq 0$, and $\mathscr{G}_k = \bigotimes_{\mathbb{R}\,i=1}^{\ k}\mathbb{R}[y_i]$.

Example 5.2: Suppose

$$\begin{aligned} c &= 1 + x_1 + x_1^{\sqcup\!\sqcup 2} + \cdots + x_1^{\sqcup\!\sqcup k} + \cdots \\ &= (1-x_1)^{\sqcup\!\sqcup -1} \\ &= \left(\frac{1}{1-y}\right)(x_1). \end{aligned}$$

Note that $\mu : X^* \longrightarrow \mathbb{R}^{2\times 2}$, where $\mu(x_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\gamma = e_2$, and $\lambda = e_1$ give a representation of x_1, that is, $x_1 = \sum_{w\in X^*} \lambda^T\mu(w)\gamma\, w$. Hence, $c = x_1^{\sqcup\!\sqcup *}$ is a shuffle-recognizable series with shuffle-representation $(1, 1-y, \{e_1\}, \{\mu\}, \{e_2\})$. □

Equation (5) states that the set of shuffle-recognizable series are generated by finite shuffle products of polynomials and their shuffle inverses. Hence, $\mathbb{R}^{\shuffle\, rec}\langle\langle X\rangle\rangle \subseteq \mathbb{R}^{\shuffle}\langle(X)\rangle$ in the category of sets. This leads to the central question of whether $\mathbb{R}^{\shuffle\, rec}\langle\langle X\rangle\rangle = \mathbb{R}^{\shuffle}\langle(X)\rangle$. The following theorem states this is the case.

Theorem 5.1: A series is shuffle-rational if and only if it is shuffle-recognizable.

Proof: It is sufficient to prove that $\mathbb{R}^{\shuffle}\langle(X)\rangle \subseteq \mathbb{R}^{\shuffle\, rec}\langle\langle X\rangle\rangle$. First it is shown that all polynomials and those that are shuffle-invertible are shuffle-recognizable. As every shuffle-rational series is in the rational closure of the polynomials, it only remains to be shown that shuffle-recognizability is preserved under the remaining shuffle-rational operations: scalar multiplication, addition, and the shuffle product.

Let $c \in \mathbb{R}\langle X\rangle$. From Lemma 5.1 there exists a nilpotent representation (λ, μ, γ) such that

$$c = \sum_{w\in X^*} \lambda^T \mu(w)\gamma\, w \quad = \frac{y}{1}\left(\sum_{w\in X^*} \lambda^T \mu(w)\gamma\, w\right).$$

Therefore, trivially, c is shuffle-recognizable with tuple $(y, 1, \{\lambda\}, \{\mu\}, \{\gamma\})$. It is equally clear that if c is non proper, and $c' = 1 - (c/(c,\emptyset))$ has a representation $(\lambda', \mu', \gamma')$, then $d = c^{\shuffle\, -1}$ is $\shuffle$-recognizable with tuple $((c,\emptyset)^{-1}, 1 - y, \{\lambda'\}, \{\mu'\}, \{\gamma'\})$.

Shuffle-recognizability is also preserved by scalar multiplication. If $\alpha \in \mathbb{R}$ and d is shuffle-recognizable with representation $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$, then αd is shuffle-recognizable with representation $(\alpha p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$.

It is next shown that shuffle-recognizability is closed under addition. Let $d, d' \in \mathbb{R}^{\shuffle}\langle(X)\rangle$ with representations $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$ and $(p', q', \{\lambda_i'\}_{i=1}^n, \{\mu_i'\}_{i=1}^n, \{\gamma_i'\}_{i=1}^n)$, respectively. Define $\lambda^T = \prod_{i=1}^k \lambda_i^T$, $\gamma = \prod_{i=1}^n \gamma_i$, $\mu = \prod_{i=1}^k \mu_i$, and likewise for λ'^T, γ' and μ'. In which case,

$$\begin{aligned} d + d' &= \frac{p}{q}(c_1) + \frac{p'}{q'}(c_2) \\ &= (p(c_1) \shuffle q(c_1)^{\shuffle\, -1}) + (p'(c_2) \shuffle q'(c_2)^{\shuffle\, -1}) \end{aligned}$$

