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Nonlinear stability and existence of compressible vortex sheets in 2D elastodynamics

Robin Ming Chen [a,*], Jilong Hu [a], Dehua Wang [a], Tao Wang [b,c], Difan Yuan [d]

[a] *Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA*
[b] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*
[c] *Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China*
[d] *School of Mathematical Sciences, Beijing Normal University and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China*

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Abstract

The nonlinear stability and local existence of compressible vortex sheets for the two-dimensional isentropic elastic fluid are established in the usual Sobolev spaces. The problem has a characteristic free boundary, and the Kreiss–Lopatinskiĭ condition is satisfied only in a weak form. This paper completes the previous works [6,7] of the first three authors where the weakly linear stability of the rectilinear vortex sheets is proved by means of an upper triangularization technique. Our proof is based on certain higher-order energy estimates and an appropriate modification of the Nash–Moser iteration. In particular, the estimate for the normal derivatives of the characteristic variables can be recovered from that for the linearized divergences and vorticities.

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* Corresponding author.
E-mail addresses: mingchen@pitt.edu (R.M. Chen), jih62@pitt.edu (J. Hu), dwang@math.pitt.edu (D. Wang), tao.wang@whu.edu.cn (T. Wang), yuandf@amss.ac.cn (D. Yuan).

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1. Introduction

This paper continues and completes the previous works [6,7] of the first three authors on the study of stability for vortex sheets in the two-dimensional compressible elastodynamics. In particular, we prove the nonlinear stability, and hence the local existence for the configuration of vortex sheets.

The physical relevance of the model, and the motivation to include elasticity and to study their stabilizing property can be found in [6,7] and the references therein. In this introduction we will recall the problem of compressible vortex sheets for elastodynamics, state the main result after transforming the free boundary problem into a fixed domain, and briefly discuss our approach.

1.1. Formation of compressible vortex sheets

The two-dimensional isentropic motion of elastodynamics can be described by the following equations (see Dafermos [14, Chapter 2]):

$$\partial_t \rho + \partial_\ell(\rho v_\ell) = 0, \tag{1.1a}$$

$$\partial_t(\rho v_i) + \partial_\ell(\rho v_\ell v_i) = \partial_\ell T_{i\ell}, \tag{1.1b}$$

$$(\partial_t + v_\ell \partial_\ell) F_{ij} = \partial_\ell v_i F_{\ell j}, \tag{1.1c}$$

for $i, j = 1, 2$, where $\partial_t := \frac{\partial}{\partial t}$ and $\partial_\ell := \frac{\partial}{\partial x_\ell}$, for $\ell = 1, 2$, denote the partial differentials, ρ is the density, $v = (v_1, v_2)^\top \in \mathbb{R}^2$ is the velocity, $\boldsymbol{F} = (F_{ij}) \in \mathbb{M}^{2\times 2}$ is the deformation gradient, and $\boldsymbol{T} = (T_{ij}) \in \mathbb{M}^{2\times 2}$ is the Cauchy stress tensor. We note that the Einstein summation convention is used in (1.1) and will also be adopted in the rest of this paper, and we denote by $\mathbb{M}^{m\times n}$ the vector space of real $m \times n$ matrices.

We consider the compressible neo-Hookean materials (see Ciarlet [9, p. 189]), for which the Cauchy stress tensor $\boldsymbol{T}$ reads

$$\boldsymbol{T} = \lambda \rho \boldsymbol{F}\boldsymbol{F}^\top - p(\rho) I_2, \tag{1.2}$$

where $\lambda > 0$ is the Hookean constant and I_m denotes the identity matrix of order m. Pressure $p(\rho)$ is a C^∞ and strictly increasing function on $(0, +\infty)$ so that the sound speed $c = c(\rho)$ satisfies

$$c(\rho) := \sqrt{p'(\rho)} > 0 \qquad \text{for} \quad \rho > 0. \tag{1.3}$$

When $\lambda = 0$, the material becomes a thermoelastic fluid (see [14, p. 39]) and equations (1.1a)–(1.1b) are reduced to the compressible isentropic Euler equations in gas dynamics. Since we are concerned with the effect of elasticity to the evolution of materials, we set without loss of generality that $\lambda = 1$.

System (1.1) is supplemented by divergence constraints

$$\operatorname{div}(\rho F_j) := \partial_\ell(\rho F_{\ell j}) = 0 \qquad \text{for} \quad j = 1, 2, \tag{1.4}$$

where F_j stands for the j-th column of $\boldsymbol{F}$. With (1.4), equations (1.1c) can be reformulated in the following divergence form:

$$\partial_t(\rho F_{ij}) + \partial_\ell(\rho F_{ij} v_\ell - v_i \rho F_{\ell j}) = 0 \qquad \text{for } i, j = 1, 2,$$

which is convenient when calculating the jump conditions for weak solutions (as in [6,7]). It is worth pointing out that constraints (1.4) are involutions to system (1.1), meaning that if constraints (1.4) hold initially, then they are preserved by the evolution; see Dafermos [13] and Hu–Wang [17]. By using (1.3)–(1.4), in smooth regions, system (1.1) can be rewritten equivalently as

$$\partial_t U + A_1(U)\partial_1 U + A_2(U)\partial_2 U = 0, \tag{1.5}$$

where $U := (\rho, v_1, v_2, F_{11}, F_{21}, F_{12}, F_{22})^\top \in \mathbb{R}^7$ is the unknown vector, and

$$A_i(U) := \begin{pmatrix} v_i & \rho e_i^\top & 0 & 0 \\ \frac{c(\rho)^2}{\rho} e_i & v_i I_2 & -F_{i1} I_2 & -F_{i2} I_2 \\ 0 & -F_{i1} I_2 & v_i I_2 & 0 \\ 0 & -F_{i2} I_2 & 0 & v_i I_2 \end{pmatrix} \qquad \text{for } i = 1, 2, \tag{1.6}$$

with $e_1 := (1, 0)^\top$ and $e_2 := (0, 1)^\top$. System (1.5) is symmetrizable hyperbolic for $\rho > 0$ due to (1.3).

Let U be smooth on each side of a smooth hypersurface $\Gamma(t) := \{x \in \mathbb{R}^2 : x_2 = \varphi(t, x_1)\}$, that is,

$$U(t,x) = \begin{cases} U^+(t,x), & \text{in } \Omega^+(t) := \{x \in \mathbb{R}^2 : x_2 > \varphi(t, x_1)\}, \\ U^-(t,x), & \text{in } \Omega^-(t) := \{x \in \mathbb{R}^2 : x_2 < \varphi(t, x_1)\}, \end{cases}$$

where $U^+(t,x)$ and $U^-(t,x)$ are smooth functions in $\Omega^+(t)$ and $\Omega^-(t)$, respectively. We are interested in *vortex sheets* for which the tangential velocity suffers a jump across $\Gamma(t)$. As in the previous paper [6], the Rankine–Hugoniot conditions of the vortex sheet solutions are reduced to

$$[v_\nu] = 0, \quad \partial_t \varphi = v_\nu^+, \quad [\rho] = 0 \qquad \text{on } \Gamma(t), \tag{1.7}$$

together with

$$F_{1\nu}^\pm = F_{2\nu}^\pm = 0 \qquad \text{on } \Gamma(t), \tag{1.8}$$

where $[f]$ denotes the jump of quantity f across $\Gamma(t)$, and

$$\nu = (-\partial_1 \varphi, 1)^\top, \quad v_\nu^\pm = v^\pm \cdot \nu, \quad F_{j\nu}^\pm = F_j^\pm \cdot \nu.$$

See Truesdell–Toupin [34, Section 185] for a thorough discussion. From (1.7) and (1.8), the boundary matrix on $\Gamma(t)$, namely

$$A_{\mathrm{bdy}} := \mathrm{diag}\big(\partial_t \varphi I_7 - \nu_\ell A_\ell(U^+), -\partial_t \varphi I_7 + \nu_\ell A_\ell(U^-)\big)\big|_{\Gamma(t)},$$

is singular, which means that the free boundary $\Gamma(t)$ is characteristic. In this sense, a vortex sheet solution is a characteristic discontinuity. Moreover, the boundary matrix A_{bdy} has 2 negative, 2 positive, and 10 zero eigenvalues. We need one boundary condition for determining the unknown front, so the correct number of boundary conditions is three, according to the well-posedness theory for hyperbolic boundary value problems. As a matter of fact, identities (1.8) are involutions inherited from the initial data (*cf.* Proposition 1.1), so they are regarded as constraints on the initial data rather than boundary conditions for the vortex sheet problem.

As discussed in [6], there exist trivial vortex sheet solutions

$$U(t,x_1,x_2)=\begin{cases}(\bar{\rho},\ \bar{v},0,\ \overline{F}_{11}^{+},\ 0,\ \overline{F}_{12}^{+},\ 0)^{\top}, & x_2>0,\\ (\bar{\rho},\ -\bar{v},0,\ \overline{F}_{11}^{-},\ 0,\ \overline{F}_{12}^{-},\ 0)^{\top}, & x_2<0,\end{cases} \tag{1.9}$$

where $\bar{\rho}>0$, $\bar{v}>0$, $\overline{F}_{11}^{\pm}$, and $\overline{F}_{12}^{\pm}$ are constants. Every rectilinear elastic vortex sheet (namely piecewise-constant vortex sheet) is of this form through the Galilean transformation. For simplicity we assume that $\overline{F}_{11}^{+}=-\overline{F}_{11}^{-}=\overline{F}_{11}$ and $\overline{F}_{12}^{+}=-\overline{F}_{12}^{-}=\overline{F}_{12}$.

A standard first step in treating a free boundary problem is to convert the problem in a fixed domain. For this purpose, we introduce

$$U_{\sharp}^{\pm}(t,x):=U(t,x_1,\Phi^{\pm}(t,x)), \tag{1.10}$$

where the lifting functions $\Phi^{\pm}$ are taken as in Francheteau–Métivier [15] to satisfy

$$\partial_t\Phi^{\pm}+v_1^{\pm}\partial_1\Phi^{\pm}-v_2^{\pm}=0,\quad \pm\partial_2\Phi^{\pm}\geq\kappa>0, \tag{1.11}$$

when $x_2\geq 0$, and

$$\Phi^{+}=\Phi^{-}=\varphi,\quad \text{when } x_2=0, \tag{1.12}$$

for some constant $\kappa>0$. Then we need to solve the following initial-boundary value problem for $U_{\sharp}^{\pm}$ in a fixed domain:

$$\mathbb{L}(U^{\pm},\Phi^{\pm}):=L(U^{\pm},\Phi^{\pm})U^{\pm}=0,\quad x_2>0, \tag{1.13a}$$

$$\mathbb{B}(U^{+},U^{-},\varphi)|_{x_2=0}=0, \tag{1.13b}$$

$$(U^{+},U^{-},\varphi)|_{t=0}=(U_0^{+},U_0^{-},\varphi_0), \tag{1.13c}$$

where we have dropped the index "$\sharp$" for convenience, $L(U,\Phi)$ and $\mathbb{B}$ are given by

$$L(U,\Phi):=I_7\partial_t+A_1(U)\partial_1+\widetilde{A}_2(U,\Phi)\partial_2, \tag{1.14}$$

$$\mathbb{B}(U^{+},U^{-},\varphi):=\begin{pmatrix}[v_1]\partial_1\varphi-[v_2]\\ \partial_t\varphi+v_1^{+}|_{x_2=0}\partial_1\varphi-v_2^{+}|_{x_2=0}\\ [\rho]\end{pmatrix}, \tag{1.15}$$

with

$$\widetilde{A}_2(U,\Phi):=\frac{1}{\partial_2\Phi}\left(A_2(U)-\partial_t\Phi I_7-\partial_1\Phi A_1(U)\right).$$

By (1.8) and (1.11), we obtain that the boundary matrix of problem (1.13), *i.e.*,

$$\mathrm{diag}\left(-\widetilde{A}_2(U^+,\Phi^+),\ -\widetilde{A}_2(U^-,\Phi^-)\right),$$

has constant rank on $\{x_2 \geq 0\}$ if and only if

$$F_{2j}^{\pm} = F_{1j}^{\pm}\partial_1\Phi^{\pm} \quad \text{for } j=1,2, \quad \text{if } x_2 \geq 0. \tag{1.16}$$

In the new variables, equations (1.4) become

$$\partial_\ell^{\Phi^\pm}(\rho^\pm F_{\ell j}^\pm) = 0 \quad \text{for } j=1,2, \quad \text{if } x_2 > 0, \tag{1.17}$$

where we denote the partial differentials with respect to the lifting function Φ by

$$\partial_t^{\Phi} := \partial_t - \frac{\partial_t\Phi}{\partial_2\Phi}\partial_2, \quad \partial_1^{\Phi} := \partial_1 - \frac{\partial_1\Phi}{\partial_2\Phi}\partial_2, \quad \partial_2^{\Phi} := \frac{1}{\partial_2\Phi}\partial_2. \tag{1.18}$$

The following proposition indicates that identities (1.16)–(1.17) are involutions for vortex sheet problem (1.11)–(1.13). The proof follows from a straightforward computation and hence is omitted.

Proposition 1.1. *For every sufficiently smooth solution of problem* (1.11)–(1.13) *on time interval* $[0, T]$*, constraints* (1.16)–(1.17) *hold for all* $t \in [0, T]$ *provided that they are satisfied initially.*

1.2. Main result and discussion

In the straightened variables, the piecewise constant vortex sheet (1.9) corresponds to

$$\overline{U}^{\pm} := \left(\bar\rho,\ \pm\bar v,\ 0,\ \pm\overline{F}_{11},\ 0,\ \pm\overline{F}_{12},\ 0\right)^{\top}, \quad \bar\varphi := 0, \quad \overline{\Phi}^{\pm} := \pm x_2. \tag{1.19}$$

For proving the nonlinear stability of elastic vortex sheets, we only need to show the existence of solutions to problem (1.11)–(1.13) on account of transform (1.10). The main result of this paper is stated as follows.

Theorem 1.1. *Let* $T > 0$ *and* $s_0 \geq 14$ *be an integer. Suppose that the background state* (1.19) *satisfies one of the following stability conditions:*

$$\bar v^2 > 2c(\bar\rho)^2 + \overline{F}_{11}^2 + \overline{F}_{22}^2, \tag{1.20}$$

or

$$\begin{cases} 0 < \bar v^2 < \overline{F}_{11}^2 + \overline{F}_{22}^2, \\ \bar v^2 \neq \dfrac{\overline{F}_{11}^2 + \overline{F}_{22}^2}{4}, \ \bar v^2 \neq \dfrac{\left((\overline{F}_{11}^2 + \overline{F}_{22}^2 + c(\bar\rho)^2)^{1/2} - (\overline{F}_{11}^2 + \overline{F}_{22}^2)^{1/2}\right)^2}{4}, \\ \bar v^2 \neq \dfrac{\overline{F}_{11}^2 + \overline{F}_{22}^2 + c(\bar\rho)^2}{4}, \ \bar v^2 \neq \dfrac{(\overline{F}_{11}^2 + \overline{F}_{22}^2)(\overline{F}_{11}^2 + \overline{F}_{22}^2 + 2c(\bar\rho)^2)}{4(\overline{F}_{11}^2 + \overline{F}_{22}^2 + c(\bar\rho)^2)}. \end{cases} \tag{1.21}$$

Suppose further that the initial data $U_0^{\pm}$ *and* φ_0 *satisfy constraints* (1.16)–(1.17) *and the compatibility conditions up to order* s_0 *(cf. Definition 4.1), and that* $(U_0^{\pm} - \bar{U}^{\pm}, \varphi_0) \in H^{s_0+1/2}(\mathbb{R}_+^2) \times H^{s_0+1}(\mathbb{R})$ *has a compact support. Then there exists a positive constant* ϵ *such that, if*

$$\|U_0^{\pm} - \bar{U}^{\pm}\|_{H^{s_0+1/2}(\mathbb{R}_+^2)} + \|\varphi_0\|_{H^{s_0+1}(\mathbb{R})} \le \epsilon,$$

then problem (1.11)–(1.13) *admits a solution* $(U^{\pm}, \Phi^{\pm}, \varphi)$ *on the time interval* $[0, T]$ *satisfying*

$$(U^{\pm} - \bar{U}^{\pm}, \Phi^{\pm} - \bar{\Phi}^{\pm}) \in H^{s_0-8}((0, T) \times \mathbb{R}_+^2), \quad \varphi \in H^{s_0-7}((0, T) \times \mathbb{R}).$$

The theorem above asserts that, unlike the two-dimensional compressible Euler flow for which the vortex sheets are stable only for large Mach numbers, the appearance of elasticity stabilizes the system even in the subsonic zone, confirming the expectation from the linear analysis [6,7]. In particular, when linearizing at the rectilinear vortex sheet, a stabilizing subsonic zone larger than the one given by (1.21) was discovered in [6] by a delicate spectral analysis of the Lopatinskiĭ determinant for the corresponding constant coefficient problem combined with an upper triangulation scheme for the energy estimates. Further perturbing away from the constant states leads to a linear problem with variable coefficients which admits a richer spectral structure. Paradifferential calculus thus becomes an effective way in place of the Fourier analysis. However, understanding the spectrum of the para-linearized system is much more challenging due to the degeneracy of the Kreiss–Lopatinskiĭ condition and the characteristic boundary. The upper triangularization method turns out to be particularly useful for treating the additional degenerate boundary points (referred to as poles) as well as gaining improved regularity of the outgoing modes; see the discussion in [7]. On the other hand, it is the complicated interaction between the poles and the other degenerate points (namely the roots) that imposes extra constraints in the subsonic region for stability.

Proceeding from linear to nonlinear stability and thus local existence can usually be achieved by an iterative argument. Our proof shall follow the general procedure (and thus format of presentation) in the spirit of Coulombel–Secchi [12]. A common feature shared by various types of compressible vortex sheets is that the free boundary is characteristic and the Kreiss–Lopatinskiĭ condition holds only in a weak sense; see, *e.g.*, [4–7,11,27]. Therefore the standard fixed-point argument cannot be applied since there is a loss of regularity from the source terms to the solution in the estimates for the linearized equations. Instead, we will appeal to the Nash–Moser iteration framework and construct solutions to the nonlinear problem (1.11)–(1.13) via the convergence of the scheme. Such type of approach has been successfully applied to other related problems [1,4,5,12,15,18,22,24,26,30–33,35]. Also refer to Alinhac–Gérard [2, Chapter III.C] and Secchi [29] for a general description.

For showing the convergence of the Nash–Moser iteration scheme, we need to establish the well-posedness of the variable coefficient linearized problem with suitable tame estimate. In [6,7], the basic *a priori* energy estimate has been derived in the weighted Sobolev space L^2_γ with one loss of derivative from the source terms. Using this estimate and the Moser-type calculus inequalities, we can control the tangential derivatives by the source terms, the coefficients, and the L^∞ norm of solutions (instead of the $W^{1,\infty}$ norm in Coulombel–Secchi [12, (37)], *cf.* (3.28)). In general one has to study characteristic hyperbolic problems in the anisotropic Sobolev spaces due to the degeneracy in the normal direction (see Secchi [28] and the references therein). Utilizing such function spaces, the iteration was carefully carried out to pass from linear to quasilinear problems in [16], resulting in the well-posedness of the full problem. However, in the present paper, instead of making use of the anisotropy in different derivatives, we will follow an idea of Trakhinin [32] and compensate the loss of normal derivatives through the estimates of the linearized divergences and vorticities (see (3.44) and (3.52)–(3.53) for the definitions). This will in turn allow us to build the well-posedness in the usual Sobolev spaces.

We remark that the recent paper [8] confirms that the elasticity can stabilize the fluids in three dimensions. Indeed, it is showed that the linear stability in three-dimensional compressible elastic fluids is more challenging and the spectrum analysis is different from the two-dimensional case due to more complicated structures of the system. We also refer the reader to the recent work [23] for the stabilization effect of elasticity in the study of the structural stability of shock waves in 2D compressible elastodynamics.

The rest of this paper is organized as follows. Section 2 is devoted to collecting several preliminaries including the notation, weighted Sobolev spaces and norms, and the Moser-type calculus inequalities in weighted spaces for later use. In Section 3, we show the well-posedness of solutions to the variable coefficient linearized problem in usual Sobolev spaces, that is, Theorem 3.1. For this purpose, we first prove the well-posedness of the linearized problem in L^2 by applying the duality argument of [10,12]. Then we show the estimate of the tangential derivatives, normal derivatives of the noncharacteristic variables, linearized divergences, and linearized vorticities in Subsections 3.3–3.6. The proof of Theorem 3.1 is given in Subsection 3.7 by finite induction. Section 4 is devoted to introducing the compatibility conditions and approximate solutions. In Section 5, we first present the Nash–Moser iteration scheme for our nonlinear problem by following [4,12]. Particularly, in Subsection 5.3, we construct and estimate a suitable modified state for deriving the convergence of the scheme.

2. Preliminaries

In this section, we shall provide the definitions of weighted Sobolev spaces and norms, and then introduce the Moser-type calculus inequalities in terms of weighted norms for later use.

First we give the following notation. Letter γ always denotes a parameter with $\gamma \geq 1$. We denote by C any universal positive constant, by $C(\cdot)$ any generic positive constant depending only on its listed arguments, and they may change from line to line. The notation $A \lesssim B$ ($B \gtrsim A$) is used if $A \leq CB$ is true for some constant $C > 0$ *independent* of γ. Symbol $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$. Set $\Omega = \{(t, x_1, x_2) \in \mathbb{R}^3 : x_2 > 0\}$, and its boundary $\partial\Omega$ is identified to $\mathbb{R}^2$. For $T \in \mathbb{R}$, write $\omega_T := (-\infty, T) \times \mathbb{R}$ and $\Omega_T := \omega_T \times \mathbb{R}_+$. We denote $\nabla := (\partial_t, \partial_1)$ when applying it to functions of (t, x_1) and $\nabla := (\partial_t, \partial_1, \partial_2)$ when applying it to functions of (t, x_1, x_2). For multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^3$, we define $\partial^\alpha := \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ and $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2$. For $m \in \mathbb{N}$, we denote $\nabla^m := \{\partial^\alpha : |\alpha| = m\}$. We remark that, besides the notation above, we also adopt the same conventional notation in many places of this paper as those in [4,7,12,16].

