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Distributionally robust facility location problem under decision-dependent stochastic demand

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ABSTRACT

While the traditional facility location problem considers exogenous demand, in some applications, locations of facilities could affect the willingness of customers to use certain types of services, e.g., carsharing, and therefore they also affect realizations of random demand. Moreover, a decision maker may not know the exact distribution of such endogenous demand and how it is affected by location choices. In this paper, we consider a distributionally robust facility location problem, in which we interpret the moments of stochastic demand as functions of facility-location decisions. We reformulate a two-stage decision-dependent distributionally robust optimization model as a monolithic formulation, and then derive exact mixed-integer linear programming reformulation as well as valid inequalities when the means and variances of demand are piecewise linear functions of location solutions. We conduct extensive computational studies, in which we compare our model with a decision-dependent deterministic model, as well as stochastic programming and distributionally robust models without the decision-dependent assumption. The results show superior performance of our approach with remarkable improvement in profit and quality of service under various settings, in addition to computational speed-ups given by formulation enhancements. These results draw attention to the need of considering the impact of location decisions on customer demand within this strategic-level planning problem.

1. Introduction

Determining facility locations has been a fundamental problem in the design of modern transportation and logistics systems. In a traditional setting, a decision maker chooses a subset of locations from candidate sites to open facilities, as well as assigning exogenous customer demand to these capacitated facilities while minimizing the total cost of building facilities and satisfying demand. The customer demand plays a critical role as it is a driving factor in determining where to open facilities. In practice, the demand values from different customers may not be fixed but are random variables following certain probability distributions. A decision maker may not have the full knowledge about the true distribution of demand. Moreover, in some applications when new services are introduced to a certain market, locations of service facilities have inherent impact on the willingness of customers to use the service and thus the demand is not entirely exogenous but endogenous, whose probability distribution will be affected by the facility-location decisions. For example, to use carsharing services like Zipcar, customers can pick up and drop off vehicles at designated parking locations and the convenience of doing so determines the service quality. Jorge and Correia (2013) and Ciari, Bock, and Balmer (2014) demonstrate that demand volumes of carsharing customers are affected mainly by their travel distances to parking locations, as their choices of whether or not to use carsharing services largely rely on service availability within their neighborhoods (Boldrini, Bruno, & Conti, 2016; Vine, Lee-Gosselin, Sivakumar, & Polak, 2011). Similar impacts and endogenous demand are also observed in problems of warehouse location selection in supply chains (Ho & Perl, 1995).

In this paper, we integrate the decision-dependent demand uncertainty in the strategic-planning phase of locating facilities and propose a distributionally robust optimization (DRO) model for locating facilities under uncertain endogenous demand, given unknown distribution of the random demand vector whose mean and variance values are functions of facility-location decisions. The goal is to minimize the sum of facility-location cost and the worst-case (maximum) expected cost of transportation and

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unsatisfied-demand penalty given any distribution of demand from a pre-defined ambiguity set that is constructed based on the decision-dependent moment information (i.e., mean and variance). Note that to handle demand uncertainty, stochastic and robust optimization approaches are widely used in the facility-location literature (see, e.g., Álvarez-Miranda, Fernández, & Ljubić, 2015; An, Zeng, Zhang, & Zhao, 2014; Baron, Milner, & Naseraldin, 2011; Cui, Ouyang, & Shen, 2010; Shen, Zhan, & Zhang, 2011; Snyder, 2006; Snyder & Daskin, 2005), depending on the amount of data we have to describe the uncertainty. Although stochastic optimization techniques are powerful in modeling specific demand realizations and the corresponding recourse decisions, their applicability is limited to the assumption that the exact demand distribution is fully known, which requires significant amounts of data to validate. In contrast, robust facility-location models minimize the worst-case (maximum) transportation and penalty cost for any possible demand value in a given uncertainty set, and could lead to over-conservative solutions and extremely high cost of building unnecessary facilities. Our proposed DRO approach utilizes the moment information from historical demand data to build an ambiguity set of the unknown demand distribution, and thus can obtain data-driven robust solutions under partial information about the true distribution. We summarize the main contributions of the paper below.

1. We derive a monolithic formulation for solving the DRO model with decision-dependent demand whose moments can be any types of functions of facility-location decisions.
2. We obtain an exact mixed-integer linear programming (MILP) reformulation of the DRO model through duality and convex envelopes, when demand mean and variance are piecewise linear functions of location solutions. We develop enhancements to strengthen the formulation, including the derivation of valid inequalities.
3. We conduct an extensive set of numerical experiments to demonstrate the effectiveness of the proposed distributionally robust decision-dependent approach. We develop both in-sample computation and out-of-sample evaluation frameworks to compare our approach with a decision-dependent deterministic formulation, as well as the traditionally stochastic programming and DRO benchmark models neglecting the decision-dependency. Our results highlight significant increase in profit and decrease in unmet demand, and its robust performance under numerous problem characteristics along with the computational efficiency brought by formulation enhancements.

The remainder of this paper is organized as follows. In Section 2, we review the most relevant studies in the facility-location literature and in related optimization methods. In Section 3, we present the decision-dependent distributionally robust facility location problem, its reformulations given moment-based ambiguity sets of the unknown demand distribution, and formulation enhancements. In Section 4, we conduct computational studies on a variety of randomly generated instances to evaluate the performance of our model and solution approaches under different parameter and problem settings. Section 5 concludes the paper and states future research directions.

2. Literature review

We review the most relevant literature in facility location (Section 2.1), stochastic programming and distributionally robust optimization (Section 2.2), and decision-dependent uncertainty modeling in stochastic and robust optimization literature (Section 2.3).

2.1. Facility location problem variants

The facility location problem (Melo, Nickel, & Gama, 2009; Owen & Daskin, 1998) has been studied for a wide variety of applications to determine optimal locations of warehouses (Ozsen, Coullard, & Daskin, 2008), distribution centers (Zhang, Berenguer, & Shen, 2015) in supply chains, and facilities to provide emergency medical services (Chen & Yu, 2016), and so on. Given the emergence of the Internet of Things (IoT), it is also considered for building smarter and connected cities via optimizing locations of sensors and devices to enhance data flows (Fischetti, Ljubic, & Sinnl, 2017). An important branch of facility location studies considers uncertainties in problem parameters such as demand volumes. Snyder (2006) conducts a thorough review of facility location problems under various types of uncertainties existing in demand parameter, cost parameter, or related to facility characteristics. Specifically, An et al. (2014), Cui et al. (2010), Shen et al. (2011), and Snyder and Daskin (2005) focus on variants of the reliable facility location problem under random supply, demand, cost of shipment or network disruptions, and develop mathematical models for enhancing the reliability and cost-performance of location design, leading to advances in supply-chain risk management. However, among all the above studies, the demand is assumed to be exogenous and its distribution does not depend on facility-location decisions.

On the other hand, the competitive facility location problem (CFLP) involves decision games between two or multiple firms that compete for customer demand of the same product or service in a shared market (Berman, Drezner, Drezner, & Krass, 2009; Freire, Moreno, & Yushimito, 2016; Mai, Lodi, & ISSN 0377–2217. URL, 2020). It extends the classical location models, including p-median and maximum coverage, to a more complex decision-making environment, where the market is not a spatial monopoly but has co-existing competitors and certain consumer patronage pattern. The total demand volume is still exogenous but is split into multiple strings of customers whose choices depend on the facilities' utilities (e.g., their locations, sizes, service prices, etc.). The maximum capture problem (MCP) constitutes an important class of CFLP where a firm decides where to locate a set of new facilities to maximize its market share, given existing facilities already set up by its competitors and utility models that represent customer preferences. Benati and Hansen (2002), Haase and Müller (2014), and Ljubić and Moreno (2018) develop different solution approaches for MCP, where they consider a probabilistic choice model for splitting customer demand over multiple facilities. In this paper, we consider a single-player strategic planning of facility locations under decision-dependent demand uncertainty, while the majority of the MCP or CFLP studies do not take into account the stochasticity of demand, nor assume the overall demand uncertainty or its volume change being dependent on newly open facilities.

2.2. Stochastic programming and DRO methods

Stochastic programming approaches can be applied to address the issue of uncertain parameter once we know the full distributional information. For example, Santoso, Ahmed, Goetschalckx, and Shapiro (2005) consider a stochastic facility location problem by sampling realizations of demand and capacity parameters from a certain distribution, and they propose an accelerated Benders decomposition algorithm. DRO provides an alternative approach to solve problems under uncertainty when the decision maker has partial information about the distribution. Although DRO yields reliable and low-cost solutions (as compared to the ones to stochastic and robust optimization models, respectively), it is less studied in the context of facility location. Lu, Ran, and Shen (2015) consider a distributionally robust reliable facility location problem by minimizing the expected cost under the worst-case distribution with given marginal disruption probabilities and information about disruption correlations. Santiváñez and Carlo (2018) generalize the study in Lu et al. (2015) by ensuring a minimum service level in satisfying demand under each disruption scenario. Liu, Li, and Zhang (2019) propose a DRO model for optimally locating emergency medical service stations under uncertain demand. To ensure the reliability of their plan, joint chance constraints are introduced and a moment-based ambiguity set is used for describing the demand uncertainty. However, these studies pose modeling limitation by not capturing the possible impact of location decisions on demand or other parameters' uncertainty.

The general DRO literature can be classified by the ambiguity set being used to describe distributional information of uncertainties. One line of research considers statistical distance-based ambiguity sets, which focuses on distributions within a certain distance to a reference distribution. Some of the most studied distance measures in this area are ϕ-divergence (Ben-Tal, Hertog, De Waegenaere, Melenberg, & Rennen, 2013; Jiang & Guan, 2016), Wasserstein distance (Esfahani & Kuhn, 2018; Gao & Kleywegt, 2016) and Levy–Prokhorov metric (Erdoğan & Iyengar, 2006). However, these approaches require a significant amount of historical data to derive a reference distribution of the uncertain parameter, which might not be available due to the lack of prior customer data during the phase of new service introduction and facility planning. In this paper, we consider moment-based ambiguity sets that have been used for deriving tractable reformulations of general DRO models (see, e.g., Delage & Ye, 2010; Tong, Sun, Luo, & Zheng, 2018). The amount of data for obtaining reliable moment information is much less than the one needed for deriving a reference distribution with high confidence, and therefore the moment-based ambiguity set is more suitable for facility planning for new services. We also assume that the mean and variance of the demand vector are functions of facility-location decision variables, to allow possible distributions of the demand and the corresponding demand realizations to be decision dependent.

2.3. Modeling decision-dependent uncertainty

Integrating decision-dependent uncertainties within an optimization framework involves modeling challenges and computational complexities. The related literature can be categorized into two groups. The first group focuses on decisions affecting the time of information discovery. Goel and Grossmann (2006) propose a mixed-integer disjunctive programming formulation for incorporating the relationship between the underlying stochastic processes and decisions affecting the time that uncertainty is revealed. To address the computational challenge, Vayanos, Kuhn, and Rustem (2011) propose a decision rule approximation to ensure solution tractability. Their work is further generalized by Vayanos, Georghiou, and Yu (2020) to handle multi-stage robust optimization models with decision-dependent information discovery and the authors derive a reformulation that can be directly solved by off-the-shelf solvers via the K-adaptability approximation. Recently, Basciftci, Ahmed, and Gebraeel (2019) study a generic mixed-integer linear program for finite stochastic processes, and derive structural results and approximation algorithms specifically on the time of information discovery for capacity expansion planning.

