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Extended decision field theory with social-learning for long-term decision-making processes in social networks

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A B S T R A C T

Modeling and analysis of human behaviors in social networks are essential in fields such as online business, marketing, and finance. However, the establishment of a generalized decision-making framework for human behavior is challenging due to different decision structures among individuals. Thus, we propose a new decision-making framework, Decision Field Theory with Learning (DFT-L), which combines the DFT model and the DeGroot model. We investigated three factors influencing preference evolution: previous experiences, current evaluations, and neighbors' preferences. The equilibrium status of social networks within this framework is obtained as an explicit formula under the independent and identically distributed (IID) conditions on weight values. This facilitates the identification of limiting expected preference values and covariance matrices. A simulation analysis using simulated and real networks is performed to validate the DFT-L framework and to demonstrate its efficiency compared with the original DFT. Our finding confirms that the diffusion process within DFT-L propagates fastest in the random network and slowest in the ring-lattice network. We also show that interactions among people affect the agent's decision within DFT-L and intensify embedded society characteristics, which helps to analyze irregular behaviors such as information cascades in social networks.

1. Introduction

There are many frameworks proposed by current researchers for modeling human behaviors in social networks. However, establishing a generalized and realistic framework has been a particular challenge since it requires consideration of the following factors: (1) an accurate understanding of individual behavior, (2) a high level of preference fluctuation in long-term decision-making processes, and (3) the interaction among people. In order to represent the individual's decision-making processes, utility function within the optimization framework has been widely used with constraints on resource limitation [20,29]. Though this approach allows modelers to take advantage of well-developed optimization theories, it presents some limitations in representing human behaviors in social networks. First, it is challenging to find generalizable utility functions for all decision-makers because every person has different objectives and distinctive perceptions of their utilities [43]. Second, even if one could find the appropriate utility function for each decision-maker, the function itself should have some good mathematical characteristics (e.g., convexity) in order to efficiently find the optimal solution, which is not always the case. Finally, the decision maker's utility function must satisfy the characteristics of completeness, transitivity, independence

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among irrelevant alternatives, and continuity; however, some researchers have pointed out that people's decision phenomena do not follow these characteristics [12]. For these reasons, a different approach to modeling individual decision-making processes in the social network is necessary.

Offering several advantages as an alternative to the utility theory, Decision Field Theory (DFT) was proposed by Busemeyer and Townsend [11,12] as a cognitive and dynamic approach to model individual decision-making processes [12,18]. One benefit of the DFT framework is that it takes into account the effect of people's previous experience and current evaluation on the preference values of alternatives. This allows DFT to achieve a general modeling framework by avoiding the use of utility functions [12]. Additionally, the non-existence of utility functions in the framework enables both a high level of freedom in modeling and efficient analysis of equilibrium states in decision-making processes [11]. A third advantage is straightforward, tracking the evolution of an individual's preference values over time [18]. Since DTF is expressed as Markov processes [18], a model's evolution over time can be tractable using traditional stochastic process theories. Finally, the DFT model carries a simple structure allowing various extensions to improve its expressiveness [11,12]. For example, DFT was initially developed for binary choice problems but was extended to multiple-choice options with multi-attribute cases [42]. Lee et al. proposed Extended Decision Field Theory to incorporate the effect of environmental change over time on preference values [35]. By using DFT as a reference model to describe individual decision-making processes, one can make the most of DFT's advantages while adjusting it to represent various scenarios in real life.

When it comes to modeling human behaviors in social networks, it is important to consider interactions among people [8]. People can share their posts and tweets on Facebook and Twitter by simply pushing the "like" button (or "retweet" button), so that one tweet uploaded by a famous person can be immediately shared among thousands of people [36]. Significant research has been devoted to studying the establishment and evolution of interactions in the social network, but there are some phenomena observed in social networks which cannot be fully understood by only analyzing individual decision-making processes [2,37]. One famous example is "information cascade," the phenomenon whereby people choose against their preference in favor of their neighbors' choices [6]. This can cause inaccurate beliefs to spread out throughout the whole social network, despite several people knowing clear evidence to the contrary. For example, in the late 2000s, the United States faced a significant economic decline caused by the collapse of the housing bubble (Subprime Mortgage) [17]. The main reason for the faulty development in the housing market was aggressive, reckless investment decisions made by financial analysts, and while only a few of them chose a risky investment option for the first time, their massive profits galvanized all play-makers in financial markets to take more risks than usual [19]. It is necessary to consider the interaction among people to enhance the reality of decision-making models.

Another way to model decision-making processes in social networks is to utilize game theory methodologies. Galeotti et al. proposed that a decision maker's choice of one option over the other is affected by its popularity among its neighbors [23]. Kalai introduced a semi-anonymous decision-making model, which represents that people's decisions are not only affected by a number of actions among the neighbors but also by the types of persons (i.e., if a close friend has chosen one alternative, then she/he is also likely to follow it) [33]. Bramoulle and Kranton introduced a public goods game that can be used to represent the consumption behavior of public goods in a social network [9]. A significant variation introduced by Ballester et al. addresses the derivation of a better pay-off structure based on their actions [3]. Along with the development of pay-off structure in social networks, Bayesian social learning has been used to demonstrate decision-making processes, and group behaviors such as herd behavior were analyzed using the sequential decision-making model [4]. However, since all of these models require the use of utility function for each alternative, the limitation of utility function still needs to be addressed.

The goal of this article is to establish a new decision-making framework, representing people's interactions through the DFT model. The present challenge is that the DFT model itself cannot convey the interactions among people because its primary focus was to represent short-term individual decision-making processes. One possible adjustment to DFT is generalizing it in long-term decision-making processes. We suggested one extension of DFT using decay theory, Decision Field Theory with Forgetting (DFT-F), which notes that people's memories soften over time when they have a longer deliberation period during their decisions [34]. Now we will show how we developed an extension of DFT by not just embodying the forgetting process but exhibiting the social learning process in order to enhance expressiveness and reality of decision-making processes in social networks.

In our research, we name the new decision-making framework Decision Field Theory with social-Learning (DFT-L). The main assumption of our model is that people's decisions are affected by three factors: previous experiences, current evaluations, and neighbors' preferences. We analyzed the dynamic evolution of people's preferences by deriving the equilibrium state of the model based on different assumptions. We also validated our model by using agent-based simulation (ABS) to confirm that DFT-L has the same characteristic as the general innovation diffusion model [2]. We then show how the population composition of different types of people affects the nature of the equilibrium state. Finally, factors influencing time until equilibrium is achieved within DFT-L are obtained by statistical analysis.

2. Proposed models

In this section, we begin by introducing two underlying models, Decision Field Theory (DFT) and Degroot's model, in Sections 2.1 and 2.2 respectively. In Section 2.3, we propose a new model, Decision Field Theory with Learning (DFT-L), with

Table 1
The example of parameter values under the DFT model.

S		***M***		***C***	
$S_{11} = 0.9$	$S_{12} = -0.05$	$M_{11} = 0.7$	$M_{12} = 0.4$	$C_{11} = 1$	$C_{12} = -1$
$S_{21} = -0.05$	$S_{12} = 0.9$	$M_{21} = 0.2$	$M_{22} = 0.6$	$C_{21} = -1$	$C_{22} = 1$

a detailed explanation of all parameters. Finally, we demonstrate the incorporation of DFT and DeGroot's model into DFT-L while addressing not only individual decision-making processes but also interaction among people in social networks.

2.1. Individual decision-making processes

As mentioned in Section 1, DFT was first introduced by Busemeyer and Townsend to model a dynamic cognitive approach for decision-making processes, which has been used and developed in various areas such as manufacturing, psychology, and supply chains [11]. From among many variants of DFT, we chose the latest model with multiple alternatives and attributes proposed by Roe et al. [11,42]. The main idea of DFT is that human decision-making processes rely on two factors: past experiences and current evaluations. If an agent has n number of alternatives with m attributes, each agent's preference values are expressed as follows:

$$\boldsymbol{P}(t+h) = \boldsymbol{SP}(t) + \boldsymbol{CMW}(t+h) = \boldsymbol{SP}(t) + \boldsymbol{V}(t+h) \tag{1}$$

$\boldsymbol{P}(t+h)$ represents the array of preference values (at time $t+h$) for the alternatives that are available to all decision-makers. $\boldsymbol{S}$ is an n by n symmetric matrix in which the diagonal elements refer to forgetting factors and the non-diagonal elements refer to competitive interactions between two alternatives. Elements of matrix $\boldsymbol{M}$ (n by m) indicate personal evaluations of options' attributes. $\boldsymbol{W}(t+h)$ is an m dimensional weight vector of the attributes. Both $\boldsymbol{M}$ and $\boldsymbol{W}$ (weight can vary from person to person [11,12]. The second term $\boldsymbol{CMW}(t+h)$ is often called a valence vector $\boldsymbol{V}(t+h)$. The DFT model can be considered a function (mapping) from multiple alternatives to the real number ($\mathbb{R}$), and the corresponding real number will represent the preference values of each alternative. Based on the assumption of DFT, the summation of all preference values of the alternatives will be $0 \in \mathbb{R}$ due to the construction of n dimensional square matrix, $\boldsymbol{C}$. This characteristic of $\boldsymbol{C}$ implies that the increase in the preference value to one option eventually decreases the preference value to the other.