We now give the definitions of weighted Sobolev spaces and norms. Let $s \in \mathbb{R}$, $m \in \mathbb{N}$, and $\gamma \geq 1$. The weighted Sobolev space

$$H^s_\gamma(\mathbb{R}^2) := \left\{ u \in \mathcal{D}'(\mathbb{R}^2) : \mathrm{e}^{-\gamma t} u(t, x_1) \in H^s(\mathbb{R}^2) \right\}$$

is defined with norm $\|u\|_{H^s_\gamma(\mathbb{R}^2)} := \|\mathrm{e}^{-\gamma t} u\|_{s,\gamma}$, where

$$\|v\|_{s,\gamma} := \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} \lambda^{2s,\gamma}(\xi)|\widehat{v}(\xi)|^2\,\mathrm{d}\xi\right)^{1/2},$$

with $\widehat{v}$ being the Fourier transform of v and $\lambda^{2s,\gamma}(\xi) := (\gamma^2 + |\xi|^2)^s$. We denote $L^2_\gamma(\mathbb{R}^2) := H^0_\gamma(\mathbb{R}^2)$ for short and obtain from the Plancherel theorem that $\|u\|_{L^2_\gamma(\mathbb{R}^2)} = \|\mathrm{e}^{-\gamma t}u\|_{L^2(\mathbb{R}^2)}$.

We abbreviate $L^2(\mathbb{R}_+; H^s_\gamma(\mathbb{R}^2))$ to $L^2(H^s_\gamma)$, which is equipped with the norm

$$\|u\|_{L^2(H^s_\gamma)} := \left(\int_{\mathbb{R}_+} \|u(\cdot, x_2)\|^2_{H^s_\gamma(\mathbb{R}^2)}\,\mathrm{d}x_2\right)^{1/2},$$

and $L^2_\gamma(\Omega) := L^2(H^0_\gamma)$, $\|u\|_{L^2_\gamma(\Omega)} = \|\mathrm{e}^{-\gamma t}u\|_{L^2(\Omega)}$. Moreover,

$$H^m_\gamma(\Omega_T) := \left\{u \in \mathcal{D}'(\Omega_T) : \mathrm{e}^{-\gamma t}u \in H^m(\Omega_T)\right\}$$

is introduced with the norm

$$\|u\|_{H^m_\gamma(\Omega_T)} := \sum_{|\alpha|\le m} \gamma^{m-|\alpha|}\|\mathrm{e}^{-\gamma t}\partial^\alpha u\|_{L^2(\Omega_T)}.$$

Similarly, the space $H^m_\gamma(\omega_T)$ and its norm are defined. Furthermore, we abbreviate $L^2(\mathbb{R}_+; H^m_\gamma(\omega_T))$ to $L^2(H^m_\gamma(\omega_T))$, which is equipped with the norm

$$\|u\|_{L^2(H^m_\gamma(\omega_T))} := \sum_{\alpha_0+\alpha_1\le m} \gamma^{m-\alpha_0-\alpha_1}\|\mathrm{e}^{-\gamma t}\partial_t^{\alpha_0}\partial_1^{\alpha_1}u\|_{L^2(\Omega_T)},$$

and $L^2_\gamma(\Omega_T) := L^2(H^0_\gamma(\omega_T))$, $\|u\|_{L^2_\gamma(\Omega_T)} = \|\mathrm{e}^{-\gamma t}u\|_{L^2(\Omega_T)}$.

In the following lemma, we present the Moser-type calculus inequalities in weighted Sobolev spaces that will be frequently adopted in proving the higher-order tame estimates and convergence of the Nash–Moser iterative scheme.

Lemma 2.1. *Let $m \in \mathbb{N}$, $\gamma \ge 1$, $T \in \mathbb{R}$, and $u, w \in H^m_\gamma(\Omega_T) \cap L^\infty(\Omega_T)$. Let b denote a C^∞–function defined in a neighborhood of the origin.*
(a) *If $|\beta_1 + \beta_2| \le m$ and $b(0) = 0$, then*

$$\left\|\partial^{\beta_1}u\partial^{\beta_2}w\right\|_{L^2_\gamma(\Omega_T)} + \left\|uw\right\|_{H^m_\gamma(\Omega_T)} \lesssim \|u\|_{L^\infty(\Omega_T)}\|w\|_{H^m_\gamma(\Omega_T)} + \|u\|_{H^m_\gamma(\Omega_T)}\|w\|_{L^\infty(\Omega_T)}, \tag{2.1}$$

$$\|b(u)\|_{H^m_\gamma(\Omega_T)} \le C\big(\|u\|_{L^\infty(\Omega_T)}\big)\|u\|_{H^m_\gamma(\Omega_T)}. \tag{2.2}$$

(b) *If $|\beta_1 + \beta_2 + \beta_3| \le m$, then*

$$\left\|\partial^{\beta_1}[\partial^{\beta_2}, b(u)]\partial^{\beta_3}w\right\|_{L^2_\gamma(\Omega_T)} \le C\big(\|u\|_{L^\infty(\Omega_T)}\big)\Big(\|w\|_{H^m_\gamma(\Omega_T)} + \|u\|_{H^m_\gamma(\Omega_T)}\|w\|_{L^\infty(\Omega_T)}\Big). \tag{2.3}$$

Furthermore, if $u \in W^{1,\infty}(\Omega_T)$, then

$$\left\|\partial^{\beta_1}[\partial^{\beta_2}, b(u)]\partial^{\beta_3} w\right\|_{L^2_\gamma(\Omega_T)} \le C\big(\|u\|_{W^{1,\infty}(\Omega_T)}\big)\Big(\|w\|_{H^{m-1}_\gamma(\Omega_T)} + \|u\|_{H^m_\gamma(\Omega_T)}\|w\|_{L^\infty(\Omega_T)}\Big). \tag{2.4}$$

Here β_i, for $i = 1, 2, 3$, are multi-indices, $[a, b]c := a(bc) - b(ac)$ denotes the usual commutator, and the increasing function C is independent of u, w, γ, and T. The same results hold with Ω_T replaced by ω_T.

We remark that the proof of the inequalities (2.1) and (2.2) can be found in [20, Section 4.5] and [12, Appendix C]. The inequalities (2.3) and (2.4) follow (2.1) and (2.2) through a straightforward calculation. We omit the proof.

3. Well-posedness of the linearized problem

In this section we shall consider the linearized problem for (1.13) and prove the well-posedness of solutions in the usual Sobolev spaces H^m for all integers m stated in Theorem 3.1 as in [12].

3.1. Variable coefficient linearized problem

Let us first perform the linearization for problem (1.13) around a basic state $(\check{U}^\pm, \check{\Phi}^\pm)$. We suppose that

$$\operatorname{supp}(\check{V}^\pm, \check{\Psi}^\pm) \subset \{-T \le t \le 2T,\ x_2 \ge 0,\ |x| \le 2\}, \tag{3.1}$$

$$\|\check{V}^\pm\|_{W^{2,\infty}(\Omega)} + \|\check{\Psi}^\pm\|_{W^{3,\infty}(\Omega)} \le K, \tag{3.2}$$

for $\check{V}^\pm := \check{U}^\pm - \overline{U}^\pm$ and $\check{\Psi}^\pm := \check{\Phi}^\pm - \overline{\Phi}^\pm$, where T and K are positive constants, and $(\overline{U}^\pm, \overline{\Phi}^\pm)$ is the background state defined by (1.19). Moreover, the basic state $(\check{U}^\pm, \check{\Phi}^\pm)$ is supposed to satisfy (1.11), (1.13b), and (1.16), *i.e.*,

$$\pm\,\partial_2\check{\Phi}^\pm \ge \kappa_0 > 0, \qquad x_2 \ge 0, \tag{3.3a}$$

$$\partial_t\check{\Phi}^\pm + \check{v}_1^\pm\partial_1\check{\Phi}^\pm - \check{v}_2^\pm = 0, \qquad x_2 \ge 0, \tag{3.3b}$$

$$\check{F}_{2j}^\pm = \check{F}_{1j}^\pm\partial_1\check{\Phi}^\pm \quad \text{for } j = 1, 2, \qquad x_2 \ge 0, \tag{3.3c}$$

$$\check{\Phi}^+ = \check{\Phi}^- = \check{\varphi}, \qquad x_2 = 0, \tag{3.3d}$$

$$\mathbb{B}\big(\check{U}^+, \check{U}^-, \check{\varphi}\big) = 0, \qquad x_2 = 0, \tag{3.3e}$$

for some positive constant κ_0. Constraints (3.3b) and (3.3c) keep the rank of the boundary matrix for the linearized problem being constant on $\overline{\Omega}$. Denote $\check{U} := (\check{U}^+, \check{U}^-)^\top$, $\check{V} := (\check{V}^+, \check{V}^-)^\top$, $\check{\Phi} := (\check{\Phi}^+, \check{\Phi}^-)^\top$, and $\check{\Psi} := (\check{\Psi}^+, \check{\Psi}^-)^\top$ for convenience.

The linearized operators read

$$\mathbb{L}'(U,\Phi)(V,\Psi) := (L(U,\Phi)+\mathcal{C}(U,\Phi))\,V - \frac{1}{\partial_2\Phi}(L(U,\Phi)\Psi)\partial_2 U, \tag{3.4}$$

$$\mathbb{B}'\big(\check{U},\check{\varphi}\big)(V,\psi) := \check{b}\nabla\psi + \check{B}V|_{x_2=0}, \tag{3.5}$$

where $V := (V^+, V^-)^\top$, and $\mathcal{C}(U,\Phi)$, $\check{b}$, and $\check{B}$ are respectively defined by

$$\mathcal{C}(U,\Phi)V := \big(\partial_{U_i}A_1(U)\partial_1 U + \partial_{U_i}\widetilde{A}_2(U,\Phi)\partial_2 U\big)\,V_i, \tag{3.6}$$

$$\check{b}(t,x_1) := \begin{pmatrix} 0 & (\check{v}_1^+ - \check{v}_1^-)|_{x_2=0} \\ 1 & \check{v}_1^+|_{x_2=0} \\ 0 & 0 \end{pmatrix}, \tag{3.7}$$

and

$$\check{B}(t,x_1) := \begin{pmatrix} 0 & \partial_1\check{\varphi} & -1 & 0 & 0 & 0 & 0 & 0 & -\partial_1\check{\varphi} & 1 & 0 & 0 & 0 & 0 \\ 0 & \partial_1\check{\varphi} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

As in Alinhac [1], we obtain

$$\mathbb{L}'(\check{U}^\pm,\check{\Phi}^\pm)(V^\pm,\Psi^\pm) = L(\check{U}^\pm,\check{\Phi}^\pm)\dot{V}^\pm + \mathcal{C}(\check{U}^\pm,\check{\Phi}^\pm)\dot{V}^\pm + \frac{\Psi^\pm}{\partial_2\check{\Phi}^\pm}\partial_2\mathbb{L}(\check{U}^\pm,\check{\Phi}^\pm), \tag{3.9}$$

where $\dot{V}^\pm$ are the "good unknowns"

$$\dot{V}^\pm := V^\pm - \frac{\partial_2\check{U}^\pm}{\partial_2\check{\Phi}^\pm}\Psi^\pm. \tag{3.10}$$

We now consider effective linear system:

$$\mathbb{L}'_e\big(\check{U}^\pm,\check{\Phi}^\pm\big)\dot{V}^\pm := L\big(\check{U}^\pm,\check{\Phi}^\pm\big)\dot{V}^\pm + \mathcal{C}(\check{U}^\pm,\check{\Phi}^\pm)\dot{V}^\pm = f^\pm, \quad x_2>0, \tag{3.11a}$$

$$\mathbb{B}'_e\big(\check{U},\check{\Phi}\big)(\dot{V},\psi) := \check{b}\nabla\psi + \check{b}_\natural\psi + \check{B}\dot{V}|_{x_2=0} = g, \quad x_2=0, \tag{3.11b}$$

$$\Psi^+ = \Psi^- = \psi, \quad x_2=0, \tag{3.11c}$$

where $L(\check{U}^\pm,\check{\Phi}^\pm)$, $\mathcal{C}(\check{U}^\pm,\check{\Phi}^\pm)$, $\check{b}$, and $\check{B}$ are given in (1.14), (3.6), (3.7), and (3.8), separately, $\dot{V} := (\dot{V}^+,\dot{V}^-)^\top$, and

$$\check{b}_\natural(t,x_1) := \check{B}(t,x_1)\begin{pmatrix} \partial_2\check{U}^+/\partial_2\check{\Phi}^+ \\ \partial_2\check{U}^-/\partial_2\check{\Phi}^- \end{pmatrix}\Bigg|_{x_2=0}. \tag{3.12}$$

Here, $\mathcal{C}(\check{U}^\pm,\check{\Phi}^\pm)$ are two smooth functions of $(\check{V}^\pm,\nabla\check{V}^\pm,\nabla\check{\Psi}^\pm)$ vanishing at the origin, $\check{b}$ is a smooth function of trace $\check{V}|_{x_2=0}$, $\check{b}_\natural$ is a smooth vector-function of $(\nabla\check{V}|_{x_2=0},\nabla\check{\Psi}|_{x_2=0})$

vanishing at the origin, and matrix $\check{B}$ is a smooth matrix-function of $\nabla\check{\varphi}$. Notice that the boundary condition (3.11b) depends on the traces of $\dot{V}$ solely through $\mathbb{P}(\check{\varphi})\dot{V}^{\pm}|_{x_2=0}$, where

$$\mathbb{P}(\check{\varphi})V^{\pm} := \big(V_1^{\pm},\ V_3^{\pm} - \partial_1\check{\varphi}V_2^{\pm}\big)^{\top}. \tag{3.13}$$

Let us convert linearized problem (3.11) into a problem with a constant diagonal boundary matrix. To this end, we define matrices

$$R(U,\Phi) := \begin{pmatrix} 0 & \langle\partial_1\Phi\rangle & \langle\partial_1\Phi\rangle & 0 & 0 & 0 & 0 \\ 1 & -\frac{c(\rho)}{\rho}\partial_1\Phi & \frac{c(\rho)}{\rho}\partial_1\Phi & 0 & 0 & 0 & 0 \\ \partial_1\Phi & \frac{c(\rho)}{\rho} & -\frac{c(\rho)}{\rho} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.14}$$

and

$$\widetilde{A}_0(U,\Phi) := \mathrm{diag}\left(1,\ \frac{\partial_2\Phi}{c(\rho)\langle\partial_1\Phi\rangle},\ -\frac{\partial_2\Phi}{c(\rho)\langle\partial_1\Phi\rangle},\ 1,\ 1,\ 1,\ 1\right), \tag{3.15}$$

where $\langle\partial_1\Phi\rangle := (1+(\partial_1\Phi)^2)^{1/2}$ and $c(\rho)$ is the sound speed given in (1.3). Then it follows from constraints (3.3b) and (3.3c) that

$$\widetilde{A}_0 R^{-1}\widetilde{A}_2 R\big(\check{U}^{\pm},\check{\Phi}^{\pm}\big) = \mathcal{I}_2 := \mathrm{diag}\big(0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0\big).$$

In terms of new unknowns

$$W^{\pm} := R^{-1}\big(\check{U}^{\pm},\check{\Phi}^{\pm}\big)\dot{V}^{\pm}, \tag{3.16}$$

the problem (3.11) can be rewritten equivalently as

$$\mathcal{A}_0^{\pm}\partial_t W^{\pm} + \mathcal{A}_1^{\pm}\partial_1 W^{\pm} + \mathcal{I}_2\partial_2 W^{\pm} + \mathcal{A}_3^{\pm}W^{\pm} = F^{\pm}, \quad x_2 > 0, \tag{3.17a}$$

$$\check{b}\nabla\psi + \check{b}_{\natural}\psi + \boldsymbol{B}W^{\mathrm{nc}} = g, \quad x_2 = 0, \tag{3.17b}$$

$$\Psi^{+} = \Psi^{-} = \psi, \quad x_2 = 0, \tag{3.17c}$$

where

$$\mathcal{A}_0^{\pm} := \widetilde{A}_0\big(\check{U}^{\pm},\check{\Phi}^{\pm}\big), \quad \mathcal{A}_1^{\pm} := \widetilde{A}_0 R^{-1}A_1 R\big(\check{U}^{\pm},\check{\Phi}^{\pm}\big), \quad F^{\pm} := \widetilde{A}_0 R^{-1}\big(\check{U}^{\pm},\check{\Phi}^{\pm}\big)f^{\pm},$$
$$\mathcal{A}_3^{\pm} := \widetilde{A}_0\big(R^{-1}\partial_t R + R^{-1}A_1\partial_1 R + R^{-1}\widetilde{A}_2\partial_2 R + R^{-1}\mathcal{C}R\big)\big(\check{U}^{\pm},\check{\Phi}^{\pm}\big).$$

In (3.17b), coefficients $\check{b}$ and $\check{b}_{\natural}$ are defined by (3.7) and (3.12), respectively,

$$\boldsymbol{B}(t,x_1) := \left.\begin{pmatrix} -\dfrac{c(\check{\rho})}{\check{\rho}}\langle\partial_1\check{\varphi}\rangle^2 & \dfrac{c(\check{\rho})}{\check{\rho}}\langle\partial_1\check{\varphi}\rangle^2 & \dfrac{c(\check{\rho})}{\check{\rho}}\langle\partial_1\check{\varphi}\rangle^2 & -\dfrac{c(\check{\rho})}{\check{\rho}}\langle\partial_1\check{\varphi}\rangle^2 \\ -\dfrac{c(\check{\rho})}{\check{\rho}}\langle\partial_1\check{\varphi}\rangle^2 & \dfrac{c(\check{\rho})}{\check{\rho}}\langle\partial_1\check{\varphi}\rangle^2 & 0 & 0 \\ \langle\partial_1\check{\varphi}\rangle & \langle\partial_1\check{\varphi}\rangle & -\langle\partial_1\check{\varphi}\rangle & -\langle\partial_1\check{\varphi}\rangle \end{pmatrix}\right|_{x_2=0}, \tag{3.18}$$

and $W^{\mathrm{nc}} := (W_+^{\mathrm{nc}}, W_-^{\mathrm{nc}})^\top$ denotes the noncharacteristic part of $W := (W^+, W^-)^\top$ with $W_\pm^{\mathrm{nc}} := (W_2^\pm, W_3^\pm)^\top$. Obviously, $\mathcal{A}_0^\pm$ and $\mathcal{A}_1^\pm$ are smooth functions of $(\check{V}^\pm, \nabla\check{\Psi}^\pm)$, $\mathcal{A}_3^\pm$ are smooth matrix-functions of $(\check{V}^\pm, \nabla\check{V}^\pm, \nabla\check{\Psi}^\pm, \nabla^2\check{\Psi}^\pm)$, and $\boldsymbol{B}$ is a smooth matrix-function of $(\check{V}|_{x_2=0}, \nabla\check{\varphi})$.

We are ready to show the following theorem in the rest of this section.

Theorem 3.1. *Let $T > 0$ and $m \in \mathbb{N}$ with $m \geq 2$ being fixed. Suppose that background state* (1.19) *satisfies* (1.20) *or* (1.21)*, and that $(\check{V}^\pm, \check{\Psi}^\pm)$ belong to $H_\gamma^{m+3}(\Omega_T)$ for all $\gamma \geq 1$, and satisfy* (3.1)–(3.3) *and*

$$\|(\check{V}^\pm, \check{\Psi}^\pm)\|_{H_\gamma^6(\Omega_T)} + \|(\check{V}^\pm, \check{\Psi}^\pm)\|_{H_\gamma^5(\omega_T)} \leq K. \tag{3.19}$$

Suppose further that source terms $(f, g) \in H^{m+1}(\Omega_T) \times H^{m+1}(\omega_T)$ vanish in the past. Then there exist constants $K_0 > 0$ and $\gamma_0 \geq 1$ such that, if $K \leq K_0$ and $\gamma \geq \gamma_0$, then problem (3.11) *has a unique solution $(\dot{V}^\pm, \psi) \in H^m(\Omega_T) \times H^{m+1}(\omega_T)$ vanishing in the past and satisfying tame estimate*

$$\begin{aligned} &\|\dot{V}\|_{H_\gamma^m(\Omega_T)} + \|\mathbb{P}(\check{\varphi})\dot{V}^\pm\|_{H_\gamma^m(\omega_T)} + \|\psi\|_{H_\gamma^{m+1}(\omega_T)} \\ &\quad \lesssim \|f\|_{H_\gamma^{m+1}(\Omega_T)} + \|g\|_{H_\gamma^{m+1}(\omega_T)} + \big(\|f\|_{H_\gamma^3(\Omega_T)} + \|g\|_{H_\gamma^3(\omega_T)}\big)\|(\check{V}^\pm, \check{\Psi}^\pm)\|_{H_\gamma^{m+3}(\Omega_T)}. \end{aligned} \tag{3.20}$$

When f and g vanish in the past (it is equivalent to zero initial data), Theorem 3.1 holds. The case of general initial data will be considered in Section 4 by constructing approximate solutions before the procedure of Nash–Moser scheme.

3.2. Well-posedness in L^2

Let us recall the following L^2 *a priori* energy estimate derived by [7] for the linearized problem (3.11).