with $c_1 = \sum_{w\in X^*} \lambda^T \mu(w)\gamma\, w$ and $c_2 = \sum_{w\in X^*} \lambda'^T \mu'(w)\gamma'\, w$. Multiplying by $q(c_1) \shuffle q(c_1)^{\shuffle\, -1} \shuffle q'(c_2) \shuffle q'(c_2)^{\shuffle\, -1}$ on both sides of the expression on the right gives

$$\begin{aligned} d + d' &= [(p(c_1) \shuffle q'(c_2)) + (p'(c_2) \shuffle q(c_1))] \shuffle \\ &\qquad [q(c_1) \shuffle q'(c_2)]^{\shuffle\, -1} \\ &= \{(p \otimes_{\mathbb{R}} q') + (q \otimes_{\mathbb{R}} p')\}(c_1, c_2) \shuffle \\ &\qquad (\{q \otimes_{\mathbb{R}} q'\}(c_1, c_2))^{\shuffle\, -1}. \end{aligned}$$

Hence, $d + d'$ is shuffle-recognizable with representation $(((p \otimes_{\mathbb{R}} q') + (q \otimes_{\mathbb{R}} p')), q \otimes_{\mathbb{R}} q', \{\Lambda_i\}_{i=1}^{k+n}, \{\Psi_i\}_{i=1}^{k+n}, \{\Gamma_i\}_{i=1}^{k+n})$, where $\Lambda_i = \lambda_i$, $\Psi_i = \mu_i$, and $\Gamma_i = \gamma_i$ if $1 \le i \le k$, and $\Lambda_i = \lambda_{i-k}'$, $\Psi_i = \mu_{i-k}'$, and $\Gamma_i = \gamma_{i-k}'$ if $(k+1) \le i \le (k+n)$.

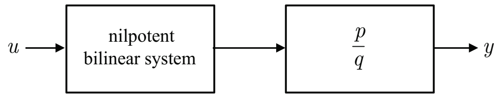

Fig. 1: Wiener-Fliess system comprised of a nilpotent bilinear system and a static rational function

Finally, the case of the shuffle product is addressed. Using the same notation as in the previous case, observe

$$\begin{aligned} d \shuffle d' &= \frac{p}{q}(c_1) \shuffle \frac{p'}{q'}(c_2) \\ &= p(c_1) \shuffle q(c_1)^{\shuffle\, -1} \shuffle p'(c_2) \shuffle q'(c_2)^{\shuffle\, -1} \\ &= (p(c_1) \shuffle p'(c_2)) \shuffle (q(c_1) \shuffle q'(c_2))^{\shuffle\, -1} \\ &= (p \otimes_{\mathbb{R}} p')(c_1, c_2) \shuffle ((q \otimes_{\mathbb{R}} q')(c_1, c_2))^{\shuffle\, -1} \\ &= (p \otimes_{\mathbb{R}} p')\left(\sum_{w\in X^*} (\lambda^T \times \lambda'^T)(\mu \times \mu')(w)(\gamma \times \gamma')w\right). \end{aligned}$$

Hence, $d \shuffle d'$ is shuffle-recognizable with representation $(p \otimes_{\mathbb{R}} p', q \otimes_{\mathbb{R}} q', \{\Lambda_i\}_{i=1}^{k+n}, \{\Psi_i\}_{i=1}^{k+n}, \{\Gamma_i\}_{i=1}^{k+n})$. ■

The final theorem is useful in the next section where shuffle-rational series are used as generating series for Fliess operators.

Theorem 5.2: A series $c \in \mathbb{R}^{\shuffle}\langle(X)\rangle$ is globally convergent if $c = p(c_1, c_2, \ldots, c_k) \shuffle q(c_1, c_2, \ldots, c_k)^{-1}$ with $\deg(q) = 0$ and locally convergent otherwise.

Proof: If $\deg(q) = 0$, then c is a polynomial, which is always globally convergent. It is known that if $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ then $c^{\shuffle\, -1} \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ [11]. Hence, if $\deg(q) \neq 0$ then, $q(c)^{\shuffle\, -1}$ is locally convergent. Therefore, $p(c) \shuffle q(c)^{\shuffle\, -1} \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. ■

VI. State Space Realizations

This section presents a realization theory for Fliess operators with shuffle-rational generating series. The classical result is given first for rational series as a point of comparison. Then it is shown that a series is shuffle-rational if and only if its corresponding Fliess operator has a certain *Wiener-Fliess* realization as shown in Figure 1 [12]. Finally, the concept is applied to bilinear systems with output saturation.