To illustrate, consider a person who wants to purchase a new car and is deciding between two alternatives, cars A and B. His evaluation is based on two attributes, cost and driving efficiency. The corresponding DFT model is expressed as Eq. (1), where P_1(t) and P_2(t) represent a preference for car A or B, respectively. In this scenario, car A has a greater cost advantage whereas car B is more driving efficient. For the construction of matrix $\boldsymbol{S}$, diagonal elements are set to be quite large absolute values (e.g., 0.9) compared to non-diagonal elements (e.g., -0.01) [11,35,42], which implies that people's impression regarding one alternative does not significantly affect their impression of the other. Finally, due to the assumption of DFT, the diagonal values of $\boldsymbol{C}$ matrix are always 1, while all the non-diagonal values have to be $-1/(n-1)$, where n represents the number of alternatives available to decision-makers. Table 1 summarizes all the parameters of the DFT model used for the example.

Now, we can assume that the decision-maker's weight on attributes can be changed over time, which enables the vector $\boldsymbol{W}$ to be the function of t, $\boldsymbol{W}(t)$. Preference values will be updated in every deliberation period, which introduces the dynamicity in analyzing decision-making processes. If we represent the example by using model (1), it will be as follows:

$$\begin{bmatrix} P_1(t+h) \\ P_2(t+h) \end{bmatrix} = \begin{bmatrix} 0.9 & -0.05 \\ -0.05 & 0.9 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} w_1(t+h) \\ w_2(t+h) \end{bmatrix} \tag{2}$$

As seen in Fig. 1, the preference value for each option within the DFT framework can be changed over time. The figure represents the dynamic evolution of preference value for car A (highlighted in yellow) and car B (highlighted in blue) over time. At the beginning of deliberation periods, neither preference value exceeds the threshold value to the person making a decision. They keep evaluating the preference values until one option is significantly preferable to the other, making them confident in their decision. The threshold value (highlighted in red) can be viewed as a confidence value for each decision-maker. Since people's preference values are more likely to be updated over time, the dynamic model is required to analyze individual decision-making processes accurately.

As described in Section 1, DFT assumes that a decision-maker updates their preference values in a short-term deliberation period. The DFT framework cannot adequately represent imperfect memory recall processes, but DFT can be extended fairly naturally by considering longer deliberation periods. The extension allows modeling of decay theory, the idea that memories weaken over time, which can be applied to DFT for increased accuracy of understanding individual decision-making processes [12]. This leads to the proposal of Decision Field Theory with Forgetting (DFT-F) to mimic long-term decision-making processes with the forgetting [34]. The main difference is the structure of matrix $\boldsymbol{S}$, as follows:

$$\boldsymbol{P}(t+h) = \boldsymbol{S}(t+h)\boldsymbol{P}(t) + \boldsymbol{CMW}(t+h) \tag{3}$$

where matrix $\boldsymbol{S}$ is replaced by $\boldsymbol{S}(t+h)$, which more accurately represents the exponential decay of human forgetting processes [10]. This accelerated memory loss, also called retroactive interference, is usually observed when people are exposed

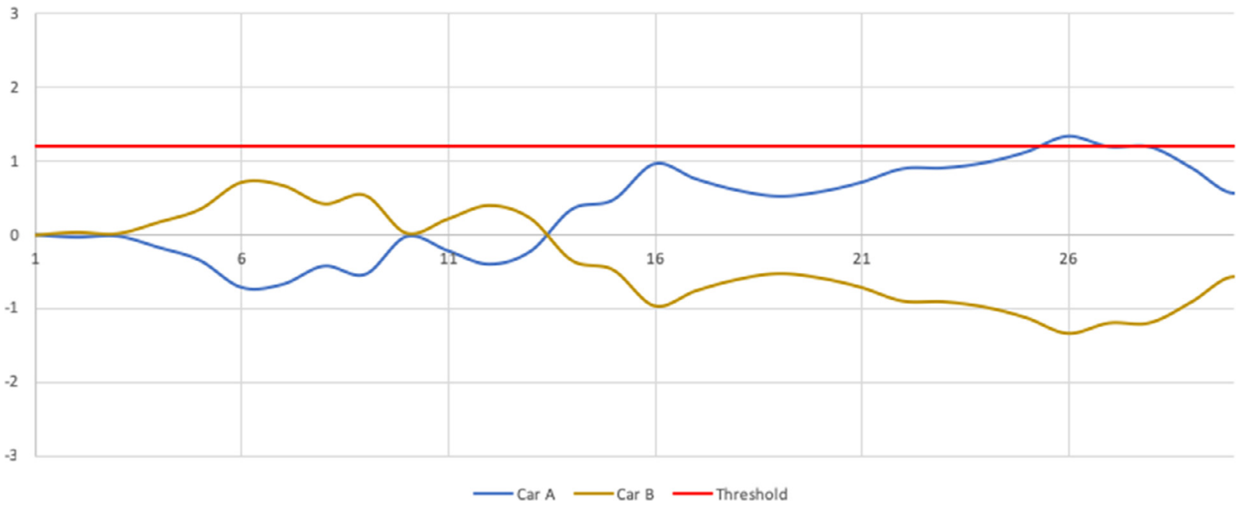

Fig. 1. Dynamic Evolution of Preference Values.

to others' opposing opinions [14,49]. Selection of the accurate forgetting function for ***S*** is a prominent research area, and Chechile and Sloboda categorized possible distributions of the forgetting process [14]. In Section 2.3, we also explain how this model (Eq. (3)) can be incorporated in the structure of the social network while deriving the asymptotic values of preferences within social networks.

2.2. Learning models in social networks

The term "learning" has various definitions and uses, but in social networks, learning generally represents the update of decision-making parameters based on the present environment [22]. Within the DFT framework, learning can refer to the process by which people can update their preferences after collecting information on their neighbors' preferences. Among various social-learning models, DeGroot's consensus model provides a simple structure to express that the relationships among agents can affect their beliefs [16]. If k number of agents living in a given social network has a belief about one option, the evolution of beliefs can be demonstrated by using the following model (4):

$$\boldsymbol{F}(t+1) \;=\; \boldsymbol{AF}(t) \tag{4}$$

Originally, DeGroot [16] used the vector ***F*** to describe the subjective probability distribution of an unknown parameter, but Hegselmann and Krause [28] interpreted it as the magnitude of a belief, which allows that belief to have any real value. In this paper, we will follow Hegselmann and Krause's extension of interpretation, which considers $\boldsymbol{F}(t++1)$ to be a k by 1 vector of quantifiable levels of the opinion of one option. Also, ***A***, which is k by k matrix, represents the magnitude at which each agent's opinion is affected by others. The element a_{ij} in matrix ***A*** indicates that agent i put a_{ij} amount of weight on j person's opinion when updating their opinion. Eq. (5), which is the i^{th} row of the model (4), infers that each agent's opinion is updated as the weighted sum of all agents' opinions, while all elements in ***A*** lie between 0 and 1.

$$\mathrm{x_i}(t+h) = \ \mathrm{a_{ik}x_1(t)} + \cdots \ + \mathrm{a_{ik}x_k(t)} \tag{5}$$

Identifying and measuring the influence of one agent on the other is another research problem [28]. If we use a graph theory to represent a social network, then nodes represent agents and edges do their connections such as Following in Twitter and Friends in Facebook. In Section 4, the detailed technique and the explanation are shown to assign weight values to the edges in given networks.

One of the variants of the Degroot's learning model (Eq. (4)) was first introduced by Abelson (1964) [1] and later extended by Hegselmann and Krause [28]. The main concept is that social factors in ***A*** alter over time, with the proposed model as follows:

$$\boldsymbol{F}(t+1) \;=\; \boldsymbol{A}(t)\boldsymbol{F}(t) \tag{6}$$

The strength of relationships among people can alter over time, especially in long-term decision processes due to the structural changes in social networks. Six types of community behavior in social networks serve as examples to demonstrate the evolution: Growth, Contraction, Merging, Splitting, Birth, and Death [41]. Growth represents that the number of edges in the network generally increases, while Contraction refers to the number of edges decreasing. Splitting describes how a huge community is divided into several groups. This process decreases the complexity of the whole network, which will increase the number of 0 values in the ***A*** matrix. Merging represents several communities combining, the opposite of the split process. Birth represents a new agent (person) entering the network and beginning to form relationships with

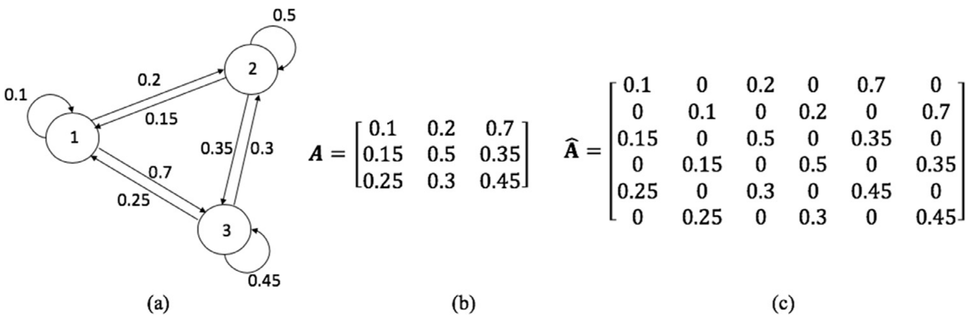

Fig. 2. Illustrative example for the DFT-L model: (a) example of social network (b) adjacency matrix for social learning (c) $\hat{\boldsymbol{A}}$ matrix for the DFT-L model.

other people in the community. Therefore, the dimension of the $\boldsymbol{A}$ matrix should also be increased. The Death process describes the disappearance of an agent in the network. We can represent the Death process without changing the dimension of the $\boldsymbol{A}$ matrix by making all the values in the ith row and column to be 0. Thus, evolution can be illustrated through the birth/death of agents, new connections, or detached connections among agents [1,28,41]. Therefore, Eq. (6) can be a substitute for decision-making processes in a longer deliberation period.