In the second group of literature, decisions can change the distribution of the underlying uncertainty, similar to the setting in our paper. Using stochastic programming approaches, Ahmed (2000) considers a network design problem under design-dependent uncertainties, whereas Basciftci, Ahmed, and Gebraeel (2020) model generators' failure probabilities dependent on their maintenance and operational plans. Hellemo, Barton, and Tomasgard (2018) conduct an overview of recent studies in this area by providing ways to model decision-dependent uncertainties in stochastic programs. The decisions affecting uncertain parameters' realizations can be also incorporated into the definition of an uncertainty set used in robust optimization (see, e.g., Zhang, Kamgarpour, Georghiou, Goulart, & Lygeros, 2017). Nohadani and Sharma (2018) consider robust linear programs where the uncertainty set is a function of decision variables, and they derive tractable reformulations for specific cases. The robust decision-dependent optimization problems have been studied in wide applications including software partitioning (Spacey, Wiesemann, Kuhn, & Luk, 2012), radiotherapy planning (Nohadani & Roy, 2017) and offshore oil planning problems (Lappas & Gounaris, 2017). However, the decision dependency of uncertain parameters has not been fully explored in the DRO framework. Among limited related studies, Zhang, Xu, and Zhang (2016) consider generic decision-dependent DRO problems with moment constraints and demonstrate the stability of the optimal solutions, whereas Royset and Wets (2017) consider these problems under a class of distance-based ambiguity sets to derive convergence results. Ryu and Jiang (2019) consider distributionally robust nurse staffing problem where the uncertainty in the absenteeism of nurses depends on the staffing level decisions, and develop a separation algorithm for solving MILP reformulations. Most recently, Noyan, Rudolf, and Lejeune (2018) and Luo and Mehrotra (2018) provide non-convex reformulations for DRO problems under various forms of decision-dependent ambiguity sets. Although these studies provide alternative reformulations, the resulting models need further analyses and require the development of efficient solution algorithms. Additionally, the effect of adopting decision-dependent DRO methods in comparison to the existing stochastic or robust methodologies are not quantitatively or numerically verified in the aforementioned studies.

In brief, our paper proposes a novel approach in determining optimal facility-location plans via modeling the decision-dependency of random demand in its moment information used in the ambiguity set of a DRO model. Therefore, it addresses various gaps in the literature of facility location and methods for optimization under decision-dependent uncertainty.

3. Problem formulation

In the distributionally robust facility location problem, facility location decisions affect the underlying demand distribution of each customer site. We first introduce the ambiguity set for describing the distributional information of demand in Section 3.1. Then, we formulate the decision-dependent DRO model and present its generic reformulation in Section 3.2. By assuming the moments being piecewise linear functions of location variables, we further derive a monolithic MILP reformulation in Section 3.3. To strengthen the obtained reformulation, we provide a polyhedral study to derive valid inequalities in Section 3.4.

3.1. Ambiguity set formulation

Consider a set of possible locations $i \in I$ for building facilities and customer sites $j \in J$ having uncertain demand. We define binary variables $y_i,\ i \in I$ to indicate location decisions, such that y_i is 1 if a facility is open at location i, and 0 otherwise. The demand at each customer site $j \in J$ is represented by a random variable $d_j(y)$ whose distribution depends on the decision vector $y = [y_i,\ i \in I]^\mathsf{T}$. We consider the case where only mean and variance information are used for constructing the ambiguity set of the unknown demand distribution.

Specifically, the true distribution of demand comes from a set of possible distributions, where the random demand at each customer site $j \in J$ can take values from a finite support set

$\mathcal{K} = \{\xi_1, \ldots, \xi_K\}$ with probabilities $\pi_{j1}, \ldots, \pi_{jK}$. Therefore, the ambiguity set $U(y)$ of the unknown probabilities $\pi_{j1}, \ldots, \pi_{jK}$ is given by:

$$U(y) = \Bigg\{ \{\pi_j\}_{j\in J} : \ \pi_j \in \mathbb{R}_+^{|K|}, \sum_{k=1}^{K} \pi_{jk} = 1 \quad \forall j \in J, \\ \left| \sum_{k=1}^{K} \pi_{jk}\xi_k - \mu_j(y) \right| \le \epsilon_j^{\mu} \quad \forall j \in J, \\ (\sigma_j^2(y) + (\mu_j(y))^2)\underline{\epsilon}_j^{\sigma} \le \sum_{k=1}^{K} \pi_{jk} \\ \xi_k^2 \le (\sigma_j^2(y) + (\mu_j(y))^2)\overline{\epsilon}_j^{\sigma} \quad \forall j \in J \Bigg\}, \tag{1}$$

where $\mu_j(y)$ and $\sigma_j^2(y)$ are the mean and variance of site j's demand being any types of functions of the location decision vector y, respectively. We describe specific function forms of $\mu_j(y)$ and $\sigma_j^2(y)$ and justify the corresponding parameter choices in these functions in Section 3.3. The constraints in set (1) guarantee that (i) the probabilities at all customer sites over the support set sum up to 1, (ii) the true mean of $d_j(y)$ is within an ℓ_1-based distance ϵ_j^{μ} to the mean $\mu_j(y)$, and (iii) the true second moment of $d_j(y)$ is bounded by the sum of $(\mu_j(y))^2$ and $\sigma_j^2(y)$ with upper- and lower-bound parameters satisfying $0 \le \underline{\epsilon}_j^{\sigma} \le 1 \le \overline{\epsilon}_j^{\sigma}$, for each $j \in J$. Parameters ϵ_j^{μ}, $\underline{\epsilon}_j^{\sigma}$, $\overline{\epsilon}_j^{\sigma}$ determine the robustness of the ambiguity set for each customer site $j \in J$. Specifically, if we have the perfect knowledge regarding the first and second moments of random demand at site j, then $\epsilon_j^{\mu} = 0$, and $\underline{\epsilon}_j^{\sigma} = \overline{\epsilon}_j^{\sigma} = 1$. Otherwise, we can adjust these parameters to consider distributions within certain proximity to the desired moment information, which consequently impacts the conservatism of facility location decisions.

3.2. DRO model and a monolithic reformulation

In addition to binary variables y_i, $\forall i \in I$, we define continuous variables x_{ij} and s_j denoting at each customer site $j \in J$, the amount of that site's demand satisfied by facility i, and unsatisfied demand of that site, respectively. Parameters f_i, c_{ij}, p_j, r_j represent the cost of opening a facility at location i, unit transportation cost from location i to site j, penalty of each unit of unsatisfied demand at site j, and revenue for satisfying each unit of demand at site j, respectively. We assume that the unit penalty of unsatisfied demand at each customer site is higher than the unit cost of transportation from any two location pairs, i.e., $p_j > c_{ij}$, $\forall i \in I$, $j \in J$. This assumption is sensible in many business settings to assure the quality of service as high as possible, via guaranteeing customer satisfaction.

Furthermore, instead of assuming a total capacity at each individual facility, we consider a relaxed capacity restriction and assume that the capacity at each facility is pre-divided for individual customer sites. For example, to prepare for shipments, different sizes of vehicle fleets are pre-booked and scheduled to serve customers in different regions. We denote the total capacity in each location i as $\sum_{j\in J} C_{ij}$, where C_{ij} is the capacity at location i dedicated to customer site j. For notation convenience, without loss of generality, we further simplify the case by assuming the same amount of capacity pre-allocated to serve each customer (i.e., C_{ij} is the same and equals to C_i for all the customer sites j).

The overall decision-dependent distributionally robust facility location problem is formulated as:

$$\min_{y\in\mathcal{Y}\subseteq\{0,1\}^{|I|}} \left\{ \sum_{i\in I} f_i y_i + \max_{\pi\in U(y)} \mathbf{E}_{\pi}[h(y, d(y))] \right\}, \tag{2}$$

$$\text{where } \ h(y, d(y)) = \min_{x,s} \ \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{j\in J}(p_j s_j - r_j d_j(y)) \tag{3a}$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij} + s_j = d_j(y) \quad \forall j \in J \tag{3b}$$

$$x_{ij} \le C_i y_i \quad \forall i \in I, j \in J \tag{3c}$$

$$s_i, x_{ij} \ge 0 \quad \forall i \in I, j \in J. \tag{3d}$$

The objective function (2) minimizes the total cost of locating facilities and the maximum expected cost of transportation and unmet demand minus revenue for any demand distribution $\pi \in U(y)$. We let the set $\mathcal{Y}$ include constraints that are solely related to facility-location decisions. Constraint (3b) ensures that demand at each customer site is either satisfied by other locations or penalized, while constraint (3c) enforces capacity restriction for each open facility $i \in I$.

To derive a single-level, monolithic reformulation of the above min-max DRO model, we first show a closed-form solution to the inner problem (3).

Proposition 1. *The optimal objective value of problem (3) can be computed by*

$$h(y, d(y)) = \sum_{j\in J}\left(\max_{i^*=0,1,\ldots,|I|} \left\{ c_{i^*j}d_j(y) + \sum_{i\in I: c_{ij}<c_{i^*j}} C_i y_i (c_{ij} - c_{i^*j}) \right\} - r_j d_j(y) \right), \tag{4}$$

where $c_{0j} := p_j$.

Proof. Note that the inner problem (3) can be decomposed with respect to each location j. Therefore, we express $h(y, d(y))$ as $\sum_{j\in I} h_j(y, d(y))$, where

$$h_j(y, d(y)) = \min_{x_{\cdot j}, s_j} \ \sum_{i\in I} c_{ij}x_{ij} + p_j s_j - r_j d_j(y) \tag{5a}$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij} + s_j = d_j(y) \tag{5b}$$

$$x_{ij} \le C_i y_i \quad \forall i \in I \tag{5c}$$

$$s_j, x_{ij} \ge 0 \quad \forall i \in I. \tag{5d}$$

Let β and υ_i be the dual variables associated with constraints (5b) and (5c), respectively. We formulate the dual of model (5) as

$$\max_{\beta, > \upsilon_i} \ \beta d_j(y) + \sum_{i\in I} C_i y_i \upsilon_i \tag{6a}$$

$$\text{s.t.} \quad \beta + \upsilon_i \le c_{ij} \quad \forall i \in I \tag{6b}$$

$$\beta \le p_j \tag{6c}$$

$$\upsilon_i \le 0 \quad \forall i \in I \tag{6d}$$

To identify the optimal objective value of model (6), we derive the extreme points of its feasible region. To this end, we examine two cases through counting the number of tight constraints.

1. $\beta = p_j$: In this case, for all $i \in I$, either $\upsilon_i = 0$ or $\upsilon_i = c_{ij} - p_j$. Due to (6b) and $p_j > c_{ij}$, we have $\upsilon_i \le c_{ij} - p_j < 0$, making the condition $\upsilon_i = 0$ redundant. Therefore, when $\beta = p_j$, υ_i equals to $c_{ij} - p_j$ in the corresponding extreme point. The value of the objective function then becomes $p_j d_j(y) + \sum_{i \in I} C_i y_i (c_{ij} - p_j)$.
2. $\beta < p_j$: In this case, for all $i \in I$, either $\upsilon_i = 0$ or $\upsilon_i = c_{ij} - \beta$. Additionally, there exists at least one location i^* such that $\upsilon_{i^*} = c_{i^*j} - \beta = 0$. Therefore, at least $|I| + 1$ constraints are satisfied at an extreme point. Thus, $\beta = c_{i^*j}$ for some $i^* \in I$. For $i \in I \setminus \{i^*\}$, we have either $\upsilon_i = 0$ or $\upsilon_i = c_{ij} - c_{i^*j}$. Since $\upsilon_i \le c_{ij} - c_{i^*j}$ and $\upsilon_i \le 0$, if $c_{ij} < c_{i^*j}$, then $\upsilon_i = c_{ij} - c_{i^*j}$. Otherwise, $\upsilon_i = 0$ because we maximize a positive number times υ_i in the objective. For a given i^* location, the objective function becomes $c_{i^*j} d_j(y) + \sum_{i \in I : c_{ij} < c_{i^*j}} C_i y_i (c_{ij} - c_{i^*j})$.

