Systems & Control Letters 147 (2021) 104827

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Generating series for networks of Chen–Fliess series

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ARTICLE INFO

Article history:
Received 2 July 2020
Received in revised form 28 October 2020
Accepted 4 November 2020
Available online xxxx

Keywords:
Nonlinear control systems
Chen–Fliess series
Networks of systems

ABSTRACT

Consider a set of single-input, single-output nonlinear systems whose input–output maps are described only in terms of convergent Chen–Fliess series without any assumption that finite dimensional state space models are available. It is shown that any additive or multiplicative interconnection of such systems always has a Chen–Fliess series representation that can be computed explicitly in terms of iterated formal Lie derivatives.

1. Introduction

The study of interconnections of nonlinear control systems is normally posed in a state space setting. Issues like controllability, observability and synchronization are natural to consider in this context [1,2]. The goal of this paper is to consider networks of nonlinear systems described only in terms of Chen–Fliess series without any assumption that finite dimensional state space models are available [3,4]. Such models are useful in the context of system identification as relatively few parameters need to be estimated to yield an accurate approximation of the input–output map [5,6]. On the other hand, it is not automatically evident that any interconnection of such systems has a Chen–Fliess series representation. Dynamic output feedback systems, for example, where both the plant and controller have Chen–Fliess series representations have been shown to always have such a representation [7,8]. The proof relies on the contraction mapping theorem applied in the ultrametric space of noncommutative formal power series. While a perfectly valid approach, it does not scale easily to complex networks. So in this paper an entirely different approach is taken based on the notion of a *universal control system* due to Kawski and Sussmann [9]. The idea is relatively straightforward in that networks of universal control systems are synthesized leading to the notion of a *formal realization* evolving on an n-fold direct product of formal Lie groups. Then the generating series for any input–output pair in the network is described using the notion of a *formal representation*, a type of infinite dimensional analogue of differential representations that are common in nonlinear control theory [10,11]. It should be stated, however, that this does *not* prove that the resulting Chen–Fliess series converges in any sense. The tools used here are purely formal and algebraic. As is often the case when working with Chen–Fliess series, the algebra and the analytic issues can be considered separately with the former providing the setting for the latter, which is actually quite convenient [12]. In particular, it will be shown that any *additive* or *multiplicative* interconnection of a set of convergent single-input, single-output Chen–Fliess series always has a Chen–Fliess series representation that can be computed explicitly in terms of iterated formal Lie derivatives. The problem of determining convergence of the network's generating series will be deferred to future work.

The paper is organized as follows: Section 2 establishes the notation and terminology of the paper. Section 3 presents the concept of a formal realization. Formal representations are described in Section 4. The main results of the paper along with several examples are given in Section 5. The conclusions are summarized in Section 6, as well as directions for future research.

2. Preliminaries

An *alphabet* $X = \{x_0, x_1, \ldots, x_m\}$ is any nonempty and finite set of noncommuting symbols referred to as *letters*. A *word* $\eta = x_{i_1}\cdots x_{i_k}$ is a finite sequence of letters from X. The number of letters in a word η, written as $|\eta|$, is called its *length*. The empty word, $\emptyset$, is taken to have length zero. The collection of all words having length k is denoted by X^k. Define $X^* = \bigcup_{k\geq 0} X^k$, which is a monoid under the concatenation (Cauchy) product. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. Often c is written as the formal sum $c = \sum_{\eta\in X^*}\langle c, \eta\rangle\eta$, where the *coefficient* $\langle c, \eta\rangle \in \mathbb{R}^\ell$ is the image of $\eta \in X^*$ under c. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. The set of all noncommutative formal power series over the alphabet X

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https://doi.org/10.1016/j.sysconle.2020.104827
0167-6911/

is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The subset of series with finite support, i.e., polynomials, is represented by $\mathbb{R}^\ell\langle X\rangle$. For any $c,d\in\mathbb{R}\langle\langle X\rangle\rangle$, the scalar product is $\langle c,d\rangle := \sum_{\eta\in X^*}\langle c,\eta\rangle\langle d,\eta\rangle$, provided the sum is finite. The set $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is an associative $\mathbb{R}$-algebra under the concatenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product uniquely specified by the shuffle product of two words

$$(x_i\eta)\sqcup\!\sqcup(x_j\xi) = x_i(\eta\sqcup\!\sqcup(x_j\xi)) + x_j((x_i\eta)\sqcup\!\sqcup\xi),$$

where $x_i,x_j\in X$, $\eta,\xi\in X^*$ and with $\eta\sqcup\!\sqcup\emptyset=\emptyset\sqcup\!\sqcup\eta=\eta$ [3]. For any letter $x_i\in X$, let x_i^{-1} denote the $\mathbb{R}$-linear *left-shift operator* defined by $x_i^{-1}(\eta)=\eta'$ when $\eta=x_i\eta'$ and zero otherwise. It acts as a derivation on the shuffle product. The Lie bracket $[x_i^{-1},x_j^{-1}]=x_i^{-1}x_j^{-1}-x_j^{-1}x_i^{-1}$ also acts as a derivation on the shuffle product. Finally, the left-shift operator is defined inductively for higher order shifts via $(x_i\eta)^{-1}=\eta^{-1}x_i^{-1}$, where $\eta\in X^*$. For $p\in\mathbb{R}\langle X\rangle$, let $p^{-1}:=\sum_{\eta\in X^*}\langle p,\eta\rangle\eta^{-1}$.

Given any $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ one can associate a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p}\geq 1$ and $t_0<t_1$ be given. For a Lebesgue measurable function $u:[t_0,t_1]\to\mathbb{R}^m$, define $\|u\|_\mathfrak{p}=\max\{\|u_i\|_\mathfrak{p}: \ 1\leq i\leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0,t_1]$. Let $L^m_\mathfrak{p}[t_0,t_1]$ denote the set of all measurable functions defined on $[t_0,t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0,t_1]:=\{u\in L^m_\mathfrak{p}[t_0,t_1]: \|u\|_\mathfrak{p}\leq R\}$. Assume $C[t_0,t_1]$ is the subset of continuous functions in $L^m_1[t_0,t_1]$. Define inductively for each word $\eta=x_i\bar\eta\in X^*$ the map $E_\eta: L^m_1[t_0,t_1]\to C[t_0,t_1]$ by setting $E_\emptyset[u]=1$ and letting

$$E_{x_i\bar\eta}[u](t,t_0)=\int_{t_0}^t u_i(\tau)E_{\bar\eta}[u](\tau,t_0)\,d\tau,$$

where $x_i\in X$, $\bar\eta\in X^*$, and $u_0=1$. The *Chen–Fliess series* corresponding to $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is

$$y(t)=F_c[u](t)=\sum_{\eta\in X^*}\langle c,\eta\rangle\,E_\eta[u](t,t_0) \tag{1}$$

[3]. If there exist real numbers $K_c,M_c>0$ such that

$$|\langle c,\eta\rangle|\leq K_cM_c^{|\eta|}|\eta|!,\quad \forall\eta\in X^*,$$

then F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0,t_0+T]$ into $B^\ell_\mathfrak{q}(S)[t_0,\ t_0+T]$ for sufficiently small $R,T>0$ and some $S>0$, where the numbers $\mathfrak{p},\mathfrak{q}\in[1,\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p}+1/\mathfrak{q}=1$ [13]. (Here, $|z|:=\max_i|z_i|$ when $z\in\mathbb{R}^\ell$.) The set of all such *locally convergent* series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, and F_c is referred to as a *Fliess operator*.