Theorem 3.2 *([7, Theorem 2.1]). Suppose that background state $(\overline{U}^\pm, \overline{\Phi}^\pm)$ defined by* (1.19) *satisfies* (1.20) *or* (1.21)*, and basic state $\big(\check{U}^\pm, \check{\Phi}^\pm\big)$ satisfies* (3.1)–(3.3)*. Then there exist constants $K_0 > 0$ and $\gamma_0 \geq 1$ such that, if $K \leq K_0$ and $\gamma \geq \gamma_0$, then*

$$\begin{aligned} &\gamma\|\dot{V}\|_{L_\gamma^2(\Omega)}^2 + \|\mathbb{P}(\check{\varphi})\dot{V}\|_{L_\gamma^2(\partial\Omega)}^2 + \|\psi\|_{H_\gamma^1(\mathbb{R}^2)}^2 \\ &\quad \lesssim \gamma^{-3}\big\|\mathbb{L}_e'\big(\check{U}^\pm, \check{\Phi}^\pm\big)\dot{V}^\pm\big\|_{L^2(H_\gamma^1)}^2 + \gamma^{-2}\big\|\mathbb{B}_e'\big(\check{U}, \check{\Phi}\big)(\dot{V}, \psi)\big\|_{H_\gamma^1(\mathbb{R}^2)}^2 \end{aligned} \tag{3.21}$$

for all $(\dot{V}, \psi) \in H^2_\gamma(\Omega) \times H^2_\gamma(\mathbb{R}^2)$, *where operators* $\mathbb{P}(\check{\varphi})$, $\mathbb{L}'_e$, *and* $\mathbb{B}'_e$ *are defined by* (3.13), (3.11a), *and* (3.11b), *respectively.*

System (3.11a) is symmetrizable hyperbolic, whose coefficients satisfy the regularity assumptions of Coulombel [10]. It implies that we need to construct a dual problem that satisfies an appropriate energy estimate. Thus, we define

$$\check{B}_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\check{\varsigma}_1^- & 0 & 0 & 0 & 0 & 0 & 0 \\ \check{\varsigma}_1^+ & 0 & 0 & 0 & 0 & 0 & 0 & \check{\varsigma}_1^- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \check{\varsigma}_2^+ & \check{\varsigma}_3^+ & 0 & 0 & 0 & 0 & 0 & -\check{\varsigma}_2^- & -\check{\varsigma}_3^- & 0 & 0 & 0 & 0 \end{pmatrix}\Bigg|_{x_2=0}, \tag{3.22}$$

$$\check{D}_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \check{\varsigma}_2^+ & \check{\varsigma}_3^+ & 0 & 0 & 0 & 0 & 0 & \check{\varsigma}_2^- & \check{\varsigma}_3^- & 0 & 0 & 0 & 0 \end{pmatrix}\Bigg|_{x_2=0}, \tag{3.23}$$

$$\check{D} := \begin{pmatrix} 0 & \partial_1\check{\varphi} & -1 & 0 & 0 & 0 & 0 & 0 & \partial_1\check{\varphi} & -1 & 0 & 0 & 0 & 0 \\ 0 & \partial_1\check{\varphi} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\check{\varsigma}_1^\pm := -\frac{\check{\rho}^\pm}{\partial_2\check{\Phi}^\pm}, \qquad \check{\varsigma}_2^\pm := -\frac{c(\check{\rho}^\pm)^2\partial_1\check{\varphi}}{2\check{\rho}^\pm\partial_2\check{\Phi}^\pm}, \qquad \check{\varsigma}_3^\pm := \frac{c(\check{\rho}^\pm)^2}{2\check{\rho}^\pm\partial_2\check{\Phi}^\pm}.$$

Use (3.3b) and (3.3c) to calculate

$$\check{B}_1^\top\check{B} + \check{D}_1^\top\check{D} = \mathrm{diag}\big(\widetilde{A}_2(\check{U}^+, \check{\Phi}^+),\ \widetilde{A}_2(\check{U}^-, \check{\Phi}^-)\big)\big|_{x_2=0},$$

where $\check{B}$ is given in (3.8). Following [20, Section 3.2], we define the dual problem for (3.11) as

$$\begin{cases} \mathbb{L}'_e\big(\check{U}^\pm, \check{\Phi}^\pm\big)^*U^\pm = f^*_\pm, & x_2 > 0, \\ \check{D}_1U = 0, \quad \mathrm{div}(\check{b}^\top\check{B}_1U) - \check{b}_\natural^\top\check{B}_1U = 0, & x_2 = 0, \end{cases} \tag{3.24}$$

where $\check{b}$, $\check{b}_\natural$, $\check{B}_1$, and $\check{D}_1$ are given in (3.7), (3.12), (3.22), and (3.23), respectively, and symbol div denotes the divergence operator in $\mathbb{R}^2$. $\mathbb{L}'_e\big(\check{U}^\pm, \check{\Phi}^\pm\big)^*$ are the adjoint operators of $\mathbb{L}'_e\big(\check{U}^\pm, \check{\Phi}^\pm\big)$. Following the same analysis as in [12, Section 3.4], we can obtain the well-posedness result in L^2 for the linearized problem (3.11).

Theorem 3.3. *Let $T > 0$ be fixed. Suppose that $f \in L^2(\mathbb{R}_+; H^1(\omega_T))$ and $g \in H^1(\omega_T)$ vanish in the past and all the hypotheses in Theorem 3.2 are satisfied. Then constants $K_0 > 0$ and $\gamma_0 \geq 1$ exist such that, if $K \leq K_0$ and $\gamma \geq \gamma_0$, then there exists a unique solution $(\dot{V}^+, \dot{V}^-, \psi) \in L^2(\Omega_T) \times L^2(\Omega_T) \times H^1(\omega_T)$ for problem (3.11a)–(3.11b) that vanishes in the past and satisfies*

$$\gamma^{1/2}\|\dot{V}\|_{L^2_\gamma(\Omega_t)} + \|\mathbb{P}(\check{\varphi})\dot{V}\|_{L^2_\gamma(\omega_t)} + \|\psi\|_{H^1_\gamma(\omega_t)} \lesssim \gamma^{-3/2}\|f\|_{L^2(H^1_\gamma(\omega_t))} + \gamma^{-1}\|g\|_{H^1_\gamma(\omega_t)} \tag{3.25}$$

for all $\gamma \geq \gamma_0$ and $t \in [0, T]$.

For the reformulated problem (3.17), Theorem 3.3 implies estimate

$$\gamma^{1/2}\|W\|_{L^2_\gamma(\Omega_T)}+\|W^{\mathrm{nc}}\|_{L^2_\gamma(\omega_T)}+\|\psi\|_{H^1_\gamma(\omega_T)}\lesssim\gamma^{-3/2}\|F^{\pm}\|_{L^2(H^1_\gamma(\omega_T))}+\gamma^{-1}\|g\|_{H^1_\gamma(\omega_T)}. \tag{3.26}$$

For any nonnegative integer m, a generic and smooth matrix-valued function of $\{(\partial^\alpha\check{V},\partial^\alpha\check{\Psi}):|\alpha|\le m\}$ is denoted by $\check{\mathrm{c}}_m$, and by $\underline{\check{\mathrm{c}}}_m$ if it vanishes at the origin. For instance, the equations for $\dot{\rho}^{\pm}$ in (3.11a) can be rewritten as

$$(\partial_t^{\check{\Phi}^\pm}+\check{v}_\ell^\pm\partial_\ell^{\check{\Phi}^\pm})\dot{\rho}^\pm+\check{\rho}^\pm\partial_\ell^{\check{\Phi}^\pm}\dot{v}_\ell^\pm=\check{\mathrm{c}}_0 f+\underline{\check{\mathrm{c}}}_1\dot{V}. \tag{3.27}$$

The exact forms of $\check{\mathrm{c}}_m$ and $\underline{\check{\mathrm{c}}}_m$ may vary from line to line.

3.3. Tangential derivatives

The following lemma provides the estimate of the tangential derivatives.

Lemma 3.1. *If the hypotheses of Theorem 3.1 hold, then there exists a constant $\gamma_m\ge 1$, independent of T, such that*

$$\begin{aligned}&\gamma^{1/2}\|W\|_{L^2(H^m_\gamma(\omega_T))}+\|W^{\mathrm{nc}}\|_{H^m_\gamma(\omega_T)}+\|\psi\|_{H^{m+1}_\gamma(\omega_T)}\\&\quad\lesssim\gamma^{-3/2}\|F^{\pm}\|_{L^2(H^{m+1}_\gamma(\omega_T))}+\gamma^{-3/2}\|W\|_{L^\infty(\Omega_T)}\|(\check{V},\check{\Psi})\|_{H^{m+3}_\gamma(\Omega_T)}\\&\qquad+\gamma^{-1}\|g\|_{H^{m+1}_\gamma(\omega_T)}+\gamma^{-1}\|(W^{\mathrm{nc}},\psi)\|_{L^\infty(\omega_T)}\|(\check{V},\check{\Psi})\|_{H^{m+2}_\gamma(\omega_T)},\end{aligned} \tag{3.28}$$

for all $\gamma\ge\gamma_m$ and solutions $(W,\psi)\in H^{m+2}_\gamma(\Omega_T)\times H^{m+2}_\gamma(\omega_T)$ of problem (3.17).

Proof. We will follow [12, Proposition 1] to consider the enlarged system, but for the estimate of the source terms we use the Moser-type calculus inequalities (2.1)–(2.4) instead of the Gagliardo–Nirenberg's and Hölder's inequalities in [12, Proposition 1].

Let $\ell\in\mathbb{N}$ with $1\le\ell\le m$. Let $\alpha=(\alpha_0,\alpha_1,0)\in\mathbb{N}^3$ with $|\alpha|=\ell$ so that $\partial^\alpha=\partial_t^{\alpha_0}\partial_1^{\alpha_1}$ is a tangential derivative satisfying $\alpha_0+\alpha_1=\ell$. Then we apply operator ∂^α to (3.17a) and get

$$\begin{aligned}&\mathcal{A}_0^\pm\partial_t\partial^\alpha W^\pm+\mathcal{A}_1^\pm\partial_1\partial^\alpha W^\pm+\mathcal{I}_2\partial_2\partial^\alpha W^\pm+\mathcal{A}_3^\pm\partial^\alpha W^\pm\\&\quad+\sum_{|\beta|=1,\,\beta\le\alpha}C_{\alpha,\beta}\big(\partial^\beta\mathcal{A}_0^\pm\partial_t\partial^{\alpha-\beta}W^\pm+\partial^\beta\mathcal{A}_1^\pm\partial_1\partial^{\alpha-\beta}W^\pm\big)=\mathscr{F}_\pm^\alpha,\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}\mathscr{F}_\pm^\alpha:=\partial^\alpha F^\pm&+\sum_{0<\beta\le\alpha}C_{\alpha,\beta}\partial^\beta\mathcal{A}_3^\pm\partial^{\alpha-\beta}W^\pm\\&+\sum_{|\beta|\ge 2,\,\beta\le\alpha}C_{\alpha,\beta}\big(\partial^\beta\mathcal{A}_0^\pm\partial_t\partial^{\alpha-\beta}W^\pm+\partial^\beta\mathcal{A}_1^\pm\partial_1\partial^{\alpha-\beta}W^\pm\big).\end{aligned}$$

Similarly, from (3.17b), we have

$$\check{b}\nabla\partial^\alpha\psi + \check{b}_\natural\partial^\alpha\psi + \boldsymbol{B}\partial^\alpha W^{\mathrm{nc}} = \mathscr{G}^\alpha \quad \text{on } \omega_T, \tag{3.30}$$

where

$$\mathscr{G}^\alpha := \partial^\alpha g - [\partial^\alpha, \check{b}]\nabla\psi - [\partial^\alpha, \check{b}_\natural]\psi - [\partial^\alpha, \boldsymbol{B}]W^{\mathrm{nc}}.$$

Since the terms involving tangential derivatives of order ℓ in (3.29) do not only contain $\partial^\alpha W^\pm$, as in [12, Proposition 1], we write an enlarged system for all the tangential derivatives of order ℓ, in order to apply the L^2 *a priori* estimate in Theorem 3.2. Note that the last term on the left-hand side of (3.29) cannot be regarded simply as source terms due to the loss of derivatives in (3.21). Defining

$$W_\pm^{(\ell)} := \left\{\partial_t^{\alpha_0}\partial_1^{\alpha_1}W^\pm : \alpha_0+\alpha_1=\ell\right\}, \quad \psi^{(\ell)} := \left\{\partial_t^{\alpha_0}\partial_1^{\alpha_1}\psi : \alpha_0+\alpha_1=\ell\right\},$$

we obtain from (3.29)–(3.30) that

$$\mathscr{A}_0^\pm\partial_t W_\pm^{(\ell)} + \mathscr{A}_1^\pm\partial_1 W_\pm^{(\ell)} + \mathscr{I}\partial_2 W_\pm^{(\ell)} + \mathscr{C}^\pm W_\pm^{(\ell)} = \mathscr{F}_\pm^{(\ell)}, \tag{3.31a}$$

$$\mathscr{B}\nabla\psi^{(\ell)} + \mathscr{B}_\natural\psi^{(\ell)} + \mathscr{M}W_{\mathrm{nc}}^{(\ell)} = \mathscr{G}^{(\ell)}, \tag{3.31b}$$

where $\mathscr{A}_0^\pm$, $\mathscr{A}_1^\pm$, and $\mathscr{I}$ are block diagonal with blocks $\mathcal{A}_0^\pm$, $\mathcal{A}_1^\pm$, and $\mathcal{I}_2$, respectively. Matrices $\mathscr{C}^\pm$ belong to $W^{1,\infty}(\Omega)$. The source terms $\mathscr{F}_\pm^{(\ell)}$ and $\mathscr{G}^{(\ell)}$ consist of $\mathscr{F}_\pm^\alpha$ and $\mathscr{G}^\alpha$ for all $\alpha = (\alpha_0, \alpha_1, 0)$ with $|\alpha| = \ell$, respectively. The enlarged problem (3.31) satisfies an energy estimate similar to (3.26), *i.e.*,

$$\begin{aligned}\gamma^{1/2}\|W^{(\ell)}\|_{L^2_\gamma(\Omega_T)} &+ \|W_{\mathrm{nc}}^{(\ell)}\|_{L^2_\gamma(\omega_T)} + \|\psi^{(\ell)}\|_{H^1_\gamma(\omega_T)} \\ &\lesssim \gamma^{-3/2}\|\mathscr{F}^{(\ell)}\|_{L^2(H^1_\gamma(\omega_T))} + \gamma^{-1}\|\mathscr{G}^{(\ell)}\|_{H^1_\gamma(\omega_T)}.\end{aligned} \tag{3.32}$$

Let us now estimate the source terms $\mathscr{F}_\pm^{(\ell)}$ and $\mathscr{G}^{(\ell)}$ by Moser-type calculus inequalities (2.1)–(2.4). First, by definition, we have

$$\|\partial^\alpha F\|_{L^2(H^1_\gamma(\omega_T))} \lesssim \|(\gamma\partial^\alpha F, \partial_t\partial^\alpha F, \partial_1\partial^\alpha F)\|_{L^2_\gamma(\Omega_T)} \lesssim \|F\|_{L^2(H^{\ell+1}_\gamma(\omega_T))}, \tag{3.33}$$

$$\|\partial^\alpha g\|_{H^1_\gamma(\omega_T)} \lesssim \|g\|_{H^{\ell+1}_\gamma(\omega_T)}. \tag{3.34}$$

For $0 < \beta \leq \alpha$, we infer

$$\|\partial^\beta\mathcal{A}_3\partial^{\alpha-\beta}W\|_{H^1_\gamma(\omega_T)} \lesssim \|(\gamma\partial^\beta\mathcal{A}_3\partial^{\alpha-\beta}W, \nabla_{t,x_1}(\partial^\beta\mathcal{A}_3\partial^{\alpha-\beta}W))\|_{L^2_\gamma(\omega_T)}. \tag{3.35}$$

Apply Moser-type calculus inequality (2.1) to deduce that

$$\begin{aligned}\|\partial^\beta\mathcal{A}_3\partial^{\alpha-\beta}W\|_{L^2_\gamma(\omega_T)} &= \|\partial^{\beta-\beta'}(\partial^{\beta'}\mathcal{A}_3)\partial^{\alpha-\beta}W\|_{L^2_\gamma(\omega_T)} \\ &\lesssim \|\partial^{\beta'}\mathcal{A}_3\|_{L^\infty(\omega_T)}\|W\|_{H^{\ell-1}_\gamma(\omega_T)} + \|\partial^{\beta'}\mathcal{A}_3\|_{H^{\ell-1}_\gamma(\omega_T)}\|W\|_{L^\infty(\omega_T)} \\ &\lesssim \|W\|_{H^{\ell-1}_\gamma(\omega_T)} + \|(\check{V}, \check{\Psi})\|_{H^{\ell+2}_\gamma(\omega_T)}\|W\|_{L^\infty(\omega_T)},\end{aligned} \tag{3.36}$$

where $\beta' \leq \beta$ with $|\beta'| = 1$. Similarly, we have

$$\|\nabla_{t,x_1}(\partial^\beta \mathcal{A}_3 \partial^{\alpha-\beta} W)\|_{L^2_\gamma(\omega_T)} \lesssim \|W\|_{H^\ell_\gamma(\omega_T)} + \|(\check{V}, \check{\Psi})\|_{H^{\ell+3}_\gamma(\omega_T)} \|W\|_{L^\infty(\omega_T)},$$

which combined with (3.35) and (3.36) implies

$$\|\partial^\beta \mathcal{A}_3 \partial^{\alpha-\beta} W\|_{L^2(H^1_\gamma(\omega_T))} \lesssim \|W\|_{L^2(H^\ell_\gamma(\omega_T))} + \|(\check{V}, \check{\Psi})\|_{H^{\ell+3}_\gamma(\Omega_T)} \|W\|_{L^\infty(\Omega_T)}. \tag{3.37}$$

For $\beta \leq \alpha$ with $|\beta| \geq 2$, similar to (3.37), we use (2.1) to derive

$$\begin{aligned} &\|\partial^\beta \mathcal{A}_0 \partial_t \partial^{\alpha-\beta} W\|_{L^2(H^1_\gamma(\omega_T))} + \|\partial^\beta \mathcal{A}_1 \partial_1 \partial^{\alpha-\beta} W\|_{L^2(H^1_\gamma(\omega_T))} \\ &\quad \lesssim \|W\|_{L^2(H^\ell_\gamma(\omega_T))} + \|(\check{V}, \check{\Psi})\|_{H^{\ell+3}_\gamma(\Omega_T)} \|W\|_{L^\infty(\Omega_T)}. \end{aligned} \tag{3.38}$$

Combining (3.33), (3.37), and (3.38) leads to

$$\|\mathscr{F}^{(\ell)}\|_{L^2(H^1_\gamma(\omega_T))} \lesssim \|F\|_{L^2(H^{\ell+1}_\gamma(\omega_T))} + \|W\|_{L^2(H^\ell_\gamma(\omega_T))} + \|(\check{V}, \check{\Psi})\|_{H^{\ell+3}_\gamma(\Omega_T)} \|W\|_{L^\infty(\Omega_T)}. \tag{3.39}$$

Using (2.3)–(2.4), we obtain

$$\begin{aligned} \|[\partial^\alpha, \check{b}]\nabla\psi\|_{H^1_\gamma(\omega_T)} &\lesssim \gamma \|[\partial^\alpha, \check{b}]\nabla\psi\|_{L^2_\gamma(\omega_T)} + \sum_{|\beta|=1} \|\partial^\beta [\partial^\alpha, \check{b}]\nabla\psi\|_{L^2_\gamma(\omega_T)} \\ &\lesssim \|\psi\|_{H^{\ell+1}_\gamma(\omega_T)} + \|\check{\underline{c}}_0\|_{H^{\ell+2}_\gamma(\omega_T)} \|\psi\|_{L^\infty(\omega_T)} \\ &\lesssim \|\psi\|_{H^{\ell+1}_\gamma(\omega_T)} + \|(\check{V}, \check{\Psi})\|_{H^{\ell+2}_\gamma(\omega_T)} \|\psi\|_{L^\infty(\omega_T)}. \end{aligned}$$

Applying Moser-type calculus inequalities (2.3)–(2.4) to the other terms in $\mathscr{G}^\alpha$, we get

$$\begin{aligned} \|\mathscr{G}^{(\ell)}\|_{H^1_\gamma(\omega_T)} &\lesssim \|g\|_{H^{\ell+1}_\gamma(\omega_T)} + \|W^{\rm nc}\|_{H^\ell_\gamma(\omega_T)} + \|\psi\|_{H^{\ell+1}_\gamma(\omega_T)} \\ &\quad + \|(\check{V}, \check{\Psi})\|_{H^{\ell+2}_\gamma(\omega_T)} \|(W^{\rm nc}, \psi)\|_{L^\infty(\omega_T)}. \end{aligned} \tag{3.40}$$

Substitute (3.39) and (3.40) into (3.32), multiply the resulting estimate by $\gamma^{m-\ell}$, sum over ℓ from 0 to m, and take γ large enough to conclude the desired tame estimate (3.28). The proof of this lemma is complete. □

3.4. Normal derivatives of the noncharacteristic variables

Following [32], we compensate the loss of normal derivatives through the estimates of the linearized divergences and vorticities. According to (3.17a), we have

$$\begin{pmatrix} 0 \\ \partial_2 W^{\rm nc}_\pm \\ 0 \end{pmatrix} = F^\pm - \mathcal{A}^\pm_0 \partial_t W^\pm - \mathcal{A}^\pm_1 \partial_1 W^\pm - \mathcal{A}^\pm_3 W^\pm, \tag{3.41}$$

which leads to

$$\|\partial_2 W^{\mathrm{nc}}\|_{L^2(H^{m-1}_\gamma(\omega_T))} \lesssim \|(F, \check{\mathrm{c}}_1\partial_t W, \check{\mathrm{c}}_1\partial_1 W, \underline{\check{\mathrm{c}}}_2 W)\|_{L^2(H^{m-1}_\gamma(\omega_T))}.$$

It follows from (2.1)–(2.2) that

$$\begin{aligned}\|\underline{\check{\mathrm{c}}}_2 W\|_{H^{m-1}_\gamma(\omega_T)} &\lesssim \|\underline{\check{\mathrm{c}}}_2\|_{L^\infty(\omega_T)}\|W\|_{H^{m-1}_\gamma(\omega_T)} + \|\underline{\check{\mathrm{c}}}_2\|_{H^{m-1}_\gamma(\omega_T)}\|W\|_{L^\infty(\omega_T)}\\ &\lesssim \|W\|_{H^{m-1}_\gamma(\omega_T)} + \|(\check{V}, \check{\Psi})\|_{H^{m+1}_\gamma(\omega_T)}\|W\|_{L^\infty(\omega_T)},\end{aligned}$$

and

$$\|\underline{\check{\mathrm{c}}}_1 W\|_{H^m_\gamma(\omega_T)} \lesssim \|W\|_{H^m_\gamma(\omega_T)} + \|(\check{V}, \check{\Psi})\|_{H^{m+1}_\gamma(\omega_T)}\|W\|_{L^\infty(\omega_T)}.$$

Since

$$\begin{aligned}\|\check{\mathrm{c}}_1\nabla_{t,x_1} W\|_{H^{m-1}_\gamma(\omega_T)} &\lesssim \|\check{\mathrm{c}}_1 W\|_{H^m_\gamma(\omega_T)} + \|\nabla_{t,x_1}\check{\mathrm{c}}_1 W\|_{H^{m-1}_\gamma(\omega_T)}\\ &\lesssim \|W\|_{H^m_\gamma(\omega_T)} + \|\underline{\check{\mathrm{c}}}_1 W\|_{H^m_\gamma(\omega_T)} + \|\underline{\check{\mathrm{c}}}_2 W\|_{H^{m-1}_\gamma(\omega_T)},\end{aligned}$$

we combine the estimates above to get

$$\begin{aligned}\|\partial_2 W^{\mathrm{nc}}\|_{L^2(H^{m-1}_\gamma(\omega_T))} &\lesssim \|F\|_{H^{m-1}_\gamma(\Omega_T)} + \|W\|_{L^2(H^m_\gamma(\omega_T))}\\ &\quad + \|(\check{V}, \check{\Psi})\|_{L^2(H^{m+1}_\gamma(\omega_T))}\|W\|_{L^\infty(\Omega_T)}.\end{aligned} \tag{3.42}$$

Next, we introduce the linearized divergences and vorticities whose estimates enable us to recover the normal derivatives of the characteristic variables

$$(W_1^\pm, W_4^\pm, W_5^\pm, W_6^\pm, W_7^\pm) = \left(\frac{\dot{v}_1^\pm + \partial_1\check{\Phi}^\pm \dot{v}_2^\pm}{\langle\partial_1\check{\Phi}^\pm\rangle^2}, \dot{F}_{11}^\pm, \dot{F}_{21}^\pm, \dot{F}_{12}^\pm, \dot{F}_{22}^\pm\right), \tag{3.43}$$

according to transformation (3.16).