Combining the above two cases, we obtain a closed-form expression for the optimal objective value of model (6). Since $p_j = c_{0j} > c_{ij},\ \forall i \in I$, the optimal objective value of the problem can be expressed as

$$\max_{i^*=0,1,\dots,|I|} \left\{ c_{i^*j} d_j(y) + \sum_{i \in I : c_{ij} < c_{i^*j}} C_i y_i (c_{ij} - c_{i^*j}) \right\}. \quad (7)$$

As the program (6) is feasible and bounded, strong duality holds between models (5) and (6). As a result, the optimal objective value of (5) equals to

$$\max_{i^*=0,1,\dots,|I|} \left\{ c_{i^*j} d_j(y) + \sum_{i \in I : c_{ij} < c_{i^*j}} C_i y_i (c_{ij} - c_{i^*j}) \right\} - r_j d_j(y), \quad (8)$$

which completes the proof. □

Theorem 1. *Model (2) is equivalent to a single-level minimization formulation given by*

$$\min_{y,\alpha,\delta^1,\delta^2,\gamma^1,\gamma^2} \quad f^\top y + \sum_{j \in J} \Big(\alpha_j + \delta_j^1 \big(\mu_j(y) + \epsilon_j^\mu\big) - \delta_j^2 \big(\mu_j(y) - \epsilon_j^\mu\big) + \gamma_j^1 (\sigma_j^2(y) + (\mu_j(y))^2) \overline{\epsilon}_j^\sigma - \gamma_j^2 (\sigma_j^2(y) + (\mu_j(y))^2) \underline{\epsilon}_j^\sigma \Big) \quad (9a)$$

$$\text{s.t.} \quad \alpha_j + \big(\delta_j^1 - \delta_j^2\big)\xi_k + \big(\gamma_j^1 - \gamma_j^2\big)\xi_k^2 \ge \theta_{jk}(y) \quad \forall j \in J, k = 1, \dots, K, \quad (9b)$$

$$y \in \mathcal{Y} \subseteq \{0,1\}^{|I|}, \delta_j^1, \gamma_j^1, \delta_j^2, \gamma_j^2 \ge 0 \quad \forall j \in J, \quad (9c)$$

where $\theta_{jk}(y) = c_{i^*_{jk} j}\xi_k + \sum_{i \in I : c_{ij} < c_{i^*_{jk} j}} C_i y_i (c_{ij} - c_{i^*_{jk} j}) - r_j \xi_k$ *and* i^*_{jk} *maximizes expression (8) with* $d_j(y)$ *being replaced by* ξ_k.

Proof. Following Proposition 1, we can reformulate the inner problem $\max_{\pi \in U(y)} \mathbf{E}[h(y, d(y))]$ for a given y as

$$\max_{\pi_{jk}, j \in J, k=1,\dots,K} \quad \sum_{j \in J} \sum_{k=1}^K \pi_{jk} \left((c_{i^*_{jk} j} - r_j)\xi_k + \sum_{i \in I : c_{ij} < c_{i^*_{jk} j}} C_i y_i (c_{ij} - c_{i^*_{jk} j}) \right) \quad (10a)$$

$$\text{s.t.} \quad \sum_{k=1}^K \pi_{jk} = 1 \quad \forall j \in J, \quad (10b)$$

$$\sum_{k=1}^K \pi_{jk} \xi_k \le \mu_j(y) + \epsilon_j^\mu \quad \forall j \in J, \quad (10c)$$

$$\sum_{k=1}^K \pi_{jk} \xi_k \ge \mu_j(y) - \epsilon_j^\mu \quad \forall j \in J, \quad (10d)$$

$$\sum_{k=1}^K \pi_{jk} \xi_k^2 \le (\sigma_j^2(y) + (\mu_j(y))^2) \overline{\epsilon}_j^\sigma \quad \forall j \in J, \quad (10e)$$

$$\sum_{k=1}^K \pi_{jk} \xi_k^2 \ge (\sigma_j^2(y) + (\mu_j(y))^2) \underline{\epsilon}_j^\sigma \quad \forall j \in J, \quad (10f)$$

$$\pi_{jk} \ge 0 \quad \forall j \in J, k = 1, \dots, K. \quad (10g)$$

Let α_j, δ_j^1, δ_j^2, γ_j^1, γ_j^2 for all $j \in J$ be the dual variables associated with all the constraints in model (10). Then, we can formulate the corresponding dual of model (10) as

$$\min_{\alpha,\delta^1,\delta^2,\gamma^1,\gamma^2} \quad \sum_{j \in J} \Big(\alpha_j + \delta_j^1 \big(\mu_j(y) + \epsilon_j^\mu\big) - \delta_j^2 \big(\mu_j(y) - \epsilon_j^\mu\big) + \gamma_j^1 (\sigma_j^2(y) + (\mu_j(y))^2) \overline{\epsilon}_j^\sigma - \gamma_j^2 (\sigma_j^2(y) + (\mu_j(y))^2) \underline{\epsilon}_j^\sigma \Big) \quad (11a)$$

$$\text{s.t.} \quad \alpha_j + \big(\delta_j^1 - \delta_j^2\big)\xi_k + \big(\gamma_j^1 - \gamma_j^2\big)\xi_k^2 \ge \theta_{jk}(y) \quad \forall j \in J, k = 1, \dots, K, \quad (11b)$$

$$\delta_j^1, \gamma_j^1, \delta_j^2, \gamma_j^2 \ge 0 \quad \forall j \in J. \quad (11c)$$

As a result, we can express model (2) in the form of (9). This completes the proof. □

3.3. Moment functions and mixed-integer linear reformulation

In this section, we specify function forms for $\mu_j(y)$ and $\sigma_j(y)$ for each $j \in J$, to further derive reformulations of Model (9) that can be directly solved by off-the-shelf solvers. We consider that the demand at site j increases from a base demand estimate $\bar{\mu}_j$ when new facilities are opened in site j's neighborhood. However, due to the size and capacity of a market, the increase in demand is restricted by an upper-bound value, denoted as μ_j^{UB} for each site j, which can be estimated by considering the growth potential of a market of interest within the planning horizon. Moreover, the highest variance of demand at a customer site occurs when there is no available facility in its neighborhood, and we can set it equal to an empirical variance $\bar{\sigma}_j^2$ that can be estimated from sample demand data. As the number of facilities in the neighborhood of a customer site increases, the variance of the demand at that site decreases. However, the variance cannot be less than a pre-determined lower-bound value, denoted as $(\sigma_j^{LB})^2$ for site j, because of the inherent uncertainty in the market.

The above assumptions are supported by Shaheen, Cohen, and Roberts (2006) and Hernndez, Jimnez, and Martn (2010), who demonstrate the increase in customers' confidence based on their past experiences with the provided service and its enhanced availability. Consequently, increased customer confidence is associated with increase in the mean and decrease in the variance of customer demand. We interpret the mean and variance information

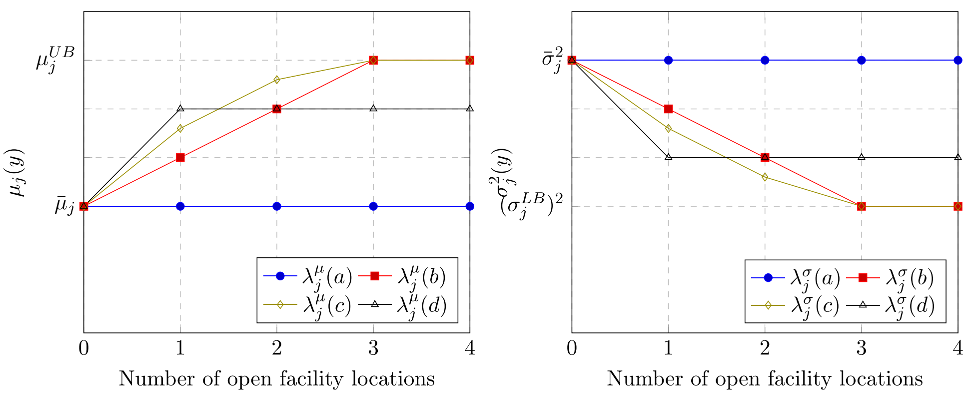

Fig. 1. Effect of the open facility locations on the moment information of demand.

using piecewise linear functions of the decision variable y as follows to indicate these relations:

$$\mu_j(y) = \min\left\{\bar{\mu}_j(1 + \sum_{i' \in I} \lambda^\mu_{ji'} y_{i'}), \mu_j^{UB}\right\},$$

$$\sigma_j^2(y) = \max\left\{\bar{\sigma}_j^2(1 - \sum_{i' \in I} \lambda^\sigma_{ji'} y_{i'}), (\sigma_j^{LB})^2\right\}. \qquad (12)$$

In (12), the effect of the distance of different facility locations on demand at a target customer site j is controlled by parameters $\lambda_j^\mu,\ \lambda_j^\sigma \in [0,1]^{|I|}$, where each element $\lambda^\mu_{ji'}$ and $\lambda^\sigma_{ji'}$ correspond to the impacts of opening location i' for customer site j on mean and variance, respectively. These parameters are represented in such a way that closer locations can have higher impacts on the first and second moments, and further locations have less effect. In practice, historical data samples can be utilized to configure the values of parameters $\bar{\mu}_j,\ \bar{\sigma}_j,\ \lambda_j^\mu,\ \lambda_j^\sigma,\ \mu_j^{UB},\ \sigma_j^{LB}$ used in (12). Take carsharing as an example and consider a problem of selecting locations for locating Zipcars in a new service region, where residents currently use other transit systems such as buses, personal vehicles, etc. First, we can use historical commute data to estimate the base mean ($\bar{\mu}_j$) and variance ($\bar{\sigma}_j^2$) of potential carsharing users in different locations, and the mean and variance upper- and lower-bounds are based on the total travel demand using all transit means who are possible to become Zipcar users. For each customer site j, we select a subset of nearby parking locations i, to which the customers living in j can travel quickly, and assign nonzero λ^μ_{ji}- and λ^σ_{ji}-values to them. Furthermore, we increase the nonzero λ^μ_{ji}- and λ^σ_{ji}-values following decreasing travel distances between customer site j and the corresponding parking location i. Moreover, we have $\sum_{i' \in I} \lambda^\sigma_{ji'} < 1$ for all $j \in J$ by assumption.

We illustrate the impacts of the above decision dependency in Fig. 1, where the first figure shows the change in the mean and the second figure depicts the change in the variance with respect to parameters λ_j^μ and λ_j^σ. For demonstration purposes, we assume the first open facility to be the closest one to customer site j, the second open facility to be the second closest, and so on. We highlight four different cases for these parameters such that in Case (a), facility location decisions have no effect on demand distribution; in Case (b) all facilities equally affect the first two moments; in Case (c) closer facilities have higher impact; and in Case (d) only the closest facility impacts customer demand. This illustration demonstrates different impacts of location decisions on customer demand, based on the dependency between moment information and customer behavior.

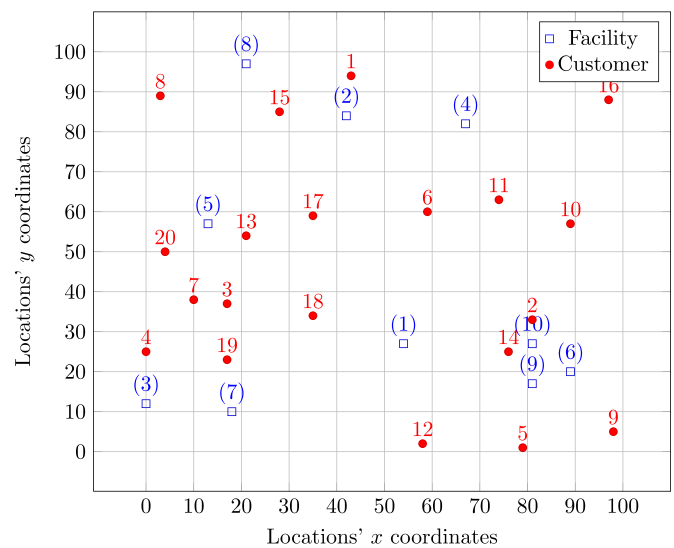

Fig. 2. Locations of customer sites and potential facilities in a specific instance.

Also, note that the presented two-piece linear functions of moments in (12) are extendable to a more generic setting. Specifically, mean and variance information can be formulated as general piecewise linear functions of the location decisions y as follows:

$$\mu_j(y) = \min_{l \in \{1,\ldots,L^\mu\}} \{a_l^\mu + {b_l^\mu}^\top y\},$$

$$\sigma_j^2(y) = \max_{l \in \{1,\ldots,L^\sigma\}} \{a_l^\sigma - {b_l^\sigma}^\top y\}, \qquad (13)$$

where parameters $L^\mu, L^\sigma \in \mathbb{Z}_+$ denote the number of pieces of linear functions considered for $\mu_j(y)$ and $\sigma_j^2(y)$, respectively; parameters $a^\mu \in \mathbb{R}^{L^\mu}$, $a^\sigma \in \mathbb{R}^{L^\sigma}$, $b^\mu \in \mathbb{R}_+^{L^\mu \times |I|}$, $b^\sigma \in \mathbb{R}_+^{L^\sigma \times |I|}$ correspond to the respective baseline mean, variance values and their slopes on each linear-function piece for $\mu_j(y)$ and $\sigma_j^2(y)$.