Given Fliess operators F_c and F_d, where $c,d\in\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c+F_d=F_{c+d}$ and $F_cF_d=F_{c\sqcup\!\sqcup d}$, respectively [3]. When Fliess operators F_c and F_d with $c\in\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ and $d\in\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c\circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where the *composition product* of c and d is given by

$$c\circ d=\sum_{\eta\in X^*}\langle c,\eta\rangle\,\psi_d(\eta)(\mathbf{1})$$

[14]. Here $\mathbf{1}$ denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to the vector space endomorphisms on $\mathbb{R}\langle\langle X\rangle\rangle$, $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$, uniquely specified by $\psi_d(x_i\eta)=\psi_d(x_i)\circ\psi_d(\eta)$ with $\psi_d(x_i)(e)=x_0(d_i\sqcup\!\sqcup e)$, $i=0,1,\ldots,m$ for any $e\in\mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the ith component series of d ($d_0:=\mathbf{1}$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. It is sometimes useful to associate a unique alphabet with each operator. For example, let $X=\{x_0,x_1,\ldots,x_m\}$ and $\tilde X=\{\tilde x_0,\tilde x_1,\ldots,\tilde x_{\tilde m}\}$. If $c\in\mathbb{R}^{\tilde\ell}_{LC}\langle\langle\tilde X\rangle\rangle$ and $d\in\mathbb{R}^{\tilde m}_{LC}\langle\langle X\rangle\rangle$, then the cascade connection $F_c\circ F_d$ has the generating series in $\mathbb{R}^{\tilde\ell}\langle\langle X\rangle\rangle$

$$c\circ d=\sum_{\tilde\eta\in\tilde X^*}\langle c,\tilde\eta\rangle\,\psi_d(\tilde\eta)(\mathbf{1}), \tag{2}$$

where now $\psi_d(\tilde x_i):\mathbb{R}\langle\langle X\rangle\rangle\to\mathbb{R}\langle\langle X\rangle\rangle$, $e\mapsto x_0(d_i\sqcup\!\sqcup e)$, $i=0,1,\ldots,\tilde m$. In this case, the letters in X are identified with the inputs of F_d, and the letters of $\tilde X$ are identified with the inputs of F_c. There is a natural isomorphism between x_0 and $\tilde x_0$ since both symbols correspond to the unity input ($\tilde u_0=u_0=1$).

Example 2.1. Suppose $X=\{x_0,x_1\}$ and $\tilde X=\{\tilde x_0,\tilde x_1\}$. Let $c=\tilde x_1\tilde x_1$ and $d=x_1$. The generating series for the series interconnected system, $c\circ d=\tilde x_1\tilde x_1\circ x_1$, can be computed directly from (2) as

$$\begin{aligned}c\circ d&=\langle c,\tilde x_1\tilde x_1\rangle\,\psi_d(\tilde x_1\tilde x_1)(\mathbf{1})=\psi_d(\tilde x_1)\circ\psi_d(\tilde x_1)(\mathbf{1})\\&=x_0(x_1\sqcup\!\sqcup(x_0(x_1\sqcup\!\sqcup\mathbf{1})))=x_0x_1x_0x_1+2x_0x_0x_1x_1.\end{aligned}$$

It will be shown later (Examples 3.1 and 4.2) that this same result can be produced using *formal realizations* and *formal representations*. □

3. Formal realizations

For any finite $T>0$, $u\in L^m_1[0,T]$ and fixed $t\in[0,T]$, one can associate the formal power series in $\mathbb{R}\langle\langle X\rangle\rangle$

$$P[u](t)=\sum_{\eta\in X^*}\eta\,E_\eta[u](t,0),$$

which is usually called a *Chen series*. If, for example, $u_i(t)=\alpha_i\in\mathbb{R}$, $i=1,2,\ldots,m$ on $[0,T]$ ($\alpha_0:=1$) then $P[u](0)=\mathbf{1}$ and

$$\begin{aligned}\frac{d}{dt}P[u](t)&=\sum_{\eta\in X^*}\eta\frac{d}{dt}E_\eta[u](t,0)\\&=\sum_{\eta\in X^*}\sum_{i=0}^m\eta u_i(t)E_{x_i^{-1}(\eta)}[u](t,0)\\&=\sum_{\eta\in X^*}\sum_{i=0}^m\alpha_ix_i\eta\,E_\eta[u](t,0)\\&=\left(\sum_{i=0}^m\alpha_ix_i\right)P[u](t).\end{aligned}$$

It follows directly that

$$\frac{d^n}{dt^n}P[u](0)=\left(\sum_{i=0}^m\alpha_ix_i\right)^n,\quad n\geq 0,$$

and, therefore

$$P[u](t)=\sum_{n=0}^\infty\left(\sum_{i=0}^m\alpha_ix_i\right)^n\frac{t^n}{n!}=\exp\left(t\sum_{i=0}^m\alpha_i\,x_i\right).$$

In general, $P[u]$ is the solution to the formal differential equation

$$\frac{d}{dt}P[u]=\left(\sum_{i=0}^m x_iu_i\right)P[u],\quad P[u](0)=\mathbf{1}, \tag{3}$$

so that $P[u]$ is always the exponential of some Lie element over X. That is, if $\mathcal{L}(X)$ is the free Lie algebra generated by X, then any $d\in\mathbb{R}\langle\langle X\rangle\rangle$ is a *Lie series* if it can be written in the form $d=\sum_{n\geq1}p_n$, where each polynomial $p_n\in\mathcal{L}(X)$ has support residing in X^n. The set of all Lie series will be denoted by $\widehat{\mathcal{L}}(X)$. An *exponential Lie series* is any series $e=\exp(d):=\sum_{n\geq0}d^n/n!$, where d is a Lie series [15, Chapter 3]. In general, (3) has a solution of the form $P[u](t) = \exp(U(t))$ with $U(t) \in \widehat{\mathcal{L}}(X)$, $t \geq 0$ [15, Corollary 3.5]. As a consequence of the Baker–Campbell–Hausdorff formula, which states that $\log(\exp(x_i)\exp(x_j))$ is a Lie series, the set of all exponential Lie series forms a group, $\mathcal{G}(X)$, under the Cauchy product with unit $\mathbf{1}$ [16, Lemma 3] and [15, Corollary 3.3].

Following the approach of Kawski and Sussmann in [9,17], $\mathcal{G}(X)$ can be viewed as a *formal Lie group* with $\widehat{\mathcal{L}}(X)$ as its corresponding Lie algebra.[1] A commutative algebra of real-valued functions on $\mathcal{G}(X)$ is defined using the shuffle algebra on the $\mathbb{R}$-vector space $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$. Specifically, for any fixed $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ define $f_c : \mathcal{G}(X) \to \mathbb{R}$ in terms of the scalar product as

$$z \mapsto f_c(z) = \langle c, z\rangle = \sum_{\eta\in X^*} \langle c, \eta\rangle\langle z, \eta\rangle. \tag{4}$$

Ree's criterion states that $p \in \mathcal{L}(X)$ if and only if $\langle \eta ⧢ \nu, p\rangle = 0$ for all nonempty words $\eta, \nu \in X^*$ [Theorem 2.2][20]. This implies that z is an exponential Lie series if and only if $\langle c ⧢ d, z\rangle = \langle c, z\rangle\langle d, z\rangle$ for all $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ [15, Theorem 3.2]. Therefore,

$$f_c(z)f_d(z) = \langle c, z\rangle\langle d, z\rangle = \langle c ⧢ d, z\rangle = f_{c ⧢ d}(z).$$