3.5. Divergences

Inspired by involutions (1.17), we introduce linearized divergences $\zeta_1^\pm$ and $\zeta_2^\pm$ by

$$\zeta_j^\pm := \partial_i^{\check{\Phi}^\pm}\left(\check{\rho}_\pm \dot{F}_{ij}^\pm + \check{F}_{ij}^\pm \dot{\rho}^\pm\right), \qquad j = 1, 2, \tag{3.44}$$

where partial differentials $\partial_i^{\check{\Phi}^\pm}$, $i = 1, 2$, are defined by (1.18). We have the following estimate for $\zeta_1^\pm$ and $\zeta_2^\pm$.

Lemma 3.2 *(Estimate of divergences). If the hypotheses of Theorem 3.1 hold, then there exists a constant $\gamma_m \geq 1$, independent of T, such that*

$$\gamma\|(\zeta_1^{\pm},\zeta_2^{\pm})\|_{H_\gamma^{m-1}(\Omega_T)} \lesssim \|(W,f)\|_{H_\gamma^m(\Omega_T)} + \|(\check{V},\check{\Psi})\|_{H_\gamma^{m+2}(\Omega_T)}\|(W,f)\|_{L^\infty(\Omega_T)}, \quad (3.45)$$

for all $\gamma \geq \gamma_m$ and solutions $(W,\psi) \in H_\gamma^{m+2}(\Omega_T) \times H_\gamma^{m+2}(\omega_T)$ of problem (3.17).

Proof. The equations for $\dot{F}_{ij}$ in (3.11a) read

$$(\partial_t^{\check{\Phi}} + \check{v}_\ell \partial_\ell^{\check{\Phi}})\dot{F}_{ij} - \check{F}_{\ell j}\partial_\ell^{\check{\Phi}}\dot{v}_i = \check{c}_0 f + \underline{\check{c}}_1 \dot{V}. \quad (3.46)$$

By using equations (3.27) and (3.46), we apply operator $\partial_i^{\check{\Phi}}$ and use

$$\check{\rho}\check{F}_{\ell 1}\partial_i^{\check{\Phi}}\partial_\ell^{\check{\Phi}}\dot{v}_i - \check{\rho}\check{F}_{i1}\partial_i^{\check{\Phi}}\partial_\ell^{\check{\Phi}}\dot{v}_\ell = \check{\rho}\check{F}_{i1}\big[\partial_\ell^{\check{\Phi}},\partial_i^{\check{\Phi}}\big]\dot{v}_\ell = \check{c}_2\nabla\dot{V}$$

to discover

$$(\partial_t^{\check{\Phi}} + \check{v}_\ell\partial_\ell^{\check{\Phi}})\zeta_j = \check{c}_1\nabla f + \check{c}_1 f + \check{c}_2\nabla W + \check{c}_2 W. \quad (3.47)$$

Applying operator $\mathrm{e}^{-\gamma t}\partial^\alpha$ with $|\alpha| \leq m-1$ to (3.47) yields

$$\begin{aligned}(\partial_t^{\check{\Phi}} &+ \check{v}_\ell\partial_\ell^{\check{\Phi}})\big(\mathrm{e}^{-\gamma t}\partial^\alpha\zeta_j\big) + \gamma\mathrm{e}^{-\gamma t}\partial^\alpha\zeta_j \\ &= \mathrm{e}^{-\gamma t}\partial^\alpha(\check{c}_1\nabla f + \check{c}_1 f + \check{c}_2\nabla W + \check{c}_2 W) - \mathrm{e}^{-\gamma t}[\partial^\alpha, \partial_t^{\check{\Phi}} + \check{v}_\ell\partial_\ell^{\check{\Phi}}]\zeta_j.\end{aligned}$$

We multiply the last identity by $\mathrm{e}^{-\gamma t}\partial^\alpha\zeta_j$ and integrate over Ω_T to infer

$$\begin{aligned}\gamma\|\partial^\alpha\zeta_j\|_{L^2_\gamma(\Omega_T)} &\lesssim \|\partial^\alpha(\check{c}_1\nabla f + \check{c}_1 f + \check{c}_2\nabla W + \check{c}_2 W)\|_{L^2_\gamma(\Omega_T)} \\ &\quad + \|[\partial^\alpha, \partial_t^{\check{\Phi}} + \check{v}_\ell\partial_\ell^{\check{\Phi}}]\zeta_j\|_{L^2_\gamma(\Omega_T)}, \end{aligned} \quad (3.48)$$

for $\gamma \geq 1$ sufficiently large, where we have used

$$\partial_t^{\check{\Phi}} + \check{v}_\ell\partial_\ell^{\check{\Phi}} = \partial_t + \check{v}_1\partial_1 \qquad \text{if } x_2 \geq 0, \quad (3.49)$$

owing to constraints (3.3b).

From Moser-type calculus inequality (2.3), we obtain

$$\begin{aligned}\|\partial^\alpha(\check{c}_1\nabla f + \check{c}_1 f)\|_{L^2_\gamma(\Omega_T)} &\lesssim \|(\check{c}_1\partial^\alpha\nabla f, \check{c}_1\partial^\alpha f)\|_{L^2_\gamma(\Omega_T)} + \|([\partial^\alpha,\check{c}_1]\nabla f, [\partial^\alpha,\check{c}_1]f)\|_{L^2_\gamma(\Omega_T)} \\ &\lesssim \|f\|_{H_\gamma^{|\alpha|+1}(\Omega_T)} + \|(\check{V},\check{\Psi})\|_{H_\gamma^{|\alpha|+2}(\Omega_T)}\|f\|_{L^\infty(\Omega_T)}. \end{aligned} \quad (3.50)$$

Since $\zeta_j = \check{c}_1 W + \check{c}_1\nabla W$, we apply Moser-type calculus inequalities (2.3)–(2.4) to deduce that

$$
\begin{aligned}
&\|\partial^\alpha(\check{c}_2\nabla W+\check{c}_2 W)\|_{L^2_\gamma(\Omega_T)}+\|[\partial^\alpha,\partial_t^{\check{\Phi}}+\check{v}_\ell\partial_\ell^{\check{\Phi}}]\zeta_j\|_{L^2_\gamma(\Omega_T)}\\
&\lesssim\|(\check{c}_2\partial^\alpha\nabla W,\check{c}_2\partial^\alpha W,[\partial^\alpha,\check{c}_2]W,[\partial^\alpha,\check{c}_2]\nabla W,[\partial^\alpha,\check{c}_1]\nabla^2 W)\|_{L^2_\gamma(\Omega_T)}\\
&\lesssim\|W\|_{H^{|\alpha|+1}_\gamma(\Omega_T)}+\|(\check{V},\check{\Psi})\|_{H^{|\alpha|+3}_\gamma(\Omega_T)}\|W\|_{L^\infty(\Omega_T)}.
\end{aligned}
\tag{3.51}
$$

Substituting (3.50) and (3.51) into (3.48) implies

$$
\gamma^{m-|\alpha|}\|\partial^\alpha\zeta_j\|_{L^2_\gamma(\Omega_T)}\lesssim\|(W,f)\|_{H^m_\gamma(\Omega_T)}+\|(\check{V},\check{\Psi})\|_{H^{m+2}_\gamma(\Omega_T)}\|(W,f)\|_{L^\infty(\Omega_T)},
$$

from which we conclude estimate (3.45) and finish the proof of this lemma. □

3.6. Vorticities

The linearized vorticities $\xi^\pm$ for velocities $\dot{v}^\pm$ and the linearized vorticities $\eta_j^\pm$ for columns $\dot{F}_j^\pm$ of the deformation gradient are defined as

$$
\xi^\pm:=\partial_1^{\check{\Phi}^\pm}\dot{v}_2^\pm-\partial_2^{\check{\Phi}^\pm}\dot{v}_1^\pm, \tag{3.52}
$$

$$
\eta_j^\pm:=\partial_1^{\check{\Phi}^\pm}\dot{F}_{2j}^\pm-\partial_2^{\check{\Phi}^\pm}\dot{F}_{1j}^\pm, \tag{3.53}
$$

for $j=1,2$. The following lemma gives the estimate of $\xi^\pm$, $\eta_1^\pm$, and $\eta_2^\pm$.

Lemma 3.3 *(Estimate of vorticities). If the hypotheses of Theorem 3.1 hold, then there exists a constant $\gamma_m\geq 1$, independent of T, such that*

$$
\gamma\|(\xi^\pm,\eta_1^\pm,\eta_2^\pm)\|_{H^{m-1}_\gamma(\Omega_T)}\lesssim\|(W,f)\|_{H^m_\gamma(\Omega_T)}+\|(\check{V},\check{\Psi})\|_{H^{m+2}_\gamma(\Omega_T)}\|(W,f)\|_{L^\infty(\Omega_T)},
\tag{3.54}
$$

for all $\gamma\geq\gamma_m$ and solutions $(W,\psi)\in H^{m+2}_\gamma(\Omega_T)\times H^{m+2}_\gamma(\omega_T)$ of problem (3.17).

Proof. The equations for $\dot{v}_1$ and $\dot{v}_2$ in (3.11a) read

$$
(\partial_t^{\check{\Phi}}+\check{v}_\ell\partial_\ell^{\check{\Phi}})\dot{v}_i-\check{F}_{\ell j}\partial_\ell^{\check{\Phi}}\dot{F}_{ij}+\frac{c(\check{\rho})^2}{\check{\rho}}\partial_i^{\check{\Phi}}\dot{\rho}=\check{c}_0 f+\underline{\check{c}}_1\dot{V}, \tag{3.55}
$$

which implies the transport equation

$$
(\partial_t^{\check{\Phi}}+\check{v}_\ell\partial_\ell^{\check{\Phi}})\xi-\check{F}_{\ell j}\partial_\ell^{\check{\Phi}}\eta_j=\check{c}_1\nabla f+\check{c}_1 f+\check{c}_2\nabla W+\check{c}_2 W. \tag{3.56}
$$

Moreover, it follows from (3.46) that

$$
(\partial_t^{\check{\Phi}}+\check{v}_\ell\partial_\ell^{\check{\Phi}})\eta_j-\check{F}_{\ell j}\partial_\ell^{\check{\Phi}}\xi=\check{c}_1\nabla f+\check{c}_1 f+\check{c}_2\nabla W+\check{c}_2 W. \tag{3.57}
$$

Apply operator $\mathrm{e}^{-\gamma t}\partial^\alpha$ with $|\alpha|\leq m-1$ to (3.56) (*resp.* (3.57)) and multiply the resulting identity by $\mathrm{e}^{-\gamma t}\partial^\alpha\xi$ (*resp.* $\mathrm{e}^{-\gamma t}\partial^\alpha\eta_j$) to obtain

$$\begin{aligned}
&\frac{1}{2}(\partial_t^{\check{\Phi}} + \check{v}_\ell \partial_\ell^{\check{\Phi}})\Big\{|\mathrm{e}^{-\gamma t}\partial^\alpha \xi|^2 + |\mathrm{e}^{-\gamma t}\partial^\alpha \eta_1|^2 + |\mathrm{e}^{-\gamma t}\partial^\alpha \eta_2|^2\Big\} \\
&\quad - \check{F}_{\ell j}\partial_\ell^{\check{\Phi}}\left(\mathrm{e}^{-2\gamma t}\partial^\alpha \xi \partial^\alpha \eta_j\right) + \gamma\Big\{|\mathrm{e}^{-\gamma t}\partial^\alpha \xi|^2 + |\mathrm{e}^{-\gamma t}\partial^\alpha \eta_1|^2 + |\mathrm{e}^{-\gamma t}\partial^\alpha \eta_2|^2\Big\} \\
&= \mathrm{e}^{-2\gamma t}\partial^\alpha \xi\left\{\partial^\alpha\left(\check{c}_1 \nabla f + \check{c}_1 f + \check{c}_2 \nabla W + \check{c}_2 W\right) - [\partial^\alpha, \partial_t^{\check{\Phi}} + \check{v}_\ell \partial_\ell^{\check{\Phi}}]\xi\right\} \\
&\quad + \mathrm{e}^{-2\gamma t}\partial^\alpha \eta_j\left\{\partial^\alpha\left(\check{c}_1 \nabla f + \check{c}_1 f + \check{c}_2 \nabla W + \check{c}_2 W\right) - [\partial^\alpha, \partial_t^{\check{\Phi}} + \check{v}_\ell \partial_\ell^{\check{\Phi}}]\eta_j\right\} \\
&\quad + \mathrm{e}^{-2\gamma t}\left\{\partial^\alpha \xi[\partial^\alpha, \check{F}_{\ell j}\partial_\ell^{\check{\Phi}}]\eta_j + \partial^\alpha \eta_j[\partial^\alpha, \check{F}_{\ell j}\partial_\ell^{\check{\Phi}}]\xi\right\}. \qquad (3.58)
\end{aligned}$$

It follows from constraints (3.3c) that

$$\check{F}_{\ell j}\partial_\ell^{\check{\Phi}} = \check{F}_{1j}\partial_1, \quad x_2 \geq 0.$$

Then we integrate identity (3.58) over Ω_T and perform the similar analysis as ζ_j in Lemma 3.2 to obtain the desired estimate (3.54). The proof of the lemma is complete. □

3.7. *Proof of Theorem 3.1*

Thanks to Lemmas 3.2 and 3.3, we can derive the estimate for the normal derivative of characteristic variables defined by (3.43). More precisely, in view of (3.43), (3.52), and (1.18), we obtain

$$\xi^\pm = -\frac{1}{\partial_2 \check{\Phi}^\pm}\partial_2\left(\langle \partial_1 \check{\Phi}^\pm \rangle^2 W_1^\pm\right) + \check{c}_1 \partial_1 W + \check{c}_2 W,$$

which implies

$$\partial_2 W_1^\pm = \check{c}_1 \xi^\pm + \check{c}_1 \partial_1 W + \check{c}_2 W. \qquad (3.59)$$

Similarly, it follows from (3.44) and (3.53) that

$$\partial_2 \dot{F}_{ij}^\pm = \check{c}_1 \zeta_j^\pm + \check{c}_1 \eta_j^\pm + \check{c}_1 \partial_1 W + \check{c}_2 W, \qquad (3.60)$$

for $i, j = 1, 2$. Thanks to identities (3.59)–(3.60), we apply Moser-type calculus inequalities (2.1)–(2.4) and use (3.42), (3.45), and (3.54) to infer that

$$\begin{aligned}
\|\partial_2^k W\|_{L^2(H_\gamma^{m-k}(\omega_T))} \lesssim{}& \|W\|_{L^2(H_\gamma^m(\omega_T))} + \gamma^{-1}\|(W, f)\|_{H_\gamma^m(\Omega_T)} \\
&+ \gamma^{-1}\|(\check{V}, \check{\Psi})\|_{H_\gamma^{m+2}(\Omega_T)}\|(W, f)\|_{L^\infty(\Omega_T)} \qquad (3.61)
\end{aligned}$$

holds for $k = 1$.

Taking advantage of identities (3.41), (3.59), and (3.60), we can combine estimates (3.45) and (3.54) to prove (3.61) by finite induction in $k = 1, \ldots, m$. Since

$$\|W\|_{H_\gamma^m(\Omega_T)} \sim \sum_{k=0}^m \|\partial_2^k W\|_{L^2(H_\gamma^{m-k}(\omega_T))},$$

we combine (3.28) and (3.61) to get for γ sufficiently large,

$$\begin{aligned}&\gamma^{1/2}\|W\|_{H^m_\gamma(\Omega_T)}+\|W^{\mathrm{nc}}|_{x_2=0}\|_{H^m_\gamma(\omega_T)}+\|\psi\|_{H^{m+1}_\gamma(\omega_T)}\\&\quad\lesssim\gamma^{-1/2}\|f\|_{H^m_\gamma(\Omega_T)}+\gamma^{-3/2}\|f\|_{L^2(H^{m+1}_\gamma(\omega_T))}+\gamma^{-1}\|g\|_{H^{m+1}_\gamma(\omega_T)}\\&\qquad+\gamma^{-1}\|(W,f)\|_{L^\infty(\Omega_T)}\big\|(\check{V},\check{\Psi})\big\|_{H^{m+3}_\gamma(\Omega_T)}+\gamma^{-1}\|(W^{\mathrm{nc}},\psi)\|_{L^\infty(\omega_T)}\big\|(\check{V},\check{\Psi})\big\|_{H^{m+2}_\gamma(\omega_T)}.\end{aligned}\tag{3.62}$$

Theorem 3.3 gives the well-posedness of the effective linear problem (3.11) for source terms $(f^\pm,g)\in L^2(H^1(\omega_T))\times H^1(\omega_T)$ vanishing in the past. Following [3,25], we can use tame estimate (3.62) to transform Theorem 3.3 into a well-posedness formulation of (3.11) in H^m. To be more precise, following Theorem 3.1, there exists a unique solution $(\dot{V}^\pm,\psi)\in H^m(\Omega_T)\times H^{m+1}(\omega_T)$ that vanishes in the past and satisfies (3.62) for all $\gamma\ge\gamma_m$.

Finally, the tame estimate (3.20) can be derived as follows. By the Sobolev embedding inequalities $\|W\|_{L^\infty(\Omega_T)}\lesssim\|W\|_{H^2_\gamma(\Omega_T)}$ and $\|\psi\|_{W^{1,\infty}(\omega_T)}\lesssim\|\psi\|_{H^3_\gamma(\omega_T)}$, as well as (3.62) with $m=2$, one has,

$$\|W\|_{L^\infty(\Omega_T)}+\|\psi\|_{W^{1,\infty}(\omega_T)}\le C_{T,\gamma}\left(\|f\|_{H^3_\gamma(\Omega_T)}+\|g\|_{H^3_\gamma(\omega_T)}\right).\tag{3.63}$$

Substituting (3.63) into (3.62) yields the tame estimate (3.20). The proof of Theorem 3.1 is completed. □

4. Compatibility conditions and approximate solutions

To apply Theorem 3.1 in the general setting, as in [12] we need to transform the original nonlinear problem (1.11)–(1.13) into the case with zero initial data. To this end, in this section the approximate solutions are introduced to incorporate the initial data into the interior equations. The necessary compatibility conditions are imposed on the initial data for the construction of smooth approximate solutions.

4.1. Compatibility conditions

Let $m\in\mathbb{N}$ with $m\ge 3$. Assume that the initial data $(U_0^\pm,\varphi_0)$ satisfy $\widetilde{U}_0^\pm:=U_0^\pm-\overline{U}^\pm\in H^{m+1/2}(\mathbb{R}^2_+)$ and $\varphi_0\in H^{m+1}(\mathbb{R})$, and $(\widetilde{U}_0^\pm,\varphi_0)$ has the following compact support,

$$\operatorname{supp}\widetilde{U}_0^\pm\subset\{x_2\ge 0,\ x_1^2+x_2^2\le 1\},\qquad\operatorname{supp}\varphi_0\subset[-1,1].\tag{4.1}$$

Taking advantage of the trace theorem, we can construct $\widetilde{\Phi}_0^+=\widetilde{\Phi}_0^-\in H^{m+3/2}(\mathbb{R}^2_+)$ satisfying

$$\widetilde{\Phi}_0^\pm|_{x_2=0}=\varphi_0,\quad\operatorname{supp}\widetilde{\Phi}_0^\pm\subset\left\{x_2\ge 0,\ x_1^2+x_2^2\le 2\right\},\tag{4.2}$$

$$\|\widetilde{\Phi}_0^\pm\|_{H^{m+3/2}(\mathbb{R}^2_+)}\le C\|\varphi_0\|_{H^{m+1}(\mathbb{R})}.\tag{4.3}$$

Define $\Phi_0^\pm:=\widetilde{\Phi}_0^\pm+\overline{\Phi}_0^\pm$, which is the initial data for the problem (1.11),

$$\Phi^{\pm}|_{t=0} = \Phi_0^{\pm}. \tag{4.4}$$

By (4.3) and the Sobolev embedding theorem, we have

$$\pm\partial_2\Phi_0^{\pm} \geq 7/8 \qquad \text{for all } x \in \mathbb{R}^2_+, \tag{4.5}$$

for sufficiently small φ_0 in $H^{m+1}(\mathbb{R})$.