Model (9) involves a nonlinear objective function (9a), binary decision vector y, continuous variables $\alpha, \delta^1, \delta^2, \gamma^1, \gamma^2$, functions $\mu_j(y)$ and $\sigma_j(y)$. In the rest of the paper, we derive solvable reformulations of Model (9) based on the specific forms of $\mu_j(y)$ and $\sigma_j(y)$ in (12). To linearize the objective function, we assume upper bounds $\overline{\delta^1}, \overline{\delta^2}, \overline{\gamma^1}, \overline{\gamma^2}$ on the variables $\delta^1, \delta^2, \gamma^1, \gamma^2$, respectively. Using these bounds, McCormick envelopes (see McCormick, 1976) can be applied for linearizing the bilinear terms in the objective function (9a). Specifically, we define set $M'_{(\underline{\eta},\overline{\eta})}$ involving the McCormick inequalities for linearizing any bilinear term $w' = \eta z$ when $\eta \in [\underline{\eta}, \overline{\eta}]$ and $z \in \{0, 1\}$ and give the details as follows.

$$M'_{(\underline{\eta},\overline{\eta})} = \Big\{(w', \eta, z) \in \mathbb{R}^3 : \eta - (1-z)\overline{\eta} \le w' \le \eta - \underline{\eta}(1-z),\ \underline{\eta}z \le w' \le \overline{\eta}z\Big\}. \quad (14)$$

Because variable z is binary valued, we have an exact reformulation in (14) for representing the bilinear terms. Similarly, define set $M''_{(\underline{\eta},\overline{\eta})}$ involving McCormick inequalities for linearizing any trilinear term $w'' = \eta z_1 z_2$ when $\eta \in [\underline{\eta}, \overline{\eta}]$ such that $\underline{\eta} \ge 0$, and $z_1, z_2 \in \{0, 1\}$.

$$\begin{aligned} M''_{(\underline{\eta},\overline{\eta})} = \Big\{&(w'', \eta, z_1, z_2) \in \mathbb{R}^4 : w'' \le \overline{\eta}z_1, w'' \le \overline{\eta}z_2, w'' \\ &\le \eta - \underline{\eta}(1-z_1), w'' \le \eta - \underline{\eta}(1-z_2), \\ &w'' \ge \underline{\eta}(-1+z_1+z_2), \\ &w'' \ge \eta + \overline{\eta}(-2+z_1+z_2), z_1 \le 1, z_2 \le 1, \underline{\eta} \le \eta \le \overline{\eta}\Big\}. \end{aligned} \quad (15)$$

The trilinear case in (15) involves two binary variables, and based on existing results, we confirm that it provides an exact reformulation of Model (9).

Proposition 2. *(Meyer & Floudas, 2004) Let $0 \le \underline{\eta} \le \overline{\eta}$. Then $M''_{(\underline{\eta},\overline{\eta})} = conv\big(\big\{(w, \eta, z_1, z_2) : w = \eta z_1 z_2, \eta \in [\underline{\eta}, \overline{\eta}], z_1, z_2 \in \{0, 1\}\big\}\big)$.*

For notation brevity, we further omit the bounds μ_j^{UB} and $(\sigma_j^{LB})^2$ in (12), and only consider $\mu_j(y)$ and $\sigma_j^2(y)$ being affine functions of y. Note that this assumption is not restrictive in terms of the complexity of the problem formulation. In the presence of these upper and lower bounds, one can model the moments in (12) as piecewise linear functions with additional binary variables. The arising nonlinear relationships can be further linearized using McCormick envelopes. Following the above assumptions, the ambiguity set $U(y)$ in (1) contains nonlinear terms in y if using mean and variance function forms defined in (12). Specifically,

$$(\mu_j(y))^2 = \bar{\mu}_j^2\left(1 + 2\sum_{i'\in I}\lambda^\mu_{ji'}y_{i'} + \sum_{i'\in I}(\lambda^\mu_{ji'})^2 y_{i'}^2 + 2\sum_{l=1}^{|I|}\sum_{m=1}^{l-1}\lambda^\mu_{jl}\lambda^\mu_{jm}y_l y_m\right) \quad (16a)$$

$$= \bar{\mu}_j^2\left(1 + \sum_{i'\in I}(2\lambda^\mu_{ji'} + (\lambda^\mu_{ji'})^2)y_{i'} + 2\sum_{l=1}^{|I|}\sum_{m=1}^{l-1}\lambda^\mu_{jl}\lambda^\mu_{jm}y_l y_m\right). \quad (16b)$$

To linearize the above expression, define a new variable $Y_{lm} := y_l y_m$ where $(Y_{lm}, y_l, y_m) \in M'_{(0,1)}$. To linearize the nonlinear terms in the objective function (9a), let $\Delta^h_{ji'} := \delta^h_j y_{i'}$, $\Gamma^h_{ji'} := \gamma^h_j y_{i'}$, $\Psi^h_{jlm} := \gamma^h_j y_l y_m$, for $h = 1,\ 2$. For any pair of $j \in J$ and $i' \in I$, denote $\Lambda_{ji'} := -\bar{\sigma}_j^2\lambda^\sigma_{ji'} + \bar{\mu}_j^2(2\lambda^\mu_{ji'} + (\lambda^\mu_{ji'})^2)$ as the parameters specific to the values of λ^μ, λ^σ, as well as empirical moment estimates for any pair $j \in J$, $i' \in I$. Combining the above result with Theorem 1, we derive an MILP reformulation (17) of model (9) under ambiguity set (1) in the following theorem.

Theorem 2. *Using specific forms of moment functions in (12), the original problem (2) is equivalent to the following MILP model (17).*

$$\begin{aligned} \min\quad & f^\top y + \sum_{j\in J}\Big(\alpha_j + \delta^1_j\big(\bar{\mu}_j + \epsilon^\mu_j\big) - \delta^2_j\big(\bar{\mu}_j - \epsilon^\mu_j\big) \\ & + \bar{\mu}_j\sum_{i'\in I}\lambda^\mu_{ji'}\big(\Delta^1_{ji'} - \Delta^2_{ji'}\big) + \big(\bar{\sigma}_j^2 + \bar{\mu}_j^2\big)\big(\overline{\epsilon}^\sigma_j\gamma^1_j - \underline{\epsilon}^\sigma_j\gamma^2_j\big) \\ & + \sum_{i'\in I}\Lambda_{ji'}\big(\overline{\epsilon}^\sigma_j\Gamma^1_{ji'} - \underline{\epsilon}^\sigma_j\Gamma^2_{ji'}\big) + 2\bar{\mu}_j^2\sum_{l=1}^{|I|}\sum_{m=1}^{l-1}\lambda^\mu_{jl}\lambda^\mu_{jm} \\ & \times\big(\overline{\epsilon}^\sigma_j\Psi^1_{jlm} - \underline{\epsilon}^\sigma_j\Psi^2_{jlm}\big)\Big) \end{aligned} \quad (17a)$$

$$\text{s.t.}\quad \alpha_j + \big(\delta^1_j - \delta^2_j\big)\xi_k + \big(\gamma^1_j - \gamma^2_j\big)\xi_k^2 \ge \big(c_{i^*j} - r_j\big)\xi_k + \sum_{i\in I: c_{ij}<c_{i^*j}} C_i y_i\big(c_{ij} - c_{i^*j}\big)\ \forall i^* \in I\cup\{0\}, j \in I, k = 1,\dots,K \quad (17b)$$

$$\big(\Delta^h_{ji'}, \delta^h_j, y_{i'}\big) \in M'_{(0,\overline{\delta^h_j})},\ \big(\Gamma^h_{ji'}, \gamma^h_j, y_{i'}\big) \in M'_{(0,\overline{\gamma^h_j})}\quad \forall j \in J, i' \in I, h = 1, 2 \quad (17c)$$

$$\big(\Psi^h_{jlm}, \gamma^h_j, y_l, y_m\big) \in M''_{(0,\overline{\gamma^h_j})}\quad \forall j \in J, l = 1,\dots,|I|, l > m \quad (17d)$$

$$y \in \mathcal{Y} \subseteq \{0, 1\}^{|I|}, \delta^1_j, \gamma^1_j, \delta^2_j, \gamma^2_j \ge 0\quad \forall j \in J. \quad (17e)$$

Proof. We linearize Model (9) obtained in Theorem 1 to derive an MILP reformulation. We first plug in the decision-dependent moment information at each site j, $\mu_j(y)$ and $\sigma_j^2(y)$, into the objective function (9a), using definitions in (12) and (16b). As the resulting objective function includes bilinear and trilinear terms, we introduce new variables to obtain the linear objective function (17a). Constraint (17b) corresponds to (9b), which is also linearized. The remaining constraints refer to definitions of the newly introduced variables, their corresponding McCormick constraints in the forms of (14) and (15), restrictions on the facility-location variable y, and the non-negativity constraints on all the decision variables. This completes the proof. □

3.4. Valid inequalities

We examine the underlying problem structure for deriving valid inequalities to obtain a stronger formulation of the MILP reformulation (17). We first present an intermediate result using the inner problem (10). Since the dual (11) of the inner problem is decomposable with respect to each location j, we study the following decomposed formulation for every $j \in J$.

$$\min_{\alpha_j,\delta^1_j,\delta^2_j,\gamma^1_j,\gamma^2_j}\quad \alpha_j + \delta^1_j\big(\mu_j(y) + \epsilon^\mu_j\big) - \delta^2_j\big(\mu_j(y) - \epsilon^\mu_j\big) + \gamma^1_j\big(\sigma_j^2(y) + (\mu_j(y))^2\big)\overline{\epsilon}^\sigma_j - \gamma^2_j\big(\sigma_j^2(y) + (\mu_j(y))^2\big)\underline{\epsilon}^\sigma_j \quad (18a)$$

$$\text{s.t.}\quad \alpha_j + (\delta^1_j - \delta^2_j)\xi_k + (\gamma^1_j - \gamma^2_j)\xi_k^2 \ge \theta_{jk}(y)\quad k = 1,\dots,K, \quad (18b)$$

$$\delta^1_j, \gamma^1_j, \delta^2_j, \gamma^2_j \ge 0. \quad (18c)$$

Lemma 1. *Extreme rays of the feasible set $\{(\alpha_j, \delta^1_j, \delta^2_j, \gamma^1_j, \gamma^2_j)$: (18b), (18c)$\}$ are*

1. $(\xi_{(1)}\xi_{(2)}, 0, \xi_{(1)} + \xi_{(2)}, 1, 0)$
2. $(\xi_{(K-1)}\xi_{(K)}, 0, \xi_{(K-1)} + \xi_{(K)}, 1, 0)$
3. $(-\xi_{(1)}\xi_{(K)}, \xi_{(1)} + \xi_{(K)}, 0, 0, 1)$

where $\xi_{(1)}, \dots, \xi_{(K)}$ represent the ordered sequence of the support of the random demand.

Proof. Since $\delta_j := \delta^1_j - \delta^2_j$ and $\gamma_j := \gamma^1_j - \gamma^2_j$ are unbounded, we can equivalently consider the following system of inequalities in place of (18b) and (18c)

$$\alpha_j + \delta_j\xi_k + \gamma_j\xi_k^2 \ge \theta_{jk}(y)\quad k = 1,\dots,K. \quad (19)$$

To identify extreme rays, we solve the inequality system (20) for $m,\ n \in \{1,\dots,K\}$;

$$\alpha_j + \delta_j\xi_m + \gamma_j\xi_m^2 = 0 \tag{20a}$$

$$\alpha_j + \delta_j\xi_n + \gamma_j\xi_n^2 = 0 \tag{20b}$$

$$\alpha_j + \delta_j\xi_k + \gamma_j\xi_k^2 \geq 0 \quad k \in \{1,\dots,K\} \setminus \{m,n\}. \tag{20c}$$

Without loss of generality, we assume that $\xi_m < \xi_n$. Solving the equalities (20a) and (20b), we obtain $\delta_j = -(\xi_m + \xi_n)\gamma_j$, and $\alpha_j = \xi_m\xi_n\gamma_j$. The next step is to ensure that the inequality system (20c) is satisfied. We study two cases with respect to the direction γ_j as follows by normalizing $|\gamma_j| = 1$.