Convergence follows from the fact that the shuffle product is known to preserve local convergence [21].[2] Often $f_c(z)$ will be abbreviated as $c(z)$, which is more natural in the present context. Analogous to standard Lie group theory, the *formal tangent space* at the unit $\mathbf{1}$, $T_{\mathbf{1}}\mathcal{G}(X)$, is identified with $\widehat{\mathcal{L}}(X)$. Thus, for any fixed $p \in \widehat{\mathcal{L}}(X)$, there is a corresponding tangent vector at $\mathbf{1}$ written as the linear functional $V_p(\mathbf{1}) : \mathbb{R}_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}$, $c \mapsto V_p(\mathbf{1})(c) := \langle c, p\mathbf{1}\rangle$ and satisfying the Leibniz rule[3]

$$\begin{aligned} V_p(\mathbf{1})(c ⧢ d) &= \langle c ⧢ d, p\mathbf{1}\rangle \\ &= \langle p^{-1}(c ⧢ d), \mathbf{1}\rangle \\ &= \langle p^{-1}(c) ⧢ d, \mathbf{1}\rangle + \langle c ⧢ p^{-1}(d), \mathbf{1}\rangle \\ &= \langle p^{-1}(c), \mathbf{1}\rangle\langle d, \mathbf{1}\rangle + \langle c, \mathbf{1}\rangle\langle p^{-1}(d), \mathbf{1}\rangle \\ &= \langle c, p\mathbf{1}\rangle\langle d, \mathbf{1}\rangle + \langle c, \mathbf{1}\rangle\langle d, p\mathbf{1}\rangle \\ &= V_p(\mathbf{1})(c)\, d(\mathbf{1}) + c(\mathbf{1})V_p(\mathbf{1})(d). \end{aligned}$$

In turn, the tangent space at $z \in \mathcal{G}(X)$, denoted as $T_z\mathcal{G}(X)$, is defined via right translation to be the vector space of linear functionals $V_p(z) : \mathbb{R}_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}$, $c \mapsto V_p(z)(c) := \langle c, pz\rangle$, $p \in \widehat{\mathcal{L}}(X)$ satisfying

$$\begin{aligned} V_p(z)(c ⧢ d) &= \langle c ⧢ d, pz\rangle \\ &= \langle c, pz\rangle\langle d, z\rangle + \langle c, z\rangle\langle d, pz\rangle \\ &= V_p(z)(c)\, d(z) + c(z)V_p(z)(d). \end{aligned} \tag{5}$$

From a Hopf algebraic viewpoint [22], elements $z \in \mathcal{G}(X)$ are group-like, that is, for $c, d \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ one has $\langle c ⧢ d, z\rangle = \langle c \otimes d, \Delta_{⧢} z\rangle = \langle c \otimes d, z \otimes z\rangle = \langle c, z\rangle\langle d, z\rangle$. Here $\Delta_{⧢}$ is the unshuffle coproduct dualizing the shuffle product. On the other hand, elements $p \in \widehat{\mathcal{L}}(X)$ are primitive, i.e., $\Delta_{⧢} p = p \otimes \mathbf{1} + \mathbf{1} \otimes p$ such that $\langle c ⧢ d, p\rangle = \langle c, p\rangle\langle d, \mathbf{1}\rangle + \langle c, \mathbf{1}\rangle\langle d, p\rangle$. Moreover, $\Delta_{⧢} pz = \Delta_{⧢} p\Delta_{⧢} z$ yields $\langle c ⧢ d, pz\rangle = \langle c, pz\rangle\langle d, z\rangle + \langle c, z\rangle\langle d, pz\rangle$. However, in this work a Hopf algebraic approach has been suppressed in favor of a purely Lie theoretic presentation.

For any $p \in \widehat{\mathcal{L}}(X)$, the mapping

$$V_p : \mathcal{G}(X) \to T_z\mathcal{G}(X),\ z \mapsto V_p(z) := pz$$

is a formal right-invariant vector field on $\mathcal{G}(X)$. Here $\mathcal{X}$ will denote the set of all such right-invariant vector fields. In addition, the *formal Lie derivative* is defined to be the mapping

$$L_p : \mathbb{R}_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}_{LC}\langle\langle X\rangle\rangle,\ c \mapsto L_pc := p^{-1}c$$

so that

$$L_pc(z) = \langle L_pc, z\rangle = \langle p^{-1}c,\ z\rangle = \langle c, pz\rangle = V_p(z)(c),$$

and, in particular,

$$\begin{aligned} L_p(c ⧢ d)(z) &= \langle L_p(c ⧢ d), z\rangle \\ &= \langle c ⧢ d, pz\rangle \\ &= (L_pc(z))\, d(z) + c(z)L_pd(z), \end{aligned}$$

which is just an alternative form of (5).

Finally, note that (1) can be written componentwise as $y_k(t) = \langle c_k, z(t)\rangle$, $k = 1, 2, \ldots, \ell$, where $c_k \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ denotes the kth component of $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $z(t) = P[u](t)$. This leads to the following definition.

Definition 3.1. For any $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, the **formal realization** of the Fliess operator $y = F_c[u]$ is

$$\begin{aligned} \dot{z} &= \sum_{i=0}^{m} x_i z u_i, \quad z(0) = \mathbf{1} \\ y_k &= \langle c_k, z\rangle, \quad k = 1, 2, \ldots, \ell. \end{aligned}$$

Observe that

$$\begin{aligned} L_{x_i}c_k(\mathbf{1}) &= x_i^{-1}c_k(\mathbf{1}) = \langle x_i^{-1}c_k, \mathbf{1}\rangle = \langle c_k, x_i\rangle \\ L_{x_j}L_{x_i}c_k(\mathbf{1}) &= x_j^{-1}x_i^{-1}c_k(\mathbf{1}) = \langle x_j^{-1}x_i^{-1}c_k, \mathbf{1}\rangle = \langle c_k, x_ix_j\rangle, \end{aligned}$$

so that the coefficients of c_k can always be written in terms of formal Lie derivatives as

$$\begin{aligned} \langle c_k, \eta\rangle &= \langle c_k, x_{i_1}\cdots x_{i_k}\rangle \\ &= L_{x_{i_k}}\cdots L_{x_{i_1}}c_k(\mathbf{1}) =: L_\eta c_k(\mathbf{1}). \end{aligned} \tag{6}$$

The notion of a formal realization in Definition 3.1 is now extended by taking a finite number of direct products of $\mathcal{G}(X)$, i.e., $\mathcal{G}^n(X) := \mathcal{G}(X) \times \mathcal{G}(X) \times \cdots \times \mathcal{G}(X)$, where $\mathcal{G}(X)$ appears n times. For any $\hat{c} = c_1 \otimes \cdots \otimes c_n \in \mathbb{R}^{\otimes n}_{LC}\langle\langle X\rangle\rangle$ define

$$\begin{aligned} &f_{\hat{c}} : \mathcal{G}^n(X) \to \mathbb{R} \\ &\quad z \mapsto (c_1 \otimes \cdots \otimes c_n)(z_1, \ldots, z_n) = \langle c_1, z_1\rangle \cdots \langle c_n, z_n\rangle. \end{aligned}$$

A commutative algebra on the $\mathbb{R}$-vector space of all such real-valued functions on $\mathcal{G}^n(X)$ is given by defining

$$\begin{aligned} f_{\hat{c}}(z)f_{\hat{d}}(z) &= [\langle c_1, z_1\rangle \cdots \langle c_n, z_n\rangle][\langle d_1, z_1\rangle \cdots \langle d_n, z_n\rangle] \\ &= \langle c_1 ⧢ d_1, z_1\rangle \cdots \langle c_n ⧢ d_n, z_n\rangle \\ &=: (\hat{c} ⧢ \hat{d})(z_1, z_2, \ldots, z_n) \\ &= f_{\hat{c} ⧢ \hat{d}}(z). \end{aligned}$$

As earlier, $f_{\hat{c}}(z)$ will often be abbreviated as $\hat{c}(z)$. The Lie algebra of $\mathcal{G}^n(X)$, denoted by $\widehat{\mathcal{L}}^n(X)$, is similarly defined as the n-fold direct sum of the Lie algebra $\widehat{\mathcal{L}}(X)$ for $\mathcal{G}(X)$ with itself. The *formal tangent space* at the unit $\mathbf{1}_n := (\mathbf{1}, \ldots, \mathbf{1})$, $T_{\mathbf{1}_n}\mathcal{G}^n(X)$, is identified with $\widehat{\mathcal{L}}^n(X)$ via the one-parameter subgroup $H(t) := (\exp(tp_1), \exp(tp_2), \ldots, \exp(tp_n))$, $p = (p_1, p_2, \ldots, p_n) \in \widehat{\mathcal{L}}^n(X)$ so that $\dot{H}(0) = p$. For any fixed $p \in \widehat{\mathcal{L}}^n(X)$, there is a corresponding tangent vector at $\mathbf{1}_n$ represented by the linear functional

$$V_p(\mathbf{1}_n) : \mathbb{R}^{\otimes n}_{LC}\langle\langle X\rangle\rangle \to \mathbb{R},\ \hat{c} \mapsto \frac{d}{dt}(\hat{c} \circ H(t))|_{t=0}.$$

1 Certain aspects of this framework can also be found in [18,19].

2 The authors of [9,17] defined their algebra on $\mathbb{R}\langle X\rangle$, which entirely avoids the convergence issue. But here $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ is more suitable for the applications to follow.