Denote the perturbation by $(\widetilde{U}^{\pm}, \widetilde{\Phi}^{\pm}) := (U^{\pm} - \overline{U}^{\pm}, \Phi^{\pm} - \overline{\Phi}^{\pm})$, and the traces of the k-th order time derivatives on $\{t = 0\}$ by

$$\widetilde{U}^{\pm}_{(k)} := \partial_t^k \widetilde{U}^{\pm}\Big|_{t=0}, \quad \widetilde{\Phi}^{\pm}_{(k)} := \partial_t^k \widetilde{\Phi}^{\pm}\Big|_{t=0} \qquad \text{for } \; k \in \mathbb{N}. \tag{4.6}$$

Note $\widetilde{U}^{\pm}_{(0)} = \widetilde{U}^{\pm}_0$ and $\widetilde{\Phi}^{\pm}_{(0)} = \widetilde{\Phi}^{\pm}_0$.

If we denote $\mathcal{W}^{\pm} := (\widetilde{U}^{\pm}, \nabla_x \widetilde{U}^{\pm}, \nabla_x \widetilde{\Phi}^{\pm})^{\top} \in \mathbb{R}^{23}$, then the first equation of (1.11) and the equation (1.13a) can be written as

$$\partial_t \widetilde{\Phi}^{\pm} = \mathbf{G}_1(\mathcal{W}^{\pm}), \qquad \partial_t \widetilde{U}^{\pm} = \mathbf{G}_2(\mathcal{W}^{\pm}), \tag{4.7}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are C^{∞}–functions vanishing at the origin. We apply ∂_t^k to (4.7), take the traces initially, and adopt the generalized Faà di Bruno's formula (see [21, Theorem 2.1]) to derive

$$\widetilde{\Phi}^{\pm}_{(k+1)} = \sum_{\alpha_i \in \mathbb{N}^{23}, |\alpha_1| + \cdots + k|\alpha_k| = k} D^{\alpha_1 + \cdots + \alpha_k} \mathbf{G}_1(\mathcal{W}^{\pm}_{(0)}) \prod_{i=1}^{k} \frac{k!}{\alpha_i!} \left(\frac{\mathcal{W}^{\pm}_{(i)}}{i!} \right)^{\alpha_i}, \tag{4.8}$$

$$\widetilde{U}^{\pm}_{(k+1)} = \sum_{\alpha_i \in \mathbb{N}^{23}, |\alpha_1| + \cdots + k|\alpha_k| = k} D^{\alpha_1 + \cdots + \alpha_k} \mathbf{G}_2(\mathcal{W}^{\pm}_{(0)}) \prod_{i=1}^{k} \frac{k!}{\alpha_i!} \left(\frac{\mathcal{W}^{\pm}_{(i)}}{i!} \right)^{\alpha_i}, \tag{4.9}$$

where $\mathcal{W}^{\pm}_{(i)}$ represent the traces $(\widetilde{U}^{\pm}_{(i)}, \nabla_x \widetilde{U}^{\pm}_{(i)}, \nabla_x \widetilde{\Phi}^{\pm}_{(i)})$. Hence, the following lemma is obtained (see [20, Lemma 4.2.1] for the details).

Lemma 4.1. *If* (4.1)–(4.5) *hold, then relations* (4.8) *and* (4.9) *determine* $\widetilde{U}^{\pm}_{(k)} \in H^{m+1/2-k}(\mathbb{R}^2_+)$ *for* $k = 1, \dots, m$, *and* $\widetilde{\Phi}^{\pm}_{(k)} \in H^{m+3/2-k}(\mathbb{R}^2_+)$ *for* $k = 1, \dots, m+1$, *which satisfy*

$$\operatorname{supp} \widetilde{U}^{\pm}_{(k)} \subset \{x_2 \geq 0,\ x_1^2 + x_2^2 \leq 1\}, \quad \operatorname{supp} \widetilde{\Phi}^{\pm}_{(k)} \subset \{x_2 \geq 0,\ x_1^2 + x_2^2 \leq 2\},$$

$$\sum_{k=0}^{m} \|\widetilde{U}^{\pm}_{(k)}\|_{H^{m+1/2-k}(\mathbb{R}^2_+)} + \sum_{k=0}^{m+1} \|\widetilde{\Phi}^{\pm}_{(k)}\|_{H^{m+3/2-k}(\mathbb{R}^2_+)} \leq C\Big(\|\widetilde{U}^{\pm}_0\|_{H^{m+1/2}(\mathbb{R}^2_+)} + \|\varphi_0\|_{H^{m+1}(\mathbb{R})} \Big), \tag{4.10}$$

for some constant $C > 0$ *depending solely upon* $\|(\widetilde{U}^{\pm}_0, \widetilde{\Phi}^{\pm}_0)\|_{W^{1,\infty}(\mathbb{R}^2_+)}$ *and* m.

To ensure the smoothness of approximate solution, we need the following compatibility conditions for the initial data.

Definition 4.1. Let $m \in \mathbb{N}$ with $m \geq 3$. Let $\widetilde{U}_0^{\pm} := U_0^{\pm} - \overline{U}_0^{\pm} \in H^{m+1/2}(\mathbb{R}^2_+)$ and $\varphi_0 \in H^{m+1}(\mathbb{R})$ satisfy (4.1). The initial data $U_0^{\pm}$ and φ_0 are said to be compatible up to order m if there exist functions $\widetilde{\Phi}_0^{\pm} \in H^{m+3/2}(\mathbb{R}^2_+)$ satisfying (4.2)–(4.5) and

$$F^{\pm}_{2j,0} = F^{\pm}_{1j,0}\partial_1 \Phi_0^{\pm} \qquad \text{for } j = 1, 2, \tag{4.11}$$

such that functions $\widetilde{U}^{\pm}_{(0)}, \dots, \widetilde{U}^{\pm}_{(m)}, \widetilde{\Phi}^{\pm}_{(0)}, \dots, \widetilde{\Phi}^{\pm}_{(m+1)}$ determined by (4.6) and (4.8)–(4.9) satisfy

$$\left(\widetilde{\Phi}^{+}_{(k)} - \widetilde{\Phi}^{-}_{(k)}\right)\big|_{x_2=0} = 0 \qquad \text{for } k = 0, \dots, m, \tag{4.12a}$$

$$\left(\tilde{\rho}^{+}_{(k)} - \tilde{\rho}^{-}_{(k)}\right)\big|_{x_2=0} = 0 \qquad \text{for } k = 0, \dots, m-1, \tag{4.12b}$$

and

$$\int_{\mathbb{R}^2_+} \left|\widetilde{\Phi}^{+}_{(m+1)} - \widetilde{\Phi}^{-}_{(m+1)}\right|^2 \mathrm{d}x_1 \frac{\mathrm{d}x_2}{x_2} < \infty, \tag{4.13a}$$

$$\int_{\mathbb{R}^2_+} \left|\tilde{\rho}^{+}_{(m)} - \tilde{\rho}^{-}_{(m)}\right|^2 \mathrm{d}x_1 \frac{\mathrm{d}x_2}{x_2} < \infty. \tag{4.13b}$$

4.2. Approximate solutions

We now start to introduce as in [12] the approximate solutions that are solutions of problem (1.11)–(1.13) in the sense of Taylor's expansions at $t = 0$.

Lemma 4.2. *Let $m \in \mathbb{N}$ with $m \geq 3$. Assume that $\widetilde{U}_0^{\pm} := U_0^{\pm} - \overline{U}_0^{\pm} \in H^{m+1/2}(\mathbb{R}^2_+)$ and $\varphi_0 \in H^{m+1}(\mathbb{R})$ satisfy (4.1), and that initial data $U_0^{\pm}$ and φ_0 are compatible up to order m. If $\widetilde{U}_0^{\pm}$ and φ_0 are sufficiently small, then there exist functions $U^{a\pm}$, $\Phi^{a\pm}$, and φ^a such that $\widetilde{U}^{a\pm} := U^{a\pm} - \overline{U}^{\pm} \in H^m(\Omega)$, $\widetilde{\Phi}^{a\pm} := \Phi^{a\pm} - \overline{\Phi}^{\pm} \in H^{m+2}(\Omega)$, $\varphi^a \in H^{m+3/2}(\partial\Omega)$, and*

$$\partial_t^j \mathbb{L}(U^{a\pm}, \Phi^{a\pm})|_{t=0} = 0, \qquad \textit{for } j = 0, \dots, m-2, \tag{4.14a}$$

$$\partial_t \Phi^{a\pm} + v_1^{a\pm}\partial_1 \Phi^{a\pm} - v_2^{a\pm} = 0, \qquad \textit{in } \Omega, \tag{4.14b}$$

$$\pm\, \partial_2 \Phi^{a\pm} \geq 3/4, \qquad \textit{in } \Omega, \tag{4.14c}$$

$$\Phi^{a+} = \Phi^{a-} = \varphi^a, \qquad \textit{on } \partial\Omega, \tag{4.14d}$$

$$\mathbb{B}(U^{a+}, U^{a-}, \varphi^a) = 0, \qquad \textit{on } \partial\Omega, \tag{4.14e}$$

$$F^{a\pm}_{2j} = F^{a\pm}_{1j}\partial_1 \Phi^{a\pm}, \qquad \textit{on } \overline{\Omega}, \quad \textit{for } j = 1, 2. \tag{4.14f}$$

Furthermore, we have

$$\operatorname{supp}\left(\widetilde{U}^{a\pm}, \widetilde{\Phi}^{a\pm}\right) \subset \left\{t \in [-T, T],\, x_2 \geq 0,\, x_1^2 + x_2^2 \leq 3\right\}, \tag{4.15}$$

$$\|\widetilde{U}^{a\pm}\|_{H^m(\Omega)} + \|\widetilde{\Phi}^{a\pm}\|_{H^{m+2}(\Omega)} + \|\varphi^a\|_{H^{m+3/2}(\partial\Omega)} \leq \varepsilon_0\Big(\|\widetilde{U}_0^{\pm}\|_{H^{m+1/2}(\mathbb{R}^2_+)} + \|\varphi_0\|_{H^{m+1}(\mathbb{R})}\Big), \tag{4.16}$$

where $\varepsilon_0(\cdot)$ denotes a generic function that tends to zero as its argument tends to zero.

Proof. The proof is divided into four steps.

Step 1. First we take $\tilde{\rho}^{a-}, \tilde{v}_1^{a\pm} \in H^{m+1}(\Omega)$ and $\widetilde{\Phi}^{a-} \in H^{m+2}(\Omega)$ to satisfy

$$\begin{aligned} \left(\partial_t^k \tilde{\rho}^{a-}, \partial_t^k \tilde{v}_1^{a\pm}\right)\big|_{t=0} &= \left(\tilde{\rho}^-_{(k)}, \tilde{v}^\pm_{1(k)}\right), && \text{for } k=0,\dots,m, \\ \partial_t^k \widetilde{\Phi}^{a-}\big|_{t=0} &= \widetilde{\Phi}^-_{(k)}, && \text{for } k=0,\dots,m+1, \end{aligned}$$

where $\tilde{\rho}^-_{(k)}$, $\tilde{v}^\pm_{1(k)}$, and $\widetilde{\Phi}^-_{(k)}$ are constructed in Lemma 4.1. Thanks to compatibility conditions (4.12)–(4.13), we can apply the lifting result in [19, Theorem 2.3] to choose $\tilde{\rho}^{a+} \in H^{m+1}(\Omega)$ and $\widetilde{\Phi}^{a+} \in H^{m+2}(\Omega)$ such that

$$\begin{aligned} \partial_t^k \tilde{\rho}^{a+}\big|_{t=0} &= \tilde{\rho}^+_{(k)}, && \text{for } k=0,\dots,m, \\ \partial_t^k \widetilde{\Phi}^{a+}\big|_{t=0} &= \widetilde{\Phi}^+_{(k)}, && \text{for } k=0,\dots,m+1, \end{aligned}$$

and

$$[\tilde{\rho}^a]=0, \qquad [\widetilde{\Phi}^a]=0 \qquad \text{on } \partial\Omega.$$

Moreover, $\tilde{\rho}^{a\pm}$, $\tilde{v}_1^{a\pm}$, and $\widetilde{\Phi}^{a\pm}$ can be taken to satisfy (4.15), because $(\widetilde{U}^\pm_{(k)}, \widetilde{\Phi}^\pm_{(k)})$ have a compact support.

Step 2. Let us define

$$\begin{aligned} \varphi^a &= \widetilde{\Phi}^{a+}\big|_{x_2=0} = \widetilde{\Phi}^{a-}\big|_{x_2=0} \in H^{m+3/2}(\partial\Omega), \\ \tilde{v}_2^{a\pm} &= \partial_t \widetilde{\Phi}^{a\pm} + (\tilde{v}_1^{a\pm} \pm \bar{v})\partial_1 \widetilde{\Phi}^{a\pm} \in H^{m+1}(\Omega). \end{aligned}$$

Hence, we deduce that functions $\tilde{v}_2^{a\pm}$ satisfy (4.15), and (4.14b), (4.14d), and (4.14e) hold.

Step 3. Note that $\tilde{v}^{a\pm} \in H^{m+1}(\Omega)$ and $\widetilde{\Phi}^{a\pm} \in H^{m+2}(\Omega)$ have been already specified. Then we take $\widetilde{F}_{ij}^{a\pm} \in H^m(\Omega)$, for $i, j = 1, 2$, as the unique solution of transport equation

$$\left(\partial_t^{\Phi^{a\pm}} + v_\ell^{a\pm}\partial_\ell^{\Phi^{a\pm}}\right)\widetilde{F}_{ij}^{a\pm} - F_{\ell j}^{a\pm}\partial_\ell^{\Phi^{a\pm}} v_i^{a\pm} = 0 \qquad \text{on } \overline{\Omega}, \tag{4.17}$$

supplemented with the initial data

$$\widetilde{F}_{ij}^{a\pm}\big|_{t=0} = \widetilde{F}^\pm_{ij(0)} \in H^{m+1/2}(\mathbb{R}^2_+). \tag{4.18}$$

It follows from (4.11) and (4.18) that constraints (4.14f) are satisfied at the initial time. Consequently, similar to the proof of Proposition 1.1, we can deduce (4.14f) for all $t \in \mathbb{R}$.

Step 4. Equations (4.8)–(4.9) imply (4.14a). Estimate (4.16) follows from (4.10) and the continuity of the lifting operator. From (4.16) and the Sobolev embedding theorem, we can obtain (4.14c) provided the initial perturbations are small enough. This finishes the proof. □

We write $U^a := (U^{a+}, U^{a-})^\top$ and $\Phi^a := (\Phi^{a+}, \Phi^{a-})^\top$ for short, and the vector (U^a, Φ^a) constructed in Lemma 4.2 is the *approximate solution* to (1.11)–(1.13). From (4.14d) and (4.15), φ^a is supported within $\{-T \le t \le T,\ x_1^2 \le 3\}$. By (4.16) and the Sobolev embedding theorem, we have

$$\|\widetilde{U}^{a\pm}\|_{W^{2,\infty}(\Omega)} + \|\widetilde{\Phi}^{a\pm}\|_{W^{3,\infty}(\Omega)} \le \varepsilon_0 \left(\|\tilde{U}_0^{\pm}\|_{H^{m+1/2}(\mathbb{R}^2_+)} + \|\varphi_0\|_{H^{m+1}(\mathbb{R})} \right)$$

for any integer $m \ge 4$. We can now transfer (1.11)–(1.13) into a problem with zero initial data as follows. Define the function f^a as: $f^a = -\mathbb{L}(U^a, \Phi^a)$ for $t > 0$, and $f^a = 0$ for $t < 0$. Then $f^a \in H^{m-1}(\Omega)$ and $\operatorname{supp} f^a \subset \left\{0 \le t \le T,\ x_2 \ge 0,\ x_1^2 + x_2^2 \le 3\right\}$ from (4.14a) and (4.15) as well as $(\tilde{U}^{a\pm}, \nabla\widetilde{\Phi}^{a\pm}) \in H^m(\Omega)$. Moreover, the Moser-type calculus inequalities and (4.16) imply

$$\|f^a\|_{H^{m-1}(\Omega)} \le \varepsilon_0 \left(\|\tilde{U}_0^{\pm}\|_{H^{m+1/2}(\mathbb{R}^2_+)} + \|\varphi_0\|_{H^{m+1}(\mathbb{R})} \right). \tag{4.19}$$

Finally, by (4.14), $(U, \Phi) = (U^a, \Phi^a) + (V, \Psi)$ is a solution to the original problem (1.11)–(1.13) on $[0, T] \times \mathbb{R}^2_+$, if $V = (V^+, V^-)^\top$ and $\Psi = (\Psi^+, \Psi^-)^\top$ solve the problem as follows:

$$\begin{cases} \mathcal{L}(V, \Psi) := \mathbb{L}(U^a + V, \Phi^a + \Psi) - \mathbb{L}(U^a, \Phi^a) = f^a, & \text{in } \Omega_T, \\ \mathcal{E}(V, \Psi) := \partial_t \Psi + (v_1^a + v_1)\partial_1 \Psi + v_1 \partial_1 \Phi^a - v_2 = 0, & \text{in } \Omega_T, \\ \mathcal{B}(V, \psi) := \mathbb{B}(U^a + V, \varphi^a + \psi) = 0, \quad \Psi^+ = \Psi^- = \psi, & \text{on } \omega_T, \\ (V, \Psi) = 0, & t < 0. \end{cases} \tag{4.20}$$

Therefore, we only need to solve the above problem (4.20) on $[0, T] \times \mathbb{R}^2_+$.

5. Nash–Moser iteration

In this section we solve the problem (4.20) by an appropriate modification of the Nash–Moser iteration scheme. We first describe the iterative scheme for problem (4.20) and present the inductive hypothesis. Then we conclude the proof of Theorem 1.1 by showing that the inductive hypothesis holds for all integers. We remark that this section follows closely the standard procedure in [12] (also see [4]).

5.1. Iterative scheme

We first recall the following result from [12, Proposition 4].

Proposition 5.1. *Let $T > 0$, $\gamma \ge 1$, and $m \in \mathbb{N}$ with $m \ge 4$. Then there exists a family $\{\mathcal{S}_\theta\}_{\theta \ge 1}$ of smoothing operators*

$$\mathcal{S}_\theta :\ \mathcal{F}^3_\gamma(\Omega_T) \times \mathcal{F}^3_\gamma(\Omega_T) \longrightarrow \bigcap_{s \ge 3} \mathcal{F}^s_\gamma(\Omega_T) \times \mathcal{F}^s_\gamma(\Omega_T),$$

where $\mathcal{F}^s_\gamma(\Omega_T) := \left\{u \in H^s_\gamma(\Omega_T) : u = 0 \text{ if } t < 0\right\}$ for $s \ge 0$, such that

$$\|\mathcal{S}_\theta u\|_{H^k_\gamma(\Omega_T)} \lesssim \theta^{(k-\ell)_+}\|u\|_{H^\ell_\gamma(\Omega_T)} \qquad \textit{for } \ell, k = 1, \dots, m, \tag{5.1a}$$

$$\|\mathcal{S}_\theta u - u\|_{H^k_\gamma(\Omega_T)} \lesssim \theta^{k-\ell}\|u\|_{H^\ell_\gamma(\Omega_T)} \qquad \textit{for } 1 \le k \le \ell \le m, \tag{5.1b}$$

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{S}_\theta u \right\|_{H^k_\gamma(\Omega_T)} \lesssim \theta^{k-\ell-1}\|u\|_{H^\ell_\gamma(\Omega_T)} \quad \textit{for } \ell, k = 1, \dots, m, \tag{5.1c}$$

and

$$\|\mathcal{S}_\theta u - \mathcal{S}_\theta w\|_{H^k_\gamma(\omega_T)} \lesssim \theta^{(k+1-\ell)_+}\|u - w\|_{H^\ell_\gamma(\omega_T)} \quad \textit{for } \ell, k = 1, \dots, m, \tag{5.2}$$

where ℓ and k are integers, and $(k-\ell)_+ := \max\{0, k-\ell\}$. In particular, if $u = w$ on ω_T, then $\mathcal{S}_\theta u = \mathcal{S}_\theta w$ on ω_T. Moreover, the smoothing operators acting on the functions defined on ω_T can be constructed analogously (still denoted by $\mathcal{S}_\theta$ for notational simplicity), which also satisfies the inequalities (5.1) with norms $\|\cdot\|_{H^\ell_\gamma(\omega_T)}$.

The next lemma provides us a lifting operator that will be used for constructing the iterative scheme and the modified state (see [15, Chapter 5] and [12] for the proof).