2.3. Details of proposed models

The proposed model in this work retains the key ideas such as forgetting and assessment processes described in DFT and the social relationship effect described in DeGroot's model while maintaining a simpler overall structure. We have titled the model Decision Field Theory with Learning (DFT-L). Block diagonal matrices are used to create a combined DFT model of the individuals in the social network, and individual DFT models are maintained as sub-matrices, while the whole framework incorporates these sub-matrices into Degroot's framework. For example, if we have k number of people, n options, and m attributes, then all parameters of DFT-L are represented as (7). The new preference matrix also becomes an extended array of preference values of all decision-makers in the network. Accordingly, the overall DFT-L model can be expressed as (8).

$$\hat{S}=\begin{bmatrix} S_1 & 0 & \cdots & 0\\ 0 & S_2 & \cdots & 0\\ \vdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & S_k \end{bmatrix}\ \hat{C}=\begin{bmatrix} C_1 & 0 & \cdots & 0\\ 0 & C_2 & \cdots & 0\\ \vdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & C_k \end{bmatrix},\ \hat{M}=\begin{bmatrix} M_1 & 0 & \cdots & 0\\ 0 & M_2 & \cdots & 0\\ \vdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & M_k \end{bmatrix}$$
$$\hat{P}=\begin{bmatrix} P_1\\ P_2\\ \cdots\\ P_k \end{bmatrix},\ \hat{W}=\begin{bmatrix} W_1\\ W_2\\ \cdots\\ W_k \end{bmatrix},\ \hat{A}=\begin{bmatrix} a_{11} & 0 & a_{12}\ 0 & \cdots & a_{1k}\ 0\\ 0 & a_{11} & 0\ a_{12} & \cdots & 0\ a_{1k}\\ \vdots & \cdots & \cdots & & \vdots\\ a_{k1} & 0 & a_{k2}\ 0 & \cdots & a_{kk}\ 0\\ 0 & a_{k1} & 0\ a_{k2} & \cdots & 0\ a_{kk} \end{bmatrix} \tag{7}$$

$$\hat{\boldsymbol{P}}\ (t+h)=\ \hat{\boldsymbol{S}}\hat{\boldsymbol{A}}\hat{\boldsymbol{P}}(t)+\hat{\boldsymbol{C}}\hat{\boldsymbol{M}}\hat{\boldsymbol{W}}(t+h)=\ \hat{\boldsymbol{S}}\hat{\boldsymbol{A}}\hat{\mathbf{P}}(\mathrm{t})+\hat{\boldsymbol{V}}(\mathrm{t}+\mathrm{h}) \tag{8}$$

Fig. 2 shows the construction of matrix $\hat{\boldsymbol{A}}$ within the social network. Suppose that two alternatives are given to three agents in social networks as shown in Fig. 2(a). Agents update their preference values based on their social factors depicted in matrix **A** (Fig. 2(b)). Then, the matrix $\hat{\boldsymbol{A}}$ can be constructed as the matrix shown in Fig. 2(c), which represents a $kn \times kn$ dimensional matrix demonstrating progression of updates in each person's preference value to the specific option based on their neighbors' preference value. In DFT-L, each decision-maker i has information others' previous preference values (at time t) but not their forgetting factors since the matrix is a block diagonal matrix. Therefore, this structure makes DFT-L more realistic because people do not generally share their internal evaluation processes with others. An agent's preference values are updated by the weighted sum of others' past preference values (including his own) as well as their own current evaluation of the alternatives. Therefore, the main differences of our proposed model from the original DFT is the existence of the matrix that can demonstrate influences by neighbors and block matrix structures of parameters for representing multiple agents in social networks.

Since the DFT-L framework is a combination of DFT and DeGroot's models, it brings all assumptions of both models. The assumptions from DFT are that the decision maker has a finite number of alternatives, new alternatives cannot be entered or removed in every deliberation period, and the number of attributes remains unchanged. DeGroot's model has the following limitations: 1) the new agent cannot be added or disposed during the deliberation periods; 2) the number of persons in the social network is finite; 3) preference updates occur at discrete points. In the DFT-L framework, these assumptions can be explained as follows: the dimension of all matrices ***S, M, W, A*** should remain the same and the preference values are updated at each discrete deliberation point. Under this assumption, the proposed DFT-L framework can be applicable to all types of social networks.

Decision Field Theory with Forgetting (DFT-F) can also be represented under a social network setting by using the block matrix defined above. Eq. (9) illustrates long-term decision-making process by considering forgetting processes and with social interactions.

$$\boldsymbol{P}(t+h) = \hat{\boldsymbol{S}}(t+h)\hat{\boldsymbol{A}}\hat{\boldsymbol{P}}(t) + \hat{\boldsymbol{C}}\hat{\boldsymbol{M}}\hat{\boldsymbol{W}}(t+h) = \hat{\boldsymbol{S}}(t+h)\hat{\boldsymbol{A}}\hat{\mathbf{P}}(t) + \hat{\boldsymbol{V}}(t+h) \tag{9}$$

The main characteristic of DFT-F within the social network is that a portion of the agent's prior preference values involved in the current decision will shift over time due to the exponential decay of memory ($\hat{\mathbf{S}}(t+h)$) while incorporating the interaction factor in the model. In Section 3, we derive the asymptotic behaviors of both Eqs. (8) and (9) and illustrate assumptions to achieve an equilibrium.

3. Asymptotic stability and its conditions in proposed models

The analysis of equilibrium is one of the most important topics in modeling decision-making processes in social networks. Researchers has found that the initial opinions of agents, network structures, and the stubbornness of agents are primary factors in achieving opinion equilibrium in social networks. A continuous coordination game is widely utilized to analyze equilibrium from iterative opinion updating processes. Jadbabaie et al. [31] examined equilibrium status under the nearest neighbor rule, while Guestrin et al. [26] and Kok et al. [32] studied and extended the coordination processes in a context-specific manner using a coordination graph (GC). Ghaderi and Srikant [24] examined the effect of the presence of stubborn agents in the social networks on the equilibrium status, while Lorenz and Lorenz [38] presented conditions for convergence to a consensus using a time-dependent dynamical model. Jackson and Wolinsky [30] utilized pairwise stability to understand stability in different network structures, which was further studied by Goyal and Joshi [25] as well as Belleflamme and Bloch [5] for its use in different applications. The equilibrium of our proposed models is explained in this section.

Using standard results from the linear systems theory, we have derived the asymptotic status of the networks under the DFT-L framework in this section. The limiting behaviors of DFT-L and DFT-F are mathematically driven and the assumptions for the stabilities are also demonstrated. Within the DFT-L framework, estimating preference values of decision-makers will be used to estimate the ultimate status of social networks. The following three lemmas represent that the mathematical approach is possible to demonstrate the eventual status of social networks.