3 Recall the definition of the scalar product in the previous section.

Observe that

$$V_p(\mathbf{1}_n)(\hat{c}) = \frac{d}{dt}\big(\langle c_1, \exp(tp_1)\rangle \cdots \langle c_i, \exp(tp_i)\rangle \cdots \langle c_n, \exp(tp_n)\rangle\big)|_{t=0} = \sum_{i=1}^{n} \langle c_1, \mathbf{1}\rangle \cdots \langle c_i, p_i\mathbf{1}\rangle \cdots \langle c_n, \mathbf{1}\rangle$$

satisfies the Leibniz rule:

$$\begin{aligned} &V_p(\mathbf{1}_n)(\hat{c} \sqcup\!\sqcup \hat{d}) \\ &= \sum_{i=1}^{n} \langle c_1 \sqcup\!\sqcup d_1, \mathbf{1}\rangle \cdots \langle c_i \sqcup\!\sqcup d_i, p_i\mathbf{1}\rangle \cdots \langle c_n \sqcup\!\sqcup d_n, \mathbf{1}\rangle \\ &= \sum_{i=1}^{n} \langle c_1 \sqcup\!\sqcup d_1, \mathbf{1}\rangle \cdots \langle p_i^{-1}(c_i \sqcup\!\sqcup d_i), \mathbf{1}\rangle \cdots \langle c_n \sqcup\!\sqcup d_n, \mathbf{1}\rangle \\ &= \sum_{i=1}^{n} \langle c_1 \sqcup\!\sqcup d_1, \mathbf{1}\rangle \cdots \langle p_i^{-1}(c_i) \sqcup\!\sqcup d_i, \mathbf{1}\rangle \cdots \langle c_n \sqcup\!\sqcup d_n, \mathbf{1}\rangle + \\ &\quad \sum_{i=1}^{n} \langle c_1 \sqcup\!\sqcup d_1, \mathbf{1}\rangle \cdots \langle c_i \sqcup\!\sqcup p_i^{-1}(d_i), \mathbf{1}\rangle \cdots \langle c_n \sqcup\!\sqcup d_n, \mathbf{1}\rangle \\ &= V_p(\mathbf{1}_n)(\hat{c})\hat{d}(\mathbf{1}_n) + \hat{c}(\mathbf{1}_n)V_p(\mathbf{1}_n)(\hat{d}). \end{aligned}$$

The tangent space at $z \in \mathcal{G}^n(X)$, denoted as $T_z\mathcal{G}^n(X)$, is defined via right translation to be the vector space of linear functionals

$$\begin{aligned} V_p(z) &: \mathbb{R}_{LC}^{\otimes n}\langle\langle X\rangle\rangle \to \mathbb{R} \\ &\hat{c} \mapsto \sum_{i=1}^{n} \langle c_1, z_1\rangle \cdots \langle c_i, p_i z_i\rangle \cdots \langle c_n, z_n\rangle \end{aligned}$$

so as to satisfy

$$V_p(z)(\hat{c} \sqcup\!\sqcup \hat{d}) = V_p(z)(\hat{c})\,\hat{d}(z) + \hat{c}(z)V_p(z)(\hat{d}).$$

For any $p \in \widehat{\mathcal{L}}^n(X)$, the mapping

$$V_p : \mathcal{G}^n(X) \to T_z\mathcal{G}^n(X),\ z \mapsto (p_1z_1, \ldots, p_nz_n)$$

is a formal right-invariant vector field on $\mathcal{G}^n(X)$. Here $\mathcal{X}^n$ will denote the set of all such right-invariant vector fields. In this context, the formal Lie derivative is defined to be the mapping

$$\begin{aligned} L_p &: \mathbb{R}_{LC}^{\otimes n}\langle\langle X\rangle\rangle \to \mathbb{R}_{LC}^{\otimes n}\langle\langle X\rangle\rangle \\ &c_1 \otimes \cdots \otimes c_n \mapsto \sum_{i=1}^{n} c_1 \otimes \cdots \otimes p_i^{-1}(c_i) \otimes \cdots \otimes c_n \end{aligned}$$

so that

$$\begin{aligned} L_p\hat{c}(z) &= \left(\sum_{i=1}^{n} c_1 \otimes \cdots \otimes p_i^{-1}(c_i) \otimes \cdots \otimes c_n\right)(z_1, \ldots, z_n) \\ &= \sum_{i=1}^{n} \langle c_1, z_1\rangle \cdots \langle c_i, p_i z_i\rangle \cdots \langle c_n, z_n\rangle \\ &= V_p(z)(\hat{c}), \end{aligned} \tag{7}$$

and directly

$$L_p(\hat{c} \sqcup\!\sqcup \hat{d})(z) = (L_p\hat{c}(z))\,\hat{d}(z) + \hat{c}(z)L_p\hat{d}(z).$$

In this generalized setting, a set of n systems with state $z = (z_1, z_2, \ldots, z_n)$ evolves on the group $\mathcal{G}^n(X)$ according to the formal state equations

$$\dot{z}_j = \sum_{i=0}^{m} x_i z_j u_{ij},\quad z_j(0) = \mathbf{1},$$

where $u_{ij} \in L_p[0, T]$ and $u_{0j} = 1$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Define ℓ outputs $y_k = \hat{c}_k(z)$, where $\hat{c}_k \in \mathbb{R}_{LC}^{\otimes n}\langle\langle X\rangle\rangle$, $k = 1, 2, \ldots, \ell$. Therefore, the corresponding input–output map $u \mapsto y$ takes an $m \times n$ matrix of inputs to ℓ outputs. Consider now the situation where a network is formed by allowing each system input to be interconnected to some function of other systems' outputs and a new external input v_{ij} to yield a new input–output map $v \mapsto y$, for example, $u_{ij} = \hat{d}_{ij}(z) + v_{ij}$, where $\hat{d}_{ij} \in \mathbb{R}_{LC}^{\otimes n}\langle\langle X\rangle\rangle$. In this case, the state equations for the interconnected system become

$$\dot{z}_j = x_0 z_j + \sum_{i=1}^{m} x_i \hat{d}_{ij}(z) z_j + x_i z_j v_{ij},\quad z_j(0) = \mathbf{1}.$$