Theorem 6.1: [4], [5] A series $c \in \mathbb{R}\langle(X)\rangle$ if and only if the Fliess operator $y = F_c[u]$ has a bilinear state space realization

$$\begin{aligned} \dot{z}(t) &= \left(A_0 + \sum_{i=1}^m A_i u_i(t)\right) z(t), \quad z(0) = \gamma \\ y(t) &= \lambda^T z(t), \end{aligned}$$

where $A_i \in \mathbb{R}^{N\times N}$ for $i = 0, 1 \ldots, m$, and $\gamma, \lambda \in \mathbb{R}^N$.

The following is the shuffle-rational analogue of this theorem. Note that convergence of the underlying operator is ensured by Theorem 5.2.

Theorem 6.2: A series $c \in \mathbb{R}^{\shuffle}\langle(X)\rangle$ if and only if the Fliess operator $y = F_c[u]$ has a state space realization consisting of a nilpotent bilinear system followed by a static rational function.

Proof: If $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$, then there exists a k order shuffle-representation $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$. From Lemma 5.1 and Theorem 6.1 it follows that each tuple $(\lambda_i, \mu_i, \gamma_i)$ corresponds to a nilpotent bilinear realization

$$\dot{z}_i(t) = \left(A_{i,0} + \sum_{j=1}^{m} A_{i,j} u_j(t)\right) z_i(t), \quad z_i(0) = \gamma_i$$
$$y_i(t) = \lambda_i^T z_i(t),$$

where $z_i(t) \in \mathbb{R}^{N_i}$, and the matrices $A_{i,j}$ are strictly upper triangular matrices. Let $d_i = \sum_{w \in X^*} \lambda_i^T \mu_i(w) \gamma_i \, w$ denote the corresponding generating polynomials so that

$$c = \prod_{i=1}^{k} p_i(d_i) \sqcup\!\sqcup\, q_i(d_i)^{\sqcup\!\sqcup\, -1}.$$

It then follows directly that

$$F_c[u] = \prod_{i=1}^{k} \frac{p_i}{q_i}(F_{d_i}[u]),$$

so that $y = F_c[u]$ has the desired realization with state $z = [z_1^T \; z_2^T \cdots \; z_k^T]^T$, and $A_i = \text{blkdiag}(A_{1,i}, \ldots, A_{k,i})$, $i = 0, 1, \ldots, m$.

Conversely, consider an arbitrary nilpotent system

$$\dot{z}(t) = \left(A_0 + \sum_{i=1}^{m} A_i u_i\right) z(t), \quad z(0) = z_0 \tag{6a}$$
$$y(t) = \frac{p}{q}(z(t)), \tag{6b}$$

where each $A_i \in \mathbb{R}^{N \times N}$ is strictly upper triangular, and $p, q \in \mathbb{R}[z_1, z_2, \ldots, z_N]$ such that $q(z) \neq 0$ on a neighborhood of z_0. First observe that the following set of N nilpotent bilinear systems

$$\dot{z}(t) = \left(A_0 + \sum_{i=1}^{m} A_i u_i(t)\right) z(t), \quad z(0) = z_0$$
$$y_i(t) = z_i(t),$$

has a corresponding representation (e_i, μ, γ) with respect to Lemma 5.1, where $\mu(x_i) = A_i$, $i = 1, 2, \ldots, N$, and $\gamma = z_0$. If $d_i \triangleq \sum_{w \in X^*} e_i^T \mu(w) \gamma$, then $F_{d_i}[u] = z_i$, $i = 1, 2, \ldots, N$. As the tensor algebras $\mathbb{R}[z_1, z_2, \ldots, z_N]$ and $\bigotimes_{i=1}^N \mathbb{R}[z_i]$ are isomorphic, there exists $p', q' \in \bigotimes_{i=1}^N \mathbb{R}[z_i]$ such that

$$\frac{p}{q}(z) = \frac{p'}{q'}(z_1, z_2, \ldots, z_N).$$

Therefore, the input-output behavior of system (6) is described by a Fliess operator F_d, where

$$d = \frac{p'}{q'}(d_1, d_2, \ldots, d_N).$$

Hence, $d \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ since it has the shuffle-representation $(p', q', \{e_i\}_{i=1}^N, \{\mu_i\}_{i=1}^N, \{\gamma_i\}_{i=1}^N)$, where $\mu_i = \mu$ and $\gamma_i = \gamma$ for $i = 1, 2, \ldots, N$. ■

The following example illustrates how a system with a shuffle-rational generating series can appear in practice.