Lemma 1. *Let* $\mathrm{E}[\hat{\mathbf{W}}(\mathrm{t})] = \hat{\mathbf{W}}_{\mathbf{E}}$, $\mathrm{Cov}[\hat{\mathbf{W}}(\mathrm{t})] = \hat{\boldsymbol{\Sigma}}$ *and both are finite. The expected and covariance value of the valence vector (*$\hat{\mathbf{V}}(\mathrm{t})$*) can be expressed as follows:*

$$E\left[\hat{\boldsymbol{V}}(t)\right] = \hat{\boldsymbol{C}}\hat{\boldsymbol{M}}\hat{\boldsymbol{W}}_{\boldsymbol{E}},\ Cov\left[\hat{\boldsymbol{V}}(t)\right] = \hat{\boldsymbol{C}}\hat{\boldsymbol{M}}\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{M}}^{\boldsymbol{T}}\hat{\boldsymbol{C}}^{\boldsymbol{T}} \tag{10}$$

Proof. Take the expectation of both sides of the equation $\hat{\mathbf{V}}(\mathrm{t}) = \hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}(\mathrm{t})$, then $\mathrm{E}[\hat{\mathbf{V}}(\mathrm{t})] = \hat{\mathbf{C}}\hat{\mathbf{M}}\mathrm{E}[\hat{\mathbf{W}}(\mathrm{t})] = \hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\mathbf{E}}$. Likewise, take the covariance of the same equation $\hat{\mathbf{V}}(\mathrm{t}) = \hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}(\mathrm{t})$, then $\mathrm{Cov}[\hat{\mathbf{V}}(\mathrm{t})] = \hat{\mathbf{C}}\hat{\mathbf{M}}\mathrm{Cov}[\hat{\mathbf{W}}(\mathrm{t})]\hat{\mathbf{M}}^{\mathbf{T}}\hat{\mathbf{C}}^{\mathbf{T}} = \hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\boldsymbol{\Sigma}}\hat{\mathbf{M}}^{\mathbf{T}}\hat{\mathbf{C}}^{\mathbf{T}}$□

Lemma 2. *Let the all initial preference values are zero (*i.e., $\mathrm{E}[\hat{\mathbf{P}}(0)] = 0$). *If the 2-norm of matrix* $\hat{\mathbf{S}}\hat{\mathbf{A}}$ *is strictly less than 1 and if* $\hat{\mathbf{W}}$ *is stationary with the finite expected value* $\mathrm{E}[\hat{\mathbf{W}}]$ *in every deliberation period, then asymptotic expected preference values within the DFT-L framework are as Eq. (9). Moreover, the asymptotic covariance preference value converges under the assumptions of iid* $\hat{\mathbf{W}}$.

$$\lim_{t\to\infty} E\left[\hat{\boldsymbol{P}}(t)\right] = \left(\boldsymbol{I} - \hat{\boldsymbol{S}}\hat{\boldsymbol{A}}\right)^{-1} E\left[\hat{\boldsymbol{V}}\right] = \left(\boldsymbol{I} - \hat{\boldsymbol{S}}\hat{\boldsymbol{A}}\right)^{-1} \hat{\boldsymbol{C}}\hat{\boldsymbol{M}}\hat{\boldsymbol{W}}_{\boldsymbol{E}} \tag{11}$$

$$\lim_{t\to\infty} Cov\left[\hat{\boldsymbol{P}}(t)\right] = \sum_{i=1}^{n} \left(\hat{\boldsymbol{S}}\hat{\boldsymbol{A}}\right)^{i-1} Cov\left[\hat{\boldsymbol{V}}(t)\right]\left(\hat{\boldsymbol{A}}^{T}\hat{\boldsymbol{S}}^{T}\right)^{i-1} = (\boldsymbol{I} - \boldsymbol{G})^{-1}\bar{\Phi} \tag{12}$$

Proof. At the first deliberation period ($t = 0$), $\mathrm{E}[\hat{\mathbf{P}}(h)] = E[\hat{\mathbf{S}}\hat{\mathbf{A}}\hat{\mathbf{P}}(0)] + E[\hat{\mathbf{V}}(\mathrm{h})] = E[\hat{\mathbf{V}}] = \hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}}$. Likewise, at $t==1$,

$\mathrm{E}[\hat{\mathbf{P}}(2\mathrm{h})] = \hat{\mathbf{S}}\hat{\mathbf{A}}\mathrm{E}[\hat{\mathbf{P}}(\mathrm{h})] + E[\hat{\mathbf{V}}(2\mathrm{h})] = \hat{\mathbf{S}}\hat{\mathbf{A}}\hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}} + \hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}} = (\mathbf{I} + \hat{\mathbf{S}}\hat{\mathbf{A}})\hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}}$.

The second last equation holds by the stationary assumption of $\hat{\mathbf{W}}$ [42].

By mathematical induction, $\mathrm{E}[\hat{\mathbf{P}}(\mathrm{nh})]$ can be expressed as $(\mathbf{I} + \hat{\mathbf{S}}\hat{\mathbf{A}} + \cdots + \hat{\mathbf{S}}\hat{\mathbf{A}}^{n-1})\hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}}$,which implies that $(\mathbf{I} - \hat{\mathbf{S}}\hat{\mathbf{A}})\mathrm{E}[\hat{\mathbf{P}}(\mathrm{nh})] = (\mathbf{I} - \hat{\mathbf{S}}\hat{\mathbf{A}})(\mathbf{I} + \hat{\mathbf{S}}\hat{\mathbf{A}} + \cdots + \hat{\mathbf{S}}\hat{\mathbf{A}}^{n-1})\hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}} = (\hat{\mathbf{S}}\hat{\mathbf{A}})^{\mathrm{n}}\ \hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}}$.

Multiplying $(\mathbf{I} - \hat{\mathbf{S}}\hat{\mathbf{A}})^{-1}$ to both sides of this equation,we can get $\mathrm{E}[\hat{\mathbf{P}}(\mathrm{nh})] = (\mathbf{I} - \hat{\mathbf{S}}\hat{\mathbf{A}})^{-1}(\hat{\mathbf{S}}\hat{\mathbf{A}})^{\mathrm{n}}\hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}} = (\mathbf{I} - \hat{\mathbf{S}}\hat{\mathbf{A}})^{-1}\hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}}$.

The last equality holds because $\lim_{\mathrm{n}\to\infty} (\hat{\mathbf{S}}\hat{\mathbf{A}})^{\mathrm{n}} = 0$ when the 2-norm of $\hat{\mathbf{S}}\hat{\mathbf{A}}$ is less than 1.

Then, $\lim_{\mathrm{n}\to\infty} \mathrm{E}[\hat{\mathbf{P}}(\mathrm{nh})] = (\mathbf{I} - \hat{\mathbf{S}}\hat{\mathbf{A}})^{-1}\hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{\boldsymbol{E}}$ is satisfied, and the last equality holds because of the finiteness of $\hat{\mathbf{W}}_{\boldsymbol{E}}$.

Likewise, take the covariance of both side of the following equation $\hat{\mathbf{P}}(\mathrm{nh}) = \sum_{i=1}^{\mathrm{n}} (\hat{\mathbf{S}}\hat{\mathbf{A}})^{i-1}\hat{\mathbf{V}}((n-i)\mathrm{h})$.

Then, by the iid assumption of $\hat{\mathbf{W}}$, $\mathrm{Cov}[\hat{\mathbf{P}}(\mathrm{nh})] = \sum_{i=1}^{\mathrm{n}} (\hat{\mathbf{S}}\hat{\mathbf{A}})^{i-1}\mathrm{Cov}[\hat{\mathbf{V}}(\mathbf{t})](\hat{\mathbf{A}}^{\mathrm{T}}\hat{\mathbf{S}}^{\mathrm{T}})^{i-1}$.

To find a closed form of an asymptotic matrix, we can use a reshaping procedure proposed by Hancock et al. [27]. Let $\mathrm{Cov}[\hat{\boldsymbol{V}}(\mathbf{t})]$ be $\boldsymbol{\Phi}$ and $(\hat{\mathbf{S}}\hat{\mathbf{A}})$ be $\mathrm{K} = [\mathrm{k}_{ij}]$ for i, $\mathrm{j} \in \{1, \cdots \mathrm{p}\}$. Also, let us define a reshaping operator. If the dimension of $\boldsymbol{\Phi}$ is p x p, then $\bar{\boldsymbol{\Phi}}$ is $1 \times \mathrm{p}^2$ matrix after rearranging values of $\boldsymbol{\Phi}$.

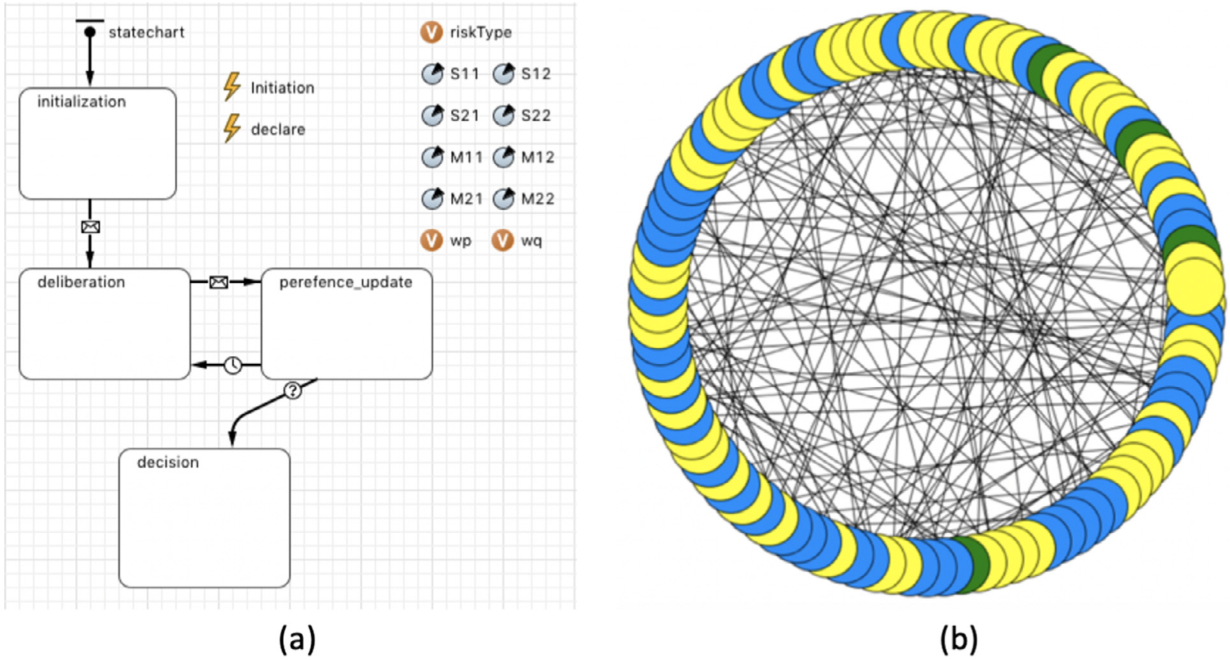

Fig. 3. Agent-based modeling techniques: (a) State diagram (b) Network representation.