Note, in particular, the appearance of state dependent vector fields $p_j z_j$ with $p_j(t) = \sum_{i=1}^{m} x_i \hat{d}_{ij}(z(t)) \in \widehat{\mathcal{L}}(X)$. The solution to $\dot{z}_j = p_j z_j$, $z_j(0) = \mathbf{1}$ has the form $z_j(t) = \exp(U_j(t))$, where $U_j(t) \in \widehat{\mathcal{L}}(X)$. The corresponding tangent vector at $z(t)$ is

$$\begin{aligned} V_{p(t)}(z(t)) &: \mathbb{R}_{LC}^{\otimes n}\langle\langle X\rangle\rangle \to \mathbb{R} \\ &\hat{c} \mapsto \frac{d}{dt}(\hat{c} \circ z(t)) \\ &\quad = \sum_{j=1}^{n} \langle c_1, z_1(t)\rangle \cdots \langle c_j, p_j(t) z_j(t)\rangle \cdots \langle c_n, z_n(t)\rangle \\ &\quad = L_{p(t)}\hat{c}(z(t)). \end{aligned} \tag{8}$$

Substituting $p_j(t) = \sum_{i=1}^{m} x_i \hat{d}_{ij}(z(t))$ on the right-hand side above, where $\hat{d}_{ij}(z(t)) = \langle d_{ij}^{(1)}, z_1(t)\rangle \cdots \langle d_{ij}^{(n)}, z_n(t)\rangle$, gives

$$\begin{aligned} L_{p(t)}\hat{c}(z(t)) &= \sum_{j=1}^{n} \langle c_1, z_1(t)\rangle \cdots \langle c_j, p_j(t) z_j(t)\rangle \cdots \langle c_n, z_n(t)\rangle \\ &= \sum_{j=1}^{n} \langle c_1, z_1(t)\rangle \cdots \sum_{i=1}^{m} \hat{d}_{ij}(z(t))\langle c_j, x_i z_j(t)\rangle \cdots \langle c_n, z_n(t)\rangle \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \langle c_1 \sqcup\!\sqcup d_{ij}^{(1)}, z_1(t)\rangle \cdots \langle x_i^{-1}(c_j \sqcup\!\sqcup d_{ij}^{(j)}), z_j(t)\rangle \\ &\qquad \cdots \langle c_n \sqcup\!\sqcup d_{ij}^{(n)}, z_n(t)\rangle \\ &=: \hat{c}'(z(t)). \end{aligned} \tag{9}$$

In this way, a second Lie derivative can now be computed directly using (8), thus circumventing the difficult task of explicitly composing time-varying vector fields. Henceforth, all such state dependent Lie series will be written as $p(z)$. No other type of state dependent series will appear in this paper. In this context, a generalization of Definition 3.1 is presented.

Definition 3.2. Let $V_i \in \mathcal{X}^n$, $i = 0, 1, \ldots, m$ with

$$\begin{aligned} V_i &: \mathcal{G}^n(X) \to T_z\mathcal{G}^n(X) \\ &z = (z_1, \ldots, z_n) \mapsto V_i(z) = (V_{i1}(z)z_1, \ldots, V_{in}(z)z_n), \end{aligned}$$

where $V_{ij}(z(t)) \in \widehat{\mathcal{L}}(X)$. The jth component of the corresponding state equation on $\mathcal{G}^n(X)$ is

$$\dot{z}_j = \sum_{i=0}^{m} V_{ij}(z) z_j u_{ij},\quad z_j(0) = z_{j0}. \tag{10}$$

Given $\hat{c}_k \in \mathbb{R}_{LC}^{\otimes n}\langle\langle X\rangle\rangle$, $k = 1, 2, \ldots, \ell$, the kth output equation is defined to be

$$y_k = \hat{c}_k(z). \tag{11}$$

Collectively, $(V, z_0, \hat{c})$ is a **formal realization** on $\mathcal{G}^n(X)$ of the formal input–output map $u \mapsto y$.

For convenience the integer n will be referred to here as the *dimension* of the realization, though this is a misnomer as the underlying group $\mathcal{G}(X)$ is *not* finite dimensional, therefore neither is the state z. The following example illustrates how the concept naturally arises when Chen–Fliess series are composed.

Example 3.1. Reconsider the systems $y_2 = F_c[u_2]$ and $y_1 = F_d[u_1]$ in Example 2.1 using the same alphabet $X = \{x_0, x_1\}$ for both series. Each has a formal realization of the form given in Definition 3.1. Setting $u_2 = y_1$ so that $y_2 = F_c \circ F_d[u_1]$ yields a formal realization of dimension two:

$$\dot z_1 = x_0 z_1 + x_1 z_1 u_1, \quad z_1(0) = \mathbf{1}$$
$$\dot z_2 = (x_0 + x_1\langle d, z_1\rangle) z_2, \quad z_2(0) = \mathbf{1}$$
$$y_2 = \langle \mathbf{1}, z_1\rangle \langle c, z_2\rangle.$$

(Note that $\langle \mathbf{1}, z_1\rangle = 1$ and $u_{11} := u_1$.) Therefore,

$$V_0(z) = \begin{bmatrix} x_0 z_1 \\ (x_0 + x_1\langle d, z_1\rangle) z_2 \end{bmatrix}, \quad V_1(z) = \begin{bmatrix} x_1 z_1 \\ 0 \end{bmatrix},$$

and $\hat c = \mathbf{1} \otimes c$. Observe that the composition $F_c \circ F_d = F_{c\circ d}$ introduces in the second component of the tangent vector $V_0(z)$ a z_1 dependence. The aim is to express $c \circ d$ directly in terms of $(V, \mathbf{1}_2, \hat c)$. This leads to the notion of a *formal representation* of a series as presented in the next section. It can be viewed as a generalization of (6). □

4. Formal representations

The following definition is a formal analog of a differential representation as appears, for example, in [10,11].

Definition 4.1. A **formal representation** of a series $d \in \mathbb{R}\langle\langle X\rangle\rangle$ is any triple $(\mu, z_0, \hat c)$, where

$$\mu : X^* \to \mathcal{X}^n, \; x_i \mapsto V_i$$

defines a monoid homomorphism, $z_0 \in \mathcal{G}^n(X)$, and $\hat c \in \mathbb{R}^{\otimes n}_{LC}\langle\langle X\rangle\rangle$, so that for any word $\eta = x_{i_k} x_{i_{k-1}} \cdots x_{i_1} \in X^*$

$$\langle d, \eta\rangle = L_{\mu(\eta)}\hat c(z_0) := L_{\mu(x_{i_1})} L_{\mu(x_{i_2})} \cdots L_{\mu(x_{i_k})}\hat c(z_0). \tag{12}$$

By definition, $\langle d, \emptyset\rangle = L_\emptyset \hat c(z_0) := \hat c(z_0)$. The integer $n \geq 1$ will be called the **dimension** of the representation.

Example 4.1. For the trivial case where $n = 1$, $\mu(x_i) = x_i$, $z_0 = \mathbf{1}$, and $d = \hat c = c$ it is immediate that (12) reduces to (6) with $\ell = 1$. □

The following lemma provides a sufficient condition under which formal representations are always well defined.