1. $\gamma_j = 1$: In this case, we need to guarantee that $(\xi_k - \xi_m)(\xi_k - \xi_n) \geq 0$ for all $k \in \{1,\dots,K\} \setminus \{m,n\}$. Consequently, we have either $\xi_k \geq \xi_m$ and $\xi_k \geq \xi_n$, or $\xi_k \leq \xi_m$ and $\xi_k \leq \xi_n$. There are only two ways to satisfy these restrictions, resulting in the following extreme ray generators of the form $(\alpha_j, \delta_j, \gamma_j)$:
   - $(\xi_{(1)}\xi_{(2)}, -(\xi_{(1)} + \xi_{(2)}), 1)$;
   - $(\xi_{(K-1)}\xi_{(K)}, -(\xi_{(K-1)} + \xi_{(K)}), 1)$.
2. $\gamma_j = -1$: In this case, we need to ensure that $(\xi_k - \xi_m)(\xi_k - \xi_n) \leq 0$ for all $k \in \{1,\dots,K\} \setminus \{m,n\}$. This requires that $\xi_m \leq \xi_k \leq \xi_n$. To satisfy this case, we have the extreme ray generator
   - $(-\xi_{(1)}\xi_{(K)}, \xi_{(1)} + \xi_{(K)}, -1)$.

Lastly, through converting the resulting extreme ray generators to the original variables of the form $(\alpha_j, \delta_j^1, \delta_j^2, \gamma_j^1, \gamma_j^2)$ using $\delta_j^1 = \max\{0, \delta_j\}$, $\delta_j^2 = \max\{0, -\delta_j\}$, $\gamma_j^1 = \max\{0, \gamma_j\}$, $\gamma_j^2 = \max\{0, -\gamma_j\}$, we obtain the desired result. This completes the proof. □

Then building on Proposition 1, we derive valid inequalities for the MILP reformulation (17) as follows.

Proposition 3. *The following inequalities are valid for Model (17):*

$$\xi_{(1)}\xi_{(2)} - (\xi_{(1)} + \xi_{(2)})(\mu_j(y) - \epsilon_j^\mu) + (\sigma_j^2(y) + (\mu_j(y))^2)\bar{\epsilon}_j^\sigma \geq 0 \quad \forall j \in J \tag{21a}$$

$$\xi_{(K-1)}\xi_{(K)} - (\xi_{(K-1)} + \xi_{(K)})(\mu_j(y) - \epsilon_j^\mu) + (\sigma_j^2(y) + (\mu_j(y))^2)\bar{\epsilon}_j^\sigma \geq 0 \quad \forall j \in J \tag{21b}$$

$$-\xi_{(1)}\xi_{(K)} + (\xi_{(1)} + \xi_{(K)})(\mu_j(y) + \epsilon_j^\mu) - (\sigma_j^2(y) + (\mu_j(y))^2)\underline{\epsilon}_j^\sigma \geq 0 \quad \forall j \in J \tag{21c}$$

Proof. First, consider the primal problem (10) and its dual problem (11). Note that the dual model (11) is always feasible as we can let values of variables α_j be arbitrarily large. To ensure the feasibility of the primal problem, it suffices to demonstrate that the dual problem is bounded. To this end, we consider the decomposed dual subproblem (18), and use the extreme ray generators in Lemma 1 by plugging them into the objective function (18a). The resulting inequalities (21) ensure that the dual problem (18) is bounded, to guarantee the feasibility of (10). This completes the proof. □

We continue to linearize nonlinear terms in (21) using (16b) and McCormick envelopes (14). As a result, inequalities (21) are equivalent to:

$$\xi_{(1)}\xi_{(2)} - (\xi_{(1)} + \xi_{(2)})(\bar{\mu}_j(1 + \sum_{i' \in I}\lambda_{ji'}^\mu y_{i'}) - \epsilon_j^\mu) + \Theta_j\bar{\epsilon}_j^\sigma \geq 0 \quad \forall j \in J \tag{22a}$$

$$\xi_{(K-1)}\xi_{(K)} - (\xi_{(K-1)} + \xi_{(K)})(\bar{\mu}_j(1 + \sum_{i' \in I}\lambda_{ji'}^\mu y_{i'}) - \epsilon_j^\mu) + \Theta_j\bar{\epsilon}_j^\sigma \geq 0 \quad \forall j \in J \tag{22b}$$

$$-\xi_{(1)}\xi_{(K)} + (\xi_{(1)} + \xi_{(K)})(\bar{\mu}_j(1 + \sum_{i' \in I}\lambda_{ji'}^\mu y_{i'}) + \epsilon_j^\mu) - \Theta_j\underline{\epsilon}_j^\sigma \geq 0 \quad \forall j \in J \tag{22c}$$

$$\Theta_j = \bar{\sigma}_j^2 + \bar{\mu}_j^2 + \sum_{i' \in I}\Lambda_{ji'}y_{i'} + 2\bar{\mu}_j^2\sum_{l=1}^{|I|}\sum_{m=1}^{l-1}\lambda_{jl}^\mu\lambda_{jm}^\mu Y_{lm} \quad \forall j \in J \tag{22d}$$

$$(Y_{lm}, y_l, y_m) \in M'_{(0,1)} \quad \forall l = 1,\dots,|I|,\ l > m. \tag{22e}$$

After integrating constraints (22) into the MILP reformulation (17), we strengthen our formulation for the original problem (2). Later our computational studies are based on Model (17) with valid inequalities (22), and we further demonstrate the efficiency of the proposed constraints in the next section.

4. Computational results

In this section, we conduct extensive numerical studies and demonstrate the efficacy of the proposed decision-dependent distributionally robust (DDDR) approach from various aspects. Specifically, we compare its solutions and performance against different facility location plans obtained from solving a deterministic formulation with decision-dependent demand, and distributionally robust (DR) and stochastic programming (SP) formulations neglecting decision-dependency in the demand parameter.

To evaluate a location plan $\hat{y}$ (given by either DDDR, DR, SP, or the deterministic model), we conduct out-of-sample tests by using model (23) (see below) to evaluate solution performance. We employ the Monte Carlo sampling and Sample Average Approximation method (see Kleywegt, Shapiro, & Mello, 2002) for generating realizations of customer demand. Specifically, we generate a set of demand realizations, denoted by $d_j^\omega(\hat{y})$, for each scenario $\omega \in \Omega$ and customer site $j \in J$. These realizations (or equivalently, the out-of-sample test scenarios) are generated based on a given solution $\hat{y}$ using the mean and variance information defined in (12). For each scenario ω, let p^ω, x_{ij}^ω and s_j^ω be the probability of realizing the scenario, the amount of demand at customer site j satisfied by facility at location i, and the unsatisfied demand at customer site j, respectively. A solution evaluation model is given by:

$$\min_{x,s} \quad \sum_{i \in I} f_i\hat{y}_i + \sum_{\omega \in \Omega} p^\omega\left(\sum_{i \in I}\sum_{j \in J} c_{ij}x_{ij}^\omega + \sum_{j \in J}\left(p_j s_j^\omega - r_j d_j^\omega(\hat{y})\right)\right) \tag{23a}$$

$$\text{s.t.} \quad \sum_{i \in I} x_{ij}^\omega + s_j^\omega = d_j^\omega(\hat{y}) \quad \forall j \in J, \omega \in \Omega \tag{23b}$$

$$x_{ij}^\omega \leq C_i\hat{y}_i \quad \forall i \in I,\ j \in J, \omega \in \Omega \tag{23c}$$

$$s_i^\omega, x_{ij}^\omega \geq 0 \quad \forall i \in I,\ j \in J, \omega \in \Omega. \tag{23d}$$

Here the objective function (23a) minimizes the total expected cost of facility location, transportation, and unmet demand minus the revenue obtained. Constraint (23b) ensures that demand is either satisfied or penalized across all the scenarios while constraint

(23c) guarantees that the capacity of each facility location is not violated. For an independently and identically distributed set of scenarios, model (23) is decomposable by scenario when the value $\hat{y}$ is given. In this case, model (23) can be solved separately for each scenario subproblem.

In the remainder of the section, we first discuss experimental settings used in our numerical studies in Section 4.1. Then we provide a comprehensive analysis of the proposed approach on various test cases in Section 4.2 including different (i) variability levels of demand, (ii) unit penalty costs, (iii) robustness levels, (iv) limits on the number of open facilities, (v) decision-dependent distribution types, and (vi) models with decision-dependent parameter. Finally, we highlight the computational efficiency of the DDDR model by conducting experiments using different sizes of instances in Section 4.3.

4.1. Experimental setup

We randomly generate a set of potential facility locations and customer sites. We first present the default settings for all the problem parameters, which remain the same throughout all the numerical studies, unless otherwise stated. Euclidean distance is used to represent the distance between each candidate facility location and customer site. These distance values are assumed to directly affect transportation cost parameters, namely c_{ij} for all $i \in I,\ j \in J$. The parameters for the fixed opening cost, f_i, and capacity, C_i, for all $i \in I$ are sampled from Uniform distributions $U(5000, 10,000)$ and $U(10, 20)$, respectively. For each $j \in J$, we set unit penalty, p_j, for the unmet demand as 225, and revenue parameter r_j as 150 for each customer site $j \in J$.

We sample the empirical mean of demand at each customer site $j \in J$, $\bar{\mu}_j$, from a Uniform distribution $U(20, 40)$. Then, we let $\bar{\sigma}_j = \bar{\mu}_j$, implying the coefficient of variation equaling to 1. We define the moment-based ambiguity set by letting $\epsilon^{\mu}_j = 0$, and $\underline{\epsilon}^{\sigma}_j = \overline{\epsilon}^{\sigma}_j = 1$ in (1), for all customer sites $j \in J$. The support size of demand values at each customer site, namely K, is taken as 100 and thus the values $\xi_1, \dots, \xi_K$ are in the range $\{1, \dots, 100\}$.

For establishing decision dependency between demand distribution and facility location decisions, we select the parameters λ^{μ}_{ji} and λ^{σ}_{ji} for all $i \in I,\ j \in J$ using the distance between each facility location and customer pair. We consider them as a decreasing function of the corresponding distance, specifically $\exp(-c_{ij}/25)$. Consequently, the effect of a facility located at i on the demand at customer site j is higher when the facility is closer to the customer. Next, the sums of the vectors λ^{μ}_j, λ^{σ}_j are normalized for each customer site $j \in J$ to adjust the effect of the location decisions on demand. Note that if λ^{μ}_{ji} and λ^{σ}_{ji} values are set to 0 for all $j \in J$ and $i \in I$ in the moment functions in (12), then the current setting reduces to a decision-independent form, i.e., a traditional distributionally robust optimization model.

To assess the performance of the proposed optimization framework by taking into account various choices of model parameters and underlying demand distribution, we provide an extensive set of numerical studies over the proposed and existing optimization approaches. We implement all the optimization models in Python using Gurobi 7.5.2 as the solver on an Intel i5-3470T 2.90 gigahertz machine with 8 gigabytes RAM.