Then, $\mathrm{Cov}[\hat{\mathbf{P}}(\mathrm{nh})] = \sum_{i=1}^{\mathrm{n}} (\hat{\mathbf{S}}\hat{\mathbf{A}})^{i-1}\mathrm{Cov}[\hat{\mathbf{V}}(\mathbf{t})](\hat{\mathbf{A}}^{\mathrm{T}}\hat{\mathbf{S}}^{\mathrm{T}})^{i-1} = \sum_{i=1}^{\mathrm{n}} \mathrm{K}^{i-1}\mathbf{\Phi}\mathrm{K}'^{i-1} = \sum_{i=1}^{n} \boldsymbol{G}^{i}\bar{\mathbf{\Phi}}$ where $\boldsymbol{G} = [\mathrm{z}_{(j-1)p+i,\ (k-1)p+i}] = \mathrm{k_{il}k_{jk}}$ for i, j, j, $\mathrm{l} \in \{1, \cdots \mathrm{p}\}$ by Hancock et al. [27].

Thus, $\lim_{\mathrm{n}\to\infty} \mathrm{Cov}[\hat{\mathbf{P}}(\mathrm{nh})] = \lim_{\mathrm{n}\to\infty} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{G}^{\mathrm{i}}\bar{\mathbf{\Phi}} = \lim_{\mathrm{n}\to\infty} (\mathbf{I}-\mathbf{G})^{-1}(\mathbf{I}-\mathbf{G}^{\mathrm{n}})\bar{\mathbf{\Phi}} = (\mathbf{I}-\mathbf{G})^{-1}\bar{\mathbf{\Phi}}$□

The assumptions used in proving Lemma 2 are three folds: (1) people' initial preference values are zero, (2) finite values for expected weight, and (3) the restriction of the 2-norm of matrix $\hat{\mathbf{S}}\hat{\mathbf{A}}$. The plausibility of the first and second assumptions can be explained by the car purchase example. At the beginning, your personal preference on each car model can be indifferent because you have no information. However, as time goes on, you will be determined in evaluating your options, so the expected weight vector becomes stable. This allows people's expected decision criteria converge to fixed values even though it can vary at some degrees. If the 2-norm of matrix $\hat{\mathbf{S}}\hat{\mathbf{A}}$ is less than 1 implying that all the eigenvalues lie within the unit circle on the complex planes, then the inverse of $(\mathbf{I}-\hat{\mathbf{S}}\hat{\mathbf{A}})$ exists. By observing the eigenvalues of $\hat{\mathbf{S}}\hat{\mathbf{A}}$, the stability of the system can be achieved. Since $||\hat{\mathbf{S}}\hat{\mathbf{A}}||_2 \le ||\hat{\mathbf{S}}||_2||\hat{\mathbf{A}}||_2$, the stability of DFT-L is achieved when all eigenvalues of both $\hat{\mathbf{S}}$ and $\hat{\mathbf{A}}$ are less than 1. The stability of $\hat{\mathbf{S}}$ is guaranteed when all $\boldsymbol{S}_{\mathbf{i}}$ for $i = \{1, \cdots, n\}$ matrices are stable (this assumption is equivalent to those used in the original DFT's, when asymptotic values are derived in analyzing the limiting status of people's choices [11,12]). Then, it suffices to show that $||\hat{\mathbf{A}}||_2 \le 1$. Because of the construction of $\hat{\mathbf{A}}$, $\hat{\mathbf{A}}$ becomes stable if its corresponding social factor matrix $\mathbf{A}$ is stable. By applying standard results in Markov Chain model, it is evident that the social factor matrix $\mathbf{A}$ has to be strongly connected and aperiodic for stability [40]. This is the same as the stability condition of the DeGroot's model. Thus, if the stability conditions of both DeGroot's model and DFT are satisfied, the stability of DFT-L is also guaranteed. Therefore, our extension does not show significant deviation from the original model. The main advantages of these two lemmas are that asymptotic preference values within the DFT-L framework can be estimated by the initial parameter values without loss of generality.

The following lemma represents asymptotic preference values in the DFT-F model in the social network.

Lemma 3. *Let the all initial preference values are 0. If agents' forgetting processes and weight vectors are independent and stationary with finite expected value* $\mathrm{E}[\hat{\mathbf{S}}] = \tilde{\boldsymbol{S}}$ *and* $\mathrm{E}[\hat{\mathbf{W}}] = \tilde{\mathbf{W}}$*, respectively, and if the 2-norm of the matrix* $\tilde{\mathbf{S}}\hat{\mathbf{A}}$ *is less than 1, then the limiting expected preference values in DFT-F are as follows:*

$$\lim_{t\to\infty} E\left[\hat{\boldsymbol{P}}(t)\right] = \left(\boldsymbol{I}-\tilde{\boldsymbol{S}}\hat{\boldsymbol{A}}\right)^{-1}\hat{\boldsymbol{C}}\hat{\boldsymbol{M}}\tilde{\boldsymbol{W}} \tag{13}$$

Proof. At $t==0$, $\mathrm{E}[\hat{\mathbf{P}}(\mathrm{h})] = E[\hat{\mathbf{S}}(\mathrm{h})\hat{\mathbf{A}}\hat{\mathbf{P}}(0)] + E[\hat{\mathbf{C}}\hat{\mathbf{M}}\hat{\mathbf{W}}(\mathrm{h})] = \hat{\mathbf{C}}\hat{\mathbf{M}}\tilde{\mathbf{W}}$, because $\hat{\mathbf{P}}(0) == \mathbf{0}$. In the next deliberation period,$\mathrm{E}[\hat{\mathbf{P}}(2\mathrm{h})] = E[\hat{\mathbf{S}}(\mathrm{h})\hat{\mathbf{A}}\hat{\mathbf{P}}(\mathrm{h}))] + \hat{\mathbf{C}}\hat{\mathbf{M}}\mathrm{E}[\hat{\mathbf{W}}(2\mathrm{h})] = \hat{\mathbf{S}}\hat{\mathbf{A}}\hat{\mathbf{C}}\hat{\mathbf{M}}\tilde{\mathbf{W}} + \hat{\mathbf{C}}\hat{\mathbf{M}}\tilde{\mathbf{W}}\ (\mathbf{I}+\hat{\mathbf{S}}\hat{\mathbf{A}})$. The second equality holds because of independence and stationary assumption of $\hat{\mathbf{S}}$ and $\tilde{\boldsymbol{W}}$. So, $(\mathbf{I}-\hat{\mathbf{S}}\hat{\mathbf{A}})\mathrm{E}[\hat{\mathbf{P}}(\mathrm{nh})] = (\hat{\mathbf{S}}\hat{\mathbf{A}})^{\mathrm{n}}\ \hat{\mathbf{C}}\hat{\mathbf{M}}\tilde{\mathbf{W}}$ by the same argument of the proof of Lemma 2. Multiplying $(\mathbf{I}-\hat{\mathbf{S}}\hat{\mathbf{A}})^{-1}$ to both sides of this equation, we can get $\lim_{\mathrm{n}\to\infty} \mathrm{E}[\hat{\mathbf{P}}(\mathrm{nh})] = (\mathbf{I}-\hat{\mathbf{S}}\hat{\mathbf{A}})^{-1}(\hat{\mathbf{S}}\hat{\mathbf{A}})^{\mathrm{n}}\hat{\mathbf{C}}\hat{\mathbf{M}}\tilde{\mathbf{W}} = (\mathbf{I}-\hat{\mathbf{S}}\hat{\mathbf{A}})^{-1}\hat{\mathbf{C}}\hat{\mathbf{M}}\tilde{\mathbf{W}}$□

Table 2
Parameter values in simulation [11,21,27,42].