Lemma 5.2. *Let $T > 0$, $\gamma \ge 1$, and $m \in \mathbb{N}_+$. Then there exists an operator $\mathcal{R}_T$ that is continuous from $\mathcal{F}^s_\gamma(\omega_T)$ to $\mathcal{F}^{s+1/2}_\gamma(\Omega_T)$ and satisfies $(\mathcal{R}_T u)|_{x_2=0} = u$ when $u \in \mathcal{F}^s_\gamma(\omega_T)$ for all $s \in [1, m]$.*

Now we follow [4,12] to describe the iteration scheme for problem (4.20). Let $N \ge 1$ be any given integer. First we set $(V_0, \Psi_0, \psi_0) = 0$ and let (V_n, Ψ_n, ψ_n) be given and satisfy

$$(V_n, \Psi_n, \psi_n)\big|_{t<0} = 0, \quad \Psi_n^+\big|_{x_2=0} = \Psi_n^-\big|_{x_2=0} = \psi_n \quad \textit{for } n = 0, \dots, N. \tag{5.3}$$

We consider

$$V_{N+1} = V_N + \delta V_N, \quad \Psi_{N+1} = \Psi_N + \delta\Psi_N, \quad \psi_{N+1} = \psi_N + \delta\psi_N, \tag{5.4}$$

where differences δV_N, $\delta\Psi_N$, and $\delta\psi_N$ will be constructed via the problem

$$\begin{cases} \mathbb{L}'_e(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2})\delta\dot{V}_N = f_N & \text{in } \Omega_T, \\ \mathbb{B}'_e(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2})(\delta\dot{V}_N, \delta\psi_N) = g_N & \text{on } \omega_T, \\ (\delta\dot{V}_N, \delta\psi_N) = 0 & \text{for } t < 0. \end{cases} \tag{5.5}$$

Here operators $\mathbb{L}'_e$ and $\mathbb{B}'_e$ are given in (3.11a) and (3.11b), respectively, $(V_{N+1/2}, \Psi_{N+1/2})$ is a modified state such that $(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2})$ satisfies constraints (3.2)–(3.3), and source term (f_N, g_N) will be determined later on. See Section 5.3 for the detailed construction of the modified state. As in (3.10), we write

$$\delta\dot{V}_N := \delta V_N - \frac{\partial_2(U^a + V_{N+1/2})}{\partial_2(\Phi^a + \Psi_{N+1/2})}\delta\Psi_N. \tag{5.6}$$

Then, we set $f_0 := \mathcal{S}_{\theta_0} f^a$ and $(e_0, \tilde{e}_0, g_0) := 0$ for $\theta_0 \geq 1$ sufficiently large, and let $(f_n, g_n, e_n, \tilde{e}_n)$ be given and vanish in the past for $n = 0, \dots, N-1$. We determine f_N and g_N by

$$\sum_{n=0}^{N} f_n + \mathcal{S}_{\theta_N} E_N = \mathcal{S}_{\theta_N} f^a, \qquad \sum_{n=0}^{N} g_n + \mathcal{S}_{\theta_N} \widetilde{E}_N = 0, \tag{5.7}$$

where

$$E_N := \sum_{n=0}^{N-1} e_n \in \mathbb{R}^{14}, \qquad \widetilde{E}_N := \sum_{n=0}^{N-1} \tilde{e}_n \in \mathbb{R}^3, \tag{5.8}$$

and $\mathcal{S}_{\theta_N}$ are the smoothing operators given in Proposition 5.1 with $\{\theta_N\}$ defined by

$$\theta_0 \geq 1, \qquad \theta_N = \sqrt{\theta_0^2 + N}. \tag{5.9}$$

As a consequence, we can use Theorem 3.1 to solve $(\delta \dot{V}_N, \delta\psi_N)$ for problem (5.5).

According to (5.6), we need to construct functions $\delta\Psi_N^+$ and $\delta\Psi_N^-$ such that $\delta\Psi_N^\pm|_{x_2=0} = \delta\psi_N$. From the boundary conditions in (5.5) (*cf.* (3.7), (3.8), and (3.12)), we obtain that $\delta\psi_N$ satisfies

$$\begin{aligned}
&\partial_t(\delta\psi_N) + U^+_{N+1/2,2}\partial_1(\delta\psi_N) + \left(\partial_1\Phi^+_{N+1/2}\frac{\partial_2 U^+_{N+1/2,2}}{\partial_2\Phi^+_{N+1/2}} - \frac{\partial_2 U^+_{N+1/2,3}}{\partial_2\Phi^+_{N+1/2}}\right)\delta\psi_N \\
&\qquad + \partial_1\Phi^+_{N+1/2}\delta\dot{V}^+_{N,2} - \delta\dot{V}^+_{N,3} = g_{N,2} \qquad \text{on } \omega_T, \\
&\partial_t(\delta\psi_N) + U^-_{N+1/2,2}\partial_1(\delta\psi_N) + \left(\partial_1\Phi^-_{N+1/2}\frac{\partial_2 U^-_{N+1/2,2}}{\partial_2\Phi^-_{N+1/2}} - \frac{\partial_2 U^-_{N+1/2,3}}{\partial_2\Phi^-_{N+1/2}}\right)\delta\psi_N \\
&\qquad + \partial_1\Phi^-_{N+1/2}\delta\dot{V}^-_{N,2} - \delta\dot{V}^-_{N,3} = g_{N,2} - g_{N,1} \qquad \text{on } \omega_T,
\end{aligned}$$

where we define $U^\pm_{N+1/2} := U^{a\pm} + V^\pm_{N+1/2}$ and $\Phi^\pm_{N+1/2} := \Phi^{a\pm} + \Psi^\pm_{N+1/2}$ for simplifying the presentation. In accordance with the identities above, we take $\delta\Psi_N^+$ and $\delta\Psi_N^-$ as the solutions to transport equations

$$\begin{aligned}
&\partial_t(\delta\Psi_N^+) + U^+_{N+1/2,2}\partial_1(\delta\Psi_N^+) + \left(\partial_1\Phi^+_{N+1/2}\frac{\partial_2 U^+_{N+1/2,2}}{\partial_2\Phi^+_{N+1/2}} - \frac{\partial_2 U^+_{N+1/2,3}}{\partial_2\Phi^+_{N+1/2}}\right)\delta\Psi_N^+ \\
&\qquad + \partial_1\Phi^+_{N+1/2}\delta\dot{V}^+_{N,2} - \delta\dot{V}^+_{N,3} = \mathcal{R}_T g_{N,2} + h_N^+,
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
&\partial_t(\delta\Psi_N^-) + U^-_{N+1/2,2}\partial_1(\delta\Psi_N^-) + \left(\partial_1\Phi^-_{N+1/2}\frac{\partial_2 U^-_{N+1/2,2}}{\partial_2\Phi^-_{N+1/2}} - \frac{\partial_2 U^-_{N+1/2,3}}{\partial_2\Phi^-_{N+1/2}}\right)\delta\Psi_N^- \\
&\qquad + \partial_1\Phi^-_{N+1/2}\delta\dot{V}^-_{N,2} - \delta\dot{V}^-_{N,3} = \mathcal{R}_T (g_{N,2} - g_{N,1}) + h_N^-,
\end{aligned} \tag{5.11}$$

where $\mathcal{R}_T$ is the lifting operator given in Lemma 5.2 and we will choose source terms $h_N^{\pm}$ through a decomposition for operator $\mathcal{E}$ defined by (4.20).

Finally, we set $(h_0^+, h_0^-, \hat{e}_0) = 0$, and let $(h_n^+, h_n^-, \hat{e}_n)$ be given and vanish in the past for $n = 0, \dots, N-1$. Under the above settings, we compute h_N^+ and h_N^- from

$$\mathcal{S}_{\theta_N}\left(\widehat{E}_N^+ - \mathcal{R}_T \widetilde{E}_{N,2}\right) + \sum_{n=0}^{N} h_n^+ = 0, \tag{5.12a}$$

$$\mathcal{S}_{\theta_N}\left(\widehat{E}_N^- - \mathcal{R}_T \widetilde{E}_{N,2} + \mathcal{R}_T \widetilde{E}_{N,1}\right) + \sum_{n=0}^{N} h_n^- = 0, \tag{5.12b}$$

where

$$\widehat{E}_N = (\widehat{E}_N^+, \widehat{E}_N^-)^\top = \sum_{n=0}^{N-1} \hat{e}_n \in \mathbb{R}^2, \tag{5.13}$$

and $h_N^{\pm} = 0$ for $t < 0$. As in [15], we can show that the traces of $h_N^{\pm}$ on ω_T vanish. Consequently, we can deduce that $\delta\Psi_N^{\pm} = 0$, for $t < 0$ and $\delta\Psi_N^{\pm}|_{x_2=0} = \delta\psi_N$. They are the unique smooth solutions satisfying transport equations (5.10)–(5.11). Hence, δV_N can be obtained from (5.6) and $(V_{N+1}, \Psi_{N+1}, \psi_{N+1})$ can be derived from (5.4).

From (5.8)–(5.7) and (5.12)–(5.13), it suffices to define the error terms e_N, $\tilde{e}_N$, and $\hat{e}_N$. To this end, by an analogous argument in [4,12], we decompose

$$\begin{aligned}
&\mathcal{L}(V_{N+1}, \Psi_{N+1}) - \mathcal{L}(V_N, \Psi_N) \\
&\quad = \mathbb{L}_e'(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2})\delta\dot{V}_N + e_N' + e_N'' + e_N''' + D_{N+1/2}\delta\Psi_N
\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}
&\mathcal{B}(V_{N+1}, \psi_{N+1}) - \mathcal{B}(V_N, \psi_N) \\
&\quad = \mathbb{B}_e'(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2})(\delta\dot{V}_N, \delta\psi_N) + \tilde{e}_N' + \tilde{e}_N'' + \tilde{e}_N''',
\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}
e_N' &:= \mathcal{L}(V_{N+1}, \Psi_{N+1}) - \mathcal{L}(V_N, \Psi_N) - \mathbb{L}'(U^a + V_N, \Phi^a + \Psi_N)(\delta V_N, \delta\Psi_N), \\
e_N'' &:= \mathbb{L}'(U^a + V_N, \Phi^a + \Psi_N)(\delta V_N, \delta\Psi_N) - \mathbb{L}'(U^a + \mathcal{S}_{\theta_N} V_N, \Phi^a + \mathcal{S}_{\theta_N}\Psi_N)(\delta V_N, \delta\Psi_N), \\
e_N''' &:= \mathbb{L}'(U^a + \mathcal{S}_{\theta_N} V_N, \Phi^a + \mathcal{S}_{\theta_N}\Psi_N)(\delta V_N, \delta\Psi_N) \\
&\qquad - \mathbb{L}'(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2})(\delta V_N, \delta\Psi_N), \\
D_{N+1/2} &:= \left(\partial_2(\Phi^a + \Psi_{N+1/2})\right)^{-1} \partial_2\mathbb{L}(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2}),
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
\tilde{e}'_N &:= \mathcal{B}(V_{N+1}, \psi_{N+1}) - \mathcal{B}(V_N, \psi_N) - \mathbb{B}'(U^a + V_N, \varphi^a + \psi_N)(\delta V_N, \delta \psi_N), \\
\tilde{e}''_N &:= \mathbb{B}'(U^a + V_N, \varphi^a + \psi_N)(\delta V_N, \delta \psi_N) \\
&\quad - \mathbb{B}'(U^a + \mathcal{S}_{\theta_N} V_N, \varphi^a + (\mathcal{S}_{\theta_N} \Psi_N)|_{x_2=0})(\delta V_N, \delta \psi_N), \\
\tilde{e}'''_N &:= \mathbb{B}'(U^a + \mathcal{S}_{\theta_N} V_N, \varphi^a + (\mathcal{S}_{\theta_N} \Psi_N)|_{x_2=0})(\delta V_N, \delta \psi_N) \\
&\quad - \mathbb{B}'_e(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2})(\delta \dot{V}_N, \delta \psi_N).
\end{aligned}$$

Take

$$e_N := e'_N + e''_N + e'''_N + D_{N+1/2}\delta\Psi_N, \qquad \tilde{e}_N := \tilde{e}'_N + \tilde{e}''_N + \tilde{e}'''_N. \tag{5.17}$$

As for error term $\hat{e}_N$, we decompose

$$\mathcal{E}(V_{N+1}, \Psi_{N+1}) - \mathcal{E}(V_N, \Psi_N) = \mathcal{E}'(V_{N+1/2}, \Psi_{N+1/2})(\delta V_N, \delta \Psi_N) + \hat{e}'_N + \hat{e}''_N + \hat{e}'''_N, \tag{5.18}$$

and set

$$\hat{e}_N := \hat{e}'_N + \hat{e}''_N + \hat{e}'''_N, \tag{5.19}$$

where

$$\begin{aligned}
\hat{e}'_N &:= \mathcal{E}(V_{N+1}, \Psi_{N+1}) - \mathcal{E}(V_N, \Psi_N) - \mathcal{E}'(V_N, \Psi_N)(\delta V_N, \delta \Psi_N), \\
\hat{e}''_N &:= \mathcal{E}'(V_N, \Psi_N)(\delta V_N, \delta \Psi_N) - \mathcal{E}'(\mathcal{S}_{\theta_N} V_N, \mathcal{S}_{\theta_N} \Psi_N)(\delta V_N, \delta \Psi_N), \\
\hat{e}'''_N &:= \mathcal{E}'(\mathcal{S}_{\theta_N} V_N, \mathcal{S}_{\theta_N} \Psi_N)(\delta V_N, \delta \Psi_N) - \mathcal{E}'(V_{N+1/2}, \Psi_{N+1/2})(\delta V_N, \delta \Psi_N).
\end{aligned}$$

It follows from (4.14b) that

$$\mathcal{E}(V, \Psi) = \partial_t(\Phi^a + \Psi) + (v_1^a + v_1)\partial_1(\Phi^a + \Psi) - (v_2^a + v_2).$$

Then we derive from (5.10)–(5.11) and (5.18) that

$$\begin{pmatrix} \mathcal{E}(V^+_{N+1}, \Psi^+_{N+1}) - \mathcal{E}(V^+_N, \Psi^+_N) \\ \mathcal{E}(V^-_{N+1}, \Psi^-_{N+1}) - \mathcal{E}(V^-_N, \Psi^-_N) \end{pmatrix} = \begin{pmatrix} \mathcal{R}_T g_{N,2} + h^+_N + \hat{e}^+_N \\ \mathcal{R}_T (g_{N,2} - g_{N,1}) + h^-_N + \hat{e}^-_N \end{pmatrix}.$$

Thus, by $\mathcal{E}(V_0, \Psi_0) = 0$, one has

$$\mathcal{E}(V^-_{N+1}, \Psi^-_{N+1}) = \mathcal{R}_T \left(\sum_{n=0}^{N} (g_{n,2} - g_{n,1}) \right) + \sum_{n=0}^{N} h^-_n + \widehat{E}^-_{N+1}. \tag{5.20}$$

Furthermore, we obtain from (5.5) and (5.15) that

$$g_N = \mathcal{B}(V_{N+1}, \psi_{N+1}) - \mathcal{B}(V_N, \psi_N) - \tilde{e}_N. \tag{5.21}$$

Denote by $\mathcal{B}(V_{N+1}, \psi_{N+1})_j$ the jth component of the vector $\mathcal{B}(V_{N+1}, \psi_{N+1})$ for $j = 1, 2$. From (4.20) and (1.15),

$$\mathcal{B}(V_{N+1}, \psi_{N+1})_2 = \mathcal{E}(V^+_{N+1}, \Psi^+_{N+1})|_{x_2=0} = \mathcal{E}(V^-_{N+1}, \Psi^-_{N+1})|_{x_2=0} + \mathcal{B}(V_{N+1}, \psi_{N+1})_1. \tag{5.22}$$

Using (5.21), we have

$$g_{N,2} - g_{N,1} = \mathcal{E}(V^-_{N+1}, \Psi^-_{N+1})|_{x_2=0} - \mathcal{E}(V^-_N, \Psi^-_N)|_{x_2=0} - \tilde{e}_{N,2} + \tilde{e}_{N,1}. \tag{5.23}$$

Then, (5.23) and (5.20) yield

$$\mathcal{E}(V^-_{N+1}, \Psi^-_{N+1}) = \mathcal{R}_T \left(\mathcal{E}\left(V^-_{N+1}, \Psi^-_{N+1}\right)|_{x_2=0} - \widetilde{E}_{N+1,2} + \widetilde{E}_{N+1,1}\right) + \sum_{n=0}^{N} h^-_n + \widehat{E}^-_{N+1}, \tag{5.24}$$

and similarly,

$$\mathcal{E}(V^+_{N+1}, \Psi^+_{N+1}) = \mathcal{R}_T \left(\mathcal{E}\left(V^+_{N+1}, \Psi^+_{N+1}\right)|_{x_2=0} - \widetilde{E}_{N+1,2}\right) + \sum_{n=0}^{N} h^+_n + \widehat{E}^+_{N+1}. \tag{5.25}$$

From (5.14) and (5.21), together with (5.5) and (5.7), one has

$$\mathcal{L}(V_{N+1}, \Psi_{N+1}) = \sum_{N=0}^{N} f_N + E_{N+1} = \mathcal{S}_{\theta_N} f^a + (I - \mathcal{S}_{\theta_N})E_N + e_N, \tag{5.26}$$

$$\mathcal{B}(V_{N+1}, \psi_{N+1}) = \sum_{N=0}^{N} g_N + \widetilde{E}_{N+1} = (I - \mathcal{S}_{\theta_N})\widetilde{E}_N + \tilde{e}_N. \tag{5.27}$$

Substituting (5.12) into (5.24)–(5.25) and using (5.22), we get

$$\begin{cases} \mathcal{E}(V^-_{N+1}, \Psi^-_{N+1}) = \mathcal{R}_T\big(\mathcal{B}(V_{N+1}, \psi_{N+1})_2 - \mathcal{B}(V_{N+1}, \psi_{N+1})_1\big) \\ \qquad\qquad + (I - \mathcal{S}_{\theta_N})\Big(\widehat{E}^-_N - \mathcal{R}_T\big(\widetilde{E}_{N,2} - \widetilde{E}_{N,1}\big)\Big) \\ \qquad\qquad + \hat{e}^-_N - \mathcal{R}_T\big(\tilde{e}_{N,2} - \tilde{e}_{N,1}\big), \\ \mathcal{E}(V^+_{N+1}, \Psi^+_{N+1}) = \mathcal{R}_T\big(\mathcal{B}(V_{N+1}, \psi_{N+1})_2\big) \\ \qquad\qquad + (I - \mathcal{S}_{\theta_N})\big(\widehat{E}^+_N - \mathcal{R}_T\widetilde{E}_{N,2}\big) + \hat{e}^+_N - \mathcal{R}_T\tilde{e}_{N,2}. \end{cases} \tag{5.28}$$

From $\mathcal{S}_{\theta_N} \to Id$ as $N \to \infty$, we conclude that if the error terms $(e_N, \tilde{e}_N, \hat{e}_N)$ tend to zero, then

$$(\mathcal{L}(V_{N+1}, \Psi_{N+1}), \mathcal{B}(V_{N+1}, \psi_{N+1}), \mathcal{E}(V_{N+1}, \Psi_{N+1})) \to (f^a, 0, 0),$$

thus, the solution to (4.20) can be obtained formally.

In order to estimate the error terms, we need to introduce the *inductive hypothesis* as follows. Let us take an integer $\mu \geq 4$, a small number $\epsilon > 0$, and another integer $\tilde{\mu} > \mu$, which will be determined later. Suppose that we have the estimate

$$\|\widetilde{U}^a\|_{H_\gamma^{\tilde{\mu}+4}(\Omega_T)} + \|\widetilde{\Phi}^a\|_{H_\gamma^{\tilde{\mu}+5}(\Omega_T)} + \|\varphi^a\|_{H_\gamma^{\tilde{\mu}+9/2}(\omega_T)} + \|f^a\|_{H_\gamma^{\tilde{\mu}+3}(\Omega_T)} \leq \epsilon, \tag{5.29}$$

then our inductive hypothesis $\mathbf{H}_{N-1}$ consists of the following four parts:

(1) $\|(\delta V_n, \delta\Psi_n)\|_{H_\gamma^m(\Omega_T)} + \|\delta\psi_n\|_{H_\gamma^{m+1}(\omega_T)} \leq \epsilon\theta_n^{m-\mu-1}\Delta_n, \quad n = 0, \ldots, N-1,\ m = 2, \ldots, \tilde{\mu},$

(2) $\|\mathcal{L}(V_n, \Psi_n) - f^a\|_{H_\gamma^m(\Omega_T)} \leq 2\epsilon\theta_n^{m-\mu-1}, \quad n = 0, \ldots, N-1,\ m = 2, \ldots, \tilde{\mu}-1,$

(3) $\|\mathcal{B}(V_n, \psi_n)\|_{H_\gamma^m(\omega_T)} \leq \epsilon\theta_n^{m-\mu-1}, \quad n = 0, \ldots, N-1,\ m = 3, \ldots, \mu,$

(4) $\|\mathcal{E}(V_n, \Psi_n)\|_{H_\gamma^3(\Omega_T)} \leq \epsilon\theta_n^{2-\mu}, \quad n = 0, \ldots, N-1,$

where θ_n is given in (5.9) and $\Delta_n := \theta_{n+1} - \theta_n$ decreases to zero with

$$\frac{1}{3\theta_n} \leq \Delta_n := \theta_{n+1} - \theta_n = \sqrt{\theta_n^2 + 1} - \theta_n \leq \frac{1}{2\theta_n}, \quad n \in \mathbb{N}. \tag{5.30}$$

We shall show that for sufficiently small ϵ and f^a, and for sufficiently large $\theta_0 \geq 1$, $\mathbf{H}_0$ is true and $\mathbf{H}_{N-1}$ implies $\mathbf{H}_N$, thus $\mathbf{H}_N$ is true for all $n \in \mathbb{N}$, which will allow us to prove Theorem 1.1 completely.

Now we assume that $\mathbf{H}_{N-1}$ holds, hence have the following estimates as in [12, Lemmas 6–7].