Lemma 4.1. *Given $(\mu, z_0, \hat c)$, if for each $x_i \in X$ $[\mu(x_i)]_j(z) := V_{ij}(z) z_j$ with $V_{ij}(z)$ being some Lie polynomial in $\mathcal{L}(X)$, then there exists a well defined $d \in \mathbb{R}\langle\langle X\rangle\rangle$ satisfying (12).*

Proof. If $(\mu, z_0, \hat c)$ is a formal representation of d then necessarily for any $\eta = x_{i_1} \cdots x_{i_k} \in X^*$

$$\langle d, x_{i_k} \cdots x_{i_1}\rangle = L_{\mu(x_{i_1})} L_{\mu(x_{i_2})} \cdots L_{\mu(x_{i_k})}\hat c(z_0),$$

where each $V_{ij}(z)$ is assumed to be a Lie polynomial. Therefore, each Lie derivative can be written as a polynomial in functions of the form $\langle e, p_i z_i\rangle$ with $p_i \in \mathcal{L}(X)$, $i = 1, 2, \ldots, n$, and $e \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$, implying that d is well defined, in fact, locally finite [23]. ■

Example 4.2. Continuing Examples 2.1 and 3.1, the claim is that $c \circ d$ has a formal representation $(\mu, \mathbf{1}_2, \hat c)$, where μ is defined in terms of the vector fields V_0 and V_1 in Example 3.1 and $\hat c = \mathbf{1} \otimes c$. Note that both vector fields satisfy the condition in Lemma 4.1. As an example, it is verified that

$$\langle x_0^2 x_1^2, c \circ d\rangle = L_{\mu(x_0^2 x_1^2)}\hat c(\mathbf{1}) = L_{V_1} L_{V_1} L_{V_0} L_{V_0}\hat c(\mathbf{1}) = 2.$$

First apply (8) (suppressing all t dependence)

$$\begin{aligned} L_{V_0}\hat c(z) &= \langle c, V_{02}(z) z_2\rangle \\ &= \langle x_1^2, (x_0 + x_1\langle x_1, z_1\rangle) z_2\rangle. \end{aligned}$$

Regarding the z_1 dependence of $V_{02}(z)$, use (9) to get

$$L_{V_0}\hat c(z) = \langle x_1, z_1\rangle \langle x_1, z_2\rangle = (x_1 \otimes x_1)(z_1, z_2) = \hat c'(z).$$

Applying (8) and (9) a second time gives:

$$\begin{aligned} & L_{V_0} L_{V_0}\hat c(z) = L_{V_0}\hat c'(z) \\ &= \langle x_1, V_{01}(z) z_1\rangle \langle x_1, z_2\rangle + \langle x_1, z_1\rangle \langle x_1, V_{02}(z) z_2\rangle \\ &= \langle x_1, x_0 z_1\rangle \langle x_1, z_2\rangle + \langle x_1, z_1\rangle \langle x_1, (x_0 + x_1\langle x_1, z_1\rangle) z_2\rangle \\ &= \langle x_1, z_1\rangle^2 \langle \mathbf{1}, z_2\rangle \\ &= \langle x_1 \sqcup\!\sqcup\, x_1, z_1\rangle \langle \mathbf{1}, z_2\rangle \\ &= (x_1 \sqcup\!\sqcup\, x_1 \otimes \mathbf{1})(z_1, z_2) \\ &= (2x_1^2 \otimes \mathbf{1})(z_1, z_2) = \hat c''(z). \end{aligned}$$

Continuing in this fashion,

$$\begin{aligned} L_{V_1} L_{V_0} L_{V_0}\hat c(z) &= L_{V_1}\hat c''(z) = \langle 2x_1, z_1\rangle \langle \mathbf{1}, z_2\rangle \\ &= (2x_1 \otimes \mathbf{1})(z_1, z_2) = \hat c'''(z) \end{aligned}$$

and

$$L_{V_1} L_{V_1} L_{V_0} L_{V_0}\hat c(z) = L_{V_1}\hat c'''(z) = \langle 2\mathbf{1}, z_1\rangle \langle \mathbf{1}, z_2\rangle.$$

Therefore, $\langle x_0^2 x_1^2, c \circ d\rangle = L_{V_1} L_{V_1} L_{V_0} L_{V_0}\hat c(\mathbf{1}) = 2$ as anticipated. □

The proposition in the previous example is established in the general case by the following theorem.

Theorem 4.1. *If $d \in \mathbb{R}\langle\langle X\rangle\rangle$ has a well defined formal representation $(\mu, z_0, \hat c_k)$, then the input–output map $u \mapsto y_k$ of the corresponding formal realization (10)–(11) has a Chen–Fliess series representation with generating series d.*

Proof. Without loss of generality, assume there is a single output so that the subscripts on $\hat c_k$ and y_k can be dropped. Likewise, assume $n = 1$ so the index on the state can be omitted. Since $\dot z(t)$ is a tangent vector at $z(t) \in \mathcal{G}(X)$ for any $t \geq 0$, it follows directly from (7) that

$$\begin{aligned} \dot z(t)(\hat c) &= \sum_{i=0}^m V_i(z(t))(\hat c) u_i(t) \\ &= \sum_{i=0}^m L_{V_i}\hat c(z(t)) u_i(t). \end{aligned}$$

Integrating both sides on $[0, t]$ and applying (9) gives

$$\begin{aligned} \hat c(z(t)) &= \hat c(z_0) + \sum_{i=0}^m \int_0^t L_{V_i}\hat c(z(\tau)) u_i(\tau)\, d\tau \\ &= \hat c(z_0) + \sum_{i=0}^m \int_0^t \hat c_i'(z(\tau)) u_i(\tau)\, d\tau, \end{aligned} \tag{13}$$

where $L_{V_i}\hat c(z(\tau)) = \hat c_i'(z(\tau)) = \langle \hat c_i', z(\tau)\rangle$. Substituting $\hat c_i'$ for $\hat c$ above yields

$$\hat c_i'(z(t)) = \hat c_i'(z_0) + \sum_{i=0}^m \int_0^t \hat c_i''(z(\tau)) u_i(\tau)\, d\tau. \tag{14}$$

Noting that $y(t)=\hat{c}(z(t))$ and substituting (14) into (13) gives

$$y(t)=\hat{c}(z_0)+\sum_{i=0}^{m}L_{V_i}\hat{c}(z_0)\int_0^t u_i(\tau)\,d\tau+$$
$$\sum_{i_1,i_2=0}^{m}\int_0^t\int_0^{\tau_1}L_{V_{i_1}}\hat{c}_{i_2}(z(\tau_2))u_{i_2}(\tau_2)\,d\tau_2\,u_{i_1}(\tau_1)\,d\tau_1.$$

Continuing in this way yields

$$\begin{aligned}y(t)&=\sum_{\eta\in X^*}L_{\mu(\eta)}\hat{c}(z_0)E_\eta[u](t)\\&=\sum_{\eta\in X^*}\langle d,\eta\rangle E_\eta[u](t),\end{aligned}$$

which proves the theorem. ■

5. Networks of Chen–Fliess series

In this section specific types of networks of Chen–Fliess series are considered for which both Lemma 4.1 and Theorem 4.1 apply. To avoid a barrage of indices, the component systems are assumed to be single-input, single-output. There is, however, no technical reason for avoiding the multivariable case. A variety of different configurations are possible. The following is perhaps the simplest.

Definition 5.1. A set of m single-input, single-output Chen–Fliess series mapping $u_i\mapsto y_i$ with generating series $c_i\in\mathbb{R}_{LC}\langle\langle X_i\rangle\rangle$, where $X_i=\{x_0,x_i\}$, and weighting matrix $M\in\mathbb{R}^{m\times m}$ is said to be **additively interconnected** if $u_i=v_i+\sum_{j=1}^m M_{ij}y_j$, $i=1,2,\ldots,m$.

In the following theorem, let $\boldsymbol{e}_i\in\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ denote the series with the ith component series being the monomial $\mathbf{1}$, and the remaining components are the series having all coefficients equal to zero. In addition, given $c_j\in\mathbb{R}_{LC}\langle\langle X\rangle\rangle$, define $\hat{c}_j=\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes c_j\otimes\mathbf{1}\cdots\otimes\mathbf{1}\in\mathbb{R}^{\otimes m}_{LC}\langle\langle X\rangle\rangle$, where c_j appears in the jth position.

Theorem 5.1. *The input–output map $v\mapsto y$ of any additive interconnection of m single-input, single-output Chen–Fliess series with generating series $c_i\in\mathbb{R}_{LC}\langle\langle X_i\rangle\rangle$ has a well defined generating series $d\in\mathbb{R}^m\langle\langle X\rangle\rangle$, where d_j has the formal representation $(\mu,\mathbf{1}_m,\hat{c}_j)$ with μ defined in terms of the vector fields*

$$V_0(z)=\begin{bmatrix}x_0z_1\\x_0z_2\\\vdots\\x_0z_m\end{bmatrix}+\mathrm{diag}(x_1z_1,\ldots,x_mz_m)M\begin{bmatrix}\langle c_1,z_1\rangle\\\langle c_2,z_2\rangle\\\vdots\\\langle c_m,z_m\rangle\end{bmatrix},$$

and $V_i(z)=x_iz_i\boldsymbol{e}_i$ for $i=1,2,\ldots,m$.