4.2. Numerical results and analyses

We first examine how facility location decisions are affected by parameter choices, robustness levels, and modeling of decision-dependency in DDDR and other benchmark approaches. In particular, we consider location solutions given by SP, DR, and DDDR models over a set of diverse instances. For obtaining in-sample location solutions of an SP model (23) where we replace the fixed $\hat{y}$ by decision variables y, we generate training samples with 20 or 100 scenarios following a Normal distribution with mean and variance of the demand at each customer site $j \in J$ being $\bar{\mu}_j$ and $\bar{\sigma}^2_j$, respectively. As scenarios are uniformly sampled from the Normal distribution following the Monte Carlo sampling method, the probability of realizing each scenario ω, i.e., p^{ω}, is taken as $1/|\Omega|$. We refer to instances of the two SP models with different training sample sizes as SP(20) and SP(100), respectively. We also utilize model (17) to develop a regular distributionally robust facility location model by setting λ^{μ}_{ji} and λ^{σ}_{ji} values to 0 for all $j \in J$ and $i \in I$ to obtain in-sample location solutions under the DR setting. For evaluating the performance of different location solutions, we generate random samples with 1000 test scenarios for the out-of-sample test. In particular, given a solution $\hat{y}$, we first obtain the values of the moment functions $\mu_j(\hat{y})$ and $\sigma^2_j(\hat{y})$ for each customer site $j \in J$ using (12), and generate test scenarios based on these values following a certain distribution. In the default setting, we test our results by considering Normal distribution as the true underlying demand distribution but vary the distribution types in one set of tests later.

Table 1 presents the average solution performance, in terms of the objective value and unmet demand of solutions given by different approaches over 10 replications of out-of-sample tests using (23) for five different sizes of instances. Specifically, we consider $|I| = 5, \dots, 10$, and $|J| = 2|I|$. Since we minimize the total cost minus revenue, smaller objective values are preferred. The results demonstrate the superior performance of DDDR solutions over the ones of SP and DR with decision-independent demand. For instance, the DDDR approach provides, on average, 18% and 12% improvements in profit, and 99% and 96% reduction in unmet demand, as compared to SP and DR approaches over instances with 10 facilities, respectively. Consequently, SP and DR approaches obtain less profit and lower quality of service by not fully satisfying the demand.

We further provide detailed results on a specific instance with $|I| = 10$ and $|J| = 20$, for which we visualize all the 20 customer sites and 10 possible facility locations in Fig. 2. The customer sites are marked by circles, and the possible facility locations are denoted by squares.

In Table 2, we present the results of DDDR, DR, and SP approaches under the default setting, where the average, standard deviation and percentile values of each solution's out-of-sample objective value (i.e., the net profit) and unmet demand are detailed. Overall, the DDDR approach yields the best results in terms of profit and unmet demand. DR is better than SP in terms of percentile values of the total profit and unmet demand. Both SP(20) and SP(100)'s solutions show similar out-of-sample performance. The results also demonstrate the importance of considering the decision-dependency in demand parameters, as the DDDR approach outperforms DR in terms of profit and unmet demand (i.e., quality of service).

4.2.1. Effect of the variability in demand

Next, we show how solutions produced by different models are affected by the variability of the underlying demand data. Fig. 3 shows average out-of-sample performance of profit and unmet demand in all the 1000 test scenarios, where the coefficient of variation is used for representing the demand variability.

In Fig. 3, as the coefficient of variation (defined as the ratio of empirical variance and mean, namely $\frac{\bar{\sigma}^2_j}{\bar{\mu}_j}$ at each customer site j) increases, the corresponding demand variability increases, assuming that the empirical mean is kept constant. Consequently, the distributionally robust approaches (DR and DDDR) become more suitable as compared to SP under higher variability as they obtain

Table 1
Average out-of-sample profit and unmet demand values given by different models' solutions for different instance sizes.

	$\|I\|$	SP(20)	SP(100)	DR	DDDR
Average profit	5	−12806.7	−12763.3	−7618.55	−22554.2
	6	−21460.7	−21483.7	−16425.1	−30298.1
	7	−28201.8	−27810.6	−24911.6	−36608.5
	8	−40914.9	−39206.7	−35810.4	−48027.2
	9	−48247.1	−48790.2	−51503.5	−59816.9
	10	−63281.7	−63337.3	−67084.4	−75164.8
Average unmet demand	5	95.4	95.9	128.2	15.1
	6	75.7	74.7	110.6	3.9
	7	71.7	75.1	86.2	0.0
	8	56.8	67.8	82.1	0.1
	9	58.1	53.9	38.7	0.2
	10	47.0	46.9	11.1	0.4

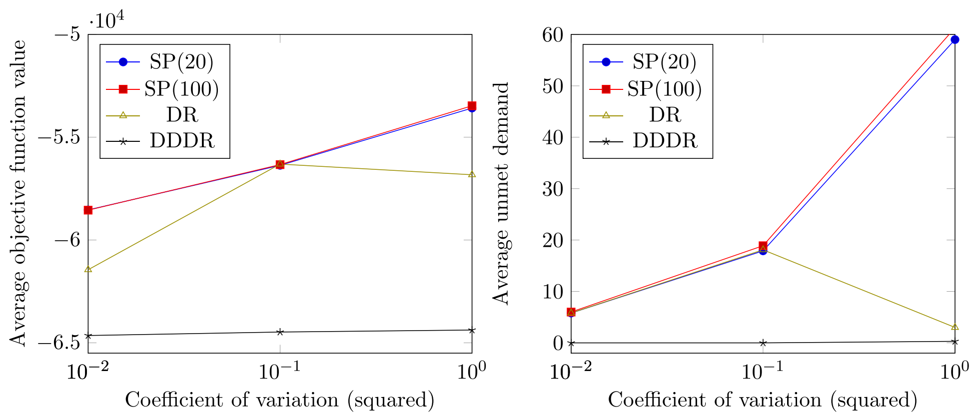

Fig. 3. Effect of the demand variability on solutions given by different approaches.

Table 2
Statistics of the out-of-sample profit and unmet-demand results given by solutions of SP, DR and DDDR models for a specific instance shown in Fig. 2.

	SP(20)	SP(100)	DR	DDDR
Average profit	−53581.0	−53468.2	−61443.7	−64375.0
Std. dev.	6457.3	6712.0	6613.8	4917.8
95%	−42448.0	−42643.8	−50474.8	−55845.2
90%	−44968.7	−44505.1	−52944.0	−58082.6
75%	−49367.8	−48939.7	−57056.9	−61178.4
50%	−53957.6	−53666.1	−61504.1	−64362.4
Average unmet demand	59.0	61.3	3.0	0.3
Std. dev.	35.1	35.6	6.8	2.3
95%	124.4	122.8	18.3	0.0
90%	105.6	107.9	11.4	0.0
75%	80.2	81.9	2.6	0.0
50%	54.4	58.0	0.0	0.0

Table 3
Statistics of profit and unmet demand given by SP, DR and DDDR solutions tested in Gamma-distribution-based out-of-sample scenarios.

	SP(20)	SP(100)	DR	DDDR
Average profit	−51962.2	−51627.2	−60575.0	−64268.5
Std. dev.	5779.2	5951.9	6145.7	4694.6
95%	−42495.9	−41383.7	−50788.8	−56590.5
90%	−44328.6	−43867.8	−52915.7	−58320.5
75%	−48110.2	−47766.7	−56590.1	−61069.6
50%	−51858.0	−51960.7	−60525.0	−64306.2
Average unmet demand	70.7	71.1	8.2	1.0
Std. dev.	47.5	46.8	16.8	5.0
95%	160.1	157.2	38.0	5.4
90%	137.4	136.3	23.9	0.0
75%	97.3	98.7	9.5	0.0
50%	61.2	62.8	0.0	0.0

location plans that are more reliable to various demand patterns in the out-of-sample test scenarios. SP is more sensitive to the demand variability as the performance of its solutions monotonically worsens as demand variance increases. As the coefficient of variation decreases, the demand variability decreases and the performance of stochastic and distributionally robust solutions become similar. Moreover, the DDDR approach performs significantly better in all the settings, highlighting the importance of considering decision dependency in parameter uncertainty quantification.

Next, we analyze the effect of misspecifying the true demand distribution by constructing a new set of test scenarios for the out-of-sample test. For each solution $\hat{y}$, instead of using the Normal distribution we follow a Gamma distribution to generate test scenarios, where the scale parameter $\hat{\theta}_j^{\gamma} = \sigma_j^2(\hat{y})/\mu_j(\hat{y})$ and the shape parameter $\hat{k}_j^{\gamma} = \mu_j(\hat{y})/\hat{\theta}_j^{\gamma}$ for each customer site j. Table 3 provides the corresponding results in comparison to Table 2 (where scenarios were sampled following a Normal distribution). As Gamma distributions are more skewed, the percentile results worsen for all approaches. Moreover, SP cannot capture the changes in the underlying distribution, whereas DR and DDDR are not much impacted by these changes. The proposed DDDR approach again yields the best results in terms of average, standard deviation and percentile values of the total profit and unmet demand.

4.2.2. Effect of unit penalty for unmet demand

We examine the effect of the parameter setting for penalizing each unit of unmet demand. Table 4 shows the facility location plans given by different approaches with unit penalty cost $p_j = 150,\ 225,\ 300$ for all $j \in J$. The case with $p_j = 225$

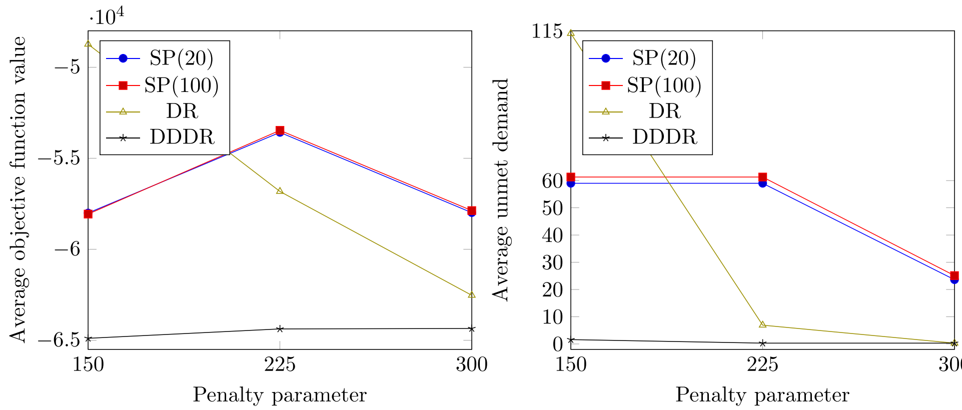

Fig. 4. Effect of the penalty parameter on solutions given by different approaches.

Table 4
Facility location solutions given by different approaches for different p_j-values.

	Open facility locations		
	$p_j = 150$	$p_j = 225$	$p_j = 300$
SP(20)	1,5,7,10	1,5,7,10	1,5,6,7,10
SP(100)	1,5,7,10	1,5,7,10	1,5,6,7,10
DR	1,7,10	1,3,5,6,7,10	1,3,4,5,6,7,10
DDDR	1,4,5,6,7,10	1,2,4,5,6,7,9,10	1,2,4,5,6,7,9,10

Table 5
Out-of-sample profit and unmet demand values of different solutions when the level of robustness $\kappa = 20\%$.

	SP(20)	SP(100)	DR	DDDR
Average profit	−51974.0	−52404.0	−60819.5	−63572.2
Std. dev.	6783.4	6819.6	7295.4	5101.9
95%	−40260.9	−41266.0	−48492.4	−54978.0
90%	−42887.1	−43644.5	−51692.9	−56970.0
75%	−47460.6	−47809.1	−55968.8	−60213.8
50%	−52170.1	−52780.4	−60927.2	−63639.1
Average unmet demand	61.6	63.5	3.6	0.5
Std. dev.	36.6	34.1	7.4	3.4
95%	129.0	127.0	20.8	1.3
90%	112.7	111.9	14.0	0.0
75%	82.7	84.7	3.9	0.0
50%	56.7	59.9	0.0	0.0

corresponds to facility location solutions in Table 2, and $p_j = 150$ represents the case when the penalty parameter is equal to the revenue amount per unit. In a decision-dependent approach, the mean of the underlying demand increases as we open new facilities. Consequently, the DDDR model enforces opening more facility locations yielding higher demand and thus higher profit. Furthermore, as unit penalty gets higher, it becomes more undesirable to have unmet demand. Thus, all approaches open more facilities when unit penalty cost increases. When we examine the actual locations of the open facilities in Fig. 2, we observe that the DDDR model opens more locations close to certain customer sites. On the other hand, SP and DR approaches open fewer locations and they are not necessarily close to the customer sites as both approaches neglect the possible boosting demand affected by location decisions.