Lemma 5.3. *If θ_0 is sufficiently large, then*

$$\|(V_n, \Psi_n)\|_{H_\gamma^m(\Omega_T)} + \|\psi_n\|_{H_\gamma^{m+1}(\omega_T)} \leq \begin{cases} \epsilon\theta_n^{(m-\mu)_+}, & \text{if } m \neq \mu, \\ \epsilon \log\theta_n, & \text{if } m = \mu, \end{cases} \tag{5.31}$$

$$\|((I - \mathcal{S}_{\theta_n})V_n, (I - \mathcal{S}_{\theta_n})\Psi_n)\|_{H_\gamma^m(\Omega_T)} \leq C\epsilon\theta_n^{m-\mu}, \tag{5.32}$$

for $n = 0, \ldots, N$, and $m = 2, \ldots, \tilde{\mu}$. Furthermore,

$$\|(\mathcal{S}_{\theta_n}V_n, \mathcal{S}_{\theta_n}\Psi_n)\|_{H_\gamma^m(\Omega_T)} \leq \begin{cases} C\epsilon\theta_n^{(m-\mu)_+}, & \text{if } m \neq \mu, \\ C\epsilon \log\theta_n, & \text{if } m = \mu, \end{cases} \tag{5.33}$$

for $n = 0, \ldots, N$, and $m = 2, \ldots, \tilde{\mu}+5$.

5.2. Estimate of the quadratic and first substitution error terms

First we rewrite quadratic error terms e_n', $\tilde{e}_n'$, and $\hat{e}_n'$, in (5.14), (5.15), and (5.18) respectively, as

$$e_n' = \int_0^1 \mathbb{L}''\big(U^a + V_n + \tau\delta V_n, \Phi^a + \Psi_n + \tau\delta\Psi_n\big)\big((\delta V_n, \delta\Psi_n), (\delta V_n, \delta\Psi_n)\big)(1-\tau)\,\mathrm{d}\tau,$$

$$\tilde{e}_n' = \int_0^1 \mathbb{B}''\big(U^a + V_n + \tau\delta V_n, \varphi^a + \psi_n + \tau\delta\psi_n\big)\big((\delta V_n, \delta\psi_n), (\delta V_n, \delta\psi_n)\big)(1-\tau)\,\mathrm{d}\tau,$$

$$\hat{e}_n' = \int_0^1 \mathcal{E}''\big(V_n + \tau\delta V_n, \Psi_n + \tau\delta\Psi_n\big)\big((\delta V_n, \delta\Psi_n), (\delta V_n, \delta\Psi_n)\big)(1-\tau)\,\mathrm{d}\tau,$$

where $\mathbb{L}''$, $\mathbb{B}''$, and $\mathcal{E}''$ are the second derivatives of operators $\mathbb{L}$, $\mathbb{B}$, and $\mathcal{E}$ respectively. More precisely, we define

$$\mathbb{L}''\big(\check{U}, \check{\Phi}\big)\big((V, \Psi), (\widetilde{V}, \widetilde{\Psi})\big) := \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{L}'\big(\check{U} + \theta\widetilde{V}, \check{\Phi} + \theta\widetilde{\Psi}\big)\big(V, \Psi\big)\Big|_{\theta=0},$$

$$\mathbb{B}''(\check{U}, \check{\varphi})\big((V, \psi), (\widetilde{V}, \tilde{\psi})\big) := \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{B}'(\check{U} + \theta\widetilde{V}, \check{\varphi} + \theta\tilde{\psi})(V, \psi)\Big|_{\theta=0},$$

$$\mathcal{E}''\big(\check{V}, \check{\Psi}\big)\big((V, \Psi), (\widetilde{V}, \widetilde{\Psi})\big) := \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{E}'\big(\check{V} + \theta\widetilde{V}, \check{\Psi} + \theta\widetilde{\Psi}\big)\big(V, \Psi\big)\Big|_{\theta=0},$$

where operators $\mathbb{L}'$ and $\mathbb{B}'$ are given in (3.4)–(3.5), and $\mathcal{E}'$ is defined by

$$\mathcal{E}'\big(\check{V}, \check{\Psi}\big)(V, \Psi) := \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{E}\big(\check{V} + \theta V, \check{\Psi} + \theta\Psi\big)\Big|_{\theta=0}.$$

In fact, in our case, we have the following:

$$\mathbb{B}''(\check{U}, \check{\varphi})\big((V, \psi), (\widetilde{V}, \tilde{\psi})\big) = \begin{pmatrix} [\tilde{v}_1]\partial_1\psi + \partial_1\tilde{\psi}[v_1] \\ \tilde{v}_1^+|_{x_2=0}\partial_1\psi + \partial_1\tilde{\psi}v_1^+|_{x_2=0} \\ 0 \end{pmatrix}, \tag{5.34}$$

$$\mathcal{E}''\big(\check{V}, \check{\Psi}\big)\big((V, \Psi), (\widetilde{V}, \widetilde{\Psi})\big) = \tilde{v}_1^+\partial_1\Psi + \partial_1\widetilde{\Psi}v_1^+. \tag{5.35}$$

A straightforward computation with an application of the Moser-type calculus inequality (2.1) yields the next proposition (see [12, Proposition 5]).

Proposition 5.4. *Let $T > 0$ and $m \in \mathbb{N}$ with $m \geq 2$. If $(\widetilde{V}, \widetilde{\Psi})$ belongs to $H_\gamma^{m+1}(\Omega_T)$ for all $\gamma \geq 1$ and satisfies $\|(\widetilde{V}, \widetilde{\Psi})\|_{W^{1,\infty}(\Omega_T)} \leq \widetilde{K}$ for some positive constant $\widetilde{K}$, then there exist two constants $\widetilde{K}_0 > 0$ and $C > 0$, independent of T and γ, such that, if $\widetilde{K} \leq \widetilde{K}_0$ and $\gamma \geq 1$, then*

$$\big\|\mathbb{L}''\big(\overline{U} + \widetilde{V}, \overline{\Phi} + \widetilde{\Psi}\big)\big((V_1, \Psi_1), (V_2, \Psi_2)\big)\big\|_{H_\gamma^m(\Omega_T)}$$
$$\leq C\|(V_1, \Psi_1)\|_{W^{1,\infty}(\Omega_T)}\|(V_2, \Psi_2)\|_{W^{1,\infty}(\Omega_T)}\big\|\big(\widetilde{V}, \widetilde{\Psi}\big)\big\|_{H_\gamma^{m+1}(\Omega_T)}$$

$$+ C \sum_{i \neq j} \|(V_i, \Psi_i)\|_{H_\gamma^{m+1}(\Omega_T)} \|(V_j, \Psi_j)\|_{W^{1,\infty}(\Omega_T)},$$

$$\left\| \mathcal{E}''\big(\widetilde{V}, \widetilde{\Psi}\big)\big((V_1, \Psi_1), (V_2, \Psi_2)\big) \right\|_{H_\gamma^m(\Omega_T)}$$

$$\leq C \sum_{i \neq j} \left\{ \|V_i\|_{H_\gamma^m(\Omega_T)} \|\Psi_j\|_{W^{1,\infty}(\Omega_T)} + \|V_i\|_{L^\infty(\Omega_T)} \|\Psi_j\|_{H_\gamma^{m+1}(\Omega_T)} \right\},$$

and

$$\left\| \mathbb{B}''\big(\bar{U} + \widetilde{V}, \tilde{\psi}\big)\big((W_1, \psi_1), (W_2, \psi_2)\big) \right\|_{H_\gamma^m(\omega_T)}$$

$$\leq C \sum_{i \neq j} \left\{ \|W_i\|_{H_\gamma^m(\omega_T)} \|\psi_j\|_{W^{1,\infty}(\omega_T)} + \|W_i\|_{L^\infty(\omega_T)} \|\psi_j\|_{H_\gamma^{m+1}(\omega_T)} \right\},$$

where $(V_i, \Psi_i) \in H_\gamma^{m+1}(\Omega_T)$ *and* $(W_i, \psi_i) \in H_\gamma^m(\omega_T) \times H_\gamma^{m+1}(\omega_T)$ *for* $i = 1, 2$*, symbol* $\tilde{\psi}$ *represents the trace of* $\widetilde{\Psi}$ *on* ω_T*, and* $(\bar{U}, \bar{\Phi})$ *is the background state defined by* (1.19)*.*

In view of (5.29)–(5.31) and the hypothesis $\mathbf{H}_{N-1}$, as in [12, Lemma 8] or [4, Lemma 8.3], we can apply Proposition 5.4, the Sobolev embedding theorem, and the trace estimate to get the following estimate.

Lemma 5.5. *If* $\mu \geq 4$*, then there exist* $\epsilon > 0$ *suitably small and* $\theta_0 \geq 1$ *large enough such that*

$$\|(e_n', \hat{e}_n')\|_{H_\gamma^m(\Omega_T)} + \|\tilde{e}_n'\|_{H_\gamma^m(\omega_T)} \leq C\epsilon^2 \theta_n^{\ell_1(m)-1} \Delta_n,$$

for $m = 2, \dots, \tilde{\mu} - 1$*, and* $n = 0, \dots, N-1$*, where* $\ell_1(m) := \max\{(m+1-\mu)_+ + 4 - 2\mu, m + 2 - 2\mu\}$*.*

For the first substitution error terms e_n'', $\tilde{e}_n''$, and $\hat{e}_n''$ defined in (5.14), (5.15), and (5.18), as in [12, Lemma 9] or [4, Lemma 8.4], we can apply Proposition 5.4 and use (5.29), (5.32)–(5.33), hypothesis (H_{n-1}), and the trace theorem to derive the next lemma.

Lemma 5.6. *If* $\mu \geq 4$*, then there exist* $\epsilon > 0$ *suitably small and* $\theta_0 \geq 1$ *large enough such that*

$$\|(e_n'', \hat{e}_n'')\|_{H_\gamma^m(\Omega_T)} \leq C\epsilon^2 \theta_n^{\ell_2(m)-1} \Delta_n \quad \text{if } m = 2, \dots, \tilde{\mu} - 1,$$

$$\|\tilde{e}_n''\|_{H_\gamma^m(\omega_T)} \leq C\epsilon^2 \theta_n^{\ell_2(m)-1} \Delta_n \quad \text{if } m = 2, \dots, \tilde{\mu} - 2,$$

for $n = 0, \dots, N-1$*, where*

$$\ell_2(m) := \max\{(m+1-\mu)_+ + 6 - 2\mu, m + 5 - 2\mu\}.$$

We emphasize that Proposition 5.4 reduces the estimate for $\|\tilde{e}_n''\|_{H_\gamma^m(\omega_T)}$ to that for the terms involving $\|(I - \mathcal{S}_{\theta_n})\Psi_n\|_{H_\gamma^{m+2}(\Omega_T)}$, which requires condition $m \leq \tilde{\mu} - 2$ in order to apply inequality (5.32).

5.3. Construction and estimate of the modified state

To control the remaining error terms, we construct and estimate the modified state $(V_{N+1/2}, \Psi_{N+1/2}, \psi_{N+1/2})$ in the following lemma.

Lemma 5.7. *If $\mu \geq 5$, then there exist functions $V_{N+1/2}$, $\Phi_{N+1/2}$, and $\psi_{N+1/2}$, which vanish in the past, such that $(U^a + V_{N+1/2}, \Phi^a + \Psi_{N+1/2}, \varphi^a + \psi_{N+1/2})$ satisfies (3.3b)–(3.3c), where (U^a, Φ^a) is the approximate solution given in Lemma 4.2. Furthermore,*

$$\Psi^{\pm}_{N+1/2} = \mathcal{S}_{\theta_n} \Psi^{\pm}_N, \quad \psi_{N+1/2} = (\mathcal{S}_{\theta_n} \Psi^{\pm}_N)|_{x_2=0}, \tag{5.36}$$

$$v^{\pm}_{N+1/2,1} = \mathcal{S}_{\theta_n} v^{\pm}_{N,1}, \tag{5.37}$$

$$\|\mathcal{S}_{\theta_n} V_N - V_{N+1/2}\|_{H^m_\gamma(\Omega_T)} \leq C\epsilon\theta_n^{m+2-\mu} \quad \text{for } m = 2, \ldots, \tilde{\mu} + 3. \tag{5.38}$$

Proof. We divide the proof into four steps.

Step 1. It follows from (5.2)–(5.3) that $(\mathcal{S}_{\theta_n} \Psi^{+}_N)|_{x_2=0} = (\mathcal{S}_{\theta_n} \Psi^{-}_N)|_{x_2=0}$, and hence we can define $\Psi^{\pm}_{N+1/2}$, $\psi_{N+1/2}$, and $v^{\pm}_{N+1/2,1}$ by (5.36)–(5.37). Thanks to (4.14d), constraint (3.3d) holds for $(\Phi^a + \Psi_{N+1/2}, \varphi^a + \psi_{N+1/2})$. As in [12, Proposition 7], we define

$$\rho^{\pm}_{N+1/2} := \mathcal{S}_{\theta_N} \rho^{\pm}_N \mp \frac{1}{2}\mathcal{R}_T \left((\mathcal{S}_{\theta_N} \rho^{+}_N)|_{x_2=0} - (\mathcal{S}_{\theta_N} \rho^{-}_N)|_{x_2=0} \right),$$

$$v^{\pm}_{N+1/2,2} := \partial_t \Psi^{\pm}_{N+1/2} + \left(v^{a\pm}_1 + v^{\pm}_{N+1/2,1} \right) \partial_1 \Psi^{\pm}_{N+1/2} + v^{\pm}_{N+1/2,1} \partial_1 \Phi^{a\pm},$$

so that $[\rho^a + \rho_{N+1/2}] = 0$ on $\partial\Omega$, and constraints (3.3b), (3.3e) hold for $(v^a + v_{N+1/2}, \Phi^a + \Psi_{N+1/2}, \varphi^a + \psi_{N+1/2})$, due to (4.14e), Lemma 5.2, and (4.14b).

Step 2. From (5.4), the trace theorem, and the hypothesis $\mathbf{H}_{N-1}$, we have

$$\begin{aligned}
\|\rho^{+}_N - \rho^{-}_N\|_{H^m_\gamma(\omega_T)} &\leq \|\rho^{+}_{N-1} - \rho^{-}_{N-1}\|_{H^m_\gamma(\omega_T)} + \|\delta\rho^{+}_{N-1} - \delta\rho^{-}_{N-1}\|_{H^m_\gamma(\omega_T)} \\
&\leq \|\mathcal{B}(V_{N-1}, \psi_{N-1})\|_{H^m_\gamma(\omega_T)} + C\|\delta\rho_{N-1}\|_{H^{m+1}_\gamma(\Omega_T)} \\
&\leq C\epsilon\theta_N^{m-\mu-1} \qquad \text{for } m \in [3, \mu].
\end{aligned} \tag{5.39}$$

Then we use Lemma 5.2, (5.2), and (5.39) to obtain

$$\begin{aligned}
&\|\rho_{N+1/2} - \mathcal{S}_{\theta_N} \rho_N\|_{H^m_\gamma(\Omega_T)} \leq C\|\mathcal{S}_{\theta_N} \rho^{+}_N - \mathcal{S}_{\theta_N} \rho^{-}_N\|_{H^m_\gamma(\omega_T)} \\
&\leq \begin{cases} C\|\rho^{+}_N - \rho^{-}_N\|_{H^{m+1}_\gamma(\omega_T)} \leq C\epsilon\theta_N^{m-\mu}, & \text{if } 2 \leq m \leq \mu - 1, \\ C\theta_N^{m+1-\mu}\|\rho^{+}_N - \rho^{-}_N\|_{H^{\mu}_\gamma(\omega_T)} \leq C\epsilon\theta_N^{m-\mu}, & \text{if } m \geq \mu. \end{cases}
\end{aligned} \tag{5.40}$$

Step 3. Using (5.36), we compute

$$\begin{aligned}
v_{N+1/2,2} - \mathcal{S}_{\theta_N} v_{N,2} = {}& \mathcal{S}_{\theta_N} \mathcal{E}(V_N, \Psi_N) + [\partial_t + v^a_1 \partial_1, \mathcal{S}_{\theta_N}]\Psi_N + [\partial_1 \Phi^a, \mathcal{S}_{\theta_N}]v_{N,1} \\
&+ \mathcal{S}_{\theta_N} v_{N,1} \partial_1 \mathcal{S}_{\theta_N} \Psi_N - \mathcal{S}_{\theta_N}(v_{N,1}\partial_1 \Psi_N).
\end{aligned} \tag{5.41}$$

Using the decomposition

$$\mathcal{E}(V_N, \Psi_N) = \mathcal{E}(V_{N-1}, \Psi_{N-1}) + \partial_t(\delta\Psi_{N-1}) + (v_1^a + v_{N-1,1})\partial_1(\delta\Psi_{N-1}) \\ + \delta v_{N-1,1}\partial_1(\Phi^a + \Psi_N) - \delta v_{N-1,2},$$

the Moser-type calculus inequality (2.1), hypothesis (H_{N-1}), and (5.31) leads to

$$\|\mathcal{E}(V_N, \Psi_N)\|_{H^3_\gamma(\Omega_T)} \le C\epsilon\theta_N^{2-\mu},$$

which together with (5.1a) implies

$$\|\mathcal{S}_{\theta_N}\mathcal{E}(V_N, \Psi_N)\|_{H^m_\gamma(\Omega_T)} \le C\epsilon\theta_N^{m-\mu}, \qquad \text{for } m \ge 2. \tag{5.42}$$

The remaining terms on the right-hand side of (5.41) are all commutators. Let us detail the estimate of $[v_1^a\partial_1, \mathcal{S}_{\theta_N}]\Psi_N$. We utilize (2.1), the Sobolev embedding theorem, (5.1a), (5.29), and (5.33) to get

$$\begin{aligned}\|[v_1^a\partial_1, \mathcal{S}_{\theta_N}]\Psi_N\|_{H^m_\gamma(\Omega_T)} &\le \|v_1^a\partial_1(\mathcal{S}_{\theta_N}\Psi_N)\|_{H^m_\gamma(\Omega_T)} + \|\mathcal{S}_{\theta_N}(v_1^a\partial_1\Psi_N)\|_{H^m_\gamma(\Omega_T)} \\ &\le C\|\mathcal{S}_{\theta_N}\Psi_N\|_{H^{m+1}_\gamma(\Omega_T)} + C\|\tilde{v}_1^a\|_{H^m_\gamma(\Omega_T)}\|\mathcal{S}_{\theta_N}\Psi_N\|_{H^3_\gamma(\Omega_T)} \\ &\quad + C\theta_N^{m-\mu}\|v_1^a\partial_1\Psi_N\|_{H^\mu_\gamma(\Omega_T)} \\ &\le C\epsilon\theta_N^{m-\mu+1} \qquad \text{for } \mu+1 \le m \le \tilde{\mu}+4.\end{aligned}$$

If $2 \le m \le \mu$, then it follows from (5.1b) and (5.31)–(5.32) that

$$\begin{aligned}\|[v_1^a\partial_1, \mathcal{S}_{\theta_N}]\Psi_N\|_{H^m_\gamma(\Omega_T)} &\le \|v_1^a\partial_1((\mathcal{S}_{\theta_N} - I)\Psi_N)\|_{H^m_\gamma(\Omega_T)} + \|(I - \mathcal{S}_{\theta_N})(v_1^a\partial_1\Psi_N)\|_{H^m_\gamma(\Omega_T)} \\ &\le C\|(\mathcal{S}_{\theta_N} - I)\Psi_N\|_{H^{m+1}_\gamma(\Omega_T)} \\ &\quad + C\theta_N^{m-\mu}\|v_1^a\partial_1\Psi_N\|_{H^\mu_\gamma(\Omega_T)} \le C\epsilon\theta_N^{m-\mu+1}.\end{aligned}$$

Performing the same analysis to the other commutators in (5.41) and using (5.42), we obtain

$$\|v_{N+1/2,2} - \mathcal{S}_{\theta_N}v_{N,2}\|_{H^m_\gamma(\Omega_T)} \le C\epsilon\theta_N^{m-\mu+1} \quad \text{for } m = 2, \dots, \tilde{\mu}+4. \tag{5.43}$$

Step 4. Let us now construct and estimate of $\boldsymbol{F}_{N+1/2}$ by following the idea of Secchi–Trakhinin [30, Proposition 28]. According to Step 1, we have already specified functions $v_{N+1/2}$ and $\Psi_{N+1/2}$. Then we can take $\boldsymbol{F}_{N+1/2}$ as the unique solution vanishing in the past of linear equations

$$\mathbb{L}_{F_{ij}}(v^a + v_{N+1/2}, \boldsymbol{F}^a + \boldsymbol{F}_{N+1/2}, \Phi^a + \Psi_{N+1/2}) = 0 \quad \text{for } i, j = 1, 2, \tag{5.44}$$

where $\mathbb{L}_{F_{ij}}$ denotes the component of operator $\mathbb{L}$ corresponding to F_{ij}, *i.e.*,

$$\mathbb{L}_{F_{ij}}(v, \boldsymbol{F}, \Phi) := \left(\partial_t^\Phi + v_\ell\partial_\ell^\Phi\right)F_{ij} - F_{\ell j}\partial_\ell^\Phi v_i. \tag{5.45}$$

Since $(v^a + v_{N+1/2}, \Phi^a + \Psi_{N+1/2})$ satisfies (3.3b), equations (5.44) do not need to be supplemented with any boundary condition.