S		M		W		a_{ij}
U(0.85, 0.95), N(0.9, 0.03^2)	-U(0.01, 0.03), -N(0.02, 0.01^2)	U(0.7, 0.9), N(0.8, 0.06^2)	U(0.2, 0.4), N(0.3, 0.06^2)	quality	$\frac{U_1}{U_1+U_2}$	Equally Weighted: $\frac{1}{d_i}$
-U(0.01,0.03), -N(0.02, 0.01^2)	U(0.85, 0.95) N(0.9, 0.03^2)	U(0.2, 0.4), N(0.3, 0.06^2)	U(0.7, 0.9) N(0.8, 0.06^2)	cost	$\frac{U_2}{U_1+U_2}$	Proportional: $\frac{d_i}{\sum_{(i,j)\in e} d_j}$

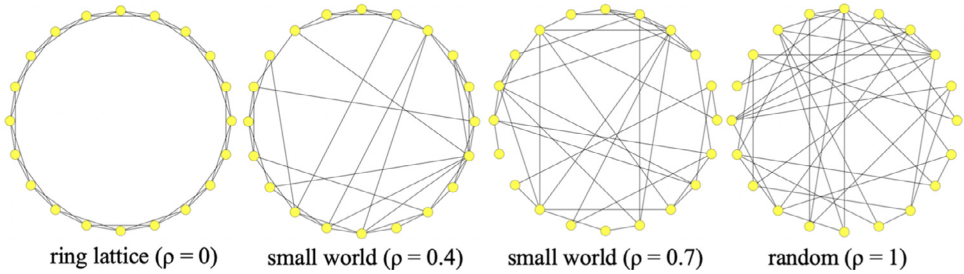

Fig. 4. Three network types based on the complexity.

All the assumptions used in Lemma 3 are the same as in Lemma 2 except the independency between forgetting factors ***S*** and weight vectors ***W***. In a real scenario, the internal forgetting process of people's experience are independent of assigning weights on criteria for decision making [22,49]. In the next section, we will illustrate and demonstrate the proposed models using the agent-based simulation and discover some interesting phenomena within DFT-L framework in social networks.

4. Simulation experiments

In this section, our main purposes are to describe the simulation configuration, validate Decision Field Theory with Learning (DFT-L), compare DFT-L and DFT, and further analyze the benefits of DFT-L. We validate DFT-L by implementing the "leader and followers" configuration widely used in the innovation diffusion model. After validation, we compare DFT-L to the original DFT model using various scenarios to highlight key differences. Finally, we describe the asymptotic status of social networks within the DFT-L framework using "opinion formation" configuration, which will help to explain some interesting phenomena (e.g., information cascade) in social networks.

4.1. Simulation configuration

We begin by describing the simulation model and its configuration. We chose Agent-Based Simulation (ABS), an acclaimed approach to modeling complex social networks [7,13], to illustrate and validate the proposed models in Sections 2 and 3, primarily because within the ABS framework, it is easy to model individual (constituent unit) decision-making processes, the environment around agents, and interaction among agents [7]. For this work, we used AnyLogic™ software as a tool of ABS.

First, individual decision-making processes were designed as shown in Fig. 3(a), with all agents in the network sharing the same reasoning process (i.e., within DFT/DFT-L framework). This process consists of four stages: (1) initialization, (2) deliberation, (3) preference updates, and (4) decision. In the initialization stage, parameters in DFT-L (elements of ***S*** and ***M***) were given to the agents. As the agents had two options to choose, they determined their weights of choice criteria in the deliberation stage according to Table 2. After that, they updated their preference values based on DFT-L/DFT in the corresponding stage. Finally, the agents made their decisions based on whether their preference values had become greater than the threshold value in the decision stage. Since all agents share the same reasoning process, we can use a state diagram to represent individual decision-making processes, represented in Fig. 3(a). In Fig. 3(b), agents are shown with different colors based on their final decisions, with blue representing the adoption of the first option while the green represents the other option; yellow agents have not chosen either of them. In this way, detailed expression of individual decision-making processes is clearly articulated with ABS.

In our second step, we chose to focus on the network's complexity among different environmental factors in social networks. The three types of networks we considered were ring lattice, small world, and random network (Fig. 4). Its complexity determines the network type; all nodes have the same degree in the ring lattice, so the complexity level is low, while nodes in the random network are likely to have different degrees which increase the complexity level, and small world networks lie between these two extremes [15,48]. We used a complexity measure (ρ) to describe this complexity. As seen in Fig. 4, random networks have the maximum complexity (while ring lattice networks have the minimum). Since

Table 3
Simulation configuration.

Number of agents	Network type	Society type	Models	Learning type	Reflecting neighbors' opinion
200	Random	Progressive	DFT	Synchronous	Proportional to neighbors' degrees
500	Small world (0.7)				
		Neutral			
700	Small world (0.4)		DFT-L	Asynchronous	Equally weighted
		Conservative			
1000	Ring				

it is well-known that the diffusion process would propagate fastest in the random network and slowest in the ring lattice network [15,48], we confirmed that our proposed model DFT-L follows the same pattern as a validation.

Third, interactions among agents are expressed by determining the values of matrix $\boldsymbol{A}$ in DFT-L. Agents may update their preference values while incorporating all neighbors' opinions equally, but they often pay more attention to a neighbor who has many friends in social networks. If a neighbor has a high level of social power (with larger degrees), then an agent will likely adjust their opinion based on the weight representing the social power of its neighbors. Equations for elements of matrix $\boldsymbol{A}$ for these two cases are explained in the last column of Table 2 (where represents degrees of agent). Based on the literature review on the application of DFT in the real field, researchers assume that the error term ($\in_i(t)$) follows Normal distribution [11,42]. Also, when they estimate DFT parameters using real data, we have found that Uniform distribution is also used to provide flexibility [21,27,45]. Thus, we decided to use both Normal and Uniform distribution for the parameter value generation in the experiment.

Finally, we considered two types of learning: synchronous and asynchronous to consider different learning processes. In synchronous learning, all agents update their preference value at the same time (e.g., in every two-time unit in the simulation). One example is how, during a presidential election period, people's preferences for presidential candidates are revealed simultaneously with the survey result. By contrast, asynchronous learning represents people updating their preference values at different deliberation periods. In this work, we set each duration to follow Uniform (0,4) time units, so that the average duration becomes two as in synchronous learning. In Section 4.4., we demonstrated that these two types of learning did not make a statistically significant difference in the time taken to reach an equilibrium using simulation analysis.

All of the other information in the simulation is summarized in Table 3. While Table 2 represents all parameter values of DFT/DFT-L used in individual reasoning processes, Table 3 shows all configurations of the environment around the agents. In Section 4.3, we explain another environmental factor, society type, which has a large effect on asymptotic status in social networks.

4.2. Validation of DFT-L

To illustrate and validate DFT-L, we used the "Leader and Followers" configuration, which has been widely applied in analyzing the diffusion process within social networks [15,44,48]. In this context, the two types of agents living in the network are the leader, the person who has the largest number of degrees in the networks (tiebreaks arbitrary), and all others who are classified as followers. All agents are presented with the option of adopting a new technology that is supposed to offer a high quality of life to the people but requires a high amount of expense to adopt (e.g., the iPhone is more expensive than other smartphones), or using an old one. At the beginning of the simulation, only the leader accepts the new technology, which makes their preference value of the new technology 10, guaranteeing that the probability of occurring reverse selection is less than 5 percent [42,46]. The followers' preference values for the new and old technology are all set to zero [11,12].

The followers update their preference values after every deliberation period by either DFT-L or DFT during the simulation; all parameter values of DFT-L and DFT are assigned according to Table 2, and followers will adopt the new technology if their preference value for the new technology exceeds the threshold. We demonstrate that the model's behaviors are consistent with two well-known results: the adoption process is fastest in the random network and slowest in the ring network; and the adoption process follows an S-curve [15,48].

As shown in Fig. 5, we confirmed that the adoption process within DFT-L aligned with generally known results regardless of population and network structures. We followed the ratio of people who adopted the technology to the total population (Y-axis) over time (X-axis). At the beginning of the simulation run, only a few agents adopted the new technology, while another portion of the population adopted it as time progressed. At some point, the adoption rate increased sharply, then slowed down after the technology was adopted by most of the agents. This 'S' curve phenomenon was observed in all types of networks with different populations.