Proof. It is straightforward to show that the set of interconnected Chen–Fliess series constitutes an m input, m output system with formal realization given by the vector fields as shown. Therefore, the claim follows directly from Lemma 4.1 and Theorem 4.1 with $\mu(x_i)=V_i$, $i=0,1,\ldots,m$, $z_0=\mathbf{1}_m$, and $\hat{c}_j\in\mathbb{R}^{\otimes m}_{LC}\langle\langle X\rangle\rangle$. ■

Example 5.1. A single system additively interconnected with itself as shown in Fig. 1 would correspond to propositional output feedback, i.e., $u=v+My$ (dropping all subscripts). Thus, the corresponding representation is given by

$$V_0(z)=(x_0+x_1M\langle c,z\rangle)z,\quad V_1(z)=x_1z,$$

$z_0=\mathbf{1}_1=\mathbf{1}$, and $\hat{c}=c$. For a unity feedback system, i.e., $M=1$, applying (12) gives the following generating series for the closed-loop system:

$$\langle d,\mathbf{1}\rangle=c(\mathbf{1})=\langle c,\mathbf{1}\rangle$$

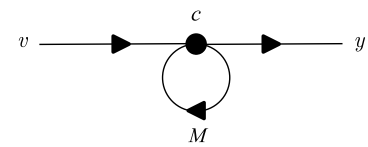

Fig. 1. Single system additively interconnected.

$$\begin{aligned}\langle d,x_1\rangle&=L_{V_1}c(\mathbf{1})=\langle c,x_1\rangle\\\langle d,x_0\rangle&=L_{V_0}c(\mathbf{1})=\langle c,x_0\rangle+\langle c,x_1\rangle\langle c,\mathbf{1}\rangle\\\langle d,x_1^2\rangle&=L_{V_1}L_{V_1}c(\mathbf{1})=\langle c,x_1^2\rangle\\\langle d,x_0x_1\rangle&=L_{V_1}L_{V_0}c(\mathbf{1})=\langle c,x_0x_1\rangle+\langle c,x_1\rangle\langle c,x_1\rangle+\\&\quad\langle c,x_1^2\rangle\langle c,\mathbf{1}\rangle\\\langle d,x_1x_0\rangle&=L_{V_0}L_{V_1}c(\mathbf{1})=\langle c,x_1x_0\rangle+\langle c,x_1^2\rangle\langle c,\mathbf{1}\rangle\\\langle d,x_0^2\rangle&=L_{V_0}L_{V_0}c(\mathbf{1})=\langle c,x_0^2\rangle+\langle c,x_1\rangle\langle c,x_0\rangle+\\&\quad\langle c,x_1x_0\rangle\langle c,\mathbf{1}\rangle+\langle c,x_0x_1\rangle\langle c,\mathbf{1}\rangle+\\&\quad\langle c,x_1\rangle\langle c,x_1\rangle\langle c,\mathbf{1}\rangle+\langle c,x_1^2\rangle\langle c,\mathbf{1}\rangle\langle c,\mathbf{1}\rangle\\&\ \ \vdots\end{aligned}$$

These expressions are consistent with those in [7], where $d=S(-c)$, and S is the antipode of the output feedback Hopf algebra. □

Example 5.2. Consider two additively interconnected systems as shown in Fig. 2, where $M_{ij}=0$ when $i=j$. Setting $M_{ij}=1$ for $i\neq j$ gives a representation of d_j specified by

$$V_0(z)=\begin{bmatrix}(x_0+x_1\langle c_2,z_2\rangle)z_1\\(x_0+x_2\langle c_1,z_1\rangle)z_2\end{bmatrix},\quad V_i(z)=x_iz_i\boldsymbol{e}_i,\quad i=1,2,$$

$z_0=\mathbf{1}_2$, and $\hat{c}_j$. For example, the generating series d_1 for the mapping $v\mapsto y_1$ is:

$$\begin{aligned}\langle d_1,\mathbf{1}\rangle&=\hat{c}_1(\mathbf{1}_2)=\langle c_1,\mathbf{1}\rangle\\\langle d_1,x_1\rangle&=L_{V_1}\hat{c}_1(\mathbf{1}_2)=\langle c_1,x_1\rangle\\\langle d_1,x_2\rangle&=L_{V_2}\hat{c}_1(\mathbf{1}_2)=0\\\langle d_1,x_0\rangle&=L_{V_0}\hat{c}_1(\mathbf{1}_2)=\langle c_1,x_0\rangle+\langle c_1,x_1\rangle\langle c_2,\mathbf{1}\rangle\\\langle d_1,x_1^2\rangle&=L_{V_1}L_{V_1}\hat{c}_1(\mathbf{1}_2)=\langle c_1,x_1^2\rangle\\\langle d_1,x_1x_2\rangle&=L_{V_2}L_{V_1}\hat{c}_1(\mathbf{1}_2)=0\\\langle d_1,x_2x_1\rangle&=L_{V_1}L_{V_2}\hat{c}_1(\mathbf{1}_2)=0\\\langle d_1,x_2^2\rangle&=L_{V_2}L_{V_2}\hat{c}_1(\mathbf{1}_2)=0\\\langle d_1,x_1x_0\rangle&=L_{V_0}L_{V_1}\hat{c}_1(\mathbf{1}_2)=\langle c_1,x_1x_0\rangle+\langle c_1,x_1^2\rangle\langle c_2,\mathbf{1}\rangle\\\langle d_1,x_0x_1\rangle&=L_{V_1}L_{V_0}\hat{c}_1(\mathbf{1}_2)=\langle c_1,x_0x_1\rangle+\langle c_1,x_1^2\rangle\langle c_2,\mathbf{1}\rangle\\&\ \ \vdots\end{aligned}$$

and similarly for d_2 corresponding to the map $v\mapsto y_2$. Unlike the first example, for networks with more than one system, there is at present no known alternative algebraic method against which to compare all of these results. Coefficient $\langle d_1,\eta\rangle$, where $\eta\in X_j^*$ and $j=1,2$ can be determined using the feedback product as described in [7], but *mixed* coefficients like $\langle d_1,x_1x_2\rangle$ cannot. □

Example 5.3. Consider three additively interconnected systems as shown in Fig. 3, where again $M_{ij}=0$ when $i=j$, and the output branches have been suppressed. For the case where $M_{ij}=1$ when $i\neq j$, a representation of d_j is given by

$$V_0(z)=\begin{bmatrix}(x_0+x_1\langle c_2,z_2\rangle+x_1\langle c_3,z_3\rangle)z_1\\(x_0+x_2\langle c_1,z_1\rangle+x_2\langle c_3,z_3\rangle)z_2\\(x_0+x_3\langle c_1,z_1\rangle+x_3\langle c_2,z_2\rangle)z_3\end{bmatrix}$$

$$V_i(z)=x_iz_i\boldsymbol{e}_i,\quad i=1,2,3,$$

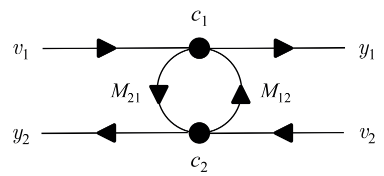

Fig. 2. Two systems additively interconnected.

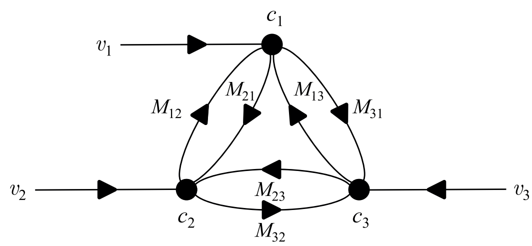

Fig. 3. Three systems additively interconnected.