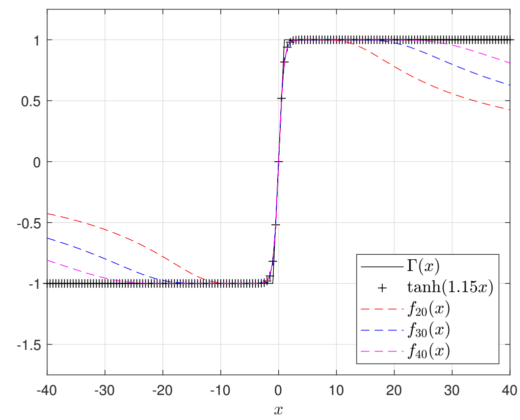

Fig. 2: Saturation function $\Gamma(x)$ and its approximations

Example 6.1: Consider a double integrator system with zero initial conditions followed by a saturation nonlinearity

$$\Gamma(x) = \begin{cases} \min(x, 1) & : x \geq 0 \\ \max(x, -1) & : x < 0. \end{cases}$$

As shown in Figure 2, Γ is well approximated by the inverse hyperbolic tangent function as

$$\tanh(1.15x) = \frac{\exp(1.15x) - \exp(-1.15x)}{\exp(1.15x) + \exp(-1.15x)}.$$

Using a Taylor series approximation of the exponential functions up to N degree gives

$$\tanh(1.15x) \approx f_n(x) \triangleq \frac{p(x)}{q(x)} = \frac{\sum_{k=0}^{n} \frac{(1.5x)^{2k+1}}{(2k+1)!}}{\sum_{k=0}^{n} \frac{(1.5x)^{2k}}{(2k)!}},$$

where $n = \lceil \frac{N}{2} \rceil$. The quality of the approximation of $\Gamma(x)$ for a few values of n is shown in Figure 2. The input-output behavior of the overall system is given by

$$y(t) = \Gamma(F_c[u]) \approx f_n(F_c[u]) = \frac{p}{q}\big(F_c[u]\big) = F_d[u],$$

where $c = x_0 x_1$ and

$$d = \left(\sum_{k=0}^{n} \frac{(1.5c)^{\sqcup\!\sqcup\,(2k+1)}}{(2k+1)!}\right) \sqcup\!\sqcup \left(\sum_{k=0}^{n} \frac{(1.5c)^{\sqcup\!\sqcup\,(2k)}}{(2k)!}\right)^{\sqcup\!\sqcup\,-1}.$$

The output response $y(t)$ of the given system and its shuffle-rational approximation $\hat{y}(t)$ when $n = 100$ is shown in Figure 3 for the applied input $u(t) = \cos(t)$. □

VII. Conclusions and Future Work

This paper introduced the notion of a shuffle-rational formal power series and two equivalent characterizations, one in terms of an analogue of recognizability of a series, and the other in the context of state space realizations for nonlinear input-output systems represented by Chen-Fliess series. An underlying computational framework was also described using the Hopf algebra of the shuffle group. As an application, it is shown how to model a bilinear system with output saturation in

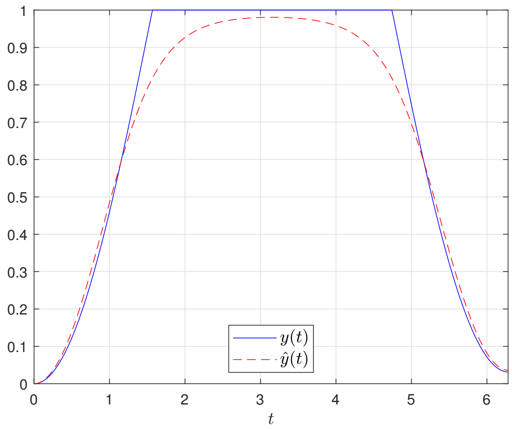

Fig. 3: Output response $y(t)$ and its shuffle-rational approximation $\hat{y}(t)$ in Example 6.1

this context. Future work will be in the direction of providing the shuffle analogues of statements 3 and 5 given in the introduction.

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