Fig. 4 shows how the average profit and unmet demand values in the out-of-sample test are affected by unit penalty cost. As the unit penalty parameter increases, the amount of unmet demand decreases for all approaches, as expected. When penalty parameter takes its smallest value, DR has the worst performance both in the profit and unmet demand results, for which we provide a detailed explanation as follows. The DR approach compares two unfavorable cases: (i) opening many locations but having few customers, and (ii) not opening many locations and missing potential customers. By favoring the latter case, the DR solution loses customers by not having enough facilities open and neglecting the increase in the demand caused by the opening of new facilities. This effect can be also seen in Table 4 as the DR approach opens fewer locations under small penalty values. On the other hand, the DDDR approach outperforms DR and SP in all settings, resulting in higher profit and less unmet demand.

4.2.3. Effect of the robustness level of the ambiguity set

In this section, we examine the effect of the robustness level of the ambiguity set (1) on facility location solutions. We adjust the parameters ϵ^μ_j, $\underline{\epsilon}^\sigma_j$, $\overline{\epsilon}^\sigma_j$ for each customer site $j \in J$. Recall that, in the default setting, $\epsilon^\mu_j = 0$, and $\underline{\epsilon}^\sigma_j = \overline{\epsilon}^\sigma_j = 1$ under the assumption of having the perfect knowledge about the underlying mean and variance for each customer site. By adjusting these parameter choices, we construct models that are robust to different levels of uncertainty in the distribution parameters.

We consider a different procedure for generating out-of-sample test scenarios to evaluate different solutions. We first compute $\mu_j(\hat{y})$ and $\sigma^2_j(\hat{y})$ for each customer site $j \in J$ given a location solution $\hat{y}$. Then, we sample the mean and variance parameters from the ranges $[(1-\epsilon^\mu_j)\mu_j(\hat{y}), (1+\epsilon^\mu_j)\mu_j(\hat{y})]$ and $[(1-\underline{\epsilon}^\sigma_j)\sigma^2_j(\hat{y}), (1+\overline{\epsilon}^\sigma_j)\sigma^2_j(\hat{y})]$, respectively. After that, we generate 100 Normally distributed scenarios using the sampled mean and variance parameters. We repeat this procedure ten times to construct the set of test scenarios of size 1000, where each subset of scenarios has its own distribution.

Table 5 shows the performance of different solutions for $\kappa = 20\%$ level of robustness, where the level of robustness parameter $0 \le \kappa \le 1$ implies $\epsilon^\mu_j = \kappa\mu_j(y)$, $\underline{\epsilon}^\sigma_j = 1-\kappa$, and $\overline{\epsilon}^\sigma_j = 1+\kappa$ for every customer site j. Therefore, the larger κ is, a wider range of ambiguity we consider in the unknown distribution and become more conservative. As a result, distributionally robust approaches (i.e., DR and DDDR) become more cautious to the increased ambiguity. On the other hand, SP solutions are not affected by these changes as they are trained with the same data and procedures. Consequently, DR and DDDR solutions perform better than the one of SP given higher κ-values. Furthermore, the DDDR's results are less affected by the increased robustness, in terms of both profit and unmet demand values, as compared to $\kappa = 0$ in the default setting with results shown in Table 2.

As we continue to increase the level of robustness, the set of test scenarios includes more variability. Due to this increased

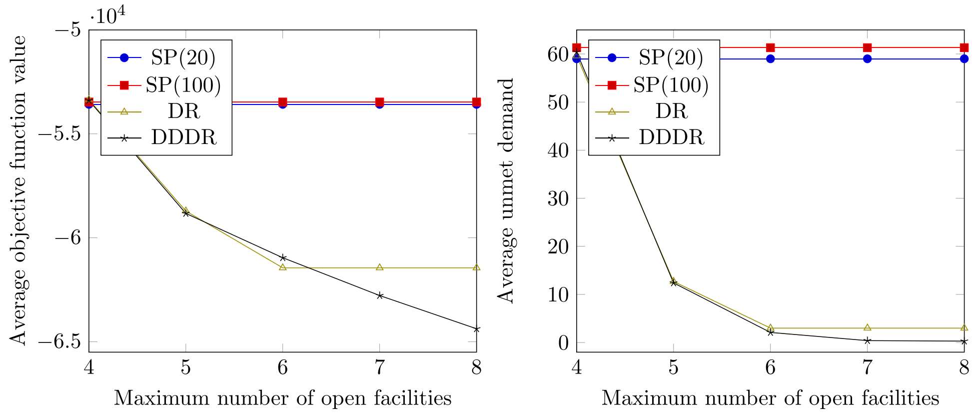

Fig. 5. Effect of varying budget for opening facilities.

variability, all approaches have higher standard deviations and worsen percentile values for the optimal objective and unmet demand values over all test scenarios. Despite of this, the distributionally robust approaches (DR and DDDR) under $\kappa = 5\%$ and 10% have the same facility location plans as in the case of $\kappa = 20\%$.

4.2.4. Effect of the budget for opening facilities

We compare solutions of DDDR, DR, and SP given a limit on the total number of facilities to open. We add a constraint to the optimization models, specifically to the set $\mathcal{Y}$, which restricts the total number of locations that can be selected. Fig. 5 summarizes the out-of-sample performance of each approach's solution in terms of average profit and unmet demand given different budgets. As the DDDR approach considers demand increase given by opening more facilities, adding such a limit hinders its capability of doing so. Consequently, the performance of DDDR, DR, SP solutions becomes similar if we only have very small budget to open few facilities. On the other hand, as we relax this limitation, the DDDR approach outperforms the others, whereas SP is not affected by the relaxation.

4.2.5. Effect of the forms of decision-dependency

We examine how DDDR results are affected by how we model the decision-dependency of random demand. Recall that, in the default setting, we consider λ^{μ}_{ji}, λ^{σ}_{ji} for each pair of customer site j and facility location i as decreasing functions of the distance between them. Alternatively, we propose a clustering-based decision-dependency for modeling the ambiguity set. In particular, in our P-*means approach*, the demand at customer site j is equally affected by the closest ρ facilities in its neighborhood. Let P^{ρ}_{j} be the set of ρ facility locations that are closest to customer site j; define $\lambda^{\mu}_{ji} = \lambda^{\sigma}_{ji} = \frac{1}{\rho}$ for each customer site $j \in J$ and facility location $i \in \mathrm{P}^{\rho}_{j}$. As locations $i \in I \setminus \mathrm{P}^{\rho}_{j}$ do not affect the demand at customer site j, their corresponding λ^{μ}_{j}- and λ^{σ}_{j}-values are set to zero.

We present the location solutions given by different approaches in Table 6. The distance-based approach corresponds to the default setting, and ρ-means approach is examined under different ρ values. As all possible facility locations are considered in the distance-based approach with inversely proportional values with respect to their corresponding distances, most facilities are opened in this setting. For ρ-means approaches, the set of facilities to be open are affected by the choice of ρ. As ρ gets larger, distances between customer and location pairs start to impact the demand less, and other factors such as opening cost of the locations may become more important. We note that $\rho = 10$ corresponds to an extreme case where all facilities equally affect the demand at any customer site.

Table 6
Facility location solutions of DDDR under different location-dependency patterns.

Decision-dependency form		Open facility locations
Distance-based		1,2,4,5,6,7,9,10
ρ-means	$\rho = 1$	1,4,5,6,7,8,10
	2	1,2,3,4,5,7,10
	3	1,2,3,5,6,7,10
	5	1,2,3,4,5,7,10
	10	1,3,4,5,6,7,10

4.2.6. Effect of decision-dependency in alternative formulations

We investigate the effect of opening locations on customer demand in a deterministic decision-dependent formulation, which replaces the demand parameter d in the original model (2) by its decision-dependent mean value $\mu(y)$. Furthermore, we obtain an MILP reformulation that can be solved in off-the-shelf solvers by replacing $\mu_j(y)$ for each site j with its counterpart in (12) and omitting the upper-bound value. We utilize the distance-based relationship to represent the impact of open locations on demand of each customer site, and adopt the base model assumptions, discussed earlier in this section. Once we obtain the optimal facility-location solution of the deterministic counterpart over the instance shown in Fig. 2, we evaluate its out-of-sample performance based on the 1000 test scenarios, to generate which we follow a Normal distribution and compute the mean and variance of demand based on the solution. We present the resulting statistics of the profit and unmet demand values in Table 7, and note that the open facility locations are 1, 2, 5, 6, 7 and 10 under this setting.

As we compare these results with the corresponding ones reported in Table 2, we observe that the solution of the decision-dependent deterministic approach has higher variance of the total profit over the set of 1000 test scenarios as compared to SP, DR and DDDR approaches, whereas on average, it performs better than the decision-independent SP and DR approaches. As we examine the unmet demand values, we observe that the decision-dependent deterministic approach outperforms SP, whereas the solutions of DR and DDDR approaches result in less unmet demand on average with also better worst-case performance. Although the deterministic approach takes into account the decision-dependency issue in the demand parameter, it does not focus on finding the best solution under the worst-case as opposed to distributionally robust optimization, resulting in such a less resilient performance. On the other hand, since the decision-dependent deterministic approach considers the effect of increased demand with enhanced availability of the open locations, it prefers to open the same number of locations as the SP and DR approaches (see Table 4 for

Table 7
Statistics of the out-of-sample profit and unmet demand values of optimal solutions to the decision-dependent deterministic model.

			Percentile values			
	Mean	Std. dev.	95%	90%	75%	50%
Profit	−62985.4	6810.3	−51590.9	−53827.2	−58120.9	−63325.1
Unmet demand	8.8	12.8	35.4	25.8	13.5	2.9

Table 8
Effect of the valid inequalities on CPU time.

	$\|I\| \times \|J\|$					
	5×10	6×12	7×14	8×16	9×18	10×20
DDDR average run time (in seconds)	2.15	3.81	8.88	17.46	46.82	111.36
Speed-up (times)	1.04	1.19	1.09	1.12	1.03	1.15

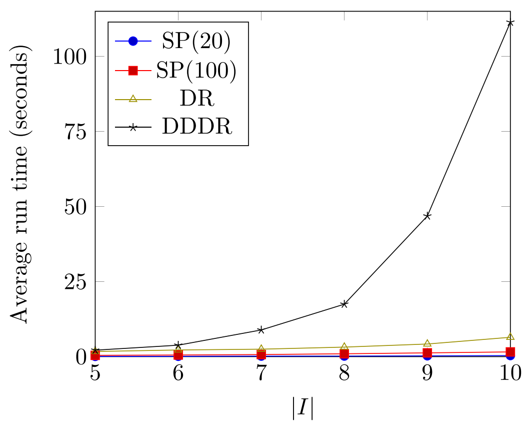

Fig. 6. Run time comparison between different approaches.

comparison), however, fewer locations than DDDR. These results highlight the efficacy of the DDDR approach by integrating decision-dependency within a distributionally robust optimization framework, to obtain facility-opening plans with better profit and higher quality of service, while further outperforming all the other approaches in terms of the worst-case performance.

4.3. Results of computational time

Lastly, we compare the computational time of SP, DR and DDDR approaches for different instance sizes. Fig. 6 provides the run time for cases having $|I| = 5, \ldots, 10$, and $|J| = 2|I|$. The run time denotes the average CPU time over 10 different randomly generated replications of each instance. In these replications, the default parameter configurations and moment-based ambiguity sets are used as described in Section 4.2. The distributionally robust approaches are more computationally expensive, whereas SP is the fastest. Furthermore, run time of the DDDR approach is more sensitive to the size of instances. Also, the computational time of DDDR model (17) depends on the upper bounds of the dual variables, which are set to 100 for all experiments.

Next we examine the effect of the inclusion of valid inequalities (21) to Model (17). Table 8 provides the average run time comparison of two formulations over 10 randomly generated instances of different sizes. We present the speed-ups in comparison to the formulation (17) without the valid inequalities (22a)–(22c) and the corresponding additional variables and constraints (22d) and (22e). These results illustrate the speed-up due to the proposed inequalities in the order of 3%–19% for different instances.