$z_0 = \mathbf{1}_3$, and $\hat{c}_j$. For example, the generating series d_1 for the mapping $v \mapsto y_1$ is:

$$\begin{aligned}
\langle d_1, \mathbf{1}\rangle &= \hat{c}_1(\mathbf{1}_3) = \langle c_1, \mathbf{1}\rangle \\
\langle d_1, x_1\rangle &= L_{V_1}\hat{c}_1(\mathbf{1}_3) = \langle c_1, x_1\rangle \\
\langle d_1, x_2\rangle &= L_{V_2}\hat{c}_1(\mathbf{1}_3) = 0 \\
\langle d_1, x_3\rangle &= L_{V_3}\hat{c}_1(\mathbf{1}_3) = 0 \\
\langle d_1, x_0\rangle &= L_{V_0}\hat{c}_1(\mathbf{1}_3) = \langle c_1, x_0\rangle + \langle c_1, x_1\rangle\langle c_2, \mathbf{1}\rangle + \\
&\qquad \langle c_1, x_1\rangle\langle c_3, \mathbf{1}\rangle \\
\langle d_1, x_1^2\rangle &= L_{V_1}L_{V_1}\hat{c}_1(\mathbf{1}_3) = \langle c_1, x_1^2\rangle \\
\langle d_1, x_1x_2\rangle &= L_{V_2}L_{V_1}\hat{c}_1(\mathbf{1}_3) = 0 \\
\langle d_1, x_1x_3\rangle &= L_{V_3}L_{V_1}\hat{c}_1(\mathbf{1}_3) = 0 \\
\langle d_1, x_1x_0\rangle &= L_{V_0}L_{V_1}\hat{c}_1(\mathbf{1}_3) = \langle c_1, x_1x_0\rangle + \langle c_1, x_1^2\rangle\langle c_2, \mathbf{1}\rangle + \\
&\qquad \langle c_1, x_1^2\rangle\langle c_3, \mathbf{1}\rangle \\
\langle d_1, x_0x_1\rangle &= L_{V_1}L_{V_0}\hat{c}_1(\mathbf{1}_3) = \langle c_1, x_0x_1\rangle + \langle c_1, x_1^2\rangle\langle c_2, \mathbf{1}\rangle + \\
&\qquad \langle c_1, x_1^2\rangle\langle c_3, \mathbf{1}\rangle \\
&\vdots
\end{aligned}$$

and similarly for d_i corresponding to the map $v \mapsto y_i$, $i = 2, 3$. □

Free from the bonds of linearity, other types of interconnections are also possible as considered next.

Definition 5.2. A set of m single-input, single-output Chen–Fliess series mapping $u_i \mapsto y_i$ with generating series $c_i \in \mathbb{R}_{LC}\langle\langle X_i\rangle\rangle$, where $X_i = \{x_0, x_i\}$, and weighting matrix $M \in \mathbb{R}^{m\times m}$ is said to be **multiplicatively interconnected** if $u_i = v_i \prod_{j=1}^{m} M_{ij}y_j$, $i = 1, 2, \ldots, m$.

Theorem 5.2. *Every input–output map $v \mapsto y$ of any multiplicative interconnection of m single-input, single-output Chen–Fliess series with generating series $c_i \in \mathbb{R}_{LC}\langle\langle X_i\rangle\rangle$ has a well defined generating series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, where d_j has the formal representation $(\mu, \mathbf{1}_m, \hat{c}_j)$ with μ defined in terms of the vector fields*

$$V_0(z) = \begin{bmatrix} x_0z_1 \\ x_0z_2 \\ \vdots \\ x_0z_m \end{bmatrix}, \quad V_i(z) = x_i \prod_{j=1}^{m} M_{ij}\langle c_j, z_j\rangle z_i\boldsymbol{e}_i.$$

Proof. The proof is perfectly analogous to that of Theorem 5.1. ■

Example 5.4. Reconsider the single system network in Example 5.1 except now multiplicatively interconnected, that is, $u = vMy$ (again dropping all subscripts). The corresponding representation is given by

$$V_0(z) = x_0z, \quad V_1(z) = x_1M\langle c, z\rangle z,$$

$z_0 = \mathbf{1}$, and $\hat{c} = c$. Setting $M = 1$ and applying (12) gives the following generating series for the closed-loop system:

$$\begin{aligned}
\langle d, \mathbf{1}\rangle &= c(\mathbf{1}) = \langle c, \mathbf{1}\rangle \\
\langle d, x_1\rangle &= L_{V_1}c(\mathbf{1}) = \langle c, x_1\rangle\langle c, \mathbf{1}\rangle \\
\langle d, x_0\rangle &= L_{V_0}c(\mathbf{1}) = \langle c, x_0\rangle \\
\langle d, x_1^2\rangle &= L_{V_1}L_{V_1}c(\mathbf{1}) = \langle c, x_1^2\rangle\langle c, \mathbf{1}\rangle\langle c, \mathbf{1}\rangle + \\
&\qquad \langle c, x_1\rangle\langle c, x_1\rangle\langle c, \mathbf{1}\rangle \\
\langle d, x_0x_1\rangle &= L_{V_1}L_{V_0}c(\mathbf{1}) = \langle c, x_0x_1\rangle\langle c, \mathbf{1}\rangle \\
\langle d, x_1x_0\rangle &= L_{V_0}L_{V_1}c(\mathbf{1}) = \langle c, x_1x_0\rangle\langle c, \mathbf{1}\rangle + \langle c, x_1\rangle\langle c, x_0\rangle \\
\langle d, x_1^3\rangle &= L_{V_1}L_{V_1}L_{V_1}c(\mathbf{1}) = \langle c, x_1^3\rangle\langle c, \mathbf{1}\rangle\langle c, \mathbf{1}\rangle\langle c, \mathbf{1}\rangle + \\
&\qquad 4\langle c, x_1^2\rangle\langle c, x_1\rangle\langle c, \mathbf{1}\rangle\langle c, \mathbf{1}\rangle + \\
&\qquad \langle c, x_1\rangle\langle c, x_1\rangle\langle c, x_1\rangle\langle c, \mathbf{1}\rangle \\
\langle d, x_0^2\rangle &= L_{V_0}L_{V_0}c(\mathbf{1}) = \langle c, x_0^2\rangle \\
&\vdots
\end{aligned}$$

Consider the particular case where $c = \sum_{k\geq 0} k!\, x_1^k$. Applying the formulas above gives the closed-loop generating series

$$d = 1 + x_1 + 3x_1^2 + 15x_1^3 + \cdots,$$

which is consistent with what was computed in [8, Example 4.10] using the antipode of the output affine feedback Hopf algebra. □

6. Conclusions and future work

Using the concept of a formal realization and a formal representation, it was shown that any additive or multiplicative interconnection of a set of convergent single-input, single-output Chen–Fliess series always has a Chen–Fliess series representation whose generating series can be computed explicitly in terms of iterated formal Lie derivatives. This of course does not exhaust the list of possible network topologies for which this method is suitable. For example, there can be mixtures of additive and multiplicative nodes in a given network. There is also no technical barrier to applying the methodology in the full multivariable setting. Finally, the issue of convergence of the network's generating series needs to be addressed in every case.

CRediT authorship contribution statement

W. Steven Gray: Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Validation, Writing - original draft, Writing - review & editing. **Kurusch Ebrahimi-Fard:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Validation, Writing - original draft, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The first author was supported by the National Science Foundation under grant CMMI-1839378. The second author was supported by the Research Council of Norway through project 302831 Computational Dynamics and Stochastics on Manifolds (CODYSMA).

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