It is also evident that the diffusion process within DFT-L propagates fastest in the random network and slowest in the ring-lattice network, as shown in Fig. 5. As the network complexity increases, the adoption rates and information diffusion

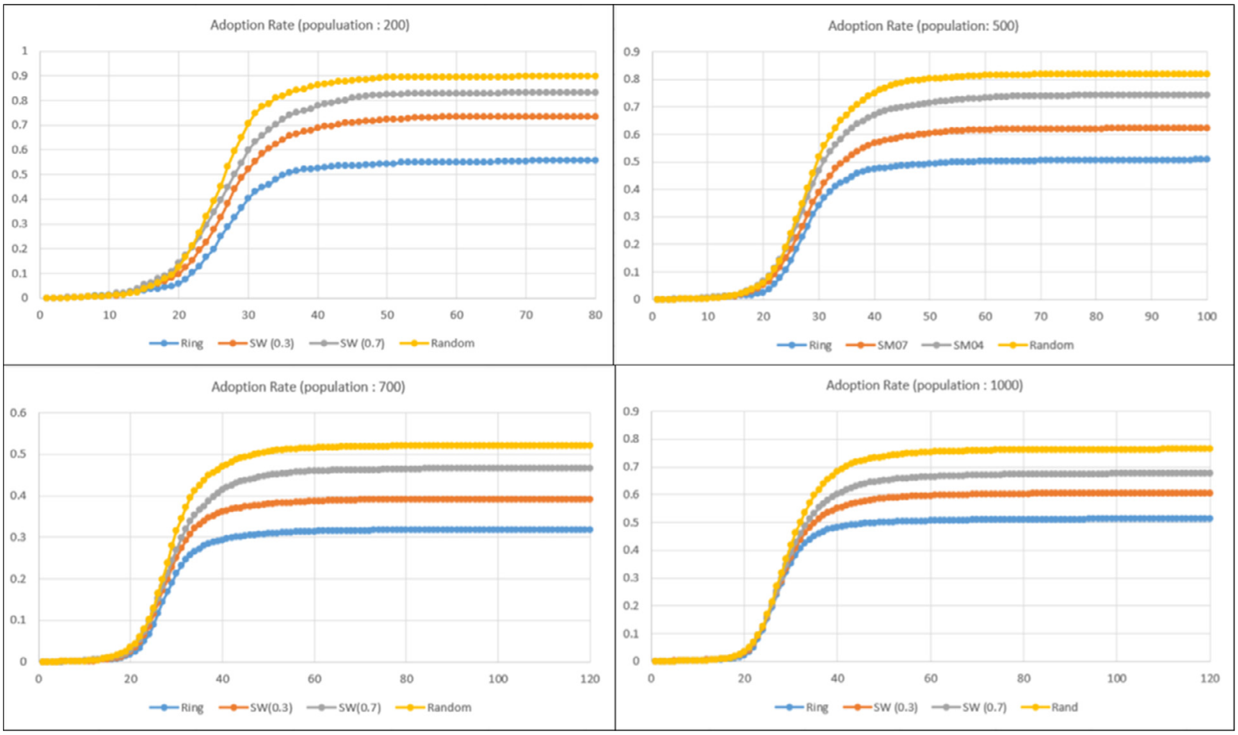

Fig. 5. Adoption process in different settings.

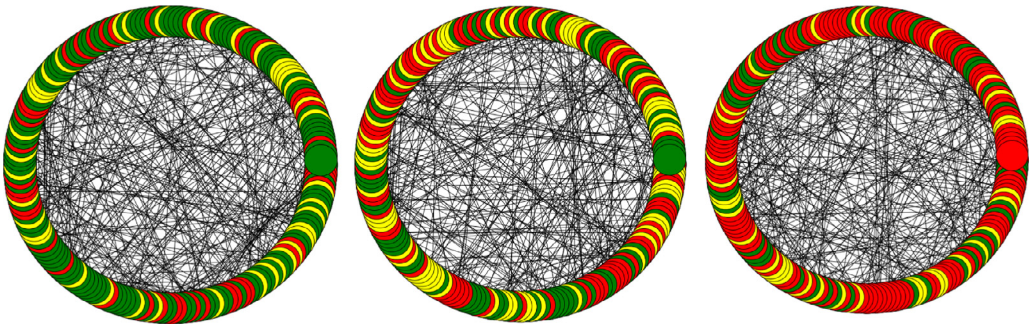

Fig. 6. Progressive, Neutral, and Conservative Society.

increase as well. These two results observed in the adoption process verify the use of the proposed framework DFT-L in analyzing the decision-making process in social networks.

4.3. Comparison between DFT-L and DFT

In this section, we detail the high validity of DFT-L compared to DFT in social networks. Through our analysis of the asymptotic status of social networks within DFT-L and DFT, we demonstrate that DFT-L more thoroughly represents characteristics of society. For the simulation configuration, we used the "opinion formation" setting instead of "leader and followers." In this setting, two alternatives for investment options (the national debt of the US and a hedge fund) are given to agents to choose based on the criteria of expected profit and level of risk. There are three types of agents in the network: risk-taking, risk-averse, and risk-neutral. Though all agents have no prior preference values for each option, each agent type weighs the two criteria differently. Risk-loving agents are assigned their weight values of the level of risk from the Uniform (0.6, 0.8) distribution, while the values of risk-neutral and risk-averse agents are assigned from the Uniform (0.4, 0.6), (0.2, 0.4) distributions, respectively [11,35]. Classifying agents based on their attitude to risk is a common approach to modeling decision-making processes in marketing, psychology, computer science, and financial engineering [7].

Taking the different components of agent types into consideration, we defined three different society types. In Fig. 6, red, yellow, and green represent risk-averse, risk-neutral, and risk-loving agents, respectively. Based on the proportion of the

Table 4
Type III ANOVA results.

Source of variation	Sum of squares	DF	*F* value	*P*- value
Society Type(ST)	0.7276	2	493.57	$<2.2 \times 10^{-6}$
Models	0.4574	1	223.67	$<2.2 \times 10^{-6}$
ST*Models	0.9689	2	281.23	$<2.2 \times 10^{-6}$
Residuals	0.2098	129	297.84	

Table 5
Confidence interval of interesting factors.

Model	DFT-L			DFT		
Statistics	Estimates	Lower CI	Upper CI	Estimates	Lower CI	Upper CI
(Intercept)	0.598	0.5508	0.6452	0.5011	0.4944	0.5078
Conservative	−0.4627	−0.5295	−0.3960	−0.1925	−0.1857	−0.1667
Progressive	0.3178	0.2510	0.3846	0.1478	0.1124	0.1315

agent type to the total population, societies are classified as follows: progressive (20%, 20%, 60%), conservative (60%, 20%, 20%) and neutral (33%, 33%, 33%). At the beginning of the simulation, no agent has any preference for national debt or hedge funds. They will update their preference values by DFT-L or DFT with parameter values as in Table 2, with the agent type determining the weights of the criteria. If either one of the preference values for the two alternatives exceeds the threshold value, then agents choose the alternative with the higher preference value. The output of the simulation is the proportion of people who finally chose the hedge fund in the equilibrium status, and the goal is to explore how three society types and different decision-making models (DFT/DFT-L) affect the response variable as it attains the equilibrium state. To accomplish this, we utilized the two-factor factorial design method, and the ANOVA (Analysis of Variance) result (Type 3) is shown in Table 4.

According to Table 4, we conclude that both factors (society type and decision-making models) significantly affect the number of people adopting the hedge funds. Furthermore, there is a significant interaction between models and society types because the P values are substantially small, which demonstrates the validity of DFT-L.

Regardless of the model implemented, the number of people that adopted the hedge fund is the largest in the progressive society and the smallest in the conservative one. If there is no interaction among people (i.e., within the DFT model), we can expect that the ratio of people who chose the hedge fund is 70% in the progressive network, since 60% of the risk-loving people and half of the risk-neutral people (10%) will choose the hedge funds. Because we found that the selection of the decision-making models between DFT-L and DFT significantly affects the adoption rate, we conducted another test to find the confidence interval of interesting factors. Under both decision-making models, it is evident that in comparison with a neutral society, adoption rates increase in a progressive society and decrease in a conservative society. However, DFT-L shows society types affecting the adoption rates even further than DFT. Within the DFT framework, a progressive society only increases the adoption rate of 15%, and a conservative society only decreases the rate of 19%; in DFT-L, these changes have become significantly higher at 46% and 31%, respectively (Table 5). Therefore, we can conclude that interaction among people affects the agent's final decision within DFT-L and intensifies embedded society characteristics at the final stage.

Another phenomenon was observed under the DFT-L model. More than 90% chose the hedge funds in the progressive network, while less than 10% of people chose them in the conservative network in the equilibrium state (see Fig. 7). The simulation results reveal that DFT-L strengthens the societal characteristics and valdity compared to DFT in equilibrium. Thus, DFT-L is able to demonstrate the bandwagon effect in social networks more thoroughly. Since the bandwagon effect is known to be the main reason for information cascade in social networks, DFT-L can be used in analyzing irregular behaviors such as information cascade in social networks while considering neighbor effects. Our observations show that the limiting behaviors within DFT-L are affected by social characteristics.

4.4. Advanced analysis of DFT-L

In this section, we demonstrate factors that affect the time until attainment of the equilibrium state. The response variable (Y) is the number of deliberation steps until equilibrium is achieved, while four types of independent variables (Society Type, Network Type, Weights, and Learning as shown in Table 3) are considered in ANOVA. In order to choose significant variables, we first conducted ANOVA with all first-order terms of the four variables. After determining significant variables, we performed 5 simulation runs for ANOVA with interaction among selected variables.