5. Conclusion

In this paper, we propose a novel framework for modeling the facility location problem under distributionally robust decision-dependent demand distributions. We consider a moment-based ambiguity set of the unknown demand distributions, using which we can formulate a monolithic model for solving the problem. We study the case when mean and variance of stochastic demand at each customer site are piecewise linear functions of facility location decisions. We benefit from linear programming duality and convex envelopes to obtain exact MILP reformulation of the monolithic model and derive valid inequalities to strengthen it. An extensive set of instances are tested to assess the performance of the proposed approach depending on various problem characteristics. Our studies indicate superior performance of the proposed approach, which results in consistently higher profit and less unmet demand, as compared to existing stochastic programming and distributionally robust methods. We also present the computational efficiency of the proposed valid inequalities with up to 19% speed-up for different instances. We believe that our study leverages a novel line of research by providing insights for the facility location and optimization under uncertainty literature, and highlighting the need to represent the dependency between customer demand and planner's decisions within various business settings.

One of the future research directions is to further extend the piecewise linear function forms for the mean and variance values used in the ambiguity set to other types of nonlinear functions, and then examine the corresponding reformulations and their numerical performance.

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References

Ahmed, S. (2000). Strategic planning under uncertainty: Stochastic integer programming approaches. Ph.D. thesis University of Illinois at Urbana-Champaign.
Álvarez-Miranda, E., Fernández, E., & Ljubić, I. (2015). The recoverable robust facility location problem. *Transportation Research Part B: Methodological, 79*, 93–120.
An, Y., Zeng, B., Zhang, Y., & Zhao, L. (2014). Reliable *p*-median facility location problem: Two-stage robust models and algorithms. *Transportation Research Part B: Methodological, 64*, 54–72. ISSN 0191–2615
Baron, O., Milner, J., & Naseraldin, H. (2011). Facility location: A robust optimization approach. *Production and Operations Management, 20*(5), 772–785.
Basciftci, B., Ahmed, S., & Gebraeel, N. (2019). Adaptive two-stage stochastic programming with an application to capacity expansion planning. https://arxiv.org/abs/1906.03513.

Basciftci, B., Ahmed, S., & Gebraeel, N. (2020). Data-driven maintenance and operations scheduling in power systems under decision-dependent uncertainty. *IISE Transactions, 52*(6), 589–602. https://doi.org/10.1080/24725854.2019.1660831. URL https://doi.org/10.1080/24725854.2019.1660831

Ben-Tal, A., Hertog, D. d., De Waegenaere, A., Melenberg, B., & Rennen, G. (2013). Robust solutions of optimization problems affected by uncertain probabilities. *Management Science, 59*(2), 341–357.

Benati, S., & Hansen, P. (2002). The maximum capture problem with random utilities: Problem formulation and algorithms. *European Journal of Operational Research, 143*(3), 518–530.

Berman, O., Drezner, T., Drezner, Z., & Krass, D. (2009). Modeling competitive facility location problems: New approaches and results. In chapter 8 (pp. 156–181). URL https://pubsonline.informs.org/doi/abs/10.1287/educ.1090.0062. 10.1287/educ.1090.0062

Boldrini, C., Bruno, R., & Conti, M. (2016). Characterising demand and usage patterns in a large station-based car sharing system. In *Proceedings of the 2016 IEEE conference on computer communications workshops (INFOCOM WKSHPS)* (pp. 572–577).

Chen, A. Y., & Yu, T. (2016). Network based temporary facility location for the emergency medical services considering the disaster induced demand and the transportation infrastructure in disaster response. *Transportation Research Part B: Methodological, 91*, 408–423.

Ciari, F., Bock, B., & Balmer, M. (2014). Modeling station-based and free-floating carsharing demand: Test case study for berlin. *Transportation Research Record, 2416*(1), 37–47.

Cui, T., Ouyang, Y., & Shen, Z. M. (2010). Reliable facility location design under the risk of disruptions. *Operations Research, 58*(4-part-1), 998–1011.

Delage, E., & Ye, Y. (2010). Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Operations Research, 58*(3), 595–612.

Erdoğan, E., & Iyengar, G. (2006). Ambiguous chance constrained problems and robust optimization. *Mathematical Programming, 107*(1), 37–61.

Esfahani, P. M., & Kuhn, D. (2018). Data-driven distributionally robust optimization using the Wasserstein metric: Performance guarantees and tractable reformulations. *Mathematical Programming, 171*(1), 115–166.

Fischetti, M., Ljubic, I., & Sinnl, M. (2017). Redesigning benders decomposition for large-scale facility location. *Management Science, 63*(7), 2146–2162.

Freire, A. S., Moreno, E., & Yushimito, W. F. (2016). A branch-and-bound algorithm for the maximum capture problem with random utilities. *European Journal of Operational Research, 252*(1), 204–212. https://doi.org/10.1016/j.ejor.2015.12.026. ISSN 0377–2217. URL http://www.sciencedirect.com/science/article/pii/S0377221715011650.

Gao, R., & Kleywegt, A. J. (2016). Distributionally robust stochastic optimization with Wasserstein distance. https://arxiv.org/abs/1604.02199.

Goel, V., & Grossmann, I. E. (2006). A class of stochastic programs with decision dependent uncertainty. *Mathematical Programming, 108*(2), 355–394. Sep

Haase, K., & Müller, S. (2014). A comparison of linear reformulations for multinomial logit choice probabilities in facility location models. *European Journal of Operational Research, 232*(3), 689–691.

Hellemo, L., Barton, P. I., & Tomasgard, A. (2018). Decision-dependent probabilities in stochastic programs with recourse. *Computational Management Science, 15*(3), 369–395.

Hernndez, B., Jimnez, J., & Martn, M. J. (2010). Customer behavior in electronic commerce: The moderating effect of e-purchasing experience. *Journal of Business Research, 63*(9), 964–971.

Ho, P., & Perl, J. (1995). Warehouse location under service-sensitive demand. *Journal of Business Logistics, 16*(1), 133.

Jiang, R., & Guan, Y. (2016). Data-driven chance constrained stochastic program. *Mathematical Programming, 158*(1), 291–327.

Jorge, D., & Correia, G. (2013). Carsharing systems demand estimation and defined operations: A literature review. *European Journal of Transport and Infrastructure Research, 13*(3), 201–220.

Kleywegt, A., Shapiro, A., & Mello, T. H.-d. (2002). The sample average approximation method for stochastic discrete optimization. *SIAM Journal on Optimization, 12*(2), 479–502.

Lappas, N. H., & Gounaris, C. E. (2017). Robust optimization for decision-making under endogenous uncertainty. http://www.optimization-online.org/DB_FILE/2017/06/6105.pdf.

Liu, K., Li, Q., & Zhang, Z. (2019). Distributionally robust optimization of an emergency medical service station location and sizing problem with joint chance constraints. *Transportation Research Part B: Methodological, 119*, 79–101.

Ljubić, I., & Moreno, E. (2018). Outer approximation and submodular cuts for maximum capture facility location problems with random utilities. *European Journal of Operational Research, 266*(1), 46–56.

Lu, M., Ran, L., & Shen, Z. M. (2015). Reliable facility location design under uncertain correlated disruptions. *Manufacturing & Service Operations Management, 17*(4), 445–455.

Luo, F., & Mehrotra, S. (2018). Distributionally robust optimization with decision dependent ambiguity sets. https://arxiv.org/abs/1806.09215.

Mai, T., Lodi, A., & ISSN 0377–2217. URL, u. (2020). A multicut outer-approximation approach for competitive facility location under random utilities. *European Journal of Operational Research, 284*(3), 874–881. https://doi.org/10.1016/j.ejor.2020.01.020.

McCormick, G. P. (1976). Computability of global solutions to factorable nonconvex programs: Part I—Convex underestimating problems. *Mathematical Programming, 10*(1), 147–175.

Melo, M., Nickel, S., & Gama, F. S. d. (2009). Facility location and supply chain management – A review. *European Journal of Operational Research, 196*(2), 401–412.

Meyer, C. A., & Floudas, C. A. (2004). Trilinear monomials with mixed sign domains: Facets of the convex and concave envelopes. *Journal of Global Optimization, 29*(2), 125–155. Jun

Nohadani, O., & Roy, A. (2017). Robust optimization with time-dependent uncertainty in radiation therapy. *IISE Transactions on Healthcare Systems Engineering, 7*(2), 81–92.

Nohadani, O., & Sharma, K. (2018). Optimization under decision-dependent uncertainty. *SIAM Journal on Optimization, 28*(2), 1773–1795.

Noyan, N., Rudolf, G., & Lejeune, M. (2018). Distributionally robust optimization with decision-dependent ambiguity set. http://www.optimization-online.org/DB_FILE/2018/09/6821.pdf.

Owen, S. H., & Daskin, M. S. (1998). Strategic facility location: A review. *European Journal of Operational Research, 111*(3), 423–447.

Ozsen, L., Coullard, C. R., & Daskin, M. S. (2008). Capacitated warehouse location model with risk pooling. *Naval Research Logistics (NRL), 55*(4), 295–312.

Royset, J., & Wets, R. (2017). Variational theory for optimization under stochastic ambiguity. *SIAM Journal on Optimization, 27*(2), 1118–1149.

Ryu, M., & Jiang, R. (2019). Nurse staffing under absenteeism: A distributionally robust optimization approach. https://arxiv.org/abs/1909.09875.

Santiváñez, J. A., & Carlo, H. J. (2018). Reliable capacitated facility location problem with service levels. *EURO Journal on Transportation and Logistics, 7*(4), 315–341.

Santoso, T., Ahmed, S., Goetschalckx, M., & Shapiro, A. (2005). A stochastic programming approach for supply chain network design under uncertainty. *European Journal of Operational Research, 167*(1), 96–115.

Shaheen, S., Cohen, A. P., & Roberts, J. D. (2006). Carsharing in north america: Market growth, current developments, and future potential. *Transportation Research Record, 1986*(1), 116–124.

Shen, Z. M., Zhan, R. L., & Zhang, J. (2011). The reliable facility location problem: Formulations, heuristics, and approximation algorithms. *INFORMS Journal on Computing, 23*(3), 470–482.

Snyder, L. V. (2006). Facility location under uncertainty: A review. *IIE Transactions, 38*(7), 547–564.

Snyder, L. V., & Daskin, M. S. (2005). Reliability models for facility location: The expected failure cost case. *Transportation Science, 39*(3), 400–416.

Spacey, S. A., Wiesemann, W., Kuhn, D., & Luk, W. (2012). Robust software partitioning with multiple instantiation. *INFORMS Journal on Computing, 24*(3), 500–515.

Tong, X., Sun, H., Luo, X., & Zheng, Q. (2018). Distributionally robust chance constrained optimization for economic dispatch in renewable energy integrated systems. *Journal of Global Optimization, 70*, 131–158.

Vayanos, P., Georghiou, A., & Yu, H. (2020). Robust optimization with decision-dependent information discovery. ArXiv preprint arXiv:2004.08490.

Vayanos, P., Kuhn, D., & Rustem, B. (2011). Decision rules for information discovery in multi-stage stochastic programming. In *Proceedings of the 50th IEEE conference on decision and control and european control conference* (pp. 7368–7373).

Vine, S. L., Lee-Gosselin, M., Sivakumar, A., & Polak, J. (2011). Design of a strategic-tactical stated-choice survey methodology using a constructed avatar. *Transportation Research Record, 2246*(1), 55–63.

Zhang, J., Xu, H., & Zhang, L. (2016). Quantitative stability analysis for distributionally robust optimization with moment constraints. *SIAM Journal on Optimization, 26*(3), 1855–1882.

Zhang, X., Kamgarpour, M., Georghiou, A., Goulart, P., & Lygeros, J. (2017). Robust optimal control with adjustable uncertainty sets. *Automatica, 75*, 249–259. https://doi.org/10.1016/j.automatica.2016.09.016. ISSN 0005–1098.URL http://www.sciencedirect.com/science/article/pii/S0005109816303600.

Zhang, Z., Berenguer, G., & Shen, Z. M. (2015). A capacitated facility location model with bidirectional flows. *Transportation Science, 49*(1), 114–129.