When estimating $\boldsymbol{F}_{N+1/2} - \mathcal{S}_{\theta_N}\boldsymbol{F}_N$, we apply standard energy method. To this end, we obtain from (5.44) that

$$\mathbb{L}_{F_{ij}}(v^a + v_{N+1/2}, \boldsymbol{F}_{N+1/2} - \mathcal{S}_{\theta_N}\boldsymbol{F}_N, \Phi^a + \Psi_{N+1/2}) = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3, \tag{5.46}$$

where

$$\begin{aligned}
\mathcal{H}_1 :=& -\mathbb{L}_{F_{ij}}(v^a + v_{N+1/2}, \boldsymbol{F}^a + \mathcal{S}_{\theta_N}\boldsymbol{F}_N, \Phi^a + \Psi_{N+1/2}) \\
&+ \mathbb{L}_{F_{ij}}(v^a + \mathcal{S}_{\theta_N}v_N, \boldsymbol{F}^a + \mathcal{S}_{\theta_N}\boldsymbol{F}_N, \Phi^a + \mathcal{S}_{\theta_N}\Psi_N), \\
\mathcal{H}_2 :=& -\mathbb{L}_{F_{ij}}(v^a + \mathcal{S}_{\theta_N}v_N, \boldsymbol{F}^a + \mathcal{S}_{\theta_N}\boldsymbol{F}_N, \Phi^a + \mathcal{S}_{\theta_N}\Psi_N) \\
&+ \mathcal{S}_{\theta_N}\mathbb{L}_{F_{ij}}(v^a + v_N, \boldsymbol{F}^a + \boldsymbol{F}_N, \Phi^a + \Psi_N),
\end{aligned}$$

and $\mathcal{H}_3 := -\mathcal{S}_{\theta_N}\mathbb{L}_{F_{ij}}(v^a + v_N, \boldsymbol{F}^a + \boldsymbol{F}_N, \Phi^a + \Psi_N)$. From (5.36), we compute

$$\begin{aligned}
\mathcal{H}_1 =& (\mathcal{S}_{\theta_N}v_{N,\ell} - v_{N+1/2,\ell})\partial_\ell^{\Phi^a+\Psi_{N+1/2}}(F^a_{ij} + \mathcal{S}_{\theta_N}F_{N,ij}) \\
&- (F^a_{\ell j} + \mathcal{S}_{\theta_N}F_{N,\ell j})\partial_\ell^{\Phi^a+\Psi_{N+1/2}}(\mathcal{S}_{\theta_N}v_{N,i} - v_{N+1/2,i}).
\end{aligned}$$

Apply Moser-type calculus inequality (2.1) to the last identity and use the Sobolev embedding, (5.36)–(5.37), (5.43), (5.29), and (5.33) to obtain

$$\begin{aligned}
\|\mathcal{H}_1\|_{H^m_\gamma(\Omega_T)} \le& C\|\mathcal{S}_{\theta_N}v_N - v_{N+1/2}\|_{H^3_\gamma(\Omega_T)}\|(\widetilde{\boldsymbol{F}}^a, \mathcal{S}_{\theta_N}\boldsymbol{F}_N, \widetilde{\Phi}^a, \mathcal{S}_{\theta_N}\Psi_N)\|_{H^{m+1}_\gamma(\Omega_T)} \\
&+ C\|\mathcal{S}_{\theta_N}v_N - v_{N+1/2}\|_{H^{m+1}_\gamma(\Omega_T)} \\
\le& C\epsilon\theta_N^{m-\mu+2} \qquad \text{for } m = 2, \ldots, \tilde{\mu}+3.
\end{aligned} \tag{5.47}$$

Regarding term $\mathcal{H}_2$, we apply the same strategy as for $[v_1^a\partial_1, \mathcal{S}_{\theta_N}]\Psi_N$ in Step 3 to derive

$$\|\mathcal{H}_2\|_{H^m_\gamma(\Omega_T)} \le C\epsilon\theta_N^{m-\mu+2} \qquad \text{for } m = 2, \ldots, \tilde{\mu}+3. \tag{5.48}$$

For term $\mathcal{H}_3$, we obtain from (5.1a), (4.17), and hypothesis $\mathbf{H}_{N-1}$ that

$$\begin{aligned}
&\|\mathcal{S}_{\theta_N}\mathbb{L}_{F_{ij}}(v^a + v_{N-1}, \boldsymbol{F}^a + \boldsymbol{F}_{N-1}, \Phi^a + \Psi_{N-1})\|_{H^m_\gamma(\Omega_T)} \\
&\quad \le C\theta_N^{m-2}\|\mathbb{L}_{F_{ij}}(v^a + v_{N-1}, \boldsymbol{F}^a + \boldsymbol{F}_{N-1}, \Phi^a + \Psi_{N-1})\|_{H^2_\gamma(\Omega_T)} \le C\epsilon\theta_N^{m-\mu-1}
\end{aligned}$$

for $m \ge 2$. Using (5.1a), (2.1), hypothesis $\mathbf{H}_{N-1}$, and (5.31) yields

$$\begin{aligned}
&\|\mathcal{S}_{\theta_N}\big(\mathbb{L}_{F_{ij}}(v^a + v_N, \boldsymbol{F}^a + \boldsymbol{F}_N, \Phi^a + \Psi_N) \\
&\quad - \mathbb{L}_{F_{ij}}(v^a + v_{N-1}, \boldsymbol{F}^a + \boldsymbol{F}_{N-1}, \Phi^a + \Psi_{N-1})\big)\|_{H^m_\gamma(\Omega_T)} \le C\epsilon\theta_N^{m-\mu+2}
\end{aligned}$$

for $m \ge 2$. We combine these two estimates with (5.47)–(5.48) to get

$$\sum_{\ell=1}^{3} \|\mathcal{H}_\ell\|_{H^m_\gamma(\Omega_T)} \le C\epsilon\theta_N^{m-\mu+2}, \qquad \text{for } m = 2, \dots, \tilde{\mu}+3. \tag{5.49}$$

Applying a standard energy argument to equations (5.46) and utilizing estimate (5.49), we infer

$$\|\boldsymbol{F}_{N+1/2} - \mathcal{S}_{\theta_N}\boldsymbol{F}_N\|_{H^m_\gamma(\Omega_T)} \le C\epsilon\theta_N^{m-\mu+2} \qquad \text{for } m = 2, \dots, \tilde{\mu}+3. \tag{5.50}$$

Estimate (5.38) follows from (5.37), (5.40), (5.43), and (5.50). The proof is complete. □

Remark 5.1. We can get constraint (3.2) from (5.29), (5.33), and (5.38) by using the Sobolev embedding theorem. Constraint (3.3a) will be obtained by taking ϵ small enough, while constraint (3.1) will follow through truncating $(V_{N+1/2}, \Psi_{N+1/2}, \psi_{N+1/2})$ by an appropriate cut-off function.

5.4. Estimate of the second substitution and last error terms

The next lemma gives the estimate of the second substitution error terms e_n''', $\tilde{e}_n'''$, and $\hat{e}_n'''$ defined in (5.14), (5.15), and (5.18).

Lemma 5.8. *If $\mu \ge 5$, then there exist $\epsilon > 0$ suitably small and $\theta_0 \ge 1$ large enough such that*

$$(\tilde{e}_n''', \hat{e}_n''') = 0, \qquad \|e_n'''\|_{H^m_\gamma(\Omega_T)} \le C\epsilon^2\theta_n^{\ell_3(m)-1}\Delta_n \quad \text{if } m = 2, \dots, \tilde{\mu}-1,$$

for $n = 0, \dots, N-1$, where $\ell_3(m) := \max\{(m+1-\mu)_+ + 9 - 2\mu, m+6-2\mu\}$.

Proof. From (5.34) and (5.36)–(5.37), we have

$$\begin{aligned}\tilde{e}_n''' &= \mathbb{B}'(U^a + \mathcal{S}_{\theta_n}V_n, \varphi^a + (\mathcal{S}_{\theta_n}\Psi_n)|_{x_2=0})(\delta V_n, \delta\psi_n) \\ &\quad - \mathbb{B}'(U^a + V_{n+1/2}, \varphi^a + (\mathcal{S}_{\theta_n}\Psi_n)|_{x_2=0})(\delta V_n, \delta\psi_n) = 0.\end{aligned}$$

Using (5.35)–(5.37) yields $\hat{e}_n''' = 0$. Thanks to (5.36), the error term e_n''' can be rewritten as

$$\begin{aligned}e_n''' = \int_0^1 &\mathbb{L}''\big(U^a + V_{n+1/2} + \tau(\mathcal{S}_{\theta_n}V_n - V_{n+1/2}),\ \Phi^a + \mathcal{S}_{\theta_n}\Psi_n\big) \\ &\times \big((\delta V_n, \delta\Psi_n), (\mathcal{S}_{\theta_n}V_n - V_{n+1/2}, 0)\big)\,\mathrm{d}\tau.\end{aligned}$$

Apply the Sobolev embedding theorem, (5.29), (5.33), and (5.38) to infer

$$\|(\widetilde{U}^a,\ V_{n+1/2},\ \mathcal{S}_{\theta_n}V_n - V_{n+1/2},\ \widetilde{\Phi}^a,\ \mathcal{S}_{\theta_n}\Psi_n)\|_{W^{1,\infty}(\Omega_T)} \le C\epsilon,$$

so that we can use Proposition 5.4 for ϵ suitably small. Note that from (5.29)–(5.31) and (5.38)

$$\|(\widetilde{U}^a,\ V_{n+1/2},\ \mathcal{S}_{\theta_n}V_n,\ \widetilde{\Phi}^a,\ \mathcal{S}_{\theta_n}\Psi_n)\|_{H^{m+1}_\gamma(\Omega_T)} \le C\epsilon\left(\theta_n^{(m+1-\mu)_+ +1} + \theta_n^{m+3-\mu}\right)$$

for $2 \le m \le \tilde{\mu} - 1$. We use Proposition 5.4, hypothesis $\mathbf{H}_{n-1}$, and (5.38) to get the estimate for term e_n''' and this finishes the proof of the lemma. □

For the last error term (5.16),

$$D_{n+1/2}\delta\Psi_n = \frac{\delta\Psi_n}{\partial_2(\Phi^a + \Psi_{n+1/2})} R_n, \quad \text{with } R_n := \partial_2\mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}),$$

we first notice that

$$|\partial_2(\Phi^{a\pm} + \Psi^{\pm}_{n+1/2})| \ge \frac{1}{2},$$

from (4.14c), (5.36), and (5.33) if ϵ is small enough. Therefore, we arrive at the following lemma analogous to [12, Lemma 8.6] or [4, Lemma 12] and the proof is omitted.

Lemma 5.9. *If $\mu \ge 5$ and $\tilde{\mu} > \mu$, then there exist $\epsilon > 0$ suitably small and $\theta_0 \ge 1$ large enough such that*

$$\|D_{n+1/2}\delta\Psi_n\|_{H^m_\gamma(\Omega_T)} \le C\epsilon^2\theta_n^{\ell_4(m)-1}\Delta_n \quad \text{if } m = 2, \dots, \tilde{\mu} - 1, \tag{5.51}$$

for $n = 0, \dots, N-1$, where

$$\ell_4(m) := \max\{(m+2-\mu)_+ + 8 - 2\mu, (m+1-\mu)_+ + 9 - 2\mu, m + 6 - 2\mu\}.$$

Lemmas 5.5–5.9 lead to the following estimates for e_n, $\tilde{e}_n$, and $\hat{e}_n$ defined in (5.17) and (5.19).

Corollary 5.10. *If $\mu \ge 5$ and $\tilde{\mu} > \mu$, then there exist $\epsilon > 0$ suitably small and $\theta_0 \ge 1$ large enough such that*

$$\|e_n\|_{H^m_\gamma(\Omega_T)} \le C\epsilon^2\theta_n^{\ell_4(m)-1}\Delta_n \quad \text{if } m = 2, \dots, \tilde{\mu} - 1, \tag{5.52}$$

$$\|\hat{e}_n\|_{H^m_\gamma(\Omega_T)} \le C\epsilon^2\theta_n^{\ell_2(m)-1}\Delta_n \quad \text{if } m = 2, \dots, \tilde{\mu} - 1, \tag{5.53}$$

$$\|\tilde{e}_n\|_{H^m_\gamma(\omega_T)} \le C\epsilon^2\theta_n^{\ell_2(m)-1}\Delta_n \quad \text{if } m = 2, \dots, \tilde{\mu} - 2, \tag{5.54}$$

for $n = 0, \dots, N-1$, where $\ell_2(m)$ and $\ell_4(m)$ are defined in Lemma 5.6 and Lemma 5.9, respectively.

5.5. *Proof of Theorem 1.1*

We first show the following lemma for accumulated error terms E_n, $\widetilde{E}_n$, and $\hat{E}_n$ that are given in (5.8) and (5.13).

Lemma 5.11. *If $\mu \ge 7$ and $\tilde{\mu} = \mu + 3$, then there exist $\epsilon > 0$ suitably small and $\theta_0 \ge 1$ large enough such that*

$$\|E_N\|_{H_\gamma^{\mu+2}(\Omega_T)} \le C\epsilon^2\theta_N, \tag{5.55}$$

$$\|\widetilde{E}_N\|_{H_\gamma^{\mu+1}(\omega_T)} + \|\widehat{E}_N\|_{H_\gamma^{\mu+1}(\Omega_T)} \le C\epsilon^2. \tag{5.56}$$

Proof. The proof follows closely [4,12]. First we note that $\ell_4(\mu+2) \le 1$ when $\mu \ge 7$. From (5.52), one has

$$\|E_N\|_{H_\gamma^{\mu+2}(\Omega_T)} \le \sum_{n=0}^{N-1} \|e_n\|_{H_\gamma^{\mu+2}(\Omega_T)} \le \sum_{n=0}^{N-1} C\epsilon^2\Delta_n \le C\epsilon^2\theta_N,$$

for $\mu \ge 7$ and $\mu+2 \le \tilde{\mu}-1$. Since $\ell_2(\mu+1) = 6-\mu \le -1$ for $\mu \ge 7$ and $\mu+1 \le \tilde{\mu}-2$, from (5.53)–(5.54), we have

$$\begin{aligned}\|\widetilde{E}_N\|_{H_\gamma^{\mu+1}(\omega_T)} + \|\widehat{E}_N\|_{H_\gamma^{\mu+1}(\Omega_T)} &\le \sum_{n=0}^{N-1} \left\{\|\tilde{e}_n\|_{H_\gamma^{\mu+1}(\omega_T)} + \|\hat{e}_n\|_{H_\gamma^{\mu+1}(\Omega_T)}\right\} \\ &\le \sum_{n=0}^{N-1} C\epsilon^2\theta_n^{-2}\Delta_n \le C\epsilon^2,\end{aligned}$$

where we have used (5.9) and (5.30) to derive the last inequality. The minimal possible $\tilde{\mu}$ is $\mu+3$. This completes the proof of the lemma. □

Based on the lemma above, we have the estimates for f_N, g_N, and $h_N^{\pm}$.

Lemma 5.12. *If $\mu \ge 7$ and $\tilde{\mu} = \mu+3$, then there exist $\epsilon > 0$ suitably small and $\theta_0 \ge 1$ large enough such that*

$$\|f_N\|_{H_\gamma^m(\Omega_T)} \le C\Delta_N\left\{\theta_N^{m-\mu-2}\left(\|f^a\|_{H_\gamma^{\mu+1}(\Omega_T)} + \epsilon^2\right) + \epsilon^2\theta_N^{\ell_4(m)-1}\right\}, \tag{5.57}$$

$$\|g_N\|_{H_\gamma^m(\omega_T)} \le C\epsilon^2\Delta_N\left(\theta_N^{m-\mu-2} + \theta_N^{\ell_2(m)-1}\right), \tag{5.58}$$

for $m = 2, \dots, \tilde{\mu}+1$, and

$$\|h_N^{\pm}\|_{H_\gamma^m(\Omega_T)} \le C\epsilon^2\Delta_N\left(\theta_N^{m-\mu-2} + \theta_N^{\ell_2(m)-1}\right) \quad \textit{for } m = 2, \dots, \tilde{\mu}. \tag{5.59}$$

Proof. Since $\theta_{N-1} \le \theta_N \le \sqrt{2}\theta_{N-1}$ and $\Delta_{N-1} \le 3\Delta_N$, from (5.1a), (5.1c), (5.52), and (5.55), we obtain

$$\begin{aligned}\|f_N\|_{H_\gamma^m(\Omega_T)} &\le \|(\mathcal{S}_{\theta_N} - \mathcal{S}_{\theta_{N-1}})f^a - (\mathcal{S}_{\theta_N} - \mathcal{S}_{\theta_{N-1}})E_{N-1} - \mathcal{S}_{\theta_N}e_{N-1}\|_{H_\gamma^m(\Omega_T)} \\ &\le C\Delta_N\theta_N^{m-\mu-2}\left(\|f^a\|_{H_\gamma^{\mu+1}(\Omega_T)} + \theta_N^{-1}\|E_{N-1}\|_{H_\gamma^{\mu+2}(\Omega_T)}\right) + \|\mathcal{S}_{\theta_N}e_{N-1}\|_{H_\gamma^m(\Omega_T)} \\ &\le C\Delta_N\left\{\theta_N^{m-\mu-2}(\|f^a\|_{H_\gamma^{\mu+1}(\Omega_T)} + \epsilon^2) + \epsilon^2\theta_N^{\ell_4(m)-1}\right\}.\end{aligned}$$

By using (5.54) and (5.56), we get

$$\begin{aligned}\|g_N\|_{H^m_\gamma(\omega_T)} &\le \|(\mathcal{S}_{\theta_N}-\mathcal{S}_{\theta_{N-1}})\widetilde{E}_{N-1}-\mathcal{S}_{\theta_N}\tilde{e}_{N-1}\|_{H^m_\gamma(\Omega_T)}\\ &\le C\Delta_N\theta_N^{m-\mu-2}\|\widetilde{E}_{N-1}\|_{H^{\mu+1}_\gamma(\Omega_T)}+\|\mathcal{S}_{\theta_N}\tilde{e}_{N-1}\|_{H^m_\gamma(\Omega_T)}\\ &\le C\epsilon^2\Delta_N\big(\theta_N^{m-\mu-2}+\theta_N^{\ell_2(m)-1}\big).\end{aligned}$$

Similarly we can deduce (5.59) for $h_N^{\pm}$ from (5.53) and (5.56). The proof is complete. □

In the next lemma, we obtain the estimate of differences δV_N, $\delta\Psi_N$, and $\delta\psi_N$ with the aid of tame estimate (3.20). See [12, Lemma 16] or [4, Lemma 8.10] for the proof.

Lemma 5.13. *Let $\mu\ge 7$ and $\tilde{\mu}=\mu+3$. If $\epsilon>0$ and $\|f^a\|_{H^{\mu+1}_\gamma(\Omega_T)}/\epsilon$ are suitably small and $\theta_0\ge 1$ is large enough, then*

$$\|(\delta V_N,\delta\Psi_N)\|_{H^m_\gamma(\Omega_T)}+\|\delta\psi_N\|_{H^{m+1}_\gamma(\omega_T)}\le\epsilon\theta_N^{m-\mu-1}\Delta_N\quad \textit{for } m=2,\dots,\tilde{\mu}. \tag{5.60}$$

Lemma 5.13 implies the first part of the hypothesis $\mathbf{H}_N$. The following lemma provides us the other parts of $\mathbf{H}_N$.

Lemma 5.14. *Let $\mu\ge 7$ and $\tilde{\mu}=\mu+3$. If $\epsilon>0$ and $\|f^a\|_{H^{\mu+1}_\gamma(\Omega_T)}/\epsilon$ are suitably small and $\theta_0\ge 1$ is large enough, then*

$$\|\mathcal{L}(V_N,\Psi_N)-f^a\|_{H^m_\gamma(\Omega_T)}\le 2\epsilon\theta_N^{m-\mu-1}\quad \textit{for } m=2,\dots,\tilde{\mu}-1, \tag{5.61}$$

$$\|\mathcal{B}(V_N,\psi_N)\|_{H^m_\gamma(\omega_T)}\le\epsilon\theta_N^{m-\mu-1}\quad \textit{for } m=3,\dots,\mu, \tag{5.62}$$

$$\|\mathcal{E}(V_N,\Psi_N)\|_{H^3_\gamma(\Omega_T)}\le\epsilon\theta_N^{2-\mu}. \tag{5.63}$$

We refer to [12, Lemmas 17–18] or [4, Lemma 8.11] for the proof of Lemma 5.14. Let us take $\mu\ge 7$, $\tilde{\mu}=\mu+3$, $\epsilon>0$ and $\|f^a\|_{H^{\mu+1}_\gamma(\Omega_T)}/\epsilon$ suitably small, and $\theta_0\ge 1$ large enough, so that the assumptions of Lemmas 5.13–5.14 are satisfied, from which we obtain the inductive hypothesis $\mathbf{H}_N$. Then, as in [12, Lemma 19] or [4, Lemma 8.12], we can prove that the hypothesis $\mathbf{H}_0$ is true.

Lemma 5.15. *If $\|f^a\|_{H^{\mu+1}_\gamma(\Omega_T)}/\epsilon$ is small enough, then the hypothesis $\mathbf{H}_0$ holds.*

We are now ready to complete the proof of Theorem 1.1. Our proof follows closely [4,12] and is still presented here for the sake of completeness.

Proof of Theorem 1.1. For $\tilde{\mu}:=s_0-4\ge 10$ and $\mu:=\tilde{\mu}-3\ge 7$, the initial data $(U_0^{\pm},\varphi_0)$ under the assumptions of Theorem 1.1 are compatible up to order $s_0=\tilde{\mu}+4$. If $(\tilde{U}_0^{\pm},\varphi_0)$ is sufficiently small in $H^{s_0+1/2}(\mathbb{R}^2_+)\times H^{s_0+1}(\mathbb{R})$ with $\widetilde{U}_0^{\pm}:=U_0^{\pm}-\overline{U}^{\pm}$, then the assumption (5.29) and all the requirements of Lemmas 5.13–5.15 are satisfied owing to (4.16) and (4.19), and hence $\mathbf{H}_N$ holds for all $N\in\mathbb{N}$. Thus, from

$$\sum_{n=0}^{\infty}\left(\|(\delta V_n, \delta\Psi_n)\|_{H^m_\gamma(\Omega_T)} + \|\delta\psi_n\|_{H^{m+1}_\gamma(\omega_T)}\right) \le C\sum_{n=0}^{\infty}\theta_n^{m-\mu-2} < \infty, \quad 3 \le m \le \mu - 1,$$

we conclude that (V_n, Ψ_n) converges to some (V, Ψ) in $H^{\mu-1}_\gamma(\Omega_T)$, and ψ_n converges to some ψ in $H^\mu_\gamma(\Omega_T)$. Then we take the limit in (5.61)–(5.62) for $m = \mu - 1 = s_0 - 8$, and in (5.63), and obtain that (V, Ψ) solves (4.20). As a consequence, $(U, \Phi) = (U^a + V, \Phi^a + \Psi)$ is a solution to (1.11)–(1.13) in Ω_T^+. The proof of Theorem 1.1 is complete. □

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