Table 6 displays the results of the ANOVA simulations, most notably that society type (described in Section 4.3) and network type (described in Section 4.2) played a significant role in the time taken to achieve an equilibrium. The interaction effect between society and network type is not significant enough to affect the average convergence time (as shown in the third row). Thus, another set of experiments was conducted to estimate the time to equilibrium based on each society and

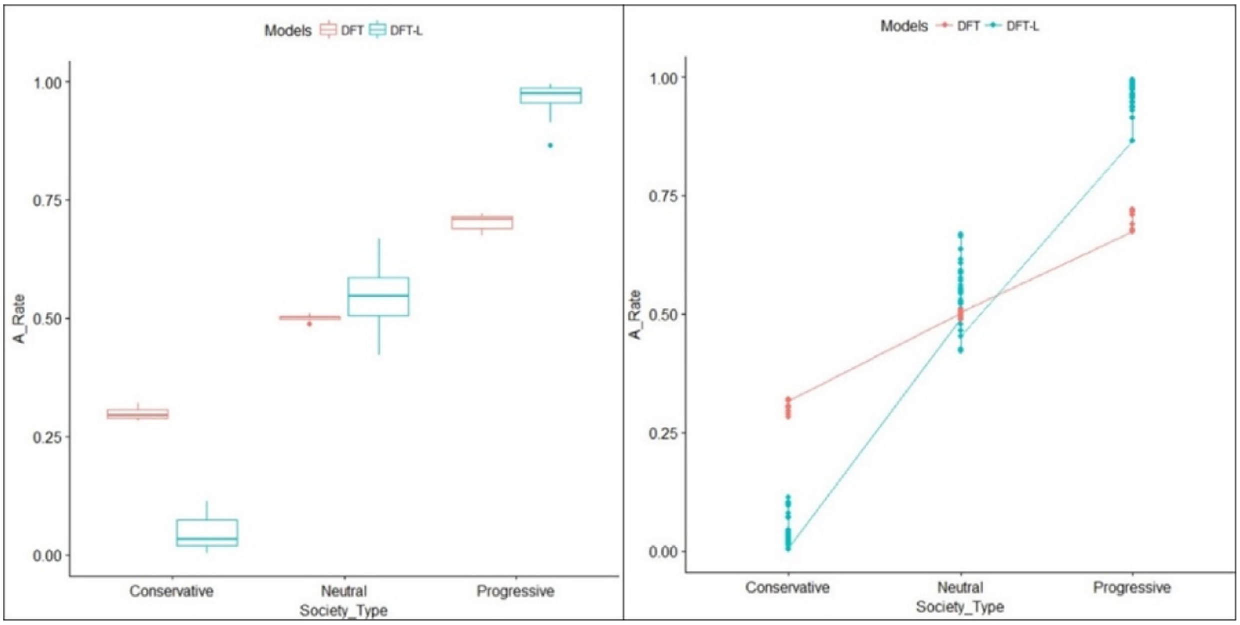

Fig. 7. Adoption rate of hedge fund in different society within DFT/DFT-L.

Table 6
Type III ANOVA results.

Factors	Sum of squares	DF	F value	P value
Society Type (ST)	9469.9	2	370.5617	$<2.2 \times 10^{-16}$
Network Type (NT)	6182.2	2	241.9113	$<2.2 \times 10^{-16}$
ST*NT	83.0	4	1.6243	0.1893
Residuals	460	36		

Table 7
Confidence interval of interesting factors.

Factors	Estimates	Lower CI	Upper CI
(Intercept)	78.911	76.4289	81.3933
Society Type: Conservative	−30.4	−33.1191	−27.6808
Society Type: Progressive	−31.133	−33.8524	−28.4142
Network Type: ring	28.467	25.7475	31.1858
Network Type: small world	11.000	8.2808	13.7191

network time, not considering those interaction effects. The output of this additional test (Table 7) shows that in both conservative and progressive networks, there exists a significant reduction by 30 deliberation periods to the convergence when compared to the neutral network. These observations demonstrate that the equilibrium is achieved faster in the progressive and conservative societies than in a neutral society. In the case of the neutral society, since the proportion of the population favoring each option is well balanced, it takes a longer time for people to make a choice. In the progressive and conservative societies with the majority of the population having a higher preference of one option over the other, the opinion of the society is formed and stabilized faster. The longest time to achieve equilibrium state is in the ring network (around 28 deliberation periods on average), and the time in the small world network is also longer (around 11 deliberation periods on average) than in the random network (See Table 7). Our findings validate that DFT-L explains the dynamic behavior of opinion formation in a more realistic way.

4.5. Validation using real data

In this section, we validate the proposed models for well-known social networks using two different types of networks: 1) Zachary's Karate network [47] and 2) American College Soccer network [39]. First, we demonstrate that the diffusion process of the Karate network lies between the diffusion behaviors of random and ring networks, as we described in Section 4.2. Since the network complexity (ρ) of the Karate network is between 0 and 1 ($0 < \rho < 1$), the diffusion behavior in the Karate

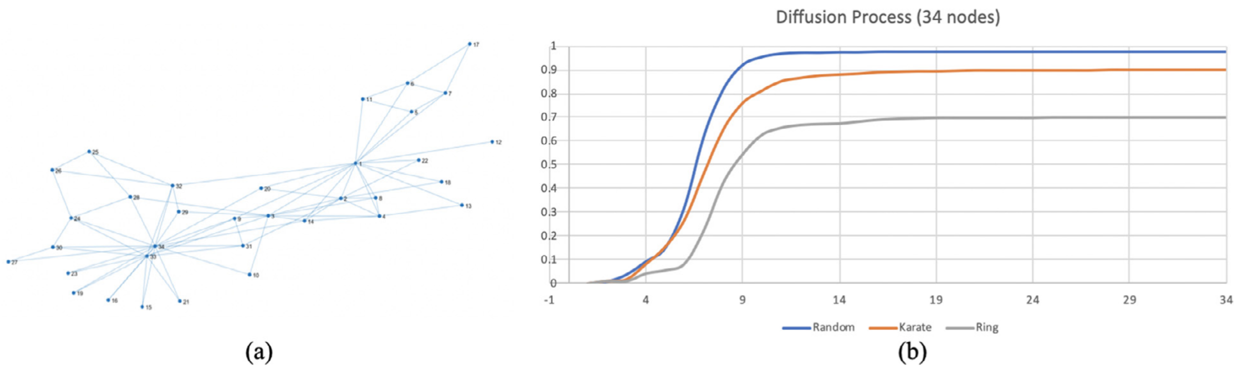

Fig. 8. (a) Karate network [47] (b) and its diffusion behavior.

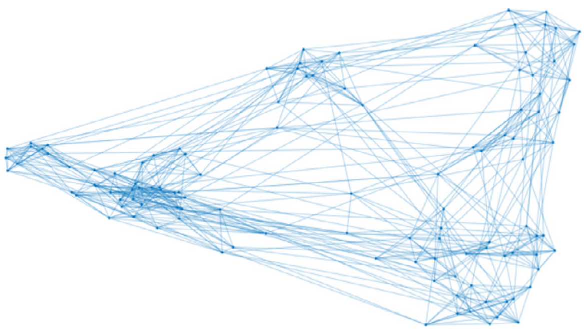

Fig. 9. American College Football Network [39].

Table 8
Type III ANOVA result.

Factors	SS	DF	*F* value	*P* value
Society Types	4.4133	2	526.78	$<2.2 \times 10^{-16}$
Models	0.0999	1	23.84	$<4.9 \times 10^{-6}$
Interaction	0.8827	2	105.36	$<2.2 \times 10^{-16}$
Residuals	0.3519	84		

network should be similar to that of the small world network. We generated a random and a ring network with 34 agents (the same number of agents in Karate network) and compared the diffusion process. As seen in Fig. 8, the diffusion process in Karate network lies between ring and random network.

Second, using the same configuration in Section 4.3., we tested the significance of effect on number of people adopting the hedge fund using factors like society type and decision-making models. This analysis was performed on American College soccer network (Fig. 9). Table 8 represents the test results.

Based on our findings, we conclude that both factors (society type and decision-making models) significantly affect the number of people adopting hedge funds within a real social network. The interaction factor is also significant, which is also shown in Table 8. Thus, the DFT-L model provides a generalized approach to describe different types of behaviors in random and real social networks.

5. Conclusions

In this paper, we proposed Decision-Field Theory with Learning (DFT-L), which is an extension to Decision Field Theory (DFT) in conjunction with DeGroot's social learning model. The extension allows us to analyze long-term deliberation decision-making processes. By proving relevant lemmas, we showed that asymptotic behavior could be derived with reasonable assumptions within DFT-L. We also demonstrated that DFT-L could exploit the advantages of both DFT and DeGroot's model without losing any validity. DFT-L is a symmetrical way to model social network behaviors combining individual decision-making processes with interaction effects in the long term. Simulation experiments were conducted to demonstrate the validity of the DFT-L and its comparison with DFT. The simulation results revealed that DFT-L is capable of representing and analyzing irregular behaviors in social networks. In summary, the contributions of this paper are the following: the approach to using Decision Field Theory in the social network setting, the proposed model having a simple structure without losing high validity and benefits of original DFT and DeGroot's model, a formal derivation of asymptotic status, and thorough validation and demonstration of the model via agent-based simulation.

Possible future works which the authors are currently pursuing are two-fold: applying network dynamics in analyzing decision-making processes and applying group detection methods to enhance the representation of people's interactions. In our setting in this paper, the social network structure is fixed over time after it is generated. This assumption can be resolved if one analyzes the network with high variability. Moreover, we only considered interactions within groups, not between groups in this paper. Therefore, if a group detection technique is incorporated into the model, multi-layer decision-making processes can also be represented in this framework.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

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